Minimal and CMC hypersurfaces of classical or diffused type: convergence properties under Morse index bounds

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Declaration

I, Myles Workman, confirm that the work presented in my thesis is my own. Where information has been derived from other sources, I confirm that this has been indicated in the thesis.

Abstract

In this thesis we investigate minimal and constant mean curvature (CMC) hypersurfaces as they arise (and converge), in two limiting procedures, of distinct type, and under a certain control for the (Morse) index.

In the first part of this thesis, which is joint work with C. Bellettini, we investigate CMC hypersurfaces which arise as the limit interface of sequences of particular (min-max, hence with index at most 1) solutions to the inhomogeneous Allen–Cahn equation. We prove that on a compact Riemannian manifold of dimension 3 or higher, with positive Ricci curvature, the Allen–Cahn min-max scheme of Bellettini–Wickramasekera [14], with prescribing function taken to be a non-zero constant λ, produces an embedded hypersurface of constant mean curvature λ. More precisely, we prove that the limit interface arising from said min-max contains no even-multiplicity minimal hypersurface and no quasi-embedded points (both of these occurrences are in principle possible in the conclusions of the aforementioned work by Bellettini–Wickramasekera).

In the second part of this thesis, we investigate sequences of bubble converging minimal hypersurfaces, or CMC hypersurfaces, in compact Riemannian manifolds without boundary, of dimension 4, 5, 6 or 7, and prove upper semi-continuity of index plus nullity, for such bubble converging sequences. This complements the previously known lower semi-continuity results for the index. The strategy of the proof is to analyse an appropriate weighted eigenvalue problem along the bubble converging sequence of hypersurfaces.

Impact Statement

This thesis lies in the area of geometric analysis, and contains results and work on minimal and constant mean curvature (CMC) hypersurfaces in Riemannian manifolds. These objects are variational in nature, in that they arise as critical points to appropriately chosen area-type functionals. Key developments in understanding their properties (regularity, compactness, index etc.) as such critical points has led to applications in many mathematical areas including Riemannian and differential geometry, and general relativity. Moreover, as these area-type functionals are some of the simplest, yet mathematically rich, non-linear geometric functionals, the study of their critical points has had immnese impacts on the calculus of variations and non-linear PDE theory.

In this thesis we produce several results concerning this variational theory for these hypersurfaces. More specifically, in Chapter 2 we further develop the min-max theory for CMC hypersurfaces, which culminates in a previously unknown existence result for embedded CMC hypersurfaces. Then, in Chapter 3 we produce results concerning the index of minimal and CMC hypersurfaces, in particular the behaviour of this variational property along certain degenerating sequences of such hypersurfaces.

Moreover, in Chapter 2 we study these hypersurfaces through the phase transition framework. As such we further develop this phase transition theory, in particular the study of min-max solutions to the inhomogeneous Allen–Cahn equation. The theory of phase transitions is widely used throughout mathematics and the natural and physical sciences, and developments of this theory have and will continue to have impacts on the wide range of areas that this theory is used in. These areas include, but are not limited to: theoretical physics, material science, mathematical biology and computer imaging.

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Introduction

Problems arising in geometry and the physical sciences are often variational in nature, and thus may be studied through analysing critical points to (often non-linear) functionals. One such class of non-linear geometric functionals is that of area-type functionals, which have minimal and constant mean curvature (CMC) hypersurfaces as examples of critical points. It is the properties of such variational hypersurfaces that are studied in this thesis.

These area-type functionals have attracted a significant portion of research attention within geometric analysis and the calculus of variations. The development of the theory of such functionals and their critical points has had significant impacts on a wide variety of mathematics and the physical sciences, including but not limited to: geometry, general relativity and material science. Moreover, as these functionals are some of the simplest, yet mathematically rich, non-linear geometric functionals that one can think up, the techniques and ideas generated through this direction of research have also had substantial influence within non-linear PDE theory and the calculus of variations.

When studying such variational problems it is of crucial importance to be able to take appropriate limits of sequences of critical points, and analyse how their properties may change under such convergence. As such, the regularity and compactness theory accompanying the variational problem is key to any developments and applications. For instance, the compactness and regularity theory for stable minimal hypersurfaces of Schoen–Simon–Yau [52] and Schoen–Simon [51] (see also the recent proof by Bellettini [10]), have become indispensable tools within geometric analysis, differential geometry and general relativity, and are crucial to the now classical existence theory of such hypersurfaces.

Theorem 1. *(Almgren [2], Pitts [45], Schoen–Simon [51]) For any compact $n+1$ dimensional Riemannian manifold (N, g), without boundary, there exists a smooth, embedded minimal hypersurface M, and,*

- *M is closed when $2 \leq n \leq 6$,*
- *$\overline{M} \setminus M$ consists of finitely many points when $n = 7$,*
- *$dim_{\mathcal{H}}(\overline{M} \setminus M) \leq n - 7$, when $n \geq 8$.*

As seen by the Simons cone ([55, Theorem 6.1.2]), the regularity of these hypersurfaces is in general optimal.

Further advancements in the regularity and compactness theory for stable minimal hypersurfaces by Wickramasekera [67], have led to a new, streamlined proof of Theorem 1 by Guaraco [30] via the theory of phase transitions (building on the previous work of Hutchinson–Tonegawa [33], Tonegawa

[59] and Tonegawa–Wickramasekera [60]). The result in Theorem 1 has also recently been extended (Marques–Neves [39], Irie–Marques–Neves [34], Chodosh–Mantoulidis [20], and Song [56]) to prove not just the existence of one, but in fact infinitely many distinct minimal hypersurfaces in every compact Riemannian manifold without boundary of dimension $3 \leq n+1 \leq 7$.

While the existence theory (and corresponding regularity and compactness theory) for minimal hypersurfaces is by now classical, the corresponding theory for CMC hypersurfaces is more recent.

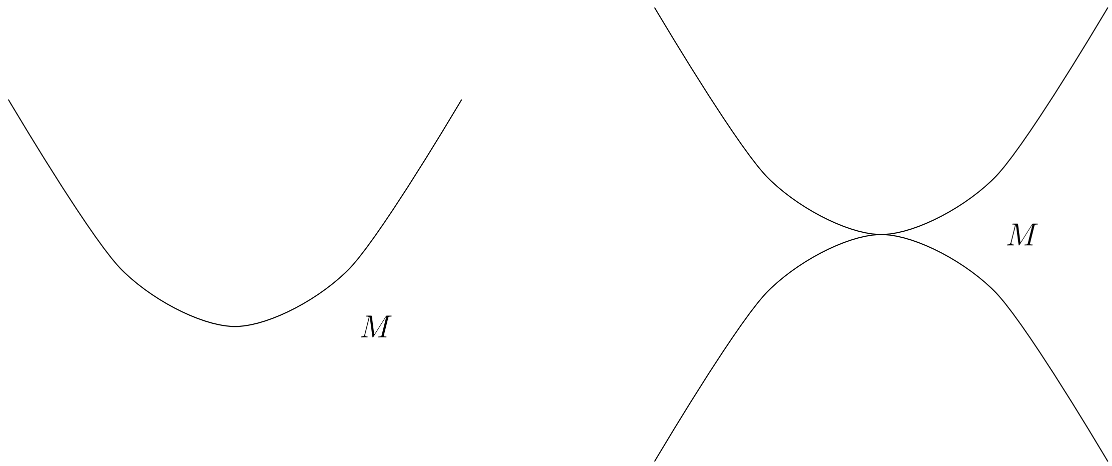
Theorem 2. *(Zhou–Zhu [71], Bellettini–Wickramasekera [12, 14], Dey [24]) For any $\lambda \in \mathbb{R} \setminus \{0\}$, and any compact $n+1$ dimensional Riemannian manifold (N, g), without boundary, there exists a smooth, two-sided, quasi-embedded hypersurface M, with constant mean curvature λ, and*

- *M is closed when $2 \leq n \leq 6$,*
- *$\overline{M} \setminus M$ consists of finitely many points when $n = 7$,*
- *$dim_{\mathcal{H}}(\overline{M} \setminus M) \leq n - 7$, when $n \geq 8$.*

Again the dimensional restrictions on $\overline{M} \setminus M$ are in general optimal.

Here we recall that a quasi-embedded hypersurface is a smooth immersion, with any self-intersections being tangential, and with the local structure being that that of two smooth embedded n-dimensional disks lying on opposite sides of each other (see Figure 1 for heuristic a picture, and Definition 1 for a detailed definition). As such, points in a quasi-embedded hypersurface can be characterised into two disjoint sets. Those points where locally M is a properly embedded hypersurface (which we call embedded points) and those where M is not (which we call non-embedded points).

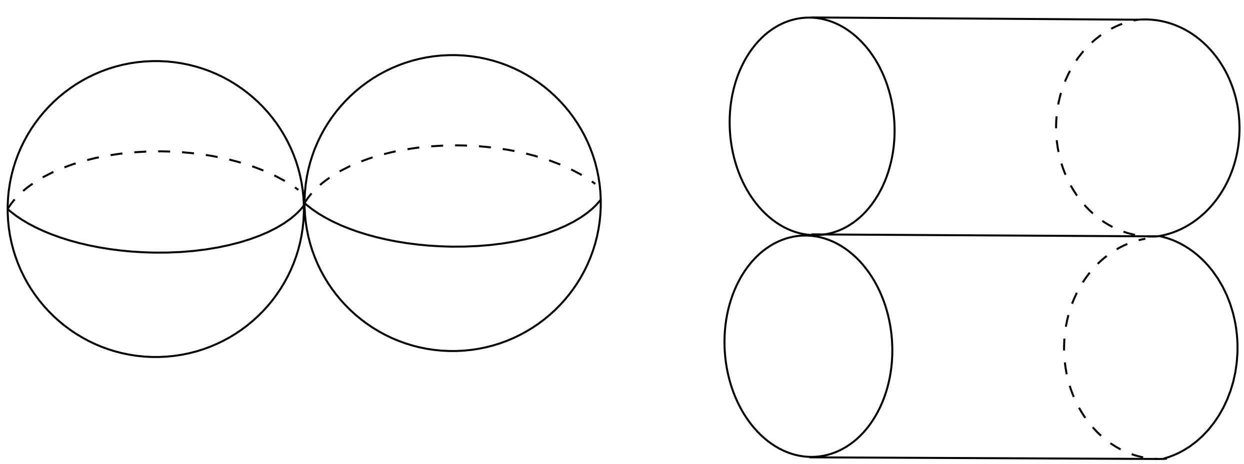
For the case of minimal hypersurfaces, $\lambda = 0$, by the one-sided maximum principle such non-embedded points cannot exist. However, for $\lambda \in \mathbb{R} \setminus \{0\}$, in general such non-embedded points need to be accounted for by considering simple examples like that of two touching spheres or two touching cylinders (Figure 2). Thus the presence of such non-embedded points is a characteristic that is unique to the theory of λ-CMC hypersurfaces for $\lambda \neq 0$, and is a major difference between these two theories with genuine technical consequences. For example, at such points our immersed hypersurface is no longer smoothly embedded, and thus such points can be viewed as a type of geometric singularity for the hypersurface. Furthermore, the size of the set of non-embedded points may be relatively large (potentially of Hausdorff dimension $n - 1$, as demonstrated by the two touching cylinders). As such this set can affect the ambient variational properties (for example index) of these hypersurfaces, as these points cannot simply be ignored through capacity arguments. To develop the theory and applications of CMC hypersurfaces it is thus necessary to develop our understanding of this set of non-embedded points. For example, some basic questions one may ask are:



(a) Local picture about an embedded point of M

(b) Local picture about a non-embedded point of M

Figure 1: The two possible local pictures about a point in a quasi-embedded hypersurface M



(a) Two spheres touching at a point

(b) Two cylinders touching along a line

Figure 2: Examples of quasi-embedded CMC hypersurfaces with non-embedded points

– What geometric and variational properties of the hypersurface and ambient space can affect how big this set can be?

– What local and global structure can this set have?

– In which scenarios can we rule out the presence of this set altogether?

In the first part of this thesis (Chapter 2), which is joint work with Bellettini (and the published version of this work may be found here [15]), we manage to answer such questions (and in fact rule out the presence of such non-embedded points) in a particular class of CMC hypersurfaces. As a direct result we obtain the following existence theory for embedded CMC hypersurfaces.

Theorem 3. *For any $\lambda \in \mathbb{R}\backslash\{0\}$, and any compact $n+1$ dimensional Riemannian manifold (N, g), without boundary, and with positive Ricci curvature, there exists a smooth, two-sided, embedded hypersurface M, with constant mean curvature λ, and*

- *M is closed when $2 \leq n \leq 6$,*
- *$\overline{M} \setminus M$ consists of finitely many points when $n = 7$,*
- *$dim_{\mathcal{H}}(\overline{M} \setminus M) \leq n - 7$, when $n \geq 8$.*

This is a new result and one of the first of its kind where we manage to prove existence of embedded CMC hypersurfaces for all values $\lambda \in \mathbb{R} \setminus \{0\}$, in a fixed class of Riemannian metrics. As a direct result of ruling out such non-embedded points we are also able to conclude that this particular class of λ-CMC hypersurfaces considered have Morse index equal to 1.

However, in general, we cannot rule out the existence of such non-embedded points. Even when considering the class of fully embedded CMC hypersurfaces it is necessary to study such non-embedded points, as they can arise in limits of objects within this class. For example, we may consider two disjoint spheres or cylinders coming together and smoothly touching (as in Figure 2), or even worse, a sequence of unduloids with singular convergence leading to a string of touching spheres (Figure 3). In this singular convergence, information (index, genus, total curvature etc.) may be lost to the touching point. In low dimensions ($2 \leq n \leq 6$), by curvature estimates one can see that non-embedded points arising through such singular convergence, must be points of index concentration along the sequence. Using this, Bourni–Sharp–Tinaglia [16] carried out a bubble analysis of sequences of CMC hypersurfaces arising as boundaries (in these low dimensions), and concluded that such singular points of convergence must be modelled on catenoids. Moreover, through this bubble analysis, Bourni–Sharp–Tinaglia were able to prove a quantisation result for the total curvature, and a lower semicontinuity result for the index. Essentially these results say that any loss in total curvature when taking this singular limit can be exactly accounted for by counting the number of catenoid bubbles, and also some of the index lost through this limit may

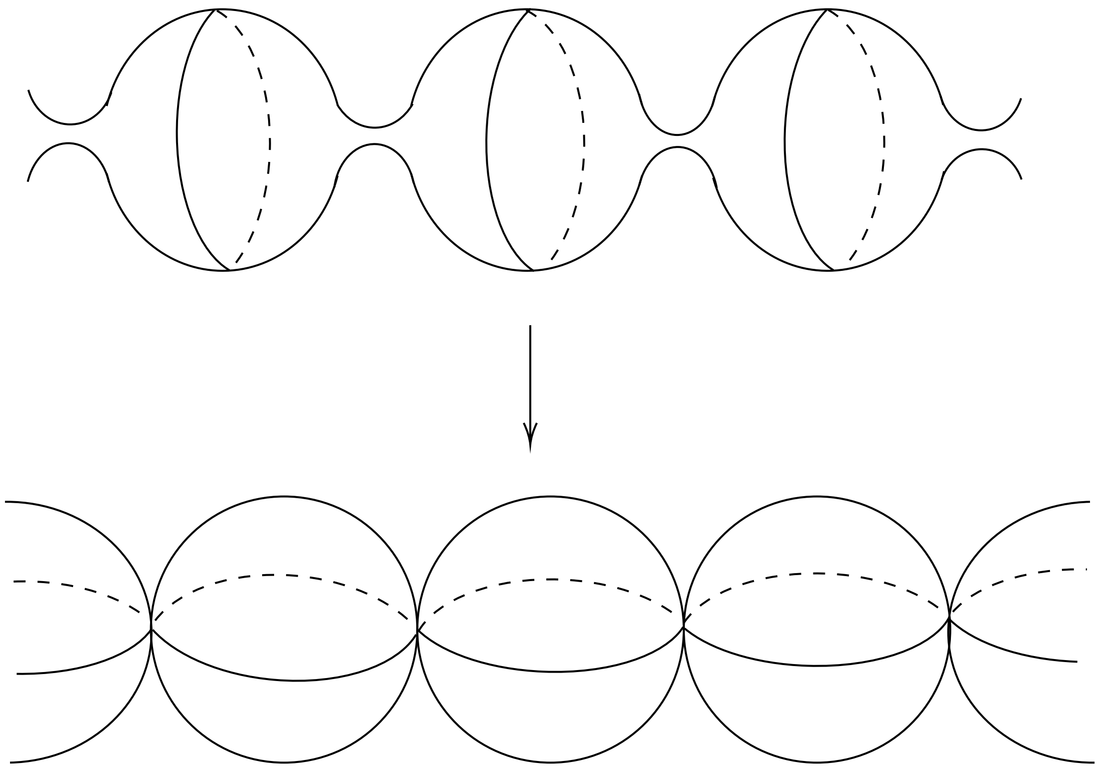


Figure 3: Unduloids exhibiting singular convergence to a line of touching spheres. Necks in the Unduloids shrink leaving the touching points between the spheres in the singular convergence.

also be accounted for by counting the index of these catenoid bubbles. However one may ask if such a bubble analysis is sharp in accounting for all the index along such a singular limit to a non-embedded point. It is this question we approach in Chapter 3 of this thesis (the pre-print of this work can be found here [68]), and answer, in dimensions $3 \leq n \leq 6$, by deducing an upper semicontinuity result for the index plus nullity (Theorem 8). In fact the method we use is rather general and equally holds for proving an appropriate upper semicontinuity of index plus nullity result for suitable sequences of bubble converging minimal hypersurfaces, as well as CMC hypersurfaces which do not necessarily arise as boundaries (again in dimension $3 \leq n \leq 6$).

A Guide to Reading this Thesis

We make a brief note on how this thesis should be read. Chapter 1 begins with some general preliminaries on minimal and CMC hypersurfaces (Section 1.1), and varifolds and Caccioppoli sets (Section 1.2). The main purpose of these preliminary sections is to set notation and definitions that will be used in both chapters of this thesis. For a detailed discussion and introduction to these areas we refer the reader to the relevant references listed in these sections. After this preliminary chapter, we recommend that the reader approaches both of the remaining Chapters (2 and 3) independently of each other. In fact both contain their own independent introductions which describe the projects in much more detail than we have went into in the above. Moreover there are some slight subtle changes in notation between both Chapters 2 and 3. These changes are to replicate the notation of each chapters own independent background references.

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Chapter 1

General Preliminaries

1.1 Minimal and CMC Hypersurfaces

Throughout this thesis (N, g) will denote a compact Riemannian manifold without boundary of dimension $n + 1 \geq 3$. We denote the open geodesic ball in N, of radius $r > 0$, about point $p \in N$ by $B_r^N(p)$. In Euclidean space $\mathbb{R}^m$, we denote the open ball of radius $r > 0$ (with respect to the standard Euclidean metric), centred at $p \in \mathbb{R}^m$ by $B_r^m(p)$. Consider an open set $\Omega \subset N$, and a smooth hypersurface $M \subset N$, such that in Ω, M is properly embedded. For a point $x \in M \cap \Omega$, we define the second fundamental form of M, at x (which is a vector valued bilinear form) by,

$$\begin{aligned} A_M(x)\colon T_xM \times T_xM &\rightarrow (T_xM)^\perp, \\ (X, Y) &\mapsto g(\nabla_X Y, \nu)\nu, \end{aligned}$$

where ν is a choice of unit normal to M at x. We may then define the mean curvature vector $\vec{H}$, at $x \in M \cap \Omega$, by,

$$\vec{H}(x) \coloneqq \mathrm{tr}_{T_xM}(A_M(x)(\cdot, \cdot)).$$

If $\vec{H}(x) = \vec{0}$, for all $x \in M \cap \Omega$, we say that M is a minimal hypersurface in Ω. For a constant $\lambda \in \mathbb{R}$, if $g(\vec{H}(x), \vec{H}(x)) = \lambda^2$, for all $x \in M \cap \Omega$, then we say that M is a constant mean curvature hypersurface, of mean curvature λ (λ-CMC) in Ω. Note that M being minimal in Ω ($\lambda = 0$) is a special case of M being a CMC hypersurface. Secondly note that if $\lambda \neq 0$, then there exists a choice of unit normal ν, for $M \cap \Omega$, such that, $\vec{H} = \lambda\nu$, on $M \cap \Omega$. Such hypersurfaces (minimal and CMC) may also be characterised as being critical points of certain area-type functionals. We now discuss this below.

Define the set $\mathfrak{S}(N)$ to be the set of smooth, properly embedded hypersurfaces of N. On this set we consider the area functional (with respect to the metric g),

$$\begin{aligned}\mathcal{A}_g(M)\colon \mathfrak{S}(N) &\to \mathbb{R},\\ M &\mapsto \mathcal{H}^n_g(M),\end{aligned}$$

where $\mathcal{H}^k_g$ denotes the k-dimensional Hausdorff measure on N, with respect to the metric g. Often we will drop the subscript g when the implied ambient metric is clear. Letting $\Gamma(TN)$ denote the space of smooth vector fields on N, and taking $X \in \Gamma(TN)$, then for small $\varepsilon > 0$, we may define a flow,

$$\Phi\colon N \times (-\varepsilon, \varepsilon) \to N,$$

such that

$$\frac{\partial}{\partial t}\Phi(\,\cdot\,, 0) = X(\,\cdot\,).$$

We define the first variation of $\mathcal{A}_g$, at M, in direction X by

$$\delta\mathcal{A}_g(M)(X) \coloneqq \frac{d}{dt}_{|t=0}\mathcal{A}_g(\Phi(M,t)),$$

and say that $\delta\mathcal{A}_g(M) = 0$ (M is a critical point of $\mathcal{A}_g$) if $\delta\mathcal{A}_g(M)(X) = 0$, for all $X \in \Gamma(TN)$. We may localise this definition to an open set $\Omega \subset N$, by only considering the subset of vector fields which have compact support in Ω, which we denote,

$$\Gamma_c(T\Omega) = \{X \in \Gamma(TN)\colon \operatorname{spt} X \subset \Omega\}.$$

In particular, we say that M is a critical point of $\mathcal{A}_g$ in Ω, if $\delta\mathcal{A}_g(M)(X) = 0$, for all $X \in \Gamma_c(T\Omega)$. Then, as shown in [22, Chapter 1, Section 1.3], if $\Omega \subset\subset N \setminus (\overline{M} \setminus M)$, then M is a critical point of $\mathcal{A}_g$ in Ω, if and only if, and M is a minimal hypersurface in Ω.

As in many problems in the calculus of variations (and even in basic calculus) one may look to study critical points to a functional by analysing the second derivative of the functional at that point. In the setting considered here, we refer to this second derivative as the second variation of $\mathcal{A}_g$ at M, which we define (in direction X) by,

$$\delta^2\mathcal{A}_g(M)(X) \coloneqq \frac{d^2}{dt^2}_{|t=0}\mathcal{A}_g(\Phi(M,t)).$$

For open sets $\Omega \subset\subset N \setminus (\overline{M} \setminus M)$, it is sufficient to only consider the following vector subspace of $\Gamma(TN)$, with an appropriate equivalence relation,

$$\begin{aligned}\Gamma_c(T(M\cap\Omega)^{\perp}) \;\coloneqq\; \{[X]\colon X \in \Gamma_c(T\Omega), &\text{ and for all } x \in M,\ X(x) \perp T_xM,\\ &\text{and } [X] = [Y] \text{ if and only if } X = Y \text{ on } M\}\end{aligned}$$

Then for $X \in \Gamma_c(T(M\cap\Omega)^\perp)$, we have that ([22, Chapter 1, Section 8.1])

$$\delta^2\mathcal{A}_g(M)(X) = -\int_M g(X, LX),$$

where L is a second order elliptic differential operator, which we call the *stability operator.* For a point $x \in M$, taking $\{E_1, \dots, E_n\}$ to be an orthonormal basis of T_xM, we may define the operator at x by

$$LX = \Delta_M^\perp X + \sum_{i,j=1}^n g(A_M(E_i,E_j),X)\,A_M(E_i,E_j) + \operatorname{Tr} R_N(\,\cdot\,,X)\,\cdot$$

where $\Delta_M^\perp$ is the normal Laplacian to M, A_M is the second fundamental form of M, and R_N is the Riemann curvature tensor of N.

We now consider the larger class of vector fields, supported in Ω, with zero Dirchlet boundary condition on $\partial\Omega$, which we denote $\Gamma_0(T\Omega)$. In particular, this is the set of vector fields on N, which are smooth in Ω, equal to 0 on $N\setminus\overline{\Omega}$, and continuous across $\partial\Omega$. Then, similarly to before we define,

$$\begin{aligned}\Gamma_0(T(M\cap\Omega)^\perp) \;&:=\; \{[X]\colon X\in\Gamma_0(T\Omega),\text{ and for all } x\in M,\ X(x)\perp T_xM,\\ &\qquad\qquad \text{and } [X]=[Y] \text{ if and only if } X=Y \text{ on } M\}\end{aligned}$$

We say that $\kappa \in \mathbb{R}$ is an eigenvalue of L, in Ω , if there exists an $X \in \Gamma_0(T(M\cap\Omega)^\perp)$, such that $X \neq 0$, and

$$LX + \kappa X = 0,$$

in $M\cap\Omega$. By standard theory [28, Section 8.12], the set of eigenvalues for L in Ω, (counted with the multiplicity given by the dimension of the linear subspace of all eigensections with such eingenvalue) is a countable, discrete set which is bounded from below, and converges to $+\infty$,

$$\kappa_1 \le \kappa_2 \le \cdots \nearrow +\infty$$

Then we define the *Morse index* of M in Ω ($\operatorname{ind}_\Omega(M)$), to be the number of negative eigenvalues of L, counted with the above notion of multiplicity. Equivalently the Morse index in Ω may also be defined by

$$\operatorname{ind}_\Omega(M) = \sup\left\{\dim\Pi\colon \text{for all } X\in\Pi\le\Gamma_0(T(M\cap\Omega)^\perp),\ \int_M g(X,LX) > 0\right\}$$

The *nullity* of M in Ω ($\operatorname{nul}_\Omega(M)$), is defined to be the number of zero eigenvalues, counted with multiplicity. We define M to be stable in Ω, if $\operatorname{ind}_\Omega(M) = 0$.

If M has trivial normal bundle in Ω, then we may identify $\Gamma_0(T(M\cap\Omega)^\perp)$ with $C_0^\infty(M\cap\Omega)$

in the following way; let ν denote a continuous choice of unit normal for $M \cap \Omega$, and for $X \in \Gamma_0(T(M\cap\Omega)^\perp)$, we may define the function,

$$f_X(x) = g(X(x), \nu(x)) \in C_0^\infty(M \cap \Omega),$$

and for $f \in C_0^\infty(M\cap\Omega)$, by a standard extension argument there exists an $X_f \in \Gamma_0(T(M\cap\Omega)^\perp)$, such that

$$f(x) = g(X_f(x), \nu(x)).$$

Then, we define the bilinear form

$$B_L[f,f] \coloneqq \delta^2\mathcal{A}_g(M)(X_f) = \int_M g(\nabla^M f, \nabla^M f) - (|A_M|^2 + \mathrm{Ric}_g(\nu,\nu))f^2\, d\mathcal{H}^n,$$

which can be extended to be a bilinear form on $W_0^{1,2}(M\cap\Omega)$. Thus, integrating this expression by parts, we see that studying the operator L on $\Gamma_0(T(M\cap\Omega)^\perp)$, is equivalent to studying the operator (which we also call the stability operator and denote by L),

$$L = \Delta_M + |A_M|^2 + \mathrm{Ric}_g(\nu,\nu),$$

on the function space $C_0^\infty(M\cap\Omega)$. Here Δ_M is the Laplace–Beltrami operator on $M\cap\Omega$ (with respect to the ambient metric g restricted to $M\cap\Omega$), $|A_M|^2$ denotes the square of the Hilbert–Schmidt norm of A_M, and Ric_g denotes the ambient Ricci curvature of (N,g).

Thus if $M\cap\Omega$ has trivial normal bundle, then the Morse index of M in Ω may be defined as the number of negative eigenvalues (counted with multiplicity) of the operator L on $C_0^\infty(M\cap\Omega)$, or

$$\mathrm{ind}_\Omega\,(M) = \sup\{\dim\Pi\colon \text{for all } f\in\Pi \le W_0^{1,2}(M\cap\Omega),\ B_L[f,f]<0\}.$$

Note that throughout the thesis (especially in Chapter 3) we will often drop the sub- and super-scripts on Δ_M and ∇^M when it is clear we are working on the hypersurface M.

Alternatively to $\mathfrak{S}(N)$, and the area-functional $\mathcal{A}_g$, we may consider the following set,

$$\mathcal{S}(N) = \{(M,E)\colon M\in\mathfrak{S}(N),\ E\subset N, \text{ open, with } \partial E = \overline{M}\},$$

and for $\lambda\in\mathbb{R}$, the following area-type functional

$$\begin{aligned} F_\lambda\colon \mathcal{S}(N) &\to \mathbb{R},\\ (M,E) &\mapsto \mathcal{H}^n(\overline{M}) - \lambda\mathcal{H}^{n+1}(E). \end{aligned}$$

As before, for $\Omega \subset\subset N\setminus(\overline{M}\setminus M)$, and taking $X\in\Gamma_c(T(M\cap\Omega)^\perp)$, we define the first variation of F_λ, at (M, E), in direction X, [7, Propositions 2.3 and 2.5]

$$\delta F_\lambda(M,E)(X) \ := \ \frac{d}{dt}_{|t=0} F_\lambda(\Phi(M,t),\Phi(E,t)).$$

By computing ([7, Propositions 2.3]) one will see that (M, E) is a critical point of F_λ in Ω, if and only if $M \cap \Omega$ is a λ-CMC hypersurface, with $\vec{H} = \lambda\nu$, where ν points into E. For a critical point (M, E) in Ω, and defining $f = g(X, \nu)$ on $M \cap \Omega$, as before we may take the second variation and compute,

$$\delta^2 F_\lambda(M,E)(X) := \frac{d^2}{d^2t}_{|t=0} \mathcal{F}_\lambda(\Phi(M,t),\Phi(E,t)) = B_M[f,f] = -\int_M f\, Lf,$$

where B_L, and $L = \Delta_M + |A_M|^2 + \mathrm{Ric}_g(\nu,\nu)$, are defined exactly as before. We then similarly define the index and nullity of M in Ω.

We shall revisit and extend this discussion on index, nullity and the stability operator in Section 3.1.2.

In studying such hypersurfaces and their variational properties (in particular analysing sequences and taking limits), it is necessary to expand the class of objects (smooth, closed, properly embedded hypersurfaces) that we consider. We define one such object below (which is at the centre of study in Chapter 2), and even weaker notions in Section 1.2.

For a point $x \in N$, an n-dimensional subspace $T \subset T_xN$, and constants $\rho, \tau \in (0, \mathrm{inj}\,(N)/2)$, we define the cylinder,

$$C_{x,T,\rho,\tau} := \{\exp_x(y + s\nu_T)\colon y \in B^{T_xN}_\rho(0) \cap T,\ s \in (-\tau,\tau)\},$$

where ν_T is a choice of unit normal to T.

Definition 1. (Quasi-embedded hypersurface) A set $M \subset N$ is defined as a quasi-embedded hypersurface if there exists a smooth manifold S, of dimension n, and a smooth proper immersion $\iota\colon S \to N$, such that $M = \iota(S)$, and for each $x \in M$, there exists $\rho, \tau \in (0, \mathrm{inj}\,(N)/2)$, a n-dimensional subspace $T \subset T_xN$ (with a choice of unit normal ν_T), along with a finite, ordered collection of distinct smooth functions $u_1 \leq u_2 \leq \ldots \leq u_k$, for some $k = k(x) \in \mathbb{Z}_{\geq 1}$,

$$u_1, \ldots, u_k\colon B^{T_xN}_\rho(0) \cap T \to (-\tau,\tau),$$

with $u_i(0) = 0$, and $\nabla u_i(0) = 0$, for all $i = 1, \ldots, k$, such that,

$$M \cap C_{x,T,\rho,\tau} = \bigcup_{i=1}^k \{\exp_x(y + u_i(y)\nu_T)\colon y \in B^{T_xN}_\rho(0) \cap T\}.$$

If $k = 1$, then we call x an embedded point of M, and if $k \geq 2$, we call x a non-embedded point of M.

Definition 2. (Quasi-embedded λ-CMC hypersurface) For $\lambda \in \mathbb{R}$, a set $M \subset N$, is defined as a quasi-embedded λ-CMC if M is a quasi-embedded hypersurface as defined in Definition , and for each $x \in M$, the graph of each function $u_1, \dots, u_k$ has constant mean curvature λ.

Remark 1. (See [12, Remark 2.6]) Note that if $\lambda = 0$ (i.e. M is minimal) then by the one-sided maximum principle for each $x \in M$, $k = 1$. Similarly, if $\lambda \neq 0$, and for $x \in M$, such that $k \geq 2$, then in fact $k = 2$. Moreover, if $\vec{H}_i$ denotes the mean curvature vector of graph$\,(u_i)$, then $g_x(\vec{H}_1(x), \nu_T) = \lambda$, and $g_x(\vec{H}_2(x), \nu_T) = -\lambda$.

It is also worth noting that in the literature these hypersurfaces are defined under different names. In [71] they are referred to as almost embedded, and in [16] they are referred to as effectively embedded. We opt for the name quasi-embedded as in [14].

1.2 Varifolds and Caccioppoli Sets

We denote n-rectifiable varifolds V, in N, by $V = (\Sigma, \theta)$, where $\Sigma \subset N$ is an n-rectifiable set, and $\theta \colon \Sigma \to \mathbb{R}_{\geq 0}$, is a $\mathcal{H}^n_g$-measureable function on Σ. To a varifold V, we define its weight measure $\|V\|$ (which is a Radon measure on N), by

$$\|V\|(A) = \int_{\Sigma \cap A} \theta \, d\mathcal{H}^n, \; A \subset N, \tag{1.1}$$

and we define spt $\|V\|$ as the support of the Radon measure $\|V\|$ on N. If θ takes positive integer values $\|V\|$-a.e. then we say that V is an integer n-rectifiable varifold. For an n-rectifiable varifold V, and a sequence of n-rectifiable varifolds $\{V_i\}$, we say that $V_i \to V$ if $\|V_i\| \rightharpoonup \|V\|$ as Radon measures.

We also note that the definition of an n-rectifiable varifold is always up to an equivalence (see the first paragraph of [54, chapter 4]), and unless otherwise stated we will take $\Sigma = \operatorname{spt}\|V\|$, and for $p \in \Sigma$,

$$\theta(p) = \liminf_{r \to 0} \frac{\mu_V(B^N_r(p))}{\omega_n r^n},$$

where we are setting, $\omega_n = \mathcal{H}^n(B^n_1(0))$.

We now look to define the first variation of a n-rectifiable varifold V. Let $f \colon N \to N$, be a proper C^1 map, then

$$f_\# V \coloneqq (f(\Sigma), \theta \circ f^{-1}),$$

defines a rectifiable n-varifold. Then, for a smooth vector field X on N (recall that X generates a flow Φ), we define the first variation of V in direction X by

$$\delta V(X) := \frac{d}{dt}_{t=0} \|\Phi(\cdot,t)_{\#}V\|(N) = \int_\Sigma (\operatorname{div}_\Sigma X)\,\theta\, d\mathcal{H}^n.$$

We then say that V has generalised mean curvature $\vec{H}$, if

$$\delta V(X) = -\int_\Sigma (X\cdot\vec{H})\,\theta\, d\mathcal{H}^n.$$

holds for all smooth vector fields X on N. If $\vec{H}=0$, we say that V is stationary.

We define the following three sets on $\operatorname{spt}\|V\|$:

1. The regular set of V (denoted $\operatorname{reg} V$) is the set of points $x \in \operatorname{spt}\|V\|$, such that there exists a $\rho \in (0, \operatorname{inj}(N))$, such that $\operatorname{spt}\|V\| \cap B^N_\rho(x)$ is a smoothly embedded hypersurface.

2. The generalised regular set (denoted gen-reg V), is the set of points $x \in \operatorname{spt}\|V\|$, such that there exists $\rho \in (0, \operatorname{inj}(N))$, such that $\operatorname{spt}\|V\| \cap B^N_\rho(x)$ is a quasi-embedded hypersurface.

3. The singular set of V (denoted $\operatorname{sing} V$), is the set of points $x \in \operatorname{spt}\|V\| \setminus \text{gen-reg}\, V$.

It is worth noting that by the one-sided maximum principle for minimal hypersurfaces, if V is stationary, then $\text{gen-reg}\, V = \operatorname{reg} V$. Moreover, with this notation, if M is a quasi-embedded hypersurface (Definition 1), with $\mathcal{H}^n(\overline{M}\setminus M) = 0$, then we may define a rectifiable n-varifold $V = (\overline{M}, 1)$, the set of embedded points of M will be contained in $\operatorname{reg} V$, the set of non-embedded points of M will be contained in $\text{gen-reg}\, V \setminus \operatorname{reg} V$, and $\operatorname{sing} V$ will be contained in $\overline{M}\setminus M$.

We refer the reader to [54, Chapter 4] for a more detailed discussion on n-rectifiable varifolds.

Another geometric object used in this thesis is that of Caccioppoli sets. Before we state the definition of Caccioppoli sets we need to define functions of bounded variation.

Definition 3. (Functions of Bounded Variation) We define the set of functions $BV(N)$, to be the set of functions $f \in L^1(N)$, such that,

$$V(f) := \sup\left\{\int_N f \operatorname{div} X : X \in \Gamma(TN),\ \|X\|_{L^\infty(N)} \le 1\right\} < +\infty$$

Equivalently, there exists a Radon measure $|Df|$, and $|Df|$-measureable vector field σ on N, with $|\sigma|(x) = 1$ for x $|Df|$-a.e. such that

$$\int_N f \operatorname{div} X = -\int_N g(X,\sigma)\, d|Df|$$

holds for all smooth vector fields X on N.

Definition 4. (Caccioppoli sets) A set $E \subset N$ is a Caccioppoli set (or a set of finite perimeter) if the characteristic function of E,

$$\chi_E(x) = \begin{cases} 1, & x \in E, \\ 0, & x \notin E, \end{cases}$$

lies in $BV(N)$.

For a Caccioppoli set E, we define the reduced boundary of E, $\partial^* E$, as the set of points x such that,

1. $|D\chi_E|(B_r^N(x)) > 0$ for all $r \in (0, \operatorname{inj}(N))$,

2. For any smooth vector field X on N

$$\lim_{r \to 0^+} \frac{1}{\omega_n r^n} \int_{B_r^N(x)} g(\sigma, X)\, d|D\chi_E| = g(\sigma, X),$$

3. $|\sigma|(x) = 1$

We refer the reader to [25, Chapter 5] for a more detailed discussion on functions of bounded variation and Caccioppoli sets.

Chapter 2

Embeddedness of Min-Max CMC Hypersurfaces in Manifolds with Positive Ricci Curvature

This chapter of the thesis contains joint work with Bellettini. The published version of this work may be found here [15]. Our main result is the following existence result,

Theorem 4. *For any $\lambda \in \mathbb{R} \setminus \{0\}$, and compact, $n+1$ dimensional ($n \geq 2$) Riemannian manifold (N, g), without boundary, and with positive Ricci curvature, there exists a smooth, embedded, two-sided hypersurface M, with constant mean curvature λ (λ-CMC), and*

1. *M is closed when $2 \leq n \leq 6$,*
2. *$\overline{M} \setminus M$ consists of finitely many points when $n = 7$,*
3. *$\dim_{\mathcal{H}} (\overline{M} \setminus M) \leq n - 7$, when $n \geq 8$.*

In Theorem 4 the emphasis is on the fact that M is embedded: this is a new result. The statement of Theorem 4 with embedded replaced by (the weaker notion of) quasi-embedded was on the other hand known, as detailed below (with two methods available). We recall that quasi-embedded means that the hypersurface is a smooth immersion, with any self-intersections being tangential, and with local structure around any point of tangential intersection being that of two embedded disks lying on opposite sides of each other (see Definitions 2 and 2).

As it will be important for our arguments, we begin by recalling that the existence result in Theorem 4, with embedded replaced by quasi-embedded, follows from the work by Bellettini–Wickramasekera in [14]. In fact, [14, Theorem 1.1] proves the following more general result. Given a compact Riemannian manifold, (N, g), $\dim N \geq 3$ (without any curvature assumptions) and a non-negative Lipschitz function $h : N \to \mathbb{R}$, there exists a quasi-embedded, two-sided C^2

hypersurface M_h such that, for each $x \in M_h$, the scalar mean curvature of M_h at x is given by $h(x)$; the singular set $\overline{M_h} \setminus M_h$ satisfies the dimensional estimates listed in Theorem 4. The construction of M_h is carried out in the Allen–Cahn min-max framework, and serves as a starting point for the present work. We briefly recall it here in the case $h = \lambda$ constant, with further details in Section 2.1.1.

Consider a sequence of functions $\{u_i\}$ in $W^{1,2}(N)$, where each u_i is the solution of the appropriate ε_i-scaled inhomogeneous Allen–Cahn equation, with $\varepsilon_i \to 0$. Assuming uniform energy bound, the works of Hutchinson–Tonegawa [33] and Röger–Tonegawa [46] give, in the $\varepsilon_i \to 0$ limit, an integral varifold V (a "limit interface"), with generalised mean curvature $H_V \in L^\infty(\text{spt}\|V\|)$, along with a Caccioppoli set E, with $\partial^* E \subset \text{spt}\,\|V\|$, such that,

$$\begin{cases} H_V(x) = \lambda,\ \theta_V(x) = 1, & \mathcal{H}^n - a.e.\ x \in \partial^* E, \\ H_V(x) = 0,\ \theta_V(x) \in 2\mathbb{Z}_{\geq 1}, & \mathcal{H}^n - a.e.\ x \in \text{spt}\,\|V\| \setminus \partial^* E. \end{cases}$$

In the presence of such a sequence $\{u_i\}$, the two major roadblocks to an existence result for a λ-CMC are (i) $\partial^* E$ may be empty, in which case the limit interface is actually minimal (ii) even if $\partial^* E \neq \emptyset$, it may not have sufficient regularity ([14, Figure 1] illustrates how lack of regularity could prevent $\partial^* E$ from being an admissible candidate). In [14] a (first) sequence $\{u_i\}$ is produced by means of a classical mountain pass lemma; the Morse index of each u_i is at most 1 (as a consequence of the fact that the min-max has one parameter). It is moreover shown (see [14, Remark 6.7]) that in the case of ambient manifold with positive Ricci curvature (and with $h = \lambda$ constant), occurrence (i) cannot arise, that is, $\partial^* E$ is non-trivial when u_i is the sequence obtained from the min-max. For arbitrary ambient manifolds, in the event that u_i leads to occurrence (i), [14] implements a gradient flow that yields a (second) sequence $\{v_i\}$, for which $\partial^* E \neq \emptyset$ and with Morse index 0. The matter is thus reduced to a regularity question for the limit interface arising from a sequence u_i with uniformly bounded Morse index. This index control is used in a key way to obtain regularity ([14, Theorem 1.2]), whose proof relies on extensions of Tonegawa's work [59] and Tonegawa–Wickramasekera's work [60], and crucially on the (non-variational) varifold regularity result of Bellettini–Wickramasekera [13, Theorem 9.1] (see also [14, Theorem 3.2]). In conclusion, [14] obtains that $V = V_\lambda + V_0$, where $\text{spt}\,\|V_\lambda\| = \partial E = \overline{M_\lambda}$ and $\text{spt}\,\|V_0\| = \overline{M_0}$; here M_λ is a two-sided, quasi-embedded λ-CMC hypersurface, and M_0 an embedded minimal hypersurface, both satisfying the dimensional estimates listed in Theorem 4. Furthermore, any intersections between M_λ and M_0, and self-intersections of M_λ, are always tangential intersections of C^2 graphs lying on one side of each other.

With this as a starting point, our first step in establishing Theorem 4 is to show that when $\text{Ric}_g > 0$, the one-parameter Allen–Cahn min-max just recalled does not produce any minimal components in the limit interface, i.e. $V_0 = 0$. (As mentioned earlier, in this case [14] establishes already that

$V_\lambda \neq 0$ for the u_i produced by min-max.)

Theorem 5. *Let (N, g) be a compact Riemannian manifold, without boundary, of dimension ≥ 3, with positive Ricci curvature, and $\lambda > 0$. The one-parameter Allen–Cahn min-max in [14], with prescribing function set to λ, produces a two-sided λ-CMC hypersurface and no minimal hypersurface.*

Theorem 5 is achieved by exhibiting a suitable continuous path, admissible in the min-max construction (which employs paths that are continuous in $W^{1,2}(N)$). This path will move through functions that are each modelled on a level set of the signed distance to M_λ. The idea is to try to place a 1-dimensional Allen–Cahn profile along the normal direction to a given level set and thus produce a function (a point in the path). This might appear problematic due to the presence of points where the level sets are not smoothly embedded in N (which, for example, may be caused by the presence of the singular set $\overline{M_\lambda} \setminus M_\lambda$, or by the fact that M_λ has quasi-embedded points). We handle this after observing that all "problematic points" are contained in a closed n-rectifiable set. The open complement (in N) of this n-rectifiable set is described (via a diffeomorphism) as an open subset of $\tilde{M} \times \mathbb{R}$, where $\tilde{M}$ is a (abstract) n-manifold whose immersion into N gives M_λ. We will refer to this open subset as the Abstract Cylinder (which is endowed with a metric pulled back from N). Each level set of the distance function becomes a subset of $\tilde{M} \times \{s\}$, where s is the chosen distance value. The sought path is then defined by "sliding" the 1-dimensional Allen–Cahn profiles in the $\mathbb{R}$-direction in the whole cylinder $\tilde{M} \times \mathbb{R}$, then restricting these functions to the Abstract Cylinder, and passing them to N. We check that this indeed produces a continuous path in $W^{1,2}(N)$. Furthermore, performing the energy calculations on the Abstract Cylinder, we see that the potentially "problematic points" do not cause any issues. The sliding argument yields a path with the (key) property that the relevant Allen-Cahn energy attains a maximum (along the path) at the function obtained in correspondence of M_λ (signed distance equal to 0); this relies on the positivity of the Ricci curvature. This property of the path easily implies that $V_0 = 0$ (no minimal component), for otherwise the min-max characterisation of V would be contradicted. Theorem 4 is then proven by showing that the λ-CMC hypersurface arising in Theorem 5 is, in fact, embedded. This is again done by exhibiting a suitable path (admissible in the min-max). This path is constructed by editing the previous one about its maximum, under the contradiction assumption that a non-embedded point exists in M_λ. The modification requires the identification of suitable hypersurfaces obtained by deforming M_λ about the non-embedded point. This construction ensures that the modified path attains a maximum that is strictly smaller than the maximum obtained for the path used in the proof of Theorem 5. This contradicts the min-max characterisation. We stress that these path constructions capitalise on the a priori knowledge (from [14]) that M_λ and M_0 are sufficiently regular.

We remark that Theorem 5 is somewhat interesting in its own sake: it is an open question whether (and under what assumptions) a sequence of solutions to the inhomogeneous Allen–Cahn equation with nowhere vanishing inhomogeneous term, and with a uniform bound on the Morse index, can produce minimal components. (The regularity result in [14] recalled earlier allows us to refer to the minimal and prescribed-mean-curvature components as hypersurfaces that are separately smooth, except for a possible small singular set when the ambient dimension is 8 or higher.) Theorem 5 rules out minimal components in the special instance in which the solutions come from a one-parameter min-max (in N compact with $\text{Ric}_N > 0$) and the inhomogeneous term is constant.

The absence of minimal components and of non-embedded points established by Theorem 4 has, among its consequences, a Morse index estimate:

Corollary 1. *The λ-CMC hypersurface in Theorem 5 has Morse index equal to 1.*

This follows directly from the work of Mantoulidis [38]. Alternatively, the arguments of Hiesmayr [31] apply verbatim.

As we recalled, [14] employs an Allen–Cahn approximation scheme to construct the λ-CMC quasi-embedded hypersurface. The statement of Theorem 4 with embedded replaced by quasi-embedded can also be obtained (without any curvature assumption on N) using the so-called Almgren–Pitts method for the min-max, see the combined works of Zhou–Zhu [71] ($2 \leq n \leq 6$) and Dey in [24] (for $n \geq 7$, relying on the compactness theory of Bellettini–Wickramasekera [12, 13]).

Regardless of the method used for the min-max construction, and without the need of curvature assumptions, if $2 \leq n \leq 6$ the λ-CMC hypersurface obtained is closed and immersed (completely smooth). In White's work [63, Theorem 35] it is proven that for each $\lambda \in \mathbb{R}$, there exists a generic set (in the sense of Baire category) of smooth metrics on the ambient manifold such that any closed, codimension-1 (completely smooth) immersion with constant mean curvature λ, is self-transverse. Therefore, combining the existence of quasi-embedded λ-CMC ([14] or [71]) with [63, Theorem 35], one obtains: when $2 \leq n \leq 6$, for any λ, there exists a generic set of metrics on N, such that each admits an embedded λ-CMC hypersurface.[1]

This argument relies however on the complete smoothness of the λ-CMC hypersurface, which is not available for $n \geq 7$ in the existence results. The flavour of Theorem 4 differs from the statement just given in that it allows a singular set and can handle all dimensions; moreover the class of metrics (Ricci positive metrics) is the same for all $\lambda \in \mathbb{R}$. We also stress that the proof of embeddedness in Theorem 4 exploits the min-max characterisation of the λ-CMC, while one can apply [63, Theorem 35] to any smooth CMC immersion, not necessarily one coming from a min-max. Theorem 4 and 5 may also hold with other assumptions on the metric on N, or other choices on the set of prescribing functions. (In these different scenarios an alternative approach to

[1] A more general version of this statement is available for h-PMC hypersurfaces, $2 \leq n \leq 6$, by again combining [63, Theorem 35] with either [14] or [72]. Note that the class of prescribing functions, h, is different in these two results.

the sliding argument mentioned above could be a gradient flow, for example, along the lines of [8, Section 5.4] and [14, Section 6.9].)

2.1 Preliminaries

2.1.1 Allen-Cahn and Construction of CMC Immersion

We recall the min-max construction in [14], of critical points to the inhomogeneous Allen–Cahn energy,

$$\mathcal{F}_{\varepsilon,\lambda}(u) = \int_N \frac{\varepsilon}{2}|\nabla u|^2 + \frac{W(u)}{\varepsilon} - \sigma \int_N \lambda u, \qquad \varepsilon \in (0,1),\, u \in W^{1,2}(N). \tag{2.1}$$

Where W is a smooth function on $\mathbb{R}$, with $W(\pm 1) = 0$ being non-degenerate minima, and $W(t) > 0$, for $t \in \mathbb{R} \setminus \{\pm 1\}$. Furthermore, we impose that W has only three critical points, $t = 0, \pm 1$, and quadratic growth outside some compact interval. For example $W(t) = (1-t^2)^2/4$, for $t \in [-2, 2]$ and has quadratic growth outside $[-3, 3]$. The constant σ is given by,

$$\sigma = \int_{-1}^{1} \sqrt{W(s)/2}\, ds.$$

Moreover, we take $\lambda > 0$.

Consider the first and second variations of (2.1) with respect to $\varphi \in C^\infty(N)$,

$$\delta \mathcal{F}_{\varepsilon,\lambda}(u)(\varphi) = \int_N \varepsilon \nabla u \cdot \nabla \varphi + \left(\frac{W'(u)}{\varepsilon} - \sigma\lambda \right) \varphi, \tag{2.2}$$

$$\delta^2 \mathcal{F}_{\varepsilon,\lambda}(u)(\varphi,\varphi) = \int_N \varepsilon |\nabla \varphi|^2 + \frac{W''(u)}{\varepsilon}\varphi^2. \tag{2.3}$$

We say that u is a critical point of (2.1), if $\delta \mathcal{F}_{\varepsilon,\lambda}(u)(\varphi) = 0$, for all $\varphi \in C^\infty(N)$, and then by standard elliptic theory we have that $u \in C^\infty(N)$, and strongly solves,

$$\varepsilon \Delta u = \frac{W'(u)}{\varepsilon} - \sigma\lambda. \tag{2.4}$$

If u is a critical point of (2.1), then (similarly to the discussion in Section 1.1) we may define the Morse index of u in $\Omega \subset N$, as

$$\operatorname{ind}_\Omega(u) \coloneqq \sup\left\{ \dim \Pi \colon \text{for all}\, \varphi \in \Pi \leq W_0^{1,2}(\Omega),\, \delta^2 \mathcal{F}_{\varepsilon,\lambda}(u)(\varphi,\varphi) < 0 \right\}. \tag{2.5}$$

If $\operatorname{ind}_\Omega(u) = 0$, then we say u is stable in Ω. By Figure 2.1, we see that there exists two stable constant solutions on N, $a_\varepsilon > -1$, and $b_\varepsilon > 1$. Furthermore, as $\varepsilon \to 0$, we have that $a_\varepsilon \to -1$, and $b_\varepsilon \to 1$. As $\mathrm{Ric}_g > 0$, [9, Proposition 7.1] shows that these are the only stable critical points

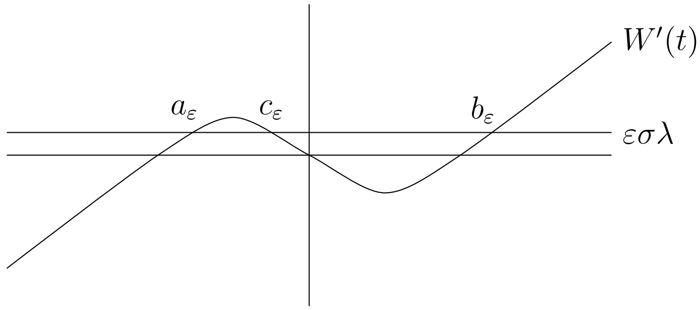


Figure 2.1: Intersection points, a_ε, b_ε, and c_ε, are the solutions to $W'(t) = \varepsilon\sigma\lambda$.

of (2.1).

The existence of these isolated, stable solutions permits us to find non-trivial critical points of (2.2) via a min-max argument.

Proposition 1. *(Existence of Min-Max Solution, [14, Proposition 5.1]) For $\varepsilon > 0$, let Γ denote the collection of all continuous paths $\gamma\colon [-1,1] \to W^{1,2}(N)$, such that $\gamma(-1) = a_\varepsilon$, and $\gamma(1) = b_\varepsilon$. Then there exists an $\varepsilon_0 > 0$, such that for all $\varepsilon < \varepsilon_0$,*

$$\inf_{\gamma\in\Gamma} \sup_{u\in\gamma([-1,1])} \mathcal{F}_{\varepsilon,\lambda} = \beta_\varepsilon > \mathcal{F}_{\varepsilon,\lambda}(a_\varepsilon) > \mathcal{F}_{\varepsilon,\lambda}(b_\varepsilon), \tag{2.6}$$

is a critical value, i.e. there exists $u_\varepsilon \in W^{1,2}(N)$, which is a critical point of $\mathcal{F}_{\varepsilon,\lambda}$, with $\mathcal{F}_{\varepsilon,\lambda}(u_\varepsilon) = \beta_\varepsilon$. Moreover, u_ε has Morse index ≤ 1.

In our Ricci positive setting, as a_ε and b_ε are the only stable critical points we actually have that u_ε has Morse index equal to 1.

Now taking a sequence $\{\varepsilon_i\}_{i\in\mathbb{N}} \subset (0, \varepsilon_0)$, with $\varepsilon_i \to 0$, and associated critical points from Proposition 1, $\{u_i = u_{\varepsilon_i}\}$, we associate the following Radon measures,

$$\mu_i := (2\sigma)^{-1} \left(\frac{\varepsilon_i}{2}|\nabla u_i|^2 + \frac{W(u_i)}{\varepsilon_i} \right) d\mu_g. \tag{2.7}$$

Where μ_g is the volume measure of (N, g). Moreover there exists constants K, $L > 0$, such that for all i,

$$\sup_N |u_i| + \mu_i(N) \leq K, \tag{2.8}$$

and

$$\mu_i(N) \geq L. \tag{2.9}$$

By the bounds of (2.8) and (2.9), there exists a subsequence $\{u_{i'}\} \subset \{u_i\}$, along with a $u_0 \in BV(N)$, with $u_0(y) \in \{+1, -1\}$ for all $y \in N$, and a non-zero Radon measure μ, such that

$u_{i'} \to u_0$ in $L^1(N)$, and $\mu_{i'} \rightharpoonup \mu$ as Radon measures. By [33, Theorem 1] and [46, Theorem 3.2], we have that μ is the weight measure of an integral n-rectifiable varifold V, with the following properties:

1. V, is an integral n-rectifiable varifold with bounded generalised mean curvature H_V.

2. The set $E := \text{interior}(\overline{\{u_0 = +1\}})$ is a Caccioppoli set, with reduced boundary $\partial^* E \subseteq \text{spt}\,\|V\| \subset N \setminus E \neq \emptyset$.

3. For $\mathcal{H}^n$-a.e. $x \in \partial^* E$, $\theta_V(x) = 1$, and $H_V(x) \cdot \nu(x) = \lambda$; where ν is the inward pointing unit normal to $\partial^* E$, i.e. $\nu = \nabla u_0 / |\nabla u_0|$.

4. For $\mathcal{H}^n$-a.e. $x \in \text{spt}\,\|V\| \setminus \partial^* E$, $\theta_V(x)$ is an even integer ≥ 2, and $H_V(x) = 0$.

Optimal regularity of V was then proven in [14] (recall definitions from Section 1.2).

1. $V = V_0 + V_\lambda$

2. V_0 is a (possibly zero) stationary integral n-varifold with singular set of Hausdorff dimension at most $n - 7$.

3. $V_\lambda = (\partial^* E, 1) \neq 0$, and $\text{gen-reg}\, V_\lambda$ is a quasi-embedded hypersurface with constant mean curvature λ, with respect to unit normal pointing into E. The singular set of V_λ has Hausdorff dimension at most $n - 7$.

Therefore, we have the following

Theorem 6. *(Theorem 1.1 [14]) Let (N, g) be a compact Riemannian manifold, without boundary, of dimension $n+1 \geq 3$, with positive Ricci curvature, and let $\lambda \in (0, \infty)$ be a fixed constant. There exists a smooth, quasi-embedded λ-CMC hypersurface (recall Definition 2 and Remark 1) $M \subset N$, with;*

1. *$\overline{M} \setminus M = \emptyset$, if $2 \leq n \leq 6$;*

2. *$\overline{M} \setminus M$ is finite if $n = 7$;*

3. *$\dim_{\mathcal{H}}(\overline{M} \setminus M) \leq n - 7$, if $n \geq 8$.*

We restate Theorems 4 and 5 with our new notation.

Theorem 7. *Let (N, g) be a compact Riemannian manifold, without boundary, of dimension $n + 1 \geq 3$, with positive Ricci curvature, and let $\lambda \in (0, \infty)$ be a fixed constant. The limiting varifold $V = V_\lambda + V_0$ from Section 2.1.1 has the following properties*

1. *$M := \text{gen-reg}\, V_\lambda$ is embedded, connected and has index 1.*

2. *$V_0 = 0$.*

2.1.2 One Dimensional Allen–Cahn Solution

We refer to [9, Section 2.2] as a reference for this section.

We define the function $\mathbb{H}$ on $\mathbb{R}$ to denote the unique, monotonically increasing, finite energy solution to the ODE $u'' - W'(u) = 0$, with the conditions $\mathbb{H}(0) = 0$ and $\lim_{t\to\pm\infty} \mathbb{H}(t) = \pm 1$. We then define $\mathbb{H}_\varepsilon(\,\cdot\,) = \mathbb{H}(\varepsilon^{-1}\,\cdot)$, which solves the ODE $\varepsilon u'' - \varepsilon^{-1}W'(u) = 0$.

We define an approximation for $\mathbb{H}_\varepsilon$. Start by considering the following bump function

$$\begin{cases} \chi \in C_c^\infty(\mathbb{R}), & \\ \chi(t) = 1, & t \in (-1,1), \\ \chi(t) = 0, & t \in \mathbb{R} \setminus (-2,2), \\ \chi(t) = \chi(-t), & t \in \mathbb{R}, \\ \chi'(t) \leq 0, & t \geq 0. \end{cases}$$

For $\varepsilon \in (0,1)$, we define the truncation of $\mathbb{H}_\varepsilon$ by

$$\overline{\mathbb{H}}_\varepsilon(t) := \begin{cases} \chi((\varepsilon\Lambda)^{-1}t)\mathbb{H}_\varepsilon(t) + 1 - \chi((\varepsilon\Lambda)^{-1}t), & t > 0, \\ \chi((\varepsilon\Lambda)^{-1}t)\mathbb{H}_\varepsilon(t) - 1 + \chi((\varepsilon\Lambda)^{-1}t), & t < 0, \end{cases}$$

where $\Lambda = 3|\log \varepsilon|$. There exists a constant $\beta = \beta(W) < +\infty$, such that for all $\varepsilon \in (0, 1/4)$,

$$2\sigma - \beta\varepsilon^2 < \int_{\mathbb{R}} \frac{\varepsilon}{2}|(\overline{\mathbb{H}}_\varepsilon)'(t)|^2 + \frac{W(\overline{\mathbb{H}}_\varepsilon(t))}{\varepsilon}\,dt < 2\sigma + \beta\varepsilon^2.$$

2.2 Idea of the Proof

We first prove Theorem 4 for the case $\lambda > 0$. To then prove for $\lambda < 0$, we take $\tilde{\lambda} = -\lambda > 0$, and reverse the direction of the unit normal on the resulting $\tilde{\lambda}$-CMC hypersurface. From here on we take $\lambda > 0$. Moreover we assume that N is connected. If not then we simply take a connected component of N.

For Caccioppoli sets $\Omega \subset N$, we define the following functional,

$$\mathcal{F}_\lambda(\Omega) := \mathcal{H}^n(\partial^*\Omega) - \lambda\mu_g(\Omega).$$

Recall our converging sequence of critical points $\{u_{\varepsilon_j}\}$, along with our limiting varifold $V = V_\lambda + V_0$, and Caccioppoli set E from Section 2.1.1. We have, as $\varepsilon_j \to 0$,

$$\mathcal{F}_{\varepsilon_j,\lambda}(u_{\varepsilon_j}) \to 2\sigma\mathcal{F}_\lambda(E) + 2\sigma\|V_0\|(N) + \sigma\lambda\mu_g(N)$$

Therefore, constructing optimal paths between $\emptyset$ and N for $\mathcal{F}_\lambda$, will provide insight into optimal paths from a_ε to b_ε for $\mathcal{F}_{\varepsilon,\lambda}$.

As N is compact, one obvious path that includes E, is $\{E_t\}$ for $t \in [-2\operatorname{diam}(N), 2\operatorname{diam}(N)]$, where,

$$E_t \coloneqq \{y \colon \tilde{d}(y) > t\}.$$

Here $\tilde{d}$ is the signed distance function to $M \coloneqq \partial^* E$, taking positive values in E, and negative values in $N \setminus \overline{E}$. We also denote,

$$\Gamma_t \coloneqq \{y \colon \tilde{d}(y) = t\} = \partial E_t.$$

Assuming sufficient regularity on the sets Γ_t and E_t, and the functions $t \mapsto \mathcal{H}^n(\Gamma_t)$ and $t \mapsto \mu_g(E_t)$, we have for $t > 0$,

$$\begin{aligned} \mathcal{F}_\lambda(E_t) - \mathcal{F}_\lambda(E) &= \int_0^t \frac{d}{ds}\mathcal{H}^n(\Gamma_s)\, ds - \lambda \int_0^t \frac{d}{ds}\mu_g(E_s)\, ds, \\ &= \int_0^t \int_{\Gamma_s} \lambda - H_{\Gamma_s}(x)\, d\mathcal{H}^n(x)\, ds, \end{aligned} \tag{2.10}$$

where H_{Γ_s} is the scalar mean curvature of Γ_s with respect to unit normal $\nabla \tilde{d}$. Recalling that $H_{\Gamma_0} = \lambda$, a straightforward calculation yields the following inequalities.

$$\begin{cases} H_{\Gamma_t} \geq \lambda + mt, & t \geq 0, \\ H_{\Gamma_t} \leq \lambda + mt, & t \leq 0, \end{cases}$$

where $m = \min_{|X|=1} Ric_g(X, X) > 0$. Therefore, by (2.10) for $t \geq 0$,

$$\mathcal{F}_\lambda(E_t) \leq \mathcal{F}_\lambda(E).$$

The same inequality holds for $t \leq 0$. Here we see the importance of the assumption on the Ricci curvature. Therefore,

$$\gamma \colon t \in [-2\operatorname{diam}(N), 2\operatorname{diam}(N)] \mapsto E_{-t} \in \{\text{Caccioppoli sets of } N\},$$

is a path from $\emptyset$ to N, that has maximum height $\mathcal{F}_\lambda(E)$.

We look to replicate this path in $W^{1,2}(N)$. Consider the Lipschitz function on N,

$$v_\varepsilon^t = \overline{\mathbb{H}}_\varepsilon(\tilde{d}(x) - t),$$

which can be thought of as placing the truncated one dimensional Allen-Cahn solution from Section

2.1.2 along the normal profile of Γ_t. By the Co-Area formula, we have,

$$\mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon^t) = \int_{\mathbb{R}} Q_\varepsilon(s-t)\mathcal{H}^n(\Gamma_s)\,ds - \sigma\lambda \int_{\mathbb{R}} \overline{\mathbb{H}}_\varepsilon(s-t)\mathcal{H}^n(\Gamma_s)\,ds,$$

were,

$$Q_\varepsilon(t) = \frac{\varepsilon}{2}|\left(\overline{\mathbb{H}}_\varepsilon\right)'(t)|^2 + \frac{W\left(\overline{\mathbb{H}}_\varepsilon(t)\right)}{\varepsilon}$$

The functions

$$t \mapsto \int_{\mathbb{R}} Q_\varepsilon(s-t)\mathcal{H}^n(\Gamma_s)\,ds, \quad \text{and} \quad t \mapsto \sigma\lambda \int_{\mathbb{R}} \overline{\mathbb{H}}_\varepsilon(s-t)\mathcal{H}^n(\Gamma_s)\,ds,$$

act as smooth approximations to $t \mapsto 2\sigma\mathcal{H}^n(\Gamma_t)$, and $t \mapsto 2\sigma\lambda\mu_g(E_t) - \sigma\lambda\mu_g(N)$, respectively.

We say that v_ε^0 is an Allen–Cahn approximation of M as,

$$\mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon^0) \to 2\sigma\mathcal{H}^n(M) - \sigma\lambda\mu_g(E) + \sigma\lambda\mu_g(N \setminus E) =: A_2,$$

as $\varepsilon \to 0$, Section 2.3.6. Carrying out a calculation which replicates the previous one, we deduce that for all $\tau > 0$, there exists an $\varepsilon_\tau > 0$, such that for all $\varepsilon \in (0, \varepsilon_\tau)$,

$$\max_{t\in[-2\operatorname{diam}(N),2\operatorname{diam}(N)]} \mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon^t) < A_2 + \tau = A_1 - 2\sigma\|V_0\|(N) + \tau,$$

where $A_1 := 2\sigma\mathcal{H}^n(M) + 2\sigma\|V_0\|(N) - \sigma\mu_g(E) + \sigma\mu_g(N \setminus E)$. Connecting $v_\varepsilon^{2\operatorname{diam}(N)} = -1$ to a_ε, and $v_\varepsilon^{-2\operatorname{diam}(N)} = +1$ to b_ε, by constant functions, we see that we have an appropriate min-max path in $W^{1,2}(N)$.

This path proves that we cannot have a minimal piece V_0. We also get criterion for M. Indeed, as there exists a 'Wall', [14, Lemma 5.1], that all min-max paths must climb over, we have that

$$2\sigma\lambda\mathcal{H}^n(M) - \sigma\lambda\mu_g(E) + \sigma\lambda\mu_g(N \setminus E) > \sigma\lambda\mu_g(N).$$

Rearranging yields,

$$\mathcal{H}^n(M) > \lambda\mu_g(E).$$

We note that the above path can be constructed for any suitable λ-CMC hypersurface which encloses a volume. Therefore, for any such pair (M, E), the above inequality holds, and our min-max must choose the pair that minimises the positive quantity $\mathcal{H}^n(M) - \lambda\mu_g(E)$. From this, one can deduce that $\overline{E}$ must be connected.

We turn our attention to proving that M is embedded. We prove by contradiction, exploiting the min-max characterisation of M. We now know that, given our sequence of critical points $\{u_{\varepsilon_j}\}$, and potentially after taking a subsequence,

$$\mathcal{F}_{\varepsilon_j,\lambda}(u_{\varepsilon_j}) \to 2\sigma\mathcal{H}^n(M) - \sigma\lambda\mu_g(E) + \sigma\lambda\mu_g(N\setminus E) = A_2,$$

as $\varepsilon_j \to 0$. Assume that M has a non-embedded point z_0. Then for every $\varepsilon_j > 0$, we construct a continuous path,

$$\gamma_{\varepsilon_j} \colon [-1,1] \to W^{1,2}(N),$$

where, $\gamma_{\varepsilon_j}(-1) = a_{\varepsilon_j}$, and $\gamma_{\varepsilon_j}(1) = b_{\varepsilon_j}$. This path satisfies the following, there exists a $J \in \mathbb{Z}_{\geq 1}$, and $\varsigma > 0$, independent of j, such that for all $j \geq J$,

$$\max_{t\in[-1,1]} \mathcal{F}_{\varepsilon_j,\lambda}(\gamma_{\varepsilon_j}(t)) < 2\sigma\mathcal{H}^n(M) - \sigma\lambda\mu_g(E) + \sigma\lambda\mu_g(N\setminus E) - \varsigma,$$

This is a contradiction of the min-max characterisation of u_{ε_j}.

We sketch the main ideas of the path in the ε-limit, Figure 2.2.

The picture at z_0 is Figure 2.3a. The limiting energy for this structure is A_2. The starting point for building this path is to construct a competitor with lower limiting energy. Then we wish to connect this competitor to $+1$ and -1, with energy always remaining a fixed amount below A_2.

***Step 1*:** Construction of Competitor, (1) $\to$ (2) in Figure 2.2, Section 2.5

The structure at z_0 is two smooth, embedded CMC disks, that touch tangentially at z_0 and lie either side of each other. To construct the competitor, we locally push these disks together, and delete portions of the disks that are pushed past each other. This reduces the area of our structure while also increasing the size of E, leading to a drop in energy. Indeed, consider open balls $B_1 \subset\subset B_2$ about z_0. We smoothly bump the disks at z_0 such that inside B_1 we move the disks distance $\rho > 0$, and outside B_2 we remain fixed. The balls B_1, and B_2, along with ρ, are chosen so that the area inside B_1 gets deleted, Figure 2.3b. Letting,

$$\varsigma = \frac{\sigma}{2}\mathcal{H}^n(B_1 \cap M),$$

we see that our competitor has energy lying below, $A_2 - \varsigma$.

***Step 2*:** Path to $+1$, Section 2.7

To construct the competitor, we only edited M locally about z_0. Therefore, pushing the competitor to the level set $\Gamma_{-\rho}$ will correspond to a similar drop in energy from pushing M to $\Gamma_{-\rho}$. This is (2) $\to$ (6) in Figure 2.2. See Figures 2.3b and 2.3f for local pictures about z_0. From $\Gamma_{-\rho}$ we can easily connect to $+1$ by pushing along level sets Γ_{-r}, as previously discussed.

***Step 3*:** Path to -1, Section 2.6

We look to follow a similar method as in ***Step 2*** by connecting our competitor to a level set Γ_{r_0}, for $r_0 > 0$, then push this along level sets Γ_r for r in $[r_0, 2\,\mathrm{diam}(N)]$ to connect it to -1. By pushing our competitor straight to Γ_{r_0} we run the risk of pushing through M and increasing our energy back up to A_2. Therefore, we carry out our path in stages, again making use of the fact that our edit about z_0 was local.

The first stage is (2) $\to$ (3) in Figure 2.2. We fix our competitor in B_2, and outside we push forward, so that outside some larger ball B_3, we line up with Γ_{r_0}. See Figures 2.3b and 2.3c for local pictures about z_0. Again, as our edit is local about z_0, this corresponds to a similar drop in energy of pushing M to Γ_{r_0}, and the drop will be of order r_0^2. For a large enough r_0 this will give us a large enough energy drop to be able to undo the edit inside B_2, and still have our energy remain below $A_2 - \varsigma$. This is the second stage from (3) $\to$ (4), in Figure 2.2. See Figure 2.3d, for local picture about non-embedded point. From here we push up inside B_3 to line up with Γ_{r_0}, (4) $\to$ (5) in Figure 2.2, Figure 2.3e. Finally, we connect to -1 by sliding along level sets as previously stated.

Path at ε Level

We carry out this 'pushing', on what we refer to as our abstract cylinder, $\tilde{M} \times \mathbb{R}$. See Section 2.3.3. Here $\tilde{M}$ is an n-dimensional manifold and $\iota\colon \tilde{M} \to M$ is a smooth immersion. We define the following map,

$$\begin{aligned} F\colon \tilde{M} \times \mathbb{R} \ &\to\ N, \\ (x,t) \ &\mapsto\ \exp_{\iota(x)}(t\nu(x))), \end{aligned}$$

with ν being a smooth choice of unit normal to immersion, pointing into E. Therefore, we view points (x,t) on our cylinder $\tilde{M} \times \mathbb{R}$ as having base point $\iota(x)$ and moving length t along the geodesic with initial direction $\nu(x)$. See Figure 2.4.

Recall our function $v_\varepsilon^0 = \overline{\mathbb{H}}_\varepsilon \circ \tilde{d}$, then by the Co-Area formula,

$$\begin{aligned} \mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon^0) &= \int_{\mathbb{R}} \left(\frac{\varepsilon}{2} |(\overline{\mathbb{H}}_\varepsilon)'(t)|^2 + \frac{W(\overline{\mathbb{H}}_\varepsilon(t))}{\varepsilon} - \sigma\lambda \overline{\mathbb{H}}_\varepsilon(t) \right) \mathcal{H}^n(\Gamma_t)\, dt, \\ &= \int_{\mathbb{R}} \int_{\tilde{M}} \left(\frac{\varepsilon}{2} |(\overline{\mathbb{H}}_\varepsilon)'(t)|^2 + \frac{W(\overline{\mathbb{H}}_\varepsilon(t))}{\varepsilon} - \sigma\lambda \overline{\mathbb{H}}_\varepsilon(t) \right) \theta_t(x)\, d\mathcal{H}^n(x)\, dt, \end{aligned}$$

where $\theta_t\colon \tilde{M} \to \mathbb{R}$, is defined in Section 2.3.4 such that for all $t \in \mathbb{R}$, and any $\mathcal{H}^n$-measurable function on N,

$$\int_{\Gamma_t} g\, d\mathcal{H}^n = \int_{\tilde{M}} (g \circ F_t)\, \theta_t\, d\mathcal{H}^n,$$

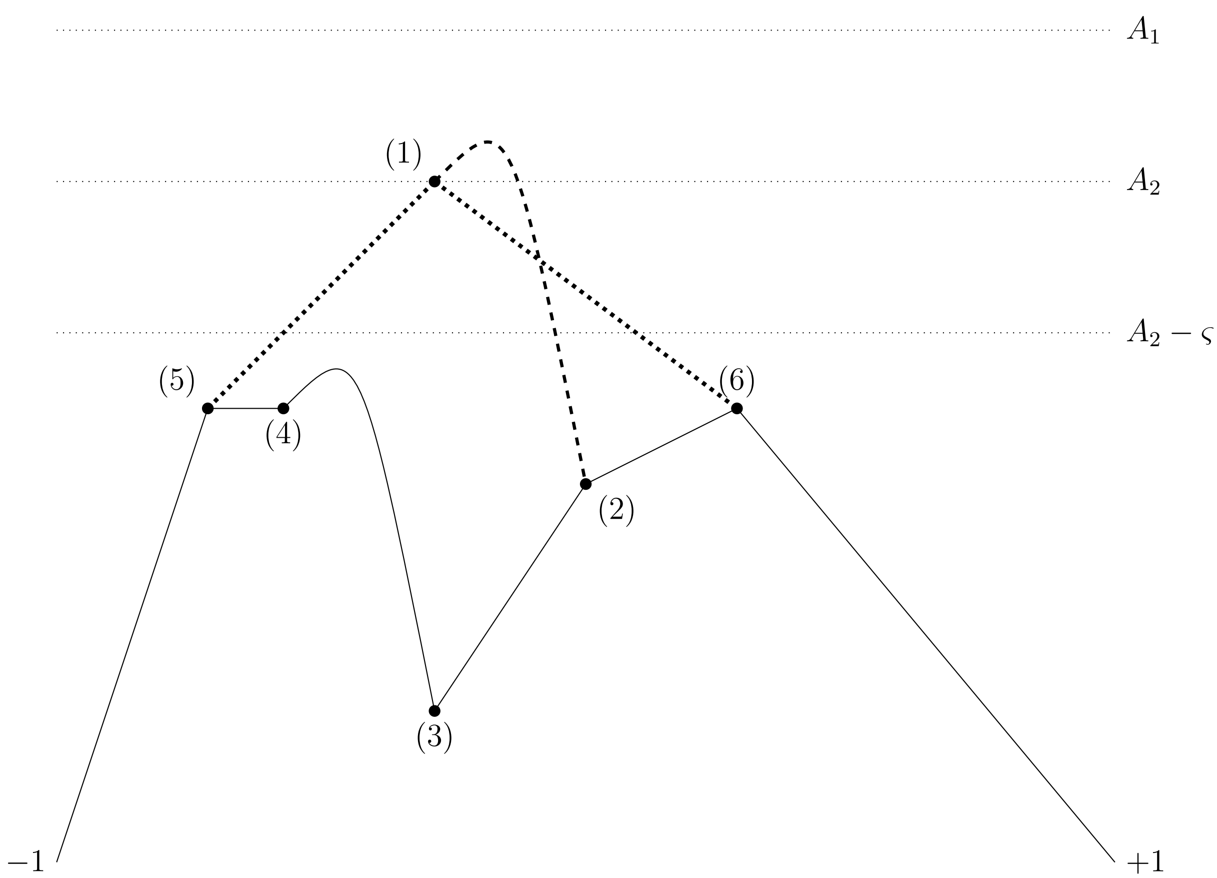
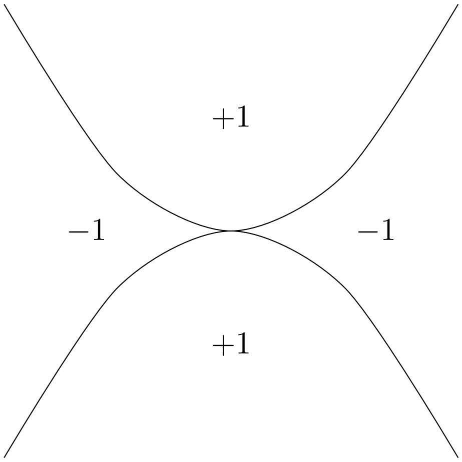
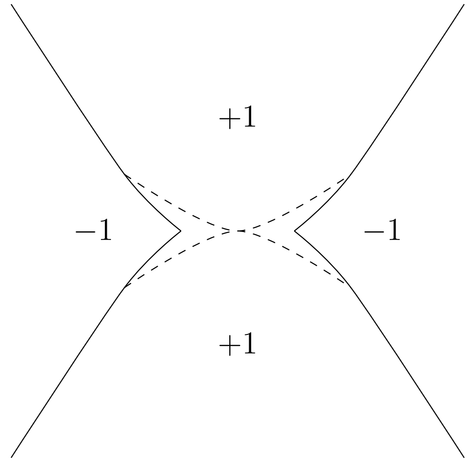


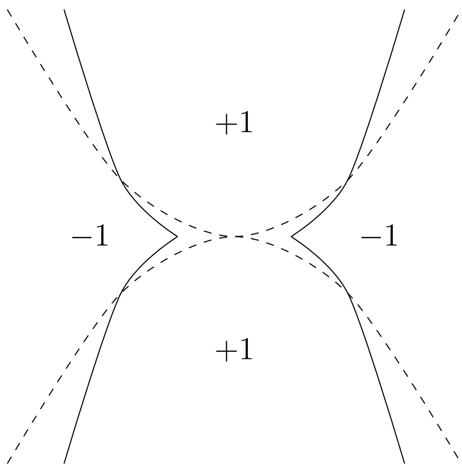
Figure 2.2: The Paths. To prove $V_0 = 0$, we follow the path from -1 to (5), then the dotted line to (1), dotted line to (6), then complete the path to $+1$. The dashed line from (1) to (2) is the construction of the competitor. Then, to prove that M is embedded, we follow the path from -1 to $+1$ given by the solid lines. Refer to Figure 2.3 for the local picture about the non-embedded point z_0 at each numbered stage on the paths.



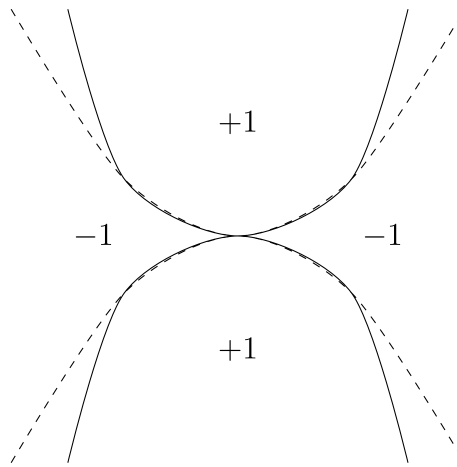
(a) (1): Non-embedded point z_0



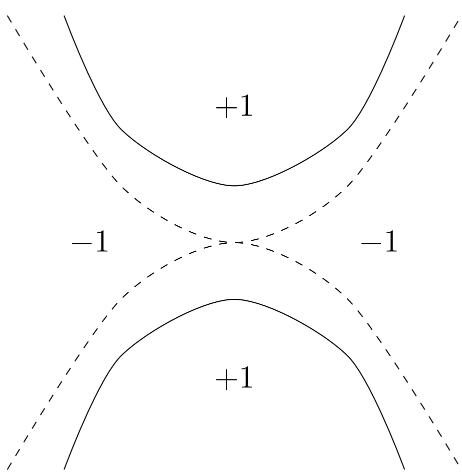
(b) (2): Competitor



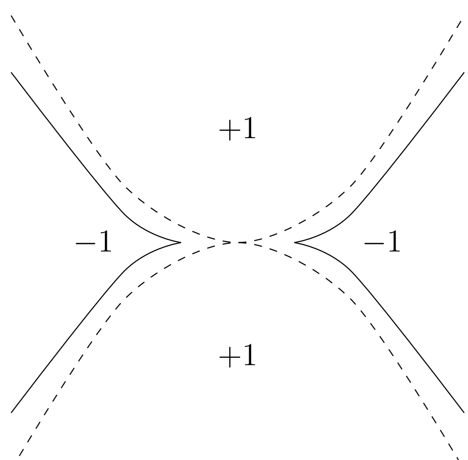
(c) (3): Move competitor to Γ_{r_0} outside ball B_3 centred at z_0.



(d) (4): Undo the edit inside B_2.

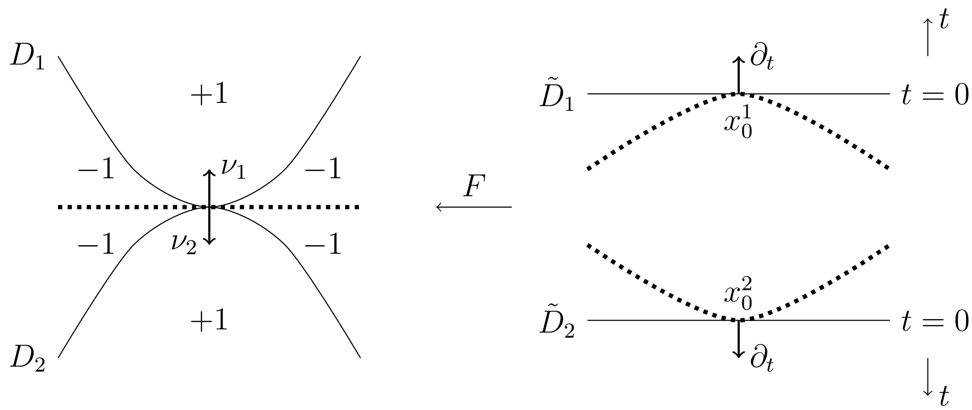


(e) (5): Push up in B_3 to come into line with Γ_{r_0}.

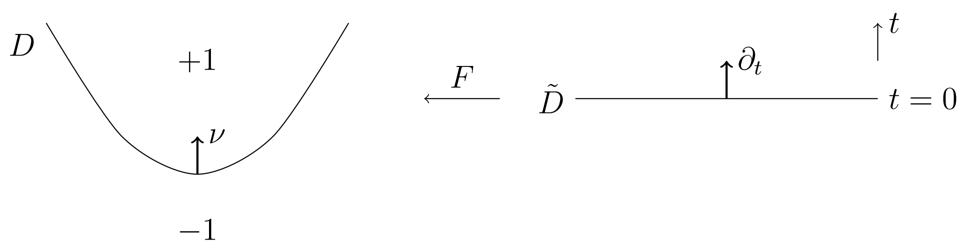


(f) (6): Push Competitor to come in line with $\Gamma_{-\rho}$.

Figure 2.3: Stages of the Path at the non-embedded point z_0. In each image, the dashed lines represent the original λ-CMC disks, as a reference to what we are changing at each step. Furthermore, in each image, it is the solid lines that are the boundaries between the '+1' and '−1' regions.



(a) On the left we have a local picture about a non-emebedded point z_0 of M. On the right the two local pictures about x_0^1 and x_0^2 in $\tilde{M} \times \mathbb{R}$, where $\iota(x_0^1) = z_0 = \iota(x_0^2)$. We have, $F(\tilde{D}_i) = D_i$, and $dF_{x_0^i}(\partial_t) = \nu_i$, for $i = 1$ and 2. The dotted line on the left picture respresents points in N which are of equal distance to D_1 and D_2. The dotted lines on the right-hand picture are the preimages of the dotted line on the left, under the map F.



(b) On the left, a local picture about an embedded point of M. On the right is its preimage in $\tilde{T}$ under the map F.

Figure 2.4: Local pictures about points in M

with $F_t(\,\cdot\,) = F(\,\cdot\,, t)$. Then we carry out the relevant 'pushings' by considering a continuous family of functions $\{g_r\}_{r\in[0,r']} \subset C(\tilde{M})$,

$$\begin{aligned}\mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon^r) \;=\; & \int_{\mathbb{R}}\int_{\tilde{M}} \Bigg(\frac{\varepsilon}{2} |(\overline{\mathbb{H}}_\varepsilon)'(t - g_r(x))|^2 + \frac{W(\overline{\mathbb{H}}_\varepsilon(t - g_r(x)))}{\varepsilon} \\ & \qquad - \sigma\lambda \overline{\mathbb{H}}_\varepsilon(t - g_r(x)) \Bigg)\, \theta_t(x)\, d\mathcal{H}^n(x,t)\, dt.\end{aligned}$$

See Figure 2.5.

2.2.1 Structure of the Paper

The paper is organised as follows. We start with setup:

- Section 2.3 is devoted to set up of objects used in the main computation.

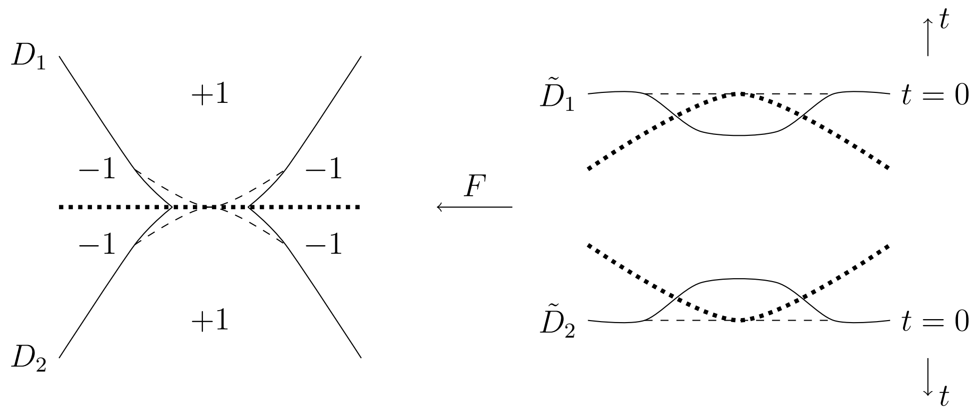


Figure 2.5: How the competitor is constructed as the graph of bump functions about points x_0^1 and x_0^2 over $\tilde{M}$. Whatever is bumped out beyond the dotted line, on the right-hand side, is not considered in N. In other words, it is deleted.

- In Section 2.4 we carry out the main computation. The constructions that follow are carried out by plugging explicitly defined functions into this computation.

To prove Theorem 5:

- In Section 2.8.2 we build the dotted path (5) $\to$ (1) $\to$ (6) in Figure 2.2. Theorem 5 then follows upon combining this with computations in Sections 2.6.4 and 2.7.2; in these sections we build the paths (5) $\to -1$, and (6) $\to +1$, in Figure 2.2.

To prove Theorem 4 we argue by contradiction, assuming that M has a non-embedded point z_0:

- In Section 2.5 we construct our competitor about z_0. This is the dashed path (1) $\to$ (2) in Figure 2.2.

- In Section 2.6 we construct a path from the competitor to the stable constant a_ε. This is the solid path (2) $\to$ (6) $\to +1$, in Figure 2.2.

- In Section 2.7 we construct a path from the competitor to the stable constant b_ε. This is the solid path (2) $\to$ (3) $\to$ (4) $\to$ (5) $\to -1$ in Figure 2.2.

- In Section 2.8.3 we piece together this continuous path from a_ε to b_ε, in $W^{1,2}(N)$. The energy $\mathcal{F}_{\varepsilon,\lambda}$, is less than $A_2 - \varsigma$ for every point along this path, Figure 2.2. This contradicts the min-max construction, proving that M is embedded.

Finally, in Section 2.9 we prove Corollary 1 (the Morse index of M is equal to 1, which also implies that M must be connected).

2.2.2 A Note on Choice of Constants

The biggest subtlety in the construction of the path in Sections 2.5, 2.6 and 2.7 is the choice of constants, and the order that we choose them in. We explicitly list the order of choices here, and reference where they have been chosen.

1. We first choose a non-embedded point z_0
2. We choose $\delta = \delta(z_0, N, M, g, \lambda, W) > 0$, in Remarks 6, 7, 8, 9, 17.
3. We choose $L = L(z_0, N, M, g, \delta, \lambda, W) > 0$, in Remarks 13, 22.
4. We choose $k = k(z_0, N, M, g, \delta, L, \lambda, W)$, in Remark 23.
5. We choose $r_0 = r_0(z_0, N, M, g, \delta, L, k, \lambda, W) > 0$, in Remarks 13, 22.
6. We choose $\rho = \rho(z_0, N, M, g, \delta, L, k, r_0, \lambda, W) > 0$, in Remarks 10, 11, 12, 15, 19, 24.
7. We define $l = l(\rho)$ in (2.11).
8. We choose $\tau > 0$ in Sections 2.8.2 and 2.8.3.
9. We finally choose $\varepsilon = \varepsilon(z_0, N, M, g, \delta, L, k, r_0, \rho, \tau, \lambda, W) > 0$, in Remarks 16, 20, 21 and Sections 2.8.2 and 2.8.3.

2.3 Construction of Objects

2.3.1 Signed Distance Function

Let $d_{\overline{M}}\colon N \to \mathbb{R}$ be the distance function to the closed set $\overline{M} \subset N$. As $\overline{M}$ is closed, and N is complete, Hopf–Rinow tells us that, for each z in N, the value, $d_{\overline{M}}(z)$, is obtained by a geodesic from z to a point on $\overline{M}$. Furthermore, $d_{\overline{M}}$ is Lipschitz, with Lipschitz constant 1.

The set $E = \operatorname{int}(\overline{\{u_0 = 1\}})$ is open in N, and $\overline{M} = \partial E$. This allows us to define the signed distance function, $\tilde{d}\colon N \to \mathbb{R}$, to $\overline{M}$, which takes positive values in E, and negative values in $N \setminus E$,

$$\tilde{d}(y) = \begin{cases} d_{\overline{M}}(z), & x \in E, \\ -d_{\overline{M}}(z), & x \notin E. \end{cases}$$

This is a Lipschitz function on N, with Lipschitz constant 1.

2.3.2 Abstract Surface

M is a quasi-embedded λ-CMC hypersurface, Definition 2.

Remark 2. (Local graphical representation of M) For a point $z \in M$, there exists an n-dimensional linear subspace $T = T_z \subset T_z N$, and a unit vector $\nu_z \in T^\perp$, along with $r = r(z) > 0$, $s = s(z) > 0$, and $S = S(z) > 0$, with $\max\{r, s\} \leq S < \operatorname{inj}(N)$, such that we can define the cylinder

$$C_{z,T,r,s} \coloneqq \exp_z \left(\{X + t\nu_z \colon X \in B_r^{T_z N}(0) \cap T,\ t \in (-s, s)\} \right) \subset B_S^N(z),$$

and, one of the following holds:

1. (See Figure 2.4b) There exists a smooth function,

$$f \colon B_{z,T,r} \coloneqq B_r^{T_z N}(0) \cap T \to (-s, s),$$

   which satisfies,

$$\begin{cases} f(0) = 0, \\ \nabla^T f(0) = 0, \\ \Delta_T f(0) = \lambda, \end{cases}$$

   and,

$$\overline{M} \cap C_{z,T,r,s} = \exp_z(\operatorname{Graph}(f)) = \exp_z(\{X + f(X)\nu_z \colon X \in B_{z,T,r}\})$$

   Furthermore, we have that,

$$E \cap C_{z,T,r,s} = \exp_z(\{X + t\nu_z \colon X \in B_{z,T,r},\ f(X) < t < s\}),$$

   and we can define a smooth choice of unit normal to $\exp_z(\operatorname{Graph}(f))$,

$$\nu \colon \exp_z(\operatorname{Graph}(f)) \to T(\exp_z(\operatorname{Graph}(f)))^\perp,$$

   such that $\nu(z) = \nu_z$.

2. (See Figure 2.4a) There exists two smooth functions,

$$f_1,\ f_2 \colon B_{z,T,r} \to (-s, s),$$

which satisfy,

$$\begin{cases} f_1(0) = 0 = f_2(0), \\ f_1 \geq f_2, \\ \nabla^T f_1(0) = 0 = \nabla^T f_2(0), \\ \Delta_T f_1(0) = \lambda = -\Delta_T f_2(0), \end{cases}$$

and,

$$\overline{M} \cap C_{z,T,r,s} = \bigcup_{i=1,2} \exp_z(\text{Graph}\,(f_i)) = \bigcup_{i=1,2} \exp_z(\{X + f_i(X)\nu_z \colon X \in B_{z,T,r}\}).$$

Furthermore, we have that,

$$\begin{aligned} E \cap C_{z,T,r,s} \;&=\; \exp_z(\{X + t\nu_z \colon X \in B_{z,T,r},\, f_1(X) < t < s\}) \\ &\qquad \cup \exp_z(\{X + t\nu_z \colon X \in B_{z,T,r},\, -s < t < f_2(X)\}), \end{aligned}$$

and we can define smooth choices of unit normals,

$$\nu_i \colon \exp_z(\text{Graph}\,(f_i)) \to T(\exp_z(\text{Graph}\,(f)))^{\perp},$$

such that $\nu_1(z) = \nu_z$, and $\nu_2(z) = -\nu_z$.

Recall that if Case 1 holds, then we say that z is an embedded point of M, and alternatively if Case 2 holds, we say that z is a non-embedded point of M. In either case, the tangent space of M at z is given by, $T_z M \coloneqq T_z$.

The set of non-embedded points of M satisfies the following

Claim 1. *([12, Remark 2.6]) The set of non-embedded points of M is locally contained in an $(n-1)$-dimensional submanifold of M, and thus this set of non-embedded points has $\mathcal{H}^n$-measure* 0.

We define our abstract surface $\tilde{M}$ by

$$\tilde{M} = \{(z, \nu) \colon z \in M,\, \nu \in T_z M^{\perp},\, \text{with}\, |\nu| = 1, \text{ and points into}\, E\}.$$

Locally $\tilde{M}$ is a smooth, embedded CMC disk in N, therefore, $\tilde{M}$ is a smooth n-dimensional manifold.

2.3.3 Abstract Cylinder

Consider x in $\tilde{M}$, then $x = (z, X)$, for some z in M and X in $T_zM^\perp$. We define two, smooth projections, first from $\tilde{M}$ to $TM^\perp$,

$$\nu\colon (z, X) \mapsto X,$$

and secondly, from $\tilde{M}$ to M,

$$\iota\colon (z, X) \mapsto z.$$

From these we define the following map,

$$\begin{aligned} F\colon \tilde{M} \times \mathbb{R} &\to N, \\ (x, t) &\mapsto \exp_{\iota(x)}(t\nu(x)), \end{aligned}$$

which, as N is complete, is well-defined. For a fixed x in $\tilde{M}$, F is a unit parametrisation of a geodesic which, at time 0, passes through $\iota(x)$, with velocity $\nu(x)$. The set $\{t\colon d_{\overline{M}}(F(x,t)) = |t|\}$, is the set of times t, at which this geodesic achieves the shortest distance from $F(x,t)$ to $\overline{M}$. Consider the subset $\{t\colon \tilde{d}(F(x,t)) = t\} \subset \{t\colon d_{\overline{M}}(F(x,t)) = |t|\}$, and its endpoints,

$$\begin{aligned} \sigma^+(x) &= \sup\{t\colon \tilde{d}(F(x,t)) = t\} \geq 0, \\ \sigma^-(x) &= \inf\{t\colon \tilde{d}(F(x,t)) = t\} \leq 0. \end{aligned}$$

These are uniformly bounded functions on $\tilde{M}$, and in fact as the next claim shows, $\{t\colon \tilde{d}(F(x,t)) = t\}$ is a closed and connected interval on $\mathbb{R}$.

Claim 2. *We have that*

$$[\sigma^-(x), \sigma^+(x)] = \{t : \tilde{d}(F(x,t)) = t\}.$$

Proof. Consider the geodesic, $\gamma : t \mapsto F(x,t)$, and define the following function,

$$f : t \mapsto \tilde{d}(F(x,t)).$$

This is a 1-Lipschitz function with $f(0) = 0$. Indeed,

$$|f(t_1) - f(t_2)| \leq |d(F(x,t_1), F(x,t_2))| \leq \text{Length}(\gamma_{|[t_1,t_2]}) = |t_1 - t_2|.$$

Thus, for $t_0 \geq 0$, such that $f(t_0) \neq t_0$, we must have $f(t_0) < t_0$. Moreover, for any $t > t_0$,

$$\begin{aligned} f(t) &= f(t) - f(t_0) + f(t_0), \\ &\leq t - t_0 + f(t_0), \\ &< t. \end{aligned}$$

Similarly, if we have $t_0 \leq 0$, such that $f(t_0) \neq t_0$, then $f(t) \neq t$, for all $t < t_0$.

By continuity, we have that $\tilde{d}(F(x, \sigma^+(x))) = \sigma^+(x)$, and therefore by above, for all $t \in [0, \sigma^+(x)]$, we must have that $\tilde{d}(F(x,t)) = t$. By definition of $\sigma^+(x)$, for all $t > \sigma^+(x)$, $\tilde{d}(F(x,t)) < t$. Therefore,

$$[0, \sigma^+(x)] = \{t \geq 0 \colon \tilde{d}(F(x,t)) = t\}.$$

Similarly, $[\sigma^-(x), 0] = \{t \leq 0 \colon \tilde{d}(F(x,t)) = t\}$. □

We define the abstract cylinder,

$$\tilde{T} = \{(x,t) \colon x \in \tilde{M},\, t \in (\sigma^-(x), \sigma^+(x))\} \subset \tilde{M} \times \mathbb{R}.$$

Defining the projection map from $\tilde{M} \times \mathbb{R}$ onto $\mathbb{R}$, $p \colon (x,t) \mapsto t$, then on $\tilde{T}$ we have that $\tilde{d} \circ F = p$.

We wish to work on $\tilde{T}$ instead of N. The following Lemma is crucial in that respect.

Lemma 1. *(Geodesic Touching Lemma) For all y in $N \setminus \overline{M}$, there exists a geodesic from y to $\overline{M}$ that achieves the length of $d_{\overline{M}}(y)$. The end point of this geodesic on $\overline{M}$ must in fact be a quasi-embedded point of M, and the geodesic will hit M orthogonally.*

Proof. Identical argument to [9, Lemma 3.1], except we replace the Sheeting Theorem of [51, Theorem 1] (alternatively [67, Theorem 3.3]) for minimal hypersurfaces, with the Sheeting Theorem of [12, Theorem 3.1] for CMC hypersurfaces. □

From this Lemma, the following result is immediate,

Proposition 2. *For all y in $N \setminus (\overline{M} \setminus M)$, there exists an x in $\tilde{M}$, such that $F(x, \tilde{d}(y)) = y$.*

Claim 3. *The functions, σ^+, $\sigma^- \colon \tilde{M} \to \mathbb{R}$, are continuous.*

Proof. We prove by contradiction. Suppose there exists an $\hat{x} \in \tilde{M}$ such that, $\liminf_{x \to \hat{x}} \sigma^+(x) = \alpha < \sigma^+(\hat{x})$. Choose $0 < \delta < \sigma^+(\hat{x}) - \alpha$, then there exists $x_n \to \hat{x}$ in $\tilde{M}$ such that $\sigma^+(x_n) < \alpha + \delta$. Now consider the points,

$$z_n = F(x_n, \alpha + \delta) \to z := F(\hat{x}, \alpha + \delta).$$

By Claim 2, $\tilde{d}(z_n) < \alpha + \delta$. By Proposition 2 there exists a sequence $\tilde{x}_n$, such that,

$$F(\tilde{x}_n, \tilde{d}(z_n)) = z_n.$$

After potentially taking a subsequence and renumerating we have that there exists a $y \in \overline{M}$, such that $\iota(\tilde{x}_n) \to y$, then note $d(y,z) = \tilde{d}(z) = \alpha + \delta$. Therefore, by Lemma 1, $y \in M$, and as

$t \mapsto F(\hat{x}, t)$ is the unique length minimising geodesic from M to z, $\tilde{x}_n \to \hat{x}$ in $\tilde{M}$. Now we have that,

$$F(x_n, \alpha + \delta) = z_n = F(\tilde{x}_n, \tilde{d}(z_n)).$$

However, $(x_n, \alpha + \delta) \neq (\tilde{x}_n, \tilde{d}(z_n))$, and

$$\lim_{n \to \infty} (x_n, \alpha + \delta) = (\hat{x}, \alpha + \delta) = \lim_{n \to \infty} (\tilde{x}_n, \tilde{d}(z_n)).$$

This implies that F is not a diffeomorphism about the point $(\hat{x}, \alpha + \delta)$, and therefore by classical theory of geodesics, [48, Lemma 2.11], $t \mapsto F(\hat{x}, t)$ is no longer length minimising to M beyond time $t = \alpha + \delta$. This contradicts $\alpha + \delta < \sigma^+(\hat{x})$.

Now suppose that $\sigma^+(\hat{x}) < \limsup_{x \to \hat{x}} \sigma^+(x) = \beta < +\infty$. Choose $0 < \delta < \beta - \sigma^+(\hat{x})$, and sequence $x_n \to \hat{x}$, such that,

$$\sigma^+(x_n) > \sigma^+(\hat{x}) + \delta.$$

Define,

$$z_n = F(x_n, \sigma^+(\hat{x}) + \delta),$$

then $\tilde{d}(z_n) = \sigma^+(\hat{x}) + \delta$. By continuity of F,

$$z_n \to z := F(\hat{x}, \sigma^+(\hat{x}) + \delta).$$

However, by definition of $\sigma^+(\hat{x})$, $\tilde{d}(z) < \sigma^+(\hat{x}) + \delta = \tilde{d}(z_n)$. This contradicts continuity of $\tilde{d}$.

Similar arguments show that σ^- is also continuous. □

We define the Cut Locus of M to be the following points in N,

$$\mathrm{Cut}\,(M) = \{F(x, \sigma^+(x)) \colon x \in \tilde{M}\} \cup \{F(x, \sigma^-(x)) \colon x \in \tilde{M}\} \subset N,$$

and by Proposition 2, we have that,

$$N \setminus (\overline{M} \setminus M) = F(\tilde{T}) \cup \mathrm{Cut}\,(M).$$

Proposition 3. *Cut$\,(M)$ is an n-rectifiable set.*

To prove Proposition 3, we first classify points in Cut(M).

Proposition 4. *A point y in $N \setminus (\overline{M} \setminus M)$, lies in Cut$(M)$ if and only if at least one of the following conditions holds:*

1. *y lies in $N \setminus \overline{M}$, and there exists an x in $\tilde{M}$ such that $F(x, \tilde{d}(y)) = y$, and $dF_{(x, \tilde{d}(y))} \colon T_x\tilde{M} \times \mathbb{R} \to T_yN$, is non-invertible.*

2. *y lies in $N \setminus \overline{M}$, and there exists at least two unique geodesics from y to $\overline{M}$ which achieve the length $d_{\overline{M}}(y)$.*

3. *y is a non-embedded point of M.*

Proof. Consider a point $y = F(x, 0) \in M$. If y is an embedded point of M, then case 1 of Remark 2 holds, and there exists an $S > 0$, such that $\overline{M} \cap B_S(y)$ is a smooth, embedded, n-dimensional CMC disk. Therefore, ([37, Proposition 4.2]) there exists an r in $(0, S/2)$, such that for all t in $(-r, r)$, $\tilde{d}(F(x,t)) = t$. Therefore, if $y \in M \cap \mathrm{Cut}(M)$, then y must be a non-embedded point.

Alternatively, if y is a non-embedded point then case 2 of Remark 2 holds, and (y, ν) and $(y, -\nu)$ both lie in $\tilde{M}$. Moreover, for $t \in (-s, 0)$, $t < f_2(0)$, implying that $F((y,\nu), t) = \exp_y(t\nu)$ lies in E. Therefore, $\tilde{d}(F((y,\nu),t)) \geq 0$, implying that $\sigma^-(y,\nu) = 0$, and thus y is a point in $\mathrm{Cut}(M)$.

For $y \in N \setminus \overline{M}$, the conclusion follows from standard theory of geodesics, see [48, Chapter II, Section 2]. As observed in [9, Proposition 3.1], thanks to Lemma 1, we may use this classical theory for minimising geodesics to smooth closed submanifolds in our setting. □

Remark 3. By standard theory of geodesics [48, Lemma 2.11], if a point y is in $\mathrm{Cut}\,(M)$, then any length minimising geodesic between y and $\overline{M}$, emanating from M, can no longer be length minimising (to $\overline{M}$), beyond length $|\tilde{d}(y)|$. Thus $F(\tilde{T})$ and $\mathrm{Cut}\,(M)$ must be disjoint. Therefore, by point 1 of Proposition 4, F must be a local diffeomorphism on $\tilde{T}$. Moreover, by point 2, $F\colon \tilde{T} \to F(\tilde{T})$ is a bijection.

Proof. (of Proposition 3) As $\mathrm{Cut}(M) \cap M$ consists of non-embedded points of M, by Claim 1 we have $\mathcal{H}^n(\mathrm{Cut}(M) \cap M) = 0$. Therefore, to prove that $\mathrm{Cut}(M)$ is n-rectifiable, we just need to concern ourselves with $\mathrm{Cut}(M) \setminus M$. Again this follows from the observation made in the proof of [9, Proposition 3.1], that as Lemma 1 holds, the arguments in [37, Theorem 4.10] hold verbatim. □

Remark 4. As M is smooth, we have that $\tilde{d}$ is smooth in $F(\tilde{T})$, [37, Proposition 4.2].

Denoting $h = F^*g$, we have that $F\colon (\tilde{T}, h) \to (F(\tilde{T}), g)$, is a bijective, local isometry.

Consider the projection map,

$$\begin{aligned} p\colon \tilde{M} \times \mathbb{R} &\to \mathbb{R}, \\ (x,t) &\mapsto t. \end{aligned}$$

In $\tilde{T}$, we have that $p = \tilde{d} \circ F$, and

$$|\nabla p(x,t)|_h = |\nabla \tilde{d}(F(x,t))|_g = 1.$$

We denote the sets,

$$\tilde{\Gamma}_t = p^{-1}(t) \cap \tilde{T},$$

and,

$$\Gamma_t = \tilde{d}^{-1}(t) \subset N.$$

Note,

$$F(\tilde{\Gamma}_t) = \begin{cases} \Gamma_t \cap F(\tilde{T}) = \Gamma_t \setminus \mathrm{Cut}(M), & t \neq 0, \\ \{\text{embedded points of } M\}, & t = 0. \end{cases}$$

Denote $H_{\tilde{\Gamma}_t}(x,t)$ as the scalar mean curvature of $\tilde{\Gamma}_t$, at (x,t), with respect to unit normal $\nabla p(x,t)$, and define the following function,

$$\begin{aligned} H_t \colon \tilde{M} &\to \mathbb{R}, \\ x &\mapsto \begin{cases} H_{\tilde{\Gamma}_t}(x,t), & (x,t) \in \tilde{T}, \\ 0, & (x,t) \notin \tilde{T}. \end{cases} \end{aligned}$$

For (x,t) in $\tilde{T}$, we have,

$$H_t(x) = -\mathrm{tr}_{T_{(x,t)}\tilde{\Gamma}_t} h(\nabla.\, \nabla p(x,t),\, \cdot\,) = -\Delta_{\tilde{\Gamma}_t} p(x,t).$$

However, as ∇p is a geodesic vector field

$$\nabla_{\nabla p} \nabla p = 0,$$

and as $|\nabla p| = 1$,

$$h(\nabla_X \nabla p, \nabla p) = \frac{1}{2} X(|\nabla p|) = 0.$$

Therefore, $\Delta_{\tilde{\Gamma}_t} p(x,t) = \Delta_{\tilde{T}} p(x,t)$, and thus for (x,t) in $\tilde{T}$,

$$H_t(x) = -\Delta p(x,t).$$

Proposition 5. *([29, Corollary 3.6]) For (x,t) in $\tilde{T}$,*

$$\partial_t H_t(x) = -\nabla p(\Delta p)(x,t) \geq m,$$

where $m = \inf_{|X|=1} Ric_g(X,X) > 0$.

Remark 5. Consider fixed x in $\tilde{M}$. For $\sigma^-(x) < 0$, we have $H_0(x) = \lambda$. If $\sigma^-(x) = 0$, we still have,

$$\lim_{t \searrow 0} H_t(x) = \lambda.$$

Thus, by Proposition 5, we have for $(x,t) \in \tilde{T}$,

$$\begin{cases} H_t(x) \geq \lambda + m\,t, & t > 0, \\ H_0(x) = \lambda, \\ H_t(x) \leq \lambda + m\,t, & t < 0. \end{cases}$$

2.3.4 Area Element

We define the function on $\tilde{M}$,

$$\theta_t(x) = \begin{cases} J_{\Pi_t}(x), & (x,t) \in \tilde{T}, \\ 0, & (x,t) \notin \tilde{T}, \end{cases}$$

where, J_{Π_t} is the Jacobian of the map $\Pi_t \colon x \in \tilde{M} \mapsto (x,t) \in \tilde{M} \times \mathbb{R}$. By the Area Formula,

$$\int_{\tilde{M}} \theta_t \, d\mathcal{H}^n = \mathcal{H}^n(\tilde{\Gamma}_t).$$

Proposition 6. *([29, Theorem 3.11]) For* (x_0, t_0) *in* $\tilde{T}$*,*

$$\partial_t \, \log(\theta_t)(x_0)_{|t=t_0} = -H_{t_0}(x_0).$$

Consider a fixed point (x_0, t_0) in $\tilde{T}$. First, consider $t_0 \geq 0$. For all t in $(0, t_0]$, (x_0, t) lies in $\tilde{T}$, which implies that the function $t \mapsto \theta_t(x_0)$ is smooth on the interval $(0, t_0]$. Furthermore, $\lim_{t \to 0^+} \theta_t(x_0) = 1$, and applying the Fundamental Theorem of Calculus,

$$\log(\theta_{t_0}(x_0)) \leq -t_0 \left(\lambda + \frac{1}{2} m t_0 \right).$$

Therefore,

$$\theta_{t_0}(x_0) \leq e^{-t_0 \left(\lambda + \frac{1}{2} m t_0\right)}.$$

Identical inequality holds for $t_0 \leq 0$.

The term $-t(\lambda + \frac{1}{2}mt)$ achieves a global maximum at $t = -\frac{\lambda}{m}$. As $\theta_{t_0}(x_0) = 0$ for all (x_0, t_0) not in $\tilde{T}$, we conclude that

$$0 \leq \theta_{t_0}(x_0) \leq e^{\frac{\lambda^2}{2m}},$$

for all (x_0, t_0) in $\tilde{M} \times \mathbb{R}$.

2.3.5 Construction About Non-Embedded point

Let z_0 in M be a non-embedded point.

Remark 6. We are in case 2 of Remark 2. We can choose $\delta = \delta(z_0, M, N, g)$ such that,

$$B_{2\delta}(z_0) \subset C_{z_0,T,r,s}.$$

We have three disjoint sets,

$$\begin{aligned}
E_1 &\coloneqq \exp_z(\{X + t\nu \colon X \in B_{z_0,T,r},\ f_1(X) < t < s\}) \cap B^N_{2\delta}(z_0),\\
F &\coloneqq \exp_z(\{X + t\nu \colon X \in B_{z_0,T,r},\ f_2(X) \le t \le f_1(X)\}) \cap B^N_{2\delta}(z_0),\\
E_2 &\coloneqq \exp_z(\{X + t\nu \colon X \in B_{z_0,T,r},\ -s < t < f_2(X)\}) \cap B^N_{2\delta}(z_0).
\end{aligned}$$

As $\partial E_i \cap B^N_{2\delta}(z_0) = \exp_z(\{\mathrm{Graph}\,(f_i)\}) \cap B^N_{2\delta}(z_0) =: D_i$, the following signed distance functions are well-defined for $i = 1, 2$,

$$\begin{aligned}
\tilde{d}_i \colon B^N_{2\delta}(z_0) &\to \mathbb{R},\\
y &\mapsto \begin{cases} d_{D_i}(y), & y \in E_i,\\ -d_{D_i}(y), & y \in B^N_{2\delta}(z_0) \setminus E_i. \end{cases}
\end{aligned}$$

For y in $B^N_{\delta}(z_0) \subset\subset B^N_{2\delta}(z_0)$,

$$\tilde{d}(y) = \max\{\tilde{d}_1(y), \tilde{d}_2(y)\}.$$

Furthermore, by [37, Proposition 4.2], we may choose $\delta > 0$, such that $\tilde{d}_1$ and $\tilde{d}_2$ will be smooth on $B^N_{2\delta}(z_0)$.

For $i = 1,\ 2$, we define

$$\tilde{D}_i \coloneqq \{(z, \nu_i(z)) \colon z \in D_i\} \subset \tilde{M},$$

and points $x^i_0 = (z_0, \nu_i(z_0))$.

Remark 7. We make a choice of $\delta = \delta(N, M, g, z_0) > 0$ small enough such that, for each $i = 1,\ 2$, we have open sets $\tilde{V}_i \subset \tilde{M} \times \mathbb{R}$, and maps,

$$F_i \colon \tilde{V}_i \to B^N_{2\delta}(z_0),$$

such that $\tilde{D}_i = \tilde{V}_i \cap \{t = 0\}$, and $F_i = F_{|\tilde{V}_i}$, is a diffeomorphism. We also insist that $\delta = \delta(N, M, g, z_0) > 0$, is chosen small enough such that the cut loci of D_1 and D_2 in $B^N_{2\delta}(z_0)$ are empty. We know we can pick such a $\delta > 0$ by [37, Proposition 4.2]

By choice of $\delta > 0$ in Remark 7, and Proposition 4,

$$\mathrm{Cut}(M) \cap B_\delta^N(z_0) = \{y \in B_\delta^N(z_0) \colon \tilde{d}_1(y) = \tilde{d}_2(y)\} \subset B_\delta^N(z_0) \setminus E.$$

Remark 8. Denote the set,

$$A = \{y \in B_{2\delta}^N(z_0) \colon \tilde{d}_1(y) = \tilde{d}_2(y)\}.$$

For $i = 1$, 2, we consider the functions,

$$\begin{aligned} \psi_i \colon \tilde{V}_i &\to \mathbb{R}, \\ (x,t) &\mapsto \tilde{d}_1(F_i(x,t)) - \tilde{d}_2(F_i(x,t)). \end{aligned}$$

Therefore, $A = F_i(\{\psi_i = 0\})$. Moreover,

$$\partial_t \psi_i(x_0^i, 0) = g(\nabla \tilde{d}_1(z_0), (dF_i)_{(x_0^i,0)}(\partial_t)) - g(\nabla \tilde{d}_2(z_0), (dF_i)_{(x_0^i,0)}(\partial_t)) = 2 \neq 0.$$

Thus, by Implicit Function Theorem we may choose $\delta = \delta(z_0, N, M, g) > 0$, such that set $A = \mathrm{Cut}\,(M) \cap B_{2\delta}^N(z_0)$ is a smooth n-submanifold in $B_{2\delta}^N(z_0)$, and σ^- is smooth on $\tilde{D}_1 \cup \tilde{D}_2$.

We now look to define the push out function to construct our competitor, Figure 2.5.

We wish to determine the amount we want to push out by, and the set we wish to push out on. Fix $\rho > 0$, and we set $l = l(\rho)$, to be,

$$l(\rho) = \sup\{t \colon \text{ for all}\, x \in\, B_t^{\tilde{M}}(x_0^1),\ |\sigma^-(x)| < \rho\}. \tag{2.11}$$

Here, $B_t^{\tilde{M}}(x)$ is the geodesic ball in $\tilde{M}$, about point x, of radius t. As σ^- is smooth about x_0^1, and $\sigma^-(x_0^1) = 0$, this implies that $l(\rho) > 0$ for all $\rho > 0$. As $\sigma^-(x) = 0$ if and only if $\iota(x)$ is a non-embedded point, and such points have $\mathcal{H}^n$-measure 0 (Claim 1) in $\tilde{M}$, we have that $\lim_{\rho \to 0^+} l(\rho) = 0$.

Remark 9. As σ^- is smooth on $\tilde{D}_1$, $\sigma^- \leq 0$, and $\sigma^-(x_0^1) = 0$, then, by considering the local Taylor expansion of σ^- about x_0^1, there exists a $C_1 = C_1(N, M, g, z_0) < +\infty$, and a $\delta = \delta(N, M, g, z_0)$, such that for all x in $\tilde{D}_1$,

$$\sigma^-(x) \geq -C_1 d_{\tilde{M}}^2(x, x_0^1).$$

As $l(\rho) \to 0$, as $\rho \to 0$, this implies that we can choose $\rho > 0$, (Remark 10), such that

$$B_{l(\rho)}^{\tilde{M}}(x_0^1) \subset\subset \tilde{D}_1.$$

Then, there exists an x' in $\tilde{D}_1$, such that $d_{\tilde{M}}(x', x_0^1) = l$, and $\sigma^-(x') = -\rho$. Therefore, by Remark

9,

$$\rho \leq C_1 l^2.$$

Remark 10. Note that we have made our first choice of $\rho = \rho(z_0, N, M, g, \delta)$.

We push out on disks D_1 and D_2 equally, so that they meet on $\mathrm{Cut}\,(M)$ in $B_\delta^N(z_0)$, which is our previously denoted set A, as seen in Figure 2.5. We consider the open sets $\tilde{W}_i \subset \tilde{D}_i$, defined by,

$$\tilde{W}_i = \{x \colon F_i(x, \sigma^-(x)) \in B_\delta(z_0)\}.$$

Clearly x_0^i lies in $\tilde{W}_i$, therefore these sets are non-empty. We can then define a diffeomorphism between $\tilde{W}_1$, and $\tilde{W}_2$.

$$\begin{aligned} \Psi \colon \tilde{W}_1 &\to \tilde{W}_2, \\ x &\mapsto (\pi \circ F_2^{-1} \circ F_1 \circ (\mathrm{id}, \sigma^-))(x), \end{aligned}$$

where we define, π by,

$$\begin{aligned} \pi \colon \tilde{M} \times \mathbb{R} &\to \tilde{M}, \\ (x, t) &\mapsto x, \end{aligned}$$

and (id, σ^-), by

$$\begin{aligned} (\mathrm{id}, \sigma^-) \colon \tilde{M} &\to \tilde{M} \times \mathbb{R}, \\ x &\mapsto (x, \sigma^-(x)). \end{aligned}$$

The function Ψ is smooth and has smooth inverse given by

$$\begin{aligned} \Psi^{-1} \colon \tilde{W}_2 &\to \tilde{W}_1, \\ x &\mapsto (\pi \circ F_1^{-1} \circ F_2 \circ (\mathrm{id}, \sigma^-))(x). \end{aligned}$$

We note that, $d\Psi_{x_0^1} = Id$.

Remark 11. We choose $\rho = \rho(z_0, N, M, g, \delta) > 0$, such that,

$$B_{2l}^{\tilde{M}}(x_0^1) \subset\subset \tilde{W}_1.$$

Consider a *push out function*, which lies in $C_c^\infty(\tilde{D}_1)$, and has the following properties,

$$f_1(x) = \begin{cases} -1, & x \in B_l^{\tilde{M}}(x_0^1), \\ [-1,0], & x \in B_{2l}^{\tilde{M}}(x_0^1) \setminus B_l^{\tilde{M}}(x_0^1), \\ 0, & x \in \tilde{D}_1 \setminus B_{2l}^{\tilde{M}}(x_0^1). \end{cases}$$

We further impose the condition,

$$|\nabla f_1| \leq \frac{2}{l}.$$

Define f_2 in $C_c^\infty(\tilde{D}_2)$, by

$$f_2(x) = \begin{cases} (f_1 \circ \Psi^{-1})(x), & x \in \tilde{W}_2, \\ 0, & x \in \tilde{D}_2 \setminus \tilde{W}_2. \end{cases}$$

The support of f_2 will lie in $\Psi(B_{2l}^{\tilde{M}}(x_0^1)) \subset\subset \Psi(\tilde{W}_1) = \tilde{W}_2$. We then define the function f in $C_c^\infty(\tilde{M})$, by $f = f_1 + f_2$.

Define the sets,

$$\begin{aligned} B_{2l} &= B_{2l}^{\tilde{M}}(x_0^1) \cup \Psi(B_{2l}^{\tilde{M}}(x_0^1)), \\ B_l &= B_l^{\tilde{M}}(x_0^1) \cup \Psi(B_l^{\tilde{M}}(x_0^1)), \\ A_l &= B_{2l} \setminus B_l. \end{aligned}$$

We will similarly define the sets,

$$B_t = B_t^{\tilde{M}}(x_0^1) \cup \Psi(B_t^{\tilde{M}}(x_0^1)),$$

for $t > 0$, such that $B_t^{\tilde{M}}(x_0^1) \subset \tilde{W}_1$.

Remark 12. We choose $\rho = \rho(z_0, M, N, g, \delta, W, \lambda) > 0$, such that

$$F_1\left(B_{2l} \times (-2\rho, 2\rho)\right) \subset\subset B_\delta^N(z_0)$$

In the final part of this section we look to define the function that will 'push out away from the non-embedded point'. This function will define the path from (2) to (3) in Figure 2.2.

Remark 13. (Choice of L and r_0) We choose $L = L(z_0, N, M, g, \delta) > 0$ and $r_0 = r_0(z_0, N, M, g, \delta) > 0$, such that,

$$B_L^{\tilde{M}}(x_0^1) \subset\subset \tilde{W}_1,$$

and,

$$F\left(B_L \times (-2r_0, 2r_0)\right) \subset\subset B_\delta^N(z_0).$$

For a sets $\tilde{\Omega}$ and Ω, were Ω is open and $\tilde{\Omega} \subset\subset \Omega$, we define the 2-Capacity of $\tilde{\Omega}$ in Ω as the value,

$$\mathrm{Cap}_2(\tilde{\Omega}, \Omega) = \inf\left\{\int_\Omega |\nabla\varphi|^2 \, d\mathcal{H}^n \colon \varphi \in C_c^\infty(\Omega),\ \varphi \geq \chi_{\tilde{\Omega}}\right\}.$$

For $n \geq 3$, by [25, Theorem 4.15 (ix), Section 4.7.1 and Theorem 4.16, Section 4.7.2],

$$\lim_{k\to\infty} \mathrm{Cap}_2(B^{\tilde{M}}_{\frac{L}{k}}(x_0^1), B^{\tilde{M}}_L(x_0^1)) = \mathrm{Cap}_2(\{x_0^1\}, B^{\tilde{M}}_L(x_0^1)) = 0.$$

Identical proofs show that this also holds for $n = 2$. Therefore, for all $\gamma > 0$, there exists a function $\varphi_{\gamma,k}$, such that,

$$\begin{cases} \varphi_{\gamma,k} \in C_c^\infty(B^{\tilde{M}}_L(x_0^1)), \\ \varphi_{\gamma,k} \colon \tilde{M} \to [0,1], \\ \varphi_{\gamma,k}(x) = 1,\ x \in B^{\tilde{M}}_{\frac{L}{k}}(x_0^1), \end{cases}$$

and, defining $\tilde{\varphi}_{\gamma,k} = \varphi_{\gamma,k} + \varphi_{\gamma,k} \circ \Psi^{-1}$, we have

$$\int_{\tilde{M}} |\nabla\tilde{\varphi}_{\gamma,k}|^2 \, d\mathcal{H}^n(x) < \gamma.$$

We consider the function $\tilde{f} = 1 - \tilde{\varphi}_{\gamma,k}$ in $C^\infty(\tilde{M})$, and $\|\nabla\tilde{f}\|^2_{L^2(\tilde{M})} < \gamma$.

Remark 14. We will later make fixed choices for $L = L(z_0, M, N, g, \delta, W, \lambda)$, $r_0 = r_0(z_0, M, N, g, \delta, W, \lambda, L)$, $\gamma = \gamma(z_0, N, M, g, \delta, r_0, L)$, and $k = k(z_0, N, M, g, \delta, L, \gamma)$.

Remark 15. We make a further choice of $\rho = \rho(z_0, N, M, g, \delta, L, r_0, k)$, such that,

$$B_{2l} \subset\subset B_{\frac{L}{k}},$$

We will make a further choice of ρ later on, so that $\rho = \rho(z_0, N, M, g, \delta, L, k, r_0)$.

2.3.6 Approximating Function for CMC

We use the tools we have constructed to give a simple proof that the function,

$$v_\varepsilon(y) = \overline{\mathbb{H}}_\varepsilon(\tilde{d}(y)),$$

is suitable approximation of M, i.e.

$$\lim_{\varepsilon\to 0} \mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon) = 2\sigma\mathcal{H}^n(M) - \sigma\lambda\mu_g(E) - \sigma\lambda\mu_g(N \setminus E).$$

By the Co-Area formula on the function $\tilde{d}$,

$$\begin{aligned}\mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon) &= \int_N \frac{\varepsilon}{2}|\nabla v_\varepsilon|^2 + \frac{W(v_\varepsilon)}{2} - \sigma\lambda \int_N v_\varepsilon, \\ &= \int_{\mathbb{R}}\int_{\Gamma_t} Q_\varepsilon(t)\, d\mathcal{H}^n\, dt - \sigma\lambda \int_{\mathbb{R}}\int_{\Gamma_t} \overline{\mathbb{H}}_\varepsilon(t)\, d\mathcal{H}^n\, dt,\end{aligned}$$

where,

$$Q_\varepsilon(t) = \frac{\varepsilon}{2}((\overline{\mathbb{H}}_\varepsilon)'(t))^2 + \frac{W(\overline{\mathbb{H}}_\varepsilon(t))}{\varepsilon}.$$

Using the fact that $N \setminus F(\tilde{T})$ is a set of 0 μ_g-measure, and that $F\colon (\tilde{T}, h) \to (F(\tilde{T}), g)$, is a bijective local isometry, we have,

$$\mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon) = \int_{\mathbb{R}} Q_\varepsilon(t)\, \mathcal{H}^n(\tilde{\Gamma}_t)\, dt - \sigma\lambda \int_{\mathbb{R}} \overline{\mathbb{H}}_\varepsilon(t)\, \mathcal{H}^n(\tilde{\Gamma}_t)\, dt.$$

From analysis of $\overline{\mathbb{H}}_\varepsilon$, we have that, $\operatorname{spt} Q_\varepsilon \subset [-2\varepsilon\Lambda, 2\varepsilon\Lambda]$, and

$$2\sigma - \beta\varepsilon^2 \leq \int_{\mathbb{R}} Q_\varepsilon(t)\, dt \leq 2\sigma + \beta\varepsilon^2.$$

Furthermore,

$$\overline{\mathbb{H}}_\varepsilon(t) \leq \begin{cases} 1, & t > -2\varepsilon\Lambda, \\ -1, & t \leq -2\varepsilon\Lambda, \end{cases}$$

and,

$$\overline{\mathbb{H}}_\varepsilon(t) \geq \begin{cases} 1, & t > 2\varepsilon\Lambda, \\ -1, & t \leq 2\varepsilon\Lambda. \end{cases}$$

Therefore,

$$\mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon) \leq (2\sigma + \beta\varepsilon^2) \operatorname*{ess\,sup}_{t\in[-2\varepsilon\Lambda, 2\varepsilon\Lambda]} \mathcal{H}^n(\tilde{\Gamma}_t) - \sigma\lambda \int_{2\varepsilon\Lambda}^{+\infty} \mathcal{H}^n(\tilde{\Gamma}_t)\, dt + \sigma\lambda \int_{-\infty}^{2\varepsilon\Lambda} \mathcal{H}^n(\tilde{\Gamma}_t)\, dt.$$

Similarly,

$$\mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon) \geq (2\sigma - \beta\varepsilon^2) \operatorname*{ess\,inf}_{t\in[-2\varepsilon\Lambda, 2\varepsilon\Lambda]} \mathcal{H}^n(\tilde{\Gamma}_t) - \sigma\lambda \int_{-2\varepsilon\Lambda}^{+\infty} \mathcal{H}^n(\tilde{\Gamma}_t)\, dt + \sigma\lambda \int_{-\infty}^{2\varepsilon\Lambda} \mathcal{H}^n(\tilde{\Gamma}_t)\, dt.$$

We have,

$$\mathcal{H}^n(\tilde{\Gamma}_t) = \int_{\tilde{M}} \theta_t(x)\, d\mathcal{H}^n(x),$$

and by applying Dominated Convergence Theorem to θ_t, we have that,

$$\lim_{t\to 0}\mathcal{H}^n(\tilde{\Gamma}_t)=\lim_{t\to 0}\int_{\tilde{M}}\theta_t(x)\,d\mathcal{H}^n(x)=\mathcal{H}^n(\tilde{M}\cap\tilde{T})=\mathcal{H}^n(\tilde{M}).$$

This implies that,

$$\lim_{\varepsilon\to 0}\operatorname*{ess\,sup}_{t\in[-2\varepsilon\Lambda,2\varepsilon\Lambda]}\mathcal{H}^n(\tilde{\Gamma}_t)=\mathcal{H}^n(\tilde{M})=\mathcal{H}^n(M),$$

and,

$$\lim_{\varepsilon\to 0}\operatorname*{ess\,inf}_{t\in[-2\varepsilon\Lambda,2\varepsilon\Lambda]}\mathcal{H}^n(\tilde{\Gamma}_t)=\mathcal{H}^n(\tilde{M})=\mathcal{H}^n(M).$$

As θ_t is bounded and lower semicontinuous in the variable t, by the Dominated Convergence Theorem we have that the function $t\mapsto\mathcal{H}^n(\tilde{\Gamma}_t)$ is also bounded and lower semicontinuous (and hence measurable). Thus,

$$\lim_{\varepsilon\to 0}\int_{\pm 2\varepsilon\Lambda}^{+\infty}\mathcal{H}^n(\tilde{\Gamma}_t)\,dt=\int_0^{+\infty}\mathcal{H}^n(\tilde{\Gamma}_t)\,dt=\mathcal{H}^{n+1}(\{y\in N\colon\tilde{d}(y)>0\})=\mu_g(E),$$

and,

$$\lim_{\varepsilon\to 0}\int_{-\infty}^{\pm 2\varepsilon\Lambda}\mathcal{H}^n(\tilde{\Gamma}_t)\,dt=\int_{-\infty}^{0}\mathcal{H}^n(\tilde{\Gamma}_t)\,dt=\mathcal{H}^{n+1}(\{y\in N\colon\tilde{d}(y)<0\})=\mu_g(N\setminus E).$$

Therefore, we have,

$$\lim_{\varepsilon\to 0}\mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon)=2\sigma\mathcal{H}^n(M)-\sigma\lambda\mu_g(E)+\sigma\lambda\mu_g(N\setminus E).$$

2.4 Base Computation

Consider a smooth function,

$$\eta\colon\mathbb{R}\times\tilde{M}\to\mathbb{R}.$$

and define the following

$$\begin{aligned} v_\varepsilon^{r,\eta}\colon\tilde{M}\times\mathbb{R} &\to \mathbb{R},\\ (x,t) &\mapsto \overline{\mathbb{H}}_\varepsilon(t-\eta(r,x)). \end{aligned}$$

Take (x,t) in $\tilde{M}\times\mathbb{R}$, such that $dF_{(x,t)}$ is invertible, then the metric $h=F^*g$, is well defined about (x,t), and by the Gauss Lemma ([29, Lemma 2.11]), it will have the form

$$h(x,t)=h_{\tilde{M}\times\{t\}}(x)+dt^2,$$

where we define,

$$h_{\tilde{M}\times\{t\}}(x) := h(x,t)_{|T_x\tilde{M}}.$$

Thus,

$$|\nabla v_\varepsilon^{r,\eta}(x,t)|^2 = ((\overline{\mathbb{H}}_\varepsilon)'(t-\eta(r,x)))^2(1+|\nabla_x\eta(r,x)|^2(x,t)),$$

where, $(\nabla_x\eta(r,x))(x,t)$, is the gradient at (x,t), of the function $(x,t)\mapsto\eta(r,x)$, with respect to the metric h. By the co-area formula on p,

$$\begin{aligned}
\mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon^{r,\eta}) &= \int_{\tilde{T}} \frac{\varepsilon}{2}|\nabla v_\varepsilon^{r,\eta}|^2 + \frac{W(v_\varepsilon^{r,\eta})}{\varepsilon} - \sigma\lambda v_\varepsilon^{r,\eta}\,d\mu_h,\\
&= \int_{\mathbb{R}}\int_{\tilde{\Gamma}_t} \frac{\varepsilon}{2}((\overline{\mathbb{H}}_\varepsilon)'(t-\eta(r,x)))^2|\nabla_x\eta(r,x)|^2(x,t)\,d\mathcal{H}^n(x,t)\,dt\\
&\quad + \int_{\mathbb{R}}\int_{\tilde{\Gamma}_t} \frac{\varepsilon}{2}((\overline{\mathbb{H}}_\varepsilon)'(t-\eta(r,x)))^2 + \frac{W(\overline{\mathbb{H}}_\varepsilon(t-\eta(r,x)))}{\varepsilon}\\
&\qquad\qquad -\sigma\lambda\overline{\mathbb{H}}_\varepsilon(t-\eta(r,x))\,d\mathcal{H}^n(x,t)\,dt,\\
&= \int_{\tilde{M}}\int_{\sigma^-(x)}^{\sigma^+(x)} \frac{\varepsilon}{2}((\overline{\mathbb{H}}_\varepsilon)'(t-\eta(r,x)))^2|\nabla_x\eta(r,x)|^2(x,t)\,\theta_t(x)\,dt\,d\mathcal{H}^n(x)\\
&\quad + \int_{\tilde{M}}\int_{\sigma^-(x)}^{\sigma^+(x)} \Bigg(\frac{\varepsilon}{2}((\overline{\mathbb{H}}_\varepsilon)'(t-\eta(r,x)))^2 + \frac{W(\overline{\mathbb{H}}_\varepsilon(t-\eta(r,x)))}{\varepsilon}\\
&\qquad\qquad -\sigma\lambda\overline{\mathbb{H}}_\varepsilon(t-\eta(r,x))\Bigg)\,\theta_t(x)\,dt\,d\mathcal{H}^n(x),
\end{aligned}$$

In the last equality we use Fubini's Theorem to switch the integrals.

We have,

$$\begin{aligned}
\mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon^{r,\eta}) - \mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon^{0,\eta}) &= \int_{\tilde{M}}\int_{\sigma^-(x)}^{\sigma^+(x)} \frac{\varepsilon}{2}((\overline{\mathbb{H}}_\varepsilon)'(t-\eta(r,x)))^2|\nabla_x\eta(r,x)|^2(x,t)\,\theta_t(x)\,dt\,d\mathcal{H}^n(x)\\
&\quad - \int_{\tilde{M}}\int_{\sigma^-(x)}^{\sigma^+(x)} \frac{\varepsilon}{2}((\overline{\mathbb{H}}_\varepsilon)'(t-\eta(0,x)))^2|\nabla_x\eta(0,x)|^2(x,t)\,\theta_t(x)\,dt\,d\mathcal{H}^n(x),\\
&\quad + \int_{\tilde{M}}\int_{\sigma^-(x)}^{\sigma^+(x)} (Q_\varepsilon(t-\eta(r,x)) - Q_\varepsilon(t-\eta(0,x)))\theta_t(x)\,dt\,d\mathcal{H}^n(x)\\
&\quad - \int_{\tilde{M}}\int_{\sigma^-(x)}^{\sigma^+(x)} \sigma\lambda(\overline{\mathbb{H}}_\varepsilon(t-\eta(r,x)) - \overline{\mathbb{H}}_\varepsilon(t-\eta(0,x)))\,\theta_t(x)\,dt\,d\mathcal{H}^n(x),
\end{aligned}$$

We have the following two terms,

$$\begin{aligned}
I_\varepsilon^{r,\eta} &= \int_{\tilde{M}}\int_{\sigma^-(x)}^{\sigma^+(x)} \frac{\varepsilon}{2}((\overline{\mathbb{H}}_\varepsilon)'(t-\eta(r,x)))^2|\nabla_x\eta(r,x)|^2(x,t)\,\theta_t(x)\,dt\,d\mathcal{H}^n(x)\\
&\quad - \int_{\tilde{M}}\int_{\sigma^-(x)}^{\sigma^+(x)} \frac{\varepsilon}{2}((\overline{\mathbb{H}}_\varepsilon)'(t-\eta(0,x)))^2|\nabla_x\eta(0,x)|^2(x,t)\,\theta_t(x)\,dt\,d\mathcal{H}^n(x),
\end{aligned}$$

and, by Fundamental Theorem of Calculus and Fubini's Theorem,

$$\begin{aligned}
II_\varepsilon^{r,\eta} &= \int_{\tilde{M}} \int_{\sigma^-(x)}^{\sigma^+(x)} (Q_\varepsilon(t-\eta(r,x)) - Q_\varepsilon(t-\eta(0,x)))\theta_t(x)\,dt\,d\mathcal{H}^n(x) \\
&\quad - \int_{\tilde{M}} \int_{\sigma^-(x)}^{\sigma^+(x)} \sigma\lambda(\overline{\mathbb{H}}_\varepsilon(t-\eta(r,x)) - \overline{\mathbb{H}}_\varepsilon(t-\eta(0,x)))\,\theta_t(x)\,dt\,d\mathcal{H}^n(x), \\
&= -\int_0^r \int_{\tilde{M}} \partial_s\eta(s,x) \int_{\sigma^-(x)}^{\sigma^+(x)} Q'_\varepsilon(t-\eta(s,x))\,\theta_t(x)\,dt\,d\mathcal{H}^n(x)\,ds \\
&\quad + \int_0^r \int_{\tilde{M}} \partial_s\eta(s,x) \int_{\sigma^-(x)}^{\sigma^+(x)} \sigma\lambda(\overline{\mathbb{H}}_\varepsilon)'(t-\eta(s,x))\,\theta_t(x)\,dt\,d\mathcal{H}^n(x)\,ds, \\
&= -\int_0^r \int_{\tilde{M}} \partial_s\eta(s,x) Q_\varepsilon(\sigma^+(x)-\eta(s,x))\,\theta^+(x)\,d\mathcal{H}^n(x)\,ds \\
&\quad + \int_0^r \int_{\tilde{M}} \partial_s\eta(s,x) Q_\varepsilon(\sigma^-(x)-\eta(s,x))\,\theta^-(x)\,d\mathcal{H}^n(x)\,ds \\
&\quad + \int_0^r \int_{\tilde{M}} \partial_s\eta(s,x) \int_{\sigma^-(x)}^{\sigma^+(x)} Q_\varepsilon(t-\eta(s,x))\,\partial_t\theta_t(x)\,dt\,d\mathcal{H}^n(x)\,ds \\
&\quad + \int_0^r \int_{\tilde{M}} \partial_s\eta(s,x) \int_{\sigma^-(x)}^{\sigma^+(x)} \sigma\lambda(\overline{\mathbb{H}}_\varepsilon)'(t-\eta(s,x))\,\theta_t(x)\,dt\,d\mathcal{H}^n(x)\,ds, \\
&= -\int_0^r \int_{\tilde{M}} \partial_s\eta(s,x) Q_\varepsilon(\sigma^+(x)-\eta(s,x))\,\theta^+(x)\,d\mathcal{H}^n(x)\,ds \\
&\quad + \int_0^r \int_{\tilde{M}} \partial_s\eta(s,x) Q_\varepsilon(\sigma^-(x)-\eta(s,x))\,\theta^-(x)\,d\mathcal{H}^n(x)\,ds \\
&\quad + \int_0^r \int_{\tilde{M}} \partial_s\eta(s,x) \int_{\sigma^-(x)}^{\sigma^+(x)} Q_\varepsilon(t-\eta(s,x))\,(\lambda - H_t(x))\,\theta_t(x)\,dt\,d\mathcal{H}^n(x)\,ds \\
&\quad + \lambda \int_0^r \int_{\tilde{M}} \Theta^1_{\varepsilon,\eta}(s,x) - \Theta^2_{\varepsilon,\eta}\,d\mathcal{H}^n(x)\,ds,
\end{aligned}$$

Where,

$$\begin{aligned}
\theta^+(x) &= \lim_{t \nearrow \sigma^+(x)} \theta_t(x), \\
\theta^-(x) &= \lim_{t \searrow \sigma^-(x)} \theta_t(x), \\
\Theta^1_{\varepsilon,\eta}(s,x) &= \sigma \int_{\sigma^-(x)}^{\sigma^+(x)} \partial_s\eta(s,x)(\overline{\mathbb{H}}_\varepsilon)'(t-\eta(s,x))\theta_t(x)\,dt, \\
\Theta^2_{\varepsilon,\eta}(s,x) &= \int_{\sigma^-(x)}^{\sigma^+(x)} \partial_s\eta(s,x) Q_\varepsilon(t-\eta(s,x))\theta_t(x)\,dt.
\end{aligned}$$

For the last equality of $II_\varepsilon^{r,\eta}$ we are using $\partial_t\theta_t(x) = -H_t(x)\theta_t(x)$, for t in $(\sigma^-(x), \sigma^+(x))$ (Proposition 6).

2.5 Competitor

2.5.1 Calculation on $\tilde{M} \times \mathbb{R}$

Here we construct the path in Figure 2.2 from (1) to (2).

Set $\eta_1(r,x) = rf(x)$, take r in $[0,\rho]$, where $\rho \in (0,1)$ will be chosen later and $f\colon \tilde{M} \to \mathbb{R}$ as defined in Section 2.3.5.

Remark 16. (Choice in ε_1) We choose $\varepsilon_1 = \varepsilon_1(\rho) \in (0,1/4)$, such that,

$$2\varepsilon_1\Lambda = 6\varepsilon_1|\log \varepsilon_1| << \rho.$$

From here we consider ε in $(0,\varepsilon_1)$.

We have,

$$\begin{aligned}
II_\varepsilon^{r,\eta_1} = &-\int_0^r \int_{\tilde{M}} f(x) Q_\varepsilon(\sigma^+(x) - sf(x))\,\theta^+(x)\,d\mathcal{H}^n(x)\,ds \\
&+\int_0^r \int_{\tilde{M}} f(x) Q_\varepsilon(\sigma^-(x) - sf(x))\,\theta^-(x)\,d\mathcal{H}^n(x)\,ds \\
&+\int_0^r \int_{\tilde{M}} f(x) \int_{\sigma^-(x)}^{\sigma^+(x)} Q_\varepsilon(t - sf(x))\,(\lambda - H_t(x))\,\theta_t(x)\,dt\,d\mathcal{H}^n(x)\,ds \\
&+\lambda \int_0^r \int_{\tilde{M}} \Theta^1_{\varepsilon,\eta_1}(s,x) - \Theta^2_{\varepsilon,\eta_1}(s,x)\,d\mathcal{H}^n(x)\,ds.
\end{aligned} \tag{2.12}$$

Concentrate on the second term of the right hand side of (2.12). As the integrand is non-positive, $f=-1$ on B_l and $\operatorname{spt} Q_\varepsilon \subset [-2\varepsilon\Lambda, 2\varepsilon\Lambda]$, we have

$$\begin{aligned}
&\int_0^r \int_{\tilde{M}} f(x) Q_\varepsilon(\sigma^-(x) - sf(x))\,\theta^-(x)\,d\mathcal{H}^n(x)\,ds \\
&\qquad \leq -(2\sigma - \beta\varepsilon^2) \int_{B_l \cap \{-r+2\varepsilon\Lambda \leq \sigma^-(x) \leq -2\varepsilon\Lambda\}} \theta^-(x)\,d\mathcal{H}^n(x)
\end{aligned}$$

We look for lower bounds on θ^-.

Remark 17. Choose $\delta = \delta(z_0, N, M, g) > 0$, such that,

$$\min_{y \in B^N_\delta(z_0)} \{\Delta\tilde{d}_1(y), \Delta\tilde{d}_2(y)\} \geq \frac{\lambda}{2},$$

and,

$$\max_{y \in B^N_\delta(z_0)} \{\Delta\tilde{d}_1(y), \Delta\tilde{d}_2(y)\} \leq \frac{3\lambda}{2}.$$

Therefore, for (x,t) in $\tilde{T}$, such that, $F(x,t)$ lies in $B^N_\delta(z_0)$, we have that,

$$\frac{\lambda}{2} \le H_t(x) \le \frac{3\lambda}{2}.$$

Thus by similar calculations carried out in Section 2.3.4, for all (x,t) in $\tilde{T}$, such that $F(x,t)$ lies in $B^N_\delta(z_0)$, we have,

$$\theta_t(x) \ge \begin{cases} e^{-\frac{3\lambda t}{2}}, & t \ge 0, \\ e^{-\frac{\lambda t}{2}}, & t \le 0. \end{cases}$$

For x in B_l, we have $\sigma^-(x) > -\rho$, and by choice of ρ in Remark 12, we have that $F(\{x\} \times (\sigma^-(x), 0)) \subset B^N_\delta(z_0)$. Thus

$$\theta^-(x) = \lim_{t \searrow \sigma^-(x)} \theta_t(x) \ge e^{-\frac{\lambda\,\sigma^-(x)}{2}} \ge 1,$$

for all x in B_l. Therefore,

$$\begin{aligned} &\int_0^r \int_{\tilde{M}} f(x) Q_\varepsilon(\sigma^-(x) - sf(x))\, \theta^-(x)\, d\mathcal{H}^n(x)\, ds \\ &\qquad \le -2\sigma \mathcal{H}^n(\{x \colon x \in B_l,\, -r + 2\varepsilon\Lambda \le \sigma^-(x) \le -2\varepsilon\Lambda\}) + C_2\varepsilon^2, \end{aligned}$$

for $C_2 = C_2(N, M, g, \lambda, W) < +\infty$. This is a lower bound for the area deleted in pushing the disks together.

Concentrate on First term on the right hand side of (2.12). By choice of $\delta > 0$ in Remark 7 and $\rho > 0$ in Remark 12 we have that for x in $\operatorname{spt} f \subset B_{2l}$, $\sigma^+(x) > 2\rho >> 2\varepsilon\Lambda$. Thus, as $\operatorname{spt} Q_\varepsilon \subset [-2\varepsilon\Lambda, 2\varepsilon\Lambda]$,

$$\int_0^r \int_{\tilde{M}} f(x) Q_\varepsilon(\sigma^+(x) - sf(x))\, \theta^+(x)\, d\mathcal{H}^n(x)\, ds = 0.$$

Concentrate on the third term on the right hand side of (2.12). Consider $s > 0$, and x in $\tilde{M}$, such that $sf(x) < -2\varepsilon\Lambda$. Again, using the fact that $\operatorname{spt} Q_\varepsilon \subset [-2\varepsilon\Lambda, 2\varepsilon\Lambda]$, and the inequalities on H_t in Remark 5,

$$\begin{aligned} \int_{\sigma^-(x)}^{\sigma^+(x)} Q_\varepsilon(t - sf(x))\,(\lambda - H_t(x))\,\theta_t(x)\,dt &= \int_{-2\varepsilon\Lambda}^{2\varepsilon\Lambda} Q_\varepsilon(\xi)(\lambda - H_{\xi + sf(x)})\theta_{\xi+sf(x)} d\xi, \\ &\ge 0. \end{aligned}$$

For $sf(x) \geq -2\varepsilon\Lambda$, we have,

$$\begin{aligned}\int_{\sigma^-(x)}^{\sigma^+(x)} Q_\varepsilon(t - sf(x))\,(\lambda - H_t(x))\,\theta_t(x)\,dt &= \int_{-2\varepsilon\Lambda}^{2\varepsilon\Lambda} Q_\varepsilon(\xi)(\lambda - H_{\xi+sf(x)})\theta_{\xi+sf(x)}d\xi,\\ &\geq C_2 \min_{t\in[-4\varepsilon\Lambda,2\varepsilon\Lambda]}(\lambda - H_t(x))\theta_t(x),\end{aligned}$$

potentially rechoosing $C_2 = C_2(M, N, g, \lambda, W)$. Therefore we have that for all r in $[0, \rho]$,

$$\begin{aligned}II_\varepsilon^{r,\eta_1} \leq\ & -2\sigma\mathcal{H}^n(\{x\colon x \in B_l,\ -r + 2\varepsilon\Lambda \leq \sigma^-(x) \leq -2\varepsilon\Lambda\})\\ &+C_2\left(r\int_{B_{2l}} q_\varepsilon^1(x)\,d\mathcal{H}^n(x) + \int_0^r\int_{\tilde{M}} \Theta^1_{\varepsilon,\eta_1}(s,x) - \Theta^2_{\varepsilon,\eta_1}(s,x)\,d\mathcal{H}^n(x)\,ds + \varepsilon^2\right),\end{aligned}$$

where,

$$q_\varepsilon^1(x) = \max_{t\in[-4\varepsilon\Lambda,2\varepsilon\Lambda]}(H_t(x) - \lambda)\theta_t(x) \geq 0,$$

and we have potentially rechosen $C_2 = C_2(M, N, g, \lambda, W)$. Therefore, for $r < 4\varepsilon\Lambda$,

$$II_\varepsilon^{r,\eta_1} \leq C_2\left(r\int_{B_{2l}} q_\varepsilon^1(x)\,d\mathcal{H}^n(x) + \int_0^r\int_{\tilde{M}} \Theta^1_{\varepsilon,\eta_1}(s,x) - \Theta^2_{\varepsilon,\eta_1}(s,x)\,d\mathcal{H}^n(x)\,ds + \varepsilon^2\right),$$

and for $r \geq 4\varepsilon\Lambda$,

$$\begin{aligned}II_\varepsilon^{r,\eta_1} \leq\ & -2\sigma\mathcal{H}^n(\{x\colon x \in B_l,\ -r + 2\varepsilon\Lambda \leq \sigma^-(x) \leq 0\})\\ &+C_2\Bigg(\mathcal{H}^n(\{x\colon x \in B_l, -2\varepsilon\Lambda < \sigma^-(x) \leq 0\})\\ &+\int_{B_{2l}} q_\varepsilon^1(x)\,d\mathcal{H}^n(x) + \int_0^r\int_{\tilde{M}} \Theta^1_{\varepsilon,\eta_1}(s,x) - \Theta^2_{\varepsilon,\eta_1}(s,x)\,d\mathcal{H}^n(x)\,ds + \varepsilon^2\Bigg).\end{aligned}$$

Again we are potentially rechoosing $C_2 = C_2(M, N, g, \lambda, W)$.

We now turn our attention to the term,

$$\begin{aligned}I_\varepsilon^{r,\eta_1} &= \int_{\tilde{M}}\int_{\sigma^-(x)}^{\sigma^+(x)} \frac{\varepsilon}{2}((\overline{\mathbb{H}}_\varepsilon)'(t - rf(x)))^2|r\nabla f|^2(x,t)\,\theta_t(x)\,dt\,d\mathcal{H}^n(x),\\ &= \int_{\tilde{M}}\int_{-2\varepsilon\Lambda}^{2\varepsilon\Lambda} \frac{\varepsilon}{2}((\overline{\mathbb{H}}_\varepsilon)'(\xi))^2|r\nabla f|^2(x, \xi + rf(x))\,\theta_{\xi+rf(x)}(x)\,d\xi\,d\mathcal{H}^n(x).\end{aligned}$$

Remark 18. Recall choice of $\delta = \delta(z_0, M, N, g)$ in Remark 7. Consider $|t| < \delta$, and x, such that $\iota(x)$ lies in $B_\delta^N(z_0)$. Take $\{E_1, \dots, E_n\}$ to be an orthonormal basis for $T_x\tilde{M}$, with respect to the metric $h_{\tilde{M}\times\{0\}}$. Then, as previously shown, we have the following metric about x on $\tilde{M}$,

$$(h_{\tilde{M}\times\{t\}})_{ij}(x) = h_{\tilde{M}\times\{t\}}(x)(E_i, E_j).$$

For a function φ on $\tilde{M}$,

$$|\nabla\varphi|^2(x,t) = (h_{\tilde{M}\times\{t\}})^{ij}(x)d\varphi(E_i)d\varphi(E_j).$$

We make a further choice $\delta = \delta(z_0, N, M, g) > 0$, such that there exists a $C_3 = C_3(z_0, \delta, M, N, g)$,

$$1 \le \sup\left\{(h_{\tilde{M}\times\{t\}})^{ij}(x)X_iX_j \colon \iota(x) \in B^N_\delta(z_0),\, |t| \le \delta/2,\, \sum_i X_i^2 = 1\right\} \le C_3 < \infty,$$

and,

$$0 < C_3^{-1} \le \inf\left\{(h_{\tilde{M}\times\{t\}})^{ij}(x)X_iX_j \colon \iota(x) \in B^N_\delta(z_0),\, |t| \le \delta/2,\, \sum_i X_i^2 = 1\right\} \le 1.$$

By choices of ρ in Remark 12, and ε in Remark 16, for all x in A_l, r in $[0, \rho]$, and ξ in $[-2\varepsilon\Lambda, 2\varepsilon\Lambda]$,

$$|r\nabla f|^2(x, \xi + rf(x)) \le C_3 r^2 |\nabla f|^2(x, 0) \le 4C_3\frac{r^2}{l^2}.$$

Note that for x in $\tilde{M} \setminus A_l$, $|\nabla f|(x,t) = 0$, for all t. We have,

$$I_\varepsilon^{r,\eta_1} \le C_3\mathcal{H}^n(A_l)\frac{r^2}{l^2},$$

where we have potentially rechosen $C_3 = C_3(z_0, N, M, g, \delta, \lambda, W) < \infty$.

For r in $[0, 4\varepsilon\Lambda)$, we have,

$$\begin{aligned} I_\varepsilon^{r,\eta_1} + II_\varepsilon^{r,\eta_1} \;\; &\le \;\; C_3\frac{(\varepsilon\Lambda)^2}{l^2} + C_2\Bigg(\varepsilon\Lambda\int_{B_{2l}} q_\varepsilon^1(x)\, d\mathcal{H}^n(x) \\ &\qquad + \int_0^r\int_{\tilde{M}} \Theta^1_{\varepsilon,\eta_1}(s,x) - \Theta^2_{\varepsilon,\eta_1}(s,x)\, d\mathcal{H}^n(x)\, ds + \varepsilon^2\Bigg). \end{aligned}$$

Again, we are potentially rechoosing $C_3 = C_3(z_0, N, M, g, \delta, \lambda, W) < \infty$.

For r in $[4\varepsilon\Lambda, \rho]$ we define the following non-decreasing function,

$$P_\varepsilon(r) := \frac{\mathcal{H}^n(\{x \colon x \in B_l,\, -r + 2\varepsilon\Lambda \le \sigma^-(x) \le 0\})}{\mathcal{H}^n(A_l)},$$

and we have,

$$\begin{aligned} I_\varepsilon^{r,\eta_1} + II_\varepsilon^{r,\eta_1} \;\; &\le \;\; \mathcal{H}^n(A_l)\left(C_3\frac{r^2}{l^2} - 2\sigma P_\varepsilon(r)\right) \\ &\quad + C_2\Bigg(\mathcal{H}^n(\{x \colon x \in B_l, -2\varepsilon\Lambda < \sigma^-(x) \le 0\}) + \int_{B_{2l}} q_\varepsilon^1(x)\, d\mathcal{H}^n(x) \\ &\quad + \int_0^r\int_{\tilde{M}} \Theta^1_{\varepsilon,\eta_1}(s,x) - \Theta^2_{\varepsilon,\eta_1}(s,x)\, d\mathcal{H}^n(x)\, ds + \varepsilon^2\Bigg). \end{aligned}$$

We now define the following function on $[0,1]$,

$$\kappa_\varepsilon(s) = \begin{cases} 0, & s \in [0,(4\varepsilon\Lambda)/\rho), \\ C_3\frac{\rho^2}{l^2}s^2 - 2\sigma P_\varepsilon(s\rho), & s \in [(4\varepsilon\Lambda)/\rho, 1]. \end{cases}$$

Note that,

$$P_\varepsilon(\rho) \xrightarrow{\varepsilon\to 0} \frac{\mathcal{H}^n(B_l)}{\mathcal{H}^n(A_l)} \xrightarrow{\rho\to 0} \frac{1}{2^n-1},$$

and furthermore, recalling the bound $\rho \leq C_1 l^2$, $C_1 = C_1(z_0, N, M, g, \delta) < +\infty$, we have,

$$0 < \frac{\rho^2}{l^2} \leq C_1\rho \xrightarrow{\rho\to 0} 0.$$

Remark 19. Choose $\rho = \rho(z_0, N, M, g, \delta, \lambda, W) > 0$, such that

$$C_3\frac{\rho^2}{l^2} < \frac{\sigma}{2(2^n-1)},$$

and,

$$\frac{\mathcal{H}^n(B_l)}{\mathcal{H}^n(A_l)} > \frac{7}{8(2^n-1)}.$$

Remark 20. There exists an $\varepsilon_2 = \varepsilon_2(z_0, M, N, g, \delta, W, \lambda, \rho) > 0$, such that, $\varepsilon_2 \leq \varepsilon_1$, and for all ε in $(0,\varepsilon_2)$,

$$P_\varepsilon(\rho) > \frac{3}{4(2^n-1)}.$$

From here we always choose ε in $(0,\varepsilon_2)$.

We have that,

$$\max_{s\in[0,1]} \kappa_\varepsilon(s) \leq C_3\frac{\rho^2}{l^2} < \frac{\sigma}{2(2^n-1)},$$

and,

$$\kappa_\varepsilon(1) < -\frac{\sigma}{2^n-1}.$$

We have, for r in $[0, 4\varepsilon\Lambda)$,

$$\mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon^{r,\eta_1}) \leq \mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon) + III_\varepsilon^{1,r},$$

where,

$$III_\varepsilon^{1,r} = C_4\left(\varepsilon\Lambda\int_{B_{2l}} q_\varepsilon^1(x)\,d\mathcal{H}^n(x) + \int_0^r\int_{\tilde{M}} \Theta^1_{\varepsilon,\eta_1}(s,x) - \Theta^2_{\varepsilon,\eta_1}(s,x)\,d\mathcal{H}^n(x)\,ds + (\varepsilon\Lambda)^2\right),$$

and $C_4 = C_4(z_0, M, N, g, \delta, W, \lambda, \rho) < +\infty$.

For r in $[4\varepsilon\Lambda, \rho]$,

$$\mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon^{r,\eta_1}) \leq \mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon) + \mathcal{H}^n(A_l)\kappa_\varepsilon\left(\frac{r}{\rho}\right) + III_\varepsilon^{2,r}, \tag{2.13}$$

where,

$$\begin{aligned} III_\varepsilon^{2,r} \;&=\; C_2\bigg(\mathcal{H}^n(\{x\colon x \in B_l, -2\varepsilon\Lambda < \sigma^-(x) \leq 0\}) + \int_{B_{2l}} q_\varepsilon^1(x)\, d\mathcal{H}^n(x) \\ &\quad + \int_0^r \int_{\tilde{M}} \Theta^1_{\varepsilon,\eta_1}(s,x) - \Theta^2_{\varepsilon,\eta_1}(s,x)\, d\mathcal{H}^n(x)\, ds + \varepsilon^2\bigg). \end{aligned}$$

Furthermore,

$$\mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon^{\rho,\eta_1}) \leq \mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon) - \frac{\sigma\mathcal{H}^n(A_l)}{2^n - 1} + III_\varepsilon^{2,\rho}. \tag{2.14}$$

2.5.2 Appropriate Function on Manifold

We wish to show that for every r in $[0, \rho]$, there exists an $\tilde{v}_\varepsilon^{r,\eta_1}$, in $W^{1,\infty}(N) \subset W^{1,2}(N)$, such that, for every (x,t) in $\tilde{T}$,

$$\tilde{v}_\varepsilon^{r,\eta_1}(F(x,t)) = v_\varepsilon^{r,\eta_1}(x,t).$$

This implies that $\mathcal{F}_{\varepsilon,\lambda}(\tilde{v}_\varepsilon^{r,\eta_1})(N) = \mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon^{r,\eta_1})(\tilde{T})$. Indeed, this follows from the fact that $\mu_g(N \setminus F(\tilde{T})) = \mu_g(\mathrm{Cut}(M) \cup (\overline{M} \setminus M)) = 0$, and $F\colon (\tilde{T}, h) \to (F(\tilde{T}), g)$ is an isometric bijection between open sets.

We have the following,

$$B_\delta^N(z_0) = \Upsilon_1 \sqcup A \sqcup \Upsilon_2,$$

where,

$$\begin{aligned} \Upsilon_1 =& \{y \in B_\delta^N(z_0)\colon \tilde{d}_1(y) > \tilde{d}_2(y)\}, \\ \Upsilon_2 =& \{y \in B_\delta^N(z_0)\colon \tilde{d}_2(y) > \tilde{d}_1(y)\}, \\ A =& \{y \in B_\delta^N(z_0)\colon \tilde{d}_1(y) = \tilde{d}_2(y)\}, \end{aligned}$$

and, recall from Remark 8, that A is a smooth n-submanifold in $B_\delta^N(z_0)$. Also, recall the diffeomorphisms, for $i = 1, 2$, defined in Remark 7,

$$F_i\colon \tilde{V}_i \subset \tilde{M} \times \mathbb{R} \to B_{2\delta}^N(z_0).$$

We then define, $\tilde{v}_\varepsilon^{r,\eta_1}$,

$$\tilde{v}_\varepsilon^{r,\eta_1}(y) = \begin{cases} \overline{\mathbb{H}}_\varepsilon(\tilde{d}(y)), & y \notin B_\delta^N(z_0), \\ v_\varepsilon^{r,\eta_1}(F_1^{-1}(y)), & y \in \overline{\Upsilon_1} \cap B_\delta^N(z_0), \\ v_\varepsilon^{r,\eta_1}(F_2^{-1}(y)), & y \in \overline{\Upsilon_2} \cap B_\delta^N(z_0). \end{cases}$$

For (x,t) in $\tilde{T}$, we have $\tilde{v}_\varepsilon^{r,\eta_1}(F(x,t)) = v_\varepsilon^{r,\eta_1}(x,t)$. Indeed, first we consider the case that $F(x,t)$ lies in $\Upsilon_1 \cup \Upsilon_2$. In Υ_i, $F = F_i$, and we have,

$$\tilde{v}_\varepsilon^{r,\eta_1}(F(x,t)) = v_\varepsilon^{r,\eta_1}(F_i^{-1}(F(x,t))) = v_\varepsilon^{r,\eta_1}(x,t).$$

As $A \subset \mathrm{Cut}(M)$, we know that $F(x,t)$ cannot lie on A. Last case to consider is $F(x,t)$ lies in $N \setminus B_\delta^N(z_0)$. By Remark 12 (x,t) must lie in $\tilde{T} \setminus (B_{2l} \times (-2\rho, 2\rho))$. If x lies in $\tilde{M} \setminus B_{2l}$, then $f(x) = 0$, and,

$$v_\varepsilon^{r,\eta_1}(x,t) = \overline{\mathbb{H}}_\varepsilon(t) = \overline{\mathbb{H}}_\varepsilon(\tilde{d}(F(x,t))) = \tilde{v}_\varepsilon^{r,\eta_1}(F(x,t)).$$

If x lies in B_{2l}, then $|t| \geq 2\rho > r|f(x)| + 2\varepsilon\Lambda$, and therefore,

$$v_\varepsilon^{r,\eta_1}(x,t) = \overline{\mathbb{H}}_\varepsilon(t - rf(x)) = \begin{cases} 1, & t \geq 2\rho > rf(x) + 2\varepsilon\Lambda, \\ -1, & t \leq -2\rho < rf(x) - 2\varepsilon\Lambda. \end{cases}$$

Also, $\tilde{d}(F(x,t)) = t$, implies that,

$$\tilde{v}_\varepsilon^{r,\eta_1}(F(x,t)) = \overline{\mathbb{H}}_\varepsilon(t) = \begin{cases} 1, & t \geq 2\rho > 2\varepsilon\Lambda, \\ -1, & t \leq -2\rho < -2\varepsilon\Lambda. \end{cases}$$

Therefore, for all (x,t) in $\tilde{T}$, we have that $v_\varepsilon^{r,\eta_1}(x,t) = \tilde{v}_\varepsilon^{r,\eta_1}(F(x,t))$.

We now just look to show that $\tilde{v}_\varepsilon^{r,\eta_1}$ lies in $W^{1,\infty}(N)$. First consider y in $N \setminus F(B_{2l} \times (-2\rho, 2\rho))$. There exists an x in $\tilde{M}$, such that, $F(x, \tilde{d}(y)) = y$, and $(x, \tilde{d}(y))$ lies in $(\tilde{M} \times \mathbb{R}) \setminus (B_{2l} \times (-2\rho, 2\rho))$. By previous argument we see that,

$$\tilde{v}_\varepsilon^{r,\eta_1}(y) = \overline{\mathbb{H}}_\varepsilon(\tilde{d}(y)).$$

and therefore, $\tilde{v}_\varepsilon^{r,\eta_1}$ is Lipschitz on the set $N \setminus F(B_{2l} \times (-2\rho, 2\rho))$.

As

$$F(B_{2l} \times (-2\rho, 2\rho)) \subset\subset B_\delta^N(z_0),$$

showing that $\tilde{v}_\varepsilon^{r,\eta_1}$ is Lipschitz on $B_\delta^N(z_0)$, implies that it is Lipschitz on N. As $\tilde{v}_\varepsilon^{r,\eta_1}$ is smooth with bounded Lipschitz constant in $\Upsilon_1 \cup \Upsilon_2 \subset B_\delta^N(z_0)$, we just need to show that it is continuous across the smooth n-submanifold $A = \partial\Upsilon_1 \cap B_\delta^N(z_0) = \partial\Upsilon_2 \cap B_\delta^N(z_0)$. Consider y on A, then $\tilde{d}_1(y) = \tilde{d}_2(y)$, and by construction of f and Ψ,

$$f(\pi(F_1^{-1}(y))) = f(\pi(F_2^{-1}(y))).$$

Therefore,

$$v_\varepsilon^{r,\eta_1}(F_1^{-1}(y)) = v_\varepsilon^{r,\eta_1}(F_2^{-1}(y)),$$

and $\tilde{v}_\varepsilon^{r,\eta_1}$ is well defined and continuous across A. Thus we have that $\tilde{v}_\varepsilon^{r,\eta_1}$ lies in $W^{1,\infty}(B_\delta^N(z_0))$.

2.5.3 Continuity of the Path

We show that the path,

$$\begin{array}{rcl} \gamma\colon [0,\rho] & \to & W^{1,2}(N), \\ r & \mapsto & \tilde{v}_\varepsilon^{r,\eta_1}, \end{array}$$

is continuous in $W^{1,2}(N)$.

Take r and s in $[0,\rho]$. Recalling that $\mu_g(N \setminus F(\tilde{T})) = \mu_g(\mathrm{Cut}(M) \cup (\overline{M} \setminus M)) = 0$,

$$\begin{array}{rcl} \|\tilde{v}_\varepsilon^{r,\eta_1} - \tilde{v}_\varepsilon^{s,\eta_1}\|_{L^2(N)}^2 & = & \displaystyle\int_{F(\tilde{T})} |\tilde{v}_\varepsilon^{r,\eta_1} - \tilde{v}_\varepsilon^{s,\eta_1}|^2, \\ & = & \displaystyle\int_{\mathbb{R}} \int_{\tilde{M}} |\overline{\mathbb{H}}_\varepsilon(t - rf(x)) - \overline{\mathbb{H}}_\varepsilon(t - sf(x))|^2 \theta_t(x)\, d\mathcal{H}^n(x)\, dt, \\ & \xrightarrow{s \to r} & 0, \end{array}$$

by Dominated Convergence Theorem.

Noting that, $\tilde{v}_\varepsilon^{r,\eta_1} = \tilde{v}_\varepsilon^{0,\eta_1}$ on $N \setminus B_\delta^N(z_0)$, for all r in $[0,\rho]$, and $\mu_g(B_\delta^N(z_0) \setminus (\Upsilon_1 \cup \Upsilon_2)) = \mu_g(A) = 0$,

$$\|\nabla \tilde{v}_\varepsilon^{r,\eta_1} - \nabla \tilde{v}_\varepsilon^{s,\eta_1}\|_{L^2(N)}^2 = \int_{\Upsilon_1 \cup \Upsilon_2} |\nabla \tilde{v}_\varepsilon^{r,\eta_1} - \nabla \tilde{v}_\varepsilon^{s,\eta_1}|\, d\mu_g.$$

As $F_i^{-1}\colon (\Upsilon_i, g) \to (F_i^{-1}(\Upsilon_i), h)$ is an isometry, we have,

$$\begin{array}{rcl} \|\nabla \tilde{v}_\varepsilon^{r,\eta_1} - \nabla \tilde{v}_\varepsilon^{s,\eta_1}\|_{L^2(N)}^2 & = & \displaystyle\int_{F_1^{-1}(\Upsilon_1) \cup F_2^{-1}(\Upsilon_2)} |\nabla v_\varepsilon^{r,\eta_1}(x,t) - \nabla v_\varepsilon^{s,\eta_1}(x,t)|^2, \\ & = & \displaystyle\int_{F_1^{-1}(\Upsilon_1) \cup F_2^{-1}(\Upsilon_2)} (\overline{\mathbb{H}}_\varepsilon'(t - rf(x)) - \overline{\mathbb{H}}_\varepsilon'(t - sf(x)))^2 \\ & & \qquad + |\nabla_x f(x)|^2 (r\overline{\mathbb{H}}_\varepsilon'(t - rf(x)) - s\overline{\mathbb{H}}_\varepsilon'(t - sf(x)))^2, \\ & \xrightarrow{s \to r} & 0, \end{array}$$

by Dominated Convergence Theorem.

2.6 Path to a_ε

2.6.1 Fixed Energy Gain Away from Non-Embedded Point

We construct the path from (2) to (3) in Figure 2.2.

Recall $\tilde{f}$ from Section 2.3.5 and set,

$$\eta_2(r,x) = \rho f(x) + r\tilde{f}(x),$$

for r in $[0, r_0]$, where $r_0 \in (0, \min\{1, \operatorname{diam}(N)/2\})$, will be chosen later. Denote, $A_L^k = B_L \setminus B_{\frac{L}{k}}$.

Remark 21. We choose $0 < \varepsilon_3 \leq \varepsilon_2$, such that $2\varepsilon_3\Lambda = 6\varepsilon_3|\log \varepsilon_3| << r_0$. From here on we consider ε on $(0, \varepsilon_3)$.

We slightly edit the Base Computation in Section 2.4. Consider $r > 2\varepsilon\Lambda$,

$$\mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon^{r,\eta_2}) - \mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon^{0,\eta_2}) = I_\varepsilon^{r,\eta_2} + (II_\varepsilon^{r,\eta_2} - II_\varepsilon^{2\varepsilon\Lambda,\eta_2}) + II_\varepsilon^{2\varepsilon\Lambda,\eta_2}.$$

We have,

$$\begin{aligned} II_\varepsilon^{r,\eta_2} - II_\varepsilon^{2\varepsilon\Lambda,\eta_2} = &- \int_{2\varepsilon\Lambda}^{r} \int_{\tilde{M}\setminus B_{\frac{L}{k}}} \tilde{f}(x) Q_\varepsilon(\sigma^+(x) - s\tilde{f}(x))\,\theta^+(x)\,d\mathcal{H}^n(x)\,ds \\ &+ \int_{2\varepsilon\Lambda}^{r} \int_{\tilde{M}\setminus B_{\frac{L}{k}}} \tilde{f}(x) Q_\varepsilon(\sigma^-(x) - s\tilde{f}(x))\,\theta^-(x)\,d\mathcal{H}^n(x)\,ds \\ &+ \int_{2\varepsilon\Lambda}^{r} \int_{\tilde{M}\setminus B_{\frac{L}{k}}} \tilde{f}(x) \int_{\sigma^-(x)}^{\sigma^+(x)} Q_\varepsilon(t - s\tilde{f}(x))\,(\lambda - H_t(x))\,\theta_t(x)\,dt\,d\mathcal{H}^n(x)\,ds \\ &+ \lambda \int_{2\varepsilon\Lambda}^{r} \int_{\tilde{M}} \Theta^1_{\varepsilon,\eta_2}(s,x) - \Theta^2_{\varepsilon,\eta_2}(s,x)\,d\mathcal{H}^n(x)\,ds. \end{aligned} \tag{2.15}$$

Considering the first term on the right hand side of (2.15),

$$- \int_{2\varepsilon\Lambda}^{r} \int_{\tilde{M}\setminus B_{\frac{L}{k}}} \tilde{f}(x) Q_\varepsilon(\sigma^+(x) - s\tilde{f}(x))\,\theta^+(x)\,d\mathcal{H}^n(x)\,ds \leq 0.$$

Considering the second term on the right hand side of (2.15), and by applying similar arguments for when we considered the corresponding term on the right-hand side of (2.12) in Section 2.5.1,

$$\begin{aligned} &\int_{2\varepsilon\Lambda}^{r} \int_{\tilde{M}\setminus B_{\frac{L}{k}}} \tilde{f}(x) Q_\varepsilon(\sigma^-(x) - s\tilde{f}(x))\,\theta^-(x)\,d\mathcal{H}^n(x)\,ds \\ &\qquad\qquad \leq C_2\mathcal{H}^n(\{x \colon x \in \tilde{M} \setminus B_{\frac{L}{k}},\, \sigma^-(x) \geq 2\varepsilon\Lambda(\tilde{f}(x) - 1)\}), \end{aligned}$$

where we are potentially rechoosing $C_2 = C_2(M, N, g, W, \lambda) < +\infty$.

Considering the third term on the right hand side of (2.15). Applying similar arguments in A_L^k from when we considered the corresponding term on the right hand side of (2.12) in Section 2.5.1,

$$\begin{aligned}\int_{2\varepsilon\Lambda}^{r}\int_{\tilde{M}\setminus B_{\frac{L}{k}}} \tilde{f}(x)\int_{\sigma^-(x)}^{\sigma^+(x)} Q_\varepsilon(t - s\tilde{f}(x))\,(\lambda - H_t(x))\,\theta_t(x)\,dt\,d\mathcal{H}^n(x)\,ds \\ \leq \int_{2\varepsilon\Lambda}^{r}\int_{\tilde{M}\setminus B_L}\int_{-2\varepsilon\Lambda}^{2\varepsilon\Lambda} Q_\varepsilon(\xi)(\lambda - H_{\xi+s}(x))\theta_{\xi+s}(x)\,d\xi\,d\mathcal{H}^n(x)\,ds \\ + C_2\int_{A_L^k} q_\varepsilon^2(x)\,d\mathcal{H}^n(x),\end{aligned} \tag{2.16}$$

where, $q_\varepsilon^2(x) \coloneqq \max_{t\in[-2\varepsilon\Lambda, 4\varepsilon\Lambda]}(\lambda - H_t(x))\theta_t(x)$, and we are potentially rechoosing $C_2 = C_2(M, N, g, \lambda, W) < \infty$. Note that for the inequality, when considering the region A_L^k, we are using the fact that $\lambda - H_t \leq 0$, for $t > 0$.

Define the following measurable set,

$$\Omega_r = \{x \in \tilde{M} \colon\ \sigma^+(x) > 2r\,\}.$$

Remark 22. We choose $L = L(z_0, N, M, g, \delta) > 0$, such that,

$$\mathcal{H}^n(\tilde{M}\setminus B_L) > \frac{3}{4}\mathcal{H}^n(\tilde{M}).$$

Then we can find an $r_0 = r_0(z_0, M, N, g, \delta, L) > 0$, such that, for all r in $[0, r_0]$,

$$\mathcal{H}^n(\{(x, 2r)\colon\ x \in \Omega_r \setminus B_L\}) > \frac{1}{2}\mathcal{H}^n(\tilde{M}).$$

For all x in Ω_r, s in $(2\varepsilon\Lambda, r)$, and ξ in $[-2\varepsilon\Lambda, 2\varepsilon\Lambda]$, $s + \xi$ lies in $(0, \sigma^+(x))$. Therefore, recalling bounds on H_t and θ_t from Remark 5 and Proposition 6, we have,

$$(\lambda - H_{\xi+s}(x))\theta_{\xi+s}(x) < -m(s+\xi)\theta_{\xi+s} \leq -m(s - 2\varepsilon\Lambda)\theta_{2r}(x).$$

Then for r in $(2\varepsilon\Lambda, r_0]$, we compute an energy decrease from the first term on the right hand side of (2.16),

$$\int_{2\varepsilon\Lambda}^{r}\int_{\tilde{M}\setminus B_L}\int_{-2\varepsilon\Lambda}^{2\varepsilon\Lambda} Q_\varepsilon(\xi)(\lambda - H_{\xi+s}(x))\theta_{\xi+s}(x)\,d\xi\,d\mathcal{H}^n(x)\,ds \leq -\frac{m\sigma}{2}\mathcal{H}^n(\tilde{M})r^2 + C_2\varepsilon\Lambda,$$

potentially rechoosing $C_2 = C_2(N, M, g, \lambda, W) < +\infty$.

For r in $[0, 2\varepsilon\Lambda]$, by repeating arguments similar to those in Section 2.3.6,

$$II_\varepsilon^{r,\eta_2} \leq C_2\Big(\int_{\tilde{M}\setminus B_{\frac{L}{k}}} m_\varepsilon^1(x)\, d\mathcal{H}^n(x) + \varepsilon\Lambda\Big),$$

where we are potentially rechoosing $C_2 = C_2(N, M, g, W, \lambda)$, and $m_\varepsilon^1(x) = \max_{t\in[-2\varepsilon\Lambda, 4\varepsilon\Lambda]} \theta_t(x) - \min_{t\in[-2\varepsilon\Lambda,4\varepsilon\Lambda]} \theta_t(x)$.

For r in $[0, r_0]$, consider the term,

$$I_\varepsilon^{r,\eta_2} = \int_{A_L^k}\int_{-2\varepsilon\Lambda}^{2\varepsilon\Lambda} \frac{\varepsilon}{2}((\overline{\mathbb{H}})'(\xi))^2 |r\nabla\tilde{f}|^2(x, r\tilde{f}(x)+\xi)\theta_{r\tilde{f}(x)+\xi}(x)\, d\xi\, d\mathcal{H}^n(x).$$

By choice of L and r_0 in Remark 13, and constant $C_3 = C_3(z_0, M, N, g, \delta, \lambda, W)$ from Remark 18, we have, for all x in $A_L^k = B_L \setminus B_{\frac{L}{k}}$, r in $[0, r_0]$, and ξ in $[-2\varepsilon\Lambda, 2\varepsilon\Lambda]$,

$$|\nabla\tilde{f}|^2(x, r\tilde{f}(x)+\xi) \leq C_3|\nabla\tilde{f}|^2(x, 0).$$

Thus we have,

$$I_\varepsilon^{r,\eta_2} \leq C_3\|\nabla\tilde{f}\|^2_{L^2(\tilde{M})} r^2.$$

Again we are potentially rechoosing $C_3 = C_3(z_0, M, N, g, \delta, W, \lambda)$.

Remark 23. Recalling definition of $\tilde{f}$ from Section 2.3.5, we may choose $k = k(z_0, M, N, g, \delta, W, \lambda, L)$ such that

$$\|\nabla\tilde{f}\|^2_{L^2(\tilde{M})} < C_3^{-1}\frac{m\sigma}{8}\mathcal{H}^n(\tilde{M}).$$

Therefore,

$$I_\varepsilon^{r,\eta_2} \leq \frac{m\sigma}{8}\mathcal{H}^n(\tilde{M}) r^2.$$

For r in $(0, 2\varepsilon\Lambda]$,

$$\mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon^{r,\eta_2}) - \mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon^{0,\eta_2}) \leq III_\varepsilon^{3,r},$$

where,

$$III_\varepsilon^{3,r} = C_2\left(\int_{\tilde{M}\setminus B_{\frac{L}{k}}} m_\varepsilon^1(x)\, d\mathcal{H}^n(x) + \varepsilon\Lambda\right),$$

where we are potentially rechoosing $C_2 = C_2(N, M, g, W, \lambda) < +\infty$. For r in $(2\varepsilon\Lambda, r_0]$,

$$\mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon^{r,\eta_2}) - \mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon^{0,\eta_2}) \leq -\frac{3m\sigma}{8}\mathcal{H}^n(\tilde{M}) r^2 + III_\varepsilon^{4,r} \tag{2.17}$$

where,

$$
\begin{aligned}
III_\varepsilon^{4,r} &= C_2\Big(\mathcal{H}^n(\{x\colon x\in\tilde{M}\setminus B_{\frac{L}{k}},\, \sigma^-(x)\geq 2\varepsilon\Lambda(\tilde{f}(x)-1)\,\})\\
&\quad + \int_{A_L^k} q_\varepsilon^2(x)\,d\mathcal{H}^n(x) + \int_{2\varepsilon\Lambda}^r\int_{\tilde{M}} \Theta^1_{\varepsilon,\eta_2}(s,x)-\Theta^2_{\varepsilon,\eta_2}\,d\mathcal{H}^n(x)\,ds+\varepsilon\Lambda\Big),
\end{aligned}
$$

again, we are potentially rechoosing $C_2 = C_2(M,N,g,W,\lambda)$.

As $\eta_2(0,x)=\eta_1(\rho,x)$, we have, for r in $(0,2\varepsilon\Lambda]$ (recalling (2.14)),

$$
\mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon^{r,\eta_2}) \leq \mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon) - \frac{\sigma\mathcal{H}^n(A_l)}{2^n-1} + III_\varepsilon^{2,\rho} + III_\varepsilon^{3,r},
$$

and for r in $(2\varepsilon\Lambda, r_0]$,

$$
\mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon^{r,\eta_2}) \leq \mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon) - \frac{\sigma\mathcal{H}^n(A_l)}{2^n-1} - \frac{3m\sigma}{8}\mathcal{H}^n(\tilde{M})r^2 + III_\varepsilon^{2,\rho} + III_\varepsilon^{4,r}.
$$

We may define the appropriate function on N, for r in $[0,r_0]$,

$$
\tilde{v}_\varepsilon^{r,\eta_2}(y) = \begin{cases} \overline{\mathbb{H}}_\varepsilon(\tilde{d}(y)-r), & y\notin B_\delta^N(z_0),\\ v_\varepsilon^{r,\eta_2}(F_1^{-1}(y)), & y\in\overline{\Upsilon_1}\cap B_\delta^N(z_0),\\ v_\varepsilon^{r,\eta_2}(F_2^{-1}(y)), & y\in\overline{\Upsilon_2}\cap B_\delta^N(z_0). \end{cases}
$$

Following similar arguments to Sections 2.5.2, and 2.5.3, we may show that $\tilde{v}_\varepsilon^{r,\eta_2}$ lies in $W^{1,\infty}(N)$, $\mathcal{F}_{\varepsilon,\lambda}(\tilde{v}_\varepsilon^{r,\eta_2})(N) = \mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon^{r,\eta_2})(\tilde{T})$ and that the path $r\mapsto\tilde{v}_\varepsilon^{r,\eta_2}$ is continuous in $W^{1,2}(N)$.

2.6.2 Reversing Construction of Competitor

We construct the path from (3) to (4) in Figure 2.2.

For r in $[0,\rho]$, we set,

$$
\eta_3(r,x) = r_0\tilde{f}(x) + (\rho-r)f(x).
$$

For x in B_{2l},

$$
\eta_3(r,x) = (\rho-r)f(x) = \eta_1(\rho-r,x),
$$

and for x in $\tilde{M}\setminus B_{2l}$,

$$
\eta_3(r,x) = r_0\tilde{f}(x) = \eta_3(0,x).
$$

Therefore,

$$
\mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon^{r,\eta_3}) - \mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon^{0,\eta_3}) = \mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon^{\rho-r,\eta_1}) - \mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon^{\rho,\eta_1}).
$$

As $\eta_1(\rho, x) = \eta_2(0, x)$, and $\eta_3(0, x) = \eta_2(r_0, x)$, we have,

$$\mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon^{r,\eta_3}) = \mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon^{\rho-r,\eta_1}) + \mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon^{r_0,\eta_2}) - \mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon^{0,\eta_2}).$$

Remark 24. We choose $\rho > 0$, such that,

$$\frac{\sigma\mathcal{H}^n(A_l)}{2^n - 1} < \frac{m\sigma}{4}\mathcal{H}^n(\tilde{M})r_0^2.$$

Therefore (recalling (2.17)), we have that

$$\mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon^{r,\eta_3}) < \mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon^{\rho-r,\eta_1}) - \frac{\sigma\mathcal{H}^n(A_l)}{2^n - 1} - \frac{m\sigma}{8}\mathcal{H}^n(\tilde{M})r_0^2 + III_\varepsilon^{4,r_0}.$$

Furthermore (recalling (2.13)), for r in $[0, \rho - 4\varepsilon\Lambda]$, we have,

$$\begin{aligned}
\mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon^{r,\eta_3}) &< \mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon) + \frac{\sigma\mathcal{H}^n(A_l)}{2(2^n - 1)} + III_\varepsilon^{2,\rho-r} - \frac{\sigma\mathcal{H}^n(A_l)}{2^n - 1} - \frac{m\sigma}{8}\mathcal{H}^n(\tilde{M})r_0^2 + III_\varepsilon^{4,r_0}, \\
&= \mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon) - \frac{\sigma\mathcal{H}^n(A_l)}{2(2^n - 1)} - \frac{m\sigma}{8}\mathcal{H}^n(\tilde{M})r_0^2 + III_\varepsilon^{2,\rho-r} + III_\varepsilon^{4,r_0}.
\end{aligned}$$

For r in $(\rho - 4\varepsilon\Lambda, \rho]$, we similarly have,

$$\mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon^{r,\eta_3}) < \mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon) - \frac{\sigma\mathcal{H}^n(A_l)}{2^n - 1} - \frac{m\sigma}{8}\mathcal{H}^n(\tilde{M})r_0^2 + III_\varepsilon^{1,\rho-r} + III_\varepsilon^{4,r_0}.$$

We define the appropriate function on N. For r in $[0, \rho]$,

$$\tilde{v}_\varepsilon^{r,\eta_3}(y) = \begin{cases} \overline{\mathbb{H}}_\varepsilon(\tilde{d}(y) - r_0), & y \notin B_\delta^N(z_0), \\ v_\varepsilon^{r,\eta_3}(F_1^{-1}(y)), & y \in \overline{\Upsilon_1} \cap B_\delta^N(z_0), \\ v_\varepsilon^{r,\eta_3}(F_2^{-1}(y)), & y \in \overline{\Upsilon_2} \cap B_\delta^N(z_0). \end{cases}$$

Following similar arguments to Sections 2.5.2, and 2.5.3, we may show that $\tilde{v}_\varepsilon^{r,\eta_3}$ lies in $W^{1,\infty}(N)$, $\mathcal{F}_{\varepsilon,\lambda}(\tilde{v}_\varepsilon^{r,\eta_3})(N) = \mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon^{r,\eta_3})(\tilde{T})$ and that the path $r \mapsto \tilde{v}_\varepsilon^{r,\eta_3}$ is continuous in $W^{1,2}(N)$.

2.6.3 Lining Up With Level Set Γ_{r_0}

We construct path from (4) to (5) in Figure 2.2

For r in $[0, r_0]$, consider,

$$\eta_4(r, x) = r_0\tilde{f}(x) + r(1 - \tilde{f}(x)) = (r_0 - r)\tilde{f}(x) + r \geq r.$$

By applying similar arguments to those in Section 2.5.1, we have

$$\begin{aligned} II_\varepsilon^{r,\eta_4} &\leq C_3\bigg(\mathcal{H}^n(\{x\in B_L\colon \sigma^-(x)\geq -2\varepsilon\Lambda\})+\varepsilon\Lambda \\ &\qquad + \int_0^r\int_{\tilde{M}}\Theta^1_{\varepsilon,\eta_4}(s,x)-\Theta^2_{\varepsilon,\eta_4}(s,x)\,d\mathcal{H}^n(x)\,ds\bigg), \end{aligned}$$

where we are potentially rechoosing $C_3 = C_3(z_0, M, N, g, \delta, W, \lambda) < +\infty$.

Similar to Section 2.6.1, and recalling Remark 23 we have

$$\begin{aligned} I_\varepsilon^{r,\eta_4} &= \int_{A_L^k}\int_{\sigma^-(x)}^{\sigma^+(x)}\frac{\varepsilon}{2}((\overline{\mathbb{H}}_\varepsilon)'(t-(r_0-r)\tilde{f}(x)-r))^2(r_0-r)^2|\nabla\tilde{f}|^2(x,t)\,\theta_t(x)\,dt\,d\mathcal{H}^n(x) \\ &\qquad - \int_{A_L^k}\int_{\sigma^-(x)}^{\sigma^+(x)}\frac{\varepsilon}{2}((\overline{\mathbb{H}}_\varepsilon)'(t-r_0\tilde{f}(x)))^2r_0^2|\nabla\tilde{f}|^2(x,t)\,\theta_t(x)\,dt\,d\mathcal{H}^n(x), \\ &\leq C_3r_0^2\|\nabla\tilde{f}\|_{L^2(\tilde{M})}, \\ &\leq \frac{m\sigma}{8}\mathcal{H}^n(\tilde{M})r_0^2. \end{aligned}$$

Therefore, for all r in $[0, r_0]$,

$$\begin{aligned} \mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon^{r,\eta_4})-\mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon^{0,\eta_4}) &= I_\varepsilon^{r,\eta_4}+II_\varepsilon^{r,\eta_4}, \\ &\leq \frac{m\sigma}{8}\mathcal{H}^n(\tilde{M})r_0^2+III_\varepsilon^{5,r}, \end{aligned}$$

where

$$\begin{aligned} III_\varepsilon^{5,r} &\leq C_3\bigg(\mathcal{H}^n(\{x\in B_{2L}\colon\ \sigma^-(x)\geq -2\varepsilon\Lambda\})+\varepsilon\Lambda \\ &\qquad + \int_0^r\int_{\tilde{M}}\Theta^1_{\varepsilon,\eta_4}(s,x)-\Theta^2_{\varepsilon,\eta_4}(s,x)\,d\mathcal{H}^n(x)\,ds\bigg). \end{aligned}$$

As $\eta_4(0, x) = \eta_3(\rho, x)$, we have, for r in $[0, r_0]$,

$$\mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon^{r,\eta_4})\leq\mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon)-\frac{\sigma\mathcal{H}^n(x)}{2(2^n-1)}+III_\varepsilon^{1,0}+III_\varepsilon^{4,r_0}+III_\varepsilon^{5,r}.$$

Consider the following function on N, for r in $[0, r_0]$,

$$\tilde{v}_\varepsilon^{r,\eta_4}(y)=\begin{cases}\overline{\mathbb{H}}_\varepsilon(\tilde{d}(y)-r_0), & y\notin B_\delta^N(z_0), \\ v_\varepsilon^{r,\eta_4}(F_1^{-1}(y)), & y\in\overline{\Upsilon_1}\cap B_\delta^N(z_0), \\ v_\varepsilon^{r,\eta_4}(F_2^{-1}(y)), & y\in\overline{\Upsilon_2}\cap B_\delta^N(z_0).\end{cases}$$

We can show, as in Section 2.5.2 and 2.5.3, that $\tilde{v}_\varepsilon^{r,\eta_3}$ lies in $W^{1,\infty}(N)$, $\mathcal{F}_{\varepsilon,\lambda}(\tilde{v}_\varepsilon^{r,\eta_4})(N) = \mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon^{r,\eta_4})(\tilde{T})$, and that, $r \mapsto \tilde{v}_\varepsilon^{r,\eta_4}$ is a continuous path in $W^{1,2}(N)$.

2.6.4 Completing Path to a_ε

We construct the path from (5) to '-1' in Figure 2.2.

Consider, for r in $[r_0, 2\operatorname{diam}(N)]$,

$$\eta_5(r,x) = r.$$

As $2\varepsilon\Lambda << r_0$, and $H_t \geq \lambda$ for $t \geq 0$ (Remark 5), we have

$$\mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon^{r,\eta_5}) - \mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon^{r_0,\eta_5}) \leq III_\varepsilon^{6,r},$$

where

$$III_\varepsilon^{6,r} = \lambda \int_\rho^r \Theta^1_{\varepsilon,\eta_5}(s,x) - \Theta^2_{\varepsilon,\eta_5}(s,x)\, d\mathcal{H}^n(x)\, ds.$$

Recalling that, $\eta_5(r_0,x) = \eta_4(r_0,x)$, we have, for all r in $[r_0, 2\operatorname{diam}(N)]$,

$$\begin{aligned}
\mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon^{r,\eta_5}) &\leq \mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon^{r_0,\eta_4}) + III_\varepsilon^{6,r}, \\
&< \mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon) - \frac{\sigma\mathcal{H}^n(A_l)}{2(2^n-1)} + III_\varepsilon^{1,0} + III_\varepsilon^{4,r_0} + III_\varepsilon^{5,r_0} + III_\varepsilon^{6,r}.
\end{aligned}$$

Define the function, $\tilde{v}_\varepsilon^{r,\eta_5}(y) = \overline{\mathbb{H}}_\varepsilon(\tilde{d}(y) - r)$, in N. This function lies in $W^{1,\infty}(N)$, $\mathcal{F}_{\varepsilon,\lambda}(\tilde{v}_\varepsilon^{r,\eta_5})(N) = \mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon^{r,\eta_5})(\tilde{T})$, and $r \mapsto \tilde{v}_\varepsilon^{r,\eta_5}$ is a continuous path in $W^{1,2}(N)$.

As $|\tilde{d}(y)| \leq \operatorname{diam}(N)$, we have that,

$$\tilde{d}(y) - 2\operatorname{diam}(N) \leq -\operatorname{diam}(N) < -2\varepsilon\Lambda.$$

Therefore,

$$\tilde{v}_\varepsilon^{2\operatorname{diam}(N),\eta_5}(y) = \overline{\mathbb{H}}_\varepsilon(\tilde{d}(x) - 2\operatorname{diam}(N)) = -1.$$

Recall that our end point is $a_\varepsilon > -1$. We connect -1 to a_ε, by constant functions,

$$u_\varepsilon^r(y) = r$$

for r in $[-1, a_\varepsilon]$. Then,

$$\mathcal{F}_{\varepsilon,\lambda}(u_\varepsilon^r) = \int_N \frac{W(r)}{\varepsilon} - \sigma\lambda r\, d\mu_g \leq \mathcal{F}_{\varepsilon,\lambda}(u_\varepsilon^{-1}).$$

As $u_\varepsilon^{-1} = \tilde{v}_\varepsilon^{2\,\mathrm{diam}(N),\eta_5}$, we have that, for all r in $[-1, a_\varepsilon]$,

$$\mathcal{F}_{\varepsilon,\lambda}(u_\varepsilon^r) < \mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon) - \frac{\sigma\mathcal{H}^n(A_l)}{2(2^n-1)} + III_\varepsilon^{1,0} + III_\varepsilon^{4,r_0} + III_\varepsilon^{5,r_0} + III_\varepsilon^{6,2\,\mathrm{diam}(N)}.$$

2.7 Path to b_ε

2.7.1 Lining Up With Level Set $\Gamma_{-\rho}$

We construct the path from (2) to (6) in Figure 2.2

We consider, for r in $[0, \rho]$, and x in $\tilde{M}$,

$$\eta_6(r, x) = \rho f(x) - r(1 + f(x)).$$

First consider r in $(2\varepsilon\Lambda, \rho]$,

$$\mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon^{r,\eta_6}) - \mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon^{0,\eta_6}) = I_\varepsilon^{r,\eta_6} + (II_\varepsilon^{r,\eta_6} - II_\varepsilon^{2\varepsilon\Lambda,\eta_6}) + II_\varepsilon^{2\varepsilon\Lambda,\eta_6}$$

Similar to Section 2.6.1 we have,

$$II_\varepsilon^{r,\eta_6} - II_\varepsilon^{2\varepsilon\Lambda,\eta_6} \leq \lambda \int_{2\varepsilon\Lambda}^{r} \int_{\tilde{M}} \Theta^1_{\varepsilon,\eta_6}(s, x) - \Theta^2_{\varepsilon,\eta_6}(s, x)\, d\mathcal{H}^n(x)\, ds.$$

For r in $[0, 2\varepsilon\Lambda]$, again by similar arguments to those in Section 2.6.1

$$II_\varepsilon^{r,\eta_6} \leq C_2 \left(\int_{\{\rho f \geq -2\varepsilon\Lambda\}} m_\varepsilon^2(x)\, d\mathcal{H}^n(x) + \varepsilon\Lambda \right),$$

where,

$$m_\varepsilon^2(x) := \max_{t\in[-6\varepsilon\Lambda.2\varepsilon\Lambda]} \theta_t(x) - \min_{t\in[-6\varepsilon\Lambda,2\varepsilon\Lambda]} \theta_t(x),$$

and we are potentially rechoosing $C_2(M, N, g, W, \lambda) < \infty$.

For r in $[0, \rho]$, we consider,

$$\begin{aligned} I_\varepsilon^{r,\eta_6} \;=\;& \int_{A_l} \int_{\sigma^-(x)}^{\sigma^+(x)} \frac{\varepsilon}{2} ((\overline{\mathbb{H}}_\varepsilon)'(t - \eta_4(r, x)))^2 (\rho - r)^2 |\nabla f|^2(x, t) \theta_t(x)\, dt\, d\mathcal{H}^n(x) \\ & - \int_{A_l} \int_{\sigma^-(x)}^{\sigma^+(x)} \frac{\varepsilon}{2} ((\overline{\mathbb{H}}_\varepsilon)'(t - \eta_4(0, x)))^2 \rho^2 |\nabla f|^2(x, t) \theta_t(x)\, dt\, d\mathcal{H}^n(x) \end{aligned}$$

Following similar arguments to Section 2.5.1, and after potentially rechoosing

$C_3 = C_3(z_0, M, N, g, \delta, W, \lambda) < \infty$, we have that,

$$I_\varepsilon^{r,\eta_6} \leq C_3 \mathcal{H}^n(A_l) \frac{\rho^2}{l^2}.$$

Therefore, recalling our choice of $\rho > 0$ in Remark 19, we have

$$I_\varepsilon^{r,\eta_6} \leq \frac{\sigma \mathcal{H}^n(A_l)}{2(2^n - 1)}.$$

Thus, for r in $[0, 2\varepsilon\Lambda]$,

$$\mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon^{r,\eta_6}) - \mathcal{F}_\varepsilon(v_\varepsilon^{0,\eta_6}) \quad < \quad \frac{\sigma \mathcal{H}^n(A_l)}{2(2^n - 1)} + III_\varepsilon^{7,r},$$

where,

$$III_\varepsilon^{7,r} = C_2 \bigg(\int_{\{\rho f \geq -2\varepsilon\Lambda\}} m_\varepsilon^2(x)\, d\mathcal{H}^n(x) + \varepsilon\Lambda \bigg).$$

For r in $(2\varepsilon\Lambda, \rho]$,

$$\mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon^{r,\eta_6}) - \mathcal{F}_\varepsilon(v_\varepsilon^{0,\eta_6}) \quad < \quad \frac{\sigma \mathcal{H}^n(A_l)}{2(2^n - 1)} + III_\varepsilon^{8,r} + III_\varepsilon^{7,2\varepsilon\Lambda},$$

where

$$\begin{aligned} III_\varepsilon^{8,r} \quad &= \quad C_2 \bigg(\int_{2\varepsilon\Lambda}^{r} \int_{\tilde{M}} \Theta^1_{\varepsilon,\eta_6}(s,x) - \Theta^2_{\varepsilon,\eta_6}(s,x)\, d\mathcal{H}^n(x) \\ &\quad + \int_{\{\rho f \geq -2\varepsilon\Lambda\}} m_\varepsilon^2(x)\, d\mathcal{H}^n(x) + \varepsilon\Lambda \bigg). \end{aligned}$$

As $\eta_6(0, x) = \eta_1(\rho, x)$, we have, for r in $[0, 2\varepsilon\Lambda]$,

$$\mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon^{r,\eta_6}) \leq \mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon) - \frac{\sigma \mathcal{H}^n(A_l)}{2(2^n - 1)} + III_\varepsilon^{2,\rho} + III_\varepsilon^{7,r},$$

and for r in $(2\varepsilon\Lambda, \rho]$, we have,

$$\mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon^{r,\eta_6}) \leq \mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon) - \frac{\sigma \mathcal{H}^n(A_l)}{2(2^n - 1)} + III_\varepsilon^{2,\rho} + III_\varepsilon^{8,r} + III_\varepsilon^{7,2\varepsilon\Lambda},.$$

For r in $[0,\rho]$, we define the following function on N,

$$\tilde{v}_\varepsilon^{r,\eta_6}(y) = \begin{cases} \overline{\mathbb{H}}_\varepsilon(\tilde{d}(y)+r), & y \notin B_\delta^N(z_0), \\ v_\varepsilon^{r,\eta_6}(F_1^{-1}(y)), & y \in \overline{\Upsilon_1} \cap B_\delta^N(z_0), \\ v_\varepsilon^{r,\eta_6}(F_2^{-1}(y)), & y \in \overline{\Upsilon_2} \cap B_\delta^N(z_0). \end{cases}$$

We can show, as in Section 2.5.2 and 2.5.3 that, $\tilde{v}_\varepsilon^{r,\eta_6}$ lies in $W^{1,\infty}(N)$, $\mathcal{F}_{\varepsilon,\lambda}(\tilde{v}_\varepsilon^{r,\eta_6})(N) = \mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon^{r,\eta_6})(\tilde{T})$, and the path $r \mapsto \tilde{v}_\varepsilon^{r,\eta_6}$ is continuous in $W^{1,2}(N)$.

2.7.2 Completing Path to b_ε

We construct the path from (6) to '+1' in Figure 2.2. This is done in an identical way to Section 2.6.4.

For r in $[\rho, 2\,\mathrm{diam}(N)]$, we define the following function on N,

$$\tilde{v}_\varepsilon^{r,\eta_7}(y) := \overline{\mathbb{H}}_\varepsilon(\tilde{d}(y)+r).$$

Similar to arguments in Section 2.6.4 we have,

$$\mathcal{F}_{\varepsilon,\lambda}(\tilde{v}_\varepsilon^{r,\eta_7}) \le \mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon) - \frac{\sigma\mathcal{H}^n(A_l)}{2(2^n-1)} + III_\varepsilon^{2,\rho} + III_\varepsilon^{8,\rho} + III_\varepsilon^{9,r},$$

where,

$$III_\varepsilon^{9,r} = \lambda \int_\rho^r \Theta^1_{\varepsilon,\eta_6}(s,x) - \Theta^2_{\varepsilon,\eta_6}(s,x)\, d\mathcal{H}^n(x)\, ds.$$

Again as in Section 2.6.4, we connect $\tilde{v}_\varepsilon^{2\,\mathrm{diam}(N),\eta_7} = 1$, to b_ε, by constant functions, $u_\varepsilon^r = r$, for r in $[1, b_\varepsilon]$. We have that for all r in $[1, b_\varepsilon]$,

$$\mathcal{F}_{\varepsilon,\lambda}(u_\varepsilon^r) \le \mathcal{F}_{\varepsilon,\lambda}(\tilde{v}_\varepsilon^{2\,\mathrm{diam}(N),\eta_7}) \le \mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon) - \frac{\sigma\mathcal{H}^n(A_l)}{2(2^n-1)} + III_\varepsilon^{2,\rho} + III_\varepsilon^{8,\rho} + III_\varepsilon^{9,2\,\mathrm{diam}(N)}.$$

Both $\tilde{v}_\varepsilon^{r,\eta_7}$, and u_ε^r give continuous paths in $W^{1,2}(N)$ with respect to r.

2.8 Conclusion of the Paths

2.8.1 Error Terms

Theta Error Terms

Consider a function $\eta\colon \mathbb{R}\times\tilde{M}\to\mathbb{R}$, and the term

$$\begin{aligned}\Theta^1_{\varepsilon,\eta}(s,x)-\Theta^2_{\varepsilon,\eta}(s,x) &= \sigma\int_{\sigma^-(x)}^{\sigma^+(x)}\partial_s\eta(s,x)(\overline{\mathbb{H}}_\varepsilon)'(t-\eta(s,x))\theta_t(x)\,dt\\ &\qquad -\int_{\sigma^-(x)}^{\sigma^+(x)}\partial_s\eta(s,x)Q_\varepsilon(t-\eta(s,x))\theta_t(x)\,dt.\end{aligned}$$

Assuming that η is monotone in the first variable, we have,

$$|\Theta^1_{\varepsilon,\eta}(s,x)-\Theta^2_{\varepsilon,\eta}(s,x)|\le 2\sigma|\partial_s\eta(s,x)|m_\varepsilon(\eta(s,x),x)+C_4\varepsilon^2,$$

where,

$$m_\varepsilon(T,x)=\max_{t\in[T-2\varepsilon\Lambda,T+2\varepsilon\Lambda]}\theta_t(x)-\min_{t\in[T-2\varepsilon\Lambda,T+2\varepsilon\Lambda]}\theta_t(x).$$

and $C_4=C_4(N,m,\lambda,W,|\eta|_{C^1})<+\infty$.

Now we assume that $\partial_s\eta\ge 0$, and $|\partial_s\eta|_{C^0(\mathbb{R}\times\tilde{M})}<+\infty$. Apply Fubini's Theorem to swap integrals,

$$\int_0^r\int_{\tilde{M}}\left|\Theta^1_{\varepsilon,\eta}(s,x)-\Theta^2_{\varepsilon,\eta}(s,x)\right|\,d\mathcal{H}^n(x)\,ds\le 2\sigma\int_{\tilde{M}}\int_{\eta(0,x)}^{\eta(r,x)}m_\varepsilon(T,x)\,dT\,\mathcal{H}^n(x)+C_4r\varepsilon^2.$$

Fixing x in $\tilde{M}$, we see that for all T in $\mathbb{R}\setminus\{\sigma^-(x),\sigma^+(x)\}$,

$$m_\varepsilon(T,x)\to 0,\ as\,\varepsilon\to 0,$$

and furthermore, we have the following bounds, $0\le m_\varepsilon(T,x)\le e^{\frac{\lambda^2}{2m}}$. Therefore we can apply Dominated Convergence Theorem for fixed x in $\tilde{M}$ and r in $[0,\infty)$,

$$\int_{\eta(0,x)}^{\eta(r,x)}m_\varepsilon(T,x)\,dT\to 0,\ as\,\varepsilon\to 0.$$

Furthermore, as $0\le\eta(r,x)-\eta(0,x)\le|\partial_s\eta|_{C^0(\mathbb{R}\times\tilde{M})}\,r$, we have the bounds,

$$0\le\int_{\eta(0,x)}^{\eta(r,x)}m_\varepsilon(T,x)\,dT\le|\partial_s\eta|_{C^0(\mathbb{R}\times\tilde{M})}re^{\frac{\lambda^2}{2m}}.$$

Therefore, again by Dominated Convergence Theorem, we have, for fixed r in $[0,\infty)$

$$\int_{\tilde{M}} \int_{\eta(0,x)}^{\eta(r,x)} m_\varepsilon(T,x)\, dT\, \mathcal{H}^n(x) \to 0,\ as\, \varepsilon \to 0.$$

Define the following continuous function on $[0,+\infty)$,

$$M_\varepsilon^\eta(r) = \int_{\tilde{M}} \int_{\eta(0,x)}^{\eta(r,x)} m_\varepsilon(T,x)\, dT\, \mathcal{H}^n(x).$$

We have that $M_\varepsilon^\eta(r) \to 0$, pointwise, as $\varepsilon \to 0$, and furthermore, as

$$0 \leq m_{\varepsilon_1}(T,x) \leq m_{\varepsilon_2}(T,x),$$

for all T in $\mathbb{R}$, x in $\tilde{M}$, and $0 < \varepsilon_1 < \varepsilon_2$, this implies that,

$$0 \leq M_{\varepsilon_1}^\eta(r) \leq M_{\varepsilon_2}^\eta(r),$$

for all r in $[0,+\infty)$. Therefore, by Dini's Theorem (a monotonic sequence of continuous functions, which converges pointwise to a continuous function, must in fact converge uniformly on compact sets), we have that,

$$M_\varepsilon^\eta \to 0,\ as\, \varepsilon \to 0,$$

uniformly on compact sets of $[0,+\infty)$. Thus,

$$\int_0^r \int_{\tilde{M}} \left|\Theta_{\varepsilon,\eta}^1(s,x) - \Theta_{\varepsilon,\eta}^2(s,x)\right|\, d\mathcal{H}^n(x)\, ds \to 0 \tag{2.18}$$

as $\varepsilon \to 0$, uniformly in r, on compact sets of $[0,+\infty)$. The same holds assuming that η satisfies $\partial_s \eta \leq 0$, on $\mathbb{R} \times \tilde{M}$, and $|\partial_s \eta|_{C^0(\mathbb{R}\times\tilde{M})} < +\infty$.

For $i = 1,\ldots,7$ our η_i's are monotone in the first variable and $|\partial_s \eta_i|_{C^0(\mathbb{R}\times\tilde{M})} < +\infty$. Therefore (2.18) holds for each i.

The Other Error Terms

We first consider,

$$\int_{B_{2l}} q_\varepsilon^1(x)\, d\mathcal{H}^n(x),$$

with,

$$q_\varepsilon^1(x) = \max_{t\in[-4\varepsilon\Lambda, 2\varepsilon\Lambda]} (H_t(x) - \lambda)\theta_t(x).$$

By choice of $\varepsilon > 0$, in Remark 16, $2\varepsilon\Lambda << \rho$. Therefore by choice of $\rho > 0$, in Remark 12, and $\delta > 0$, from Remark 17, we have

$$0 \le \max_{x \in B_{2l}} q^1_\varepsilon(x) \le \frac{\lambda}{2} e^{\frac{\lambda^2}{2m}}.$$

Fixing x' in $B_{2l} \setminus \{x\colon \sigma^-(x) = 0\}$, we see that there exists an $\varepsilon' = \varepsilon'(x') > 0$, such that for all $0 < \varepsilon \le \varepsilon'$,

$$[-4\varepsilon\Lambda, 2\varepsilon\Lambda] \subset (\sigma^-(x'), \sigma^+(x')).$$

Therefore, $(H_t(x') - \lambda)\theta_t(x')$, is a smooth function in t on $[-4\varepsilon\Lambda, 2\varepsilon\Lambda]$, and clearly,

$$\max_{t \in [-4\varepsilon\Lambda, 2\varepsilon\Lambda]} (H_t(x') - \lambda)\theta_t(x') \to 0, \, as \, \varepsilon \to 0.$$

Thus $q^1_\varepsilon \to 0$, $\mathcal{H}^n$–a.e in B_{2l}, and we can apply Dominated Convergence Theorem to say that

$$\int_{B_{2l}} q^1_\varepsilon(x)\, d\mathcal{H}^n(x) \to 0, \, as \, \varepsilon \to 0.$$

Identically we also have,

$$\int_{A^k_L} q^2_\varepsilon(x)\, d\mathcal{H}^n(x) \to 0, \, as \, \varepsilon \to 0,$$

recalling $q^2_\varepsilon(x) = \max_{t \in [-2\varepsilon\Lambda, 4\varepsilon\Lambda]} (\lambda - H_t(x))\theta_t(x)$.

For the remaining error terms, as $\mathcal{H}^n(\{x \in \tilde{M}\colon\ \sigma^-(x) = 0\}) = 0$, by Dominated Convergence Theorem, we have that,

$$\mathcal{H}^n(\{x \in \tilde{M}\colon\ \sigma^-(x) \ge -2\varepsilon\Lambda\}) \to 0,$$

and

$$\int_{\tilde{M}} m^i_\varepsilon(x)\, d\mathcal{H}^n(x) \to 0,$$

where,

$$\begin{aligned} m^1_\varepsilon(x) &= \max_{t \in [-2\varepsilon\Lambda, 4\varepsilon\Lambda]} \theta_t(x) - \min_{t \in [-2\varepsilon\Lambda, 4\varepsilon\Lambda]} \theta_t(x), \\ m^2_\varepsilon(x) &= \max_{t \in [-6\varepsilon\Lambda, 2\varepsilon\Lambda]} \theta_t(x) - \min_{t \in [-6\varepsilon\Lambda, 2\varepsilon\Lambda]} \theta_t(x). \end{aligned}$$

2.8.2 Path for Theorem 5

Consider the following continuous path in $W^{1,2}(N)$, for $\varepsilon > 0$,

$$\gamma_\varepsilon(t) = \begin{cases} -1 - 2\mathrm{diam}\,(N) - t, & t \in [-2\,\mathrm{diam}\,(N) - a_\varepsilon - 1, -2\,\mathrm{diam}\,(N)], \\ \overline{\mathbb{H}}_\varepsilon(\tilde{d} - t), & t \in [-2\,\mathrm{diam}\,(N), 2\,\mathrm{diam}\,(N)], \\ 1 - 2\,\mathrm{diam}\,(N) + t, & t \in [2\,\mathrm{diam}\,(N), 2\,\mathrm{diam}\,(N) + b_\varepsilon - 1], \end{cases}$$

which satisfies $\gamma_\varepsilon(-1-2\text{diam}\,(N)-a_\varepsilon)=a_\varepsilon$, and $\gamma_\varepsilon(1-2\,\text{diam}\,(N)+b_\varepsilon)=b_\varepsilon$.

Replacing $r_0=2\varepsilon\Lambda$, in Section 2.6.4, and $\rho=2\varepsilon\Lambda$, in Section 2.7.2, we see that, for all ε in $(0,\tilde{\varepsilon})$, for some $\tilde{\varepsilon}=\tilde{\varepsilon}(N,M,g,\lambda,W)>0$, fixed,

$$\begin{cases} \mathcal{F}_{\varepsilon,\lambda}(\gamma_\varepsilon(t))<\mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon)+III_\varepsilon^{6,2\,\text{diam}\,(N)}, & t\in[-2\,\text{diam}\,(N)-a_\varepsilon-1,2\,\text{diam}\,(N)],\\ \mathcal{F}_{\varepsilon,\lambda}(\gamma_\varepsilon(t))<\mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon)+III_\varepsilon^{6,-t}, & t\in[-2\text{diam}\,(N),-2\varepsilon\Lambda],\\ \mathcal{F}_{\varepsilon,\lambda}(\gamma_\varepsilon(t))<\mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon)+III_\varepsilon^{9,t}, & t\in[2\varepsilon\Lambda,2\text{diam}\,(N)],\\ \mathcal{F}_{\varepsilon,\lambda}(\gamma_\varepsilon(t))<\mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon)+III_\varepsilon^{9,2\,\text{diam}\,(N)}, & t\in[2\,\text{diam}\,(N),2\,\text{diam}\,(N)+b_\varepsilon-1]. \end{cases}$$

Recalling from Section 2.3.6

$$\mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon)\to 2\sigma\mathcal{H}^n(M)-\sigma\lambda\mu_g(E)+\sigma\lambda\mu_g(N\setminus E),$$

as $\varepsilon\to 0$, and Section 2.8.1,

$$\max_{t\in[2\varepsilon\Lambda,2\,\text{diam}\,(N)]}\left(III_\varepsilon^{6,t}+III^{9,t}\right)\to 0,$$

as $\varepsilon\to 0$. Therefore, for $\tau>0$, there exists a $0<\varepsilon_\tau=\varepsilon_\tau(N,M,g,\lambda,W)\leq\tilde{\varepsilon}$, such that for all ε in $(0,\varepsilon_\tau)$ and t in $[-2\,\text{diam}\,(N)-a_\varepsilon-1,2\,\text{diam}\,(N)+b_\varepsilon-1]\setminus(-2\varepsilon\Lambda,2\varepsilon\Lambda)$,

$$\mathcal{F}_{\varepsilon,\lambda}(\gamma_\varepsilon(t))<2\sigma\mathcal{H}^n(M)-\sigma\lambda\mu_g(E)+\sigma\lambda\mu_g(N\setminus E)+\tau.$$

Furthermore by similar arguments to those in Section 2.3.6, and after potentially rechoosing $\varepsilon_\tau>0$, we have that for all ε in $(0,\varepsilon_\tau)$

$$\max_{t\in[-2\varepsilon\Lambda,2\varepsilon\Lambda]}\mathcal{F}_{\varepsilon,\lambda}(\gamma_\varepsilon(t))<2\sigma\mathcal{H}^n(M)-\sigma\lambda\mu_g(E)+\sigma\lambda\mu_g(N\setminus E)+\tau.$$

Therefore this is an admissible path in $W^{1,2}(N)$, that proves that for the limiting varifold $V=V_\lambda+V_0$, we must have that $V_0=0$. This completes the proof of Theorem 5.

Remark 25. Note that we can build the path γ_ε, for any suitable Caccioppoli set E. The suitable properties are the following:

1. $\partial^*E\neq\emptyset$, and gen-reg$\,(\partial^*E,1)$ is a quasi embedded λ-CMC hypersurface, with respect to unit normal pointing into E.

2. ∂E satisfies the Geodesic Touching Lemma (Lemma 1).

From Remark 25 we can deduce that E must be a single connected component and minimises the value

$$F_\lambda(E)=\mathcal{H}^n(\partial^*E)-\lambda\mu_g(E)>0,$$

among all suitable competitors satisfying the two properties of Remark 25.

2.8.3 Contradiction Path for Theorem 4

Recall all the error terms from Sections 2.5, 2.6 and 2.7. By Section 2.3.6 and 2.8.1, for $\tau > 0$, there exists an $\varepsilon_\tau = \varepsilon(z_0, M, N, g, \delta, W, \lambda, L, k, r_0, \rho, \tau) \in (0, \varepsilon_3)$, such that for all ε in $(0, \varepsilon_\tau)$, we have that

$$\begin{aligned}\mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon) &+ \max_{r\in[0,4\varepsilon\Lambda)} III_\varepsilon^{1,r} + \max_{r\in[4\varepsilon\Lambda,\rho]} III_\varepsilon^{2,r} + \max_{r\in[0,2\varepsilon\Lambda]} III_\varepsilon^{3,r} \\ &+ \max_{r\in(2\varepsilon\Lambda,r_0]} III_\varepsilon^{4,r} + \max_{r\in[0,r_0]} III_\varepsilon^{5,r} + \max_{r\in[r_0,2\operatorname{diam}(N)]} III_\varepsilon^{6,r} \\ &+ \max_{r\in[0,2\varepsilon\Lambda]} III_\varepsilon^{7,r} + \max_{r\in(2\varepsilon\Lambda,\rho]} III_\varepsilon^{8,r} + \max_{r\in[\rho,2\operatorname{diam}(N)]} III_\varepsilon^{9,r} \\ &< 2\sigma\mathcal{H}^n(M) - \sigma\lambda\mu_g(E) + \sigma\lambda\mu_g(N\setminus E) + \tau.\end{aligned}$$

Therefore, for any $\tau > 0$, there exists an $\varepsilon_\tau > 0$, such that for any ε in $(0, \varepsilon_\tau)$, we can define the continuous path,

$$\gamma_\varepsilon \colon [-1 - a_\varepsilon, 4\operatorname{diam}(N) + r_0 + \rho + b_\varepsilon - 1] \to W^{1,2}(N),$$

by

$$\gamma_\varepsilon(t) = \begin{cases} -1 - t, & t \in [-1 - a_\varepsilon, 0], \\ \overline{\mathbb{H}}_\varepsilon(\tilde{d} + t - 2\operatorname{diam}(N)), & [0, 2\operatorname{diam}(N) - r_0], \\ \tilde{v}_\varepsilon^{2\operatorname{diam}(N)-t,\eta_4}, & [2\operatorname{diam}(N) - r_0, 2\operatorname{diam}(N)], \\ \tilde{v}_\varepsilon^{2\operatorname{diam}(N)+\rho-t,\eta_3}, & [2\operatorname{diam}(N), 2\operatorname{diam}(N) + \rho], \\ \tilde{v}_\varepsilon^{2\operatorname{diam}(N)+\rho+r_0-t,\eta_2}, & [2\operatorname{diam}(N) + \rho, 2\operatorname{diam}(N) + \rho + r_0], \\ \tilde{v}_\varepsilon^{t-(2\operatorname{diam}(N)+\rho+r_0),\eta_6}, & [2\operatorname{diam}(N) + \rho + r_0, 2\operatorname{diam}(N) + 2\rho + r_0], \\ \overline{\mathbb{H}}_\varepsilon(\tilde{d} + t - (2\operatorname{diam}(N) + \rho + r_0)), & [2\operatorname{diam}(N) + 2\rho + r_0, 4\operatorname{diam}(N) + \rho + r_0], \\ 1 + t - (4\operatorname{diam}(N) + \rho + r_0), & [4\operatorname{diam}(N) + \rho + r_0, 4\operatorname{diam}(N) + \rho + r_0 + b_\varepsilon - 1] \end{cases}$$

This path satisfies the following; $\gamma_\varepsilon(-1 - a_\varepsilon) = a_\varepsilon$, $\gamma_\varepsilon(4\operatorname{diam}(N) + r_0 + \rho + b_\varepsilon - 1) = b_\varepsilon$, and

$$\gamma_\varepsilon(t) < 2\sigma\mathcal{H}^n(M) - \sigma\lambda\mu_g(E) + \sigma\lambda\mu_g(N\setminus E) - \frac{\sigma\mathcal{H}^n(A_l)}{2(2^n - 1)} + \tau,$$

for all t in $[-1 - a_\varepsilon, 4\operatorname{diam}(N) + r_0 + \rho + b_\varepsilon - 1]$. This contradicts the min-max construction of M, implying that M must be embedded, and therefore completing Theorem 4.

2.9 Morse Index

Recall the discussion from Section 1.1 that, for an embedded CMC hypersurface M, with $\overline{M} = \partial E$, for an open set $E \subset N$, then for an open set $\Omega \subset N$, such that $\Omega \subset\subset N \setminus (\overline{M} \setminus M)$, we define the index of M in Ω ($\mathrm{ind}_\Omega(M)$), as the largest dimensional subspace of $W^{1,2}_0(M \cap \Omega)$, such that the bilinear form,

$$B_{L_M}[\varphi, \psi] = \int_M \nabla^M \varphi \cdot \nabla^M \psi - (|A_M|^2 + \mathrm{Ric}_g(\nu, \nu))\varphi\,\psi\,d\mathcal{H}^n,$$

is negative definite. We then define,

$$\mathrm{ind}(M) = \sup_{\Omega \subset\subset N \setminus (\overline{M} \setminus M)} (\mathrm{ind}_\Omega(M)).$$

Now consider our M from Theorem 5. Thus, as M is embedded, and our sequence of critical points $\{u_i\}$ from Section 2.1.1 has $\mathrm{ind}\, u_i \leq 1$ (Proposition 1), by [38, Theorem 1a.], we have that $\mathrm{ind}(M) \leq 1$.

Remark 26. As M is two-sided and embedded, and the inhomogeneous term is a constant, we may also apply the ideas and arguments of [31] verbatim to conclude that $\mathrm{ind}(M) \leq 1$.

Claim 4. *$\mathrm{ind}\, M = 1$.*

Proof. We only need to show a lower bound, which follows from the Ricci positivity on N. We construct an appropriate function on M, using a similar argument to [9, Lemma 5.1].

We wish to prove that we can find a set $\Omega \subset\subset N \setminus (\overline{M} \setminus M)$, and a function φ in $W^{1,2}_0(M \cap \Omega)$ such that,

$$B_M(\varphi, \varphi) < 0.$$

By the Ricci positivity of N, for any $\Omega \subset\subset N \setminus (\overline{M} \setminus M)$, and φ in $W^{1,2}_0(M \cap \Omega)$, we have

$$B_M(\varphi, \varphi) \leq \int_M |\nabla^M \varphi|^2 - |A_M|^2 \varphi^2\,d\mathcal{H}^n.$$

If $\overline{M} \setminus M = \emptyset$, we set $\Omega = N$, and $\varphi = 1$,

$$B_M(\varphi, \varphi) \leq -\int_M |A_M|^2\,d\mathcal{H}^n < 0.$$

For $\overline{M} \setminus M \neq \emptyset$, we first we note that we must have $n \geq 7$, and $\mathcal{H}^{n-2}(\overline{M} \setminus M) = 0$. Furthermore, as $(\overline{M}, 1)$ is a multiplicity 1 integral n-varifold with uniformly bounded generalised mean curvature, we have a monotonicity formula [54, Corollary 17.8], which implies Euclidean volume growth about each point in $\overline{M}$. Both of these facts combined imply that for each $\delta > 0$, we can construct a function φ_δ, on N, with the following properties (see [66, pp. 89-90])

$$\begin{cases} \varphi_\delta \in W_c^{1,\infty}(N), \\ \operatorname{spt} \varphi_\delta \subset\subset N \setminus (\overline{M} \setminus M), \\ \varphi_\delta(y) \in [0,1], \; y \in N, \\ \|\nabla^M \varphi_\delta\|_{L^2(M)} \leq \sqrt{\delta}, \\ \mathcal{H}^n(M \cap \{\varphi_\delta = 1\}) > \mathcal{H}^n(M) - \delta, \end{cases}$$

Thus, taking $\delta > 0$ small enough we may set $\Omega = \operatorname{spt} \varphi_\delta \subset\subset N \setminus (\overline{M} \setminus M)$, and have that

$$B_M(\varphi_\delta, \varphi_\delta) \leq \delta - n^{-2}\lambda^2 \mathcal{H}^n(\{\varphi_\delta = 1\} \cap M) < 0.$$

This implies that $\operatorname{ind} M \geq 1$. □

The fact that M is connected immediately follows from this, as on each connected component we could construct a function as in Claim 4. Therefore each connected component adds atleast 1 to the index.

Chapter 3

Upper Semicontinuity of Index Plus Nullity for Minimal and CMC Hypersurfaces

As previously, let (N, g) be a compact Riemannian manifold, with no boundary, of dimension $n+1$ (in this Chapter we will mostly focus on $2 \leq n \leq 6$), and $H > 0$, be a fixed constant. In this chapter we investigate two classes, $\mathfrak{M}(N, g)$, and $\mathfrak{C}_H(N, g)$. Here, $\mathfrak{M}(N, g)$ is the class of smooth, closed, properly embedded, minimal hypersurfaces of N, with respect to the metric g, and $\mathfrak{C}_H(N, g)$ is the class of smooth, closed, properly embedded hypersurfaces in N, of constant mean curvature $H > 0$, with respect to the metric g.

As these hypersurfaces arise as critical points to appropriately chosen area-type functionals, a natural property to study is their Morse index (with respect to the associated functional). For two fixed numbers $\Lambda > 0$, and $I \in \mathbb{Z}_{\geq 0}$, we define the subclasses,

$$\mathfrak{M}(N, g, \Lambda, I) = \{M \in \mathfrak{M}(N, g) \colon \mathcal{H}^n_g(M) \leq \Lambda,\ \mathrm{ind}\,(M) \leq I\},$$
$$\mathfrak{C}_H(N, g, \Lambda, I) = \{M \in \mathfrak{C}_H(N, g) \colon \mathcal{H}^n_g(M) \leq \Lambda,\ \mathrm{ind}\,(M) \leq I\}.$$

Making use of the curvature estimates for stable minimal hypersurfaces by Schoen–Simon–Yau [52] and Schoen–Simon [51] (see also the recent proof by Bellettini [10]), for $2 \leq n \leq 6$, compactness properties have been proven for $\mathfrak{M}(N, g, \Lambda, I)$ by Sharp [53], and for $\mathfrak{C}_H(N, g, \Lambda, I)$ by Bourni–Sharp–Tinaglia [16]. In dimension $n + 1 = 3$, it is worth noting that various other compactness results have been shown for minimal surfaces by Choi–Schoen [21], Anderson [6], Ros [47], and White [65], and for H-CMC surfaces by Sun [57]. We note that $\mathfrak{M}(N, g, \Lambda, I)$ is sequentially compact (under the correct notion of convergence), whereas for $\mathfrak{C}_H(N, g, \Lambda, I)$, we must expand our class to quasi-embedded, H-CMC hypersurfaces (see Definition 1). We denote this enlarged class by $\overline{\mathfrak{C}_H}(N, g, \Lambda, I)$. We briefly describe this notion of convergence, with full details described in point 1 of Definition 5. Consider a sequence $\{M_k\} \subset \mathfrak{M}(N, g, \Lambda, I)$ (resp. $\mathfrak{C}_H(N, g, \Lambda, I)$), then after potentially taking a subsequence and renumerating, there is a smooth, closed, embedded minimal hypersurface (resp. H-CMC quasi-embedded hypersurface) M_∞, and a finite set of points $\mathcal{I} \subset M_\infty$, where $|\mathcal{I}| \leq I$, such that on compact subsets of $N \setminus \mathcal{I}$, M_k will converge to M_∞, smoothly and locally graphically, with integer multiplicity potentially greater than 1. It is then shown that $M_\infty \in \mathfrak{M}(N, g, \Lambda, I)$ (resp. $\overline{\mathfrak{C}_H}(N, g, \Lambda, I)$). The set of points $\mathcal{I} \subset M_\infty$, is defined by the condition that for each $y \in \mathcal{I}$, there exists a sequence points $\{x_k^y \in M_k\}_{k \in \mathbb{N}}$, such that $x_k^y \to y$, and the curvature $|A_{M_k}(x_k^y)|$, blows up as $k \to \infty$. Thus we call $\mathcal{I}$ the singular set of the convergence.

In a bid to understand the formation of such singularities, a bubble analysis was carried out by Chodosh–Ketover–Maximo [19], Buzano–Sharp [17], and Bourni–Sharp–Tinaglia [16]. Zooming in at appropriate rates, along particular sequences of points converging onto $\mathcal{I}$, yields a complete, embedded, non-planar, minimal hypersurface in $\mathbb{R}^{n+1}$, of finite index, with Euclidean volume growth at infinity. These minimal hypersurfaces in $\mathbb{R}^{n+1}$ are referred to as the 'bubbles', and they are the singularity models at the singular points of the convergence. The hypersurface M_∞ is referred to as the 'base'. This terminology is borrowed from other non-linear geometric problems. See Figure 3.1 for a heuristic picture. In the case of $n = 2$, a bubble analysis has been carried out by Ros [47], in $\mathbb{R}^3$ with the standard Euclidean metric, assuming uniform bounds on the total curvature instead of the Morse index, and by White [65], in general 3-manifolds, assuming uniform bounds on genus instead of Morse index.

One may be interested in certain information about the hypersurfaces along these sequences, for example; genus ([19]), index and total curvature ([17, 16]). Understanding the formation of these singularities through this bubble analysis allows us to track this information along the sequence, and how it behaves when taking the limit $M_k \to M_\infty$. For example, if we have our sequence $\{M_k\}$ as above, and we know all our 'bubbles' are given by $\Sigma^1, \ldots, \Sigma^J \subset \mathbb{R}^{n+1}$, then we say that $M_k \to (M_\infty, \Sigma^1, \ldots, \Sigma^J)$ 'bubble converges' (see Definition 5 for a detailed definition), and [17, 16],

$$\operatorname{ind}(M_\infty) + \sum_{j=1}^{J} \operatorname{ind}(\Sigma^j) \leq \liminf_{k \to \infty} \operatorname{ind}(M_k). \tag{3.1}$$

This inequality gives a quantitative way of accounting for some of the index lost when taking the limit $M_k \to M_\infty$. In this work we are interested in proving an opposite inequality, which will give a finer analysis and description of the index along such a converging sequence, and shows that in certain situations, this bubble analysis will account for all the index in the limit.

Our main result is Theorem 8 below, however we first illustrate the conclusions in a simplified setting with the following special case. We delay the statement of Theorem 8 until the end of this section.

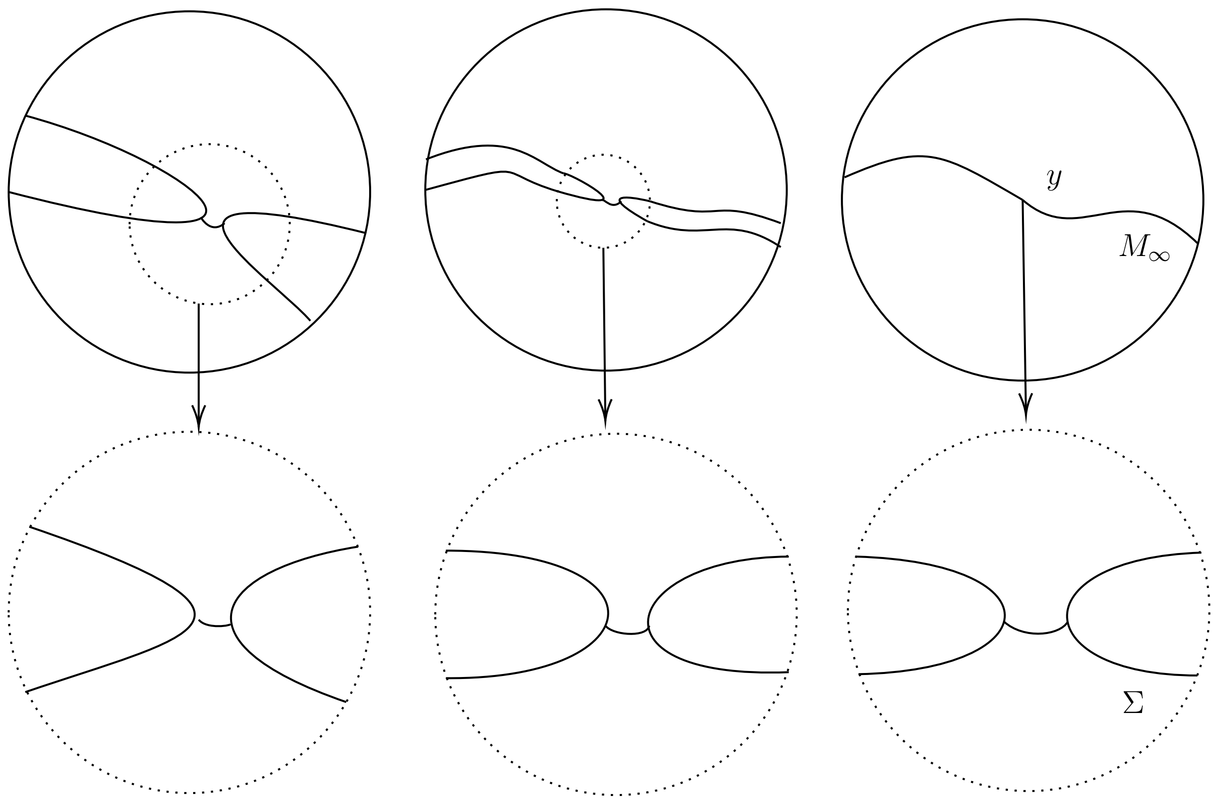


Figure 3.1: The top row depicts a local picture of a converging sequence about a point $y \in \mathcal{I}$, which converges with multiplicity 2 on the base M_∞. The second row then depicts a dilation of the dotted circles in the top row, which when we take a limit, as seen in the last column, allows us to see a catenoid as the bubble Σ.

Corollary 2. *Consider a compact Riemannian manifold (N, g), without boundary and of dimension $n+1$, $3 \leq n \leq 6$. Let $\{M_k\}$ be a sequence of smooth, closed, embedded minimal hypersurfaces of (N, g), such that $M_k \to (M_\infty, \Sigma^1, \ldots, \Sigma^J)$ bubble converges as in Definition 5, with $M_\infty \subset N$ being a smooth, connected, two-sided, closed, embedded minimal hypersurface. Then:*

$$\begin{aligned} \limsup_{k\to\infty}(ind\,(M_k) + nul\,(M_k)) \quad &\leq \quad m(ind\,(M_\infty) + nul\,(M_\infty)) \\ &\qquad + \sum_{j=1}^{J} ind\,(\Sigma^j) + nul_{\omega_{\Sigma^j,R}}\,(\Sigma^j), \end{aligned}$$

where $m \in \mathbb{Z}_{\geq 1}$ is the multiplicity of the convergence onto M_∞. Here,

$$ind\,(\Sigma^j) = \lim_{S\to\infty} ind_{B_S^{n+1}(0)}\,(\Sigma^j),$$

and R may be chosen to be any finite positive real number greater than some $R_0 = R_0(\Sigma^1, \ldots, \Sigma^J) \in [1, \infty)$, and

$$\omega_{\Sigma^j,R}(x) = \begin{cases} R^{-2},\ x \in B_R^{n+1}(0) \cap \Sigma^j, \\ |x|^{-2},\ x \in \Sigma^j \setminus B_R^{n+1}(0). \end{cases}$$

and,

$$nul_{\omega_{\Sigma^j,R}}\,(\Sigma^j) = dim\,\{f \in C^\infty(\Sigma^j)\colon \Delta f + |A_{\Sigma^j}|^2 f = 0,\ f^2\,\omega_{\Sigma^j,R} \in L^1(\Sigma^j),\ |\nabla f|^2 \in L^1(\Sigma^j)\}.$$

The inequality in Corollary 2 can be further strengthened by noting that in this situation if $m \geq 2$, then $\operatorname{ind}(M_\infty) = 0$ ([53, Claim 6]), and $\operatorname{nul}(M_\infty) = 1$ (as the first eigenvalue of the stability operator will be simple).

Results on the lower semicontinuity of index along converging sequences (3.1), are common in the literature. For certain classes of minimal hypersurfaces see Sharp [53], Buzano–Sharp [17], Ambrozio–Carlotto–Sharp [5] and Ambrozio–Buzano–Carlotto–Sharp [4], and for certain classes of CMC hypersurfaces see Bourni–Sharp–Tinaglia [16]. In the setting of Allen–Cahn solutions see Le [35], Hiesmayr [31], Gaspar [27] and Mantoulidis [38]. For the setting of of bubble converging harmonic maps see Moore–Ream [42, Theorem 6.1], and Hirsch–Lamm [32, Theorem 1.1].

The opposite upper semicontinuity inequality (Theorem 8) is more intricate. We recall a few examples of such results from the literature. When convergence happens with multiplicity one for sequences of critical points of the Allen–Cahn functional, upper semicontinuity of the index plus nullity has been established by Chodosh–Mantoulidis [20, Theorem 1.9] and Mantoulidis [38, Theorem 1 (c)]. In the case of bubble converging harmonic maps such an inequality was first established by Yin [69, 70], and then by Da Lio–Gianocca–Rivière [23] and Hirsch–Lamm [32].

Note that in [23] and [32, Section 6] the proofs are for a bubble converging sequence of critical points for a general class of conformally invariant lagrangians (fixed along the sequence).

When combined with the lower semicontinuity of index, the inequality in Theorem 8 shows that in the case of the limiting hypersurface being two-sided and minimal (as is the case of Corollary 2), the index along the sequence can be *fully* accounted for in the limit. Thus we should view Theorem 8 as saying that we cannot lose index to the neck, or index cannot merely just disappear in the bubble convergence of Chodosh–Ketover–Maximo [19] and Buzano–Sharp [17] for minimal hypersurfaces (in dimensions $3 \leq n \leq 6$).

In order to conclude that the inequality in Theorem 8 is non-trivial, we must show that for each bubble, Σ^j, of finite index, $\mathrm{nul}_{\omega_{\Sigma^j,R}}(\Sigma^j) < +\infty$. This is shown in Proposition 16. Proposition 16 also has the following Corollary which may be of interest to some readers (and may be compared with [36, Corollary 2]).

Corollary 3. *Let Σ be a complete, connected, n-dimensional manifold, $n \geq 3$, and $\iota\colon \Sigma \to \mathbb{R}^{n+1}$ be a two-sided, proper, minimal immersion, with finite total curvature,*

$$\int_\Sigma |A_\Sigma|^n < +\infty,$$

and Euclidean volume growth at infinity,

$$\limsup_{r\to\infty} \frac{\mathcal{H}^n(\iota(\Sigma) \cap B_r^{n+1}(0))}{r^n} < +\infty.$$

Then:

$$\textit{anl-nul}(\Sigma) \coloneqq \mathrm{dim}\{f \in W^{1,2}(\Sigma)\colon \Delta f + |A_\Sigma|^2 f = 0\} < +\infty.$$

We note that as Σ is not compact, Corollary 3 does not follow from analysing the spectrum of a compact operator.

We briefly remark on the strategy of the proof for Theorem 8, which is close to the strategy of Da Lio–Gianocca–Rivière [23]. We prove Theorem 8 by reframing the problem in terms of a weighted eigenvalue problem. The weight is specifically chosen so that sequences of normalised weighted eigenfunctions $\{f_k\}$, along the sequence $\{M_k\}$, with non-positive weighted eigenvalues, exhibit good convergence on the base M_∞, and the bubbles, $\Sigma^1, \ldots, \Sigma^J$. The key steps for the proof are showing the equivalence of the weighted and unweighted eigenvalue problems (Section 3.3), the convergence on the base M_∞ (Section 3.2.2), and the convergence on the bubbles $\Sigma^1, \ldots, \Sigma^J$ (Section 3.2.3), along with a Lorentz–Sobolev inequality on the neck, which shows that the normalised weighted eigenfunctions cannot concentrate on the neck (Section 3.2.4).

Due to the different settings, there are several key differences between our work and that of [23]. One such difference is our use of a Lorentz–Sobolev inequality to deduce strict stability on the neck. Another major difference is that in our setting the bubbles are non-compact. This poses complications in the theory of the elliptic operator on the bubble. In particular its spectrum may not be discrete, and thus effectively analysing the index and nullity of these bubbles is subtle.

It is worth pointing out that the method used in [23], and in Theorem 8 is rather general. In the proof of Theorem 8, only a few aspects rely specifically on the mean curvature assumptions of the submanifolds. Thus it is plausible that the ideas and techniques could be applied to a large range of problems in which one wishes to study how an elliptic PDE behaves along a sequence of (sub)manifolds which 'bubble converge' in an appropriate sense.

Theorem 8. *For a compact Riemannian manifold (N, g) without boundary, of dimension $n+1$, $3 \leq n \leq 6$, if we have a sequence $\{M_k\} \subset \mathfrak{M}(N, g)$ ($\{M_k\} \subset \mathfrak{C}_H(N, g)$), such that $M_k \to (M_\infty, \Sigma^1, \ldots, \Sigma^J)$ bubble converges as in Definition 5, with $M_\infty = \cup_{i=1}^{l} M^i$, where each M^i is a closed minimal hypersurface (resp. closed, quasi-embedded H-CMC hypersurface, with $co(M^i)$ connected) and $\theta_{|M^i} = m_i \in \mathbb{Z}_{\geq 1}$ ($\theta^i = m_i \in \mathbb{Z}_{\geq 1}$), then*

$$\begin{aligned}\limsup_{k\to\infty}(ind(M_k) + nul(M_k)) \;\leq\; & \sum_{i=1}^{l} co(m)_i(anl\text{-}ind(co(M^i_\infty)) + anl\text{-}nul(co(M^i_\infty))) \\ & + \sum_{j=1}^{J} ind(\Sigma^j) + nul_{\omega_{\Sigma^j,R}}(\Sigma^j),\end{aligned}$$

where for each $i = 1, \ldots, l$, $co(m)_i \in \mathbb{Z}_{\geq 1}$, is such that $co(m)_i \leq m_i$ if M^i is one-sided, and $co(m)_i = m_i$ if M^i is two-sided. Here,

$$ind(\Sigma^j) = \lim_{S\to\infty} ind(\Sigma^j \cap B^{n+1}_S(0)),$$

and R may be chosen to be any finite positive real number greater than some $R_0 = R_0(\Sigma^1, \ldots, \Sigma^J) \in [1, \infty)$, and

$$\omega_{\Sigma^j,R}(x) = \begin{cases} R^{-2},\ x \in B^{n+1}_R(0) \cap \Sigma^j, \\ |x|^{-2},\ x \in \Sigma^j \setminus B^{n+1}_R(0). \end{cases}$$

and,

$$nul_{\omega_{\Sigma^j,R}}(\Sigma^j) = dim\{f \in C^\infty(\Sigma^j)\colon \Delta f + |A_{\Sigma^j}|^2 f = 0,\ f^2\,\omega_{\Sigma^j,R} \in L^1(\Sigma^j),\ |\nabla f|^2 \in L^1(\Sigma^j)\}.$$

The exact method of proof we employ does not extend to the case of $n = 2$ (2 dimensional surfaces in 3-manifolds). Two key reasons are the choice of weight (Remark 31), and the criticality of the Lorentz–Sobolev inequality (Proposition 10) for $n = p = 2$.

We take a moment to comment on the terms anl-ind (M^i_∞) and anl-nul (M^i_∞), that appear in the statement of Theorem 8. These terms respectively stand for the *analytic index* and *analytic nullity* of M^i_∞. This refers to the index and nullity of the stability operator acting on the function space $C^\infty(co(M^i_\infty))$, where $co(M^i_\infty)$ is a connected component of the two-sided double cover of $M^i_\infty \subset N$. We explain the reasoning behind this with the following example, which is also demonstrated in Figure 3.2. Consider a unit hypersphere in $\mathbb{R}^{n+1}$ (this is a CMC hypersurface), then the function $f = 1$ on the hypersphere, is an eigenfunction of the stability operator, with negative eigenvalue, and corresponds to shrinking the hypersphere. Now consider a sequence of two disjoint unit hyperspheres in $\mathbb{R}^{n+1}$, such that in the limit they touch at a point. In order to account for these eigenfunctions in the limit, we must allow for variations that act on the hyperspheres independently, even at the touching point. Thus we view the hyperspheres as immersions, and allow variations which 'shrink' the hyperspheres separately. This type of variation cannot arise through an ambient vector field due to the behaviour at the touching point. Thus in general, the analytic index and analytical nullity of M_∞, will not be equivalent to the Morse index and nullity of M_∞, which is customarily defined through ambient vector fields. See Section 3.1.2 for further details.

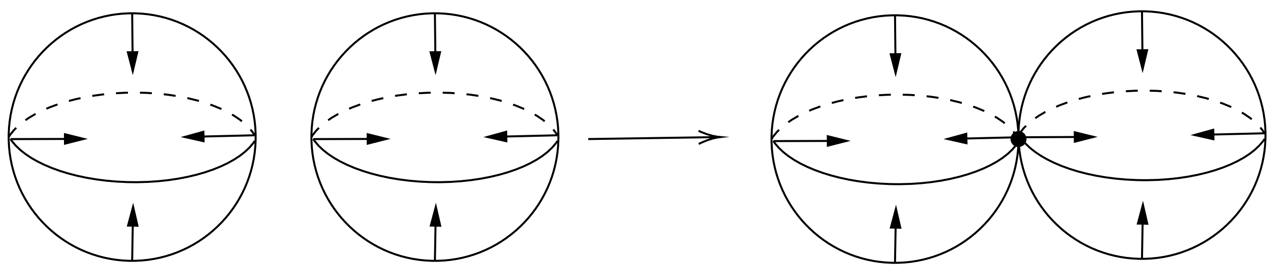


Figure 3.2: Sequence of two spheres coming together to touch at a point. The arrows attached to the spheres demonstrate the ambient vector field needed to give rise to the eigenfunction which corresponds to 'shrinking' of these spheres. Notice that at the non-embedded point this vector field is not well defined as an ambient one.

Finally we make note of another special case, by considering a sequence of H-CMC hypersurfaces $\{M_k\}$, which bubble converge $M_k \to (\cup_{i=1}^l M^i, \Sigma^1, \dots, \Sigma^J)$, such that each M_k arises as the boundary of some open set $E_k \subset N$. This particular setting has been analysed by Bourni–Sharp–Tinaglia [16], were they showed that for each i, $co(m)_i = 1$, and, by applying a uniqueness result of Schoen [50, Theorem 3], that each bubble Σ^j, will be given by a catenoid $\mathcal{C}$. Thus, as catenoids have index 1 ([58, Theorem 2.1]), we have that

$$\limsup_{k\to\infty}(\text{ind}\,(M_k) + \text{nul}\,(M_k)) \leq \sum_{i=1}^{l}(\text{anl-ind}\,(co(M^i_\infty)) + \text{anl-nul}\,(co(M^i_\infty))) + J(1 + \text{nul}_{\omega_{\mathcal{C},R}}\,(\mathcal{C})).$$

In Section 3.6 we investigate $\text{nul}_{\omega_{\mathcal{C},R}}(\mathcal{C})$, and show that it has a lower bound of n. In particular, in Section 3.6 we analyse Jacobi fields on the n-dimensional catenoid $\mathcal{C} \subset \mathbb{R}^{n+1}$ (for $n \geq 3$), which arise from rigid motions of $\mathbb{R}^{n+1}$ (translations, rotations and scalings). We show that the only non-trivial such Jacobi fields which lie in $W^{1,2}(\mathcal{C})$, or the weighted space $W^{1,2}_{\omega_{\mathcal{C},R}}(\mathcal{C})$ are those generated by translations which are parallel to the ends of $\mathcal{C}$.

3.1 Preliminaries

3.1.1 Bubble Convergence Preliminaries

In this section we give a precise definition of bubble convergence (Definition 5), and prove some technical lemmas (Lemmas 2 and 3), which describe the structure of the neck regions in the bubble convergence, as well as the ends of the bubbles.

In this chapter, for a quasi-embedded hypersurface M, we denote $e(M)$ to be the set of embedded points of M, and $t(M)$ to be the set of non-embedded points of M. This is a slight change of notation to that of Chapters 1 and 2. The reason for this is to keep notation consistent with that of [16], which is one of the main background references for this chapter.

Definition 5. Consider a Riemannian manifold (N, g), of dimension $n+1$, $n \geq 2$, (and $H > 0$) along with a sequence $\{M_k\}_{k\in\mathbb{N}} \subset \mathfrak{M}(N,g)$ ($\{M_k\}_{k\in\mathbb{N}} \subset \mathfrak{C}_H(N,g)$), an $M_\infty \in \mathfrak{M}(N,g)$ ($M_\infty \in \overline{\mathfrak{C}_H}(N,g)$), and a collection of non-planar, complete, properly embedded, minimal hypersurfaces $\{\Sigma_j\}_{j=1}^J$ in $\mathbb{R}^{n+1}$, with $J \in \mathbb{Z}_{\geq 1}$. Then, we say that

$$M_k \to (M_\infty, \Sigma_1, \ldots, \Sigma_J),$$

bubble converges, if:

1. For the case of minimal hypersurfaces; $(M_k, 1) \to (M_\infty, \theta)$ as varifolds, where $\theta : M_\infty \to \mathbb{Z}_{\geq 1}$, and is constant on connected components of M_∞. Moreover, there exists an at most finite collection of points $\mathcal{I} \subset M_\infty$, such that locally on $M_\infty \setminus \mathcal{I}$, M_k converges smoothly and graphically, with multiplicity θ (see Remark 27 for a precise definition).

   For the case of H-CMC hypersurfaces; $(M_k, 1) \to (M_\infty, \theta)$ as varifolds, where $M_\infty = \cup_{i=1}^a M^i_\infty$, and each M^i_∞ is a distinct, closed, quasi-embedded H-CMC hypersurface, such that for its respective immersion, $\iota^i \colon S^i \to M^i_\infty$, S^i is connected, and there exists a $\theta^i \in \mathbb{Z}_{\geq 1}$, such that $\theta(y) = \sum_{i=1}^a (|(\iota^i)^{-1}(y)|\theta^i)$. Moreover, there exists an at most finite collection of points $\mathcal{I} \subset M_\infty$, such that locally on $M_\infty \setminus \mathcal{I}$, M_k converges smoothly and graphically, with multiplicity θ (see Remark 27 for a precise definition).

2. For each $i \in \{1, \ldots, J\}$, there exist point-scale sequences $\{(p^i_k, r^i_k)\}_{k\in\mathbb{N}}$, such that for each $k \in \mathbb{Z}_{\geq 1}$, $p^i_k \in M_k$, and there exists a $y^i \in \mathcal{I}$, such that $p^i_k \to y^i$, $r^i_k \to 0$. Moreover, for each $R \in (0, \infty)$, and large enough k, the connected component of $M_k \cap B^N_{Rr^i_k}(p^i_k)$, through p^i_k, denoted $\Sigma^{i,R}_k$, is such that, if we rescale the geodesic ball $B^N_{Rr^i_k}(p^i_k)$ by r^i_k, and denote

$$\tilde{\Sigma}^{i,R}_k \cap B^{n+1}_R(0) \coloneqq \frac{1}{r^i_k} \exp^{-1}_{p^i_k}(\Sigma^{i,R}_k \cap B^N_{Rr^i_k}(p^i_k)) \subset B^{n+1}_R(0) \subset \mathbb{R}^{n+1},$$

then $\tilde{\Sigma}^{i,R}_k \to \Sigma^i \cap B_R(0)$ smoothly and graphically, and hence with multiplicity one. Furthermore, for $i \neq j$, either

$$\lim_{k\to\infty} \frac{r^i_k}{r^j_k} + \frac{r^j_k}{r^i_k} + \frac{\operatorname{dist}^N_g(p^i_k, p^j_k)}{r^i_k + r^j_k} = \infty,$$

or for each $R \in (0, \infty)$, and then large enough $k \in \mathbb{N}$, $p^j_k \notin \Sigma^{i,R}_k$.

3. Defining,

$$d_k(x) \coloneqq \min_{i=1,\ldots,J} \operatorname{dist}^N_g(x, p^i_k),$$

then,

$$\lim_{\delta\to 0} \lim_{R\to\infty} \lim_{k\to\infty} \sup_{x\in M_k\cap(\cup_{y\in\mathcal{I}} B^N_\delta(y)\setminus\cup^J_{i=1}\Sigma^{i,R}_k)} \int_{M^x_k\cap B^N_{d_k(x)/2}(x)} |A_k|^n = 0,$$

where $M^x_k \cap B^N_{d_k(x)/2}(x)$ is the connected component of $M_k \cap B^N_{d_k(x)/2}(x)$, that contains x.

Remark 27. We now remark on exactly what we mean by M_k converging smoothly and graphically, with multiplicity θ, locally on $M_\infty \setminus \mathcal{I}$.

1. First we consider $y \in M_\infty \setminus \mathcal{I}$ being an embedded point (as is always the case for minimal hypersurfaces). Then, there exists $\rho, \tau \in (0, \operatorname{inj}(N)/2)$, and an n-dimensional subspace $T \subset T_yN$, such that, defining the cylinder,

$$C_{y,T,\rho,\tau} \coloneqq \{\exp_y(X + s\nu_T) \colon X \in B^{T_yN}_\rho(0) \cap T,\ s \in (-\tau, \tau)\},$$

(where ν_T is a choice of unit normal to T) we have that $C_{y,T,\rho,\tau} \subset\subset N \setminus \mathcal{I}$, and there exists a smooth function,

$$u \colon B^{T_yN}_\rho(0) \cap T \to (-\tau/2, \tau/2),$$

such that,

$$M_\infty \cap C_{y,T,\rho,\tau} = \operatorname{graph}(u) \coloneqq \{\exp_y(X + u(X)\nu_T) \colon X \in B^{T_yN}_\rho(0) \cap T\}.$$

Moreover, for large enough k, there exists $\theta(y)$ distinct smooth functions,

$$v_{1,k}, \ldots, v_{\theta(y),k} \colon B^{T_yN}_\rho(0) \cap T \to (-\tau, \tau),$$

such that,

$$M_k \cap C_{y,T,\rho,\tau} = \bigcup_{l=1}^{\theta(y)} \operatorname{graph}(v_{l,k}),$$

and for each $l = 1, \ldots, \theta(y)$, $v_{l,k} \to u$ smoothly.

2. For the case of $y \in M_\infty \setminus \mathcal{I}$ being a non-embedded point of a quasi-embedded H-CMC hypersurface, then, there exists $\rho, \tau \in (0, \operatorname{inj}(N)/2)$, and an n-dimensional subspace $T \subset T_yN$, along with two distinct smooth functions,

$$u_1,\ u_2 \colon B_\rho^{T_yN}(0) \cap T \to (-\tau/2, \tau/2),$$

such that $C_{y,T,\rho,\tau} \subset\subset N \setminus \mathcal{I}$, and

$$M_\infty \cap C_{y,T,\rho,\tau} = \operatorname{graph}(u_1) \cup \operatorname{graph}(u_2),$$

with $u_1(0) = 0 = u_2(0)$. Moreover, there exists $i_1,\ i_2 \in \{1, \ldots, a\}$ (potentially equal), such that $\operatorname{graph}(u_j) \subset M_\infty^{i_j}$. Then, for large enough k, there exists $\theta(y)$ smooth functions,

$$v_{1,k}, \ldots, v_{\theta(y),k} \colon B_\rho^{T_yN} \cap T \to (-\tau, \tau),$$

such that,

$$M_k \cap C_{y,T,\rho,\tau} = \bigcup_{l=1}^{\theta(y)} \operatorname{graph}(v_{l,k}),$$

and for $l = 1, \ldots, \theta^{i_1}$, $v_{l,k} \to u_1$, smoothly, and for $l = \theta_{i_1} + 1, \ldots, \theta_{i_1} + \theta_{i_2} = \theta(y)$, $v_{l,k} \to u_2$, smoothly.

Consider such a convergence $M_k \to (M_\infty, \Sigma^1, \ldots, \Sigma^J)$, as in Definition 5, then we may remark:

- The convergence considered in the bubble analysis of Chodosh–Ketover–Maximo [19] and Buzano–Sharp [17] for minimal hypersurfaces, and Bourni–Sharp–Tinaglia [16] for CMC boundaries, satisfies Definition 5.

- As the multiplicity function $\theta \colon M_\infty \to \mathbb{Z}_{\geq 1}$ is uniformly bounded, and $\mathcal{H}^n(M_\infty \cap U) < +\infty$, for $U \subset N$ compact, then we may deduce by the varifold convergence that $\sup_k \mathcal{H}^n(M_k \cap U) < +\infty$. Applying this fact, and using the monotonicity formula for varifolds with bounded mean curvature ([54, Theorem 17.7]), we may deduce that each Σ^j must have Euclidean volume growth at infinity (see [17, Corollary 2.6]).

- The final sentence of point 2 in Definition 5 guarantees that all of these bubbles are distinct.

- For each $j = 1, \dots, J$, as Σ^j is complete and properly embedded, by [49] Σ^j is two-sided in $\mathbb{R}^{n+1}$.
- Without loss of generality we may assume that $|A_{\Sigma^j}|(0) > 0$, for each $j = 1, \dots, J$.

We also now assume that each Σ^j has finite index. This is a reasonable assumption for us to make, as if it does not hold then the result that we are interested in (Theorem 8) would hold trivially. Thus, for $3 \le n \le 6$, by a result of Tysk [61], the finite index and Euclidean volume growth at infinity, imply that each bubble Σ^j will have finite total curvature;

$$\int_{\Sigma^j} |A_{\Sigma^j}|^n < +\infty.$$

The following curvature estimate (Proposition 7) is important in our analysis. First we list some notation. If $\iota\colon M \to N$ is a proper immersion, and $S \subset N$, $x \in M$, and $\iota(x) \in S$, then we denote $\iota^{-1}(S)_x$ as the connected component of $\iota^{-1}(S)$ which contains x.

Proposition 7. *Consider $(B_1^{n+1}(0), g)$, where g is a Riemannian metric (a constant $H > 0$), and a proper, g-minimal (g-H-CMC) immersion $\iota\colon M \to B_1^{n+1}(0)$, such that $\iota(\partial M) \subset \partial B_1^{n+1}(0)$. There exists an $\varepsilon_0 = \varepsilon_0(g)(= \varepsilon_0(g, H)) > 0$, such that for $x \in \iota^{-1}(B_{1/2}^{n+1}(0))$, $r \in (0, 1/4)$, and $\varepsilon \in (0, \varepsilon_0)$, if*

$$\int_{\iota^{-1}(B_r^{n+1}(\iota(x)))_x} |A|^n \le \varepsilon,$$

then,

$$\sup_{y \in B_r^{n+1}(\iota(x)))_x} dist_g^{B_1^{n+1}(0)}(\iota(y), \partial B_r^{n+1}(\iota(x)))\, |A|(y) \le C_\varepsilon,$$

with $C_\varepsilon = C(g, \varepsilon)(= C(g, H, \varepsilon)) < +\infty$. Moreover, $C_\varepsilon \to 0$ as $\varepsilon \to 0$.

Proof. The proof, detailed below, follows from a standard point-picking argument, which may be found in [64, Lecture 3]. In the proof we will denote $\mathrm{dist}_g^{B_1^{n+1}(0)}$ by dist_g.

Assume that the bound is not true. Then there exists a sequence of proper, g-minimal (g-H-CMC) immersions $\{\iota_k\colon M_k \to B_1^{n+1}(0)\}_{k\in\mathbb{N}}$ (from here on we will drop the subscript k from the maps ι_k) with points $x_k \in M_k$, radii $\{r_k\}_{k\in\mathbb{N}}$, with $r_k < 1/4$, such that,

$$\int_{\iota^{-1}(B_{r_k}^{n+1}(\iota(x_k)))_{x_k}} |A_k|^n \to 0,$$

but a fixed $\delta > 0$, and points $y_k \in \overline{\iota^{-1}(B_{r_k}^{n+1}(\iota(x_k)))_{x_k}} \subset\subset M_k$, such that

$$\mathrm{dist}_g(\iota(y_k), \partial B_{r_k}^{n+1}(\iota(x_k)))|A_k|(y_k) = \sup_{y \in \iota^{-1}(B_{r_k}^{n+1}(\iota(x_k)))_{x_k}} \mathrm{dist}_g(\iota(y), \partial B_{r_k}^{n+1}(x_k))|A_k|(y) \ge \delta.$$

We can see that $y_k \notin \iota^{-1}(\partial B^{n+1}_{r_k}(\iota(x_k)))_{x_k}$. Denote,

$$\begin{aligned} T_k &= |A_k|(y_k), \\ \rho_k &= \mathrm{dist}_g(\iota(y_k), \partial B^{n+1}_{r_k}(\iota(x_k))) \leq r_k < 1/4. \end{aligned}$$

Translate $\iota(y_k)$ to the origin, rescale by T_k, and denote our new immersion by,

$$\begin{aligned} \tilde{\iota}\colon M_k &\to B^{n+1}_{T_k}(-\iota(y_k)), \\ x &\mapsto T_k(\iota(x) - \iota(y_k)). \end{aligned}$$

This will be a proper minimal ($T_k^{-1}H$-CMC) immersion in the appropriately translated and rescaled metric, which we denote $\tilde{g}_k$. After taking an appropriate rotation we may also assume that $\tilde{\iota}$ maps the tangent space $T_{y_k}M_k$ to $\{x_{n+1} = 0\}$.

Denoting $\tilde{A}_k$ as the second fundamental form of M_k with respect to immersion $\tilde{\iota}$, we have

$$|\tilde{A}_k|(y_k) = T_k^{-1}|A_k|(y_k) = 1,$$

Moreover, $s_k = T_k\rho_k \geq \delta$, and we have that for $z \in M_k \cap \tilde{\iota}^{-1}(B^{n+1}_{s_k/2}(0))_{y_k} \subset\subset M_k$,

$$\begin{aligned} \frac{s_k}{2}|\tilde{A}_k|(z) &\leq C\mathrm{dist}_{\tilde{g}_k}(z, \partial B_{s_k}(0))|\tilde{A}_k|(z), \\ &\leq C\mathrm{dist}_{\tilde{g}_k}(0, \partial B_{s_k}(0))|\tilde{A}_k|(0), \\ &= Cs_k, \end{aligned}$$

where $C = C(g) < +\infty$, will come from the fact that we can find a $C' = C'(g) < +\infty$, such that

$$\frac{1}{C'}|p - q| \leq \mathrm{dist}_g(p, q) \leq C'|p - q|,$$

for $p, q \in B^{n+1}_{5/6}(0)$. Thus, $|\tilde{A}_k|(z) \leq 2C$, and there will exist a constant $l = l(g)$, such that for,

$$2\tau = \min\{\delta/2, l\},$$

we have that there is a smooth function,

$$u_k\colon B^{n+1}_{\tau}(0) \cap \{x_{n+1} = 0\} \to \mathbb{R},$$

with $u_k = 0$, $\nabla u_k = 0$, and for some connected set $S \subset M_k$, containing y_k, $\tilde{\iota}(S) = \mathrm{graph}(u_k)$ (see [22, Chapter 2, Lemma 2.4]). Moreover, this bound on $|\tilde{A}_k|$ translates to a uniform bound (only dependent on g) on the C^2-norm of these functions $\{u_k\}$. Moreover, these functions solve the $\tilde{g}_k$-minimal surface equation ($\tilde{g}_k$-$T_k^{-1}H$-CMC equation). Thus by Schauder theory (noting that $T_k^{-1} \leq$

$(4\delta)^{-1})$ [28, Section 6.1], we gain a uniform bound on the $C^{2,\alpha}$-norm of these functions. Therefore, potentially after taking a subsequence and renumerating, there exists a $u_\infty \in C^{2,\alpha}(B^{n+1}_{\tau/2}(0) \cap \{x_{n+1} = 0\})$ such that $u_k \to u_\infty$ in $C^2(B^{n+1}_{\tau/2}(0) \cap \{x_{n+1} = 0\})$, a Riemannian metric $\tilde{g}_\infty$ on $B^{n+1}_{2\tau}(0)$, such that $\tilde{g}_k \to \tilde{g}_\infty$ smoothly on $B^{n+1}_{2\tau}(0)$, and a constant $\Lambda \in [0, (4\delta)^{-1}]$, such that $T_k^{-1} \to \Lambda$. Again, by Schauder Theory we may upgrade this to smooth convergence and $u_\infty \in C^\infty(B^{n+1}_{\tau/2}(0) \cap \{x_{n+1} = 0\})$. Then u_∞ solves the $\tilde{g}_\infty$-minimal surface equation ($\tilde{g}_\infty$-ΛH-CMC equation), and denoting $M_\infty = \text{graph}(u_\infty)$, we have that $0 \in M_\infty$, and $|A_\infty|(0) = 1$. Thus,

$$\int_{M_\infty} |A_\infty|^n \neq 0.$$

However this contradicts the assumption that,

$$\int_{M_k \cap B_{r_k}(x_k)} |A_k|^n \to 0,$$

due to the quantity's scale invariance. □

Proposition 7 implies that if we take a sequences $\delta_k \to 0$ and $R_k \to \infty$, such that $R_k r_k^j \to 0$ for all $j = 1, \dots, J$, and $\delta_k \geq 4R_k r_k^j$ for all $k \in \mathbb{Z}_{\geq 1}$ and $j = 1, \dots, J$, and pick a sequence of points,

$$x_k \in M_k \cap (\cup_{y \in \mathcal{I}} B^N_{\delta_k}(y) \setminus \cup_{i=1}^J \Sigma_k^{i,R_k}),$$

and denote, $s_k = d_k(x_k)/2$, and

$$\tilde{M}_k \cap B^{n+1}_1(0) \coloneqq \frac{1}{s_k} \exp^{-1}_{x_k}(M_k \cap B^N_{s_k}(x_k)),$$

then, after potentially taking a subsequence and renumerating, the component of $\tilde{M}_k$ through the origin must smoothly converge to a plane through the origin, on compact sets of $B^{n+1}_1(0)$.

Lemma 2. *Consider a point-scale sequence $\{(p_k^i, r_k^i)\}$ from Definition 5, and a sequence $s_k \to 0$, such that $s_k/r_k^i \to \infty$. Then if we denote $\tilde{M}'_k \subset B^{n+1}_1(0)$ to be the connected component of*

$$\tilde{M}_k \cap B^{n+1}_1(0) \coloneqq \frac{1}{s_k} \exp^{-1}_{p_k^i}(M_k \cap B^N_{s_k}(p_k^i)),$$

that passes through the origin, we have that after potentially taking a subsequence and renumerating, there exists a hyperplane $E \subset \mathbb{R}^{n+1}$, and an $m \in \mathbb{Z}_{\geq 1}$, such that,

$$(\tilde{M}'_k, 1) \to (E, m),$$

in $B^{n+1}_1(0)$ as varifolds, and smoothly, as m disjoint graphs, on open sets of $B^{n+1}_1(0)$ which are compactly contained away from finitely many points in $\overline{B^{n+1}_1(0)}$.

Proof. For any $R \in (2, \infty)$, and large enough k, consider the map,

$$\begin{aligned} G_k^R \colon B_R^{n+1}(0) &\to B_{Rs_k}^N(p_k^i), \\ x &\mapsto \exp_{p_k^i}(s_k x), \end{aligned}$$

and the metric $g_k = s_k^{-2}(G_k^R)^* g$, on $B_R^{n+1}(0)$. We have that $(g_k)_{\alpha,\beta} \to \delta_{\alpha,\beta}$, smoothly as $k \to \infty$. Denote $\tilde{M}_k = (G_k^R)^{-1}(M_k \cap B_{Rs_k}^N(p_k^i))$. By the monotonicity formula, for large enough k we deduce,

$$\mathcal{H}_{g_k}^n(\tilde{M}_k \cap B_R^{n+1}(0)) \leq C\, R^n\, (\sup_{M_\infty} \theta)\, (\mathcal{H}_g^n(M_\infty) + 1)$$

with $C = C(N, g)(= C(N, g, H)) < +\infty$ (and moreover, mean curvature $s_k H$). Denote $\tilde{M}_k'$ to be the connected component of $\tilde{M}_k$ that passes through the origin. Thus, by Allard's Compactness Theorem [1, Theorem 6.4] (we use the theorem as stated in [54, Theorem 42.7 and Remark 42.8]) we know that, after potentially taking a subsequence and renumerating, there exists an integral, n-rectifiable, stationary varifold V in $B_R^{n+1}(0)$, such that $(\tilde{M}_k', 1) \to V$, as varifolds. By the monotonicity formula, as for each k, $0 \in \tilde{M}_k'$, and $\tilde{M}_k'$ is connected, then, $0 \in \operatorname{spt} \|V\| \neq \emptyset$, and $\operatorname{spt} \|V\|$ is a closed connected set. Thus, if we can show that $\operatorname{spt} \|V\|$ is a plane, then by the Constancy Theorem [54, Theorem 41.1] we may conclude that $\theta \equiv m \in \mathbb{Z}_{\geq 1}$.

After potentially taking a subsequence and renumerating, we may assume that for each $j = 1, \ldots, J$, the following sequence has a well defined limit (potentially $+\infty$),

$$\frac{\operatorname{dist}_g^N(p_k^i, p_k^j)}{s_k}.$$

We denote the set,

$$B_{i,\{s_k\}} = \{j \colon \lim_{k\to\infty} \operatorname{dist}_g^N(p_k^i, p_k^j)/s_k \leq 2R\},$$

and for each $j \in B_{i,\{s_k\}}$, and large enough k, we denote,

$$q_k^j := (G_k^{3R})^{-1}(p_k^j) \in B_{3R}^{n+1}(0),$$

and, again after potentially taking a subsequence and renumerating,

$$q^j := \lim_{k\to\infty} q_k^j \in \overline{B_{2R}^{n+1}(0)}.$$

Claim 5. *For each $r > 0$, there exists a sequence $R_k \to \infty$, such that for large enough k,*

$$G_k^R((\tilde{M}_k' \cap B_R^{n+1}(0)) \setminus \cup_{j\in B_{i,\{s_k\}}} B_r^{n+1}(q^j)) \subset (M_k \setminus \cup_{j=1}^J \Sigma_k^{j,R_k}) \cap B_{Rs_k}^N(p_k^i).$$

We prove this claim by contradiction. We assume there exists an $r > 0$, $S < +\infty$, and a subsequence, which we may renumerate by such that there is an $l \in \{1, \ldots, J\}$, and points,

$$z_k \in G_k^R((\tilde{M}_k' \cap B_R^{n+1}(0)) \setminus \cup_{j \in B_{i,\{s_k\}}} B_r^{n+1}(q^j)) \cap \Sigma_k^{l,S}.$$

Thus,

$$\operatorname{dist}_g^N(z_k, p_k^l) < S r_k^l,$$

and,

$$\operatorname{dist}_g^N(z_k, p_k^i) < R s_k.$$

Moreover, for large enough k,

$$\min_{j=1,\ldots,J} \operatorname{dist}_g^N(z_k, p_k^j) \geq \frac{r}{2} s_k,$$

which implies that,

$$s_k < \frac{2S}{r} r_k^l,$$

and thus,

$$\operatorname{dist}_g^N(p_k^i, p_k^l) < \left(\frac{2R}{r} + 1\right) S r_k^l.$$

Then as $\Sigma_k^{l,2(r^{-1}+R^{-1})SR}$ and $\tilde{M}_k'$ are both connected and contain z_k, we have that,

$$p_k^i \in \Sigma_k^{l,2(r^{-1}+R^{-1})SR}.$$

However, by smooth convergence on the bubble,

$$s_k |A_k|(p_k^i) \leq \frac{2S}{r} r_k^l |A_k|(p_k^i) \leq 2\frac{S}{r} \sup_{\Sigma^l \cap B^{n+1}_{2(r^{-1}+R^{-1})SR}(0)} |A_{\Sigma^l}| < +\infty$$

which contradicts fact that,

$$s_k |A_k|(p_k^i) = \frac{s_k}{r_k^i} r_k^i |A_k|(p_k^i) \to \infty.$$

Therefore we have proven the claim.

Thus, for each $r > 0$, small enough so that for each k, $\tilde{M}_k' \cap B_R^{n+1}(0) \setminus \cup_{j \in B_{i,\{s_k\}}} B_r^{n+1}(q^j)$ is non-empty, we have by Point 3 of Definition 5, and Proposition 7, that $\operatorname{spt}\|V\| \setminus \cup_{j \in B_{i,\{s_k\}}} B_r^{n+1}(q^j)$ must be a collection of hyperplanes. Moreover, by uniform bounds on the second fundamental form, the convergence on $B_R^{n+1}(0) \setminus \cup_{j \in B_{i,\{s_k\}}} B_r^{n+1}(q^j)$, must be smooth, and thus by the embeddedness assumption along the sequence, these hyperplanes cannot intersect transversely on $B_R^{n+1}(0) \setminus \cup_{j \in B_{i,\{s_k\}}} B_r^{n+1}(q^j)$. Therefore, after potentially taking R larger, and renumerating along a further subsequence, we see that these hyperplanes must be parallel. Then taking $r \to 0$ we see that $\operatorname{spt} V$ must be this collection of parallel hyperplanes. To conclude we note that as $\operatorname{spt} V$ is connected, it must just be a single hyperplane through the origin, which we converge to smoothly, away from the finite set of points $\{q^j : j \in B_{i,\{s_k\}}\}$. $\square$

Lemma 3. *Take $\rho > 0$, $a > 0$, and $n \geq 2$. Consider a sequence of smooth, embedded, minimal (H_k-CMC) hypersurfaces, $\{M_k \subset B^n_\rho(0) \times (-a, a)\}$, with respect to Riemannian metrics $\{g_k\}$, which pass through 0, and $\partial M_k \subset \partial(B^n_\rho(0) \times (-a, a))$. Suppose that g_k smoothly converge to the standard Euclidean metric on $B^n_\rho(0) \times (-a, a)$, ($H_k \to 0$), and*

$$(M_k, 1) \to (\{x_{n+1} = 0\}, m),$$

as varifolds, with $m \in \mathbb{Z}_{\geq 1}$, and that for compact sets $K \subset B^n_\rho(0) \setminus \{0\} \subset \{x_{n+1} = 0\}$, the convergence is as m disjoint graphs over K, smoothly converging to 0. Moreover, suppose we have a sequence $r_k \to 0$, such that for any other convergent sequence $\{t_k\} \subset (0, \rho]$, (say $t_k \to t_\infty \in [0, \rho]$), which satisfies,

$$\frac{t_k}{r_k} \to \infty$$

we have the condition that, after potentially taking a subsequence and renumerating,

$$\tilde{M}_k := \frac{M_k}{t_k} \subset B^n_{\rho/t_k}(0) \times (-a/t_k, a/t_k),$$

converges (potentially with multiplicity) to some plane $E_{\{t_k\}}$, through the origin, with the convergence being smooth and graphical on compact sets of $(B^n_{\rho/t_\infty}(0) \times (-a/t_\infty, a/t_\infty)) \setminus \{0\}$, ($\mathbb{R}^{n+1} \setminus \{0\}$ if $t_\infty = 0$). Then for any sequence $\{t_k\}$ as above, $E_{\{t_k\}} = \{x_{n+1} = 0\}$, and there exists an R_0, such that if C_k is a connected component of $M_k \cap (B^n_\rho(0) \times (-a, a)) \setminus B^{n+1}_{R_0 r_k}(0))$, then, $\tilde{C}_k := C_k/t_k$ converges to $\{x_{n+1} = 0\}$ with multiplicity one.

Proof. We first prove the multiplicity one covergence of $\tilde{C}_k$. The argument used is similar to that in [11, Proposition 3], and we only include a sketch, referring the reader to [11] and the references contained there for more details.

First we show that there exists an $R_0 \in (1, \infty)$, such that, taking k large enough, for all $t \in [R_0 r_k, 3\rho/4)$, M_k intersects $\partial B_t(0)$ transversely. Indeed, if not we can produce a subsequence, that we may renumerate along, with $t_k \in (r_k, 3\rho/4)$, such that,

$$\frac{t_k}{r_k} \to \infty,$$

and points $x_k \in M_k \cap \partial B^{n+1}_{t_k}(0)$, such that

$$\nu_k(x_k) \perp \partial B^{n+1}_{t_k}(0), \tag{3.2}$$

with respect to the metric g_k. Here, ν_k denotes a choice of normal to M_k at x_k, with respect to the metric g_k. Rescaling by t_k, and potentially taking a subsequence and renumerating, we get, that $t_k \to t_\infty \in [0, 3\rho/4]$, and we denote

$$\tilde{M}_k := \frac{M_k}{t_k} \subset B^n_{\rho/t_k}(0) \times (-a/t_k, a/t_k),\ \tilde{x}_k := \frac{x_k}{t_k} \to \tilde{x}_\infty \in \partial B^{n+1}_1(0).$$

Furthermore, we denote the appropriately rescaled metrics on $B^n_{\rho/t_k}(0) \times (-a/t_k, a/t_k)$, by $\tilde{g}_k$, and we see that $\tilde{g}_k$ will smoothly converge to the standard Euclidean metric on compact sets of $B^n_{\rho/t_\infty}(0) \times (-a/t_\infty, a/t_\infty)$ ($\mathbb{R}^{n+1}$, if $t_\infty = 0$). By the assumption in the statement of the Lemma, $\tilde{M}_k$ converges on compact sets of $B^n_{\rho/t_\infty}(0) \times (-a/t_\infty, a/t_\infty)$ ($\mathbb{R}^{n+1}$, if $t_\infty = 0$), as varifolds to a plane E (potentially with multiplicity) through the origin. Furthermore, the convergence is smooth on compact sets away from the origin, which when combined with the smooth convergence of the rescaled metrics, implies that

$$\nu_k(x_k) \to \nu_E \in T_{\tilde{x}_\infty} \partial B^{n+1}_1(0),$$

where ν_E, denotes a choice of unit normal to the plane E, with respect to the standard Euclidean metric. However this contradicts the condition (3.2) along the sequence.

Take a connected component of $M_k \cap B^n_\rho(0) \times (-a, a) \setminus B^{n+1}_{R_0 r_k}(0)$, and call it C_k. We now look to show that for all $t \in [R_0 r_k, 3\rho/4)$, $C_k \cap \partial B_t(0)$ is a single connected component Define the following function,

$$\begin{aligned} h_k \colon C_k &\to \mathbb{R}, \\ x &\mapsto |x|. \end{aligned}$$

This is a smooth function on C_k, and by the transversality statement, and smooth convergence of the metrics, for large enough k, h_k will not have any critical points on C_k. Thus, by standard Morse Theory ([41, Theorem 3.1]), we may use h_k to construct a continuous deformation retraction of C_k, onto $\{h_k = t\} = C_k \cap \partial B^{n+1}_t(0)$. Therefore, as C_k is connected, this implies that $C_k \cap \partial B^{n+1}_t(0)$ is also connected.

We may now prove the multiplicity one of the convergence. Now take any sequence $\{t_k\} \subset (0, 3\rho/4)$, with $t_k \to t_\infty \in [0, 3\rho/4]$, such that

$$\frac{t_k}{r_k} \to \infty,$$

and define,

$$\tilde{C}_k := \frac{C_k}{t_k} \subset (B^n_{\rho/t_k}(0) \times (-a/t_k, a/t_k)) \setminus B^{n+1}_{R_0 r_k/t_k}(0).$$

Again, we denote the appropriately rescaled metrics by $\tilde{g}_k$. We know that on compact sets of $B^n_{\rho/t_\infty}(0) \times (-a/t_\infty, a/t_\infty)$ ($\mathbb{R}^{n+1}$ if $t_\infty = 0$), $\tilde{g}_k$ smoothly converges to the Euclidean metric, and

$$\tilde{C}_k \to (E, \tilde{m}),$$

as varifolds, with convergence being smooth on such compact sets away from the origin. As previously, E is some hyperplane that passes through the origin. If we consider,

$$\tilde{S}_k := \tilde{C}_k \cap \partial B_1(0) = \frac{1}{t_k}(C_k \cap \partial B_{t_k}(0)),$$

by the previous discussion, $\tilde{S}_k$ is a smooth, connected and embedded hypersurface of $\partial B_1^{n+1}(0)$, that smoothly converges to $\tilde{S}_\infty := E \cap \partial B_1(0)$, with multiplicity $\tilde{m}$. However, as $\partial B_1(0)$ is simply connected, each $\tilde{S}_k$, and $\tilde{S}_\infty$, must be two sided in $\partial B_1(0)$. Thus the smooth convergence implies that $\tilde{m} = 1$.

The fact that $E = \{x_{n+1} = 0\}$ follows from a foliation and maximum principle argument in [17, Claim 1 of Lemma 4.1] and [16, Lemma 5.6, Proposition 5.7]. For completeness we include a sketch of this argument, however we note that it is identical to arguments contained in [17, Claim 1 of Lemma 4.1] and [16, Lemma 5.6, Proposition 5.7] and we refer the reader to them for further details.

Choose $\tau \in (0, \rho/4)$, and then we know from above that,

$$C_k \cap ((B^n_{\rho/2}(0) \setminus \overline{B^n_\tau(0)}) \times (-a, a)),$$

consists of a single connected component, and there will be a function,

$$u_k \colon B^n_{\rho/2}(0) \setminus \overline{B^n_\tau(0)} \to \mathbb{R},$$

such that,

$$C_k \cap ((B^n_{\rho/2}(0) \setminus \overline{B^n_\tau(0)}) \times (-a, a)) = \{(x, u_k(x)) \colon x \in B^n_{\rho/2}(0) \setminus \overline{B^n_\tau(0)}\},$$

and $u_k \to 0$, smoothly. If we are in the case of H_k-CMC hypersurfaces, we may assume that the mean curvature vector points in the positive x_{n+1} direction. By [16, Proposition 5.7] we have the following graphical foliation on an open neighbourhood of $B^n_{\rho/2}(0) \times \{0\}$, which we briefly describe. Taking k large enough, there exists a $b \in (0, a/2)$, such that for each $h \in (-b, b)$, there exists a function,

$$v^h_k \colon B^n_{\rho/2}(0) \to \mathbb{R}.$$

which is a g_k-minimal graph (g_k-H_k-CMC graph, with mean curvature pointing in the positive x_{n+1} direction), and,

$$v^h_k = u_k + h, \text{ on } \partial B^n_{\rho/2}(0).$$

Moreover, we have that the functions v^h_k vary smoothly with h, and $v^h_k \to h$, smoothly as $k \to 0$.

We denote the open set,

$$C_{\rho/2,b} = \{(x, v_k^h(x)) \colon x \in B^n_{\rho/2}(0),\, h \in (-b,b)\},$$

and for large enough k,

$$B^n_{\rho/2}(0) \times [-b/2, b/2] \subset C_{\rho/2,b}.$$

and we may define the following diffeomorphism,

$$\begin{aligned} F_k \colon B^n_{\rho/2}(0) \times (-b,b) &\to C_{\rho/2,b}, \\ (x,y) &\mapsto (x, v_k^{y+h_k}(x)), \end{aligned}$$

where $h_k \to 0$, is uniquely chosen such that $v_k^{h_k}(0) = 0$. We then have that $F_k \to Id$ smoothly, and we define the metric, $\hat{g}_k = F_k^* g_k$, which smoothly converges to the Euclidean metric.

Then, we have that in $B^n_{\rho/2}(0) \times (-b,b)$, the horizontal slices $\{y = c\}$ are $\hat{g}_k$-minimal ($\hat{g}_k$-H_k-CMC, with mean curvature vector pointing in the positive y direction), and $\hat{C}_k = F_k^{-1}(C_k)$, is $\hat{g}_k$-minimal ($\hat{g}_k$-H_k-CMC, with mean curvature vector pointing in positive y direction). Moreover,

$$\partial \hat{C}_k \cap \partial(B^n_{\rho/2}(0) \times (-b,b)) = \{y = -h_k\} \cap (\partial B^n_{\rho/2}(0) \times (-b,b)),$$

and,

$$\partial \hat{C}_k \cap (B^n_{\rho/2}(0) \times (-b,b)) \subset F_k^{-1}(B^{n+1}_{R_0 r_k}(0)).$$

After taking a subsequence and renumerating, we may assume, without loss of generality, that $-h_k \geq 0$.

Now suppose that $E_{\{t_k\}} \neq \{x_{n+1} = 0\}$. We rescale by t_k,

$$\tilde{C}_k = \frac{\hat{C}_k}{t_k} \subset B^n_{\rho/(2t_k)}(0) \times (-b/t_k, b/t_k),$$

and denote our appropriately rescaled metrics by $\tilde{g}_k$. Then, as $F_k \to Id$, we have that $\tilde{g}_k$ converges to the standard Euclidean metric, and,

$$\tilde{C}_k \to (E, 1),$$

smoothly on compact sets of $B^n_{\rho/(2t_\infty)}(0) \times (-b/t_\infty, b/t_\infty)$ ($\mathbb{R}^{n+1}$, if $t_\infty = 0$), away from the origin. As $E \neq \{y = 0\}$, then, if we consider the smooth function

$$\begin{aligned} f_k \colon \tilde{C}_k &\to \mathbb{R}, \\ (x,y) &\mapsto y, \end{aligned}$$

there will exist points $z_k \in \tilde{C}_k \cap \partial B_r^{n+1}(0)$, with $r = \min\{1/4, b/(4t\infty)\}$, such that,

$$f_k(z_k) \to \Lambda < 0,$$

$\Lambda = \Lambda(b, t_\infty, E)$. However, for $z \in \partial\tilde{C}_k$, either $z \in \partial(B^n_{\rho/(2t_k)} \times (-b/t_k, b/t_k))$, and thus

$$f_k(z) = -h_k/t_k \geq 0,$$

, or, $z \in F_k^{-1}(B^{n+1}_{R_0 r_k/t_k}(0))$, and as $r_k/t_k \to 0$, we have that

$$f_k(z) > \Lambda/2,$$

for large enough k. Therefore, for large k, f_k achieves an interior minimum, at say z_k, $f_k(z_k) = \gamma_k$. Therefore, C_k lies to one side of $\{y = \gamma_k\}$, and they touch tangentially at interior point z_k. However, as each horizontal slice is a $\tilde{g}_k$-minimal ($\tilde{g}_k$-H_k/t_k-CMC) hypersurface, and by the boundary condition for $\tilde{C}_k$, $\tilde{C}_k \not\subset \{y = \gamma_k\}$, we derive a contradiction to the one-sided maximum principle for $\tilde{g}_k$-minimal ($\tilde{g}_k$-H_k/t_k-CMC) hypersurfaces. □

Remark 28. ([17, Claim 1 of Lemma 4.1] and [16, Lemma 5.6]) We remark that combining point 3 in Definition 5, along with a contradiction argument that makes use of Proposition 7 and Lemma 3, implies that there exists positive constants S_0 and s_0, such that for each $y \in \mathcal{I}$, $s \in (0, s_0)$, and $S \in [S_0, +\infty)$, and taking large enough k, if we let C_k denote a connected component of $M_k \cap (B_s^N(y) \setminus \cup_{i=1}^J \overline{\Sigma_k^{i,S}})$, then there exists a non-empty open set $A_k = A_k(C_k, s, S) \subset T_y M_\infty$, and a smooth function,

$$u_k \colon A_k \to \mathbb{R},$$

such that,

$$C_k = \{\exp_y(x + u_k(x)\nu(y)) \colon x \in A_k\},$$

where $\nu(y)$ is a choice of unit normal to M_∞ at y. Moreover, we have that

$$\lim_{s\to 0} \lim_{S\to\infty} \lim_{k\to\infty} \|u_k\|_{C^1} = 0.$$

Remark 29. Consider Σ, a connected, complete n-dimensional manifold ($n \geq 3$), and $\iota \colon \Sigma \to \mathbb{R}^{n+1}$, a proper, minimal immersion, with finite total curvature,

$$\int_\Sigma |A_\Sigma|^n < +\infty,$$

and Euclidean volume growth at infinity,

$$\limsup_{R\to\infty} \frac{\mathcal{H}^n(\iota(\Sigma)\cap B_R^{n+1}(0))}{R^n} < +\infty$$

Given a compact exhaustion $K_1 \subset K_2 \subset \cdots$, of Σ, we define an end E, of Σ, to be a nested sequence of open sets of Σ,

$$U_1 \supset U_2 \supset U_3 \supset \cdots ,$$

such that each $U_i \subset \Sigma \setminus K_i$, is connected. We now look to show that Σ has finitely many such ends, and each one may be represented by the graph of a function, with small gradient, over a hyperplane of $\mathbb{R}^{n+1}$ minus a compact set. The arguments in the proof are very similar to that of Lemma 3, and in fact the general idea of the argument is somewhat standard. Therefore we only present a sketch of these facts and refer the reader to [61, Lemma 3 and Lemma 4] and [11, Proposition 3] for details.

Consider a sequence $R_i \to \infty$. Applying classical geometric measure theory arguments (monotonicity formula for stationary varifolds, and Allard's compactness Theorem), which make use of the minimality of the proper immersion, and the Euclidean volume growth at infinity, we see that the sequence of blow downs of $\iota(\Sigma)$, given by

$$V_i = \left(\frac{\iota(\Sigma)}{R_i}, 1\right),$$

will, after potentially taking a subsequence and renumerating, converge as varifolds, on compact subsets of $\mathbb{R}^{n+1}$, to a stationary, non-empty, n-rectifiable, integral varifold V (with locally finite mass). Moreover, as this V arises from a sequence of blown downs, its support, which we denote by C, will be a cone in $\mathbb{R}^{n+1}$ (i.e. C is invariant under rescalings centred at the origin).

Now, the finite total curvature assumption along with the curvature estimate of Proposition 7 imply that, for each $\varepsilon > 0$, there exists an R_ε, such that for all $x \in \Sigma \setminus \iota^{-1}(\overline{B_{R_\varepsilon}^{n+1}(0)})$,

$$|\iota(x)|^2\, |A_\Sigma|^2(x) \leq \varepsilon, \tag{3.3}$$

Therefore, following arguments similar to those in Lemma 2, we have that the blow downs of our immersions, will converge smoothly and graphically to C (potentially with multiplicity), on compact subsets of $\mathbb{R}^{n+1} \setminus \{0\}$. Moreover we will also have that,

$$C = \cup_{i=1}^{L} P_i,$$

where each P_i is a hyperplane of $\mathbb{R}^{n+1}$, which passes through the origin, and $L \in \mathbb{Z}_{\geq 1}$ is finite.

Then, applying identical arguments to Lemma 3 (cf. [11, Proposition 3]), we have that there exists an r_0, such that for all $R \in [r_0, \infty)$, $\iota(\Sigma)$ intersects $\partial B^{n+1}_R(0)$ transversely. Then again applying identical arguments to those in Lemma 3 (cf. [11, Proposition 3]) we have that for all $S_2 > S_1 \geq r_0$, if we consider a connected component U of

$$\Sigma \cap \iota^{-1}(B^{n+1}_{S_2}(0) \setminus \overline{B^{n+1}_{S_1}(0)}),$$

then, ∂U consists of two connected components, one of which is smoothly immersed, by ι, into $\partial B^{n+1}_{S_1}(0)$, and one which is smoothly immersed, by ι, into $\partial B^{n+1}_{S_2}(0)$.

Now consider an end E,

$$U_1 \supset U_2 \supset U_3 \supset \cdots$$

given by this compact exhaustion, $\{\iota^{-1}(\overline{B^{n+1}_{R_i}(0)})\}$. From above, we have that for large enough i, ∂U_i will be connected, and

$$\frac{\iota(\partial U_i)}{R_i} \subset \partial B^{n+1}_1(0),$$

will be a smooth immersion, that smoothly and graphically converges to a finite collection of equators $\cup_{i=1}^{L'} Q_i \cap \partial B^{n+1}_1(0)$, with finite multiplicity. Where each Q_i is a distinct hyperplane through the origin, which is contained in our cone C. As ∂U_i is connected for large i, we see that in fact $L' = 1$, and moreover (as S^{n-1} is its own universal conver for $n \geq 3$), conclude that this convergence on $\partial B^{n+1}_1(0)$ must happen with multiplicity one. Thus, $R_i^{-1}\iota(U_i)$ converges smoothly and graphically on compact subsets on $\mathbb{R}^{n+1} \setminus B^{n+1}_1(0)$, to a hyperplane Q, with multiplicity one. Therefore, the number of ends of Σ, given by the compact exhaustion $\{\iota^{-1}(\overline{B^{n+1}_{R_i}(0)})\}$, is finite and equal to the integer

$$m = \frac{\|V\|(B^{n+1}_1(0))}{\mathcal{H}^n(B^n_1(0))}.$$

Moreover, by the monotonicity formula for stationary varifolds, the integer m is independent of the sequence $R_i \to \infty$. Thus, Σ has finitely many ends $E^1, \ldots, E^m$, and therefore, each end E^i, may be represented by an open set W^i. By this we mean that there exists a compact set K, and m disjoint open sets $W^1, \ldots, W^m$, such that,

$$\Sigma \setminus K = \cup_{i=1}^m W^i,$$

and given any compact exhaustion $K_1 \subset K_2 \subset \cdots$, then if the representation of end E^i with respect to this compact exhaustion is given by,

$$U^i_1 \supset U^i_2 \supset U^i_3 \supset \cdots$$

then for large enough j, $U^i_j \subset W^i$. With a slight abuse of notation we will simply denote this set W^i by E^i, and when we refer to an end, we will in fact be referring to this open set.

Finally, consider any other sequence $R_i' \to \infty$, and our previously considered end E. Then, by the foliation and maximum principle argument in Lemma 3 (cf. [16, Proposition 5.7] and [17, Claim 1 of Lemma 4.1]) implies that, after potentially taking subsequence and renumerating $(R_i')^{-1}\iota(E)$, will also converge smoothly with multiplicity one, on compact subsets of $\mathbb{R}^{n+1} \setminus \{0\}$, to the same plane Q.

Thus, applying the same argument as in Remark 28, we see that the ends of Σ, $E^1, \ldots, E^m$, are given as the graphs of a smooth function. By this we mean, for any end E^i, there exists a rotation r_i about the origin, a compact subset $B_i \subset \mathbb{R}^n$, and a smooth function

$$u_i \colon \mathbb{R}^n \setminus B_i \to \mathbb{R},$$

such that,

$$r_i(\iota(E^i)) = \{(x, u_i(x)) \colon x \in \mathbb{R}^n \setminus B_i\}.$$

Moreover, for each $\eta > 0$, there exists a compact set $B_i^\eta \,(\supset B_i)$, such that $|\nabla u_i| < \eta$, on $\mathbb{R}^n \setminus B_i^\eta$.

3.1.2 Stability Operator, Index and Nullity

We continue the discussion on the stability operator, index and nullity from Section 1.1. In Section 1.1 we discussed how one may look to study minimal and CMC hypersurfaces by studying the vector valued stability operator on $\Gamma(TM^\perp)$, and if the hypersurface is embedded and two-sided, instead of studying this vector valued operator, we may alternatively study a scalar valued operator on appropriate function spaces on our hypersurface. We now look to define appropriate function spaces on connected covers of our hypersurfaces to allow us to extend this to one-sided and quasi-embedded hypersurfaces.

Consider a properly embedded, closed, minimal hypersurface $M \subset N$. We define $o(M)$ to be the two-sided double cover of M,

$$o(M) \coloneqq \{(x, \nu) \colon x \in M,\ \nu \in T_x^\perp M, \text{ and } |\nu| = 1\},$$

and $co(M)$ to be a connected component of $o(M)$. Note that if M is two-sided in N, then we identify $co(M)$ with M, and if M is one-sided in N, then $co(M) = o(M)$. We define the obvious continuous projections, $\iota \colon o(M) \to M$, and $\nu \colon o(M) \to T^\perp M$, and note that $z = (\iota(z), \nu(z)) \in o(M)$.

For the case of $M \subset N$ being a quasi-embedded H-CMC hypersurface we define,

$$\begin{aligned} co(M) \coloneqq \{(x, \nu) \quad : \quad & x \in M,\ \nu \in T_x^\perp M,\ |\nu| = 1, \\ & \text{and points in direction of the mean curvature vector}\}. \end{aligned}$$

Thus at non-embedded point $x \in M$ we have (x, ν), and $(x, -\nu) \in co(M)$.

For $X \in \Gamma(TM^{\perp})$ we define a function $f_X \in C^{\infty}(co(M))$ by,

$$f_X(z) = g(X(\iota(z)), \nu(z)).$$

and note that if z_1, $z_2 \in co(M)$ such that $z_1 \neq z_2$ but $\iota(z_1) = \iota(z_2)$, then $f_X(z_2) = -f_X(z_1)$. Thus,

$$f_X \in C^{\infty}(co(M))^- := \{f \in C^{\infty}(co(M)) \quad : \quad if\,(x, \nu)\,\text{and}\,(x, -\nu)\,\text{are both in}\,co(M), \\ \text{then}\,f((x, -\nu)) = -f((x, \nu))\}.$$

Moreover, by a standard extension argument, for each $f \in C^{\infty}(co(M))^-$, there exists an $X_f \in \Gamma(TM^{\perp})$ such that

$$f(z) = g(X_f(\iota(z)), \nu(z)).$$

Elements of the function space $C^{\infty}(co(M))^-$ should be thought of as '*ambient variations*'.

Thus, instead of considering the vector valued stability operator on $\Gamma(TM^{\perp})$, it is equivalent to consider the scalar valued stability operator,

$$Lf = \Delta f + (|A_M|^2 \circ \iota + \mathrm{Ric}_N\,(\nu, \nu))f,$$

on the function space $C^{\infty}(co(M))^-$. We associate to L the bilinear form,

$$B_L[f, h] := \int_{co(M)} \nabla f \cdot \nabla h - (|A_M|^2 \circ \iota + \mathrm{Ric}_N\,(\nu, \nu))\, f\, h,$$

for f, $h \in C^{\infty}(co(M))^-$, which we may extend to the space,

$$W^{1,2}(co(M))^- = \overline{C^{\infty}(co(M))^-}^{\|\cdot\|_{W^{1,2}(co(M))}}$$

For M being a properly embedded, one-sided hypersurface we have that $co(M) = o(M)$, and

$$W^{1,2}(co(M))^- = \{f \in W^{1,2}(co(M))\colon \text{for a.e.}\,(x, \nu) \in co(M),\ f((x, \nu)) = -f((x, -\nu))\}.$$

For M being a properly embedded, two-sided hypersurface we may simply identify $co(M)$ with M, and $W^{1,2}(co(M))^-$ with $W^{1,2}(M)$. Then, for M properly embedded, it may be shown (an argument is contained in the proof of Proposition 14 and Remark 33), that

$$\mathrm{ind}\,(M) + \mathrm{nul}\,(M) = \sup\{\dim \Pi\colon \Pi \subset W^{1,2}(co(M))^-\,\text{is a linear space}, \\ \text{on which}\,B_L\,\text{is negative semi-definite}\}. \tag{3.4}$$

As Ric_N is bilinear, the function $x \mapsto \mathrm{Ric}_N(\nu,\nu)(x)$, for $\nu \in T_x^\perp M$, $|\nu| = 1$, is a well defined function on M, even if M is one-sided. Thus we may consider the operator L, and its associated bilinear form B_L, on the function space $W^{1,2}(M)$, and its index and nullity on this function space. We define this as the *analytic index* (anl-ind$\,(M)$) and *analytic nullity* (anl-nul$\,(M)$) of M. Note that if M is two-sided and embedded then these two notions of index and nullity will coincide. We also similarly refer to the *analytic index* and *analytic nullity* of $co(M)$, and refer to elements of the function space $W^{1,2}(co(M))$ as '*analytic variations*'.

It is also worth noting that for H-CMC quasi-embedded hypersurfaces ($H \neq 0$), the mean curvature vector gives a global choice of unit normal, and thus such immersions are two-sided. Moreover, as seen from above, we can still define the stability operator on M even if the hypersurface does not arise as the boundary of an open set. Thus we do not need M to arise as the boundary of a set to define its index and nullity (as was the motivation in Section 1.1).

3.1.3 Lorentz Spaces

Let (M,g) be a Riemannian manifold and μ be the volume measure associated to g. For a μ-measureable function $f\colon M \to \mathbb{R}$, we define the function

$$\alpha_f(s) \coloneqq \mu(\{x \in M\colon |f(x)| > s\}).$$

We may then define the decreasing rearrangement f^*, of f by,

$$f^*(t) \coloneqq \begin{cases} \inf\{s > 0\colon \alpha_f(s) \leq t\}, & t > 0, \\ \operatorname{ess\,sup}|f|, & t = 0. \end{cases}$$

For $p \in [1,\infty)$, and $q \in [1,\infty]$, and a μ-measureable function f on M, we define,

$$\|f\|_{(p,q)} \coloneqq \begin{cases} \left(\int_0^\infty t^{q/p} f^*(t)^q \frac{dt}{t}\right)^{1/q}, & 1 \leq q < \infty, \\ \sup_{t>0} t^{1/p} f^*(t), & q = \infty. \end{cases}$$

The Lorentz space $L(p,q)(M,g)$ is then defined to be the space of μ-measureable functions f such that $\|f\|_{(p,q)} < +\infty$. It is worth noting that $\|\cdot\|_{(p,q)}$ is not a norm on $L(p,q)(M,g)$ as it does not generally satisfy the triangle inequality. However it is possible to define an appropriate norm, $\|\cdot\|_{p,q}$ (see [18, Definition 2.10]), on the space $L(p,q)(M,g)$, so that the normed space $(L(p,q)(M,g), \|\cdot\|_{p,q})$ is a Banach Space [18, Theorem 2.19]. Moreover, for $1 < p < \infty$, and $1 \leq q \leq \infty$, we have the following equivalence ([18, Proposition 2.14]),

$$\|\cdot\|_{(p,q)} \leq \|\cdot\|_{p,q} \leq \frac{p}{p-1}\|\cdot\|_{(p,q)}. \tag{3.5}$$

Proposition 8. *(Hölder–Lorentz inequality, [18, Theorem 2.9]) Take* $p_1,\, p_2 \in (1,\infty)$ *and* $q_1,\, q_2 \in [1,\infty]$ *such that* $1/p_1+1/p_2 = 1/q_1+1/q_2 = 1$*. Then for* $f \in L(p_1,q_1)(M,g)$ *and* $h \in L(p_2,q_2)(M,g)$*, we have,*

$$\int_M |f\, h|\, d\mu \le \|f\|_{(p_1,q_1)}\, \|h\|_{(p_2,q_2)}.$$

The following fact can be easily derived from the definition of $\|\cdot\|_{(p,q)}$.

Proposition 9. *Take* $1 < p < +\infty$, $1 \le q \le \infty$, *and* $\gamma > 0$, *then for* $f \in L(p,q)(M,g)$, *we have,*

$$\| |f|^\gamma \|_{(\frac{p}{\gamma},\frac{q}{\gamma})} = \|f\|^\gamma_{(p,q)}.$$

The following Lorentz–Sobolev inequality on $\mathbb{R}^n$ is crucial in Section 3.2.4. For a proof see [3, Appendix]

Proposition 10. *(Lorentz–Sobolev inequality on* $\mathbb{R}^n$*) Take* $1 < p < n$, *and* $p^* = np/(n-p)$, *then there exists a constant* $C = C(n,p)$ *such that for all* $u \in C_c^\infty(\mathbb{R}^n)$,

$$\|u\|_{(p^*,p)} \le C\|\nabla u\|_{L^p}.$$

By standard covering and partitions of unity arguments, from Proposition 10 we may also obtain a Lorentz–Sobolev inequality on a bounded subset of a Riemannian manifold.

Proposition 11. *(Lorentz–Sobolev inequality on manifolds) Let* (M,g) *be a complete Riemannian manifold of dimension* n*. Take* $1 < p < n$, $p^* = np/(n-p)$, *and an open, bounded set* $\Omega \subset M$*. Then there exists a constant* $C = C(\Omega, g, n, p) < +\infty$, *such that for all* $u \in C_c^\infty(\Omega)$,

$$\|u\|_{(p^*,p)} \le C\|u\|_{W^{1,p}(M)}.$$

Remark 30. We may extend the inequalties in Propositions 10 and 11 to $u \in W^{1,p}(\mathbb{R}^n)$ and $W_0^{1,p}(\Omega)$ respectively, by using a standard density argument, the fact $(L(p,q)(M,g), \|\cdot\|_{p,q})$ is a Banach space, and the equivalence in (3.5).

Proposition 12. *Let* (M,g) *be a compact Riemannian manifold of dimension* $n \ge 3$, *with* $\omega \in L(n/2,\infty)(M,g)$*. Then for any* $f_1,\, f_2 \in W^{1,2}(M)$, *we have that there exists a* $C = C(M,g) < \infty$, *such that,*

$$\left|\int_M f_1 f_2 \omega\right| \le C\|\omega\|_{(n/2,\infty)}\|f_1\|_{W^{1,2}(M)}\|f_2\|_{W^{1,2}(M)}.$$

Proof. Using Propositions 8, 9 and 11,

$$\begin{aligned} \left|\int_M f_1 f_2 \omega\right| &\leq \left(\int_M |\omega|\, f_1^2\right)^{1/2} \left(\int_M |\omega|\, f_2^2\right)^{1/2}, \\ &\leq \|\omega\|_{(n/2,\infty)} \, \|f_1^2\|_{(2^*/2,1)}^{1/2} \, \|f_2^2\|_{(2^*/2,1)}^{1/2}, \\ &\leq \|\omega\|_{(n/2,\infty)} \, \|f_1\|_{(2^*,2)} \, \|f_2\|_{(2^*,2)}, \\ &\leq C\|\omega\|_{(n/2,\infty)} \, \|f_1\|_{W^{1,2}(M)} \, \|f_2\|_{W^{1,2}(M)}. \end{aligned}$$

□

3.2 Weighted Eigenfunctions

We proceed with the proof of Theorem 8, taking a sequence $M_k \to (M_\infty, \Sigma^1, \ldots, \Sigma^J)$ as in Definition 5. We assume that for each $k \in \mathbb{Z}_{\geq 1}$, M_k is a single connected component. Thus for the case of minimal hypersurfaces, $l = 1$. The general statement is proven by applying the argument to each individual connected component of M_k.

We also note that the reader may find it easier to follow the rest of this chapter by only considering the case where each M_k, and M_∞, are two-sided minimal hypersurfaces. By doing so much of the notation introduced in Section 3.1.2 can be ignored, and we can just consider M_k instead of $co(M_k)$, and $W^{1,2}(M_k)$ instead of $W^{1,2}(co(M_k))^-$.

3.2.1 The Weight

Take, $R \geq 4$, and $\delta > 0$, such that for large enough k, and all $j = 1, \ldots, J$,

$$4Rr_k^j < \delta < \min\left\{\frac{\text{inj}(N)}{8}, \min_{y_1, y_2 \in \mathcal{I}, y_1 \neq y_2} \frac{\text{dist}_g^N(y_1, y_2)}{8}\right\}.$$

We first define our weight, on $co(M_k)$, about the point scale sequence $\{(p_k^j, r_k^j)\}_{k \in \mathbb{N}}$,

$$\omega_{k,\delta,R}^j(x) := \begin{cases} \max\{\delta^{-2}, \text{dist}_g^N(\iota(x), p_k^j)^{-2}\}, & \iota_k(x) \in M_k \setminus B^N_{Rr_k^j}(p_k^j), \\ (Rr_k^j)^{-2}, & \iota(x) \in B_{Rr_k^j}(p_k^j) \cap M_k, \end{cases}$$

We consider the weight,

$$\omega_{k,\delta,R}(x) := \max_{j=1,\ldots J} \omega_{k,\delta,R}^j(x).$$

We also define $\omega_\delta \in W^{1,\infty}_{\text{loc}}(co(M_\infty) \setminus \iota_\infty^{-1}(\mathcal{I})) \cap L(n/2,\infty)(co(M_\infty))$, by

$$\omega_\delta(x) := \max\{\delta^{-2}, \text{dist}_g^N(\iota_\infty(x), \mathcal{I})^{-2}\}.$$

The fact that $\omega_\delta \in L(n/2,\infty)(co(M_\infty))$ will follow from a similar calculation to that in Claim 6.

Recall the stability operator on $co(M_k)$,

$$L_k \coloneqq \Delta + |A_k|^2 + R_k,$$

where $A_k(x) = A_{M_k}(\iota_k(x))$, and $R_k(x) = \mathrm{Ric}_N(\nu_k(x),\nu_k(x))(\iota_k(x))$, and the associated bilinear form, B_k, acting on $W^{1,2}(co(M_k))$,

$$B_k[\varphi,\psi] \coloneqq \int_{M_k} \nabla^k\varphi \cdot \nabla^k\psi - (|A_k|^2 + R_k)\,\varphi\,\psi.$$

We define the unweighted eigenspace for an eigenvalue λ of L_k by

$$\begin{aligned}\mathcal{E}(\lambda; L_k, W^{1,2}(co(M_k))^-) \coloneqq \{f \in W^{1,2}(co(M_k))^- \colon B_k[f,\psi] = \lambda \int_{M_k} f\,\psi,\\ \mathit{for\,all}\, \psi \in W^{1,2}(co(M_k))^-\},\end{aligned} \tag{3.6}$$

and the weighted eigenspace for a weighted eignevalue λ of L_k by,

$$\begin{aligned}\mathcal{E}_{\omega_{k,\delta,R}}(\lambda; L_k, W^{1,2}(co(M_k))^-) \coloneqq \{f \in W^{1,2}(co(M_k))^- \colon B_k[f,\psi] = \lambda \int_{M_k} f\,\psi\,\omega_{k,\delta,R},\\ \mathit{for\,all}\, \psi \in W^{1,2}(co(M_k))^-\}.\end{aligned} \tag{3.7}$$

Identical definitions hold for $\mathcal{E}(\lambda; L_\infty, W^{1,2}(co(M_\infty))$ and $\mathcal{E}_{\omega_\delta}(\lambda; L_\infty, W^{1,2}(co(M_\infty)))$. Recall that when we refer to function space $W^{1,2}(co(M))^-$ we are considering *'ambient variations'*, and when we consider the function space $W^{1,2}(co(M))$ we are considering *'analytic variations'*. In general, for $\lambda \in \mathbb{R}$, a Riemannian manifold (M,g), with a second order, linear, elliptic operator L, with associated bilinear form B_L, and $\omega \colon M \to \mathbb{R}$, we define the function space,

$$\mathcal{E}_\omega(\lambda; L, A, B) \coloneqq \{f \in A \colon B_L[f,\varphi] = \lambda \int_M f\,\varphi\,\omega,\ \mathit{for\,all}\,\varphi \in B\},$$

where A, and B are some function spaces on M. If $A = B$, we write $\mathcal{E}_\omega(\lambda; L, A)$, and if $\omega \equiv c$, for some constant c, we write, $\mathcal{E}_c(\lambda; L, A, B) = \mathcal{E}(c\lambda; L, A, B)$.

Lemma 4. *There exists a* $C = C(N, g, M_\infty, \Sigma^1, \ldots, \Sigma^J, \delta, R) < +\infty$ *such that for all* k*,*

$$\frac{|A_k|^2 + |R_k|}{\omega_{k,\delta,R}} \leq C. \tag{3.8}$$

Proof. We assume the statement does not hold and prove by contradiction. After potentially choosing a subsequence and renumerating we have that there exists a sequence of points $x_k \in$

$co(M_k)$, such that

$$\omega^{-1}_{k,\delta,R}(|A_k|^2 + |R_k|)(x_k) \to \infty.$$

Note that there exists a $C = C(N, g)$, such that $|R_k| \le C$, for all k, thus,

$$\omega^{-1}_{k,\delta,R}(|A_k|^2 + C)(x_k) \to \infty.$$

Again potentially after taking a subsequence and renumerating we may assume that $\iota_k(x_k) \to y \in M_\infty$.

If $y \in M_\infty \setminus \mathcal{I}$, then for large enough k,

$$\omega_{k,\delta,R} \ge (2\,\mathrm{dist}^N_g(y, \mathcal{I}))^{-2},$$

which implies that,

$$(2\mathrm{dist}^N_g(y, \mathcal{I}))^2(|A_k|^2(x_k) + C) \to \infty.$$

Thus, $|A_k|^2(x_k) \to \infty$, which contradicts the smooth convergence away from $\mathcal{I}$.

We have that $y \in \mathcal{I}$, and thus $\omega^{-1}_{k,\delta,R}(x_k) \to 0$, thus from here we just consider the term $\omega^{-1}_{k,\delta,R}(x_k)|A_k|^2(x_k)$. There exists an $i = 1, \dots, J$, such that after potentially taking a subsequence and renumerating

$$\omega_{k,\delta,R}(x_k) = \omega^i_{k,\delta,R}(x_k).$$

We split into different cases.

Case 1: There exists a $j = 1, \dots, J$, an $S < +\infty$, and a subsequence we can renumerate along to get that $\iota_k(x_k) \in \Sigma^{S,j}_k$ for all k.

Then we have,

$$\omega^{-1}_{k,\delta,R}(x_k) = (\omega^i_{k,\delta,R}(x_k))^{-1} \le (\omega^j_{k,\delta,R}(x_k))^{-1} \le \max\{S, R\}^2(r^j_k)^2.$$

By the smooth convergence,

$$\omega^{-1}_{k,\delta,R}(x_k)|A_k|^2(x_k) \le 2\max\{S, R\}^2 \max_{B_S(0)\cap\Sigma^j} |A_{\Sigma^j}|^2,$$

which clearly contradicts the assumption that $\omega^{-1}_{k,\delta,R}(x_k)|A_k|^2(x_k) \to \infty$.

Case 2: There are sequences $\delta_k \to 0$, and $R_k \to \infty$, such that, for all $k \in \mathbb{N}$,

$$\iota_k(x_k) \in M_k \cap B^N_{\delta_k}(y) \setminus \cup^L_{j=1}\Sigma^{j,R_k}_k.$$

There exists a $j = 1, \dots, J$, such that after potentially picking a subsequence and renumerating,

$$d_k(x_k) = \mathrm{dist}_g^N(\iota_k(x_k), p_k^j).$$

We have,

$$\omega_{k,\delta,R}^{-1}(x_k) \leq (\omega_{k,\delta,R}^j(x_k))^{-1}.$$

Case 2.1: Potentially after picking a subsequence and renumerating,

$$\omega_{k,\delta,R}^j(x_k) = \mathrm{dist}_g^N(\iota_k(x_k), p_k^j)^{-2}.$$

Then,

$$\omega_{k,\delta,R}^{-1}(x_k)|A_k|^2(x_k) \leq d_k^2(x_k)|A_k|^2(x_k).$$

By Point 3 of Definition 5 and Proposition 7, we have that the right hand side converges to 0. This clearly contradicts assumption that $\omega_{k,\delta,R}^{-1}(x_k)|A_k|^2(x_k) \to \infty$.

Case 2.2: We have that,

$$\omega_{k,\delta,R}^j(x_k) = R^2(r_k^j)^2.$$

We claim there exists an $S > 0$, such that for large enough k,

$$\frac{\mathrm{dist}_g^N(\iota_k(x_k), p_k^j)}{r_k^j} \geq S, \tag{3.9}$$

Assuming (3.9), we have

$$\omega_{k,\delta,R}^{-1}(x_k)|A_k|^2(x_k) \leq \frac{2R^2}{S^2} d_k^2(x_k)|A_k|^2(x_k),$$

which converges to zero by Point 3 of Definition 5 and Proposition 7, again contradicting our assumption.

We now prove (3.9) by contradiction. If (3.9) does not hold then after potentially taking a subsequence and renumerating,

$$\frac{\mathrm{dist}_g^N(\iota_k(x_k), p_k^j)}{r_k^j} \to 0.$$

Repeating arguments similar to those in Lemma 2, we may conclude that the connected component of

$$\frac{1}{r_k^j} \exp_{p_k^j}^{-1}(M_k \cap B_{r_k^j}(p_k^j)) \subset B_1^{n+1}(0),$$

through $(r_k^j)^{-1} \exp_{p_k^j}^{-1}(\iota_k(x_k))$ must converge to a plane Π, through the origin, with convergence as described in Lemma 2. As Σ^j is a non-trivial, minimal, embedded hypersurface through the origin, by the one sided maximum principle, this plane cannot lie on one side of Σ^j. However, by the smooth convergence away from finitely many points, this contradicts the embeddedness along the sequence. ☐

Claim 6. *We have that there exists a* $C = C(N, g, M_\infty, m, \delta, J)(= C(N, g, M_\infty, m_1, \ldots, m_a, H, \delta, J)) < +\infty$ *such that for large enough* $k \in \mathbb{Z}_{\geq 1}$, *and* $R \geq 1$,

$$\|\omega_{k,\delta,R}\|_{(n/2,\infty)} \leq C.$$

Proof. For a $j = 1, \ldots, J$, consider $f = \omega^j_{k,\delta,R}$. We have,

$$\alpha_f(s) = \begin{cases} \mathcal{H}^n(co(M_k)), & s \in [0, \delta^{-2}), \\ \mathcal{H}^n(\iota_k^{-1}(B^N_{1/\sqrt{s}}(p^j_k)) \cap co(M_k)), & \delta^{-2} \leq s < (Rr^j_k)^{-2}, \\ 0, & s \geq (Rr^j_k)^{-2}. \end{cases}$$

We may choose k large enough such that $\sup_k \mathcal{H}^n(M_k) \leq m\mathcal{H}^n(M_\infty) + 1$. Then, by the monotonicity formula ([54, Theorem 17.7]), we have that there exists a uniform $C = C(N, g, m, M_\infty)(= C(N, g, M_\infty, m_1, \ldots, m_a, H)) < +\infty$, such that for $r \in (0, \text{inj}(N)/2)$,

$$\mathcal{H}^n(co(M_k) \cap \iota_k^{-1}(B^N_r(p^j_k))) \leq Cr^n$$

Then,

$$\begin{cases} f^*(t) = 0, & t \geq \mathcal{H}^n(co(M_k)), \\ f^*(t) = \delta^{-2}, & \mathcal{H}^n(co(M_k) \cap \iota_k^{-1}(B^N_\delta(p^j_k))) \leq t < \mathcal{H}^n(co(M_k)), \\ f^*(t) \leq (C/t)^{2/n}, & \mathcal{H}^n(co(M_k) \cap \iota_k^{-1}(B^N_{Rr^j_k}(p^j_k))) \leq t \leq \mathcal{H}^n(co(M_k) \cap \iota_k^{-1}(B^N_\delta(p^j_k))), \\ f^*(t) = (Rr^j_k)^{-2}, & 0 \leq t \leq \mathcal{H}^n(co(M_k) \cap \iota_k^{-1}(B^N_{Rr^j_k}(p^j_k))). \end{cases}$$

Thus,

$$\|f\|_{(n/2,\infty)} = \sup_{t>0} t^{2/n} f^*(t) \leq C,$$

for $C = C(N, g, m, M_\infty, \delta)(= C(N, g, M_\infty, m_1, \ldots, m_a, H, \delta)) < +\infty$. This in turn implies,

$$\|\omega_{k,\delta,R}\|_{(n/2,\infty)} \leq \left(\frac{n}{n-2}\right) \sum_{j=1}^J \|\omega^j_{k,\delta,R}\|_{(n/2,\infty)} \leq C,$$

for $C = C(N, g, m, M_\infty, \delta, J)(= C(N, g, M_\infty, m_1, \ldots, m_a, H, \delta, J)) < +\infty$. The $n/(n-2)$ factor is coming from (3.5). ☐

Remark 31. This choice of weight $\omega_{k,\delta,R}$ fails to work for the case of $n = 2$. In [23] (in which $n = 2$) the choice of weight is subtle, and relies on improved estimates on the neck region of the bubbles. We were unable to derive appropriate corresponding estimates on the neck regions in the setting of this chapter.

3.2.2 Convergence on the Base

Consider a sequence of functions $\{f_k\}_{k\in\mathbb{N}}$, $f_k \in W^{1,2}(co(M_k))^-$, which satisfy the following weighted eigenvalue problem,

$$\int_{co(M_k)} \nabla f_k \cdot \nabla \varphi - (|A_k|^2 + R_k)\, f_k\, \varphi = \lambda_k \int_{co(M_k)} f_k\, \varphi\, \omega_{k,\delta,R}, \tag{3.10}$$

for all $\varphi \in W^{1,2}(co(M_k))^-$, with $\lambda_k \leq 0$, for all k. We take

$$\int_{co(M_k)} f_k^2\, \omega_{k,\delta,R} = 1,$$

and by Lemma 4,

$$\begin{aligned} \int_{co(M_k)} |\nabla f_k|^2 &\leq \int_{co(M_k)} (|A_k|^2 + |R_k|) f_k^2, \\ &\leq C \int_{co(M_k)} f_k^2\, \omega_{k,\delta,R}, \\ &= C. \end{aligned}$$

Furthermore,

$$\delta^{-2} \|f_k\|^2_{L^2(co(M_k))} \leq \int_{co(M_k)} f_k^2\, \omega_{k,\delta,R} = 1.$$

Thus, for all $k \in \mathbb{N}$.,

$$\|f_k\|_{W^{1,2}(co(M_k))} \leq C = C(N, g, \delta, R, m, M_\infty, \Sigma^1, \dots, \Sigma^J) < +\infty.$$

Using Lemma 4 we may also obtain a lower bound on the negative eigenvalues,

$$\lambda_k = \lambda_k \int_{co(M_k)} f_k^2\, \omega_{k,\delta,R} = \int_{co(M_k)} |\nabla f_k|^2 - (|A_k|^2 + R_k) f_k^2 \geq -C \int_{co(M_k)} f_k^2\, \omega_{k,\delta,R} = -C. \tag{3.11}$$

Thus, after potentially taking a subsequence and renumerating, we may assume that $\lambda_k \to \lambda_\infty \leq 0$.

We define the map,

$$\begin{aligned} F\colon co(M_\infty)\times\mathbb{R} &\to N,\\ (x,t) &\mapsto \exp_{\iota_\infty(x)}(t\nu_\infty(x)). \end{aligned}$$

Note, as M_∞ is smooth, and properly embedded, ([37, Proposition 4.2]) there exists a $\tau = \tau(N, M_\infty, g) > 0$, such that,

$$F\colon co(M_\infty)\times(-\tau,\tau)\to F(co(M_\infty)\times(-\tau,\tau))\subset N,$$

is a smooth, local diffeomorphism. We define the metric $\tilde{g} = F^*g$, on $co(M_\infty)\times(-\tau,\tau)$, and assume that for all $k\in\mathbb{N}$, $M_k\subset F(co(M_\infty)\times(-\tau,\tau))$.

First we consider the case of minimal hypersurfaces. As M_∞ is properly embedded, and N is compact, we may take $\tau>0$, such that, $F^{-1}(M_\infty) = co(M_\infty)\times\{0\}$. For $r>0$, define the open set $\Omega_r\subset co(M_\infty)$, by,

$$\Omega_r := \iota_\infty^{-1}\left(M_\infty\setminus\bigcup_{y\in\mathcal{I}}\overline{B_r^N(y)}\right)\subset co(M_\infty).$$

We define, $M_k^r := M_k\cap F(\Omega_r\times(-\tau,\tau))$, and $\tilde{M}_k^r = F^{-1}(M_k^r)\cap(\Omega_r\times(-\tau,\tau))$. By the convergence described in Definition 5 (and Remark 27), along with the the fact that $co(M_\infty)$ is two-sided in $co(M_\infty)\times(-\tau,\tau)$, and $\theta_{|M_\infty}\equiv m$, for large enough k, there exists m smooth functions,

$$u_k^{i,r}\colon\Omega_r\to(-\tau,\tau),$$

such that, $u_k^{1,r}<u_k^{2,r}<\cdots<u_k^{m,r}$, and

$$\tilde{M}_k^r = \bigcup_{i=1}^m\{(x,u_k^{i,r}(x))\colon x\in\Omega_r\}\subset\Omega_r\times(-\tau,\tau).$$

We also note that $u_k^{i,r}\to 0$ in $C^l(\Omega_r)$ for all $l\in\mathbb{N}$, and for $0<r<s$, $u_k^{i,r}=u_k^{i,s}$ on $\Omega_s\subset\Omega_r$. Moreover, we define the metric $g_k=\iota_k^*(g_{|M_k})$ on $co(M_k)$, and the metric $g_\infty=\iota_\infty^*(g_{|M_\infty})$ on $co(M_\infty)$.

First we consider the case in which M_k is one-sided. For each connected component of $\tilde{M}_k^r$,

$$\tilde{M}_k^{i,r} := \{(x,u_k^{i,r}(x))\colon x\in\Omega_r\},$$

we denote the $\nu_k^{i,r}$ to be the choice of unit normal to $\tilde{M}_k^{i,r}$ (with respect to $\tilde{g}$) which points in the positive τ direction. Through this choice of unit normal we identify Ω_r as a subset of $co(M_k)$, by the map (which is a diffeomorphism onto its image)

$$\begin{aligned} F_k^{i,r} \colon \Omega_r &\to co(M_k), \\ x &\mapsto (F(x, u_k^{i,r}(x)), dF(\nu_k^{i,r})), \end{aligned} \tag{3.12}$$

and define,

$$\tilde{f}_k^{i,r}(x) = (f_k \circ F_k^{i,r})(x).$$

We do note that $\tilde{f}_k^{i,r}$ depends on the choice of unit normal to $\tilde{M}_k^{i,r}$ that we pick, however as $f_k \in W^{1,2}(co(M_k))^-$, this choice is only up to a sign.

For the case of M_k being two-sided, we simply define,

$$\begin{aligned} F_k^{i,r} \colon \Omega_r &\to co(M_k) = M_k, \\ x &\mapsto F(x, u_k^{i,r}(x)), \end{aligned}$$

and define,

$$\tilde{f}_k^{i,r}(x) = (f_k \circ F_k^{i,r})(x).$$

For the case of H-CMC hypersurfaces, we have that

$$co(M_\infty) = \sqcup_{i=1}^a co(M_\infty^i),$$

where each M_∞^i is a distinct, closed, quasi-embedded H-CMC hypersurface such that $co(M_\infty^i)$ is connected. We have that $\theta^i = m_i \in \mathbb{Z}$, and we denote, $\Omega_{r/2}^i \subset co(M_\infty^i)$ as before. Then similarly to before, for each $i = 1, \ldots, a$, and large enough k, there exists m_i smooth graphs ($j = 1, \ldots, m_i$),

$$u_k^{j,i,r} \colon \Omega_{r/2}^i \to (-\tau, \tau),$$

such that, $u_k^{1,i,r} < u_k^{2,i,r} < \cdots < u_k^{m_i,i,r}$, $u_k^{j,i,r} \to 0$ in $C^l(\Omega_r^i)$ for all $l \in \mathbb{Z}_{\geq 1}$, and, for large enough k,

$$M_k \setminus \bigcup_{y \in \mathcal{I}} B_r^N(y) \subset \bigcup_{i=1}^a \bigcup_{j=1}^{m_i} \{F(x, u_k^{j,i,r/2}(x)) \colon x \in \Omega_{r/2}^i\}.$$

Define $\tilde{M}_k^{j,i,r} = \{(x, u_k^{j,i,r}(x)) \colon x \in \Omega_r^i\}$, and as before we identify this as a subset of $co(M_k)$, and similarly define the map $F_k^{j,i,r}$, and the function $\tilde{f}_k^{j,i,r} \in W^{1,2}(\Omega_r^i)$.

For ease of notation we just consider the case of minimal hypersurfaces. For an open set $\Omega \subset\subset co(M_\infty) \setminus \iota_\infty^{-1}(\mathcal{I})$, we may take $r > 0$, such that $\Omega \subset\subset \Omega_r$, and then define, for large enough k,

$$\tilde{f}_k^i(x) = \tilde{f}_k^{i,r}(x),\ x \in \Omega.$$

Note that this definition is independent of the choice of $0 < r < r_0$, for $\Omega \subset \Omega_{r_0}$. When dealing with a fixed open set $\Omega \subset\subset co(M_\infty) \setminus \iota_\infty^{-1}(\mathcal{I})$, for appropriate choices of r, we drop the superscript r in the notation of the maps $F_k^{i,r}$, and functions $u_k^{i,r}$. Then, choosing k large enough (so that $\|u_k^i\|_{C^1(\Omega_r)}$ is small enough), we have that,

$$\|\tilde{f}_k^i\|_{W^{1,2}(\Omega, g_\infty)} \leq 2\|f_k\|_{W^{1,2}(co(M_k, g_k))} \leq C.$$

and thus for each $i = 1, \dots, m$, there exists an $\tilde{f}_\infty^i \in W^{1,2}_{\text{loc}}(co(M_\infty) \setminus \iota_\infty^{-1}(\mathcal{I}))$, such that, after potentially picking a subsequence and renumerating,

$$\begin{cases} \tilde{f}_k^i \rightharpoonup \tilde{f}_\infty^i, & W^{1,2}_{\text{loc}}(co(M_\infty) \setminus \iota_\infty^{-1}(\mathcal{I})), \\ \tilde{f}_k^i \to \tilde{f}_\infty^i, & L^2_{\text{loc}}(co(M_\infty) \setminus \iota_\infty^{-1}(\mathcal{I})), \end{cases} \tag{3.13}$$

Note that by lower semicontinuity of the $W^{1,2}$ norm for (3.13), for all open $\Omega \subset co(M_\infty) \setminus \iota_\infty^{-1}(\mathcal{I})$, we have a uniform bound $\|\tilde{f}_\infty^i\|_{W^{1,2}(\Omega)} \leq C$. Multiplying $\tilde{f}_\infty^i$, and $|\nabla \tilde{f}_\infty^i|$, by sequences of appropriately chosen characteristic functions on $co(M_\infty)$, we may apply monotone convergence theorem to deduce that $\tilde{f}_\infty^i$, and $|\nabla \tilde{f}_\infty^i|$, lie in $L^2(co(M_\infty))$. Then by a standard point removal argument (similar to that in Proposition 13) we may show that $\nabla \tilde{f}_\infty^i$ extends to be the distributional gradient of $\tilde{f}_\infty^i$, on $co(M_\infty)$. Thus, $\tilde{f}_\infty^i \in W^{1,2}(co(M_\infty))$. Moreover, we have that

$$\int_{co(M_\infty)} (\tilde{f}_\infty^i)^2 \, (\omega_\delta \circ \iota_\infty) \leq 1,$$

For $i = 1, \dots, m$, and large enough k, we define the metric, $\tilde{g}_k^i := (F_k^i)^* g_k$, and its associated gradient $\tilde{\nabla}_k^i$, on Ω. Let J_k^i denote the Jacobian of the map F_k^i with respect to the metric g_∞ on Ω. For a point $x_0 \in \Omega$, we may choose $s > 0$, small enough so that $B_s^{co(M_\infty)}(x_0) \subset\subset \Omega$,

$$F(B_s^{co(M_\infty)}(x_0) \times (-\tau, \tau)) \subset B^N_{\text{inj}\,(N)/2}(\iota(x_0)).$$

Consider $\varphi \in C_c^\infty(B_s^{co(M_\infty)}(x_0))$, and for each $i = 1, \dots, m$ and $x \in B_s^{co(M_\infty)}(x_0)$, we define the function

$$\varphi_k^i(F_k^i(x)) = \varphi(x),$$

on $C_c^\infty(F_k^i(B_s^{co(M_\infty)}(x_0))) \subset C^\infty(co(M_k))$. As each M_k is properly embedded, by [19, Lemma C.1] (cf. [49]),

$$\{\iota_k(F_k^i(x)) \colon x \in B_s^{co(M_\infty)}(x_0)\} \subset M_k \cap B^N_{\text{inj}\,(N)/2}(\iota(x_0))$$

is two-sided, and thus we can extend φ_k^i to a vector field on N, and thus to a function in

$C^\infty(co(M_k))^-$. Thus we may plug φ_k^i into (3.10) and obtain,

$$\int_\Omega g_k^i(\tilde{\nabla}_k^i \tilde{f}_k^i, \tilde{\nabla}_k^i \varphi) J_k^i = \lambda_k \int_\Omega \tilde{f}_k^i \, \varphi \, (\omega_{k,\delta,R} \circ F_k^i) \, J_k^i + \int_\Omega ((|A_k|^2 + R_k) \circ F_k^i) \, \tilde{f}_k^i \, \varphi \, J_k^i.$$

Hence, by (3.13), convergence of $\omega_{k,\delta,R} \circ F_k^i \to \omega_\delta \circ \iota_\infty$, in $W^{1,\infty}(\Omega)$, and smooth convergence of, $F_k^i \to id$, on Ω, we have that,

$$\int_{co(M_\infty)} \nabla \tilde{f}_\infty^i \cdot \nabla \varphi - ((|A_\infty|^2 + R_\infty) \circ \iota_\infty) \, \tilde{f}_\infty^i \, \varphi = \lambda_\infty \int_{co(M_\infty)} \tilde{f}_\infty^i \, \varphi \, (\omega_\delta \circ \iota_\infty), \tag{3.14}$$

holds for all $\varphi \in C_c^\infty(B_s^{co(M_\infty)}(x_0))$. Thus by standard regularity theory for linear elliptic PDEs we have that $\tilde{f}_\infty^i \in W^{2,2}(\Omega)$ and $\Delta \tilde{f}_\infty^i + ((|A_\infty|^2 + R_\infty) \circ \iota_\infty) \, \tilde{f}_\infty^i + \lambda_\infty \tilde{f}_\infty^i \, (\omega_\delta \circ \iota_\infty) = 0$ a.e. on Ω. This then implies that (3.14) holds for all $\varphi \in C_c^\infty(co(M_\infty) \setminus \iota_\infty^{-1}(\mathcal{I}))$.

Proposition 13. *Let (M, g) be a compact, n-dimensional, Riemannian manifold, with $n \geq 3$. Consider $V \in L^\infty(M)$, and $\omega \in L(n/2, \infty)(M)$. Suppose that we have $u \in W^{1,2}(M)$, and a finite set of points $\mathcal{J} \subset M$, such that, for all $\varphi \in C_c^\infty(M \setminus \mathcal{J})$,*

$$\int_M \nabla u \cdot \nabla \varphi - V u \varphi - \omega u \varphi = 0, \tag{3.15}$$

then in fact (3.15) holds for all $\varphi \in W^{1,2}(M)$.

Proof. As M is compact there exists an $r_0 = r_0(M, g) > 0$, such that for all $r \in (0, r_0]$, and $x \in M$,

$$\mathcal{H}^n(B_r^M(x)) \leq (\omega_n + 1) r^n,$$

where $\omega_n = \mathcal{H}^n(B_1^n(0))$. Now, consider a smooth function on $\mathbb{R}$ with the following properties,

$$\begin{cases} \chi(t) = 1, & t \leq 1, \\ \chi(t) = 0, & t \geq 2, \\ -3 \leq \chi'(t) \leq 0. & \end{cases}$$

Then for any positive $\varepsilon < \min\{\mathrm{inj}(M)/4, \min_{y_1, y_2 \in \mathcal{J}} \mathrm{dist}(y_1, y_2)/4, r_0/4\}$, we define the following smooth function on M,

$$\chi_\varepsilon(x) = \chi\left(\frac{d_{\mathcal{J}}(x)}{\varepsilon}\right),$$

where we define,

$$d_{\mathcal{J}}(x) = \mathrm{dist}_g(x, \mathcal{J}).$$

We note that,

$$\int |\nabla \chi_\varepsilon|^2 \leq 9(\omega_n + 1)\varepsilon^{n-2}.$$

Take $\varphi \in C^\infty(M)$, and we define,

$$\begin{aligned}\varphi_{1,\varepsilon} &:= (1-\chi_\varepsilon)\varphi,\\ \varphi_{2,\varepsilon} &:= \chi_\varepsilon\varphi.\end{aligned}$$

Then, $\varphi = \varphi_{1,\varepsilon} + \varphi_{2,\varepsilon}$, and $\varphi_{1,\varepsilon} \in C_c^\infty(M \setminus \mathcal{J})$, which implies that,

$$\int_M \nabla u \cdot \nabla \varphi - Vu\varphi - \omega u\varphi = \int_M \nabla u \cdot \nabla \varphi_{2,\varepsilon} - Vu\varphi_{2,\varepsilon} - \omega u\varphi_{2,\varepsilon}.$$

We have,

$$\begin{aligned}\|\varphi_{2,\varepsilon}\|^2_{W^{1,2}(M)} &= \int_M |\varphi\,\nabla\chi_\varepsilon + \chi_\varepsilon\,\nabla\varphi|^2 + \chi_\varepsilon^2\,\varphi^2,\\ &\leq 2\int_M \varphi^2|\nabla\chi_\varepsilon|^2 + 2\int_M \chi_\varepsilon^2|\nabla\varphi|^2 + \int_M \chi_\varepsilon^2\,\varphi^2,\\ &\leq \|\varphi\|^2_{C^1(M)}(\omega_n+1)(18+3\varepsilon^2)\varepsilon^{n-2}\end{aligned}$$

Thus, by the above and Proposition 12,

$$\begin{aligned}&\left|\int_M \nabla u \cdot \nabla \varphi - Vu\varphi - \omega u\varphi\right|\\ &= \left|\int_M \nabla u \cdot \nabla \varphi_{2,\varepsilon} - Vu\varphi_{2,\varepsilon} - \omega u\varphi_{2,\varepsilon}\right|\\ &\leq C\|\varphi\|_{C^1(M)}\|u\|_{W^{1,2}(M)}(1+\|V\|_{L^\infty(M)}+\|\omega\|_{(n/2,\infty)(M)})\left((\omega_n+1)(18+3\varepsilon^2)\varepsilon^{n-2}\right)^{1/2},\end{aligned}$$

with $C = C(M,g) < +\infty$. Therefore, letting $\varepsilon \to 0$, we conclude that (3.15) holds for all $\varphi \in C_c^\infty(M)$.

The fact that (3.15) holds for all $\varphi \in W^{1,2}(M)$, follows from a standard density argument, with one small subtlety in that if $\varphi_k \to \varphi$ in $W^{1,2}(M)$, then by Proposition 12,

$$\left|\int_M u\,(\varphi-\varphi_k)\,\omega\right| \leq C\|\omega\|_{(n/2,\infty)}\|u\|_{W^{1,2}(M)}\|\varphi-\varphi_k\|_{W^{1,2}(M)} \to 0.$$

□

By Proposition 13 we have that (3.14) holds for all $\varphi \in W^{1,2}(co(M_\infty))$. Thus,

$$\tilde{f}^i_\infty \in \mathcal{E}_{\omega_\delta}(\lambda_\infty; L_\infty, W^{1,2}(co(M_\infty))).$$

It is worth noting that we could have $\tilde{f}^i_\infty = 0$.

3.2.3 Convergence on the Bubble

For $S > 0$ fixed, and $i \in \{1, \ldots, J\}$, consider the bubble $\Sigma_k^{i,S}$, and its associated point-scale sequence, $\{(p_k^i, r_k^i)\}_{k\in\mathbb{N}}$. Let $\{\partial_1, \ldots, \partial_{n+1}\}$ be an orthonormal basis for $T_{p_k^i}N$, with respect to the metric g, and define the map,

$$\begin{aligned} G_k^i \colon \mathbb{R}^{n+1} = \operatorname{span}\{\partial_1, \ldots, \partial_{n+1}\} &\to N, \\ x &\mapsto \exp_{p_k^i}(r_k^i x), \end{aligned}$$

then on $B_{2S}^{n+1}(0)$, for large enough k we define the metric, $\tilde{g}_k = (r_k^i)^{-2}(G_k^i)^* g$, and we have that,

$$(\tilde{g}_k)_{\alpha,\beta}(x) \coloneqq g_{\alpha,\beta}(r_k^i x) \to \delta_{\alpha,\beta},$$

and for our bubble,

$$\tilde{\Sigma}_k^{i,2S} \coloneqq \frac{1}{r_k^i} \exp_{p_k^i}^{-1}(\Sigma_k^{i,S} \cap B_{2Sr_k^i}^N(p_k^i)) \to \Sigma^i \cap B_{2S}^{n+1}(0) \eqqcolon \Sigma^{i,2S},$$

smoothly. As Σ^i is two-sided there is a choice of unit normal ν, and a $\tau > 0$, such that the map,

$$\begin{aligned} F \colon \Sigma^{i,S} \times (-\tau, \tau) &\to \mathbb{R}^{n+1}, \\ (x, t) &\mapsto x + t\nu(x), \end{aligned}$$

is a diffeomorphism onto its image. Then for large enough k, there exists a smooth function,

$$v_k^{i,S} \colon \Sigma^{i,S} \to (-\tau, \tau),$$

such that,

$$\tilde{\Sigma}_k^{i,2S} \cap F(\Sigma^{i,S} \times (-\tau, \tau)) = \{F(x, v_k^{i,S}(x)) \colon x \in \Sigma^{i,S}\}.$$

As before, for large enough k, define the smooth map,

$$\begin{aligned} F_k^{i,S} \colon \Sigma^{i,S} &\to \mathbb{R}^{n+1}, \\ x &\mapsto F(x, v_k^{i,S}(x)). \end{aligned}$$

From ν, we get a choice of unit normal ν_k^i (which points in the $dF(\partial_t)$ direction), to $\tilde{\Sigma}_k^{i,2S} \cap F(\Sigma^{i,S} \times (-\tau, \tau))$, with respect to $\tilde{g}_k$.

If M_k is one-sided, then we define the following functions on $\Sigma^{i,S}$ (recalling our functions f_k from Section 3.2.2),

$$\tilde{f}_k^{i,S}(x) \coloneqq (r_k^i)^{n/2-1} f_k((G_k^i \circ F_k^i)(x), (r_k^i)^{-1} dG_k^i(\nu_k^i(F_k^i(x))))$$

and,

$$\omega_k^{\Sigma^i,R,S}(x) := (r_k^i)^2\omega_{k,\delta,R}((G_k^i \circ F_k^i)(x), (r_k^i)^{-1}dG_k^i(\nu_k^i(F_k^i(x)))).$$

We note that while $\tilde{f}_k^{i,S}$ depends on our choice of unit normal ν to Σ^i, this dependence is only up to a choice in sign.

If M_k is two sided, the we define the following functions on $\Sigma^{i,S}$,

$$\tilde{f}_k^{i,S}(x) := (r_k^i)^{n/2-1}f_k((G_k^i \circ F_k^i)(x))$$

and,

$$\omega_k^{\Sigma^i,R,S}(x) := (r_k^i)^2\omega_{k,\delta,R}((G_k^i \circ F_k^i)(x)).$$

For each $j = 1, \ldots, J$, such that $\mathrm{dist}_g^N(p_k^j, p_k^i) \to 0$, we denote,

$$q_k^j = (r_k^i)^{-1}\exp_{p_k^i}^{-1}(p_k^j),$$

and, after potentially taking a subsequence and renumerating, either $|q_k^j| \to \infty$, or there exists a $q^j \in \mathbb{R}^{n+1}$, such that $q_k^j \to q^j$. If $q_k^j \to q^j$, then there exists $\alpha_j \in [0, \infty]$, such that

$$\frac{r_k^j}{r_k^i} \to \alpha_j.$$

If $\alpha_j = +\infty$, we may rescale about p_k^j at rate r_k^j, and use Lemma 1 on the component passing through the rescaled point of p_k^j to deduce a contradiction in a similar way in which we proved (3.9). Thus $\alpha_j \in [0, +\infty)$.

For each such j we define the following functions on $\mathbb{R}^{n+1}$. For the case $\alpha_j \neq 0$,

$$\omega^{i,j,R}(x) = \begin{cases} |x - q^j|^{-2}, & x \notin B_{R\alpha_j}^{n+1}(q^j), \\ (R\alpha_j)^{-2}, & x \in B_{R\alpha_j}^{n+1}(q^j), \end{cases}$$

and for $\alpha_j = 0$,

$$\omega^{i,j,R}(x) = |x - q^j|^{-2}.$$

For all other such $j = 1, \ldots, J$, $j \neq i$, we simply set $\omega^{i,j,R} = 0$. For $j = i$ we set,

$$\omega^{i,i,R}(x) = \begin{cases} |x|^{-2}, & x \notin B_R^{n+1}(0), \\ R^{-2}, & x \in B_R^{n+1}(0). \end{cases}$$

If $\alpha_j = 0$, then we have that $q^j \notin \Sigma^i$, again using Lemma 1 and a similar contradiction argument to that used to show (3.9).

We now define the following function on Σ^i,

$$\omega^{\Sigma^i,R}(x) = \max_{j=1,\dots,L} \omega^{i,j,R}(x), \tag{3.16}$$

and note that, $\omega^{\Sigma^i,R} \in W^{1,\infty}(\Sigma^i)$, and $\omega_k^{\Sigma^i,R,S} \to \omega^{\Sigma^i,R}$ in $W^{1,\infty}(\Sigma^{i,S})$. We have, for large k,

$$\int_{\Sigma^{i,S}} (\tilde{f}_k^{i,S})^2 \omega_k^{\Sigma^i,R,S} \le 2 \int_{co(M_k)} f_k^2 \omega_{k,\delta,R} = 2,$$

implying that, for large enough k,

$$\int_{\Sigma^{i,S}} (\tilde{f}_k^{i,S})^2 \le \frac{1}{\min_{\Sigma^{i,S}} \omega^{\Sigma^i,R}} \int_{\Sigma^{i,S}} (\tilde{f}_k^{i,S})^2 \omega_{k,R}^{\Sigma^i,S} \le C(S, R, \Sigma^1, \dots, \Sigma^J) < +\infty.$$

Moreover,

$$\int_{\Sigma^{i,S}} |\nabla \tilde{f}_k^{i,S}|^2 \le 2 \int_{co(M_k)} |\nabla f_k|^2 \le C(N, g, \delta, R, M_\infty, \Sigma^1, \dots, \Sigma^J) < +\infty.$$

Similar to previous, for $0 < S_1 < S_2 < +\infty$, and large enough k, we have that $\tilde{f}_k^{i,S_1} = \tilde{f}_k^{i,S_2}$ on Σ^{i,S_1}. Thus, for any open, bounded set $\Omega \subset \Sigma^i$, we may take any $S > 0$ such that $\Omega \subset \Sigma^{i,S}$, then for large enough k, the function $\tilde{f}_k^i = \tilde{f}_k^{i,S}$ is well defined on Ω, with,

$$\int_\Omega (\tilde{f}_k^i)^2 \le C(\Omega, R, M_\infty, \Sigma^1, \dots, \Sigma^J),$$

and,

$$\int_\Omega |\nabla \tilde{f}_k^i|^2 \le C(N, g, \delta, R, M_\infty, \Sigma^1, \dots, \Sigma^J).$$

We may conclude that there exists an $\tilde{f}_\infty^i \in W^{1,2}_{\text{loc}}(\Sigma^i)$,

$$\begin{cases} \tilde{f}_k^i \to \tilde{f}_\infty^i, & L^2_{\text{loc}}(\Sigma^i), \\ \tilde{f}_k^i \rightharpoonup \tilde{f}_\infty^i, & W^{1,2}_{\text{loc}}(\Sigma^i), \end{cases}$$

and, we have

$$\int_{\Sigma^i} |\nabla \tilde{f}_\infty^i|^2 < +\infty,$$

and,

$$\int_{\Sigma^i} (\tilde{f}_\infty^i)^2 \omega^{\Sigma^i,R} \le 1.$$

Thus,

$$\tilde{f}_\infty^i \in W^{1,2}_{\omega^{\Sigma^i,R}}(\Sigma^i) := \left\{ f \in W^{1,2}_{\text{loc}}(\Sigma^i) \colon \int_{\Sigma^i} |\nabla f|^2 < +\infty, \int_{\Sigma^i} \omega^{\Sigma^i,R} f^2 < +\infty \right\}$$

Similar to before, and the fact that our metric on $B_S^{n+1}(0)$ converges to the standard Euclidean one, we deduce that for all $\varphi \in C_c^\infty(\Sigma^i)$,

$$\int_{\Sigma^i} \nabla \tilde{f}^i_\infty \cdot \nabla \varphi - |A_{\Sigma^i}|^2 \, \tilde{f}^i_\infty \, \varphi - \lambda_\infty \, \omega^{\Sigma^i,R} \, \tilde{f}^i_\infty \, \varphi = 0. \tag{3.17}$$

As Σ^i has finite index and Euclidean volume growth at infinity, we may deduce that there exists an $S > 0$, such that $\Sigma^i \setminus B_S^{n+1}(0)$ is stable [26, Proposition 1]. Thus, noting that Σ^i is embedded, by [51, Theorem 3] (alternatively, [52, Theorem 3] for $2 \leq n \leq 5$, and [10, Theorem 2] for $2 \leq n \leq 6$), for all $x \in \Sigma^i \setminus B_{2S}^{n+1}(0)$, we have that there exists a $C = C(\Sigma^i) < +\infty$, such that,

$$|A_{\Sigma^i}|^2(x) \leq \frac{C}{(|x| - S)^2} \leq \frac{C}{|x|^2}. \tag{3.18}$$

Moreover, we may choose S large enough so that, $|x|^{-2} \leq 2\omega^{\Sigma^i,R}$ on $\Sigma^i \setminus B_S^{n+1}(0)$. Thus, for $x \in \Sigma^i \setminus B_{2S}^{n+1}(0)$,

$$|A_{\Sigma^i}|^2(x) \leq C\omega^{\Sigma^i,R}.$$

Thus by Hölder's inequality we have, for $\varphi \in L^2(\Sigma^i)$,

$$\left| \int_{\Sigma^i} |A_{\Sigma^i}|^2 \, \tilde{f}^i_\infty \, \varphi + \omega^{\Sigma^i,R} \, \tilde{f}^i_\infty \, \varphi \right| \leq C \|\omega^{\Sigma^i,R}\|^{1/2}_{L^\infty(\Sigma^i)} \, \|\varphi\|_{L^2(\Sigma^i)}.$$

This allows us to apply a standard density argument to deduce that (3.17) holds for all $\varphi \in W^{1,2}(\Sigma^i)$. Thus we have that

$$\tilde{f}^i_\infty \in \mathcal{E}_{\omega^{\Sigma^i,R}}(\lambda_\infty; L_{\Sigma^i}, W^{1,2}_{\omega^{\Sigma^i,R}}(\Sigma^i), W^{1,2}(\Sigma^i)).$$

Again it is worth noting that we could have $\tilde{f}^i_\infty = 0$.

Remark 32. We may also deduce $\tilde{f}_\infty \in L^{2^*}(\tilde{\Sigma})$, for $2^* = 2n/(n-2)$. Indeed, take $S > R$, large enough so that $|x|^{-2} \leq 2\omega_R^{\Sigma^i}$, and define a function χ_S on Σ^i (see Proposition 15), with the following properties

$$\begin{cases} \chi_S(x) = 1, & x \in \Sigma^{i,S}, \\ \chi_S(x) = 0, & x \in \Sigma^{i,2S}, \\ |\nabla \chi_S| \leq 3/S. \end{cases}$$

We plug the function $\chi_S \tilde{f}_\infty$ into the Michael–Simon–Sobolev inequality ([40, Theorem 2.1]) on Σ^i

(noting that Σ^i is an embedded, minimal hypersurface in $\mathbb{R}^{n+1}$),

$$\begin{aligned}\left(\int_{\Sigma^{i,S}} |\tilde{f}_\infty|^{2^*}\right)^{1/2^*} &\leq \left(\int_{\Sigma^i} |\chi_S\, \tilde{f}_\infty|^{2^*}\right)^{1/2^*}, \\ &\leq C(n)\Bigg(\int_{\Sigma^i} |\nabla \tilde{f}_\infty|^2\, |\chi_S|^2 + 2|\nabla \tilde{f}_\infty|\, |\chi_S|\, |\tilde{f}_\infty|\, |\nabla \chi_S| \\ &\qquad\qquad + |\tilde{f}_\infty|^2\, |\nabla \chi_S|^2\Bigg)^{1/2}, \\ &\leq C\left(\int_{\Sigma^i} |\nabla \tilde{f}_\infty|^2 + \int_{\Sigma^i} \omega^{\Sigma^i,R} \tilde{f}_\infty^2\right)^{1/2}.\end{aligned}$$

The bound is independent of S, and thus we may deduce that $\tilde{f}_\infty \in L^{2^*}(\Sigma^i)$.

3.2.4 Strict Stability of the Neck

For a Riemannian manifold (M, g), we define $W^{1,2}_0(M, g)$ to be the closure of $C^1_c(M)$, with respect to the standard norm on $W^{1,2}(M, g)$.

Lemma 5. *For $n \geq 3$, consider the cylinder $(A \times \mathbb{R}, g)$, where $A \subset \mathbb{R}^n$ is a non-empty, open set, and g is a smooth Riemannian metric, such that there exists a constant $K \in [1, \infty)$ such that for $x \in A \times \mathbb{R}$, and $X \in \mathbb{R}^{n+1}$,*

$$\frac{1}{K}\langle X, X \rangle \leq g_x(X, X) \leq K\langle X, X \rangle, \tag{3.19}$$

where $\langle \cdot, \cdot \rangle$ is the standard metric on $\mathbb{R}^n$. Now suppose we have a smooth function

$$u\colon A \to (-T, T),$$

such that $\|\nabla^{\mathbb{R}^n} u\|_{L^\infty(A)} \leq 1/2$ ($\nabla^{\mathbb{R}^n}$ denotes the gradient on $\mathbb{R}^n$ with respect to the standard Euclidean metric), and denote $M := graph(u) \subset A \times (-2T, 2T)$. For fixed $W \in (0, \infty)$ suppose we have functions, $\omega \in L(n/2, \infty)(M, g)$, and $V \in L^\infty(M)$, such that $\|\omega\|_{L(n/2,\infty)(M,g)} \leq W$, and $\operatorname{ess\,inf} \omega > 0$. *Then, there exists an $\varepsilon = \varepsilon(n, K, W) > 0$, and $C = C(n, K) \in (0, +\infty)$ such that if $|V| \leq \varepsilon\omega$ on M, then*

$$0 < \frac{1}{C\,W} \leq \inf\left\{\int_M |\nabla f|^2 - V f^2 \colon f \in W^{1,2}_0(M),\ \int_M f^2\omega = 1\right\}.$$

Proof. First, we recall that positive-definite real symmetric matrices are always diagonalisable with positive eigenvalues, thus (3.19) implies that,

$$K^{-(n+1)/2} \leq \sqrt{|g|} \leq K^{(n+1)/2}.$$

We define the following map, $F(x) \coloneqq (x, u(x))$, and the metric $\tilde{g} = F^*g$ on A. For $x \in A$, and $X \in \mathbb{R}^n$, we have,

$$\begin{aligned}
\tilde{g}_x(X,X) &= g_{(x,u(x))}(dF_x(X), dF_x(X)),\\
&= g_{(x,u(x))}(X + du_x(X)\partial_{n+1},\, X + du_x(X)\partial_{n+1}),\\
&= g_{(x,u(x))}(X,X) + 2du_x(X) g_{(x,u(x))}(X, \partial_{n+1}) + du_x(X)^2 g_{(x,u(x))}(\partial_{n+1}, \partial_{n+1}),\\
&\leq K\langle X, X\rangle + 2K\|\nabla^{\mathbb{R}^n} u\|_{L^\infty(A)}\langle X, X\rangle + K\|\nabla^{\mathbb{R}^n} u\|^2_{L^\infty(A)}\langle X, X\rangle,\\
&= C\langle X, X\rangle,
\end{aligned}$$

with $C = C(n, K)$, which from here on may be rechosen at each step. Thus, for $x \in A$, and $f \in C^1(M)$,

$$\begin{aligned}
|\nabla^{\mathbb{R}^n}(f \circ F)(x)| &= \sup_{\langle X,X\rangle \leq 1} |d(f\circ F)(x)(X)|,\\
&\leq \sup_{\tilde{g}_x(X,X)\leq C} |d(f\circ F)(x)(X)|,\\
&\leq C|\nabla^{\tilde{g}}(f\circ F)|(x).
\end{aligned}$$

Moreover, if we consider the metric on A, $g_1 = F^*\langle\cdot,\cdot\rangle$, then $g_1(X,X) \leq K\tilde{g}(X,X) \leq C\langle X, X\rangle$, and thus

$$1 \leq 1 + |\nabla^{\mathbb{R}^n} u(x)|^2 = |g_1| \leq K^n|\tilde{g}| \leq C.$$

Now, take an $f \in C^1_c(M)$, that satisfies,

$$\int_M f^2\,\omega\,d\mu = 1,$$

where μ denotes the volume measure of (M, g). We have, by the above, and applying Propositions

8, 9 and 10, for large enough k (see [62, Theorem 1.1] for a similar computation)

$$\begin{aligned}
1 &= \int_A (f\circ F)^2\,(\omega\circ F)\sqrt{|\tilde{g}|}\,dx,\\
&\le C\int_A (f\circ F)^2\,(\omega\circ F)\,dx,\\
&\le C\|\omega\circ F\|_{(n/2,\infty)(A,\langle\cdot\,,\cdot\rangle)}\|f\circ F\|^2_{(2^*,2)(A,\langle\cdot\,,\cdot\rangle)},\\
&\le C\|\omega\circ F\|_{(n/2,\infty)(A,\langle\cdot\,,\cdot\rangle)}\int_A |\nabla^{\mathbb{R}^n}(f\circ F)|^2\,dx,\\
&\le C\|\omega\circ F\|_{(n/2,\infty)(A,\langle\cdot\,,\cdot\rangle)}\int_M |\nabla f|^2 - Vf^2\,d\mu\\
&\qquad +C\|\omega\circ F\|_{(n/2,\infty)(A,\langle\cdot\,,\cdot\rangle)}\int_M Vf^2\,d\mu,\\
&\le C\|\omega\circ F\|_{(n/2,\infty)(A,\langle\cdot\,,\cdot\rangle)}\left(\varepsilon+\int_M |\nabla f|^2 - Vf^2\,d\mu\right),
\end{aligned}$$

were again we are potentially rechoosing $C = C(n,K)$ at each line. Moreover,

$$\|\omega\circ F\|_{(n/2,\infty)(A,\langle\cdot\,,\cdot\rangle)} \le K\|\omega\|_{(n/2,\infty)(M,g)} \le K\,W,$$

and thus, again rechoosing $C = C(n,K)$, and choosing $\varepsilon = (2CW)^{-1}$,

$$\int_M |\nabla f|^2 - Vf^2 \ge \frac{1}{2\,C\,W} > 0,$$

for all $f \in C^1_c(M)$.

Now the Lemma may be concluded by a standard density argument. We point out one small subtlety in this density argument (similar to the end of Proposition 13), in that we need to make use of the following Lorentz–Sobolev inequality on (M,g),

$$\begin{aligned}
\|f\|_{(2^*,2)(M,g)} &\le C\|f\circ F\|_{(2^*,2)(A,\langle\cdot,\cdot\rangle)},\\
&\le C\|\nabla^{\mathbb{R}^n}(f\circ F)\|_{L^2(A,\langle\cdot,\cdot\rangle)},\\
&\le C\|\nabla f\|_{L^2(M,g)},
\end{aligned}$$

for $f \in W^{1,2}_0(M,g)$, with $C = C(n,K)$. The reason for this being that if we have $f_k \to f$ in $W^{1,2}_0(M,g)$, then we may deduce that

$$\begin{aligned}
\left|\int_M f^2\omega\,d\mu - \int_M f_k^2\omega\,d\mu\right| &\le \|\omega\|_{(n/2,\infty)(M,g)}\|f-f_k\|_{(2^*,2)(M,g)}\|f+f_k\|_{(2^*,2)(M,g)},\\
&\le C\|\omega\|_{(n/2,\infty)(M,g)}\|\nabla f\|_{L^2(M,g)}\|\nabla f - \nabla f_k\|_{L^2(M,g)},\\
&\to 0.
\end{aligned}$$

□

As previously, for ease of notation we only consider the case of minimal hypersurfaces, however the argument for H-CMC hypersurfaces is identical.

In Sections 3.2.2 and 3.2.3 we showed that if we have a sequence,

$$f_k \in \mathcal{E}_{\omega_{k,\delta,R}}(\lambda_k; L_k, W^{1,2}(co(M_k))^-),$$

with $\lambda_k \leq 0$, and

$$\int_{co(M_k)} f_k^2 \, \omega_{k,\delta,R} = 1,$$

for all k, then after potentially taking a subsequence and renumerating we have that $\lambda_k \to \lambda_\infty \leq 0$, and,

$$f_k \to ((f_\infty^1, \ldots, f_\infty^m), f_\infty^{\Sigma^1}, \ldots, f_\infty^{\Sigma^J}),$$

where, for $i = 1, \ldots, m$, $f_\infty^i \in W^{1,2}(co(M_\infty))$, and for $j = 1, \ldots, J$, $f_\infty^{\Sigma^j} \in W^{1,2}_{\omega^{\Sigma^j},R}(\Sigma^j) \subset L^{2^*}(\Sigma^j)$. By this convergence we mean, for $i = 1, \ldots, m$ (recalling notation from Section 3.2.2), $\tilde{f}_k^i = (f_k \circ F_k^i)$,

$$\begin{cases} \tilde{f}_k^i \rightharpoonup f_\infty^i, & W^{1,2}_{\text{loc}}(co(M_\infty) \setminus \iota_\infty^{-1}(\mathcal{I}))), \\ \tilde{f}_k^i \to f_\infty^i, & L^2_{\text{loc}}(co(M_\infty) \setminus \iota_\infty^{-1}(\mathcal{I}))), \end{cases} \tag{3.20}$$

and for $j = 1, \ldots, J$, setting $\tilde{f}_k^{\Sigma^j} = \tilde{f}_k^j$ (this time $\tilde{f}_k^j$ as defined in Section 3.2.3),

$$\begin{cases} \tilde{f}_k^{\Sigma^j} \rightharpoonup f_\infty^{\Sigma^j}, & W^{1,2}_{\text{loc}}(\Sigma^j), \\ \tilde{f}_k^{\Sigma^j} \to f_\infty^{\Sigma^j}, & L^2_{\text{loc}}(\Sigma^j). \end{cases} \tag{3.21}$$

Furthermore, we are able to deduce that for $i = 1, \ldots, m$,

$$f_\infty^i \in \mathcal{E}_{\omega_\delta}(\lambda_\infty; L_\infty, W^{1,2}(co(M_\infty))),$$

and $j = 1, \ldots, J$,

$$f_\infty^{\Sigma^j} \in \mathcal{E}_{\omega^{\Sigma^j},R}(\lambda_\infty; L_{\Sigma^j}, W^{1,2}_{\omega^{\Sigma^j},R}(\Sigma^j), W^{1,2}(\Sigma^j)).$$

Claim 7. *If $\{f_{1,k}, \ldots, f_{b,k}\}$, is an orthonormal collection, with respect to the $\omega_{k,\delta,R}$-weighted L^2 norm, of $\omega_{k,\delta,R}$-weighted eigenfunctions, with non-positive eigenvalues, and $\sum_{i=1} a_i^2 = 1$, then,*

$$h_k \coloneqq \sum_{i=1}^b a_i f_{i,k} \to \sum_{i=1}^b a_i((f_{i,\infty}^1, \ldots, f_{i,\infty}^m), f_{i,\infty}^{\Sigma^1}, \ldots, f_{i,\infty}^{\Sigma^J}) \neq ((0, \ldots, 0), 0, \ldots, 0).$$

Proof. This claim is proven by a contradiction argument similar to that in [23, Claim 1 of Lemma IV.6].

By Remark 28, for all $\eta \in (0, 1/2]$, $\tau \in (0, 1)$, and $y \in \mathcal{I}$, there exists an $r_0 \in (0, \delta/4)$, and $R_0 \in (4R, \infty)$, such that, taking geodesic normal coordinates about $y \in N$, such that $T_y M_\infty = \{x_{n+1} = 0\}$, for large enough k, if we denote C_k to be a connected component of

$$\exp_y^{-1}(M_k \cap B^N_{r_0}(y) \setminus \cup_{j=1}^J \overline{\Sigma_k^{j,R_0}}) \subset \mathbb{R}^{n+1},$$

then there exists a non-empty open set $A(C_k) \subset \{x_{n+1} = 0\}$, and a smooth function,

$$u_k \colon A(C_k) \to (-\tau, \tau),$$

such that, $C_k = \operatorname{graph}(u_k)$, and $\|\nabla^{\mathbb{R}^n} u_k\|_{L^\infty} \leq \eta$. Moreover, for each $\varepsilon > 0$, we may make further choices of r_0 and R_0, such that for large enough k, on each C_k, if we denote $V_k = (|A_k|^2 + R_k)_{|C_k} \in L^\infty(C_k)$, and $\omega_k = (\omega_{k,\delta,R})_{|C_k} \in L(n/2, \infty)(C_k, g) \cap L^\infty(C_k)$, then

$$\|\omega_k\|_{L(n/2,\infty)(C_k,g)} \leq W, \qquad \text{and} \qquad |V_k| \leq \varepsilon \omega_k,$$

with $W = W(N, g, M_\infty, m, \delta, J) < +\infty$ (coming from Claim 6). Thus, fixing $\varepsilon = \varepsilon(N, g, M_\infty, m, \delta, J) > 0$ small enough, by Lemma 5, for large enough k

$$\inf\left\{ \int_{C_k} |\nabla f|^2 - V_k f^2 \colon f \in W_0^{1,2}(C_k),\ \int_{C_k} f^2 \omega_k = 1 \right\} \geq \gamma > 0, \tag{3.22}$$

with $\gamma = \gamma(N, g, M_\infty, m, \delta, J) > 0$. Now assuming Claim 7 does not hold, we will provide a contradiction to (3.22), on at least one connected component C_k of $M_k \cap (\cup_{y \in \mathcal{I}} B^N_{r_0}(y) \setminus \cup_{j=1}^J \overline{\Sigma_k^{j,R_0}})$. Note that there is a uniform bound, $m|\mathcal{I}| < +\infty$, on the number of such connected components.

Consider the following smooth cutoff,

$$\begin{cases} \chi \in C^\infty(\mathbb{R}; [0, 1]), & \\ \chi(t) = 1, & t \in (-\infty, 1], \\ \chi(t) = 0, & t \in [2, \infty), \\ -3 \leq \chi'(t) \leq 0, & \end{cases}$$

and the following distance functions defined on $co(M_k)$,

$$d_k(x) = \operatorname{dist}_g^N(\iota_k(x), \mathcal{I}),$$

and for $j = 1, \ldots, J$,

$$d_k^j(x) = \operatorname{dist}_g^N(\iota_k(x), p_k^j).$$

We then define the following function

$$H_k(x) = \begin{cases} 0, & x \in co(M_k) \setminus \cup_{y\in\mathcal{I}} \iota_k^{-1}(B^N_{r_0}(y)), \\ \chi(2r_0^{-1}d_k(x))h_k, & x \in \iota_k^{-1}(\cup_{y\in\mathcal{I}} B^N_{r_0}(y) \setminus \cup_{j=1}^J \Sigma_k^{j,2R_0}), \\ h_k(1-\chi((R_0r_k^j)^{-1}d_k^j(x))), & x \in \iota_k^{-1}(\Sigma_k^{j,2R_0}),\ j=1,\dots,J. \end{cases}$$

We compute the gradient of H_k. For $x \in co(M_k) \setminus \cup_{y\in\mathcal{I}} \iota_k^{-1}(B^N_{r_0}(y))$, or $x \in \iota_k^{-1}(\Sigma_k^{j,R_0})$, $j=1,\dots,J$, we have, $\nabla H_k = 0$. For $x \in co(M_k) \cap \iota_k^{-1}(B^N_{r_0}(y) \setminus \cup_{j=1}^J \Sigma_k^{j,2R_0})$, we have,

$$\nabla H_k(x) = \nabla h_k(x)\chi(2r_0^{-1}d_k(x)) + h_k(x)2r_0^{-1}\chi'(2r_0^{-1}d_k(x))\nabla d_k(x).$$

Finally, for $x \in \iota_k^{-1}(\Sigma_k^{j,2R_0} \setminus \Sigma_k^{j,R_0})$, $j=1,\dots,J$, we have,

$$\nabla H_k(x) = \nabla h_k(x)(1-\chi((Rr_k^j)^{-1}d_k^j(x))) - h_k(x)(R_0r_k^j)^{-1}\chi'((R_0r_k^j)^{-1}d_k^j(x))\nabla d_k^j(x).$$

Noting that $H_k^2 \leq h_k^2$, we have

$$|B_k(h_k,h_k) - B_k(H_k,H_k)| \leq \left| \int_{co(M_k)} |\nabla h_k|^2 - |\nabla H_k|^2 \right| + C\int_{co(M_k)} \omega_{k,\delta,R}(h_k^2 - H_k^2), \tag{3.23}$$

with $C = C(N,g,M_\infty,m,\Sigma^1,\dots,\Sigma^J,\delta,R) < +\infty$. We split the first term on the right hand side of (3.23) into separate domains,

$$\left| \int_{co(M_k)} |\nabla h_k|^2 - |\nabla H_k|^2 \right| \leq I + II + \sum_{y\in\mathcal{I}} III^y + \sum_{j=1}^J (IV^j + V^j), \tag{3.24}$$

where,

$$I = \int_{co(M_k)\setminus\cup_{y\in\mathcal{I}}\iota_k^{-1}(B^N_{r_0}(y))} |\nabla h_k|^2,$$

$$\begin{aligned} II &\leq \int_{\cup_{y\in\mathcal{I}}\iota_k^{-1}(B^N_{r_0}(y)\setminus B^N_{r_0/2}(y))} |\nabla h_k|^2 + 12r_0^{-1}|h_k|\,|\nabla h_k| + 36r_0^{-2}h_k^2, \\ &\leq C\int_{\cup_{y\in\mathcal{I}}\iota_k^{-1}(B^N_{r_0}(y)\setminus B^N_{r_0/2}(y))} |\nabla h_k|^2 + r_0^{-2}h_k^2 \end{aligned}$$

For $y \in \mathcal{I}$,

$$III^y = \left| \int_{\iota_k^{-1}(B^N_{r_0/2}(y)\setminus(\cup_{j=1}^J \Sigma_k^{j,2R_0}))} |\nabla h_k|^2 - |\nabla H_k|^2 \right| = 0,$$

and $j=1,\dots,J$,

$$
\begin{aligned}
IV^j &\leq \int_{\iota_k^{-1}(\Sigma_k^{j,2R_0}\setminus\Sigma_k^{j,R_0})} |\nabla h_k|^2 + 6(R_0 r_k^j)^{-1}|h_k|\,|\nabla h_k| + 9(R_0 r_k^j)^{-2}h_k^2,\\
&\leq C\left(\int_{\iota_k^{-1}(\Sigma_k^{j,2R_0}\setminus\Sigma_k^{j,R_0})} |\nabla h_k|^2 + (R_0 r_k^j)^{-2}h_k^2\right)
\end{aligned}
$$

$$
V^j = \int_{\iota_k^{-1}(\Sigma_k^{j,R_0})} |\nabla h_k|^2.
$$

Along our sequence we have (in a weak sense),

$$
\Delta h_k + (|A_k|^2 + R_k)\, h_k = -\left(\sum_{i=1}^{b} a_i \lambda_{i,k} f_{i,k}\right)\omega_{k,\delta,R} = -P_k\, \omega_{k,\delta,R},
$$

and we may note that,

$$
\|h_k\|_{L^2(co(M_k))} + \|P_k\|_{L^2(co(M_k))} \leq C\sum_{i=1}^{b} \|f_{i,k}\|_{L^2(co(M_k))} \leq C,
$$

with $C = C(N, g, M_\infty, m, \Sigma^1, \ldots, \Sigma^J, \delta, R) < +\infty$. Recalling notation from Section 3.2.2, for each $l = 1, \ldots, m$, we denote,

$$
\tilde{h}_k^l := \sum_{i=1}^{b} a_i \tilde{f}_{i,k}^l, \quad \text{and,} \quad \tilde{P}_k^l := \sum_{i=1}^{b} a_i \lambda_{i,k} \tilde{f}_{i,k}^l,
$$

and then by standard interior estimates,

$$
\|\tilde{h}_k^l\|_{W^{2,2}(\Omega_{r_0/4})} \leq \tilde{C}(\|\tilde{h}_k^l\|_{L^2(\Omega_{r_0/8})} + r_0^{-2}\|\tilde{P}_k^l\|_{L^2(\Omega_{r_0/8})}) \leq \tilde{C},
$$

for $\tilde{C} = \tilde{C}(N, g, M_\infty, m, \Sigma^1, \ldots, \Sigma^J, \delta, R, r_0) < +\infty$. Thus, by our assumption that

$$
h_k \to ((0, \ldots, 0), 0 \ldots, 0),
$$

after potentially taking a subsequence and renumerating we have,

$$
\begin{cases}
\tilde{h}_k^l \rightharpoonup 0, & \text{weakly}\, W^{2,2}(\Omega_{r_0/4}),\\
\tilde{h}_k^l \to 0, & \text{strongly}\, W^{1,2}(\Omega_{r_0/4}).
\end{cases}
$$

By identical arguments, for each $j = 1, \dots, J$ denoting,

$$\tilde{h}_k^{\Sigma^j} = \sum_{i=1}^{b} a_i \tilde{f}_{i,k}^{\Sigma^j},$$

we have that, after potentially taking a subsequence and renumerating,

$$\begin{cases} \tilde{h}_k^{\Sigma^j} \rightharpoonup 0, & \text{weakly } W^{2,2}(\Sigma^{j,4R_0}), \\ \tilde{h}_k^{\Sigma^j} \to 0, & \text{strongly } W^{1,2}(\Sigma^{j,4R_0}). \end{cases}$$

Therefore, we have that for all $\zeta > 0$, and then large enough k,

$$\left| \int_{co(M_k)} |\nabla h_k|^2 - |\nabla H_k|^2 \right| \leq \hat{C} \left(\sum_{l=1}^{m} \|\tilde{h}_k^l\|^2_{W^{1,2}(\Omega_{r_0/4})} + \sum_{j=1}^{J} \|\tilde{h}_k^{\Sigma^j}\|^2_{W^{1,2}(\Sigma^{j,4R_0})} \right) < \zeta,$$

with $\hat{C} = \hat{C}(N, g, M_\infty, m, \Sigma^1, \dots, \Sigma^J, \delta, R, r_0, R_0) < +\infty$. Similarly we have,

$$\int_{co(M_k)} \omega_{k,\delta,R}(h_k^2 - H_k^2) \leq \hat{C} \left(\sum_{l=1}^{m} \|\tilde{h}_k\|^2_{L^2(\Omega_{r_0/4})} + \sum_{j=1}^{J} \|\tilde{h}_k^{\Sigma^j}\|^2_{L^2(\Sigma^{j,4R_0})} \right) < \zeta.$$

Thus, for large k, there exists a connected component,

$$C_k \subset M_k \cap (\cup_{y \in \mathcal{I}} B^N_{r_0}(y) \setminus (\cup_{j=1}^{J} \Sigma_k^{j,R_0})),$$

such that, denoting $\tilde{H}_k = (H_k)_{|\iota_k^{-1}(C_k)}$, we have that $\tilde{H}_k \in W_0^{1,2}(\iota_k^{-1}(C_k))$, and

$$\begin{aligned} \int_{\iota_k^{-1}(C_k)} \tilde{H}_k^2 \, \omega_{k,\delta,R} \quad &\geq \quad \frac{1-\zeta}{m|\mathcal{I}|}, \\ B_k[\tilde{H}_k, \tilde{H}_k] \quad &< \quad \zeta. \end{aligned}$$

Thus, choosing

$$\zeta < \frac{\gamma}{2(2m|\mathcal{I}| + \gamma)},$$

we derive a contradiction to (3.22) on C_k. □

3.3 Equivalence of Weighted and Unweighted Eigenspaces

Proposition 14. *Let (M, g) be a compact Riemannian manifold of dimension $n \geq 3$, with $V \in L^\infty(M)$, $\omega \in L(n/2, \infty)(M, g)$, and $\operatorname{ess\,inf} \omega > 0$. Define the elliptic operator,*

$$L := \Delta + V.$$

Then,

$$\mathit{span}\left\{\cup_{\lambda\le 0}\mathcal{E}(\lambda;L,W^{1,2}(M))\right\} = \oplus_{\lambda\le 0}\mathcal{E}(\lambda;L,W^{1,2}(M)), \tag{3.25}$$
$$\mathit{span}\left\{\cup_{\lambda\le 0}\mathcal{E}_\omega(\lambda;L,W^{1,2}(M))\right\} = \oplus_{\lambda\le 0}\mathcal{E}_\omega(\lambda;L,W^{1,2}(M)), \tag{3.26}$$

and

$$\mathit{dim}\left(\mathit{span}\left\{\cup_{\lambda\le 0}\mathcal{E}_\omega(\lambda;L,W^{1,2}(M))\right\}\right) = \mathit{dim}\left(\mathit{span}\left\{\cup_{\lambda\le 0}\mathcal{E}(\lambda;L,W^{1,2}(M))\right\}\right).$$

Proof. First we note by Proposition 12, and fact that $\operatorname{ess\,inf}\omega > 0$,

$$\langle f_1, f_2\rangle_\omega := \int_M f_1\, f_2\, \omega,$$

is a well-defined inner product on $W^{1,2}(M)$. Consider the bilinear form on $f_1,\ f_2 \in W^{1,2}(M)$, corresponding to our elliptic operator L,

$$B_L[f_1,f_2] := \int_M \nabla f_1\cdot\nabla f_2 - V f_1 f_2.$$

If $f_i \in \mathcal{E}_\omega(\lambda_i;L,W^{1,2}(M))$, for $i = 1,\ 2$, with $\lambda_1\ne\lambda_2$, we have,

$$\lambda_1\langle f_1,f_2\rangle_\omega = B_L[f_1,f_2] = \lambda_2\langle f_1,f_2\rangle_\omega.$$

Thus, $\langle f_1,f_2\rangle_\omega = 0$, and so (3.26) follows. An identical argument will also conclude (3.25).

We now show that,

$$\begin{aligned}\dim\left(\oplus_{\lambda\le 0}\mathcal{E}(\lambda;L,W^{1,2}(M))\right) &= \sup\{\dim\Pi\colon \Pi\le W^{1,2}(M),\ \text{is a linear space such that, } (B_L)_{|\Pi}\le 0\}\\ &= \dim\left(\oplus_{\lambda\le 0}\mathcal{E}_\omega(\lambda;L,W^{1,2}(M))\right).\end{aligned}$$

Indeed, we begin by showing that,

$$\begin{aligned}\dim\left(\oplus_{\lambda\le 0}\mathcal{E}_\omega(\lambda;L,W^{1,2}(M))\right)\le \sup\{\dim\Pi\colon \Pi\le W^{1,2}(M),&\\ \text{is linear space such that, } (B_L)_{|\Pi}\le 0\}.&\end{aligned}\tag{3.27}$$

If the left-hand side is equal to 0 then the inequality is trivial. Thus, take $b\in\mathbb{Z}_{\ge 1}$, such that

$$b\le \dim\left(\oplus_{\lambda\le 0}\mathcal{E}_\omega(\lambda;L,W^{1,2}(M))\right),$$

then we have b, ω-weighted eigenvectors,

$$\{f_1,\ldots,f_b\}\subset W^{1,2}(M),$$

with respective non-positive eigenvalues $\{\lambda_1, \dots, \lambda_b\}$. By the argument at the begining of the proof, we may take the set $\{f_1, \dots, f_b\}$, to be orthonormal with respect to $\langle \cdot, \cdot \rangle_\omega$. Thus,

$$\Pi = \operatorname{span}\{f_1, \dots, f_b\}, \tag{3.28}$$

is a b-dimensional vector space, and for $a_1, \dots, a_b \in \mathbb{R}$, we have,

$$\begin{aligned} B_L\left[\sum_i a_i f_i, \sum_j a_j f_j\right] &= \sum_{i,j} a_i a_j \lambda_i \langle f_i, f_j \rangle_\omega, \\ &= \sum_i \lambda_i a_i^2, \\ &\leq 0. \end{aligned}$$

Thus, (3.27) holds. Identical argument shows that same inequality holds for unweighted eigenspaces.

We look to show reverse inequality of (3.27). As we know that (3.27) holds, if the left-hand side of (3.27) is unbounded, then equality holds trivially. Thus, we consider, $b \in \mathbb{Z}_{\geq 0}$,

$$b = \dim\left(\oplus_{\lambda \leq 0} \mathcal{E}(\lambda; L, W^{1,2}(M))\right) < \infty,$$

and prove reverse of (3.27) by contradiction. Indeed, assume that we have a linear subspace $\tilde{\Pi} \leq W^{1,2}(M)$, of dimension $b+1$, such that B_L is non-positive on $\tilde{\Pi}$. Consider the b dimensional linear subspace Π, identically defined as (3.28). Note that,

$$W^{1,2}(M) = \Pi \oplus \Pi^{\perp_\omega}.$$

The projection map $P_\Pi \colon \tilde{\Pi} \to \Pi$, must have a non-trivial kernel, implying that there exists a $v \in \tilde{\Pi} \cap \Pi^{\perp_\omega}$, with $\langle v, v \rangle_\omega = 1$. Thus,

$$\tilde{\lambda} = \inf\{B_L[f, f] \colon f \in \Pi^{\perp_\omega},\ \langle f, f \rangle_\omega = 1\} \leq 0.$$

Take $\tilde{f}_k \in \Pi^{\perp_\omega}$, $\langle \tilde{f}_k, \tilde{f}_k \rangle_\omega = 1$, such that

$$\tilde{\lambda} = \lim_{k \to \infty} B_L[\tilde{f}_k, \tilde{f}_k].$$

Thus, noting that $\operatorname{ess\,inf} \omega > 0$, and $|V| < +\infty$, we may deduce a lower bound on

$$\tilde{\lambda} \geq -\frac{\|V\|_{L^\infty(M)}}{\operatorname{ess\,inf} \omega},$$

and uniform $W^{1,2}(M)$ bounds for $\{\tilde{f}_k\}$, and conclude that there exists an $\tilde{f} \in W^{1,2}(M)$, such that after potentially taking a subsequence and renumerating,

$$\begin{cases} \tilde{f}_k \rightharpoonup \tilde{f}, \; in \, W^{1,2}(M), \\ \tilde{f}_k \to \tilde{f}, \; in \, L^2(M). \end{cases}$$

Denote,

$$\begin{aligned} h_1 &:= P_\Pi(\tilde{f}), \\ h_2 &:= \tilde{f} - h_1 \in \Pi^{\perp_\omega}. \end{aligned}$$

There exists $a_1, \ldots, a_b \in \mathbb{R}$, such that,

$$h_1 = \sum_{i=1}^{b} a_i f_i.$$

Then, as $\tilde{f}_k \in \Pi^{\perp_\omega}$,

$$\begin{aligned} B_L[\tilde{f}_k, \tilde{f}] &= \sum_{i=1}^{m} a_i B_L[\tilde{f}_k, f_i] + B_L[\tilde{f}_k, h_2], \\ &= \sum_{i=1}^{m} a_i \lambda_i \langle \tilde{f}_k, f_i \rangle_\omega + B_L[\tilde{f}_k, h_2], \\ &= B_L[\tilde{f}_k, h_2]. \end{aligned}$$

Thus, by weak convergence,

$$\begin{aligned} B_L[\tilde{f}, \tilde{f}] &= \lim_{k\to\infty} B_L[\tilde{f}_k, \tilde{f}], \\ &= \lim_{k\to\infty} B_L[\tilde{f}_k, h_2], \\ &= B_L[h_1 + h_2, h_2], \\ &= \sum_{i=1}^{m} a_i \lambda_i \langle f_i, h_2 \rangle_\omega + B_L[h_2, h_2], \\ &= B_L[h_2, h_2]. \end{aligned}$$

After potentially taking a further subsequence and renumerating so that, $\tilde{f}_k \to \tilde{f}$ pointwise a.e., by Fatou's Lemma we have the following,

$$0 \leq \alpha = \langle h_2, h_2 \rangle_\omega \leq \langle h_2, h_2 \rangle_\omega + \langle h_1, h_1 \rangle_\omega = \langle \tilde{f}, \tilde{f} \rangle_\omega \leq \liminf_{k\to\infty} \langle \tilde{f}_k, \tilde{f}_k \rangle_\omega = 1,$$

and we reduce our argument to three cases.

First consider, $\alpha = 0$. Thus, $h_2 = 0$, and by lower semicontinuity of the $W^{1,2}(M)$-norm under weak convergence,

$$0 = B_L[h_2, h_2] = B_L[\tilde{f}, \tilde{f}] \leq \lim_{k\to\infty} B_L[\tilde{f}_k, \tilde{f}_k] = \tilde{\lambda} \leq 0.$$

Therefore, $\tilde{\lambda} = 0$, and recalling the function $v \in \tilde{\Pi} \cap \Pi^{\perp_\omega}$, with $\langle v, v\rangle_\omega = 1$, we must have

$$B_L[v, v] = \inf\{B_L[f, f] \colon f \in \Pi^{\perp_\omega},\ \langle f, f\rangle_\omega = 1\} = 0.$$

Standard variational arguments then show that $v \in \mathcal{E}_\omega(0; L, W^{1,2}(M))$, which contradicts the definition b.

If $\alpha = 1$, then,

$$B_L[h_2, h_2] = \inf\{B_L[f, f] \colon f \in \Pi^{\perp_\omega},\ \langle f, f\rangle_\omega = 1\} = \tilde{\lambda}.$$

Again, standard variational arguments then show that $h_2 \in \mathcal{E}_\omega(\tilde{\lambda}; L, W^{1,2}(M))$, which contradicts the definition of b.

The final case is $0 < \alpha < 1$. Note that if $B_L[h_2, h_2] = 0$, then we may apply a similar argument to that in case $\alpha = 0$. Therefore, we may assume that $B_L[h_2, h_2] < 0$. Define,

$$h := \alpha^{-1/2} h_2.$$

Thus,

$$B_L[h, h] = \alpha^{-1} B_L[h_2, h_2] < B_L[h_2, h_2] \leq \tilde{\lambda}.$$

However, as $h \in \Pi^{\perp_\omega}$, and $\langle h, h\rangle_\omega = 1$, this is a contradiction. □

Remark 33. For an embedded hypersurface $M \subset N$, the method of proof in Proposition 14 may be applied to show that,

$$\operatorname{span}\left\{\cup_{\lambda\leq 0}\,\mathcal{E}_\omega(\lambda; L, W^{1,2}(co(M))^-)\right\} = \oplus_{\lambda\leq 0}\,\mathcal{E}_\omega(\lambda; L, W^{1,2}(co(M))^-),$$

and,

$$\dim\left(\oplus_{\lambda\leq 0}\,\mathcal{E}_{\omega_{k,\delta,R}}(\lambda; L, W^{1,2}(co(M))^-)\right) = \dim\left(\oplus_{\lambda\leq 0}\,\mathcal{E}(\lambda; L, W^{1,2}(co(M))^-)\right).$$

When reapplying this method the two major things to note is that $W^{1,2}(co(M))^-$ is a linear space, and that when applying Rellich–Kondrachov, we have that the limit will also lie $W^{1,2}(co(M))^-$.

Let (M, g) be a complete but not necessarily compact Riemannian manifold. Recall the following function space,

$$W^{1,2}_\omega(M) := \{f \in L^1_{\mathrm{loc}}(M) \colon |\nabla f| \in L^2(M), \text{ and } f^2\omega \in L^1(M)\}.$$

Proposition 15. *Let Σ be a, connected, complete n-dimensional manifold, $n \geq 3$, and $\iota : \Sigma \to \mathbb{R}^{n+1}$ be a two-sided, proper, minimal immersion, with finite total curvature,*

$$\int_\Sigma |A_\Sigma|^n < +\infty,$$

and Euclidean volume growth at infinity. Consider a function $\omega \in L^\infty(\Sigma)$, such that there exists $\Lambda \in [1, \infty)$, and $R \in (0, \infty)$, such that $\operatorname{ess\,inf} \omega > 0$ *on $\Sigma \cap \iota^{-1}(\overline{B_R^{n+1}(0)})$, and*

$$\frac{1}{\Lambda |\iota(x)|^2} \leq \omega(x) \leq \frac{\Lambda}{|\iota(x)|^2},$$

for $x \in \Sigma \setminus \iota^{-1}(B_R^{n+1}(0))$. Then

$$\textit{span}\,\{\cup_{\lambda \leq 0} \mathcal{E}_\omega(\lambda; L_\Sigma, W^{1,2}_\omega(\Sigma), W^{1,2}(\Sigma))\} = \oplus_{\lambda \leq 0} \mathcal{E}_\omega(\lambda; L_\Sigma, W^{1,2}_\omega(\Sigma), W^{1,2}(\Sigma)), \tag{3.29}$$

and,

$$\begin{aligned} \textit{dim}\,(\oplus_{\lambda < 0} \mathcal{E}_\omega(\lambda; L_\Sigma, W^{1,2}_\omega(\Sigma), W^{1,2}(\Sigma))) &= \textit{anl-ind}\,(\Sigma) \\ &:= \lim_{S \to \infty} \textit{anl-ind}_{\iota^{-1}(B_S^{n+1}(0))}(\Sigma) \\ &:= \lim_{S \to \infty} \textit{dim}\,(\oplus_{\lambda < 0} \mathcal{E}(\lambda; L_\Sigma, W^{1,2}_0(\Sigma \cap \iota^{-1}(B_S^{n+1}(0)))) \end{aligned}$$

Remark 34. In the literature, for a two-sided, properly immersed minimal hypersurface Σ, the analytic index (anl-ind$\,(\Sigma)$) and analytic nullity (anl-nul$\,(\Sigma)$), are just referred to as the index and nullity of Σ. We choose to maintain the terms of analytic index and analytic nullity to keep the notation and definitions consistent throughout the chapter.

Proof. Denote the stability operator on Σ by $L = L_\Sigma$. As the immersion is proper, $\operatorname{ess\,inf} \omega > 0$, on compact sets of Σ, and thus $W^{1,2}_\omega(\Sigma) \subset W^{1,2}_{\text{loc}}(\Sigma)$. Moreover, as Σ has finite total curvature, this implies that,

$$\lim_{S \to \infty} \int_{\Sigma \cap \iota^{-1}(B_S^{n+1}(0))} |A_\Sigma|^n \to 0.$$

Thus, by applying the curvature estimate of Proposition 7, we have that there exists an $S_0 > 0$, such that for $x \in \Sigma \setminus \iota^{-1}(B_{2S_0}^{n+1}(0))$,

$$|A_\Sigma|^2(x) \leq \frac{1}{(|\iota(x)| - S_0)^2} \leq \frac{4}{|\iota(x)|^2}. \tag{3.30}$$

Moreover, choosing $S_0 \geq R$, we have that for $x \in \Sigma \setminus \iota^{-1}(B_{2S_0}^{n+1}(0))$,

$$|A_\Sigma|^2(x) \leq 4\Lambda\omega.$$

Thus, for $f,\, h\in W^{1,2}_\omega(\Sigma)$,

$$\left|\int_\Sigma f\,h\,\omega\right| + \left|\int_\Sigma |A_\Sigma|^2\,f\,h\right| < (1+4\Lambda)\left(\int_\Sigma f^2\,\omega\right)^{1/2}\left(\int_\Sigma h^2\,\omega\right)^{1/2} < +\infty.$$

Therefore the quantities, $B_L[f,h]$, and $\langle f,h\rangle_\omega$, are finite and well defined for $f,\, h\in W^{1,2}_\omega(\Sigma)$. We also note that for $\lambda\in\mathbb{R}$, such that

$$\dim\left(\mathcal{E}_\omega(\lambda; L_\Sigma, W^{1,2}_\omega(\Sigma), W^{1,2}(\Sigma))\right)\neq 0,$$

then, similar to (3.11) we may deduce $\lambda\geq -C$, with $C=C(\Sigma,\iota,R)<+\infty$.

Recall the following function,

$$\begin{cases}\chi\in C^\infty(\mathbb{R};[0,1]),\\ \chi(t)=1, & t\in(-\infty,1],\\ \chi(t)=0, & t\in[2,\infty),\\ -3\leq\chi'(t)\leq 0.\end{cases}$$

For large $S>0$, we then define the following smooth function,

$$\chi_S(x)=\chi\left(\frac{|\iota(x)|}{S}\right).$$

For $f\in W^{1,2}_\omega(\Sigma)$, we define $f_S=f\chi_S\in W^{1,2}_0(\Sigma\cap\iota^{-1}(B^{n+1}_{2S}(0)))$. Again using the estimate (3.30), we may deduce that for $f,\, h\in W^{1,2}_\omega(\Sigma)$,

$$\begin{aligned}\lim_{S\to\infty} B_L[f,h_S] &= B_L[f,h],\\ \lim_{S\to\infty}\langle f,h_S\rangle_\omega &= \langle f,h\rangle_\omega.\end{aligned}$$

Thus if $f\in(\mathcal{E}_\omega(\lambda; L, W^{1,2}_\omega(\Sigma), W^{1,2}(\Sigma)))$, then in fact, for all $h\in W^{1,2}_\omega(\Sigma)$, $B_L[f,h]=\lambda\langle f,h\rangle_\omega$. This allows us to conclude (3.29) in an identical way to (3.26).

We now proceed similarly to Proposition 14. First recall that as Σ has finite total curvature and Euclidean volume growth at infinity, its (analytic) index is finite [61, Section 3] (cf. [36]). Thus we may pick an $S_0>0$, such that

$$\text{anl-ind}_{\iota^{-1}(B^{n+1}_{S_0}(0))}(\Sigma)=\text{anl-ind}(\Sigma). \tag{3.31}$$

First we show that,

$$\dim\left(\oplus_{\lambda<0}\,\mathcal{E}_\omega(\lambda; L_\Sigma, W^{1,2}_\omega(\Sigma), W^{1,2}(\Sigma))\right) \leq \text{anl-ind}\,(\Sigma) = I. \tag{3.32}$$

If the left hand side is equal to 0, then the inequality is trivial, thus assume we have $b \in \mathbb{Z}_{\geq 1}$, such that,

$$b \leq \dim\left(\oplus_{\lambda<0}\,\mathcal{E}_\omega(\lambda; L_\Sigma, W^{1,2}_\omega(\Sigma), W^{1,2}(\Sigma))\right).$$

Therefore, as in Proposition 14, we may pick a set of eigenfunctions,

$$\{f_1, \ldots, f_b\} \subset \oplus_{\lambda<0}\,\mathcal{E}_\omega(\lambda; L, W^{1,2}_\omega(\Sigma), W^{1,2}(\Sigma)),$$

which are orthonormal with respect to $\langle\,\cdot,\cdot\,\rangle_\omega$. The inequality then follows noting that for large enough $S > S_0$,

$$\text{span}\{(f_1)_S, \ldots, (f_b)_S\} \subset W^{1,2}_0(\Sigma \cap B^{n+1}_{2S}(0)),$$

is a b-dimensional subspace on which B_L is negative definite.

For the reverse of (3.32), we take $I \geq 1$ (note that if $I = 0$ then (3.32) implies equality), and an increasing sequence $S_k \to \infty$. For large enough S_k, as $\Sigma \cap \iota^{-1}(B^{n+1}_{S_k}(0))$ is compact with smooth boundary, by identical arguments to those contained in Proposition 14, there exists a sequences $\lambda^k_1 \leq \cdots \leq \lambda^k_I < 0$ and a set,

$$\{f^k_1, \ldots, f^k_I\} \subset W^{1,2}_0(\Sigma \cap \iota^{-1}(B^{n+1}_{S_k}(0))),$$

which is orthonormal with respect to $\langle\,\cdot,\cdot\,\rangle_\omega$, such that for all $\varphi \in W^{1,2}_0(\Sigma \cap \iota^{-1}(B^{n+1}_{S_k}(0)))$,

$$B_L[f^k_i, \varphi] = \lambda^k_i \langle f^k_i, \varphi\rangle_\omega.$$

By standard theory (see [31, Lemma 3.7]), for each $i = 1, \ldots, I$, $\lambda^k_i \geq \lambda^{k+1}_i$. Then recalling our uniform bound $\lambda^k_i \geq -C$, for each $i = 1, \ldots, I$, there exists a $\lambda^\infty_i \in (0, -C]$, such that $\lambda^k_i \to \lambda^\infty_i$. By similar arguments contained in Section 3.2.3 and Claim 7, we may deduce that after potentially taking a subsequence and renumerating, for each $i = 1, \ldots, I$, there exists an $f^\infty_i \in W^{1,2}_\omega(\Sigma)$, such that,

$$\begin{cases} f^k_i \rightharpoonup f^\infty_i, & W^{2,2}_{\text{loc}}(\Sigma), \\ f^k_i \to f^\infty_i, & W^{1,2}_{\text{loc}}(\Sigma), \end{cases}$$

and $f^\infty_i \in \mathcal{E}_\omega(\lambda^\infty_i; L_\Sigma, W^{1,2}_\omega(\Sigma), W^{1,2}(\Sigma))$. We now look to show that

$$\dim\left(\text{span}\{f^\infty_1, \ldots, f^\infty_I\}\right) = I,$$

which will complete the proof.

For a contradiction, assume not. Then there exists $a_1, \dots, a_I \in \mathbb{R}$, with $\sum a_i^2 = 1$, such that,

$$h \coloneqq \sum_{i=1}^{I} a_i\, f_i^\infty = 0.$$

We then define,

$$h_k \coloneqq \sum_{i=1}^{I} a_i\, f_i^k \in W_0^{1,2}(\Sigma \cap \iota^{-1}(B_{S_k}^{n+1}(0))),$$

and note that $\langle h_k, h_k \rangle_\omega = 1$ for all k, and $h_k \to 0$ in $W^{1,2}_{\mathrm{loc}}(\Sigma)$. By our choice of S_0 (3.31), we have that $\Sigma \setminus \iota^{-1}(\overline{B_{S_0}^{n+1}(0)})$ is stable, however, $\chi_{S_0} h_k \in W_0^{1,2}(\Sigma \setminus \iota^{-1}(\overline{B_{S_0}^{n+1}(0)}))$, and for large enough k,

$$B_L[\chi_{S_0} h_k, \chi_{S_0} h_k] < 0,$$

which clearly contradicts the stability of $\Sigma \setminus \iota^{-1}(\overline{B_{S_0}^{n+1}(0)})$. □

3.4 Proof of the Theorem

Again, for ease of notation we only write the proof of Theorem 8 for the case of minimal hypersurfaces, however the identical argument works for the case of H-CMC hypersurfaces.

We follow similar arguments to those in [23, Lemma IV.6] and [32, Theorem 1.2].

If $\limsup_{k\to\infty}(\mathrm{ind}(M_k) + \mathrm{nul}(M_k)) = 0$, then the conclusion of Theorem 8 is trivial. Suppose for $b \in \mathbb{Z}_{\geq 1}$,

$$b \leq \limsup_{k\to\infty}(\mathrm{ind}(M_k) + \mathrm{nul}(M_k)) = \limsup_{k\to\infty} \left(\dim\left(\oplus_{\lambda \leq 0}\, \mathcal{E}_{\omega_{k,\delta,R}}(\lambda; L_k, W^{1,2}(co(M_k))^-)\right)\right),$$

where the equality comes from equivalence of considering the weighted and unweighted eigenvalue problems along our sequence (Proposition 14 and Remark 33). After potentially taking a subsequence and renumerating we have that for each k, there exists a linear subspace,

$$W_k \coloneqq \mathrm{span}\{f_{i,k}\}_{i=1}^b \subset W^{1,2}(co(M_k))^-,$$

where, for each $i = 1, \dots, b$, there is a $\lambda_{i,k} \leq 0$, such that,

$$f_{i,k} \in \mathcal{E}_{\omega_{k,\delta,R}}(\lambda_{i,k}; L_k, W^{1,2}(co(M_k))^-),$$

and the set $\{f_{i,k}\}_{i=1}^b$, is orthonormal with respect to the $\omega_{k,\delta,R}$-weighted L^2 inner product, i.e. for

$i,\, j = 1, \ldots, b,$

$$\int_{M_k} f_{i,k}\, f_{j,k}\, \omega_{k,\delta,R} = \delta_{ij}.$$

We may assume this by the argument used to prove (3.26). Thus, as outlined in Section 3.2.4, for each $i = 1, \ldots, b$, after potentially taking a subsequence and renumerating,

$$f_{i,k} \to ((f^1_{i,\infty}, \ldots, f^m_{i,\infty}), f^{\Sigma^1}_{i,\infty}, \ldots, f^{\Sigma^J}_{i,\infty}) \in E_\infty$$

where we are defining,

$$\begin{aligned} E_\infty \;&:=\; \left(\times_{j=1}^m \left(\oplus_{\lambda \le 0} \mathcal{E}_{\omega_\delta}(\lambda; L_\infty, W^{1,2}(co(M_\infty)))\right)\right) \\ &\quad \times \left(\times_{j=1}^J \left(\oplus_{\lambda \le 0} \mathcal{E}_{\omega_{\Sigma^j,R}}(\lambda; L_{\Sigma^j}, \dot{W}^{1,2}_{\omega_{\Sigma^j,R}}(\Sigma^j), W^{1,2}(\Sigma^j))\right)\right). \end{aligned}$$

We define the linear map,

$$\begin{aligned} \Pi_k \colon W_k \;&\to\; E_\infty, \\ f_{i,k} \;&\mapsto\; ((f^1_{i,\infty}, \ldots, f^m_{i,\infty}), f^{\Sigma^1}_{i,\infty}, \ldots, f^{\Sigma^J}_{i,\infty}). \end{aligned}$$

Thus $W_\infty := \Pi_k(W_k)$ is a linear subspace of E_∞.

Define the integer

$$co(m) = \liminf_{r \to 0} \liminf_{k \to \infty} |\{\text{connected components of } M_k \setminus \cup_{y \in \mathcal{I}} B^N_r(y)\}| \le m.$$

If M_∞ is two-sided, by the graphical convergence on sets, compactly contained away from the finite collection of points $\mathcal{I}$, we have that $co(m) = m$. If M_∞ is one-sided, taking r small enough and k large enough such that,

$$co(m) = |\{\text{connected components of } M_k \setminus \cup_{y \in \mathcal{I}} B^N_r(y)\}|,$$

we recall notation from Section 3.2.2, and we have

$$M^r_k = \bigcup_{l=1}^{co(m)} M^{l,r}_k,$$

where each $M^{l,r}_k$ is a connected hypersurface. Then, $\cup_{j=1}^m \{(x, u^{j,r}_k(x)) \colon x \in \Omega_r\}$ is a double cover of M^r_k with trivial normal bundle, implying that we identify (as in Section 3.2.2)

$$\cup_{j=1}^m \{(x, u^{j,r}_k(x)) \colon x \in \Omega_r\} = \cup_{l=1}^{co(m)} o(M^{l,r}_k).$$

If there is an $l \in \{1, \ldots, co(m)\}$, and $j \neq J \in \{1, \ldots, m\}$, such that for all large enough k,

$$o(M_k^{l,r}) = \{(x, u_k^{j,r}(x)) \colon x \in \Omega_r\} \cup \{(x, u_k^{J,r}(x)) \colon x \in \Omega_r\},$$

then (depending on the choice of unit normal in Section 3.2.2), for each $i = 1, \ldots, m$, either

$$f_{i,\infty}^j((x,\nu)) = -f_{i,\infty}^J((x,-\nu)), \text{ for all } (x,\nu) \in co(M_\infty) \setminus \iota^{-1}(\mathcal{I}),$$

or,

$$f_{i,\infty}^j((x,\nu)) = f_\infty^J((x,-\nu)), \text{ for all } (x,\nu) \in co(M_\infty) \setminus \iota^{-1}(\mathcal{I}).$$

Thus we may define an injective map

$$P \colon W_\infty \to F_\infty,$$

where,

$$\begin{aligned} F_\infty &= \left(\times_{l=1}^{co(m)} \left(\oplus_{\lambda \leq 0} \mathcal{E}_{\omega_\delta}(\lambda; L_\infty, W^{1,2}(co(M_\infty)))\right)\right) \\ &\quad \times \left(\times_{j=1}^{L} \left(\oplus_{\lambda \leq 0} \mathcal{E}_{\omega^{\Sigma^j},R}(\lambda; L_{\Sigma^j}, W^{1,2}_{\omega^{\Sigma^j},R}(\tilde{\Sigma}^j), W^{1,2}(\Sigma^j))\right)\right) \end{aligned}$$

Claim 8. Π_k *is injective*

Proof. We prove by contradiction. Assume we have an h_k,

$$h_k = \sum_{i=1}^{b} a_i f_{i,k},$$

with $\sum_{i=1}^{b} a_i^2 = 1$, and $\Pi_k(h_k) = 0$. This implies that for all k,

$$\int_{co(M_k)} h_k^2 \, \omega_{k,\delta,R} = 1,$$

and,

$$h_k \to ((0, \ldots, 0), 0, \ldots, 0),$$

which contradicts Claim 7. □

Thus,

$$\dim W_\infty = \dim W_k = b,$$

which implies that, $b \leq \dim F_\infty$. We conclude Theorem 8 by combining the results in Section 3.3

(Proposition 14 and Proposition 15), and noting that

$$\mathcal{E}_{\omega_{\Sigma^j,R}}(0; L_{\Sigma^j}, W^{1,2}_{\omega_{\Sigma^i,R}}(\Sigma^j), W^{1,2}(\Sigma^j)) = \mathcal{E}_{\omega^{\Sigma^j,R}}(0; L_{\Sigma^j}, W^{1,2}_{\omega^{\Sigma^j,R}}(\Sigma^j), W^{1,2}(\Sigma^j)).$$

for $\omega_{\Sigma^j,R}$ as defined in the statement of Theorem 8, and $\omega^{\Sigma^j,R}$ as defined in (3.16). Lastly, by standard regularity theory for elliptic PDEs, we note that,

$$\begin{aligned} \mathrm{nul}_{\omega_{\Sigma^j,R}}(\Sigma^j) &:= \dim\left(\{\psi \in C^\infty(\Sigma) \cap W^{1,2}_{\omega_{\Sigma^j,R}}(\Sigma^j) \colon L_{\Sigma^j}\psi = 0\}\right), \\ &= \dim\left(\mathcal{E}_{\omega^{\Sigma^i,R}}(0; L_{\Sigma^i}, W^{1,2}_{\omega_{\Sigma^i,R}}(\Sigma^j), W^{1,2}(\Sigma^j))\right). \end{aligned}$$

3.5 Finiteness of the Nullity

Proposition 16. *Let Σ be a complete, connected, n-dimensional manifold, $n \geq 3$, and*

$$\iota \colon \Sigma \to \mathbb{R}^{n+1}$$

be a proper, two-sided, minimal immersion, of finite total curvature

$$\int_\Sigma |A_\Sigma|^n < +\infty,$$

and Euclidean volume growth at infinity

$$\limsup_{R\to\infty} \frac{\mathcal{H}^n(\iota(\Sigma) \cap B^{n+1}_R(0))}{R^n} < +\infty.$$

Consider a function $\omega \in L^\infty(\Sigma)$, such that there exists an $R > 0$, and $\Lambda \geq 1$, such that $\operatorname{ess\,inf} \omega > 0$ *in $\iota^{-1}(B^{n+1}_R(0))$, and for $x \in \Sigma \setminus \iota^{-1}(B^{n+1}_R(0))$,*

$$\frac{1}{\Lambda|\iota(x)|^2} \leq \omega \leq \frac{\Lambda}{|\iota(x)|^2}.$$

Then

$$\textit{anl-nul}_\omega(\Sigma) := \dim\{\psi \in W^{1,2}_\omega(\Sigma) \colon L_\Sigma\psi = 0\} < +\infty.$$

Proof. We assume the statement does not hold and prove by contradiction. There exists a set, $\{\psi_1, \psi_2, \ldots\} \subset W^{1,2}_\omega(\Sigma) \cap C^\infty(\Sigma)$, such that for all $k, l \in \mathbb{Z}_{\geq 1}$,

$$\Delta\psi_k + |A_\Sigma|^2\psi_k = 0,$$

and,

$$\int_\Sigma \psi_k\, \psi_l\, \omega = \delta_{kl}.$$

Claim. For any $\delta > 0$, there exists k, $l \in \mathbb{Z}_{\geq 1}$, $k \neq l$, such that

$$\|\psi_k - \psi_l\|^2_{W^{1,2}(\Sigma\cap\iota^{-1}(B^{n+1}_{2S}(0)))} < \delta.$$

Proof. (of Claim) Fix $S > 0$, for all $k \in \mathbb{Z}_{\geq 1}$,

$$\int_{\Sigma\cap\iota^{-1}(B^{n+1}_{3S}(0))} \psi_k^2 \leq C(S) < +\infty.$$

By standard interior estimates for linear elliptic PDEs we have,

$$\|\psi_k\|_{W^{2,2}(\Sigma\cap\iota^{-1}(B^{n+1}_{2S}(0))} \leq C(S) < +\infty.$$

Therefore, there exists a subsequence $\{\psi_{k'}\} \subset \{\psi_k\}$, and a function $\psi_\infty \in W^{2,2}(\Sigma \cap \iota^{-1}(B^{n+1}_{2S}(0)))$ such that,

$$\begin{cases} \psi_{k'} \rightharpoonup \psi_\infty, & \text{weakly in } W^{2,2}(\Sigma \cap \iota^{-1}(B^{n+1}_{2S}(0))), \\ \psi_{k'} \to \psi_\infty, & \text{strongly in } W^{1,2}(\Sigma \cap \iota^{-1}(B^{n+1}_{2S}(0))). \end{cases}$$

Thus this subsequence is Cauchy, so for any $\delta > 0$, there exists l, $k \in \mathbb{Z}_{\geq 1}$, $l \neq k$, such that

$$\|\psi_k - \psi_l\|^2_{W^{1,2}(\Sigma\cap\iota^{-1}(B^{n+1}_{2S}(0)))} < \delta$$

□

For $\delta > 0$ fixed we denote,

$$\psi_\delta = \psi_l - \psi_k \in W^{1,2}_\omega(\Sigma),$$

and note that, $\Delta\psi_\delta + |A_\Sigma|^2\psi_\delta = 0$, and,

$$\int_\Sigma \psi_\delta^2\,\omega = 2.$$

Also, $B_\Sigma[\psi_\delta, \psi_\delta] = 0$, which may be seen by following the argument at the begining of Proposition 15.

Recall from Remark 29, that Σ has finitely many ends (say $m \in \mathbb{Z}_{\geq 1}$), which, for large enough $S \geq R$, may be denoted by,

$$\sqcup_{i=1}^m E^i = \Sigma \setminus \iota^{-1}(B^{n+1}_S(0)).$$

Moreover, each end E^i is graphical over some hyperplane minus a compact set B_i (with the graphing function having small gradient), and for each $\varepsilon > 0$, we may further choose $S = S(\Sigma, \iota, R, \Lambda, \varepsilon) < +\infty$, such that,

$$|A_\Sigma|^2(x) \leq \varepsilon\omega,$$

for $x \in \Sigma \setminus \iota^{-1}(\overline{B}_S^{n+1}(0))$. We also remark that $\omega \in L(n/2, \infty)(\Sigma)$ (this can be shown similarly to Claim 6).

Fixing a choice of $S = S(\Sigma, \iota, \Lambda, R, \varepsilon) < +\infty$, for any $\zeta > 0$, we may pick $\delta = \delta(S, \zeta) > 0$, then $T = T(\delta) > 4S$, such that

$$\|\psi_\delta\|_{W^{1,2}\left(\Sigma \cap \iota^{-1}(B_{2S}^{n+1}(0))\right)} < \zeta, \quad \text{and,} \quad \|\nabla \psi_\delta\|_{L^2\left(\Sigma \setminus \iota^{-1}(B_T^{n+1}(0))\right)} + \left(\int_{\Sigma \setminus \iota^{-1}(B_T^{n+1}(0))} \psi_\delta^2 \, \omega \right)^{1/2} < \zeta.$$

Recalling definition of functions χ_S and χ_T from Proposition 15, we define

$$\Psi_\delta = \chi_T(1 - \chi_S)\psi_\delta \in W_0^{1,2}(\iota^{-1}(B_{2T}^{n+1}(0) \setminus \overline{B_S^{n+1}(0)})).$$

Performing similar computations to those in Claim 7 we deduce,

$$\int_\Sigma (\psi_\delta^2 - \Psi_\delta^2)\omega \leq C \left(\int_{\Sigma \cap \iota^{-1}(B_{2S}^{n+1}(0))} \psi_\delta^2 + \int_{\Sigma \setminus \iota^{-1}(B_T^{n+1}(0))} \psi_\delta^2 \, \omega \right) < C\zeta,$$

and,

$$\begin{aligned} |B_L[\psi_\delta, \psi_\delta] - B_L[\Psi_\delta, \Psi_\delta]| \;&\leq\; C \Bigg(\|\psi_\delta\|^2_{W^{1,2}\left(\Sigma \cap \iota^{-1}(B_{2S}^{n+1}(0))\right)} \\ &\qquad + \|\nabla \psi_\delta\|^2_{L^2\left(\Sigma \setminus \iota^{-1}(B_T^{n+1}(0))\right)} + \int_{\Sigma \setminus \iota^{-1}(B_T^{n+1}(0))} \psi_\delta^2 \, \omega \Bigg) \\ &<\; C\zeta, \end{aligned}$$

with $C = C(\Sigma, \iota, \omega, R) < +\infty$. Now, choosing small enough $\varepsilon = \varepsilon(\omega) > 0$, large enough $S = S(\Sigma, \iota, \Lambda, R, \varepsilon)$, then small enough $\zeta = \zeta(\Sigma, \iota, \omega, R, \varepsilon) > 0$, $\delta = \delta(S, \zeta) > 0$, and large enough $T = T(\delta) > 4S$, Ψ_δ will derive a contradiction to Lemma 5 (in a similar fashion to Claim 7) on at least one of the ends $E^1, \ldots, E^m$. $\square$

Corollary 3 may then be concluded by noting that,

$$\text{anl-nul}\,(\Sigma) \coloneqq \dim \{\psi \in W^{1,2}(\Sigma) \colon L_\Sigma \psi = 0\} \leq \dim \{\psi \in W_\omega^{1,2}(\Sigma) \colon L_\Sigma \psi = 0\} < +\infty.$$

3.6 Jacobi Fields on the Higher Dimensional Catenoid

In this section we analyse Jacobi fields on the n-dimensional catenoid, $n \geq 3$. First, we briefly recall the definition of the n-dimensional catenoid for $n \geq 3$ (as in [58, Section 2]). For $h_0 > 0$, consider the following integral, for $n \geq 3$,

$$s(h) = \int_{h_0}^{h} \frac{d\tau}{(a\tau^{2(n-1)} - 1)^{1/2}}, \tag{3.33}$$

with $a = h_0^{-2(n-1)}$. Then the function $s(h)$ is increasing and maps $[h_0, +\infty)$ to $[0, s_\infty)$, with

$$s_\infty = \int_{h_0}^{\infty} \frac{d\tau}{(a\tau^{2(n-1)} - 1)^{1/2}} < \infty.$$

Thus, the inverse of s, $h\colon [0, s_\infty) \to [h_0, +\infty)$ is well defined, with $h(0) = h_0$, and $h'(0) = 0$. We then smoothly extend h as an even function across $(-s_\infty, s_\infty)$. Now, letting S^{n-1} denote the unit sphere in $\mathbb{R}^n$, we define the catenoid, $\mathcal{C}$, in $\mathbb{R}^{n+1}$, $n \geq 3$, by the embedding,

$$\begin{aligned} F\colon (-s_\infty, s_\infty) \times S^{n-1} &\to \mathbb{R}^{n+1}, \\ (s, w) &\mapsto (h(s)w, s). \end{aligned}$$

For a point $y = F(s, w)$, the unit normal to $\mathcal{C}$ at y is given by,

$$\nu(y) = \frac{(w, -h'(s))}{(1 + (h')^2)^{1/2}}.$$

It was shown by Schoen [50, Theorem 3], up to rotations, tranlations and scalings, the catenoid $\mathcal{C}$, is the unique complete, non-flat minimal hypersurface in $\mathbb{R}^{n+1}$, with two ends, which is regular at infinity (for a definition of regular at infinity see [50, pp. 800]). Recall our weight, $\omega_{\mathcal{C},R}$, which is given by

$$\omega_{\mathcal{C},R}(s, w) = \begin{cases} (h(s_1)^2 + s_1^2)^{-1}, & s \in (-s_1, s_1), \\ (h(s)^2 + s^2)^{-1}, & s \in (-s_\infty, s_1] \cup [s_1, s_\infty), \end{cases}$$

for $s_1 \in (0, s_\infty)$, given by $R^2 = h(s_1)^2 + s_1^2$.

We now look at jacobi fields on $\mathcal{C}$,

$$JF_{\mathcal{C}} \coloneqq \{f \in C^\infty(\mathcal{C})\colon \Delta_{\mathcal{C}} f + |A_{\mathcal{C}}|^2 f = 0\}.$$

In particular we will focus on elements of $JF_{\mathcal{C}}$ which are generated through rigid motions in $\mathbb{R}^{n+1}$ (translations, scalings and rotations of $\mathcal{C}$), and look to see which of them lie in $W^{1,2}(\mathcal{C})$ and $W^{1,2}_{\omega_{\mathcal{C},R}}(\mathcal{C})$.

We note that it is still an open question whether these Jacobi fields generated through rigid motions account for all the Jacobi fields on $\mathcal{C}$. For the case of $n = 2$, this is known to be true for the Costa–Hoffman–Meeks minimal surfaces [43, Theorem 2 and Corollary 3].

We will denote the Jacobi fields on $\mathcal{C}$ defined through rigid motions by $RMJF_{\mathcal{C}}$. Those arising through translations are generated by the span of $f_1, \ldots, f_{n+1}$, where

$$f_i(y) \coloneqq \langle \nu(y), e_i \rangle = \begin{cases} \langle (w,0), e_i \rangle \, (1+(h')^2)^{-1/2}, & i = 1, \ldots, n, \\ -h' \, (1+(h')^2)^{-1/2}, & i = n+1. \end{cases} \tag{3.34}$$

The 1-dimensional subspace of $RMJF_{\mathcal{C}}$ generated through scaling has a basis element given by,

$$f_d(y) \coloneqq \langle \nu(y), y \rangle = \frac{h - sh'}{(1+(h')^2)^{1/2}}. \tag{3.35}$$

The last group to consider are those generated through rotations. Rotations of $\mathbb{R}^{n+1}$ about the origin are given by the special orthogonal group,

$$SO(n+1) = \{R \in GL(n+1) \colon \det R = 1,\ R^{-1} = R^T\},$$

where $GL(n+1)$ denotes the general linear group of all $(n+1) \times (n+1)$ real matrices. Then a smooth rotation of $\mathbb{R}^{n+1}$, is given by a smooth curve,

$$\gamma \colon [0,T] \to SO(n+1),$$

with $\gamma(0) = Id$ (the identity). Then the Jacobi field on $\mathcal{C}$ generated by the 1-parameter family of catenoids, $\gamma(t)(\mathcal{C})$, is

$$f_\gamma(y) = \langle \nu(y), \gamma'(0) y \rangle,$$

where $\gamma'(0) \in T_{Id}SO(n+1)$. One may show that,

$$T_{Id}SO(n+1) = \{R \in GL(n+1) \colon R^T = -R\}.$$

Taking $R = (R_{ij})_{ij} \in T_{Id}SO(n+1)$, we have that,

$$\begin{aligned} Ry &= R(hw, s), \\ &= \sum_{i=1}^{n+1} \left(\sum_{j=1}^{n} h R_{ij} w_j + R_{i,n+1} s \right) e_i, \end{aligned}$$

and,

$$\langle \nu(y), \gamma'(0) y \rangle = (1+(h')^2)^{-1/2} \left(h \sum_{i,j=1}^{n} R_{ij} w_i w_j + \sum_{i=1}^{n} R_{i,n+1} s w_i - hh' \sum_{j=1}^{n} R_{n+1,j} w_j - R_{n+1,n+1} h' s \right).$$

As $R^T = -R$, this implies, $R_{ii} = 0$, and

$$\sum_{i,j=1}^{n} R_{ij} w_i, w_j = 0.$$

Thus, for $y = F(s, w)$ we have

$$f_\gamma(y) = (1 + (h')^2)^{-1/2} \left((s + hh') \sum_{i=1}^{n} R_{i,n+1} w_i \right)$$

We now look to see which elements of $RMJF_{\mathcal{C}}$ lie in either $W^{1,2}(\mathcal{C})$, or $W^{1,2}_{\omega_{\mathcal{C},R}}(\mathcal{C})$. The following proposition tells us that to check if an element of $JF_{\mathcal{C}}$ lies in $W^{1,2}(\mathcal{C})$ (resp. $W^{1,2}_{\omega_{\mathcal{C},R}}(\mathcal{C})$), we only need to check if it lies in $L^2(\mathcal{C})$ (resp. $L^2_{\omega_{\mathcal{C},R}}(\mathcal{C})$).

Proposition 17. *For $n \geq 3$, let Σ be a complete n-dimensional manifold, and $\iota \colon \Sigma \to \mathbb{R}^{n+1}$ be a two-sided, proper, minimal immersion, with finite total curvature, and Euclidean volume growth at infinity. Consider a positive continuous function $\omega \in L^\infty(\Sigma)$, such that there exists an $R > 0$, $\Lambda \geq 1$, such that for $x \in \Sigma \setminus \iota^{-1}(B^{n+1}_R(0))$,*

$$\frac{1}{\Lambda |\iota(x)|^2} \leq \omega(x).$$

Then, for $f \in C^\infty(\Sigma)$, with $\Delta_\Sigma f + |A_\Sigma|^2 f = 0$, if $f \in L^2_\omega(\Sigma)$, then $f \in W^{1,2}_\omega(\Sigma)$.

In particular, if we set $\omega = 1$, then if a Jacobi field of Σ lies in $L^2(\Sigma)$, then it in fact lies in $W^{1,2}(\Sigma)$.

Proof. As previously discussed in Remark 29, the finite total curvature assumption, along with the Euclidean volume growth at infinity implies that Σ has regular ends, and that there exists a $C < +\infty$, such that

$$|A_\Sigma|^2 \leq C\omega.$$

Now fix any $S > R > 0$, and recall the function χ_S from Proposition (15). Then we have that,

$$\begin{aligned} \int \chi_S^2 |\nabla f|^2 &= \int |A|^2 \chi_S^2 f^2 - \int 2(\chi_S \nabla f) \cdot (f \nabla \chi_S), \\ &\leq C \int f^2 \omega + \frac{1}{2} \int \chi_S^2 |\nabla f|^2 + 2 \int f^2 |\nabla \chi_S|^2. \end{aligned}$$

Thus,

$$\int_{\iota^{-1}(B^{n+1}_S(0))} |\nabla f|^2 \leq (2C + 36\Lambda) \int f^2 \omega.$$

As the upper bound is finite and independent of S, we have that $|\nabla f| \in L^2(\Sigma)$. ☐

Consider the pull back of the Euclidean metric to $(-s_\infty, s_\infty) \times S^{n-1}$ by F,

$$g = (1 + (h')^2)\, ds^2 + h^2\, g_{S^{n-1}},$$

where $g_{S^{n-1}}$ is the standard round metric on S^{n-1}. We have,

$$\sqrt{|g|} = h^{n-1}\sqrt{1 + (h')^2}.$$

We now compute the L^2-norm of our elements of $RMJF_{\mathcal{C}}$.

Starting with the translations (3.34), for $i = 1, \ldots, n$,

$$\begin{aligned} \int f_i^2 &= \int_{-s_\infty}^{s_\infty} \int_{S^{n-1}} f_i^2 \sqrt{|g|}\, dw\, ds, \\ &= \left(\int_{-s_\infty}^{s_\infty} \frac{h^{n-1}}{(1 + (h')^2)^{1/2}}\, ds \right) \left(\int_{S^{n-1}} w_i\, dw \right). \end{aligned}$$

Differentiating (3.33), we have that

$$h' = (a h^{2(n-1)} - 1)^{1/2}, \tag{3.36}$$

which implies that,

$$\int f_i^2 = \frac{2 s_\infty}{a^{1/2}} \int_{S^{n-1}} w_i^2\, dw < +\infty.$$

Thus by Proposition 17 we have that $f_i \in W^{1,2}(\mathcal{C}) \subset W^{1,2}_{\omega_{\mathcal{C},R}}(\mathcal{C})$. For f_{n+1}, we have that

$$|f_{n+1}|(s) \to 1,$$

as $|s| \to s_\infty$, implying that there exists an $s_1 \in (0, s_\infty)$, such that for $(s, w) \in (-s_\infty, -s_1) \cup (s_1, s_\infty) \times S^{n-1}$,

$$|f_{n+1}|^2(s, w) \geq \frac{1}{2}.$$

Thus, choosing $s_1 \geq s_0$, we have that

$$\int_{\mathcal{C}} f_{n+1}^2\, \omega_{\mathcal{C},R} \geq \int_{s_1}^{s_\infty} \int_{S^{n-1}} (h^2 + s^2)^{-1} \sqrt{|g|}\, dw\, ds = \mathcal{H}^{n-1}(S^{n-1}) \int_{s_1}^{s_\infty} \frac{h^{n-1}\sqrt{1 + (h')^2}}{h^2 + s^2}\, ds.$$

As $h' > 0$ on the interval $[s_1, s_\infty)$, we can make the change of variables $\tau = h(s)$, (and recalling (3.36)),

$$\int_{s_1}^{s_\infty} \frac{h^{n-1}\sqrt{1 + (h')^2}}{h^2 + s^2}\, ds = a^{1/2} \int_{h(s_1)}^{\infty} \frac{\tau^{2(n-1)}}{(\tau^2 + s(\tau)^2) h'}\, d\tau = a^{1/2} \int_{h(s_1)}^{\infty} \frac{\tau^{2(n-1)}}{(\tau^2 + s(\tau)^2)(a\tau^{2(n-1)} - 1)^{1/2}}\, d\tau. \tag{3.37}$$

Therefore, as $n \geq 3$, and noting that $s(\tau)$ is a positive increasing function in τ, which is bounded from above by $s_\infty < +\infty$, the integrand (which is positive)

$$\frac{\tau^{2(n-1)}}{(\tau^2 + s(\tau)^2)(a\tau^{2(n-1)} - 1)^{1/2}} = \frac{\tau^{n-3}}{(1 + s(\tau)^2\tau^{-2})(a - \tau^{-2(n-1)})^{1/2}}$$

is bounded from below by a positive constant for large τ. Thus the integral in (3.37) is unbounded, implying that $f_{n+1} \notin L^2_{\omega_{\mathcal{C},R}}(\mathcal{C})$, and thus $f_{n+1} \notin W^{1,2}_{\omega_{\mathcal{C},R}}(\mathcal{C})$, and hence also not in $W^{1,2}(\mathcal{C})$.

For the Jacobi field f_d, generated by scaling, we have,

$$|f_d| \;=\; \frac{|h - sh'|}{(1 + (h')^2)^{1/2}} = \frac{|h(h')^{-1} - s|}{((h')^{-2} + 1)^{1/2}}.$$

Thus as $|s| \to s_\infty$, we have that $|h'| \to \infty$, $h \to \infty$, and, recalling (3.36)

$$\frac{h}{|h'|} = \frac{h}{(ah^{2(n-1)} - 1)^{1/2}} \to 0, \tag{3.38}$$

which implies that $|f_d| \to s_\infty$. Thus, repeating the same arguments as above for f_{n+1}, we see that f_d does not lie in $W^{1,2}(\mathcal{C})$ or $W^{1,2}_{\omega_{\mathcal{C},R}}(\mathcal{C})$.

Finally we look at elements of $RMJF_{\mathcal{C}}$ which are generated by rotations. Recall from above that all such Jacobi fields are of the form

$$f(s, w) = (1 + (h')^2)^{-1/2} \left((s + hh') \sum_{i=1}^{n} R_{i,n+1} w_i \right).$$

If $R_{i,n+1} = 0$ for all $i = 1, \ldots, n$, we have that $f = 0$. Assuming that $R_{i,n+1} \neq 0$ for some $i = 1, \ldots, n$, then as,

$$\sum_{i=1}^{n} R_{i,n+1} w_i$$

is a smooth function on S^{n-1}, and is non-zero at,

$$w_f = \left(\sum_{i=1}^{n} R^2_{i,n+1} \right)^{-1} (R_{1,n+1}, \ldots, R_{n,n+1}) \in S^{n-1},$$

we may conclude that there exists an $\alpha > 0$, and set $U_\alpha \subset S^{n-1}$ of positive $\mathcal{H}^{n-1}$ measure, such that

$$\left| \sum_{i=1}^{n} R_{i,n+1} w_i \right| \geq \alpha.$$

Thus for $w \in U_\alpha$, we have that,

$$|f|^2 \geq \alpha^2 \frac{(s + hh')^2}{1 + (h')^2} = \alpha^2 \frac{(s(h')^{-1} + h)^2}{(h')^{-2} + 1},$$

implying that $|f|^2$ becomes unbounded as $|s| \to s_\infty$. Again repeating arguments above we deduce that f does not lie in $W^{1,2}(\mathcal{C})$, or $W^{1,2}_{\omega_{\mathcal{C},R}}(\mathcal{C})$.

Therefore, we have shown that the only non-trivial Jacobi fields on $\mathcal{C}$, which are generated through rigid motions and lie in $W^{1,2}(\mathcal{C})$ or $W^{1,2}_{\omega_{\mathcal{C},R}}(\mathcal{C})$ are those spanned by the translations in the $\{x_{n+1} = 0\}$ hyperplane, i.e. Jacobi fields in span $(\{f_1, \dots, f_n\})$. This implies that,

$$n \leq \mathrm{nul}\,(\mathcal{C}) \leq \mathrm{nul}_{\omega_{\mathcal{C},R}}(\mathcal{C}).$$

Bibliography

[1] W. K. Allard, *On the First Variation of a Varifold*, Annals of Mathematics, vol. 95, no. 3, pp. 417 – 491 (1972)

[2] F. J. Almgren, *The Theory of Varifolds*, Mimeographed notes, Princeton (1965).

[3] A. Alvino, P. L. Lions and G. Trombetti, *On Optimization Problems with Prescribed Rearrangements*, Nonlinear Analysis, Theory, Methods and Applications, vol. 13, no. 2, pp. 185 – 220 (1989).

[4] L. Ambrozio, R. Buzano, A. Carlotto and B. Sharp, *Bubbling Analysis and Geometric Convergence Results for Free Boundary Minimal Surfaces*, Journal de l'École Polytechnique — Mathématiques, vol. 6, pp. 621 – 624 (2019).

[5] L. Ambrozio, A. Carlotto and B. Sharp, *Compactness of the Space of Minimal Hypersurfaces with Bounded Volume and p-th Jacobi Eigenvalue*, Journal of Geometric Analysis, vol. 26, no. 4, pp. 2591 – 2601 (2016).

[6] M. Anderson, *Curvature Estimates for Minimal Surfaces in 3-Manifolds*, Annales scientifiques de l'École Normale Supérieure, Series 4, vol. 18, no. 1, pp. 89 – 105 (1985).

[7] J. Barbosa, M. do Carmo and J. Eschenburg, *Stability of hypersurfaces of constant mean curvature in Riemannian manifolds*, Mathematische Zeitschrift, vol. 197, pp. 123 – 138, (1988).

[8] C. Bellettini, *Generic Existence of Multiplicity-1 Minmax Minimal Hypersurfaces via Allen–Cahn*, Calculus of Variations and Partial Differential Equations **61** (2022), ar. 149.

[9] C. Bellettini, *Multiplicity-1 Minmax Minimal Hypersurfaces in Manifolds with Positive Ricci Curvature*, Communications in Pure and Applied Mathematics, vol. 77, no. 3, pp. 2081 – 2137 (2023).

[10] C. Bellettini, *Extensions of Schoen–Simon–Yau and Schoen–Simon Theorems via Iteration à la De Giorgi* (2023), arXiv:2310.01340.

[11] C. Bellettini, O. Chodosh and N. Wickramasekera, *Curvature Estimates and Sheeting Theorems for Weakly Stable CMC Hypersurfaces*, Advances in Mathematics, vol. 352, pp. 133 - 157 (2019).

[12] C. Bellettini and N. Wickramasekera, *Stable CMC Integral Varifolds of Codimension 1: Regularity and Compactness* (2018), arXiv:1802.00377.

[13] C. Bellettini and N. Wickramasekera, *Stable Prescribed-Mean-Curvature Integral Varifolds of Codimension 1: Regularity and Compactness* (2020), arXiv:1902.09669.

[14] C. Bellettini and N. Wickramasekera, *The Inhomogeneous Allen–Cahn Equation and the Existence of Prescribed-Mean-Curvature Hypersurfaces* (2020), arXiv:2010.05847.

[15] C. Bellettini and M. Workman, *Embeddedness of Min-Max CMC Hypersurfaces in Manifolds with Positive Ricci Curvature*, Nonlinear Differential Equations and Applications NoDEA, vol. 31, no. 27 (2024).

[16] T. Bourni, B. Sharp and G. Tinaglia, *CMC Hypersurfaces with Bounded Morse Index*, Journal für die reine und angewandte Mathematik (Crelles Journal), no. 786, pp. 175 – 203 (2022).

[17] R. Buzano and B. Sharp, *Qualitative and Quantitative Estimates for Minimal Hypersurfaces with Bounded Index and Area*, Transactions of the American Mathematical Society, vol. 370, no. 6, pp. 4373 – 4399 (2018).

[18] R. E. Castillo and H. C. Chaparro, *Classical and Multidimensional Lorentz Spaces*, De Gruyter (2021).

[19] O. Chodosh, D. Ketover and D. Maximo, *Minimal Hypersurfaces with Bounded Index*, Inventiones Mathematicae, vol. 209, no. 3, pp. 617 – 664 (2017).

[20] O. Chodosh and C. Mantoulidis, *Minimal Surfaces and the Allen–Cahn Equation on 3 Manifolds* Annals of Mathematics, vol. 191, no. 1, pp. 213 – 328 (2020)

[21] H. I. Choi and R. Schoen, *The Space of Minimal Embeddings of a Surface into a Three-Dimensional Manifold of Positive Ricci Curvature*, Inventiones Mathematicae, vol. 81, pp. 387 – 398 (1985).

[22] T. H. Colding and W. P. Minicozzi II, *A Course in Minimal Surfaces*, Graduate Studies in Mathematics, vol. 121, American Mathematical Society (2011).

[23] F. Da Lio, M. Gianocca and T. Rivière, *Morse Index Stability for Critical Points to Conformally Invariant Lagrangians* (2022), arXiv:2212.03124.

[24] A. Dey, *Existence of Multiple Closed CMC Hypersurfaces with Small Mean Curvature*, Journal of Differential Geometry, vol. 125, pp. 379 – 403 (2023).

[25] L.C. Evans and R.F. Gariepy, *Measure Theory and Fine Properties of Functions*, Studies in Advanced Mathematics, Taylor and Francis (1992).

[26] D. Fischer-Colbrie, *On Complete Minimal Surfaces with Finite Index in Three Manifolds*, Inventiones Mathematicae, vol. 82, pp. 121 – 132 (1985).

[27] P. Gaspar, *The Second Inner Variation of Energy and the Morse Index of Limit Interfaces*, The Journal of Geometric Analysis, vol. 30, pp. 69 – 85 (2020).

[28] D. Gilbarg and N. Trudinger, *Elliptic Partial Differential Equations of Second Order*, 2nd ed., Springer-Verlag (1983).

[29] A. Gray, *Tubes*, 2nd ed. Progress in Mathematics, vol. 221, Birkhäuser Basel (2004).

[30] M. A. M. Guaraco, *Min-Max for Phase Transistions and Existence of Embedded Minimal Hypersurfaces*, Journal of Differential Geometry, vol. 108, pp. 91 – 133 (2018).

[31] F. Hiesmayr, *Spectrum and Index of Two-Sided Allen-Cahn Minimal Hypersurfaces*, Communications in Partial Differential Equations vol. 43, pp. 1541 – 1565 (2017)

[32] J. Hirsch and T. Lamm, *Index Estimates for Sequences of Harmonic Maps*, (to appear in) Communications in Analysis and Geometry, arXiv:2212.13808

[33] J. Hutchinson and Y. Tonegawa, *Convergence of Phase Interfaces in the Van der Waals-Cahn-Hilliard Theory*, Calculus of Variations and Partial Differential Equations vol. 10, no. 1, pp. 49 – 84 (2000)

[34] K. Irie, F. C. Marques and A. Neves, *Density of Minimal Hypersurfaces for Generic Metrics*, Annals of Mathematics, vol. 187, issue 3, pp. 963 – 972 (2018).

[35] N. Q. Le, *On the Second Inner Variation of the Allen–Cahn Functional and its Applications*, Indiana University Mathematics Journal, vol. 60, no. 6, pp. 1843 – 1856 (2011).

[36] P. Li and S.-T. Yau, *On the Schrödinger Equation and the Eigenvalue Problem*, Communications in Mathematical Physics, vol. 88, no. 3 (1983).

[37] C. Mantegazza and A. C. G. Mennucci, *Hamilton–Jacobi Equations and Distance Functions on Riemannian Manifolds*, Applied Mathematics and Optimization vol. 47, pp. 1 – 25 (2002).

[38] C. Mantoulidis, *Variational Aspects of Phase Transitions with Prescribed Mean Curvature*, Calculus of Variations and Partial Differential Equations, vol, 61, no. 2, ar. 43 (2022).

[39] F. C. Marques and A. Neves, *Existence of Infinitely Many Minimal Hypersurfaces in Positive Ricci Curvature*, Inventiones Mathematicae, vol. 209, pp. 577 – 616 (2017)

[40] J. H. Michael and L. Simon, *Sobolev and Mean Value Inequalities on Generalized Submanifolds of* $\mathbb{R}^n$, Communications on Pure and Applied Mathematics, vol. 26, pp. 361 – 379 (1973).

[41] J. Milnor, *Morse Thoery*, Based off Lecture Notes by M. Spivak and R. Wells, Princeton University Press (1963).

[42] J. D. Moore and R. Ream, *Minimal Two Spheres of Low Index in Manifolds with Positive Complex Sectional Curvature*, Mathematische Zeitschrift, vol. 219, pp. 1295 – 1335 (2019).

[43] F. Morabito, *Index and Nullity of the Gauss Map of the Costa–Hoffman–Meeks Surfaces*, Indiana University Mathematics Journal, vol. 58, no. 2, pp. 677 – 707 (2009).

[44] S. Nayatani and H. Urakawa, *Spectrum of the Schrödinger Operator on a Complete Manifold*, Journal of Functional Analysis, vol 112, pp. 459 – 479 (1993).

[45] J. Pitts, *Existence and Regularity of Minimal Surfaces on Riemannian Manifolds*, Princeton University Press (1981).

[46] M. Röger and Y. Tonegawa, *Convergence of Phase-Field Approximations to the Gibbs–Thomson Law*, Calculus of Variations and Partial Differential Equations, vol. 32, no. 1, pp. 111 – 136 (2008).

[47] A. Ros, *Compactness of Spaces of Properly Embedded Minimal Surfaces with Finite Total Curvature*, Indiana University Mathematics Journal, vol. 44, no. 1, pp. 139 – 152 (1995)

[48] T. Sakai, *Riemannian Geometry*, Translations of Mathematical Monographs, Fields Institute Communications, American Mathematical Society (1996).

[49] H. Samelson, *Orientability of Hypersurfaces in* $\mathbb{R}^n$, Proceedings of the American Mathematical Society, vol. 22, pp. 301 – 302 (1969).

[50] R. Schoen, *Uniqueness, Symmetry and Embeddedness of Minimal Surfaces*, Journal of Differential Geometry, vol. 18, pp. 791 – 809 (1983).

[51] R. Schoen and L. Simon, *Regularity of Stable Minimal Hypersurfaces*, Communications on Pure and Applied Mathematics, vol. 34, issue 6, pp. 741 – 797 (1981)

[52] R. Schoen, L. Simon and S.-T. Yau, *Curvature Estimates for Minimal Hypersurfaces*, Acta Mathematica, vol. 134, pp. 275 – 288 (1975).

[53] B. Sharp, *Compactness of Minimal Hypersurfaces with Bounded Index*, Journal of Differential Geometry, vol. 106, no. 2, pp. 317 – 339 (2017).

[54] L. Simon, *Lectures on Geometric Measure Theory*, Centre for Mathematical Analysis, Australian National University (1983).

[55] J. Simons, *Minimal Varieties in Riemannian Manifolds*, Annals of Mathematics, vol. 88, no. 1, pp. 62 – 105 (1968).

[56] A. Song, *Existence of Infinitely Many Minimal Hypersurfaces in Closed Manifolds*, Annals of Mathematics, vol. 197, pp. 859 – 895 (2023).

[57] A. Sun, *Compactness of Constant Mean Curvature Surfaces in a Three-Manifold with Positive Ricci Curvature*, Pacific Journal of Mathematics, vol. 305, no. 2, pp. 735 – 756 (2020).

[58] L.-F. Tam and D. Zhou, *Stability Properties for the Higher Dimensional Catenoid in* $\mathbb{R}^{n+1}$, Proceedings of the American Mathematical Society, vol. 137, no. 10, pp. 3451 – 3461 (2009).

[59] Y. Tonegawa, *On Stable Critical Points for a Singular Perturbation Problem*, Communications in Analysis and Geometry, vol. 13, pp. 439 – 459 (2005).

[60] Y. Tonegawa and N. Wickramasekera, *Stable Phase Interfaces in the Van der Waals–Cahn–Hilliard Theory*, Journal fur die Reine und Angewandte Mathematik, no. 668, pp. 191 – 210 (2012).

[61] J. Tysk, *Finiteness of Index and Total Scalar Curvature of Minimal Hypersurfaces*, Proceedings of the American Mathematical Society, vol. 105, no. 2, pp. 429 - 435 (1989).

[62] N. Visciglia, *A Note About the Generalized Hardy–Sobolev Inequality with Potential in* $L^{p,d}(\mathbb{R}^N)$, Calculus of Variations and Partial Differential Equations, vol. 24, pp. 167 – 184 (2005).

[63] B. White, *Generic Transversality of Minimal Submanifolds and Generic Regularity of Two-Dimensional Area-Minimizing Integral Currents*, (2019), arXiv: 1901.05148.

[64] B. White, *Lectures on Minimal Surface Theory*, IAS/Park City Mathematics Series, vol. 22, pp. 387 – 438 (2016).

[65] B. White, *On the Compactness Theorem for Embedded Minimal Surfaces in 3-Manifolds with Locally Bounded Area and Genus*, Communications in Analysis and Geometry, vol. 26, no. 3, pp. 659 – 678 (2018).

[66] N. Wickramasekera, *A Regularity and Compactness Theory for Immersed Stable Minimal Hypersurfaces og Multiplicity at Most 2*, Journal of Differential Geometry, vol. 80, pp. 79 – 173 (2008),

[67] N. Wickramasekera, *A General Regularity Theory for Stable Codimension 1 Integral Varifolds*, Annals of Mathematics vol. 179, no. 3, pp. 843 – 1007 (2014).

[68] M. Workman, *Upper Semicontinuity of Morse Index Plus Nullity for Minimal and CMC Hypersufaces* (2023), arXiv:2312.09227

[69] H. Yin, *Generalized Neck Analysis of Harmonic Maps from Spheres*, Calculus of Variations and Partial Differential Equations, vol. 60, no. 117 (2021).

[70] H. Yin, *Higher-Order Neck Analysis of Harmonic Maps and its Applications*, Annals of Global Analysis and Geometry, vol. 62, pp. 457 – 477 (2022).

[71] X. Zhou and J. Zhu, *Min-Max Theory for Constant Mean Curvature Hypersurfaces*, Inventiones mathematicae **218** (2019), pp. 441 – 490.

[72] X. Zhou and J. Zhu, *Existence of Hypersurfaces with Prescribed Mean Curvature I - Generic Min-Max*, Cambridge Journal of Mathematics **8** (2020), pp. 311 – 362.