TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 00, Number 0, Pages 000–000
S 0002-9947(XX)0000-0

MARKOV ADDITIVE FRIENDSHIPS

LEIF DÖRING, LUKAS TROTTNER, AND ALEXANDER R. WATSON

Abstract. The Wiener–Hopf factorisation of a Lévy or Markov additive process describes the way that it attains new extrema in terms of a pair of so-called ladder height processes. Vigon's theory of friendship for Lévy processes addresses the inverse problem: when does a process exist which has certain prescribed ladder height processes? We give a complete answer to this problem for Markov additive processes, provide simpler sufficient conditions for constructing processes using friendship, and address in part the question of the uniqueness of the Wiener–Hopf factorisation for Markov additive processes.

1. Introduction

Lévy processes and Markov additive processes are a staple of applied probability and have found a home in areas as diverse as queueing theory, stochastic finance and fragmentation modelling. For many applications, it is beneficial to know how the processes make new maxima (or minima), and a key tool for Lévy processes is the theory of friendship, which makes it possible to build models which cross levels in a prescribed way. In this article, we approach the theory of friendship for Markov additive processes, beginning with a review of the situation for Lévy processes, before exploring how this changes with the introduction of a Markov component.

Lévy processes. The Wiener–Hopf factorisation is one of the central results in fluctuation theory for Lévy processes. Its spatial version tells us that, for a Lévy process ξ with characteristic exponent $\psi(\theta) \coloneqq \log \mathbb{E}[\mathrm{e}^{\mathrm{i}\theta\xi_1}]$, we have the identity

$$\psi(\theta) = -c\psi^-(-\theta)\psi^+(\theta), \quad \theta \in \mathbb{R}, \tag{1.1}$$

where the functions $\psi^\pm$ are the characteristic exponents of the ascending and descending ladder height processes $H^\pm$. The processes H^+ and H^- are (killed) subordinators whose ranges are the set of new suprema and infima of ξ, respectively, and $c > 0$ is a constant whose value influences the time scale of $H^\pm$. We refer to [30, Chapter 6] for details. H^+ and H^- are of central importance for both theoretical and practical considerations since they are the building block for first passage identities of ξ [13]. However, for any given Lévy process ξ, the factorisation (1.1), and the identities derived from it, are in most cases not explicit. Vigon flipped the perspective on the Wiener–Hopf factorisation in his pioneering thesis [47]. Instead of considering some fixed Lévy process ξ and looking for its Wiener–Hopf factors,

2020 *Mathematics Subject Classification.* Primary 60G51, 60J25, 47A68.
LT gratefully acknowledges financial support of Carlsberg Foundation Young Researcher Fellowship grant CF20-0640 "Exploring the potential of nonparametric modelling of complex systems via SPDEs".

Vigon starts with two subordinators H^+ and H^- and finds necessary and sufficient criteria on their characteristics such that the right hand side of (1.1) determines the Lévy–Khintchine exponent of some Lévy process ξ. Vigon calls such subordinators *friends* (*amis*) and refers to the resulting Lévy process ξ as *le fruit de l'amitié*, which we translate (more conservatively) as the *bonding process*.

We recall his results explicitly at this point. Let H^+ and H^- be two Lévy subordinators with drifts and Lévy measures (d^+, Π^+) and (d^-, Π^-), respectively. Let moreover $\dagger^\pm = -\psi^\pm(0)$ be their respective killing rates and denote by $\overline{\Pi}^\pm(x) = \Pi^\pm(x,\infty)$ the tails of $\Pi^\pm$, where $x > 0$. We call H^+ and H^- *compatible* if $d^\mp > 0$ implies that $\Pi^\pm$ has a càdlàg Lebesgue density $\partial\Pi^\pm$ on $(0,\infty)$, which can be expressed as the right tail of a signed measure $\eta^\pm$, i.e., $\partial\Pi^\pm(x) = \eta^\pm(x,\infty)$ for $x > 0$. For the sake of notational consistency, if $d^\mp = 0$ we let $\partial\Pi^\pm$ be some version of the Lebesgue density of the absolutely continuous part of Π_i^+. Then, we have the following result.

Theorem 1.1 (Vigon's theorem of friends)**.** *H^+ and H^- are friends if, and only if, they are compatible and the function*

$$\Upsilon(x) = \mathbf{1}_{(0,\infty)}(x)\Big(\int_{x+}^{\infty} \big(\overline{\Pi}^-(y-x) - \psi^-(0)\big)\,\Pi^+(\mathrm{d}y) + d^-\partial\Pi^+(x)\Big)$$
$$+ \mathbf{1}_{(-\infty,0)}(x)\Big(\int_{(-x)+}^{\infty} \big(\overline{\Pi}^+(y+x) - \psi^+(0)\big)\,\Pi^-(\mathrm{d}y) + d^+\partial\Pi^-(-x)\Big),$$

is a.e. equal to a function decreasing on $(0,\infty)$ and increasing on $(-\infty,0)$. Moreover, when H^+ and H^- are friends, we have for a.e. $x \in \mathbb{R}$ the identity

$$\mathbf{1}_{(0,\infty)}(x)\Pi(x,\infty) + \mathbf{1}_{(-\infty,0)}(x)\Pi(-\infty,x) = \Upsilon(x), \tag{1.2}$$

for the Lévy measure Π of the bonding Lévy ξ.

Remark 1.2. In Vigon's original formulation, (1.2) holds everywhere rather than almost everywhere, the reason being that it is claimed that the convolution $x \mapsto \overline{\overline{\Pi}}^- * \Pi^+(x)$ is a càdlàg function. However, this fails, for example, when $H^\pm$ are pure jump processes with jumps of size $\{1,2\}$ and $\Pi^\pm(\{1\}) > \Pi^\pm(\{2\}) > 0$, even though these processes can be shown to be friends (they are *philanthopes discrètes* [47]). In this example, the statement can be easily repaired by considering the closed tails $\Pi^\pm([x,\infty))$ instead of the open tails $\overline{\Pi}^\pm(x) = \Pi^\pm((x,\infty))$, but it is not clear that such an approach can work in general, for instance, when one of the Lévy measures is singular continuous. We arrived at the above statement of the result after consultation with Vigon, but we remark that the original formulation remains valid when $\Pi^\pm$ are absolutely continuous.

Equation (1.2) is called *équation amicale* by Vigon and characterises the Lévy measure of ξ in terms of the characteristics of its ascending and descending ladder height processes. Aside from [47], this equation is proved in [48, Proposition 3.3] and [14, Theorem 16]. A textbook treatment of the theorem of friends appears in [30, Theorem 6.22].

The monotonicity conditions on Υ do not appear easy to handle at first sight. However, they simplify dramatically for a certain class of processes. Vigon calls a subordinator with decreasing Lévy density a *philanthopist*, and develops the following neat result:

Theorem 1.3 (Vigon's theorem of philanthropy)**.** *Two philanthropists are always friends.*

For a direct proof of this result under under additional second moment assumptions we refer to [32, Theorem 4.4]. Let us also remark that the class of Laplace exponents of philanthropists (or equivalently, Bernstein functions with decreasing Lévy density) is known as the *Jurek class* of Bernstein functions [45].

The class of philanthropists is quite large and contains many tractable examples. As a consequence, the theorem of philanthropy offers a new tool for the construction of Lévy processes, quite different from the classical techniques of specifying the Lévy triplet, transition semigroup or characteristic exponent. However, the form of the équation amicale means that the Lévy measure of the bonding process may not be simple to express, and there is a tradeoff between explicitness of the bonding process and explicitness of the corresponding friends, regardless of which side of the Wiener–Hopf factorisation we start on. One of the major achievements of this approach is the class of *hypergeometric Lévy processes*, built from friendships of β-subordinators [28, 26, 30] and motivated by the relation with killed and conditioned stable processes (see [9] and [26, Theorem 1].)

Markov additive processes. Coming from this well-established theory for Lévy processes, our goal in this paper is to extend Vigon's theory of friends and philanthropy to the class of Markov additive processes (MAPs). A MAP (ξ, J) with state space $\mathbb{R} \times \{1, \dots, n\}$ can be thought of as a regime-switching Lévy process: depending on the state (or phase) of a Markov chain J, the process ξ follows a different Lévy process; see Section 2 for a full description. Many concepts in Lévy process theory have direct analogues in the theory of MAPs. For any MAP (ξ, J) there exists a matrix form of the characteristic exponent, which we call the MAP exponent, $\mathbf{\Psi}\colon \mathbb{R} \to \mathbb{C}^{n\times n}$, such that

$$\big(\mathbb{E}^{0,i}[\exp(\mathrm{i}\theta\xi_t);\ J_t = j]\big)_{i,j\in[n]} = \mathrm{e}^{t\mathbf{\Psi}(\theta)}, \quad \theta \in \mathbb{R}, t \geq 0,$$

where $\mathbb{P}^{x,i}$ indicates that the process (ξ, J) is a.s. started in $(x, i) \in \mathbb{R} \times [n]$. We will sometimes call the range of J the *phase space*. The Lévy measure Π of a Lévy process ξ has a natural analogue in the *Lévy measure matrix* $\mathbf{\Pi}$ of a MAP (ξ, J), which describes the jump structure of ξ in the Markovian environment governed by J. Much as in the Lévy case, the new suprema of ξ can be described by MAP subordinator (H^+, J^+), referred to as the *ascending ladder height MAP*; likewise, the new infima of the time-reversed process can be related to the *descending ladder height MAP* (H^-, J^-). Again, we make these statements precise in Section 2.

Our question arises from considering the Wiener–Hopf factorisation of the MAP exponent $\mathbf{\Psi}$ into the MAP exponents $\mathbf{\Psi}^+, \mathbf{\Psi}^-$ of (H^+, J^+) and (H^-, J^-), respectively. This states that, if ξ is nonlattice and J is irreducible with stationary distribution $\boldsymbol{\pi}$, then, for an appropriate time scaling of H^+ and H^-, we have the matrix identity

$$\mathbf{\Psi}(\theta) = -\mathbf{\Delta}_{\boldsymbol{\pi}}^{-1}\mathbf{\Psi}^-(-\theta)^\top \mathbf{\Delta}_{\boldsymbol{\pi}} \mathbf{\Psi}^+(\theta), \quad \theta \in \mathbb{R}, \tag{1.3}$$

where $\mathbf{\Delta}_{\boldsymbol{\pi}}$ is the diagonal matrix with entries given by the vector $\boldsymbol{\pi}$. Note that when the modulating space is one-dimensional, i.e., $J_t = 1$ for all $t \geq 0$, the above equality reduces to (1.1). The factorisation (1.3) was shown in matrix form in [22, Theorem 1] for spectrally negative MAPs; in [12, Theorem 26] more generally, under the condition that ξ is killed at the same (possibly zero) rate in all phases;

and in [19, equation (19)] for ξ killed with positive rate in every phase. We will show in Theorem 2.1 that (1.3) holds for any irreducible MAP, regardless of the lifetimes of the Lévy components.

The Wiener–Hopf factorisation of MAPs has a long history. Of particular note is [21, Theorem 3.28], which is the analogue of (1.3) for MAPs killed at constant, positive rate, with a description given in terms of generators rather than matrix exponents. This followed a series of investigations into discrete-time MAPs, some of the earliest being [41, 2] for general phase space. In the setting of finite phase space in discrete time, where things can be made more explicit, textbook treatments of these problems can be found in [3, Chapter XI.2f] and [40, Chapter 5]. It is also worth mentioning the works [5, 42], which investigated factorisations of Markov chains in terms of embedded Markov chains obtained from time changes via additive functionals. As a special case, [42, Theorem 1] can be applied to obtain a factorisation of the generator matrix $\mathbf{\Psi}(0)$ of the modulator of a pure drift MAP in terms of the diagonal drift matrix and intensity matrices associated to the sub chains corresponding to positive and negative drift phases of the MAP. However, this type of factorisation is structurally different from the Wiener–Hopf factorisation (1.3) at $\theta = 0$.

Even when compared to the already quite scarce number of explicit Wiener–Hopf Lévy factorisations, the situation for MAPs is even more tenuous. Apart from the *deep factorisation* of the stable process in [31], to the best of our knowledge there is no known example of an explicit MAP Wiener–Hopf factorisation. As will become apparent from our analysis this is not a mere artifact of the youth of the MAP Wiener–Hopf factorisation, but also a consequence of the increased complexity due to phase transitions.

Despite these difficulties, the theorem below, which is our main result, provides a complete picture of friendship of MAPs. This is established in Section 3 in the form of Theorems 3.10 and 3.11. The notation $\partial\mathbf{\Pi}^{\pm}$ is a matrix of densities of the elements of $\mathbf{\Pi}^{\pm}$, the appropriate elements of which exist under the conditions of the theorem.

Theorem 1.4. *Two MAP subordinators (H^+, J^+) and (H^-, J^-) are $\boldsymbol{\pi}$-friends (in the sense of Definition 3.1) if, and only if, they are $\boldsymbol{\pi}$-compatible (in the sense of Definition 3.7) and the matrix-valued function*

$$\begin{aligned}\mathbf{\Upsilon}(x) = \Big\{ \int_{x+}^{\infty} \mathbf{\Delta}_{\boldsymbol{\pi}}^{-1}\Big(\overline{\mathbf{\Pi}}^{-}(y-x) - \mathbf{\Psi}^{-}(0)\Big)^{\top} \mathbf{\Delta}_{\boldsymbol{\pi}}\, \mathbf{\Pi}^{+}(\mathrm{d}y) + \mathbf{\Delta}_{\boldsymbol{d}}^{-}\partial\mathbf{\Pi}^{+}(x)\Big\}\mathbf{1}_{(0,\infty)}(x) \\ + \Big\{ \int_{(-x)+}^{\infty} \mathbf{\Delta}_{\boldsymbol{\pi}}^{-1}\big(\mathbf{\Pi}^{-}(\mathrm{d}y)\big)^{\top} \mathbf{\Delta}_{\boldsymbol{\pi}}\, \big(\overline{\mathbf{\Pi}}^{+}(y+x) - \mathbf{\Psi}^{+}(0)\big) \\ + \mathbf{\Delta}_{\boldsymbol{\pi}}^{-1}\big(\mathbf{\Delta}_{\boldsymbol{d}}^{+}\partial\mathbf{\Pi}^{-}(-x)\big)^{\top}\mathbf{\Delta}_{\boldsymbol{\pi}}\Big\}\mathbf{1}_{(-\infty,0)}(x),\end{aligned}$$

is a.e. equal to a function decreasing on $(0,\infty)$ and increasing on $(-\infty,0)$. Moreover, when (H^+, J^+) is a $\boldsymbol{\pi}$-friend of (H^-, J^-), then for a.e. $x \in \mathbb{R}$,

$$\mathbf{1}_{(0,\infty)}(x)\mathbf{\Pi}(x,\infty) + \mathbf{1}_{(-\infty,0)}(x)\mathbf{\Pi}(-\infty,x) = \mathbf{\Upsilon}(x),$$

for the Lévy measure matrix of the bonding MAP (ξ, J).

The notions of $\boldsymbol{\pi}$-friendship and $\boldsymbol{\pi}$-compatibility, which we have not yet defined, are made precise in Section 3. $\boldsymbol{\pi}$-friendship is the obvious counterpart to friendship of Lévy processes, meaning that the matrix Wiener–Hopf factorisation (1.3) holds.

Meanwhile, $\boldsymbol{\pi}$-compatibility is partly the analogue of Vigon's compatibility of Lévy processes, but also places more stringent requirements on the Lévy measure matrices at and near zero and conditions on the rates of the Markov chains.

The second part of our work seeks an extension of Vigon's theory of philanthropy; that is, sufficient conditions that allow one to more easily establish that two MAP subordinators are friends. The additional challenges of $\boldsymbol{\pi}$-friendship make it difficult to find 'unilateral' conditions, which can be verified separately for each of two subordinators and lead to $\boldsymbol{\pi}$-friendship between them. Instead, we introduce the notion of $\boldsymbol{\pi}$-fellowship, and Theorem 4.4 states that two mutual $\boldsymbol{\pi}$-fellows are indeed $\boldsymbol{\pi}$-friends. Although this theory is harder to apply than in the Lévy case, we use it to give general sufficient conditions for mutual $\boldsymbol{\pi}$-fellowship (Theorems 4.8 and 4.12) which in turn lead to explicit examples of spectrally positive MAPs with completely monotone ascending ladder jump measures, as well as a class of MAPs with double exponential jump structure and known Wiener–Hopf factorisation.

Finally, since our approach is to analyse the MAP Wiener–Hopf factorisation equation (1.3), we need to know about the uniqueness of this in order to draw probabilistic conclusions, and in Section 5 we prove this under a wide range of assumptions, the main result being Theorem 5.7.

Context and applications. The theory of friendship is strongly connected with research on self-similar Markov processes. Since Lamperti's pioneering work [37] in 1972, it has been understood that the class of Lévy processes and that of self-similar Markov processes with state space $[0, \infty)$ are in bijection by means of a sample path transformation. It was this that inspired the construction, via friendship, of the hypergeometric class of Lévy processes by Kuznetsov and Pardo [26]. This class and its extensions [29, 33] can be used to characterise a number of modifications of stable processes, among them killing and conditioning [8, 26], path-censoring [34] and ricochet [7, 33].

Over the past two decades, Lamperti's results have been extended to cover self-similar Markov processes with more general state space, and the Lévy processes in Lamperti's representation are replaced by Markov additive processes [11, 10, 1]. For instance, a $\mathbb{R}^d$-valued self-similar Markov process corresponds to a Markov additive process whose phase space is S^{d-1}. In $d = 1$, this is the discrete phase space $S^0 = \{-1, 1\}$, and this is the context of the Wiener-Hopf factorisations obtained in [31, 35]. In higher dimensions, the MAP representation has still proved valuable and factorisations can be found under isotropy assumptions, despite the challenges of a continuous state space [36]. We expect that the theory of friendship developed in this article, and extensions accounting for more general phase space, will be an important tool in the study of self-similar Markov processes.

Outline. The remainder of the paper is structured as follows. Section 2 is devoted to introducing notation and the most important facts on MAPs that we need in the rest of the paper. Section 3 deals with friendship of MAPs and general consequences thereof, while in Section 4 we explore $\boldsymbol{\pi}$-fellowship and develop constructive criteria for $\boldsymbol{\pi}$-compatibility in order to generate examples of $\boldsymbol{\pi}$-friendship, and thereby of MAPs with explicit Wiener–Hopf factorisation. The final Section 5 discusses the uniqueness of the matrix Wiener–Hopf factorisation, and verifies that this applies to the examples found in previous sections.

Notation. We collect some notational conventions that are used throughout the paper. Small bold letters $\boldsymbol{a}$ represent column vectors in $\mathbb{R}^n$ and capitalized letters $\boldsymbol{A}$ refer to matrices in $\mathbb{R}^{n\times n}$. The vectors containing only zeros and only ones are denoted by $\mathbf{0}$ and $\mathbf{1}$, respectively. For measures $\mu^\pm$ concentrated on $\mathbb{R}_\pm \setminus \{0\}$, where $\mathbb{R}_+ := [0,\infty), \mathbb{R}_- := (-\infty,0]$, we let $\bar{\mu}^+(x) := \mu^+((x,\infty)) \equiv \mu(x,\infty)$ be the right tail of μ^+ for $x>0$ and $\bar{\mu}^-(x) := \mu^-((-\infty,x)) \equiv \mu^-(-\infty,x)$ be the left tail of μ^- for $x<0$. For a (signed) finite measure μ on $\mathbb{R}$, its distribution function is represented by $\underline{\mu}(x) := \mu((-\infty,x])$, $x\in\mathbb{R}$. Moreover, for a given (signed) measure μ on $\mathbb{R}$, $\widetilde{\mu}(\mathrm{d}x) := \mu(-\mathrm{d}x)$ denotes the reflected measure and for a function $f\colon \mathbb{R}\to\mathbb{R}$ we let $\widetilde{f} := f(-\cdot)$. Then, given a finite measure μ, we interpret $\underline{\widetilde{\mu}}$ in the sense of reflecting the distribution function of μ, i.e., $\underline{\widetilde{\mu}}(x) = \underline{\mu}(-x)$. For $n\in\mathbb{N}$ we let $[n] := \{1,\dots,n\}$.

2. Fundamentals on Markov additive processes

A *Markov additive process (MAP)* with n phases is a strong Markov process $(\xi,J) = (\xi_t,J_t)_{t\geq 0}$ with state space $\mathbb{R}\times[n]$ and lifetime $\zeta\in(0,\infty]$, with probability measures $(\mathbb{P}^{x,i})_{x\in\mathbb{R},i\in[n]}$ and associated expectations $(\mathbb{E}^{x,i})_{x\in\mathbb{R},i\in[n]}$, adapted to a filtration $\mathcal{F} = (\mathcal{F}_t)_{t\geq 0}$ satisfying the usual hypotheses, with the property that

$$\mathbb{E}^{x,i}\big[f(\xi_{t+s}-\xi_t, J_{t+s})\mathbf{1}_{\{\zeta>t\}} \,\big|\, \mathcal{F}_t\big] = \mathbb{E}^{0,J_t}\big[f(\xi_s,J_s)\big]\mathbf{1}_{\{\zeta>t\}}, \quad s,t\geq 0,$$

for all bounded measurable f. The process ξ is called the *ordinator* of the MAP, and the process J the *modulator*. Any such MAP can be decomposed into a Markov process on $[n]$ with generator matrix $\boldsymbol{Q} = (q_{i,j})_{i,j\in[n]}$ and killing rate $\dagger_i$ in state i, a collection of Lévy processes $(\xi^{(i)})_{i\in[n]}$, and a collection of probability distributions $(F_{i,j})_{i,j\in[n]}$ with the convention $F_{i,i} = \delta_0$. Roughly speaking, the MAP evolves as follows. J evolves as a Markov process with generator matrix $\boldsymbol{Q}$. When $J_t = i$, ξ_t evolves as $\xi^{(i)}$, and the process is killed at rate $\dagger_i$. When J jumps to state j, which occurs with rate $q_{i,j}$, ξ experiences a jump whose distribution is $F_{i,j}$, and the evolution of ξ then begins to follow $\xi^{(j)}$.

The distribution of a MAP can be described more formally using the *MAP exponent* of (ξ,J), which is a matrix-valued function $\boldsymbol{\Psi}$ such that

$$(\mathrm{e}^{t\boldsymbol{\Psi}(\theta)})_{i,j} = \mathbb{E}^{0,i}\big[\mathrm{e}^{\mathrm{i}\theta\xi_t}; J_t = j\big].$$

The preceding description of the MAP then gives rise to the structure

$$\boldsymbol{\Psi}(\theta) = \boldsymbol{\Delta}_{\boldsymbol{\psi}(\theta)} + \boldsymbol{Q}\odot\boldsymbol{G}(\theta) - \boldsymbol{\Delta}_{\boldsymbol{\dagger}},$$

where $\boldsymbol{\Delta}_{\boldsymbol{v}}$ is the diagonal matrix whose (i,i)th entry is v_i; ψ_i is the characteristic exponent of the Lévy process $\xi^{(i)}$, satisfying $\mathrm{e}^{t\psi_i(\theta)} = \mathbb{E}^0[\mathrm{e}^{\mathrm{i}\theta\xi^{(i)}_t}]$; $G_{i,j}(\theta) = \int \mathrm{e}^{\mathrm{i}\theta x}\,F_{i,j}(\mathrm{d}x)$; and $\odot$ is the Hadamard (elementwise) product.

The jump behaviour of the MAP is encoded in its *Lévy measure matrix* $\boldsymbol{\Pi}$, defined by taking $\Pi_{i,i}$ to be equal to Π_i, the Lévy measure (in the usual sense) of the process $\xi^{(i)}$, and taking $\Pi_{i,j} = q_{i,j}F_{i,j}$ for $i\neq j$.

This decomposition reveals that we can view (ξ,J) either as a killed Markov process J on top of which we run unkilled Lévy processes, or as an unkilled Markov process J on which we run killed Lévy processes, killing the whole MAP when one of the component processes dies. We will typically take the former perspective.

We remark that, given only $\boldsymbol{\Psi}$, one can readily obtain $\boldsymbol{\dagger} = -\boldsymbol{\Psi}(0)\mathbf{1}$ and $\boldsymbol{Q} = \boldsymbol{\Psi}(0) + \boldsymbol{\Delta}_{\boldsymbol{\dagger}}$.

If J is irreducible and $\boldsymbol{\pi} \in \mathbb{R}^n$ is a stochastic vector on $[n]$ with full support, meaning that $\sum_{i=1}^n \pi(i) = 1$ and $\pi(i) > 0$ for all i, then we say that $\boldsymbol{\pi}$ is *invariant* for J if

$$\boldsymbol{\pi}^\top \boldsymbol{Q} = \mathbf{0}^\top;$$

that is, $\boldsymbol{\pi}$ is an invariant distribution for an unkilled version of J. Under this condition, we can speak about $\boldsymbol{\pi}$-duality for the MAP [12, section A.2]. The function

$$\widehat{\boldsymbol{\Psi}}(\theta) = \boldsymbol{\Delta}_{\boldsymbol{\pi}}^{-1} \boldsymbol{\Psi}(-\theta)^\top \boldsymbol{\Delta}_{\boldsymbol{\pi}}, \quad \theta \in \mathbb{R},$$

is the MAP exponent of some MAP $(\widehat{\xi}, \widehat{J})$ which can be obtained by time-reversal and reflection:

$$\mathbb{E}^{0,\boldsymbol{\pi}}\big[f(\xi_{(t-s)-} - \xi_t, J_{(t-s)-}; s \le t)\mathbf{1}_{\{\zeta > t\}}\big] = \mathbb{E}^{0,\boldsymbol{\pi}}\big[f(\widehat{\xi}_s, \widehat{J}_s; s \le t)\mathbf{1}_{\{\widehat{\zeta} > t\}}\big]$$

holds for any functional f and $t \ge 0$; here, $\mathbb{P}^{0,\boldsymbol{\pi}} = \sum_{i=1}^n \pi(i)\mathbb{P}^{0,i}$. Moreover, $\boldsymbol{\pi}$ is invariant for $\widehat{J}$, and $\widehat{J}$ is also killed with rates $\boldsymbol{\dagger}$. We say that $(\widehat{\xi}, \widehat{J})$ is the $\boldsymbol{\pi}$-*dual* of (ξ, J).

A key aspect of the theory of MAPs is the *Wiener–Hopf factorisation*, which expresses the characteristics of a MAP in terms of its ladder processes. We begin with the *local time at the supremum*, which we define by stitching together the local times of the constituent Lévy processes. During a given phase i, if $\xi^{(i)}$ is such that 0 is regular for $(0, \infty)$, it is a continuous increasing functional L which increases precisely at the time at which ξ is at its running supremum. On the other hand, if $\xi^{(i)}$ is such that 0 is irregular for $(0, \infty)$, it is an increasing jump process L whose jumps occur at the isolated times at which ξ makes new suprema (in the strict sense), and whose jump sizes are independent with a standard exponential distribution. At a phase switch occuring at time T, if $\xi_{T-} < \xi_T = \sup_{t \le T} \xi_t$ and $\xi^{(J_T)}$ is such that 0 is irregular for $(0, \infty)$, it is necessary to introduce an additional jump of L with an exponential distribution occurring at time T.

The *inverse local time at the supremum* is defined by $L_t^{-1} = \inf\{s \ge 0 : L_s > t\}$, and the phase at this time is $J_t^+ = J_{L_t^{-1}}$. If we further let $\boldsymbol{L}_t^{-1}$ represent the vector whose i-th entry is $\int_0^t \mathbf{1}_{\{J_s^+ = i\}} \, \mathrm{d}s$, and define $H_t^+ = \xi_{L_t^{-1}}$, then both $(\boldsymbol{L}^{-1}, H^+, J^+)$ and (H^+, J^+) are MAPs, the former having an $(n+1)$-dimensional ordinator. These two MAPs are called the *ascending ladder process* and the *ascending ladder height process*, respectively.

The descending ladder processes may be defined similarly by considering the local time at the supremum of the dual process $(\widehat{\xi}, \widehat{J})$, and we write (H^-, J^-) for the descending ladder height process. Some care is required here when some components of the MAP are compound Poisson processes. We adopt the convention that the ascending ladder process records strict new suprema, and the descending ladder process weak new dual suprema; see [30, section 6.2] for a discussion of this distinction in the setting of Lévy processes. Likewise, the case where a phase switch occurs at the time of a new dual supremum must be handled correctly. For a further discussion of this, we refer to [19, section 5.1], and caution that in [19], the '$\widehat{\cdot}$' notation refers only to time-reversal (without spatial reflection).

Denote the MAP exponent of (H^+, J^+) by $\boldsymbol{\Psi}^+$ and that of (H^-, J^-) by $\boldsymbol{\Psi}^-$. The Wiener–Hopf factorisation can be expressed in the following result, which is really a corollary of the more general Theorem B.1 proven in Appendix B.

Theorem 2.1. *Suppose that J is irreducible with invariant distribution $\boldsymbol{\pi}$. For suitable normalisation of the local times of the process and its dual at the supremum,*

$$-\boldsymbol{\Psi}(\theta) = \boldsymbol{\Delta}_{\boldsymbol{\pi}}^{-1}\boldsymbol{\Psi}^{-}(-\theta)^{\top}\boldsymbol{\Delta}_{\boldsymbol{\pi}}\boldsymbol{\Psi}^{+}(\theta), \quad \theta \in \mathbb{R}.$$

The local times of (ξ, J) and $(\widehat{\xi}, \widehat{J})$ at the supremum are defined up to a multiplicative constant, possibly differing in each phase. Suppose that, due to this effect, we see ladder height processes given for $t \geq 0$ by $(\widetilde{H}_t^{\pm}, \widetilde{J}_t^{\pm}) = (H^{\pm}_{A_t^{\pm}}, J^{\pm}_{A_t^{\pm}})$, where $A_t^{\pm} = \int_0^t a^{\pm}_{J_s^{\pm}}\,\mathrm{d}s$ and $\boldsymbol{a}^{\pm}$ are positive vectors. It is not hard to see that these processes have exponents $\widetilde{\boldsymbol{\Psi}}^{\pm}(\theta) = \boldsymbol{\Delta}_{\boldsymbol{a}^{\pm}}\boldsymbol{\Psi}^{\pm}(\theta)$, for $\theta \in \mathbb{R}$, and so the Wiener-Hopf factorisation above can be expressed in the equation

$$-\boldsymbol{\Psi}(\theta) = \boldsymbol{\Delta}_{\boldsymbol{\pi}}^{-1}\widetilde{\boldsymbol{\Psi}}^{-}(-\theta)^{\top}\boldsymbol{\Delta}_{\boldsymbol{\pi}}\boldsymbol{\Delta}_{\boldsymbol{b}}\widetilde{\boldsymbol{\Psi}}^{+}(\theta), \quad \theta \in \mathbb{R}, \tag{2.1}$$

where $b_i = \frac{1}{a_i^+ a_i^-}$. On the other hand, if one starts from a factorisation of form (2.1), then by choosing local times with a different normalisation, one can recover Theorem 2.1.

Our analytical approach to study the MAP Wiener–Hopf factorisation fundamentally relies on the theory developed by Vigon in [47, 48], which allows to translate the Wiener–Hopf factorisation of a characteristic Lévy exponent into a convolution identity on the space of tempered distributions. Denote by $\mathcal{S}(\mathbb{R})$ the usual Schwartz space of rapidly decreasing functions $\varphi\colon \mathbb{R} \to \mathbb{C}$. Tempered distributions are the elements of its dual space $\mathcal{S}'(\mathbb{R})$. We define the Fourier transform of a function $\varphi \in \mathcal{S}(\mathbb{R})$ by $\mathscr{F}\varphi(x) \coloneqq \int_{\mathbb{R}} \varphi(y)\mathrm{e}^{\mathrm{i}xy}\,\mathrm{d}y$, $x \in \mathbb{R}$. The Fourier transform is an isomporphism on $\mathcal{S}(\mathbb{R})$, which is extended on the space of tempered distributions via the duality relation $\langle \mathscr{F}T, \varphi\rangle = \langle T, \mathscr{F}\varphi\rangle$, $\varphi \in \mathcal{S}(\mathbb{R})$, for a given tempered distribution $T \in \mathcal{S}'(\mathbb{R})$. We also recall that for $T \in \mathcal{S}'(\mathbb{R})$, its distributional derivative is specified by $\langle T', \varphi\rangle = -\langle T, \varphi'\rangle$ for $\varphi \in \mathcal{S}(\mathbb{R})$.

To make the connection to the MAP Wiener–Hopf factorisation, let ψ be the characteristic exponent of a Lévy process with characteristic triplet (a, σ, Π) and killing rate $\dagger$ and κ be the characteristic exponent of a finite variation Lévy process with drift d, Lévy measure Λ and killing rate $\bar{\dagger}$, i.e., for $x \in \mathbb{R}$,

$$\psi(x) = -\dagger + \mathrm{i}ax - \frac{\sigma^2}{2}x^2 + \int_{\mathbb{R}} (\mathrm{e}^{\mathrm{i}xy} - 1 - \mathbf{1}_{[-1,1]}(y)\mathrm{i}xy)\,\Pi(\mathrm{d}y),$$
$$\kappa(x) = -\bar{\dagger} + \mathrm{i}dx + \int_{\mathbb{R}} (\mathrm{e}^{\mathrm{i}xy} - 1)\,\Lambda(\mathrm{d}y).$$

By the integrability properties of Π and Λ, for any $\varphi \in \mathcal{S}(\mathbb{R})$, the compensated integrals

$$\langle \Gamma^2\Pi, \varphi\rangle \coloneqq \int_{\mathbb{R}} (\varphi(x) - 1 - \mathbf{1}_{[-1,1]}(x)\varphi'(x))\,\Pi(\mathrm{d}x), \quad \langle \Gamma\Lambda, \varphi\rangle \coloneqq \int_{\mathbb{R}} (\varphi(x) - 1)\,\Lambda(\mathrm{d}x),$$

are well defined. In fact, these relations define tempered distributions $\Gamma^2\Pi \in \mathcal{S}'(\mathbb{R})$ and $\Gamma\Lambda \in \mathcal{S}'(\mathbb{R})$, which now allow us to express ψ, κ (interpreted as tempered distributions induced by the slowly growing functions ψ, κ via $\langle \psi, \varphi\rangle \coloneqq \int \psi\varphi$ and $\langle \kappa, \varphi\rangle \coloneqq \int \kappa\varphi$) as Fourier transforms of tempered distributions:

$$\mathscr{F}\Big\{ -\dagger\delta - a\delta' + \frac{\sigma^2}{2}\delta'' + \Gamma^2\Pi \Big\} = \psi, \quad \mathscr{F}\Big\{ -\bar{\dagger}\delta - d\delta' + \Gamma\Lambda \Big\} = \kappa. \tag{2.2}$$

The elements of the matrix product in the rhs of (1.3) are given as linear combinations of products fg of (i) a subordinator exponent f with a negative subordinator exponent g, or (ii) a (negative or positive) subordinator exponent f and the Fourier transform of a finite measure $g = \mathscr{F}\mu$, or (iii) Fourier transforms of finite measures. Since all of these factors are locally bounded and have polynomial growth, their products again induce a tempered distribution, which by (2.2) is given by the Fourier transform of two tempered distributions. Moreover, for any characteristic Lévy exponent ψ, by splitting $\mathbb{I}^2\Pi$ into its restrictions on $[-1,1]$ and $[-1,1]^c$, it is clear from (2.2) that the tempered distribution $\mathscr{F}^{-1}\psi$ can be split into the sum of a tempered distribution with compact support $[-1,1]$ and a tempered distribution induced by the finite measure $\Pi|_{[-1,1]^c}$, see also [48, Proprieté 3.9].

Consequently, standard convolution theorems for tempered distributions show that for factors f, g as above with $f = \mathscr{F}T, g = \mathscr{F}S$ and S, T as in (2.2) or induced by a finite measure it holds that $\mathscr{F}^{-1}(fg) = S * T$, with the usual definition of convolutions of appropriate tempered distributions. Thus, by taking inverse Fourier transforms, the MAP Wiener–Hopf factorisation can be interpreted component wise in the sense of equalities of tempered distributions associated either to characteristic Lévy exponents or finite measures with a linear combination of convolved tempered distributions having factors of these types.

3. Friendship of MAPs

The matrix Wiener–Hopf factorisation (1.3) motivates the following definition.

Definition 3.1. Let $\boldsymbol{\pi} \in (0,1]^n$ be a stochastic vector and (H^+, J^+) and (H^-, J^-) be Markov additive subordinators. We say that (H^+, J^+) is a *$\boldsymbol{\pi}$-friend* of (H^-, J^-) if there exists a Markov additive process (ξ, J), such that

$$\boldsymbol{\Psi}(\theta) = -\boldsymbol{\Delta}_{\boldsymbol{\pi}}^{-1}\boldsymbol{\Psi}^-(-\theta)^\top \boldsymbol{\Delta}_{\boldsymbol{\pi}} \boldsymbol{\Psi}^+(\theta), \quad \theta \in \mathbb{R}, \tag{3.1}$$

and $\boldsymbol{\pi}^\top \boldsymbol{\Psi}(0) \le \mathbf{0}^\top$. Here, $\boldsymbol{\Delta}_{\boldsymbol{\pi}} = \operatorname{diag}(\boldsymbol{\pi})$ and $\boldsymbol{\Psi}, \boldsymbol{\Psi}^+$ and $\boldsymbol{\Psi}^-$ are the MAP exponents of (ξ, J), (H^+, J^+) and (H^-, J^-), respectively. In this case, we call (ξ, J) the *bonding MAP* of the $\boldsymbol{\pi}$-friends (H^+, J^+) with (H^-, J^-).

While the requirement (3.1) is motivated by the matrix Wiener–Hopf factorisation, the additional condition $\boldsymbol{\pi}^\top \boldsymbol{\Psi}(0) \le \mathbf{0}^\top$ makes sure that $\boldsymbol{\pi}$ is a valid candidate for an invariant distribution of the modulator J. This also entails that, as suggested by the name, $\boldsymbol{\pi}$-friendship is a symmetric relation:

Proposition 3.2. *(H^+, J^+) is a $\boldsymbol{\pi}$-friend of (H^-, J^-) if, and only if, (H^-, J^-) is a $\boldsymbol{\pi}$-friend of (H^+, J^+).*

Proof. Consider the matrix function

$$\widehat{\boldsymbol{\Psi}}(\theta) = \boldsymbol{\Delta}_{\boldsymbol{\pi}}^{-1}\boldsymbol{\Psi}(-\theta)^\top \boldsymbol{\Delta}_{\boldsymbol{\pi}} = -\boldsymbol{\Delta}_{\boldsymbol{\pi}}^{-1}\boldsymbol{\Psi}^+(-\theta)^\top \boldsymbol{\Delta}_{\boldsymbol{\pi}} \boldsymbol{\Psi}^-(\theta). \tag{3.2}$$

This function meets the conditions of Lemma A.1. In particular, the inequality $\boldsymbol{\pi}^\top \boldsymbol{\Psi}(0) \le \mathbf{0}^\top$ ensures that condition (iii) of that lemma is satisfied for $\widehat{\boldsymbol{\Psi}}$; and in turn, the condition $\boldsymbol{\pi}^\top \widehat{\boldsymbol{\Psi}}(0) \le \mathbf{0}^\top$ is implied by (iii) for $\boldsymbol{\Psi}$. □

Proposition 3.2 says that the bonding MAP of (H^-, J^-) with (H^+, J^+) is equal to the $\boldsymbol{\pi}$-dual of the bonding MAP of (H^+, J^+) with (H^-, J^-). As a consequence of this result, we will often refer to two MAPs as being *$\boldsymbol{\pi}$-friends*, rather than imposing an order.

Our goal in this section is two-fold:

(1) given two $\boldsymbol{\pi}$-friends, express the Lévy measure matrix of the bonding MAP in terms of $\boldsymbol{\pi}$ and the Lévy measure matrices of the friends;
(2) find necessary and sufficient criteria for two MAP subordinators to be $\boldsymbol{\pi}$-friends.

3.1. **The Lévy measure matrix of the bonding MAP.** Let us now investigate the relationship between the Lévy measure matrices of friends and their bonding MAP, complementing the équations amicales inversés for MAPs derived in [16] and extending the équations amicales for Lévy processes in [48].

The following results show that the Lévy characteristics and transitional jumps of friends must satisfy certain compatibility conditions. The proofs of the propositions will be given along the proof of Theorem 3.10. Proposition 3.3, describing the existence of densities of the jump measures associated to $\boldsymbol{\pi}$-friends subject to existence of Lévy drifts of the respective friend is the natural generalization of Proposition 3.1 in [48]. Proposition 3.4 and the following corollary demonstrate that transitional jumps and Lévy jumps in a friendship must be finely synchronized. This aspect has no counterpart for friendships of Lévy processes and is among the reasons why constructing explicit examples of MAP friendships is far from being a trivial extension of the strategy known for Lévy friendships.

The measures

$$\chi_i^+(\mathrm{d}x) = d_i^+\delta_0(\mathrm{d}x) + \mathbf{1}_{(0,\infty)}(x)\overline{\Pi}_i^+(x)\,\mathrm{d}x,$$

and

$$\widetilde{\chi}_i^-(\mathrm{d}x) = d_i^-\delta_0(\mathrm{d}x) + \mathbf{1}_{(-\infty,0)}(x)\overline{\overline{\Pi}}_i^-(x)\,\mathrm{d}x,$$

defined for $x \in \mathbb{R}$ and $i \in [n]$, will play an important role throughout the section. Probabilistically, χ_i^+ can be interpreted as the invariant measure of the overshoot process associated to (an unkilled version of) $H^{+,(i)}$, and $\widetilde{\chi}_i^-(-\,\mathrm{d}x)$ as the same quantity for $H^{-,(i)}$; see [16, Theorem 3.10].

Proposition 3.3. *Suppose that (H^+, J^+) is a $\boldsymbol{\pi}$-friend of (H^-, J^-). Then, for any $i \in [n]$, $\Pi_i^\pm$ has a density $\partial\Pi_i^\pm$ on $(0,\infty)$ if $d_i^\mp > 0$, which has a càdlàg version. Moreover, for any $i, j \in [n]$ with $i \neq j$, $F_{i,j}^\pm$ restricted to $(0,\infty)$ has a density $f_{i,j}^\pm$ if $d_i^\mp > 0$, which can also be chosen càdlàg.*

Proposition 3.4. *Suppose that (H^+, J^+) is a $\boldsymbol{\pi}$-friend of (H^-, J^-). Then, for any $i, j \in [n]$ with $i \neq j$, it holds that*

$$q_{i,j}^+ F_{i,j}^+ * \widetilde{\chi}_i^-(\mathrm{d}x) - \frac{\pi(j)}{\pi(i)} q_{j,i}^- \widetilde{F}_{j,i}^- * \chi_j^+(\mathrm{d}x) = \underline{\nu}_{i,j}(x)\,\mathrm{d}x, \quad x \in \mathbb{R},$$

for some finite signed measure $\nu_{i,j}$ with $\nu_{i,j}(\mathbb{R}) = 0$. Moreover,

$$q_{i,j}F_{i,j}(\{0\}) = (\dagger_i^- - q_{i,i}^-)q_{i,j}^+F_{i,j}^+(\{0\}) + (\dagger_j^+ - q_{j,j}^+)\frac{\pi(j)}{\pi(i)}q_{j,i}^-F_{j,i}^-(\{0\}) - \nu_{i,j}(\{0\})$$
$$- \sum_{k \neq i,j} \frac{\pi(k)}{\pi(i)} q_{k,i}^- q_{k,j}^+ \widetilde{F}_{k,i}^- * F_{k,j}^+(\{0\}).$$

Corollary 3.5. *Suppose that (H^+, J^+) is a $\boldsymbol{\pi}$-friend of (H^-, J^-) and that the measures $F_{i,j}^\pm$ are continuous on $(0,\infty)$ for distinct $i, j \in [n]$. Then,*

$$\boldsymbol{\Delta}_{\boldsymbol{d}}^- \boldsymbol{\Pi}^+(\{0\}) = \boldsymbol{\Delta}_{\boldsymbol{\pi}}^{-1}\big(\boldsymbol{\Delta}_{\boldsymbol{d}}^+ \boldsymbol{\Pi}^-(\{0\})\big)^\top \boldsymbol{\Delta}_{\boldsymbol{\pi}}.$$

Proof. By Proposition 3.4, the measure $q^+_{i,j}F^+_{i,j} * \widetilde{\chi}^-_i - \frac{\pi(j)}{\pi(i)} q^-_{j,i}\widetilde{F}^-_{j,i} * \chi^+_j$ is absolutely continuous. Using that the measures $F^{\pm}_{i,j}$ are continuous on $(0,\infty)$, we obtain for $i \neq j$ that

$$\begin{aligned} 0 &= q^+_{i,j}F^+_{i,j} * \widetilde{\chi}^-_i(\{0\}) - \frac{\pi(j)}{\pi(i)} q^-_{j,i}\widetilde{F}^-_{j,i} * \chi^+_j(\{0\}) \\ &= d^-_i q^+_{i,j}F^+_{i,j}(\{0\}) - d^+_j \frac{\pi(j)}{\pi(i)} q^-_{j,i}F^-_{j,i}(\{0\}), \end{aligned}$$

which implies

$$d^-_i q^+_{i,j}F^+_{i,j}(\{0\}) = d^+_j \frac{\pi(j)}{\pi(i)} q^-_{j,i}F^-_{j,i}(\{0\}),$$

proving the claim. □

The following statement refines the characterisation of the Lévy measures Π^+_i and Π^-_i of $\boldsymbol{\pi}$-friends in terms of càdlàg densities whenever $d^-_i > 0$ and $d^+_i > 0$, resp., as stated above. The proof is almost identical to that of Théorème 6.4.1 in [47] if we take into account the équations amicales of MAPs given in Theorem 3.10 below and is therefore omitted.

Proposition 3.6. *Suppose that (H^+, J^+) is a $\boldsymbol{\pi}$-friend of (H^-, J^-). Then, for any $i \in [n]$, if $d^{\mp}_i > 0$ the measure $\Pi^{\pm}_i$ restricted to $(0,\infty)$ has a càdlàg density $\partial\Pi^{\pm}_i = \overline{\eta}^{\pm}_i$, where $\eta^{\pm}_i$ is a signed measure on $(0,\infty)$.*

The necessary conditions for friendship given in Proposition 3.3, Proposition 3.4 and Proposition 3.6 together with the characterisation of MAP exponents in Lemma A.1 motivate the following definition.

Definition 3.7. A MAP subordinator (H^+, J^+) is called $\boldsymbol{\pi}$-compatible with a MAP subordinator (H^-, J^-) if the following conditions are satisfied:

(i) for any distinct $i, j \in [n]$, if $d^{\mp}_i > 0$, the measures $\Pi^{\pm}_i$ and $F^{\pm}_{i,j}$ restricted to $(0,\infty)$ have càdlàg densities $\partial\Pi^{\pm}_i$ and $f^{\pm}_{i,j}$, respectively, where $\partial\Pi^{\pm}_i$ is given as the right tail of signed measures $\eta^{\pm}_i$ on $(0,\infty)$;

(ii) for any distinct $i, j \in [n]$, there exists a finite signed measure $\nu_{i,j}$ such that $\nu_{i,j}(\mathbb{R}) = 0$ and

$$q^+_{i,j}F^+_{i,j} * \widetilde{\chi}^-_i(\mathrm{d}x) - \frac{\pi(j)}{\pi(i)} q^-_{j,i}\widetilde{F}^-_{j,i} * \chi^+_j(\mathrm{d}x) = \underline{\nu}_{i,j}(x)\,\mathrm{d}x, \quad x \in \mathbb{R}, \tag{3.3}$$

and moreover

$$\begin{aligned} 0 \leq{}& (\dagger^-_i - q^-_{i,i})q^+_{i,j}F^+_{i,j}(\{0\}) + (\dagger^+_j - q^+_{j,j})\frac{\pi(j)}{\pi(i)} q^-_{j,i}F^-_{j,i}(\{0\}) - \nu_{i,j}(\{0\}) \\ &- \sum_{k\neq i,j} \frac{\pi(k)}{\pi(i)} q^-_{k,i}q^+_{k,j}\widetilde{F}^-_{k,i} * F^+_{k,j}(\{0\}); \end{aligned} \tag{3.4}$$

(iii) the vector $\boldsymbol{\Delta}^{-1}_{\boldsymbol{\pi}}\boldsymbol{\Psi}^-(0)^\top\boldsymbol{\Delta}_{\boldsymbol{\pi}}\boldsymbol{\Psi}^+(0)\mathbf{1}$ is nonnegative.

(iv) the vector $\boldsymbol{\pi}^\top\boldsymbol{\Delta}^{-1}_{\boldsymbol{\pi}}\boldsymbol{\Psi}^-(0)^\top\boldsymbol{\Delta}_{\boldsymbol{\pi}}\boldsymbol{\Psi}^+(0)$ is nonnegative.

Remark 3.8.

(i) When (H^+, J^+) is unkilled, (iii) is satisfied and the vector in question is zero; and when (H^-, J^-) is unkilled, the same applies to (iv).

(ii) If condition (i) of $\boldsymbol{\pi}$-compatibility holds, then condition (ii) is equivalent to:

(a) $q^+_{i,j} d^-_i F^+_{i,j}(\{0\}) - \frac{\pi(j)}{\pi(i)} q^-_{j,i} d^+_i F^-_{j,i}(\{0\}) = 0$,

(b) the expression

$$q^+_{i,j}\left(\int_{\mathbb{R}} \mathbf{1}_{\{y\geq 0, y>x\}} \overline{\Pi}^-_i(y-x) F^+_{i,j}(\mathrm{d}y) + d^-_i f^+_{i,j}(x)\right) \\ - \frac{\pi(j)}{\pi(i)} q^-_{j,i}\left(\int_{\mathbb{R}} \mathbf{1}_{\{y\geq 0, -x<y\}} \overline{\Pi}^+_j(x+y)\, F^-_{j,i}(\mathrm{d}y) + d^+_j f^-_{j,i}(-x)\right)$$

is a.e. equal to a right-continuous, bounded variation function of x, denoted $n_{i,j}(x)$, whose limit as $x \to \pm\infty$ is 0, and

(c) the inequality

$$(\dagger^-_i - q^-_{i,i}) q^+_{i,j} F^+_{i,j}(\{0\}) + (\dagger^+_j - q^+_{j,j}) \frac{\pi(j)}{\pi(i)} q^-_{j,i} F^-_{j,i}(\{0\}) \\ - \sum_{k\neq i,j} \frac{\pi(k)}{\pi(i)} q^-_{k,i} q^+_{k,j} \widetilde{F}^-_{k,i} * F^+_{k,j}(\{0\}) - \lim_{x\downarrow 0} n_{i,j}(x) + \lim_{x\uparrow 0} n_{i,j}(x) \geq 0$$

holds.

Lemma 3.9. *(H^+, J^+) is $\boldsymbol{\pi}$-compatible with (H^-, J^-) if, and only if, (H^-, J^-) is $\boldsymbol{\pi}$-compatible with (H^+, J^+).*

Proof. We prove one direction of the equivalence, the other following immediately by swapping (H^+, J^+) and (H^-, J^-). Clearly, condition (i) for $\boldsymbol{\pi}$-compatibility between (H^-, J^-) and (H^+, J^+) is satisfied since (H^+, J^+) is assumed to be $\boldsymbol{\pi}$-compatible with (H^-, J^-). Next, we have

$$\begin{aligned} &- \frac{\pi(i)}{\pi(j)}\left(q^-_{i,j} F^-_{i,j} * \widetilde{\chi}^+_i(\mathrm{d}x) - \frac{\pi(j)}{\pi(i)} q^+_{j,i} \widetilde{F}^+_{j,i} * \chi^-_j(\mathrm{d}x)\right) \\ &= q^+_{j,i} \widetilde{F}^+_{j,i} * \chi^-_j(\mathrm{d}x) - \frac{\pi(i)}{\pi(j)} q^-_{i,j} F^-_{i,j} * \widetilde{\chi}^+_i(\mathrm{d}x) \\ &= q^+_{j,i} F^+_{j,i} * \widetilde{\chi}^-_j(-\mathrm{d}x) - \frac{\pi(i)}{\pi(j)} q^-_{i,j} \widetilde{F}^-_{i,j} * \chi^+_i(-\mathrm{d}x) \\ &= \underline{\nu}_{j,i}(-x)\,\mathrm{d}x \\ &= -\bar{\nu}_{j,i}(-x)\,\mathrm{d}x \\ &= -\widetilde{\nu}_{j,i}((-\infty, x])\,\mathrm{d}x, \end{aligned}$$

where for the penultimate line we used $\nu_{j,i}(\mathbb{R}) = 0$. Thus, (3.3) holds with with the role of $(H^\pm, J^\pm)$ reversed and $\nu_{i,j}$ replaced by the finite signed measure $\rho_{i,j}(\mathrm{d}x) = \frac{\pi(j)}{\pi(i)} \nu_{j,i}(-\mathrm{d}x)$, for which $\rho_{i,j}(\mathbb{R}) = 0$. Moreover,

$$\begin{aligned} &\frac{\pi(i)}{\pi(j)}\Big((\dagger^+_i - q^+_{i,i}) q^-_{i,j} F^-_{i,j}(\{0\}) + (\dagger^-_j - q^-_{j,j}) \frac{\pi(j)}{\pi(i)} q^+_{j,i} F^+_{j,i}(\{0\}) - \rho_{i,j}(\{0\}) \\ &\quad - \sum_{k\neq i,j} \frac{\pi(k)}{\pi(i)} q^+_{k,i} q^-_{k,j} \widetilde{F}^+_{k,i} * F^-_{k,j}(\{0\})\Big) \\ &= (\dagger^-_j - q^-_{j,j}) q^+_{j,i} F^+_{j,i}(\{0\}) + \frac{\pi(i)}{\pi(j)} (\dagger^+_i - q^+_{i,i}) q^-_{i,j} F^-_{i,j}(\{0\}) - \nu_{j,i}(\{0\}) \\ &\quad - \sum_{k\neq i,j} \frac{\pi(k)}{\pi(j)} q^+_{k,i} q^-_{k,j} \widetilde{F}^+_{k,i} * F^-_{k,j}(\{0\}) \end{aligned}$$

$\geq 0,$

by the assumption that (H^+, J^+) is $\boldsymbol{\pi}$-compatible with (H^-, J^-). Consequently, (3.4) is satisfied as well with the role of $(H^\pm, J^\pm)$ reversed and $\rho_{i,j}$ in place of $\nu_{i,j}$. We next need to verify that (a) $\boldsymbol{\Delta}_{\boldsymbol{\pi}}^{-1}\boldsymbol{\Psi}^+(0)^\top \boldsymbol{\Delta}_{\boldsymbol{\pi}} \boldsymbol{\Psi}^-(0)\mathbf{1}$ using hypothesis (b) $\boldsymbol{\pi}^\top \boldsymbol{\Delta}_{\boldsymbol{\pi}}^{-1}\boldsymbol{\Psi}^-(0)^\top \boldsymbol{\Delta}_{\boldsymbol{\pi}} \boldsymbol{\Psi}^+(0) \geq \mathbf{0}^\top$. Statement (b) is equivalent to $\mathbf{0} \leq \mathbf{1}^\top \boldsymbol{\Psi}^-(0)^\top \boldsymbol{\Delta}_{\boldsymbol{\pi}} \boldsymbol{\Psi}^+(0)$ and statement (a) is equivalent to

$$\mathbf{0}^\top \leq \boldsymbol{\Delta}_{\boldsymbol{\pi}}\big(\boldsymbol{\Delta}_{\boldsymbol{\pi}}^{-1}\boldsymbol{\Psi}^+(0)^\top \boldsymbol{\Delta}_{\boldsymbol{\pi}} \boldsymbol{\Psi}^-(0)\mathbf{1}\big) = \big(\mathbf{1}^\top \boldsymbol{\Psi}^-(0)^\top \boldsymbol{\Delta}_{\boldsymbol{\pi}} \boldsymbol{\Psi}^+(0)\big)^\top.$$

It follows that (b) implies (a). The proof that $\boldsymbol{\pi}^\top \boldsymbol{\Delta}_{\boldsymbol{\pi}}^{-1}\Psi^+(0)^\top \boldsymbol{\Delta}_{\boldsymbol{\pi}} \boldsymbol{\Psi}^-(0)$ is nonnegative is analogous. ☐

Given two $\boldsymbol{\pi}$-compatible MAP subordinators (H^+, J^+) and (H^-, J^-), let

$$\partial\boldsymbol{\Pi}^+(x) := (\partial\Pi_i^+(x)\mathbf{1}_{\{i=j\}} + q_{i,j}^+ f_{i,j}^+(x)\mathbf{1}_{\{i\neq j\}})_{i,j=1,\dots,n}, \quad x > 0,$$

where $\partial\Pi_i^+$ is the absolutely continuous part of the measure Π_i^+ and $f_{i,j}^+$ the absolutely continuous part of $F_{i,j}^+$, and in case $d_i^- > 0$ we use the càdlàg versions guaranteed by the definition of $\boldsymbol{\pi}$-compatibility. We call $\partial\boldsymbol{\Pi}^+$ the *Lévy density matrix* of (H^+, J^+), and define $\partial\boldsymbol{\Pi}^-$ for (H^-, J^-) analogously.

Theorem 3.10 (Équations amicales for MAPs). *Suppose that (H^+, J^+) is a $\boldsymbol{\pi}$-friend of (H^-, J^-). Then, for a.e. $x > 0$*

$$\boldsymbol{\Pi}(x,\infty) = \int_{x+}^{\infty} \boldsymbol{\Delta}_{\boldsymbol{\pi}}^{-1}\Big(\overline{\boldsymbol{\Pi}}^-(y-x) - \boldsymbol{\Psi}^-(0)\Big)^\top \boldsymbol{\Delta}_{\boldsymbol{\pi}}\, \boldsymbol{\Pi}^+(\mathrm{d}y) + \boldsymbol{\Delta}_{\boldsymbol{d}}^- \partial\boldsymbol{\Pi}^+(x), \tag{3.5}$$

and for a.e. $x < 0$

$$\boldsymbol{\Pi}(-\infty,x) = \int_{(-x)+}^{\infty} \boldsymbol{\Delta}_{\boldsymbol{\pi}}^{-1}\big(\boldsymbol{\Pi}^-(\mathrm{d}y)\big)^\top \boldsymbol{\Delta}_{\boldsymbol{\pi}} \big(\overline{\boldsymbol{\Pi}}^+(y+x) - \boldsymbol{\Psi}^+(0)\big) + \boldsymbol{\Delta}_{\boldsymbol{\pi}}^{-1}\big(\boldsymbol{\Delta}_{\boldsymbol{d}}^+ \partial\boldsymbol{\Pi}^-(-x)\big)^\top \boldsymbol{\Delta}_{\boldsymbol{\pi}}. \tag{3.6}$$

Proof of Propositions 3.3 and 3.4 and Theorem 3.10. Let us define the measures $\widetilde{\Pi}_i^-(\mathrm{d}x) = \Pi_i^-(-\mathrm{d}x)$ and $\widetilde{F}_{i,j}^-(\mathrm{d}x) = F_{i,j}^-(-\mathrm{d}x)$ and recall (2.2). Taking inverse Fourier transforms on both sides of (3.1) (intepreted in the sense of distributions) we obtain for $i = j$,

$$\begin{aligned}
&(q_{i,i} - \dagger_i)\delta - a_i\delta' + \frac{1}{2}\sigma_i^2\delta'' + \mathbb{F}^2\Pi_i \\
&= -\Big(\big((q_{i,i}^- - \dagger_i^-)\delta + d_i^-\delta' + \mathbb{F}\widetilde{\Pi}_i^-\big) * \big((q_{i,i}^+ - \dagger_i^+)\delta - d_i^+\delta' + \mathbb{F}\Pi_i^+\big) \\
&\quad + \sum_{k\neq i} \frac{\pi(k)}{\pi(i)} q_{k,i}^- q_{k,i}^+ \widetilde{F}_{k,i}^- * F_{k,i}^+\Big).
\end{aligned} \tag{3.7}$$

Above, all convolutions are well defined since $\mathbb{F}\widetilde{\Pi}_i^-$ and $\mathbb{F}\Pi_i^+$ can both be decomposed into the sum of a distribution with compact support and a distribution induced by a finite measure, respectively (see also Proprieté 3.9 in [48]). Let $\varrho(x) = \mathbf{1}_{(0,\infty)}(x) - \mathbf{1}_{(-\infty,0)}(x)$, $\overline{\Pi}_i(x) = \mathbf{1}_{(0,\infty)}(x)\Pi_i(x,\infty) + \mathbf{1}_{(-\infty,0)}(x)\Pi_i(-\infty,x)$ and $\overline{\widetilde{\Pi}}_i^-(x) = \mathbf{1}_{(-\infty,0)}(x)\widetilde{\Pi}_i^-(-\infty,x)$. Then, $(\mathbb{F}\varrho\overline{\Pi}_i)' = -\mathbb{F}^2\Pi_i + c_i\delta'$ for some constant $c_i \in \mathbb{R}$ and $(\overline{\widetilde{\Pi}}_i^-)' = \mathbb{F}\widetilde{\Pi}_i^-$ by Lemma 3.12 in [48]. Moreover, it is easily shown that for $\underline{\widetilde{F}}^-_{i,j}(x) = \mathbf{1}_{\mathbb{R}_-}(x)\widetilde{F}^-_{i,j}([x,0])$ we have $(\underline{\widetilde{F}}^-_{i,j})' = -\widetilde{F}^-_{i,j}$. Thus, taking primitives on both sides of (3.7) we obtain

$$\begin{aligned}&(q_{i,i} - \dagger_i)\mathbf{1}_{\mathbb{R}_-} + a_i\delta - \tfrac{1}{2}\sigma_i^2\delta' + \Gamma\varrho\overline{\Pi}_i \\ &\quad = \big((\dagger_i^- - q^-_{i,i})\mathbf{1}_{\mathbb{R}_-} + d_i^-\delta + \overline{\overline{\Pi}}{}_i^-\big) * \big((q^+_{i,i} - \dagger_i^+)\delta - d_i^+\delta' + \Gamma\Pi_i^+\big) \\ &\qquad - \sum_{k\neq i}\frac{\pi(k)}{\pi(i)}q^-_{k,i}q^+_{k,i}\underline{\widetilde{F}}^-_{k,i} * F^+_{k,i} + c_i\delta + c_{\text{int}},\end{aligned}$$

for some integration constant $c_{\text{int}} \in \mathbb{R}$. By resticting to $(0,\infty)$ this implies that we have the following equality of distributions in $\mathcal{D}'_{(0,\infty)}$:

$$\begin{aligned}&\overline{\Pi}_i|_{(0,\infty)} \\ &\quad = \big((\dagger_i^- - q^-_{i,i})\mathbf{1}_{\mathbb{R}_-} + d_i^-\delta + \overline{\overline{\Pi}}{}_i^-\big) * \Pi_i^+|_{(0,\infty)} \\ &\qquad - \sum_{k\neq i}\frac{\pi(k)}{\pi(i)}q^-_{k,i}q^+_{k,i}\underline{\widetilde{F}}^-_{k,i} * F^+_{k,i}|_{(0,\infty)} + c_{\text{int}} \\ &\quad = (\dagger_i^- - q^-_{i,i})\overline{\Pi}_i^+|_{(0,\infty)} + d_i^-\Pi_i^+|_{(0,\infty)} + \overline{\overline{\Pi}}{}_i^- * \Pi_i^+|_{(0,\infty)} \\ &\qquad - \sum_{k\neq i}\frac{\pi(k)}{\pi(i)}q^-_{k,i}q^+_{k,i}\underline{\widetilde{F}}^-_{k,i} * F^+_{k,i}|_{(0,\infty)} + c_{\text{int}}.\end{aligned} \tag{3.8}$$

Here we used Proprieté 3.8 in [48], telling us that for tempered distributions S, T, where T is supported on $\mathbb{R}_-$ and the convolution $T*S$ is well defined as a tempered distribution, it holds $(T*S)|_{(0,\infty)} = (T*S|_{(0,\infty)})|_{(0,\infty)}$. Since all other terms are distributions induced by some function on $(0,\infty)$, it follows that if $d_i^- > 0$, $\Pi_i^+|_{(0,\infty)}$ is also induced by a function, i.e. Π_i^+ possesses a Lebesgue density $\partial\Pi_i^+$ on $(0,\infty)$. Let us first show that $c_{\text{int}} = 0$. Let $\varphi \in \mathcal{D}_{(0,\infty)}$ be non-negative with $\operatorname{supp}(\varphi) \subset (0,1)$, $\|\varphi\|_\infty \leq 1$, $\int\varphi = 1/2$ and $\varphi_z = \varphi(\cdot - z)$ for $z > 0$. Then, with multiple uses of Fubini,

$$\begin{aligned}&\int_{\mathbb{R}}\varphi_z(x)\overline{\overline{\Pi}}{}_i^- * \Pi_i^+(x)\,\mathrm{d}x \\ &\quad \leq \int_z^{z+1}\overline{\overline{\Pi}}{}_i^- * \Pi_i^+(x)\,\mathrm{d}x \\ &\quad = \int_{-\infty}^0\int_0^\infty \mathbf{1}_{(z,z+1)}(x+y)\,\Pi_i^+(\mathrm{d}x)\overline{\overline{\Pi}}{}_i^-(y)\,\mathrm{d}y \\ &\quad = \int_0^\infty \Pi_i^+((z+y,z+y+1))\overline{\Pi}_i^-(y)\,\mathrm{d}y \\ &\quad \leq \Pi_i^+((z,z+2))\int_0^1\overline{\Pi}_i^-(y)\,\mathrm{d}y + \int_1^\infty \Pi_i^+((z+y,z+y+1))\overline{\Pi}_i^-(y)\,\mathrm{d}y \\ &\quad \leq \Pi_i^+((z,z+2))\int_0^1\overline{\Pi}_i^-(y)\,\mathrm{d}y + \overline{\Pi}_i^-(1)\int_{1+z}^\infty\int_{(u-(z+2))\vee 1}^{u-(z+1)}\mathrm{d}y\,\Pi_i^+(\mathrm{d}u) \\ &\quad \leq \Pi_i^+((z,z+2))\int_0^1\overline{\Pi}_i^-(y)\,\mathrm{d}y \\ &\qquad + \overline{\Pi}_i^-(1)\big(\Pi_i^+(z+2,\infty) + \int_{z+1}^{z+2}(u-z-1)\Pi_i^+(\mathrm{d}u)\big)\end{aligned}$$

$$\leq \Pi_i^+((z,z+2))\int_0^1 \overline{\Pi}_i^-(y)\,\mathrm{d}y + \overline{\Pi}_i^-(1)\Pi_i^+(z+1,\infty),$$

which establishes

$$\int_{\mathbb{R}} \varphi_z(x)\overline{\overline{\Pi}}_i^- * \Pi_i^+(x)\,\mathrm{d}x \underset{z\to\infty}{\longrightarrow} 0. \tag{3.9}$$

Writing

$$\begin{aligned}\mu = \overline{\Pi}_i|_{(0,\infty)} - (\dagger_i^- - q_{i,i}^-)\overline{\Pi}_i^+|_{(0,\infty)} - d_i^-\Pi_i^+|_{(0,\infty)} - \overline{\overline{\Pi}}_i^- * \Pi_i^+|_{(0,\infty)} \\ + \sum_{k\neq i}\frac{\pi(k)}{\pi(i)} q_{k,i}^- q_{k,i}^+ \underline{\widetilde{F}}_{k,i}^- * F_{k,i}^+|_{(0,\infty)},\end{aligned}$$

it therefore follows from (3.8) that

$$\frac{c_{\text{int}}}{2} = c_{\text{int}}\int \varphi_z(x)\,\mathrm{d}x = \langle \mu, \varphi_z\rangle \underset{z\to\infty}{\longrightarrow} 0,$$

hence $c_{\text{int}} = 0$. Next, let us show that if $d_i^- > 0$, $\partial\Pi_i^+$ has a càdlàg version. Reordering (3.8) using $c_{\text{int}} = 0$ we obtain

$$\begin{aligned}\overline{\Pi}_i|_{(0,\infty)} - (\dagger_i^- - q_{i,i}^-)\overline{\Pi}_i^+|_{(0,\infty)} + \sum_{k\neq i}\frac{\pi(k)}{\pi(i)} q_{k,i}^- q_{k,i}^+ \underline{\widetilde{F}}_{k,i}^- * F_{k,i}^+|_{(0,\infty)} \\ = d_i^-\Pi_i^+|_{(0,\infty)} + \overline{\overline{\Pi}}_i^- * \Pi_i^+|_{(0,\infty)}.\end{aligned} \tag{3.10}$$

Since the left hand side is a distribution induced by a function that is bounded away from 0 it follows that the Lebesgue density of Π_i^+ has a version g_i^+ that is bounded away from zero as well, i.e., for any $x > 0$ it holds that

$$\sup_{z\geq x} g_i^+(z) < \infty. \tag{3.11}$$

Integration by parts then shows

$$\overline{\overline{\Pi}}_i^- * \Pi_i^+(x) = \int_{(0,\infty)}\int_x^{x+y} g_i^+(z)\,\mathrm{d}z\,\Pi_i^-(\mathrm{d}y),$$

such that dominated convergence in conjunction with (3.11) and the integrability properties of the Lévy measure Π_i^- readily imply that $x \mapsto \overline{\overline{\Pi}}_i^- * \Pi_i^+(x)$ is continuous on $(0,\infty)$. Hence, the function

$$\begin{aligned}\partial\Pi_i^+(x) \coloneqq \frac{1}{d_i^-}\Big(\overline{\Pi}_i|_{(0,\infty)}(x) - (\dagger_i^- - q_{i,i}^-)\overline{\Pi}_i^+|_{(0,\infty)}(x) \\ + \sum_{k\neq i}\frac{\pi(k)}{\pi(i)} q_{k,i}^- q_{k,i}^+ \int_{(x,\infty)} F_{k,i}^-([0,y-x))\,F_{k,i}^+(\mathrm{d}y) - \overline{\overline{\Pi}}_i^- * g_i^+(x)\Big),\ x > 0,\end{aligned}$$

is càdlàg and by (3.10) is the desired càdlàg version of the Lebesgue density of Π_i^+ on $(0,\infty)$.

It now follows from above that the equality (3.8) of distributions in $\mathcal{D}'_{(0,\infty)}$ translates to the equality of functions

$$\begin{aligned}\Pi_i(x,\infty) = (\dagger_i^- - q_{i,i}^-)\overline{\Pi}_i^+(x) + d_i^-\partial\Pi_i^+(x) + \int_{x+}^{\infty} \overline{\Pi}_i^-(y-x)\,\Pi_i^+(\mathrm{d}y) \\ - \sum_{k\neq i}\frac{\pi(k)}{\pi(i)} q_{k,i}^- q_{k,i}^+ \int_{x+}^{\infty} F_{k,i}^-([0,y-x])\,F_{k,i}^+(\mathrm{d}y),\end{aligned} \tag{3.12}$$

which holds for a.e. $x > 0$.

Next, for $i \neq j$, it follows again by taking inverse Fourier transforms on the (i,j)th entry of (3.1) that

$$\begin{aligned} q_{i,j} F_{i,j} = -\Big\{ & q^+_{i,j} \big((q^-_{i,i} - \dagger^-_i)\delta + d^-_i \delta' + \Gamma \widetilde{\Pi}^-_i \big) * F^+_{i,j} \\ & + \frac{\pi(j)}{\pi(i)} q^-_{j,i} \widetilde{F}^-_{j,i} * \big((q^+_{j,j} - \dagger^+_j)\delta - d^+_j \delta' + \Gamma \Pi^+_j \big) \\ & + \sum_{k \neq i,j} \frac{\pi(k)}{\pi(i)} q^-_{k,i} q^+_{k,j} \widetilde{F}^-_{k,i} * F^+_{k,j} \Big\}. \end{aligned} \tag{3.13}$$

From this it follows that

$$\begin{aligned} & q^+_{i,j} F^+_{i,j} * \big(d^-_i \delta' + \Gamma \widetilde{\Pi}^-_i \big) + \frac{\pi(j)}{\pi(i)} q^-_{j,i} \widetilde{F}^-_{j,i} * \big(- d^+_j \delta' + \Gamma \Pi^+_j \big) \\ & = q^+_{i,j} F^+_{i,j} * \big(\widetilde{\chi}^-_i \big)' - \frac{\pi(j)}{\pi(i)} q^-_{j,i} \widetilde{F}^-_{j,i} * \big(\chi^+_j \big)' \\ & = \Big(q^+_{i,j} F^+_{i,j} * \widetilde{\chi}^-_i - \frac{\pi(j)}{\pi(i)} q^-_{j,i} \widetilde{F}^-_{j,i} * \chi^+_j \Big)', \end{aligned}$$

must be induced by a finite signed measure. By Lemma A.2, this implies the existence of a finite signed measure $\nu_{i,j}$ such that

$$q^+_{i,j} F^+_{i,j} * \widetilde{\chi}^-_i - \frac{\pi(j)}{\pi(i)} q^-_{j,i} \widetilde{F}^-_{j,i} * \chi^+_j = \underline{\nu}_{i,j} + c \tag{3.14}$$

for some $c \in \mathbb{R}$. Letting φ_z as before and arguing as in (3.9), we obtain

$$\lim_{z \to \infty} \int \varphi_z(x) \overline{\overline{\Pi}}^-_i * F^+_{i,j}(x)\, \mathrm{d}x = 0, \qquad \lim_{z \to -\infty} \int \varphi_z(-x) q^-_{j,i} \widetilde{F}^-_{j,i} * \overline{\Pi}^+_j(x)\, \mathrm{d}x = 0. \tag{3.15}$$

Since for $x > z > 0$ we have $\widetilde{F}^-_{j,i} * \overline{\Pi}^+_j(x) \leq \overline{\Pi}^+_j(z)$ and for $x < z < 0$ it holds $\overline{\overline{\Pi}}^-_i * F^+_{i,j}(x) \leq \overline{\Pi}^-_i(-z)$, we obtain

$$\lim_{z \to -\infty} \int \varphi_z(-x) \overline{\overline{\Pi}}^-_i * F^+_{i,j}(x)\, \mathrm{d}x = 0, \qquad \lim_{z \to \infty} \int \varphi_z(x) q^-_{j,i} \widetilde{F}^-_{j,i} * \overline{\Pi}^+_j(x)\, \mathrm{d}x = 0. \tag{3.16}$$

Consequently, using also $\lim_{x \to -\infty} \underline{\nu}_{i,j}(x) = 0$, it follows

$$\frac{c}{2} = \int \varphi_z(-x)\, \mathrm{d}x = \Big\langle q^+_{i,j} F^+_{i,j} * \widetilde{\chi}^-_i - \frac{\pi(j)}{\pi(i)} q^-_{j,i} \widetilde{F}^-_{j,i} * \chi^+_j - \underline{\nu}_{i,j}, \varphi_z(-\cdot) \Big\rangle \underset{z \to -\infty}{\longrightarrow} 0,$$

whence, $c = 0$. Since $\nu_{i,j}$ is finite we may now write

$$q^+_{i,j} F^+_{i,j} * \widetilde{\chi}^-_i - \frac{\pi(j)}{\pi(i)} q^-_{j,i} \widetilde{F}^-_{j,i} * \chi^+_j = \nu_{i,j}(\mathbb{R}) - \bar{\nu}_{i,j}.$$

Hence, from $\bar{\nu}_{i,j}(x) \underset{x \to \infty}{\longrightarrow} 0$ and (3.15), (3.16) it follows

$$\begin{aligned} \frac{\nu_{i,j}(\mathbb{R})}{2} = \nu_{i,j}(\mathbb{R}) \int \varphi_z(x)\, \mathrm{d}x &= \Big\langle q^+_{i,j} F^+_{i,j} * \widetilde{\chi}^-_i - \frac{\pi(j)}{\pi(i)} q^-_{j,i} \widetilde{F}^-_{j,i} * \chi^+_j + \bar{\nu}_{i,j}, \varphi_z \Big\rangle \\ & \underset{z \to \infty}{\longrightarrow} 0, \end{aligned}$$

whence, $\nu_{i,j}(\mathbb{R}) = 0$. We now also obtain from (3.13),

$$q_{i,j} F_{i,j}(\{0\}) = (\dagger^-_i - q^-_{i,i}) q^+_{i,j} F^+_{i,j}(\{0\}) + (\dagger^+_j - q^+_{j,j}) \frac{\pi(j)}{\pi(i)} q^-_{j,i} F^-_{j,i}(\{0\}) - \nu_{i,j}(\{0\})$$

$$- \sum_{k\neq i,j} \frac{\pi(k)}{\pi(i)} q^-_{k,i} q^+_{k,j} \widetilde{F}^-_{k,i} * F^+_{k,j}(\{0\}).$$

Taking everything together establishes Proposition 3.4. Let now

$$\overline{F}_{i,j}(x) = F_{i,j}((x,\infty))\mathbf{1}_{(0,\infty)}(x) + F_{i,j}((-\infty,x))\mathbf{1}_{(-\infty,0)}(x).$$

Then, $(\varrho\overline{F}_{i,j})' = -F_{i,j} + \delta$, and hence, together with the considerations above we obtain by taking primitives on (3.13)

$$\begin{aligned}
\varrho q_{i,j}\overline{F}_{i,j} = {} & q^+_{i,j}\big((\dagger^-_i - q^-_{i,i})\mathbf{1}_{\mathbb{R}_-} + d^-_i\delta + \overline{\overline{\widetilde{\Pi}}}{}^-_i\big) * F^+_{i,j} \\
& - \frac{\pi(j)}{\pi(i)} q^-_{j,i} \underline{\widetilde{F}}^-_{j,i} * \big((q^+_{j,j} - \dagger^+_j)\delta - d^+_j\delta' + \mathbb{\Gamma}\Pi^+_j\big) \\
& - \sum_{k\neq i,j} \frac{\pi(k)}{\pi(i)} q^-_{k,i} q^+_{k,j} \underline{\widetilde{F}}^-_{k,i} * F^+_{k,j} - q_{i,j}\mathbf{1}_{\mathbb{R}_-} + c_{\mathrm{int}},
\end{aligned}$$

where c_{int} is an integration constant. Similarly to the on-diagonal case above, restricting to $(0,\infty)$ shows that $F^+_{i,j}$ possesses a càdlàg density $f^+_{i,j}$ whenever $d^-_i > 0$ (recall that $F^+_{i,j}$ was chosen to be trivial on $(0,\infty)$ when $q^+_{i,j} = 0$) and that for a.e. $x > 0$

(3.17)

$$\begin{aligned}
q_{i,j}F_{i,j}(x,\infty) = {} & q^+_{i,j}\Big((\dagger^-_i - q^-_{i,i})\overline{F}^+_{i,j}(x) + d^-_i f^+_{i,j}(x) + \int_{x+}^\infty \overline{\Pi}^-_i(y-x)\, F^+_{i,j}(\mathrm{d}y)\Big) \\
& - \frac{\pi(j)}{\pi(i)} q^-_{j,i} \int_{x+}^\infty F^-_{j,i}([0,y-x])\, \Pi^+_j(\mathrm{d}y) \\
& - \sum_{k\neq i,j} \frac{\pi(k)}{\pi(i)} q^-_{k,i} q^+_{k,j} \int_{x+}^\infty F^-_{k,i}([0,y-x])\, F^+_{k,j}(\mathrm{d}y) \\
= {} & q^+_{i,j}\Big((\dagger^-_i - q^-_{i,i})\overline{F}^+_{i,j}(x) + d^-_i f^+_{i,j}(x) + \int_{x+}^\infty \overline{\Pi}^-_i(y-x)\, F^+_{i,j}(\mathrm{d}y)\Big) \\
& + \frac{\pi(j)}{\pi(i)} \Big(q^-_{j,i} \int_{x+}^\infty \overline{F}^-_{j,i}(y-x)\, \Pi^+_j(\mathrm{d}y) - q^-_{j,i}\overline{\Pi}^+_j(x)\Big) \\
& + \sum_{k\neq i,j} \frac{\pi(k)}{\pi(i)} \Big(q^-_{k,i} q^+_{k,j} \int_{x+}^\infty \overline{F}^-_{k,i}(y-x)\, F^+_{k,j}(\mathrm{d}y) - q^-_{k,i} q^+_{k,j}\overline{F}^+_{k,j}(x)\Big).
\end{aligned}$$

Above, $c_{\mathrm{int}} = 0$ follows by arguing as in the on-diagonal case and using (3.15), (3.16). Combining (3.12) and (3.17) yields (3.5). Relation (3.6) and the claims on existence of càdlàg densities of $F^-_{i,j}$ and Π^-_i whenever $d^+_i > 0$ are proved analogously. □

3.2. **Characterisation of friendship.** We are now ready to fully characterise friendships of MAPs. Combining Theorem 3.11 with Theorem 3.10 yields our main result Theorem 1.4.

Theorem 3.11 (Theorem of friends for MAPs)**.** *Two MAP subordinators (H^+, J^+) and (H^-, J^-) are $\boldsymbol{\pi}$-friends if, and only if, they are $\boldsymbol{\pi}$-compatible and the matrix-valued function*

$$\boldsymbol{\Upsilon}(x) = \Big\{ \int_{x+}^\infty \boldsymbol{\Delta}_{\boldsymbol{\pi}}^{-1}\Big(\overline{\boldsymbol{\Pi}}^-(y-x) - \boldsymbol{\Psi}^-(0)\Big)^\top \boldsymbol{\Delta}_{\boldsymbol{\pi}}\, \boldsymbol{\Pi}^+(\mathrm{d}y) + \boldsymbol{\Delta}^-_{\boldsymbol{d}}\partial\boldsymbol{\Pi}^+(x)\Big\}\mathbf{1}_{(0,\infty)}(x)$$

$$+\Big\{\int_{(-x)+}^{\infty}\mathbf{\Delta}_{\boldsymbol{\pi}}^{-1}\big(\mathbf{\Pi}^{-}(\mathrm{d}y)\big)^{\top}\mathbf{\Delta}_{\boldsymbol{\pi}}\big(\overline{\mathbf{\Pi}}^{+}(y+x)-\mathbf{\Psi}^{+}(0)\big)$$
$$+\mathbf{\Delta}_{\boldsymbol{\pi}}^{-1}\big(\mathbf{\Delta}_{\boldsymbol{d}}^{+}\partial\mathbf{\Pi}^{-}(-x)\big)^{\top}\mathbf{\Delta}_{\boldsymbol{\pi}}\Big\}\mathbf{1}_{(-\infty,0)}(x),$$

where $x\in\mathbb{R}$, *is a.e. equal to a function decreasing on* $(0,\infty)$ *and increasing on* $(-\infty,0)$.

Proof. By symmetry, we need only to prove that (H^+,J^+) is a $\boldsymbol{\pi}$-friend of (H^-,J^-) if, and only if, (H^+,J^+) is $\boldsymbol{\pi}$-compatible with (H^-,J^-) and $\mathbf{\Upsilon}$ is a.e. equal to a function decreasing on $(0,\infty)$ and increasing on $(-\infty,0)$.

Necessity of $\boldsymbol{\pi}$-compatibilty is an immediate consequence of the combined conclusions of Lemma A.1, Proposition 3.3, Proposition 3.4 and Proposition 3.6. Necessity of the monotonocity assumptions on $\mathbf{\Upsilon}$ follows from Theorem 3.10 once we notice that when (H^+,J^+) and (H^-,J^-) are $\boldsymbol{\pi}$-friends with bonding MAP (ξ,J), we have

$$\mathbf{\Upsilon}(x)=\mathbf{\Pi}(x,\infty)\mathbf{1}_{(0,\infty)}(x)+\mathbf{\Pi}(-\infty,x)\mathbf{1}_{(-\infty,0)}(x),\quad x\in\mathbb{R},$$

by the équations amicales. Let us therefore turn to sufficiency.

Condition (iii) of $\boldsymbol{\pi}$-compatibility ensures that the right hand side of (3.1) satisfies condition (iii) of Lemma A.1, so according to the same lemma, it suffices to check the following two properties:

(A) The diagonal elements of the right-hand side of (3.1) can be written as the Lévy–Khintchine exponent of a (killed) Lévy process, which is equivalent to requiring that for any $i\in[n]$, there exists a (positive) measure μ_i integrating $x\mapsto 1\wedge x^2$ and constants $c_i\in\mathbb{R},\mathtt{k}_i,\tau_i\in\mathbb{R}_+$ such that

$$\begin{aligned}&-\Big(\big((q^-_{i,i}-\dagger^-_i)\delta+d^-_i\delta'+\mathbb{\Gamma}\widetilde{\Pi}^-_i\big)*\big((q^+_{i,i}-\dagger^+_i)\delta-d^+_i\delta'+\mathbb{\Gamma}\Pi^+_i\big)\\&\qquad+\sum_{k\neq i}\frac{\pi(k)}{\pi(i)}q^-_{k,i}q^+_{k,i}\widetilde{F}^-_{k,i}*F^+_{k,i}\Big)\\&=\mathbb{\Gamma}^2\mu_i-\mathtt{k}_i\delta-c_i\delta'+\tau_i\delta''.\end{aligned}\tag{3.18}$$

(B) The off-diagonal elements of (3.1) constitute the Fourier transform of a finite measure, which is equivalent to requiring that for any $i,j\in[n]$ with $i\neq j$, there exists a finite measure $\mu_{i,j}$ such that

$$\begin{aligned}&\mu_{i,j}\\&=-\Big(q^+_{i,j}\big((q^-_{i,i}-\dagger^-_i)\delta+d^-_i\delta'+\mathbb{\Gamma}\widetilde{\Pi}^-_i\big)*F^+_{i,j}\\&\qquad+\frac{\pi(j)}{\pi(i)}q^-_{j,i}\widetilde{F}^-_{j,i}*\big((q^+_{j,j}-\dagger^+_j)\delta-d^+_j\delta'+\mathbb{\Gamma}\Pi^+_j\big)+\sum_{k\neq i,j}\frac{\pi(k)}{\pi(i)}q^-_{k,i}q^+_{k,j}\widetilde{F}^-_{k,i}*F^+_{k,j}\Big).\end{aligned}\tag{3.19}$$

Let us start with (A). According to the first step of the proof of the Théorème des amis in [47] (at this point (i) of $\boldsymbol{\pi}$-compatibility comes into play), there exists a signed measure $\widetilde{\mu}_i$ without an atom at 0 integrating $x\mapsto 1\wedge x^2$ and a constant $\widetilde{c}_i\in\mathbb{R}_+$ such that

$$\begin{aligned}&-\big((q^-_{i,i}-\dagger^-_i)\delta+d^-_i\delta'+\mathbb{\Gamma}\widetilde{\Pi}^-_i\big)*\big((q^+_{i,i}-\dagger^+_i)\delta-d^+_i\delta'+\mathbb{\Gamma}\Pi^+_i\big)\\&=\mathbb{\Gamma}^2\widetilde{\mu}_i-(q^-_{i,i}-\dagger^-_i)(q^+_{i,i}-\dagger^+_i)\delta-\widetilde{c}_i\delta'+d^-_id^+_i\delta''.\end{aligned}$$

Moreover, $\widetilde{\nu}_i = -\sum_{k\neq i} \frac{\pi(k)}{\pi(i)} q^-_{k,i} q^+_{k,i} \widetilde{F}^-_{k,i} * F^+_{k,i}(\cdot \cap \{0\}^{\mathsf{c}})$ is a signed finite measure without atom at 0 and mass

$$\widetilde{\nu}_i(\mathbb{R}) = -\sum_{k\neq i} \frac{\pi(k)}{\pi(i)} q^-_{k,i} q^+_{k,i} (1 - \widetilde{F}^-_{k,i} * F^+_{k,i}(\{0\})),$$

and thus, $\mu_i \coloneqq \widetilde{\mu}_i + \widetilde{\nu}_i$ is a signed measure without atom at 0, integrating $x \mapsto 1 \wedge x^2$. Define

$$\mathtt{k}_i = (q^-_{i,i} - \dagger^-_i)(q^+_{i,i} - \dagger^+_i) + \sum_{k\neq i} \frac{\pi(k)}{\pi(i)} q^-_{k,i} q^+_{k,i} \geq 0.$$

Letting further $\tau_i = d^-_i d^+_i \geq 0$ and

$$c_i = \widetilde{c}_i - \int_{[-1,1]} x \, \widetilde{\nu}_i(\mathrm{d}x) \in \mathbb{R}$$

we obtain (3.18). It remains to check that μ_i is a positive measure. To this end, observe that taking primitives (compare to the proof of Theorem 3.10), we find

$$\begin{aligned} &\mathtt{k}_i \mathbf{1}_{\mathbb{R}_-} + (c_i + b_i)\delta - \tau_i \delta' + \Gamma \varrho \bar{\mu}_i \\ &\quad = \big((\dagger^-_i - q^-_{i,i})\mathbf{1}_{\mathbb{R}_-} + d^-_i \delta + \overline{\overline{\Pi}}{}^-_i\big) * \big((q^+_{i,i} - \dagger^+_i)\delta - d^+_i \delta' + \Gamma \Pi^+_i\big) \\ &\qquad - \sum_{k\neq i} \frac{\pi(k)}{\pi(i)} q^-_{k,i} q^+_{k,i} \underline{\widetilde{F}}{}^-_{k,i} * F^+_{k,i} + c_{\mathrm{int}}, \end{aligned}$$

where $b_i \in \mathbb{R}$ and c_{int} is an integration constant. Restricting to $(0,\infty)$ and letting $x \to \infty$ shows that $c_{\mathrm{int}} = 0$ and that for a.e. $x > 0$,

$$\mu_i(x, \infty) = \Upsilon_{i,i}(x).$$

Similarly, we find for a.e. $x < 0$

$$\mu_i(-\infty, x) = \Upsilon_{i,i}(x).$$

Since $\Upsilon_{i,i}$ is a.e. increasing on $(0,\infty)$ and a.e. decreasing on $(-\infty, 0)$ it follows that the tails of μ_i are a.e. increasing on $(-\infty,0)$ and a.e. decreasing on $(0,\infty)$. Since the tails of μ_i are càglàd on $(-\infty, 0)$ and càdlàg on $(0,\infty)$, this establishes that the tails are increasing on all of $(-\infty, 0)$ and decreasing on all of $(0,\infty)$. Thus, $\overline{\mu}_i$ is the tail of a positive measure, i.e., μ_i is a positive measure.

We proceed with (B). Combining condition (ii) of $\boldsymbol{\pi}$-compatibility with Lemma A.2, it follows that

$$\begin{aligned} &\big(d^-_i \delta' + \Gamma \widetilde{\Pi}^-_i\big) * q^+_{i,j} F^+_{i,j} + \big(-d^+_j \delta' + \Gamma \Pi^+_j\big) * \frac{\pi(j)}{\pi(i)} q^-_{j,i} \widetilde{F}^-_{j,i} \\ &\quad = \Big(\widetilde{\chi}^-_i * q^+_{i,j} F^+_{i,j} - \chi^+_j * \frac{\pi(j)}{\pi(i)} q^-_{j,i} \widetilde{F}^-_{j,i}\Big)' = \nu_{i,j}, \end{aligned}$$

and thus, (3.19) indeed defines a finite signed measure. It remains to show that $\mu_{i,j}$ is positive. It follows, again with condition (ii) of $\boldsymbol{\pi}$-compatibility,

$$\begin{aligned} \mu_{i,j}(\{0\}) &= (\dagger^-_i - q^-_{i,i}) q^+_{i,j} F^+_{i,j}(\{0\}) + (\dagger^+_j - q^+_{j,j}) \frac{\pi(j)}{\pi(i)} q^-_{j,i} F^-_{j,i}(\{0\}) - \nu_{i,j}(\{0\}) \\ &\quad - \sum_{k\neq i,j} \frac{\pi(k)}{\pi(i)} q^-_{k,i} q^+_{k,j} \widetilde{F}^-_{k,i} * F^+_{k,j}(\{0\}) \geq 0. \end{aligned}$$

Moreover, taking primitives and restricting to $(0,\infty)$ we find (compare this to (3.17)) for a.e. $x>0$

$$\mu_{i,j}(x,\infty) = \Upsilon_{i,j}(x).$$

Similarly, for a.e. $x<0$,

$$\mu_{i,j}(-\infty,x) = \Upsilon_{i,j}(x).$$

Thus, our assumptions on $\mathbf{\Upsilon}$ guarantee that the tails of $\mu_{i,j}$ are a.e. decreasing on $(0,\infty)$ and a.e. increasing on $(-\infty,0)$. Since the tails are càglàd on $(-\infty,0)$ and càdlàg on $(0,\infty)$ this establishes that the tails are increasing on all of $(-\infty,0)$ and decreasing on all of $(0,\infty)$. Together with $\mu_{i,j}(\{0\}) \geq 0$ this establishes that $\mu_{i,j}$ is a finite positive measure. □

In general, not only the central monotonicity conditions on $\mathbf{\Upsilon}$ but also the necessary requirement of $\boldsymbol{\pi}$-compatibility makes engineering MAP friendships significantly harder than for Lévy processes. Especially condition (ii) of Definition 3.7 appears rather cumbersome and requires skilled matching of the Lévy measure matrices of potential friends. In particular the possible existence of atoms at 0 for the transitional jumps poses a significant challenge in constructing explicit examples of friendships. We describe some ways to deal with this effect in Section 4, where we develop an extension of Vigon's theory of philanthropy.

3.3. **Other properties of friendship.** In this section we collect some simple implications of $\boldsymbol{\pi}$-friendship. In particular, in order to interpret the equation (3.1), it is essential that the vector $\boldsymbol{\pi}$ be the invariant distribution of (an unkilled version of) the bonding modulator J. The results in this section give some sufficient conditions for this to hold.

Lemma 3.12. *If (H^+,J^+) and (H^-,J^-) are $\boldsymbol{\pi}$-friends and the bonding process (ξ,J) is unkilled, then $\boldsymbol{\pi}$ is invariant for J.*

Proof. When J is unkilled, $\mathbf{\Psi}(0)\mathbf{1} = \mathbf{0}$. Together with this, the condition $\boldsymbol{\pi}^\top \mathbf{\Psi}(0) \leq \mathbf{0}^\top$ of $\boldsymbol{\pi}$-friendship actually implies more: $\boldsymbol{\pi}^\top \mathbf{\Psi}(0) = \mathbf{0}^\top$. This means that $\boldsymbol{\pi}$ is invariant for J. □

Lemma 3.13. *Suppose that (H^+,J^+) is a $\boldsymbol{\pi}$-friend of (H^-,J^-). Then, the bonding MAP (ξ,J) is unkilled if, and only if, either*

$$-\mathbf{\Delta}_{\boldsymbol{\pi}}^{-1}\mathbf{\Psi}^-(0)^\top \mathbf{\Delta}_{\boldsymbol{\pi}} \boldsymbol{\dagger}^+ = \mathbf{0} \tag{3.20}$$

or

$$-\mathbf{\Delta}_{\boldsymbol{\pi}}^{-1}\mathbf{\Psi}^+(0)^\top \mathbf{\Delta}_{\boldsymbol{\pi}} \boldsymbol{\dagger}^- = \mathbf{0}. \tag{3.21}$$

Moreover, if both (H^+,J^+) and (H^-,J^-) are irreducible, then the bonding MAP is unkilled if, and only if, at least one of (H^+,J^+) or (H^-,J^-) is unkilled.

Proof. On the one hand, we have

$$\boldsymbol{\dagger} = -\mathbf{\Delta}_{\boldsymbol{\pi}}^{-1}\mathbf{\Psi}^-(0)^\top \mathbf{\Delta}_{\boldsymbol{\pi}} \mathbf{\Psi}^+(0)\mathbf{1} = -\mathbf{\Delta}_{\boldsymbol{\pi}}^{-1}\mathbf{\Psi}^-(0)^\top \mathbf{\Delta}_{\boldsymbol{\pi}} \boldsymbol{\dagger}^+.$$

On the other hand, considering the bonding MAP $(\widehat{\xi},\widehat{J})$ of (H^-,J^-) with (H^+,J^+), and denoting its killing rates by $\widehat{\boldsymbol{\dagger}}$, we see by the same argument that

$$\widehat{\boldsymbol{\dagger}} = -\mathbf{\Delta}_{\boldsymbol{\pi}}^{-1}\mathbf{\Psi}^+(0)^\top \mathbf{\Delta}_{\boldsymbol{\pi}} \boldsymbol{\dagger}^-.$$

If this is the zero vector, then by the preceding lemma, this implies that $\boldsymbol{\pi}$ is invariant for $\widehat{J}$. Taking the $\boldsymbol{\pi}$-dual of $(\widehat{\xi}, \widehat{J})$, as explained at the start of section 3, we obtain (ξ, J) which is also killed at rate $\widehat{\boldsymbol{\dagger}} = \mathbf{0}$.

We have proved that the bonding MAP is unkilled if, and only if, either (3.20) or (3.21) holds. It follows immediately that, if either (H^+, J^+) or (H^-, J^-) is unkilled, then so is the bonding MAP. For the converse, suppose that both (H^+, J^+) and (H^-, J^-) are irreducible and killed. Then $\boldsymbol{\Psi}^+(0)$ and $\boldsymbol{\Psi}^-(0)$ are invertible by [17, Theorem 6.2.26].

If the bonding MAP were unkilled, then one of (3.20) or (3.21) would have to be true; but the former implies $\boldsymbol{\dagger}^+ = \mathbf{0}$, and the latter implies $\boldsymbol{\dagger}^- = \mathbf{0}$, both of which are contradictions. Hence, (ξ, J) is killed. □

Lemma 3.14. *Suppose that (H^+, J^+) and (H^-, J^-) are two unkilled MAP subordinators such that (3.1) is the matrix exponent of a MAP (ξ, J). Then (ξ, J) is unkilled and $\boldsymbol{\pi}$ is invariant for J.*

Proof. We observe that $\boldsymbol{\Psi}(0)\mathbf{1} = \boldsymbol{\Delta}_{\boldsymbol{\pi}}^{-1}\boldsymbol{\Psi}^-(0)^\top \boldsymbol{\Delta}_{\boldsymbol{\pi}} \boldsymbol{\dagger}^+ = \mathbf{0}$ and that $\boldsymbol{\pi}^\top \boldsymbol{\Psi}(0) = (\boldsymbol{\dagger}^-)^\top \boldsymbol{\Delta}_{\boldsymbol{\pi}} \boldsymbol{\Psi}^+(0) = \mathbf{0}$, and the claim follows. □

The significance of the (admittedly straightforward) result above is that it allows one to ignore the condition $\boldsymbol{\pi}^\top \boldsymbol{\Psi}(0) \le \mathbf{0}^\top$ of $\boldsymbol{\pi}$-friendship.

Lemma 3.15. *Suppose that (H^+, J^+) and (H^-, J^-) are two subordinator MAP exponents such that (3.1) is the matrix exponent of a MAP (ξ, J), and let $\boldsymbol{Q}$ be the generator matrix of (the unkilled version of) the bonding modulator J; that is,*

$$-\boldsymbol{\Delta}_{\boldsymbol{\pi}}^{-1}\boldsymbol{\Psi}^-(0)^\top \boldsymbol{\Delta}_{\boldsymbol{\pi}} \boldsymbol{\Psi}^+(0) = \boldsymbol{Q} - \boldsymbol{\Delta}_{\boldsymbol{\dagger}},$$

where $\dagger_i$ is the killing rate of J in state i. Then, $\boldsymbol{\pi}^\top \boldsymbol{Q} = \mathbf{0}^\top$ if, and only if,

$$\boldsymbol{\pi}^\top \big(\boldsymbol{\Delta}_{\boldsymbol{\dagger}}^- \boldsymbol{\Psi}^+(0) - \boldsymbol{\Delta}_{\boldsymbol{\dagger}}^+ \boldsymbol{\Psi}^-(0)\big) = \mathbf{0}^\top. \tag{3.22}$$

Proof. Since $\boldsymbol{Q}$ is a generator matrix we have

$$\dagger_i = \sum_{j=1}^n \sum_{k=1}^n \frac{\pi(k)}{\pi(i)} (\boldsymbol{\Psi}^-(0))_{k,i} (\boldsymbol{\Psi}^+(0))_{k,j}$$

and hence

$$\begin{aligned}(\boldsymbol{\pi}^\top \boldsymbol{\Delta}_{\boldsymbol{\dagger}})_i = \pi(i)\dagger_i &= \sum_{j=1}^n \sum_{k=1}^n \pi(k) (\boldsymbol{\Psi}^-(0))_{k,i} (\boldsymbol{\Psi}^+(0))_{k,j} \\ &= \sum_{k=1}^n \pi(k) (\boldsymbol{\Psi}^-(0))_{k,i} \sum_{j=1}^n (\boldsymbol{\Psi}^+(0))_{k,j} \\ &= -\sum_{k=1}^n \pi(k) (\boldsymbol{\Psi}^-(0))_{k,i} \dagger_k^+,\end{aligned}$$

where the last line follows from $\sum_{j=1}^n q_{k,j}^+ = 0$ by definition of a generator matrix. Moreover,

$$\big(\boldsymbol{\pi}^\top (-\boldsymbol{\Delta}_{\boldsymbol{\pi}}^{-1}\boldsymbol{\Psi}^-(0)^\top \boldsymbol{\Delta}_{\boldsymbol{\pi}} \boldsymbol{\Psi}^+(0))\big)_i = -\sum_{j=1}^n \big(\boldsymbol{\Psi}^-(0)^\top \boldsymbol{\Delta}_\pi \boldsymbol{\Psi}^+(0)\big)_{j,i}$$

$$
\begin{aligned}
&= -\sum_{j=1}^{n}\sum_{k=1}^{n} \pi(k)(\mathbf{\Psi}^-(0))_{k,j}(\mathbf{\Psi}^+(0))_{k,i} \\
&= -\sum_{k=1}^{n} \pi(k)(\mathbf{\Psi}^+(0))_{k,i}\sum_{j=1}^{n}(\mathbf{\Psi}^-(0))_{k,j} \\
&= \sum_{k=1}^{n} \pi(k)(\mathbf{\Psi}^+(0))_{k,i}\dagger_k^-,
\end{aligned}
$$

where the last line is again a consequence of $\sum_{j=1}^n q^-_{k,j} = 0$ by definition of a generator matrix. Thus, $\boldsymbol{\pi}^\top \boldsymbol{Q} = \mathbf{0}^\top$ if, and only if, for any $i \in [n]$

$$
0 = (\boldsymbol{\pi}^\top \boldsymbol{Q})_i = \sum_{k=1}^{n} \pi(k)\Big(- \mathbf{\Psi}^-(0)_{k,i}\ \dagger_k^+ + \mathbf{\Psi}^+(0)_{k,i}\ \dagger_k^- \Big),
$$

which is satisfied if, and only if,

$$
\boldsymbol{\pi}^\top\big(\mathbf{\Delta}^-_{\boldsymbol{\dagger}}\mathbf{\Psi}^+(0) - \mathbf{\Delta}^+_{\boldsymbol{\dagger}}\mathbf{\Psi}^-(0)\big) = \mathbf{0}^\top,
$$

that is, if, and only if, (3.22) is satisfied. □

4. Markov additive fellowship

Having found a characterisation of Markov additive friendships, our aim is now to find a version of Vigon's theory of philanthropy, as summarized in [30, Section 6.6]. It emerges that the situation is rather more complicated in the Markov additive world, and we instead term our relationship *fellowship*.

For Lévy processes, a philanthropist is a subordinator which is friends with an unkilled pure drift [47], and it emerges that this is equivalent to having a decreasing Lévy density.

Definition 4.1. We say that a MAP (ξ, J) is pure drift, if

$$
\xi_t = \int_0^t d_{J_s}\,\mathrm{d}s, \quad t \in [0, \zeta).
$$

Friendship of a MAP subordinator with a pure drift MAP subordinator can be characterised as follows with Theorem 3.11.

Lemma 4.2. *A MAP subordinator (H^+, J^+) is the $\boldsymbol{\pi}$-friend of a pure drift subordinator (H^-, J^-) if, and only if, (H^+, J^+) is $\boldsymbol{\pi}$-compatible with (H^-, J^-) and the matrix function*

$$
-\mathbf{\Delta}_{\boldsymbol{\pi}}^{-1}\mathbf{\Psi}^-(0)^\top\mathbf{\Delta}_{\boldsymbol{\pi}}\overline{\mathbf{\Pi}}^+(x) + \mathbf{\Delta}^-_{\boldsymbol{d}}\partial\mathbf{\Pi}^+(x), \quad x > 0,
$$

is decreasing.

Proof. This is a direct consequence of Theorem 3.11 once we observe that since $\mathbf{\Pi}^-$ is trivial on $(0, \infty)$ we have

$$
\mathbf{\Upsilon}(x) = -\mathbf{\Delta}_{\boldsymbol{\pi}}^{-1}\mathbf{\Psi}^-(0)^\top\mathbf{\Delta}_{\boldsymbol{\pi}}\overline{\mathbf{\Pi}}^+(x) + \mathbf{\Delta}^-_{\boldsymbol{d}}\partial\mathbf{\Pi}^+(x), \quad x > 0.
$$

□

For MAPs, neither a decreasing Lévy measure matrix nor $\boldsymbol{\pi}$-friendship with a pure drift is a workable criterion for $\boldsymbol{\pi}$-friendship with another process, or even with another pure drift. This is already suggested by necessity of $\boldsymbol{\pi}$-compatibility for $\boldsymbol{\pi}$-friendship, which entails specific balance conditions between the characteristics of two friends, and cannot easily be reduced to a condition on only one of the pair.

But even when $\boldsymbol{\pi}$-compatibility holds, $\boldsymbol{\pi}$-friendship may not. To be specific, take $n=2$, $\boldsymbol{\pi}=(1/2,1/2)^\top$, and consider the example of a MAP subordinator (H^+,J^+) with transition rate matrix $\boldsymbol{Q}^+=\left(\begin{smallmatrix}-1 & 1\\ 1 & -1\end{smallmatrix}\right)$, drift $+1$ in all phases, and jumps given by $\Pi_i^+(\mathrm{d}x)=e^{-x}\mathbf{1}_{\{x>0\}}\,\mathrm{d}x$, for $i=1,2$ and $F_{i,j}^+(\mathrm{d}x)=\frac{1}{2}(\delta_{\{0\}}(\mathrm{d}x)+e^{-x}\mathbf{1}_{\{x>0\}}\,\mathrm{d}x)$, for $i,j=1,2,i\neq j$. Let (H^-,J^-) be the pure drift subordinator with $\boldsymbol{Q}^-=\boldsymbol{Q}^+$ and $H_t^-=at$. By directly checking the conditions of Theorem 3.11, one sees that (H^+,J^+) is $\boldsymbol{\pi}$-friends with (H^-,J^-) when $a=2$ but not when $a=1/2$.

We are motivated, therefore, to find conditions not on a single MAP subordinator, but on a pair, which leads us to the notion of $\boldsymbol{\pi}$-fellowship developed below.

We say that a MAP subordinator has a continuous, decreasing, differentiable or convex Lévy density matrix $\partial\boldsymbol{\Pi}^+$ on $(0,\infty)$ if for every $i,j\in[n]$, $\boldsymbol{\Pi}_{i,j}^+$ has a continuous, decreasing, differentiable or convex density, respectively, on $(0,\infty)$. In case of differentiability we denote by $\partial^2\boldsymbol{\Pi}^+$ the matrix-valued function on $(0,\infty)$ defined by

$$(\partial^2\boldsymbol{\Pi}^+(x))_{i,j}=\frac{\partial}{\partial x}(\partial\boldsymbol{\Pi}^+(x))_{i,j},\quad i,j\in[n],x>0.$$

Definition 4.3. We say that a MAP subordinator (H^+,J^+) is a $\boldsymbol{\pi}$-fellow of another MAP subordinator (H^-,J^-) if they have decreasing Lévy density matrices $\partial\boldsymbol{\Pi}^+$ and $\partial\boldsymbol{\Pi}^-$ on $(0,\infty)$, and the matrix functions

$$-\boldsymbol{\Delta}_{\boldsymbol{\pi}}^{-1}\boldsymbol{\Psi}^-(0)^\top\boldsymbol{\Delta}_{\boldsymbol{\pi}}\overline{\boldsymbol{\Pi}}^+(x)+\boldsymbol{\Delta}_{\boldsymbol{d}}^-\partial\boldsymbol{\Pi}^+(x),\quad x>0, \tag{4.1}$$

and

$$-\boldsymbol{\Delta}_{\boldsymbol{\pi}}^{-1}\boldsymbol{\Psi}^+(0)^\top\boldsymbol{\Delta}_{\boldsymbol{\pi}}\overline{\boldsymbol{\Pi}}^-(x)+\boldsymbol{\Delta}_{\boldsymbol{d}}^+\partial\boldsymbol{\Pi}^-(x),\quad x>0, \tag{4.2}$$

are decreasing.

Evidently, $\boldsymbol{\pi}$-fellowship is a symmetric relation. We also note that any two Lévy philanthropists are automatically fellows. With this terminology, Lemma 4.2 can be reformulated in the following form:

A MAP subordinator (H^+,J^+) with a decreasing Lévy density matrix on $(0,\infty)$ is a $\boldsymbol{\pi}$-friend of a pure drift MAP subordinator (H^-,J^-) if, and only if, it is a $\boldsymbol{\pi}$-compatible $\boldsymbol{\pi}$-fellow of (H^-,J^-).

Theorem 4.4. *Two $\boldsymbol{\pi}$-compatble MAP subordinators that are $\boldsymbol{\pi}$-fellows of each other are $\boldsymbol{\pi}$-friends.*

Proof. By assumption, (H^+,J^+) is $\boldsymbol{\pi}$-compatible with (H^-,J^-). Therefore, it follows from Theorem 3.11 that if we can show that $\boldsymbol{\Upsilon}$ is decreasing on $(0,\infty)$ and increasing on $(-\infty,0)$, then (H^+,J^+) is a $\boldsymbol{\pi}$-friend of (H^-,J^-). For $x>0$ we have

$$\begin{aligned}&\boldsymbol{\Upsilon}(x)\\&=\int_{0+}^\infty\boldsymbol{\Delta}_{\boldsymbol{\pi}}^{-1}\overline{\boldsymbol{\Pi}}^-(y)^\top\boldsymbol{\Delta}_{\boldsymbol{\pi}}\partial\boldsymbol{\Pi}^+(x+y)\,\mathrm{d}y-\boldsymbol{\Delta}_{\boldsymbol{\pi}}^{-1}\boldsymbol{\Psi}^-(0)^\top\boldsymbol{\Delta}_{\boldsymbol{\pi}}\overline{\boldsymbol{\Pi}}^+(x)+\boldsymbol{\Delta}_{\boldsymbol{d}}^-\partial\boldsymbol{\Pi}^+(x)\\&=:\boldsymbol{\Upsilon}^{(1)}(x)+\boldsymbol{\Upsilon}^{(2)}(x),\end{aligned}$$

where $\boldsymbol{\Upsilon}^{(1)}(x)=\int_{0+}^\infty\boldsymbol{\Delta}_{\boldsymbol{\pi}}^{-1}\overline{\boldsymbol{\Pi}}^-(y)^\top\boldsymbol{\Delta}_{\boldsymbol{\pi}}\partial\boldsymbol{\Pi}^+(x+y)\,\mathrm{d}y$ and $\boldsymbol{\Upsilon}^{(2)}(x)=\boldsymbol{\Upsilon}(x)-\boldsymbol{\Upsilon}^{(1)}(x)$. Since $\partial\boldsymbol{\Pi}^+$ is decreasing by assumption, it follows that $\boldsymbol{\Upsilon}^{(1)}$ is decreasing as well. Moreover, since (H^+,J^+) is a $\boldsymbol{\pi}$-fellow of (H^-,J^-), $\boldsymbol{\Upsilon}^{(2)}$ is decreasing. Hence, $\boldsymbol{\Upsilon}$

is decreasing on $(0,\infty)$. For $x<0$ we have

$$\begin{aligned}\boldsymbol{\Upsilon}(x) &= \int_{0+}^{\infty} \boldsymbol{\Delta}_{\boldsymbol{\pi}}^{-1}\partial\boldsymbol{\Pi}^-(y-x)^\top\boldsymbol{\Delta}_{\boldsymbol{\pi}}\overline{\boldsymbol{\Pi}}^+(y)\,\mathrm{d}y - \boldsymbol{\Delta}_{\boldsymbol{\pi}}^{-1}\big(\overline{\boldsymbol{\Pi}}^-(-x)\big)^\top\boldsymbol{\Delta}_{\boldsymbol{\pi}}\boldsymbol{\Psi}^+(0)\\ &\quad + \boldsymbol{\Delta}_{\boldsymbol{\pi}}^{-1}\big(\boldsymbol{\Delta}_{\boldsymbol{d}}^+\partial\boldsymbol{\Pi}^-(-x)\big)^\top\boldsymbol{\Delta}_{\boldsymbol{\pi}}\\ &=: \boldsymbol{\Upsilon}^{(3)}(x)+\boldsymbol{\Upsilon}^{(4)}(x),\end{aligned}$$

where $\boldsymbol{\Upsilon}^{(3)}(x)=\int_{0+}^{\infty}\boldsymbol{\Delta}_{\boldsymbol{\pi}}^{-1}\partial\boldsymbol{\Pi}^-(y-x)^\top\boldsymbol{\Delta}_{\boldsymbol{\pi}}\overline{\boldsymbol{\Pi}}^+(y)\,\mathrm{d}y$ and $\boldsymbol{\Upsilon}^{(4)}(x)=\boldsymbol{\Upsilon}(x)-\boldsymbol{\Upsilon}^{(3)}(x)$. By assumption, $\partial\boldsymbol{\Pi}^-$ is decreasing, which implies that $\boldsymbol{\Upsilon}^{(3)}$ is increasing on $(-\infty,0)$. Finally, observe that on $(-\infty,0)$,

$$\boldsymbol{\Delta}_{\boldsymbol{\pi}}^{-1}\boldsymbol{\Upsilon}^{(4)}(x)^\top\boldsymbol{\Delta}_{\boldsymbol{\pi}} = -\boldsymbol{\Delta}_{\boldsymbol{\pi}}^{-1}\boldsymbol{\Psi}^+(0)^\top\boldsymbol{\Delta}_{\boldsymbol{\pi}}\overline{\boldsymbol{\Pi}}^-(-x)+\boldsymbol{\Delta}_{\boldsymbol{d}}^+\partial\boldsymbol{\Pi}^-(-x),\quad x<0,$$

is increasing by (4.2). Since $\boldsymbol{\Delta}_{\boldsymbol{\pi}}$ is a strictly positive diagonal matrix, this shows that $\boldsymbol{\Upsilon}^{(4)}(x)$ is increasing on $(-\infty,0)$. With the above, this now implies that $\boldsymbol{\Upsilon}$ is increasing on $(-\infty,0)$ and it follows that (H^+,J^+) is a $\boldsymbol{\pi}$-friend of (H^-,J^-). □

4.1. **Construction of spectrally one-sided MAPs.** In the setting of Lévy processes, the concept of philanthropy makes the construction of Lévy processes out of friends simple, since we can simply combine any two Lévy subordinators with decreasing Lévy density. For MAPs, the story is much more involved.

The major stumbling block in proving $\boldsymbol{\pi}$-friendship is not in satisfying the decreasing matrix condition of either Theorem 3.11 or the notion of $\boldsymbol{\pi}$-fellowship, but rather in proving $\boldsymbol{\pi}$-compatibility. In order to provide simpler constructions for MAPs, we focus on finding sufficient conditions for $\boldsymbol{\pi}$-compatibility, beginning with the case of spectrally one-sided MAPs. We set $\boldsymbol{F}^\pm := (F_{i,j}^\pm)_{i,j\in[n]}$ and say that a matrix $\boldsymbol{A}$ is an ML-matrix, if it has nonnegative off-diagonal entries.

Definition 4.5. Let $(H^\pm,J^\pm)$ be MAP subordinators with decreasing Lévy density matrices $\boldsymbol{\Pi}^\pm$ on $(0,\infty)$ and let $f_{i,j}^\pm$ be càdlàg versions of the Lebesgue densities of $F_{i,j}^\pm$ on $(0,\infty)$. A MAP subordinator (H^+,J^+) is $\boldsymbol{\pi}$-quasicompatible with another MAP subordinator (H^-,J^-) if:

(i) for all $i,j\in[n]$ with $i\neq j$, the functions $\psi_{i,j}^\pm$ defined by

$$\psi_{i,j}^+(x)=\Big(d_i^-q_{i,j}^+f_{i,j}^+(x)-\frac{\pi(j)}{\pi(i)}q_{j,i}^-F_{j,i}^-(\{0\})\overline{\Pi}_j^+(x)\Big)\mathbf{1}_{(0,\infty)}(x),\quad x\in\mathbb{R},\tag{4.3}$$

$$\psi_{i,j}^-(x)=\Big(\overline{\Pi}_i^-(-x)q_{i,j}^+F_{i,j}^+(\{0\})-\frac{\pi(j)}{\pi(i)}q_{j,i}^-f_{j,i}^-(-x)d_j^+\Big)\mathbf{1}_{(0,\infty)}(-x),\quad x\in\mathbb{R},\tag{4.4}$$

are of bounded variation on $\mathbb{R}$. Moreover, for

$$\alpha_{i,j}^+:=\lim_{x\downarrow0}\psi_{i,j}^+(x),\quad \alpha_{i,j}^-:=\lim_{x\uparrow0}\psi_{i,j}^-(x),\tag{4.5}$$

the matrix

$$-\boldsymbol{\Delta}_{\boldsymbol{\pi}}^{-1}\big(\boldsymbol{\Psi}^-(0)\odot\boldsymbol{F}^-(\{0\})\big)^\top\boldsymbol{\Delta}_{\boldsymbol{\pi}}\big(\boldsymbol{\Psi}^+(0)\odot\boldsymbol{F}^+(\{0\})\big)-\boldsymbol{A}\tag{4.6}$$

is an ML-matrix, where $\boldsymbol{A}_{i,j}=\alpha_{i,j}^+-\alpha_{i,j}^-$ for $i\neq j$ and $\boldsymbol{A}_{i,i}=0$;

(ii) it holds that

$$\boldsymbol{\Delta}_{\boldsymbol{d}}^-\boldsymbol{\Pi}^+(\{0\})=\boldsymbol{\Delta}_{\boldsymbol{\pi}}^{-1}\big(\boldsymbol{\Delta}_{\boldsymbol{d}}^+\boldsymbol{\Pi}^-(\{0\})\big)^\top\boldsymbol{\Delta}_{\boldsymbol{\pi}};\tag{4.7}$$

(iii) the vector $\boldsymbol{\Delta}_{\boldsymbol{\pi}}^{-1}\boldsymbol{\Psi}^-(0)^\top\boldsymbol{\Delta}_{\boldsymbol{\pi}}\boldsymbol{\Psi}^+(0)\mathbf{1}$ is nonnegative.

(iv) the vector $\boldsymbol{\pi}^\top \boldsymbol{\Delta}_{\boldsymbol{\pi}}^{-1} \boldsymbol{\Psi}^-(0)^\top \boldsymbol{\Delta}_{\boldsymbol{\pi}} \boldsymbol{\Psi}^+(0)$ is nonnegative.

Remark 4.6. If $f_{i,j}^+$ is also differentiable on $(0,\infty)$, it is not hard to show that $\psi_{i,j}^+$ has bounded variation if, for some $\varepsilon > 0$, the function $x \mapsto d_i^- q_{i,j}^+ \frac{\partial}{\partial x} f_{i,j}^+(x) + \frac{\pi(j)}{\pi(i)} q_{j,i}^- F_{j,i}^-(\{0\}) \partial \Pi_j^+(x)$ is integrable on $(0,\varepsilon)$. An analogous statement is true for $\psi_{i,j}^-$. These conditions should therefore be understood as a way to say that $d_i^- f_{i,j}^+$ and $F_{j,i}^-(\{0\}) \overline{\Pi}_j^+$ must compensate each other appropriately.

Again, $\boldsymbol{\pi}$-quasicompatibility is a symmetric relation:

Lemma 4.7. *If (H^+, J^+) is $\boldsymbol{\pi}$-quasicompatible with (H^-, J^-), then (H^-, J^-) is $\boldsymbol{\pi}$-quasicompatible with (H^+, J^+).*

Proof. It is clearly enough to verify (4.6) and (4.7) with swapped roles of $\boldsymbol{\Psi}^+$ and $\boldsymbol{\Psi}^-$. Since (H^+, J^+) is $\boldsymbol{\pi}$-quasicompatible with (H^-, J^-), it follows from (4.5) that

$$\lim_{x \downarrow 0} d_i^+ q_{i,j}^- f_{i,j}^-(x) - \frac{\pi(j)}{\pi(i)} q_{j,i}^+ F_{j,i}^+(\{0\}) \overline{\Pi}_j^-(x) = -\frac{\pi(j)}{\pi(i)} \alpha_{j,i}^- =: \widetilde{\alpha}_{i,j}^+$$

and

$$\lim_{x \uparrow 0} \overline{\Pi}_i^+(-x) q_{i,j}^- F_{i,j}^-(\{0\}) - \frac{\pi(j)}{\pi(i)} q_{j,i}^+ f_{j,i}^+(-x) d_j^- = -\frac{\pi(j)}{\pi(i)} \alpha_{j,i}^+ =: \widetilde{\alpha}_{i,j}^-.$$

Thus, if we denote $\widetilde{\boldsymbol{A}} = (\widetilde{\alpha}_{i,j}^+ - \widetilde{\alpha}_{i,j}^-)_{i,j \in [n]}$ with $\widetilde{\alpha}_{i,i}^+ = \widetilde{\alpha}_{i,i}^- = 0$, we have $\widetilde{\boldsymbol{A}} = \boldsymbol{\Delta}_{\boldsymbol{\pi}}^{-1} \boldsymbol{A}^\top \boldsymbol{\Delta}_{\boldsymbol{\pi}}$ and therefore

$$\begin{aligned} &\boldsymbol{\Delta}_{\boldsymbol{\pi}}^{-1} \Big(- \boldsymbol{\Delta}_{\boldsymbol{\pi}}^{-1} \big(\boldsymbol{\Psi}^+(0) \odot \boldsymbol{F}^+(\{0\} \big)^\top \boldsymbol{\Delta}_{\boldsymbol{\pi}} \big(\boldsymbol{\Psi}^-(0) \odot \boldsymbol{F}^-(\{0\}) \big) - \widetilde{\boldsymbol{A}} \Big)^\top \boldsymbol{\Delta}_{\boldsymbol{\pi}} \\ &\quad = -\boldsymbol{\Delta}_{\boldsymbol{\pi}}^{-1} \big(\boldsymbol{\Psi}^-(0) \odot \boldsymbol{F}^-(\{0\} \big)^\top \boldsymbol{\Delta}_{\boldsymbol{\pi}} \big(\boldsymbol{\Psi}^+(0) \odot \boldsymbol{F}^+(\{0\}) \big) - \boldsymbol{A}. \end{aligned}$$

Since a matrix $\boldsymbol{M}$ is an ML-matrix iff $\boldsymbol{\Delta}_{\boldsymbol{\pi}}^{-1} \boldsymbol{M}^\top \boldsymbol{\Delta}_{\boldsymbol{\pi}}$ is an ML-matrix and

$$-\boldsymbol{\Delta}_{\boldsymbol{\pi}}^{-1} \big(\boldsymbol{\Psi}^-(0) \odot \boldsymbol{F}^-(\{0\} \big)^\top \boldsymbol{\Delta}_{\boldsymbol{\pi}} \big(\boldsymbol{\Psi}^+(0) \odot \boldsymbol{F}^+(\{0\}) \big) - \boldsymbol{A}$$

is an ML-matrix by assumption, it follows that

$$-\boldsymbol{\Delta}_{\boldsymbol{\pi}}^{-1} \big(\boldsymbol{\Psi}^+(0) \odot \boldsymbol{F}^+(\{0\} \big)^\top \boldsymbol{\Delta}_{\boldsymbol{\pi}} \big(\boldsymbol{\Psi}^-(0) \odot \boldsymbol{F}^-(\{0\}) \big) - \widetilde{\boldsymbol{A}}$$

is an ML-matrix. Moreover, rearranging (4.7) yields that

$$\boldsymbol{\Delta}_{\boldsymbol{d}}^+ \boldsymbol{\Pi}^-(\{0\}) = \boldsymbol{\Delta}_{\boldsymbol{\pi}}^{-1} \big(\boldsymbol{\Delta}_{\boldsymbol{d}}^- \boldsymbol{\Pi}^+(\{0\}) \big)^\top \boldsymbol{\Delta}_{\boldsymbol{\pi}}$$

and hence (H^-, J^-) is $\boldsymbol{\pi}$-quasicompatible with (H^+, J^+). □

This yields the following characterisation of $\boldsymbol{\pi}$-friendship with a pure drift.

Theorem 4.8. *Let (H^+, J^+) be a MAP subordinator with decreasing Lévy density matrix on $(0,\infty)$ and (H^-, J^-) be a pure drift MAP subordinator. Then, (H^+, J^+) and (H^-, J^-) are $\boldsymbol{\pi}$-compatible if, and only if, they are $\boldsymbol{\pi}$-quasicompatible. In particular, they are $\boldsymbol{\pi}$-friends if, and only if, they are $\boldsymbol{\pi}$-quasicompatible $\boldsymbol{\pi}$-fellows.*

Proof. Given Lemma 4.2 it is enough to show that, under the conditions in the result, the two processes are $\boldsymbol{\pi}$-quasicompatible if, and only if, they are $\boldsymbol{\pi}$-compatible. To this end, first note that condition (i) of Definition 3.7 is automatically satisfied by assumption, and that (iii) and (iv) of the definition of $\boldsymbol{\pi}$-quasicompatibility are the same as conditions (iii) and (ii) in Definition 3.7. Moreover, (ii) is necessary for $\boldsymbol{\pi}$-compatibility by Corollary 3.5. Let us therefore assume that (ii) holds. It now remains to show that condition (i) is equivalent to condition (ii) of Definition 3.7 in the special case of (H^+, J^+) having a decreasing and differentiable Lévy density matrix on $(0,\infty)$ and (H^-, J^-) being pure drift. In this scenario, with (ii) in place we have

$$\begin{aligned}\vartheta_{i,j}(\mathrm{d}x) &:= q^+_{i,j} F^+_{i,j} * \widetilde{\chi}^-_i(\mathrm{d}x) - \frac{\pi(j)}{\pi(i)} q^-_{j,i} \widetilde{F}^-_{j,i} * \chi^+_j(\mathrm{d}x)\\ &= \mathbf{1}_{(0,\infty)}(x)\big(d^-_i q^+_{i,j} f^+_{i,j}(x) - \tfrac{\pi(j)}{\pi(i)} q^-_{j,i} \overline{\Pi}^+_j(x)\big)\,\mathrm{d}x,\end{aligned}$$

for $x \in \mathbb{R}$. Thus, $\vartheta_{i,j}$ is absolutely continuous and its density is equal to $\psi^+_{i,j}$. Since $\lim_{x\to\infty} f^+_{i,j}(x) = 0$ by assumption and $\psi^+_{i,j}$ is right-continuous a.e., it follows that $\psi^+_{i,j} = \nu_{i,j}((-\infty,\cdot])$ holds a.e. for a finite signed measure $\nu_{i,j}$ if, and only if, $\psi^+_{i,j}$ has bounded variation. Taking into account that $\psi^-_{i,j} \equiv 0$, this shows that (3.3) is satisfied if, and only if, $\psi^+_{i,j}$ has bounded variation. Finally, $\nu_{i,j}(\{0\}) = \psi^+_{i,j}(0+) = \alpha^+_{i,j}$ and $F^-_{i,j}(\{0\}) = 1$ for all $i \neq j$ yields

$$\begin{aligned}&(\dagger^-_i - q^-_{i,i}) q^+_{i,j} F^+_{i,j}(\{0\}) + (\dagger^+_j - q^+_{j,j}) \frac{\pi(j)}{\pi(i)} q^-_{j,i} F^-_{j,i}(\{0\}) - \nu_{i,j}(\{0\})\\ &\quad - \sum_{k\neq i,j} \frac{\pi(k)}{\pi(i)} q^-_{k,i} q^+_{k,j} F^-_{k,i}(\{0\}) F^+_{k,j}(\{0\})\\ &= (\dagger^-_i - q^-_{i,i}) q^+_{i,j} F^+_{i,j}(\{0\}) + (\dagger^+_j - q^+_{j,j}) \frac{\pi(j)}{\pi(i)} q^-_{j,i} - \alpha_{i,j} - \sum_{k\neq i,j} \frac{\pi(k)}{\pi(i)} q^-_{k,i} q^+_{k,j} F^+_{k,j}(\{0\}),\end{aligned}$$

showing that (3.4) is satisfied if, and only if, the off-diagonal elements of (4.6) are nonnegative. □

As a simple consequence of this result we obtain simple criteria for two drifts to be friends, which yields a blueprint for constructing Markov modulated Brownian motions, i.e., MAPs whose Lévy components are potentially killed linear Brownian motions, having an explicit Wiener–Hopf factorisation.

Corollary 4.9. *Two pure drift MAPs (H^+, J^+) and (H^-, J^-) are $\boldsymbol{\pi}$-friends if, and only if, the matrix $\boldsymbol{\Xi} := -\boldsymbol{\Delta}_{\boldsymbol{\pi}}^{-1} \boldsymbol{\Psi}^-(0)^\top \boldsymbol{\Delta}_{\boldsymbol{\pi}} \boldsymbol{\Psi}^+(0)$ is an ML-matrix satisfying $\boldsymbol{\Xi}\mathbf{1} \leq \mathbf{0}$, $\boldsymbol{\pi}^\top \boldsymbol{\Xi} \leq \mathbf{0}^\top$, and*

$$\pi(i) q^+_{i,j} d^-_i = \pi(j) q^-_{j,i} d^+_j, \quad i,j \in [n], i \neq j.$$

Proof. It is clear that (H^+, J^+) and (H^-, J^-) are $\boldsymbol{\pi}$-fellows. Checking the meaning of $\boldsymbol{\pi}$-quasicompatibility, we see that there $\psi^\pm_{i,j} \equiv 0$, so this is equivalent to the conditions listed in the statement. The result follows from Theorem 4.8. □

More generally, Theorem 4.8 can be thought of as a construction principle for spectrally one-sided MAPs with known Wiener–Hopf factorisation. Such MAPs have seen numerous applications in the modeling of insurance risk, storage models and queuing theory [4]. Having access to the ladder height processes is useful in these cases since the distributional properties of first passage events can be expressed using their characteristics. Let us investigate a specific class of examples. Recall that a function $f\colon (0,\infty) \to \mathbb{R}$ is called *completely monotone* if $f \in \mathcal{C}^\infty((0,\infty))$ and for any $n \in \mathbb{N}_0$ it holds that

$$(-1)^n \frac{\mathrm{d}^n}{\mathrm{d}x^n} f(x) \geq 0, \quad x > 0.$$

By Bernstein's Theorem, see [45, Theorem 1.4], $f \in \mathcal{C}^\infty((0,\infty))$ is completely monotone if, and only if, there exists a *representing measure* μ on $(\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+))$ such that

$$f(x) = \int_{\mathbb{R}_+} \mathrm{e}^{-xy}\, \mu(\mathrm{d}y), \quad x > 0,$$

i.e., f is given as the Laplace transform of μ. A completely monotone function f is the Lévy density of some subordinator if, and only if,

$$\mu(\{0\}) = 0 \quad \text{and} \quad \int_{(0,1)} y^{-1}\, \mu(\mathrm{d}y) + \int_{(1,\infty)} y^{-2}\, \mu(\mathrm{d}y) < \infty \tag{4.8}$$

hold [45, Theorem 6.2].

Examples giving rise to completely monotone Lévy density include jumps of exponential size (μ a multiple of a Dirac mass), jumps of mixed exponential size (μ a discrete measure with finitely many atoms), and subordinators in the 'meromorphic class' of Lévy processes [25] (μ a discrete measure); see [45, chapter 16] for an extensive list.

Example 4.10. Let $\boldsymbol{Q}^-, \boldsymbol{Q}^+ \in \mathbb{R}^{2\times 2}$ be irreducible generator matrices (of unkilled Markov processes) and $\boldsymbol{\pi} \in (0,1]^2$ be a stochastic vector. Let Π_i^+ be absolutely continuous with completely monotone densities with representing measures μ_i^+ for $i = 1,2$ each satisfying (4.8), i.e.,

$$\Pi_i^+(\mathrm{d}x) = \int_{\mathbb{R}_+} \mathrm{e}^{-xy}\, \mu_i^+(\mathrm{d}y)\, \mathrm{d}x, \quad x > 0, i = 1,2,$$

and $\boldsymbol{d}^+, \boldsymbol{d}^- \in (0,\infty)^2$ such that

(i) for $i = 1,2$,

$$\int_{(0,1)} y^{-3}\, \mu_i^+(\mathrm{d}y) < \infty;$$

(ii)

$$d_1^+ + \int_{0+}^\infty \overline{\Pi}_1^+(x)\,\mathrm{d}x = \frac{\pi(2)}{\pi(1)} \frac{q_{2,1}^+ d_2^-}{q_{1,2}^-}, \quad d_2^+ + \int_{0+}^\infty \overline{\Pi}_2^+(x)\,\mathrm{d}x = \frac{\pi(1)}{\pi(2)} \frac{q_{1,2}^+ d_1^-}{q_{2,1}^-};$$

(iii) for any $x > 0$, and $i \neq j$,

$$\int_{\mathbb{R}_+} \Big(1 + \frac{d_i^-}{q_{i,j}^-} y - \frac{q_{j,i}^-}{d_j^-} \frac{1}{y}\Big) \mathrm{e}^{-xy}\, \mu_i^+(\mathrm{d}y) > 0.$$

For example, when the representing measures are supported away from zero, i.e., $\mathrm{supp}(\mu_i^+) \subset (a_i^+, \infty)$ for some $a_i^+ > 0$, then (iii) is satisfied whenever

$$a_i^+ \Big(1 + \frac{d_i^-}{q_{i,j}^-} a_i^+\Big) > \frac{q_{j,i}^-}{d_j^-}$$

and thus examples fufilling both of the above conditions can be easily constructed by choosing large enough and matching drifts $\boldsymbol{d}^+, \boldsymbol{d}^-$.

Choose now probability measures $F_{1,2}^+, F_{2,1}^+$ with support $\mathbb{R}_+$ and decreasing and differentiable densities $f_{1,2}^+, f_{2,1}^+$ on $(0,\infty)$ such that

$$f_{1,2}^+(x) = \frac{\pi(2)}{\pi(1)} \frac{q_{2,1}^-}{q_{1,2}^+ d_1^-} \overline{\Pi}_2^+(x), \quad f_{2,1}^+(x) = \frac{\pi(1)}{\pi(2)} \frac{q_{1,2}^-}{q_{2,1}^+ d_2^-} \overline{\Pi}_1^+(x),$$

for $x > 0$. Then, condition (i) of Definition 4.5 is fulfilled with $\psi_{1,2} = \psi_{2,1} \equiv 0$ and $\alpha_{1,2} = \alpha_{2,1} = 0$ and condition (ii) of Definition 4.5 holds with

$$F^+_{1,2}(\{0\}) = \frac{\pi(2)}{\pi(1)} \frac{q^-_{2,1} d^+_2}{q^+_{1,2} d^-_1}, \quad F^+_{2,1}(\{0\}) = \frac{\pi(1)}{\pi(2)} \frac{q^-_{1,2} d^+_1}{q^+_{2,1} d^-_2}. \tag{4.9}$$

Our assumption (ii) ensures that $F^+_{i,j}$ are probability measures for each $i \neq j$. Moreover, it is easy to see that (4.6) is satisfied since $\boldsymbol{A} = \boldsymbol{0}_{2\times 2}$ and the first term is always an ML-matrix when the matrices have dimension 2×2. Let now (H^+, J^+) be a MAP subordinator with characteristic matrix exponent $\boldsymbol{\Psi}^+$ associated to the drift vector $\boldsymbol{d}^+$ and Lévy measure matrix

$$\boldsymbol{\Pi}^+ = \begin{bmatrix} \Pi^+_1 & q^+_{1,2} F^+_{1,2} \\ q^+_{2,1} F^+_{2,1} & \Pi^+_2 \end{bmatrix}$$

and (H^-, J^-) the pure drift MAP associated to $\boldsymbol{d}^-$ and generator matrix $\boldsymbol{Q}^-$. Straightforward calculations show that for any $x > 0$,

$$\begin{aligned} &\frac{\partial}{\partial x}\Big\{\boldsymbol{\Delta}_{\boldsymbol{\pi}}^{-1}\boldsymbol{\Psi}^-(0)^\top \boldsymbol{\Delta}_{\boldsymbol{\pi}} \overline{\boldsymbol{\Pi}}^+(x) - \boldsymbol{\Delta}^-_{\boldsymbol{d}} \partial \boldsymbol{\Pi}^+(x)\Big\} \\ &= -\boldsymbol{\Delta}_{\boldsymbol{\pi}}^{-1}\boldsymbol{\Psi}^-(0)^\top \boldsymbol{\Delta}_{\boldsymbol{\pi}} \partial\boldsymbol{\Pi}^+(x) - \boldsymbol{\Delta}^-_{\boldsymbol{d}} \partial^2 \boldsymbol{\Pi}^+(x) \\ &= \begin{bmatrix} q^-_{1,2} \int_{\mathbb{R}_+} \Big(1 + \frac{d^-_1}{q^-_{1,2}} y - \frac{q^-_{2,1}}{d^-_2} \frac{1}{y}\Big) \mathrm{e}^{-xy}\, \mu^+_1(\mathrm{d}y) & \frac{\pi(2)}{\pi(1)} \frac{q^-_{1,2} q^-_{2,1}}{d^-_1} \int_{\mathbb{R}_+} \frac{1}{y} \mathrm{e}^{-xy}\, \mu^+_2(\mathrm{d}y) \\ \frac{\pi(1)}{\pi(2)} \frac{q^-_{1,2} q^-_{2,1}}{d^-_2} \int_{\mathbb{R}_+} \frac{1}{y} \mathrm{e}^{-xy}\, \mu^+_1(\mathrm{d}y) & q^-_{2,1} \int_{\mathbb{R}_+} \Big(1 + \frac{d^-_2}{q^-_{2,1}} y - \frac{q^-_{1,2}}{d^-_1} \frac{1}{y}\Big) \mathrm{e}^{-xy}\, \mu^+_2(\mathrm{d}y) \end{bmatrix}, \end{aligned} \tag{4.10}$$

which is nonnegative by (iii), directly implying that our two processes are $\boldsymbol{\pi}$-fellows. Property (iii) of $\boldsymbol{\pi}$-quasicompatibility is satisfied since (H^+, J^+) is unkilled, and property (iv) is satisfied since (H^-, J^-) is unkilled. Taking all this into account, we see that (H^+, J^+) is a $\boldsymbol{\pi}$-quasicompatible $\boldsymbol{\pi}$-fellow of (H^-, J^-) and hence, with Proposition 4.8, that the MAP subordinator (H^+, J^+) is a $\boldsymbol{\pi}$-friend of the pure drift (H^-, J^-). Moreover, since $q^+_{i,j} = -q^+_{i,i}$ and $q^-_{i,j} = -q^-_{i,i}$ for $i, j \in \{1, 2\}$ with $i \neq j$, it follows that for the bonding MAP (ξ, J), the matrix

$$\boldsymbol{Q} = -\boldsymbol{\Delta}_{\boldsymbol{\pi}}^{-1} (\boldsymbol{Q}^-)^\top \boldsymbol{\Delta}_{\boldsymbol{\pi}} \boldsymbol{Q}^+ = \begin{bmatrix} -q^+_{1,1} q^-_{1,1} - \frac{\pi(2)}{\pi(1)} q^+_{2,2} q^-_{2,2} & q^+_{1,1} q^-_{1,1} + \frac{\pi(2)}{\pi(1)} q^+_{2,2} q^-_{2,2} \\ q^+_{2,2} q^-_{2,2} + \frac{\pi(1)}{\pi(2)} q^+_{1,1} q^-_{1,1} & -q^+_{2,2} q^-_{2,2} - \frac{\pi(1)}{\pi(2)} q^+_{1,1} q^-_{1,1} \end{bmatrix}$$

is an irreducible generator matrix, and hence $\boldsymbol{\pi}$ is the unique invariant distribution of J. From [16, Lemma 3.20] we know that the Lévy components $\xi^{(1)}, \xi^{(2)}$ of ξ have non-trivial Brownian part with scaling factor $\sigma_i^2 = 2 d^+_i d^-_i$ for $i = 1, 2$. Furthermore, combining (4.9) and (4.10) we conclude with Proposition 3.4 and Theorem 3.10 that the Lévy measure matrix of (ξ, J) is given by

$$\begin{aligned} &\boldsymbol{\Pi}(\mathrm{d}x) \\ &= \begin{bmatrix} \mathbf{1}_{\{x>0\}} q^-_{1,2} \int_{\mathbb{R}_+} \Big(1 + \frac{d^-_1 y}{q^-_{1,2}} - \frac{q^-_{2,1}}{d^-_2 y}\Big) \mathrm{e}^{-xy}\, \mu^+_1(\mathrm{d}y)\, \mathrm{d}x & \frac{\pi(2)}{\pi(1)} q^-_{2,2} \Big\{ \Big(q^+_{2,2} + q^-_{1,1} \frac{d^+_2}{d^-_1}\Big) \delta_0(\mathrm{d}x) + \mathbf{1}_{\{x>0\}} \frac{q^-_{1,1}}{d^-_1} \int_{\mathbb{R}_+} \frac{\mathrm{e}^{-xy}}{y}\, \mu^+_2(\mathrm{d}y)\, \mathrm{d}x \Big\} \\ \frac{\pi(1)}{\pi(2)} q^-_{1,1} \Big\{ \Big(q^+_{1,1} + q^-_{2,2} \frac{d^+_1}{d^-_2}\Big) \delta_0(\mathrm{d}x) + \mathbf{1}_{\{x>0\}} \frac{q^-_{2,2}}{d^-_2} \int_{\mathbb{R}_+} \frac{\mathrm{e}^{-xy}}{y}\, \mu^+_1(\mathrm{d}y)\, \mathrm{d}x \Big\} & \mathbf{1}_{\{x>0\}} q^-_{2,1} \int_{\mathbb{R}_+} \Big(1 + \frac{d^-_2 y}{q^-_{2,1}} - \frac{q^-_{1,2}}{d^-_1 y}\Big) \mathrm{e}^{-xy}\, \mu^+_2(\mathrm{d}y)\, \mathrm{d}x \end{bmatrix}. \end{aligned}$$

As outlined in Remark 5.8, the MAP Wiener–Hopf factorisation is unique for this process, and so the ladder height processes for the bonding MAP (ξ, J) are indeed (H^+, J^+) and (H^-, J^-).

4.2. **Construction of MAPs jumping in both directions.** Moving away from the spectrally one-sided assumption, we structure our analysis as follows. Firstly, we concentrate on finding explicit criteria for $\boldsymbol{\pi}$-compatibility of two pure jump MAP subordinators with diffuse jump structure. Secondly, we show that the $\boldsymbol{\pi}$-friendship of two general MAP subordinators can be studied by splitting them into their diffuse pure jump parts and the remainder, for which the notion of $\boldsymbol{\pi}$-quasicompatibility can be used to simplify conditions.

We begin with the diffuse pure jump part, as follows.

Proposition 4.11. *Let (H^+, J^+) and (H^-, J^-) be MAP subordinators with decreasing, differentiable and convex Lévy density matrices on $(0,\infty)$ such that*

$$\boldsymbol{\Pi}^+(\{0\}) = \boldsymbol{\Pi}^-(\{0\}) = \boldsymbol{\Delta}_{\boldsymbol{d}}^+ = \boldsymbol{\Delta}_{\boldsymbol{d}}^- = \mathbf{0}_{n\times n}.$$

Then, (H^+, J^+) is $\boldsymbol{\pi}$-compatible with (H^-, J^-) if

(i) for any $i \in [n]$ we have $\mathbb{E}^0\big[H_1^{+,(i)}\big] < \infty$ and $\mathbb{E}^0\big[H_1^{-,(i)}\big] < \infty$;

(ii) for $i \neq j$ we define

$$v_{i,j}(x) := q_{i,j}^+ f_{i,j}^+(x)\overline{\Pi}_i^-(x), \quad w_{i,j}(x) := \frac{\pi(j)}{\pi(i)} q_{j,i}^- f_{j,i}^-(x)\overline{\Pi}_j^+(x),$$

for $x > 0$, then $v_{i,j}, w_{i,j} \in L^1((0,\infty))$;

(iii) the vector $\boldsymbol{\Delta}_{\boldsymbol{\pi}}^{-1}\boldsymbol{\Psi}^-(0)^\top\boldsymbol{\Delta}_{\boldsymbol{\pi}}\boldsymbol{\Psi}^+(0)\mathbf{1}$ is nonnegative;

(iv) the vector $\boldsymbol{\pi}^\top\boldsymbol{\Delta}_{\boldsymbol{\pi}}^{-1}\boldsymbol{\Psi}^-(0)^\top\boldsymbol{\Delta}_{\boldsymbol{\pi}}\boldsymbol{\Psi}^+(0)$ is nonnegative.

Proof. Since the transitional jumps have no atom at 0, this boils down to showing that under (i) and (ii), (3.3) holds for some finite signed measure $\nu_{i,j}$ with $\nu_{i,j}(\{0\}) = \nu_{i,j}(\mathbb{R}) = 0$. Fix $i, j \in [n]$ with $i \neq j$. Since $F_{i,j}^+(\{0\}) = 0$ and $d_i^- = 0$, we obtain

$$\begin{aligned}
\vartheta_{i,j}^{(1)}(\mathrm{d}x) &:= q_{i,j}^+ F_{i,j}^+ * \widetilde{\chi}_i^-(\mathrm{d}x) \\
&= \mathbf{1}_{\mathbb{R}\setminus\{0\}}(x) \int_{0+}^\infty \mathbf{1}_{(0,\infty)}(y-x) q_{i,j}^+ f_{i,j}^+(y)\overline{\Pi}_i^-(y-x)\,\mathrm{d}y\,\mathrm{d}x \\
&= \Big[\mathbf{1}_{(-\infty,0)}(x)\int_{0+}^\infty q_{i,j}^+ f_{i,j}^+(y)\overline{\Pi}_i^-(y-x)\,\mathrm{d}y \\
&\qquad + \mathbf{1}_{(0,\infty)}(x)\int_{x+}^\infty q_{i,j}^+ f_{i,j}^+(y)\overline{\Pi}_i^-(y-x)\,\mathrm{d}y\Big]\,\mathrm{d}x \\
&= \Big[\mathbf{1}_{(-\infty,0)}(x)\int_{0+}^\infty q_{i,j}^+ f_{i,j}^+(y)\overline{\Pi}_i^-(y-x)\,\mathrm{d}y \\
&\qquad + \mathbf{1}_{(0,\infty)}(x)\int_{0+}^\infty q_{i,j}^+ f_{i,j}^+(y+x)\overline{\Pi}_i^-(y)\,\mathrm{d}y\Big]\,\mathrm{d}x \\
&=: \big(\mathbf{1}_{(-\infty,0)}(x)\theta_{i,j}^-(x) + \mathbf{1}_{(0,\infty)}(x)\theta_{i,j}^+(x)\big)\,\mathrm{d}x.
\end{aligned}$$

Now, since $f_{i,j}^+$ is decreasing by assumption and $\overline{\Pi}^-$ is decreasing as well, it follows by monotone convergence that

$$\lim_{x\uparrow 0}\theta_{i,j}^-(x) = \lim_{x\downarrow 0}\theta_{i,j}^+(x) = \int_{0+}^\infty v_{i,j}(y)\,\mathrm{d}y < \infty.$$

Thus, if we define

$$\theta_{i,j}(x) = \begin{cases} \theta^-_{i,j}(x), & x < 0, \\ \int_{0+}^\infty v_{i,j}(y)\,\mathrm{d}y, & x = 0, \\ \theta^+_{i,j}(x), & x > 0, \end{cases}$$

then $\vartheta^{(1)}_{i,j}$ is absolutely continuous with continuous density $\theta_{i,j}$. Using that $\partial\mathbf{\Pi}^-$ decreases on $(0,\infty)$, we find for any $y > 0$ and $x < 0$ that

$$\frac{\partial}{\partial x} q^+_{i,j} f^+_{i,j}(y)\overline{\Pi}^-_i(y-x) = q^+_{i,j} f^+_{i,j}(y)\partial\Pi^-_i(y-x) \leq q^+_{i,j} f^+_{i,j}(y)\partial\Pi^-_i(-x).$$

The Lévy density $\partial\Pi^-_i$ is bounded away from zero by our assumptions and $f^+_{i,j} \in L^1((0,\infty))$, implying that we may differentiate under the integral such that $\theta_{i,j}$ is differentiable on $(-\infty, 0)$ with

$$\theta'_{i,j}(x) = \int_{0+}^\infty q^+_{i,j} f^+_{i,j}(y)\partial\Pi^-_i(y-x)\,\mathrm{d}y, \quad x < 0.$$

Moreover, using convexity of $\mathbf{\Pi}^+$ on $(0,\infty)$, it follows for any $x > 0$ and $y > 0$ that

$$\Big|\frac{\partial}{\partial x} q^+_{i,j} f^+_{i,j}(x+y)\overline{\Pi}^-_i(y)\Big| = -q^+_{i,j}\tfrac{\partial}{\partial x} f^+_{i,j}(x+y)\overline{\Pi}^-_i(y) \leq -q^+_{i,j}\tfrac{\partial}{\partial x} f^+_{i,j}(x)\overline{\Pi}^-_i(y).$$

Local boundedness of $\frac{\partial}{\partial x} f^+_{i,j}(x)$ and $\overline{\Pi}^-_i \in L^1((0,\infty))$ thanks to (i) now also imply that $\theta_{i,j}$ is differentiable on $(0,\infty)$ with derivative

$$\theta'_{i,j}(x) = \int_{0+}^\infty q^+_{i,j}\tfrac{\partial}{\partial x} f^+_{i,j}(x+y)\overline{\Pi}^-_i(y)\,\mathrm{d}y, \quad x > 0.$$

Thus, if we let $\nu^{(1)}_{i,j}$ be the signed measure with density $\theta'_{i,j}\mathbf{1}_{\mathbb{R}\setminus\{0\}}$, it follows that $\theta_{i,j}(x) = \nu^{(1)}_{i,j}((-\infty, x])$ for $x \in \mathbb{R}$. Finiteness of $\nu^{(1)}_{i,j}$ follows now from the fact that

$$|\nu^{(1)}_{i,j}|(\mathbb{R}_-) = \theta_{i,j}(0) = |\nu^{(1)}_{i,j}|(\mathbb{R}_+)$$

and $\theta_{i,j}(0) = \int_{0+}^\infty v_{i,j}(y)\,\mathrm{d}y < \infty$. We therefore conclude that there exists a finite signed measure $\nu^{(1)}_{i,j}$ without atom at 0 such that $\vartheta^{(1)}_{i,j}$ is absolutely continuous with density $\nu^{(1)}_{i,j}((-\infty,\cdot])$. Similarly, it follows that for

$$\vartheta^{(2)}_{i,j}(\mathrm{d}x) = \frac{\pi(j)}{\pi(i)} q^-_{j,i} F^-_{j,i} * \chi^+_j(\mathrm{d}x), \quad x \in \mathbb{R},$$

there exists some finite signed measure $\nu^{(2)}_{i,j}$ without atom at 0 such that $\vartheta^{(2)}_{i,j}$ is absolutely continuous with density $\nu^{(2)}_{i,j}((-\infty,\cdot])$. Finally, since

$$q^+_{i,j} F^+_{i,j} * \widetilde{\chi}^-_i - \frac{\pi(j)}{\pi(i)} q^-_{j,i} \widetilde{F}^-_{j,i} * \chi^+_j = \vartheta^{(1)}_{i,j} - \vartheta^{(2)}_{i,j},$$

it follows that (3.3) is satisfied for the finite signed measure $\nu_{i,j} = \nu^{(1)}_{i,j} - \nu^{(2)}_{i,j}$, which has no atom at 0 and satisfies $\nu_{i,j}(\mathbb{R}) = 0$ since by monotone convergence $\lim_{x\to\infty} \nu^{(1)}_{i,j}((-\infty,x]) = \lim_{x\to\infty} \theta^+_{i,j}(x) = 0$, and similarly $\lim_{x\to\infty} \nu^{(2)}_{i,j}((-\infty,x]) = 0$. This proves the assertion. □

We can use Theorem 4.8 to give simpler conditions for $\boldsymbol{\pi}$-compatibility with a drift, and Proposition 4.11 to deal with $\boldsymbol{\pi}$-compatibility of processes with diffuse pure jump structure. In general, we can reduce the $\boldsymbol{\pi}$-compatibility condition to these special cases by splitting up the structure of the MAPs. To this end, let $(H^{\pm}, J^{\pm})$ MAPs such that $F^{\pm}_{i,j}(\{0\}) \neq 1$ for all $i \neq j$ and define MAPs $(H^{\pm,\circ}, J^{\pm,\circ})$ by setting $\boldsymbol{\Psi}^{\pm,\circ}(0) = \boldsymbol{\Psi}^{\pm}(0)$, $\boldsymbol{\Delta}_{\boldsymbol{d}^{\pm,\circ}} = \boldsymbol{\Pi}^{\pm,\circ}(\{0\}) = \mathbf{0}_{n\times n}$ and

$$\boldsymbol{\Pi}^{\pm,\circ}(\mathrm{d}x) = \boldsymbol{\Theta}^{\pm} \odot \boldsymbol{\Pi}^{\pm}(\mathrm{d}x), \quad x > 0,$$

where $\boldsymbol{\Theta}^{\pm}_{i,i} = 1$ for any $i \in [n]$ and

$$\boldsymbol{\Theta}^{\pm}_{i,j} = \frac{1}{F^{+}_{i,j}((0,\infty))}, \quad i,j \in [n], i \neq j.$$

In other words, $(H^{\pm,\circ}, J^{\pm,\circ})$ are MAPs obtained from $(H^{\pm}, J^{\pm})$ by eliminating the drift part and conditioning the transitional jumps to be strictly positive. Moreover, if the Lévy density matrices of (H^{+}, J^{+}) and (H^{-}, J^{-}) are decreasing, differentiable and convex on $(0,\infty)$, the MAPs $(H^{\pm,\circ}, J^{\pm,\circ})$ fall into the class of MAPs considered in Proposition 4.11, where constructive criteria for $\boldsymbol{\pi}$-compatibility are established.

Theorem 4.12. *Let (H^{+}, J^{+}) and (H^{-}, J^{-}) be MAP subordinators with continuous Lévy density matrices on $(0,\infty)$ and such that $F^{\pm}_{i,j}(\{0\}) \neq 1$ for all $i, j \in [n]$ with $i \neq j$. If (H^{+}, J^{+}) and (H^{-}, J^{-}) are $\boldsymbol{\pi}$-quasicompatible $\boldsymbol{\pi}$-fellows, and either*

(i) the conditions of Proposition 4.11 hold for $(H^{+,\circ}, J^{+,\circ})$ and $(H^{-,\circ}, J^{-,\circ})$, or

(ii) $\boldsymbol{F}^{+}(\{0\}) = (\boldsymbol{F}^{-}(\{0\}))^{\top}$ and $(H^{+,\circ}, J^{+,\circ})$, $(H^{-,\circ}, J^{-,\circ})$ are $\boldsymbol{\pi}$-compatible,

then (H^{+}, J^{+}) and (H^{-}, J^{-}) are $\boldsymbol{\pi}$-friends.

The following lemma will be used in the proof of this theorem.

Lemma 4.13. *Suppose that (H^{+}, J^{+}) and (H^{-}, J^{-}) are MAP subordinators with decreasing, continuous Lévy density matrices on $(0,\infty)$ such that*

$$\boldsymbol{\Pi}^{+}(\{0\}) = \boldsymbol{\Pi}^{-}(\{0\}) = \boldsymbol{\Delta}^{+}_{\boldsymbol{d}} = \boldsymbol{\Delta}^{-}_{\boldsymbol{d}} = \mathbf{0}_{n\times n}.$$

If (H^{+}, J^{+}) is $\boldsymbol{\pi}$-compatible with (H^{-}, J^{-}), then $\underline{\nu}_{i,j}$ from Definition 3.7 is a continuous function for any $i, j \in [n]$ with $i \neq j$.

Proof. The $\boldsymbol{\pi}$-compatibility of the two MAP subordinators implies that there is a finite signed measure $\nu_{i,j}$ such that $\underline{\nu}_{i,j}(x) = \vartheta^{(1)}_{i,j}(\mathrm{d}x) - \vartheta^{(2)}_{i,j}(\mathrm{d}x)$, in the sense of distributions and using notation from the proof of Proposition 4.11. Considering the representation of $\vartheta^{(1)}_{i,j}$ given in said proof, we see that, under our assumptions, it is absolutely continuous with continuous density. The same holds for $\vartheta^{(2)}_{i,j}$, and this shows that $\underline{\nu}_{i,j}$ is continuous. □

Proof of Theorem 4.12. We will show that the processes (H^{+}, J^{+}) and (H^{-}, J^{-}) are $\boldsymbol{\pi}$-compatible $\boldsymbol{\pi}$-fellows, and the conclusion will then follow from Theorem 4.4. It is enough to establish $\boldsymbol{\pi}$-compatibility of (H^{+}, J^{+}) with (H^{-}, J^{-}), which under the given assumptions boils down to showing that condition (ii) of Definition 3.7 holds. To this end, let us write

$$\vartheta_{i,j}(\mathrm{d}x) := q^{+}_{i,j} F^{+}_{i,j} * \widetilde{\chi}^{-}_{i}(\mathrm{d}x) - \frac{\pi(j)}{\pi(i)} q^{-}_{j,i} \widetilde{F}^{-}_{j,i} * \chi^{+}_{j}(\mathrm{d}x), \quad x \in \mathbb{R},$$

and notice that since (H^+, J^+) is $\boldsymbol{\pi}$-quasicompatible with (H^-, J^-) we have

$$d_i^- q_{i,j}^+ F_{i,j}^+(\{0\}) = \tfrac{\pi(j)}{\pi(i)} d_j^+ q_{j,i}^- F_{j,i}^-(\{0\}),$$

such that we may write $\vartheta_{i,j}(\mathrm{d}x) = \vartheta^\circ_{i,j}(\mathrm{d}x) + \vartheta^{\rightsquigarrow}_{i,j}(\mathrm{d}x)$, where, for $x \in \mathbb{R}$,

$$\begin{aligned}\vartheta^{\rightsquigarrow}_{i,j}(\mathrm{d}x) = \mathbf{1}_{(0,\infty)}(x)\big(d_i^- q_{i,j}^+ f_{i,j}^+(x) - \tfrac{\pi(j)}{\pi(i)} q_{j,i}^- F_{j,i}^-(\{0\})\overline{\Pi}_j^+(x)\big)\,\mathrm{d}x \\ + \mathbf{1}_{(-\infty,0)}(x)\big(\overline{\Pi}_i^-(-x) q_{i,j}^+ F_{i,j}^+(\{0\}) - \tfrac{\pi(j)}{\pi(i)} q_{j,i}^- f_{j,i}^-(-x) d_j^+\big)\,\mathrm{d}x,\end{aligned}$$

and

$$\vartheta^\circ_{i,j}(\mathrm{d}x) = F_{i,j}^+((0,\infty)) q_{i,j}^+ F_{i,j}^{+,\circ} * \widetilde{\chi}_i^{-,\circ}(\mathrm{d}x) - F_{j,i}^-((0,\infty)) \frac{\pi(j)}{\pi(i)} q_{j,i}^- \widetilde{F}_{j,i}^{-,\circ} * \chi_j^{+,\circ}(\mathrm{d}x),$$

where $\chi^{\pm,\circ}$ play the same role for $(H^{\pm,\circ}, J^{\pm,\circ})$ as do the measures $\chi^\pm$ for $(H^\pm, J^\pm)$. If hypothesis (i) holds, then both summands in $\vartheta^\circ_{i,j}$ represent distribution functions of a finite signed measure, and checking the proof of said proposition, we can see that these distribution functions are continuous; hence, there exists some finite signed measure $\nu^\circ_{i,j}$ with continuous distribution function $\underline{\nu}^\circ_{i,j}$ such that

$$\vartheta^\circ_{i,j}(\mathrm{d}x) = \underline{\nu}^\circ_{i,j}(x)\,\mathrm{d}x, \quad x \in \mathbb{R}. \tag{4.11}$$

On the other hand, if hypothesis (ii) holds, then we can write

$$\vartheta^\circ_{i,j}(\mathrm{d}x) = F_{i,j}^+((0,\infty))\Big(q_{i,j}^+ F_{i,j}^{+,\circ} * \widetilde{\chi}_i^{-,\circ}(\mathrm{d}x) - \frac{\pi(j)}{\pi(i)} q_{j,i}^- \widetilde{F}_{j,i}^{-,\circ} * \chi_j^{+,\circ}(\mathrm{d}x)\Big), \quad x \in \mathbb{R}.$$

Since $(H^{+,\circ}, J^{+,\circ})$ is $\boldsymbol{\pi}$-compatible with $(H^{-,\circ}, J^{-,\circ})$ and both processes have zero drift vectors and their Lévy measure matrices have no atoms at 0 it follows with Lemma 4.13 that for any $i \neq j$ there exists some finite signed measure $\nu^\circ_{i,j}$ with continuous distribution function $\underline{\nu}^\circ_{i,j}$ once again satisfying (4.11).

Following the arguments from the proof of Theorem 4.8, the assumption that (H^+, J^+) is $\boldsymbol{\pi}$-quasicompatible with (H^-, J^-) guarantees the existence of a finite signed measure $\nu^{\rightsquigarrow}_{i,j}$ such that for a.e. $x \in \mathbb{R}$,

$$\vartheta^{\rightsquigarrow}_{i,j}(\mathrm{d}x) = \underline{\nu}^{\rightsquigarrow}_{i,j}(x)\,\mathrm{d}x.$$

To conclude, $\nu_{i,j} := \nu^\circ_{i,j} + \nu^{\rightsquigarrow}_{i,j}$ is a finite signed measure such that $\vartheta_{i,j}(\mathrm{d}x) = \underline{\nu}_{i,j}(x)\,\mathrm{d}x$, i.e. (3.3) is satisfied, and since $\nu^\circ_{i,j}(\{0\}) = 0$ it follows that (4.6) being an ML-matrix guarantees that (3.4) holds as well. Therefore, (H^+, J^+) is $\boldsymbol{\pi}$-compatible with (H^-, J^-). □

Example 4.14. Let irreducible generator matrices $\boldsymbol{Q}^+, \boldsymbol{Q}^- \in \mathbb{R}^{2\times 2}$ (of unkilled Markov processes), a stochastic vector $\boldsymbol{\pi} \in (0,1)^2$ and $\boldsymbol{\gamma}^+, \boldsymbol{\gamma}^-, \boldsymbol{\beta}^+, \boldsymbol{\beta}^-, \boldsymbol{d}^+, \boldsymbol{d}^- \in (0,\infty)^2$ be given such that the following conditions are satisfied:

(i)

$$\frac{\pi(i)}{\pi(j)} q_{i,j}^- \Big(d_i^+ + \frac{\gamma_i^+}{(\beta_i^+)^2}\Big) = q_{j,i}^+ \Big(d_j^- + \frac{\gamma_j^-}{(\beta_j^-)^2}\Big), \quad i, j \in \{1,2\}, i \neq j; \tag{4.12}$$

(ii)

$$d_1^+ > \frac{\pi(2)}{\pi(1)} \frac{q_{2,1}^+}{q_{1,2}^-} \frac{\gamma_2^-}{(\beta_2^-)^2} \quad \text{and} \quad d_2^+ > \frac{\pi(1)}{\pi(2)} \frac{q_{1,2}^+}{q_{2,1}^-} \frac{\gamma_1^-}{(\beta_1^-)^2} \tag{4.13}$$

(iii)

$$0 < 1 + \frac{\beta_i^{\pm} d_i^{\mp}}{q_{i,j}^{\mp}} - q_{j,i}^{\mp}\beta_i^{\pm} \frac{d_i^{\pm}(\beta_j^{\mp})^2 - \frac{\pi(j)}{\pi(i)}\frac{q_{j,i}^{\pm}}{q_{i,j}^{\mp}}\gamma_j^{\mp}}{d_i^{\pm}d_j^{\mp}(\beta_i^{\pm})^2(\beta_j^{\mp})^2 - \gamma_j^{\mp}\gamma_i^{\pm}}, \quad i,j \in \{1,2\}, i \neq j. \tag{4.14}$$

Note that this can, e.g., easily be achieved by fixing all vectors but $\boldsymbol{d}^+, \boldsymbol{d}^-$ and then choosing $\boldsymbol{d}^+, \boldsymbol{d}^-$ large enough such that (4.13) and (4.14) are satisfied, while simultaneosuly ensuring that $\boldsymbol{d}^+$, $\boldsymbol{d}^-$ solve (4.12). Let now

$$\Pi_i^{\pm}(\mathrm{d}x) = \gamma_i^{\pm}\mathrm{e}^{-\beta_i^{\pm}x}\,\mathrm{d}x, \quad x > 0, i = 1,2,$$

and $F_{i,j}^{\pm}$ be measures on $\mathbb{R}_+$ with density $f_{i,j}^{\pm}$ on $(0,\infty)$ satisfying

$$f_{i,j}^{\pm}(x) = \frac{\pi(j)}{\pi(i)}\frac{q_{j,i}^{\mp}}{q_{i,j}^{\pm}d_i^{\mp}}F_{j,i}^{\mp}(\{0\})\overline{\Pi}_j^{\pm}(x) = \frac{\pi(j)}{\pi(i)}\frac{q_{j,i}^{\mp}}{q_{i,j}^{\pm}d_i^{\mp}}F_{j,i}^{\mp}(\{0\})\frac{\gamma_j^{\pm}}{\beta_j^{\pm}}\mathrm{e}^{-\beta_j^{\pm}x}, \tag{4.15}$$

for $x > 0, i,j \in \{1,2\}, i \neq j$. Integrating (4.15) under the restriction that $F_{i,j}^{\pm}$ as defined above are probability distributions, we obtain the system of linear equations

$$1 - F_{i,j}^{\pm}(\{0\}) = \frac{\pi(j)}{\pi(i)}\frac{q_{j,i}^{\mp}}{q_{i,j}^{\pm}d_i^{\mp}}F_{j,i}^{\mp}(\{0\})\frac{\gamma_j^{\pm}}{(\beta_j^{\pm})^2}, \quad i \in \{1,2\}, i \neq j,$$

for $(F_{1,2}^{+}(\{0\}), F_{2,1}^{+}(\{0\}), F_{1,2}^{-}(\{0\}), F_{2,1}^{-}(\{0\}))$. Solving the system yields

$$F_{i,j}^{\pm}(\{0\}) = \frac{d_j^{\pm}d_i^{\mp} - d_j^{\pm}\frac{\pi(j)}{\pi(i)}\frac{q_{j,i}^{\mp}}{q_{i,j}^{\pm}}\frac{\gamma_j^{\pm}}{(\beta_j^{\pm})^2}}{d_j^{\pm}d_i^{\mp} - \frac{\gamma_j^{\pm}\gamma_i^{\mp}}{(\beta_j^{\pm})^2(\beta_i^{\mp})^2}}, \quad i,j \in \{1,2\}, i \neq j. \tag{4.16}$$

Note that combining (4.12) and (4.13) shows that indeed $F_{i,j}^{\pm}(\{0\}) \in (0,1)$ and thus $F_{i,j}^{\pm}$ are probability distributions as desired. Let now $(H^{\pm}, J^{\pm})$ be unkilled MAP subordinators with modulator generator matrices $\boldsymbol{Q}^{\pm}$, Lévy measure matrices

$$\boldsymbol{\Pi}^{\pm} = \begin{bmatrix} \Pi_1^{\pm} & q_{1,2}^{\pm}F_{1,2}^{\pm} \\ q_{2,1}^{\pm}F_{2,1}^{\pm} & \Pi_2^{\pm} \end{bmatrix}$$

and drifts $\boldsymbol{d}^{\pm}$. For

$$\zeta_{i,j}^{\pm} = q_{i,j}^{\mp}q_{j,i}^{\mp}\frac{d_j^{\pm}(\beta_i^{\mp})^2 - \frac{\pi(i)}{\pi(j)}\frac{q_{i,j}^{\pm}}{q_{j,i}^{\mp}}\gamma_i^{\mp}}{d_j^{\pm}d_i^{\mp}(\beta_j^{\pm})^2(\beta_i^{\mp})^2 - \gamma_i^{\mp}\gamma_j^{\pm}},$$

we then have

$$\begin{aligned} &-\boldsymbol{\Delta}_{\boldsymbol{\pi}}^{-1}\boldsymbol{\Psi}^{\mp}(0)^{\top}\boldsymbol{\Delta}_{\boldsymbol{\pi}}\partial\boldsymbol{\Pi}^{\pm}(x) - \boldsymbol{\Delta}_{\boldsymbol{d}}^{\mp}\partial^2\boldsymbol{\Pi}^{\pm}(x) \\ &= \begin{bmatrix} \big(q_{1,2}^{\mp} + (d_1^{\mp} - \zeta_{2,1}^{\pm})\beta_1^{\pm}\big)\gamma_1^{\pm}\mathrm{e}^{-\beta_1^{\pm}x} & \frac{\pi(2)}{\pi(1)}\zeta_{1,2}^{\pm}\beta_2^{\pm}\gamma_2^{\pm}\mathrm{e}^{-\beta_2^{\pm}x} \\ \frac{\pi(1)}{\pi(2)}\zeta_{2,1}^{\pm}\beta_1^{\pm}\gamma_1^{\pm}\mathrm{e}^{-\beta_1^{\pm}x} & \big(q_{2,1}^{\mp} + (d_2^{\mp} - \zeta_{1,2}^{\pm})\beta_2^{\pm}\big)\gamma_2^{\pm}\mathrm{e}^{-\beta_2^{\pm}x} \end{bmatrix}, \end{aligned} \tag{4.17}$$

which is nonnegative for any $x > 0$ by (4.12)-(4.14). This shows that (H^+, J^+) and (H^-, J^-) are $\boldsymbol{\pi}$-fellows. Moreover, (4.15) implies (i) of Definition 4.5, and a short calculation reveals that the matrices $-\boldsymbol{\Delta}_{\boldsymbol{\pi}}^{-1}(\boldsymbol{Q}^{\mp})^{\top}\boldsymbol{\Delta}_{\boldsymbol{\pi}}\boldsymbol{Q}^{\pm}$ are generators of unkilled Markov processes for which $\boldsymbol{\pi}$ is invariant. Also observe that (4.12) together with (4.16) shows that (4.7) is satisfied, while the choice (4.15) ensures that condition (i) of $\boldsymbol{\pi}$-quasicompatibility holds. Consequently, (H^+, J^+) and (H^-, J^-) are $\boldsymbol{\pi}$-quasicompatible $\boldsymbol{\pi}$-fellows. Moreover, it is obvious that $(H^{\pm,\circ}, J^{\pm,\circ})$ satisfy the conditions of Proposition 4.11 and are thus $\boldsymbol{\pi}$-compatible with one another. Theorem 4.12 thus demonstrates that (H^+, J^+) and (H^-, J^-) are $\boldsymbol{\pi}$-friends.

Let us now calculate the Lévy measure matrix

$$\mathbf{\Pi} = \begin{bmatrix} \Pi_1 & q_{1,2}F_{1,2} \\ q_{2,1}F_{2,1} & \Pi_2 \end{bmatrix}$$

of the bonding MAP (ξ, J). Plugging into the équations amicales from Theorem 3.10 and using the expression for the transitional atoms at 0 from Proposition 3.4 in conjunction with (4.16) we obtain

$$\begin{aligned} &\Pi_i(\mathrm{d}x) \\ &= \Bigg\{ \mathbf{1}_{\{x<0\}} \Bigg\{ \frac{\gamma_i^+}{\beta_i^+ + \beta_i^-} \left(1 + \frac{\pi(j)}{\pi(i)} \frac{\zeta_{j,i}^+ \zeta_{j,i}^- \beta_i^+ \beta_i^-}{q_{i,j}^+ q_{j,i}^+ q_{i,j}^- q_{j,i}^-} \right) + q_{i,j}^+ + (d_i^+ - \zeta_{j,i}^-)\beta_i^- \Bigg\} \gamma_i^- \mathrm{e}^{-\beta_i^- |x|} \\ &\quad + \mathbf{1}_{\{x>0\}} \Bigg\{ \frac{\gamma_i^-}{\beta_i^+ + \beta_i^-} \left(1 + \frac{\pi(j)}{\pi(i)} \frac{\zeta_{j,i}^+ \zeta_{j,i}^- \beta_i^+ \beta_i^-}{q_{i,j}^+ q_{j,i}^+ q_{i,j}^- q_{j,i}^-} \right) + q_{i,j}^- + (d_i^- - \zeta_{j,i}^+)\beta_i^+ \Bigg\} \gamma_i^+ \mathrm{e}^{-\beta_i^+ x} \Bigg\} \mathrm{d}x \end{aligned}$$

and

$$\begin{aligned} &q_{i,j}F_{i,j}(\mathrm{d}x) \\ &= \Bigg\{ \mathbf{1}_{\{x<0\}} \Bigg\{ \frac{\gamma_j^+}{\beta_j^+ + \beta_i^-} \left(\frac{\zeta_{i,j}^+ \beta_j^+}{q_{i,j}^- q_{j,i}^-} + \frac{\pi(i)}{\pi(j)} \zeta_{j,i}^- \beta_i^- \left(\frac{\beta_j^+ + \beta_i^-}{\gamma_j^+} + \frac{1}{q_{i,j}^+ q_{j,i}^+} \right) \right) \Bigg\} \gamma_i^- \mathrm{e}^{-\beta_i^- |x|} \\ &\quad + \mathbf{1}_{\{x>0\}} \Bigg\{ \frac{\gamma_i^-}{\beta_j^+ + \beta_i^-} \left(\frac{\zeta_{j,i}^- \beta_i^-}{q_{i,j}^+ q_{j,i}^+} + \frac{\pi(j)}{\pi(i)} \zeta_{i,j}^+ \beta_j^+ \left(\frac{\beta_j^+ + \beta_i^-}{\gamma_i^-} + \frac{1}{q_{i,j}^- q_{j,i}^-} \right) \right) \Bigg\} \gamma_j^+ \mathrm{e}^{-\beta_j^+ x} \Bigg\} \mathrm{d}x \\ &\quad + \left(\frac{q_{i,j}^-}{q_{j,i}^+} d_j^+ \zeta_{j,i}^- + \frac{\pi(j)}{\pi(i)} \frac{q_{j,i}^+}{q_{i,j}^-} d_i^- \zeta_{i,j}^+ \right) \delta_0(\mathrm{d}x), \end{aligned}$$

for $i, j \in \{1, 2\}$ with $i \neq j$. Moreover, we know from [16, Lemma 3.20] that the Lévy components $\xi^{(1)}, \xi^{(2)}$ of ξ have non-trivial Brownian part with scaling factor $\sigma_i^2 = 2d_i^+ d_i^-$ for $i = 1, 2$. Finally, (ξ, J) is unkilled since the same is true for $(H^\pm, J^\pm)$ by construction, see Lemma 3.13.

As explained in Remark 5.8, the Wiener–Hopf factorisation is unique in this case.

The Lévy processes belonging to the bonding MAP (ξ, J) belong to the family of *double exponential jump diffusions* [23] and the transitional jumps form a mixture distribution of a two-sided exponential distribution and a point mass at 0. These processes can be interpreted as a natural extension of double exponential jump diffusions, which we call *Markov modulated double exponential jump diffusions*. In [23] the overshoot distribution of double exponential jump diffusions is calculated, from which the characteristics of the ascending ladder height process—which is a subordinator with strictly positive drift and exponentially distributed jumps—can be inferred via overshoot convergence. Our approach therefore allows to go the inverse route for the Markov modulated version by constructing the bonding MAP for a given parametrization of the ascending/descending ladder height processes.

Remark 4.15. In principle, the above construction can be carried out for arbitrary ladder height Lévy measures as long as the integrability conditions of Proposition 4.11 are satisfied and (4.12)-(4.14) are replaced by appropriate conditions ensuring that the analogue of (4.17) is nonnegative. As in Example 4.10, promising candidates for this purpose are ladder height Lévy measures with completely monotone densities.

4.3. **An example with non-irreducible factors.** Consider the MAP with $\xi^{(1)}_t = -t$, $\xi^{(2)}$ a standard Brownian motion, $F_{12} = \delta$ and F_{21} the distribution of the negative of an exponential random variable of rate 1. Let the phase transition rates be given by the matrix $Q = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$. This process has MAP exponent

$$\mathbf{\Psi}(\theta) = \begin{pmatrix} -\mathrm{i}\theta - 1 & 1 \\ \frac{1}{1+\mathrm{i}\theta} & -\frac{\theta^2}{2} - 1 \end{pmatrix}$$

Observe that $\boldsymbol{\pi} = (1/2, 1/2)^T$.

Now, as possible ladder height processes, consider the exponents

$$\mathbf{\Psi}^+(\theta) = \begin{pmatrix} -(q_1^+ + k_1^+) & q_1^+ \\ 0 & b^+\mathrm{i}\theta - k_2^+ \end{pmatrix}$$

and

$$\mathbf{\Psi}^-(\theta) = \begin{pmatrix} b_1^-\mathrm{i}\theta - q_1^- & \frac{q_1^-}{1-\mathrm{i}\theta} \\ q_2^- & b_2^-\mathrm{i}\theta + \lambda_2^-\left(\frac{1}{1-\mathrm{i}\theta} - 1\right) - q_2^- \end{pmatrix}$$

It is relatively straightforward to see that the factors must be of this form, but computing the coefficients is more difficult. We first define

$$\lambda_2^- = \frac{q_1^+ q_1^-}{k_2^+ + b^+}. \tag{4.18}$$

Let b_1^- be the unique real solution of the equation

$$2x^3 + 2x^2 + x - 1 = 0,$$

which is approximately 0.44, define

$$\begin{aligned} b^+ &= 1/2, & b_2^- &= 1, \\ q_1^+ &= 1, & q_1^- &= b_1^-, \\ k_1^+ &= 2b_1^-(1 + b_1^-), & q_2^- &= 2b_1^-, \\ k_2^+ &= \frac{1}{2b_1^-} - \frac{1}{2}. \end{aligned}$$

With this choice of coefficients, $\mathbf{\Psi}^\pm$ form a Wiener-Hopf factorisation of the form (1.3) for the exponent $\mathbf{\Psi}$.

We found λ_2^- using the équations amicales, as described below, and obtained the rest of the coefficients and verified the factorisation using Mathematica.

This factorisation is interesting from two perspectives. The first is that it is an instance in which the modulator is irreducible for the bonding MAP but not for the ladder heights, since phase 1 is transient in $\mathbf{\Psi}^+$.

The second is that we can observe an interesting feature of the équations amicales (or, equivalently in this situation, the $\boldsymbol{\pi}$-fellowship condition (4.1)). Consider the general forms of $\mathbf{\Psi}^\pm$ without selecting the particular coefficients above. In the bonding MAP, $\xi^{(2)}$ should have no jumps. However, for $x < 0$, we have

$$\begin{aligned} \Upsilon_{22}(x) &= -\Psi_{12}^+(0)\overline{\Pi}_{12}^-(-x) - \Psi_{22}^+(0)\overline{\Pi}_{22}^-(-x) + d_2^+\partial\Pi_{22}^-(-x) \\ &= -q_1^+ q_1^- e^x + k_2^+\lambda_2^- e^x + b^+\lambda_2^- e^x. \end{aligned}$$

It is only by defining λ_2^- as in (4.18) that we can ensure $\Upsilon_{22} \equiv 0$.

When starting with the bonding MAP, it is not at all surprising that element $(2,2)$ of $\boldsymbol{\Psi}^-$ should contain a jump component, since the dual bonding MAP may start out at the maximum in phase 2, move away, switch to phase 1, and then switch back to phase 2, incurring a new maximum in the process. This manifests as a jump in phase 2 without phase change. However, this kind of consideration is much less clear when one begins instead from $\boldsymbol{\Psi}^{\pm}$, and this highlights the delicacy of MAP friendship, both in terms of the particular form of the équations amicales with their mixed signs, and in terms of the conditions for $\boldsymbol{\pi}$-compatibility.

5. Uniqueness of the Wiener–Hopf factorisation

Throughout this section $\boldsymbol{\Psi}$ is the exponent of an irreducible MAP with invariant modulating distribution $\boldsymbol{\pi}$. Let $\mathbb{C}_+ := \{z \in \mathbb{C} : \operatorname{Re} z \geq 0\}$. For a given MAP subordinator (ξ, J), its Laplace exponent is the unique matrix valued function $\boldsymbol{\Phi}\colon \mathbb{C}_+ \to \mathbb{C}^{n\times n}$ such that

$$\mathbb{E}^{0,i}[\exp(-z\xi_1)\,;\,J_1 = j] = (\mathrm{e}^{-\boldsymbol{\Phi}(z)})_{i,j}, \quad i,j \in [n], z \in \mathbb{C}_+.$$

Let $\mathcal{A}$ be a class of MAP subordinator Laplace exponents and suppose that $\boldsymbol{\Psi}$ has a Wiener–Hopf factorisation in $\mathcal{A}$, i.e., for some $\boldsymbol{F}, \widehat{\boldsymbol{F}} \in \mathcal{A}$ it holds

$$\boldsymbol{\Psi}(\theta) = -\boldsymbol{\Delta}_{\boldsymbol{\pi}}^{-1}\widehat{\boldsymbol{F}}(\mathrm{i}\theta)^{\top}\boldsymbol{\Delta}_{\boldsymbol{\pi}}\boldsymbol{F}(-\mathrm{i}\theta), \quad \theta \in \mathbb{R}.$$

In our language, this means that $\boldsymbol{\Psi}$ is a bonding MAP of two friends belonging to $\mathcal{A}$. We say that $\boldsymbol{\Psi}$ *has a unique MAP Wiener–Hopf factorisation in* $\mathcal{A}$ if, for any other pair of MAP subordinator exponents $\boldsymbol{G}, \widehat{\boldsymbol{G}} \in \mathcal{A}$ such that

$$\boldsymbol{\Psi}(\theta) = -\boldsymbol{\Delta}_{\boldsymbol{\pi}}^{-1}\widehat{\boldsymbol{G}}(\mathrm{i}\theta)^{\top}\boldsymbol{\Delta}_{\boldsymbol{\pi}}\boldsymbol{G}(-\mathrm{i}\theta), \quad \theta \in \mathbb{R},$$

there exists some constant, invertible real matrices $\boldsymbol{C}, \widehat{\boldsymbol{C}}$ such that

$$\boldsymbol{G} = \boldsymbol{C}\boldsymbol{F} \quad \text{and} \quad \widehat{\boldsymbol{G}} = \widehat{\boldsymbol{C}}\widehat{\boldsymbol{F}}.$$

When the Laplace exponents are invertible away from 0 (which will always be the true for the cases considered here, see below), it must hold $\widehat{\boldsymbol{C}} = \boldsymbol{\Delta}_{\boldsymbol{\pi}}^{-1}(\boldsymbol{C}^{-1})^{\top}\boldsymbol{\Delta}_{\boldsymbol{\pi}}$. In the case $\boldsymbol{C} = \boldsymbol{\Delta}$ for some diagonal matrix $\boldsymbol{\Delta}$ with strictly positive diagonal entries, a MAP (ξ^G, J^G) corresponding to $\boldsymbol{G}$ is obtained from the MAP (ξ^F, J^F) corresponding to $\boldsymbol{F}$ by performing the linear time changes $\xi_t^{(i),G} = \xi_{\boldsymbol{\Delta}_{i,i}t}^{(i),F}$ and $\boldsymbol{Q}^G = \boldsymbol{\Delta}\boldsymbol{Q}^F$. Suppose that we can prove uniqueness in $\mathcal{A}$, and assume that the ascending, resp. descending ladder height MAP Laplace exponents $\boldsymbol{\Phi}^{\pm}$ belong to $\mathcal{A}$. Then, if for $\boldsymbol{F} = \boldsymbol{\Phi}^+$ we can argue that $\boldsymbol{C}$ must be diagonal with strictly positive diagonal entries, it follows that any MAP Wiener–Hopf factorisation in $\mathcal{A}$ represents ascending and descending ladder height processes with different scaling of local times expressed through arbitrary choices of diagonal matrices $\boldsymbol{\Delta}$. In other words, the friends giving rise to the bonding MAP $\boldsymbol{\Psi}$ carry the probabilistic interpretation of ladder height subordinators.

It is essential to be able to prove uniqueness of the MAP Wiener–Hopf factorisation in order to endow $\boldsymbol{\pi}$-friendship, which is an essentially analytic condition, with probabilistic meaning. Turning to Lévy processes, uniqueness has long been known in the case of a killed process; there is a probabilistic proof in [43, pp. 583–4], and this is mirrored by an analytic argument based on Liouville's theorem (see, for example, [27, Theorem 1(e,f)].) However, without imposing further conditions, these techniques do not extend to the case of an unkilled Lévy process. This case was finally settled by the authors and Mladen Savov in [15].

As a consequence, we approach the uniqueness question for MAPs in two ways: firstly by assuming the MAP in question is killed, in which situation uniqueness follows relatively straightforwardly; and secondly by assuming not, in which case we need to impose some additional conditions reflecting the requirements of Liouville's theorem.

The idea for killed MAPs is to follow Vigon's distributional approach.

Theorem 5.1. *If $\boldsymbol{\Psi}$ is killed and has a MAP Wiener–Hopf factorisation in the class of irreducible MAP subordinators, then the factorisation is unique in this class.*

Proof. Let $\boldsymbol{F}, \widehat{\boldsymbol{F}}, \boldsymbol{G}, \widehat{\boldsymbol{G}}$ be MAP subordinator Laplace exponents such that

$$\boldsymbol{\Psi}(\theta) = -\boldsymbol{\Delta}_{\boldsymbol{\pi}}^{-1}\widehat{\boldsymbol{F}}(\mathrm{i}\theta)^\top \boldsymbol{\Delta}_{\boldsymbol{\pi}} \boldsymbol{F}(-\mathrm{i}\theta) = -\boldsymbol{\Delta}_{\boldsymbol{\pi}}^{-1}\widehat{\boldsymbol{G}}(\mathrm{i}\theta)^\top \boldsymbol{\Delta}_{\boldsymbol{\pi}} \boldsymbol{G}(-\mathrm{i}\theta), \quad \theta \in \mathbb{R}.$$

We first note that this implies that $\boldsymbol{F}, \widehat{\boldsymbol{F}}, \boldsymbol{G}, \widehat{\boldsymbol{G}}$ are all killed MAP exponents as well: Since $\boldsymbol{\Psi}$ is killed it holds that

$$\boldsymbol{\pi}^\top \boldsymbol{\Psi}(0)\mathbf{1} = -\sum_{i=1}^n \pi(i)\dagger_i < 0$$

whence,

$$0 > -\boldsymbol{\pi}^\top \boldsymbol{\Delta}_{\boldsymbol{\pi}}^{-1}\widehat{\boldsymbol{F}}(0)^\top \boldsymbol{\Delta}_{\boldsymbol{\pi}} \boldsymbol{F}(0)\mathbf{1} = -(\widehat{\boldsymbol{F}}(0)\mathbf{1})^\top \boldsymbol{\Delta}_{\boldsymbol{\pi}} \boldsymbol{F}(0)\mathbf{1},$$

which shows that $\widehat{\boldsymbol{F}}, \boldsymbol{F}$ are killed MAP exponents as well. The statement for $\widehat{\boldsymbol{G}}, \boldsymbol{G}$ follows in the same way.

Since $\boldsymbol{\Psi}$ represents an irreducible and killed MAP, $\boldsymbol{\Psi}(\theta) \in \mathrm{GL}_n(\mathbb{C})$ for any $\theta \in \mathbb{R}$. To see this, note that if we denote by ζ the a.s. finite killing time of (ξ, J) we have

$$\begin{aligned} \int_0^\infty \left|(\mathrm{e}^{t\boldsymbol{\Psi}(\theta)})_{i,j}\right| \mathrm{d}t &= \int_0^\infty \left|\mathbb{E}^{0,i}[\exp(\mathrm{i}\theta\xi_t)\,;\, J_t = j, t < \zeta]\right| \mathrm{d}t \\ &\le \int_0^\infty \mathbb{P}^{0,i}(t < \zeta)\,\mathrm{d}t = \mathbb{E}^{0,i}[\zeta] < \infty, \end{aligned} \tag{5.1}$$

where finiteness of the mean comes from the fact that under $\mathbb{P}^{0,i}$, ζ has a phase-type distribution. Thus, the integral $\int_0^\infty \mathrm{e}^{t\boldsymbol{\Psi}(\theta)}\,\mathrm{d}t$ converges absolutely and we obtain from [18, Lemma A.9] that the maximal real part of the eigenvalues of $\boldsymbol{\Psi}(\theta)$ is strictly smaller than 0, whence $\boldsymbol{\Psi}(\theta)$ is invertible.

Consequently, the factorisations imply that also $\boldsymbol{G}(z), \widehat{\boldsymbol{F}}(z) \in \mathrm{GL}_n(\mathbb{C})$ for all $z \in \mathrm{i}\mathbb{R}$. We may therefore write

$$\boldsymbol{F}(-\mathrm{i}\theta)\boldsymbol{G}(-\mathrm{i}\theta)^{-1} = \boldsymbol{\Delta}_{\boldsymbol{\pi}}^{-1}\Big(\widehat{\boldsymbol{G}}(\mathrm{i}\theta)\widehat{\boldsymbol{F}}(\mathrm{i}\theta)^{-1}\Big)^\top \boldsymbol{\Delta}_{\boldsymbol{\pi}}, \quad \theta \in \mathbb{R}. \tag{5.2}$$

Since $\boldsymbol{\dagger}^G, \boldsymbol{\dagger}^{\widehat{F}} \neq \mathbf{0}$ and $\boldsymbol{G}$ and $\widehat{\boldsymbol{F}}$ are irreducible, the measures $U^G_{i,j}, U^{\widehat{F}}_{i,j}$ are finite for any $i, j \in [n]$ and their Fourier transforms are well defined. Performing the calculation in (5.1) for these MAP subordinators and using [18, Lemma A.9] then shows $\mathscr{F}U^G_{i,j}(\theta) = -(\boldsymbol{G}(-\mathrm{i}\theta))^{-1}_{i,j}$, $\mathscr{F}U^{\widehat{F}}_{i,j}(\theta) = -(\widehat{\boldsymbol{F}}(-\mathrm{i}\theta))^{-1}_{i,j}$. By uniqueness of the Fourier transform of tempered distributions, this gives the following equalities in $\mathcal{S}'(\mathbb{R})$: for all $i, j \in [n]$,

$$\begin{aligned} &\big((\dagger^F_i - q^F_{i,i})\delta + d^F_i \delta' - \Gamma\Pi^F_i\big) * U^G_{i,j} - \sum_{k\neq j} q^F_{i,k} F^F_{i,k} * U^G_{k,j} \\ &= \frac{\pi(j)}{\pi(i)}\big((\dagger^{\widehat{G}}_j - q^{\widehat{G}}_{j,j})\delta - d^{\widehat{G}}_j \delta' - \Gamma\widetilde{\Pi}^{\widehat{G}}_j\big) * \widetilde{U}^{\widehat{F}}_{j,i} - \frac{\pi(j)}{\pi(i)} \sum_{k\neq j} q^{\widehat{G}}_{j,k} \widetilde{F}^{\widehat{G}}_{j,k} * \widetilde{U}^{\widehat{F}}_{k,i}, \end{aligned}$$

where all convolutions are well defined since the potential measures $U^G_{i,j}, U^{\widehat{F}}_{i,j}$ are finite for all $i,j \in [n]$. Taking primitives, we obtain

$$\eta^+_{i,j} + \eta^-_{i,j} = c_{i,j}, \tag{5.3}$$

where

$$\eta^+_{i,j} \coloneqq \big((\dagger^F_i - q^F_{i,i})\mathbf{1}_{\mathbb{R}_+} + d^F_i \delta + \overline{\Pi}^F_i\big) * U^G_{i,j} - \sum_{k\neq j} q^F_{i,k} \underline{F}^F_{i,k} * U^G_{k,j},$$

and

$$\eta^-_{i,j} \coloneqq \frac{\pi(j)}{\pi(i)}\big((\dagger^{\widehat{G}}_j - q^{\widehat{G}}_{j,j})\mathbf{1}_{\mathbb{R}_-} + d^{\widehat{G}}_j \delta + \overline{\overline{\Pi}}^{\widehat{G}}_j\big) * \widetilde{U}^{\widehat{F}}_{j,i} - \frac{\pi(j)}{\pi(i)} \sum_{k\neq j} q^{\widehat{G}}_{j,k} \underline{\widetilde{F}}^{\widehat{G}}_{j,k} * \widetilde{U}^{\widehat{F}}_{k,i},$$

and $c_{i,j} \in \mathbb{R}$ is some integration constant. Note that this implies that the measures $d^F_i U^G_{i,j}$ and $d^{\widehat{G}}_i \widetilde{U}^{\widehat{F}}_{i,j}$ are absolutely continuous and thus both $\eta^\pm_{i,j}$ are induced by a function. Further, since $\eta^+_{i,j}$ is a tempered distribution concentrated on $\mathbb{R}_+$ and $\eta^-_{i,j}$ is a tempered distribution concentrated on $\mathbb{R}_-$, (5.3) forces

$$\eta^+_{i,j} = c_{i,j}\mathbf{1}_{\mathbb{R}_+} \tag{5.4}$$

and

$$\eta^-_{i,j} = c_{i,j}\mathbf{1}_{\mathbb{R}_-}. \tag{5.5}$$

Consequently, taking Laplace transforms on the implied equality of measures

$$\eta^+_{i,j}(\mathrm{d}x) = \eta^+_{i,j}(x)\,\mathrm{d}x = c_{i,j}\mathbf{1}_{\mathbb{R}_+}(x)\,\mathrm{d}x$$

yields

$$\begin{aligned}\frac{c_{i,j}}{\lambda} = \mathscr{L}(\eta^+_{i,j})(\lambda) = \Big((\dagger^F_i - q^F_{i,i})/\lambda + d^F_i + \int_0^\infty (1-\mathrm{e}^{-\lambda x})\,\Pi^F_i(\mathrm{d}x)/\lambda\Big)\cdot(\boldsymbol{G}(\lambda))^{-1}_{i,j}\\ - \frac{1}{\lambda}\sum_{k\neq j} q^F_{i,k}\int_0^\infty \mathrm{e}^{-\lambda x}\,F^F_{i,k}(\mathrm{d}x)\cdot(\boldsymbol{G}(\lambda))^{-1}_{k,j}, \quad \lambda>0,\end{aligned}$$

where we used invertibility of $\boldsymbol{G}(\lambda)$ giving $\mathscr{L}(U^G_{i,j})(\lambda) = (\boldsymbol{G}(\lambda))^{-1}_{i,j}$. Multiplying both sides of the equality by λ yields

$$\boldsymbol{C}\boldsymbol{G}(\lambda) = \boldsymbol{F}(\lambda), \quad \lambda>0,$$

for $\boldsymbol{C} \coloneqq (c_{i,j})_{i,j\in[n]}$. This implies $\boldsymbol{F}(z) = \boldsymbol{C}\boldsymbol{G}(z)$ for $z \in \mathrm{i}\mathbb{R}$ and (5.2) yields $\widehat{\boldsymbol{F}}(z) = \boldsymbol{\Delta}_{\boldsymbol{\pi}}^{-1}(\boldsymbol{C}^{-1})^\top \boldsymbol{\Delta}_{\boldsymbol{\pi}}\widehat{\boldsymbol{G}}(z)$ for $z \in \mathrm{i}\mathbb{R}$. □

In the non-killed case, the singularity of at least one of the MAP friends at 0 (which translates to the potential measures being infinite) prevents us from pursuing the same strategy. Instead, we proceed with a proof that is complex analytic in nature and follows ideas that have been successfully employed in the literature for different kinds of Wiener–Hopf type equations. See, e.g., [24] or [27], with the latter essentially dealing with uniqueness of the Wiener–Hopf factorisation of killed Lévy processes. To illustrate the idea in the Lévy case, suppose that we are given two Lévy Wiener–Hopf factorisations

$$\psi(\theta) = -f(-\mathrm{i}\theta)\widehat{f}(\mathrm{i}\theta) = -g(-\mathrm{i}\theta)\widehat{g}(\mathrm{i}\theta), \quad \theta\in\mathbb{R},$$

and let us define a function

$$h(z) := \begin{cases} f(z)/g(z), & z \in \mathbb{C}_+ \setminus \{0\} \\ \widehat{g}(-z)/\widehat{f}(-z), & z \in \mathbb{C}_- \setminus \{0\}, \end{cases}$$

where we use the unique analytic extensions of the functions $f, g, \widehat{f}, \widehat{g}$ to $\mathbb{C}_+ := \{z \in \mathbb{C} : \operatorname{Re} z \geq 0\}$. Here, dividing by g on $\mathbb{C}_+ \setminus \{0\}$ and $\widehat{f}(-\cdot)$ on $\mathbb{C}_- \setminus \{0\}$ is well-defined when the corresponding Lévy processes have non-lattice support due to the following classical result, which can be traced back at least to [50, 38].

Proposition 5.2. *Let ξ be an unkilled Lévy process with characteristic exponent ψ. Then, for any $\theta \neq 0$,*

$$\psi(\theta) = 0 \iff \forall t \geq 0 : \mathbb{P}\Big(\xi_t \in \frac{2\pi}{\theta}\mathbb{Z}\Big) = 1.$$

With this definition of h, the goal is to show that h can be extended to an analytic function of $\mathbb{C}$ and to establish sublinear growth of h at ∞ such that by the extended Liouville theorem we can conclude that h is in fact equal to some constant c, which implies that $f = cg$ and $\widehat{f} = c^{-1}\widehat{g}$.

To follow such analytic approach in the more general MAP context we must first deal with the open question of invertibility of unkilled characteristic MAP exponents away from 0, i.e., we want to find a natural analogue to Proposition 5.2 in the MAP context. A related question was pursued in [20], where invertibility of the analytic extension of the matrix exponents of spectrally one-sided MAPs (including MAPs with monotone paths) away from the real axis was studied.

Proposition 5.3. *Let (ξ, J) be an unkilled MAP with characteristic exponent $\boldsymbol{\Psi}$. If J is irreducible and none of the Lévy components has lattice support, then $\boldsymbol{\Psi}(\theta) \in \mathrm{GL}_n(\mathbb{C})$ for any $\theta \in \mathbb{R} \setminus \{0\}$.*

Proof. We argue by contradiction. Suppose that $\det \boldsymbol{\Psi}(\theta) = 0$ for some $\theta \neq 0$. Then $\lambda = 0$ is a left-eigenvalue of $\boldsymbol{\Psi}(\theta)$ and there is a left eigenvector $\boldsymbol{v} \in \mathbb{C}^n$ such that $\sum_{i=1}^n |v_i| = 1$. Then, for any $t > 0$, $\boldsymbol{v}$ is a left-eigenvector with eigenvalue $\widetilde{\lambda} = 1$ for the matrix $\mathrm{e}^{t\boldsymbol{\Psi}(\theta)}$. Hence,

$$\forall j \in [n] : \quad v_j = \sum_{i=1}^n v_i \mathbb{E}^{0,i}[\exp(\mathrm{i}\theta\xi_t)\,;\ J_t = j].$$

Writing $v_i = |v_i|\mathrm{e}^{\mathrm{i}\varphi_i}$ it follows that

$$\forall j \in [n] : \quad |v_j| = \sum_{i=1}^n |v_i| \mathbb{E}^{0,i}[\exp(\mathrm{i}(\theta\xi_t + \varphi_i - \varphi_j))\,;\ J_t = j]$$

and therefore, by summing over j,

$$1 = \sum_{j=1}^n \sum_{i=1}^n |v_i| \mathbb{E}^{0,i}[\exp(\mathrm{i}(\theta\xi_t + \varphi_i - \varphi_j))\,;\ J_t = j].$$

By taking the real part of the right-hand side it follows that

$$1 = \sum_{i=1}^n |v_i| \sum_{j=1}^n \mathbb{E}^{0,i}[\cos(\theta\xi_t + \varphi_i - \varphi_j)\,;\ J_t = j].$$

Since

$$\sum_{j=1}^{n} \mathbb{E}^{0,i}[\cos(\theta\xi_t + \varphi_i - \varphi_j)\,;\ J_t = j] \leq \sum_{j=1}^{n} \mathbb{P}^{0,i}(J_t = j) \leq 1,$$

and $\sum_{i=1}^{n}|v_i| = 1$, it follows that for any $i \in [n]$ such that $v_i \neq 0$ we have

$$\sum_{j=1}^{n} \mathbb{E}^{0,i}[\cos(\theta\xi_t + \varphi_i - \varphi_j)\,;\ J_t = j] = 1.$$

Pick such $i \in [n]$. Noting that

$$\sum_{j=1}^{n} \mathbb{E}^{0,i}[\cos(\theta\xi_t+\varphi_i-\varphi_j)\,;\ J_t = j] = \sum_{j=1}^{n} \mathbb{E}^{0,i}[\cos(\theta\xi_t+\varphi_i-\varphi_j) \mid J_t = j]\mathbb{P}^{0,i}(J_t = j),$$

we now obtain from $\sum_{j=1}^{n} \mathbb{P}^{0,i}(J_t = j) = 1$, $\mathbb{P}^{0,i}(J_t = j) > 0$ and $\mathbb{E}^{0,i}[\cos(\theta\xi_t + \varphi_i - \varphi_j) \mid J_t = j] \leq 1$ that

$$\forall j \in [n]: \qquad \mathbb{E}^{0,i}[\cos(\theta\xi_t + \varphi_i - \varphi_j) \mid J_t = j] = 1.$$

In particular,

$$\mathbb{E}^{0,i}[\cos(\theta\xi_t) \mid J_t = i] = 1,$$

which shows that ξ_t is supported on $\frac{2\pi}{\theta}\mathbb{Z}$ under $\mathbb{P}^{0,i}(\cdot \mid J_t = i)$. Noting that, for σ_1 denoting the first jump time of J, we have $\{\sigma_1 > t, J_0 = i\} \subset \{J_0 = i, J_t = i\}$ and under $\mathbb{P}^{0,i}$ we have $\xi_t \overset{\mathrm{d}}{=} \xi_t^{(i)}$ on $\{\sigma_1 > t\}$ it follows that for any $t > 0$, $\xi_t^{(i)}$ is supported on the lattice $\frac{2\pi}{\theta}\mathbb{Z}$. Proposition 24.14 in [44] therefore yields that $\xi^{(i)}$ has lattice support. □

Let $\mathcal{A}_0$ be the class of finite mean MAP subordinator Laplace exponents with non-trivial Lévy components and irreducible modulators. Moreover define $\mathcal{A}_1$ to be the class of MAP subordinator Laplace exponents $\mathbf{\Phi}$ such that

$$\begin{aligned}&\forall i \in [n]:\\ &\lim_{|z|\to\infty,\mathrm{Re}\,z\geq 0}|\phi_i(z)| = \infty\\ &\quad \vee \big(\phi_i \text{ is a compound Poisson Laplace exponent and } \forall j \in [n]: \mathbf{\Pi}_{i,j} \ll \mathrm{Leb}\big).\end{aligned}$$

Note that $\lim_{|z|\to\infty,\mathrm{Re}\,z\geq 0}|\phi_i(z)| = \infty$ whenever $d_i > 0$ or Π_i is non-finite and absolutely continuous, cf. Lemma A.4. Moreover, if we define the extremal classes $\mathcal{A}_\infty, \mathcal{A}_\ll$ to be the respective families of MAP subordinator Laplace exponents such that for all $i \in [n]$, $\lim_{|z|\to\infty,\mathrm{Re}\,z\geq 0}|\phi_i(z)| = \infty$ or such that the associated Lévy measure matrices are absolutely continuous, then clearly $\mathcal{A}_\infty \cup \mathcal{A}_\ll \subset \mathcal{A}_1$.

Theorem 5.4. *If $\mathbf{\Psi}$ has a MAP Wiener–Hopf factorisation in $\mathcal{A}_0 \cap \mathcal{A}_1$, then the factorisation is unique in this class.*

Proof. By Theorem 5.1 we only have to deal with unkilled MAPs, i.e., $\mathbf{\Psi}(0) \notin \mathrm{GL}_n(\mathbb{C})$. Let $\boldsymbol{F}, \widehat{\boldsymbol{F}}, \boldsymbol{G}, \widehat{\boldsymbol{G}}$ be MAP subordinator exponents that all belong to $\mathcal{A}_0 \cap \mathcal{A}_1$ such that

$$\mathbf{\Psi}(\theta) = -\mathbf{\Delta}_{\boldsymbol{\pi}}^{-1}\widehat{\boldsymbol{F}}(\mathrm{i}\theta)^\top\mathbf{\Delta}_{\boldsymbol{\pi}}\boldsymbol{F}(-\mathrm{i}\theta) = -\mathbf{\Delta}_{\boldsymbol{\pi}}^{-1}\widehat{\boldsymbol{G}}(\mathrm{i}\theta)^\top\mathbf{\Delta}_{\boldsymbol{\pi}}\boldsymbol{G}(-\mathrm{i}\theta), \quad \theta \in \mathbb{R},$$

for a given MAP exponent $\mathbf{\Psi}$. Clearly, the assumptions imply that all Lévy components of $\boldsymbol{F}, \widehat{\boldsymbol{F}}, \boldsymbol{G}, \widehat{\boldsymbol{G}}$ have non-lattice support. Hence, the combined conclusions of Proposition 5.3 and [20, Theorem 1] together with Hurwitz' theorem (see also Remark 2.2 and the remarks following Theorem 9 of the same paper) yield that all of $\boldsymbol{F}(z), \widehat{\boldsymbol{F}}(z), \boldsymbol{G}(z), \widehat{\boldsymbol{G}}(z)$ are non-singular in $\mathbb{C}_+ \setminus \{0\}$. Therefore, $\boldsymbol{H}\colon \mathbb{C}\setminus\{0\} \to \mathbb{C}^n$ given by

$$\boldsymbol{H}(z) = \begin{cases} \boldsymbol{F}(-z)\boldsymbol{G}(-z)^{-1}, & \operatorname{Re} z \leq 0, \\ \boldsymbol{\Delta}_{\boldsymbol{\pi}}^{-1}\big(\widehat{\boldsymbol{G}}(z)\widehat{\boldsymbol{F}}(z)^{-1}\big)^{\top}\boldsymbol{\Delta}_{\boldsymbol{\pi}}, & \operatorname{Re} z \geq 0, \end{cases}$$

is well-defined. Moreover, $\boldsymbol{H}$ is holomorphic in $\{z \in \mathbb{C} : \operatorname{Re} z \neq 0\}$ and continuous on $\{z \in \mathbb{C} \setminus \{0\} : \operatorname{Re} z = 0\}$. Consequently, by a classical result of Walsh [49], for any $i, j \in [n]$, the integral of $H_{i,j}$ over a rectifiable Jordan curve contained in $\{z \in \mathbb{C} : \operatorname{Re} z \geq 0, z \neq 0\}$ is zero (and analogously in the left half-plane).

Take any triangle in $\mathbb{C}\setminus\{0\}$ not enclosing 0. If it lies in one of the half-planes, the integral of $H_{i,j}$ over the triangle is zero. If it crosses $\mathrm{i}\mathbb{R}$, then it can be decomposed into two quadrangles, each lying in one of the closed half-planes excluding 0, and the integral of $H_{i,j}$ over each of these is zero.

Hence, the integral of $H_{i,j}$ over any triangle in $\mathbb{C} \setminus \{0\}$ is zero, and by Morera's theorem, $H_{i,j}$ is holomorphic on $\mathbb{C} \setminus \{0\}$. Thus, $\boldsymbol{H}$ is holomorphic on $\mathbb{C} \setminus \{0\}$.

By Riemann's theorem on removable singularities, $\boldsymbol{H}$ can be uniquely extended to a holomorphic function on $\mathbb{C}$ if, and only if, $\lim_{z\to 0} z\boldsymbol{H}(z) = \mathbf{0}_{n\times n}$. Recall that we assumed $\boldsymbol{\Psi}(0) \notin \mathrm{GL}_n(\mathbb{C})$ and assume initially that also (H^G, J^G) is unkilled. Observe that if $(U^G_{i,j})_{i,j\in[n]}$ denote the potential measures associated to $\boldsymbol{G}$ with $U^G_{i,j}(y) \coloneqq U^G_{i,j}([0,y])$, we have

$$\boldsymbol{G}(z)^{-1} = \Big(\int_0^\infty \mathrm{e}^{-zy}\, U^G_{i,j}(\mathrm{d}y)\Big)_{i,j\in[n]}, \quad z \in (0,\infty).$$

Hence, for $z > 0$,

(5.6)
$$z\boldsymbol{G}(z)^{-1}_{i,j} = z\int_0^\infty \mathrm{e}^{-zy}\, U^G_{i,j}(\mathrm{d}y) = z^2\int_0^\infty \mathrm{e}^{-zy} U^G_{i,j}(y)\,\mathrm{d}y = \int_0^\infty \mathrm{e}^{-y} z U^G_{i,j}(y/z)\,\mathrm{d}y.$$

By the Markov renewal theorem, see Theorem 28 in [12], we have

$$\lim_{z\downarrow 0} zU^G_{i,j}(y/z) = y\lim_{x\to\infty}\frac{U^G_{i,j}(x)}{x} = y\frac{\pi^G(j)}{\mathbb{E}^{0,\boldsymbol{\pi}^{\boldsymbol{G}}}[H^G_1]}, \tag{5.7}$$

where H^G is the ordinator and $\boldsymbol{\pi}^{\boldsymbol{G}}$ the invariant distribution of the modulator associated to $\boldsymbol{G}$. Again, by the Markov renewal theorem, there exits $c, a > 0$ such that for $x > c$, $U^G_{i,j}(x) \leq ax$. Hence, for $z \leq 1$

$$zU^G_{i,j}(y/z) \leq ay\mathbf{1}_{\{y/z>c\}} + U^G_{i,j}(y/z)\mathbf{1}_{\{y/z\leq c\}} \leq ay + \sup_{x\in[0,c]} U^G_{i,j}(x) = ay + U^G_{i,j}(c) \eqqcolon f_{i,j}(y).$$

Thus, the function $y \mapsto \mathrm{e}^{-y} f_{i,j}(y)$ is an integrable majorant of $y \mapsto z\mathrm{e}^{-y}U_{i,j}(y/z)\,\mathrm{d}y$ for any $z \leq 1$. We can therefore apply dominated convergence in (5.6) to obtain with (5.7) that

$$\lim_{z\downarrow 0} z\boldsymbol{G}(z)^{-1}_{i,j} = \frac{\pi^G(j)}{\mathbb{E}^{0,\boldsymbol{\pi}^{\boldsymbol{G}}}[H^G_1]}\int_0^\infty y\mathrm{e}^{-y}\,\mathrm{d}y = \frac{\pi^G(j)}{\mathbb{E}^{0,\boldsymbol{\pi}^{\boldsymbol{G}}}[H^G_1]}.$$

Since the determinant of a matrix is a polynomial of its entries and the one-sided derivatives

$$\lim_{z\to 0, \operatorname{Re} z\geq 0}\frac{\boldsymbol{G}_{i,j}(z) - \boldsymbol{G}_{i,j}(0)}{z} = \begin{cases} \mathbb{E}[H^{G,(i)}_1], & i = j, \\ q_{i,j}\mathbb{E}[\Delta^G_{i,j}], & i \neq j, \end{cases}$$

exist and are finite by assumption for all $i,j \in [n]$, it follows that

$$\lim_{z\to 0,\operatorname{Re} z\geq 0} \frac{\det \boldsymbol{G}(z)}{z} = \lim_{z\to 0,\operatorname{Re} z\geq 0} \frac{\det \boldsymbol{G}(z) - \det \boldsymbol{G}(0)}{z},$$

exists and is equal to

$$\lim_{z\downarrow 0, z\in\mathbb{R}} \frac{\det \boldsymbol{G}(z)}{z} = -\mathbb{E}^{0,\boldsymbol{\pi}^{\boldsymbol{G}}}[H_1^G]\prod_{i=1}^{n-1}(-\lambda_i) \in (-\infty,0),$$

where λ_i denote the eigenvalues of $\boldsymbol{Q}^{\boldsymbol{G}}$ with principal eigenvalue $\lambda_n = 0$, see [20, Lemma 10]. Using that

$$\boldsymbol{G}(z)^{-1} = \operatorname{adj}(\boldsymbol{G}(z))/\det \boldsymbol{G}(z), \quad z \in \{z \in \mathbb{C}\setminus\{0\} : \operatorname{Re} z \geq 0\},$$

and that $\operatorname{adj}(\boldsymbol{G}(\cdot))$ is continuous on $\mathbb{C}_+$, it therefore follows that

$$\lim_{z\to 0,\operatorname{Re} z\geq 0} z\boldsymbol{G}^{-1}(z)$$

exists and is given by

$$\lim_{z\to 0,\operatorname{Re} z\geq 0} z\boldsymbol{G}(z)^{-1}_{i,j} = \lim_{z\downarrow 0} z\boldsymbol{G}(z)^{-1}_{i,j} = \frac{\pi^G(j)}{\mathbb{E}^{0,\boldsymbol{\pi}^{\boldsymbol{G}}}[H_1^G]}, \quad i,j\in[n].$$

Note that $-\boldsymbol{F}(0)$ is a generator matrix iff none of the Lévy components is killed, which in turn holds iff $\boldsymbol{F}(0) \notin \mathrm{GL}_n(\mathbb{C})$, see Corollary 1.4 in [46]. Thus, if $\boldsymbol{F}(0) \notin \mathrm{GL}_n(\mathbb{C})$, we have

$$\lim_{z\to 0,\operatorname{Re} z\geq 0} z\boldsymbol{F}(z)\boldsymbol{G}(z)^{-1} = \boldsymbol{F}(0)\cdot \mathbf{1}\cdot \boldsymbol{\pi}^G/\mathbb{E}^{0,\boldsymbol{\pi}^G}[H_1^G] = \mathbf{0}_{n\times n},$$

since $\mathbf{1}$ is a right eigenvector for the eigenvalue $\lambda = 0$ of the generator matrix $-\boldsymbol{F}(0)$. Thus, we have shown that

$$\begin{aligned}\lim_{z\to 0,\operatorname{Re} z\leq 0} z\boldsymbol{H}(z) &= -\lim_{z\to 0,\operatorname{Re} z\geq 0} z\boldsymbol{F}(z)\boldsymbol{G}(z)^{-1}\\ &= \mathbf{0}_{n\times n}, \quad \text{if } \boldsymbol{F}(0)\notin \mathrm{GL}_n(\mathbb{C}) \text{ or } \boldsymbol{G}(0)\in \mathrm{GL}_n(\mathbb{C})\end{aligned}$$

and analogously we find

$$\lim_{z\to 0,\operatorname{Re} z\geq 0} z\boldsymbol{H}(z) = \mathbf{0}_{n\times n}, \quad \text{if } \widehat{\boldsymbol{G}}(0)\notin \mathrm{GL}_n(\mathbb{C}) \text{ or } \widehat{\boldsymbol{F}}(0)\in \mathrm{GL}_n(\mathbb{C}),$$

We now show that the remaining cases cannot occur. Suppose initially that $\boldsymbol{F}(0) \in \mathrm{GL}_n(\mathbb{C})$ and $\boldsymbol{G}(0) \notin \mathrm{GL}_n(\mathbb{C})$. Since $\boldsymbol{\Psi}(0) \notin \mathrm{GL}_n(\mathbb{C})$, it follows from the Wiener–Hopf factorisation that $\widehat{\boldsymbol{F}}(0) \notin \mathrm{GL}_n(\mathbb{C})$. Hence, using again Lemma 10 in [20], it follows that

$$\lim_{z\to 0,\operatorname{Re} z=0} \frac{1}{z}\det \widehat{\boldsymbol{F}}(z) \neq 0,$$

with

$$\operatorname{sgn}\Big(\lim_{z\to 0,\operatorname{Re} z=0} \frac{1}{z}\det \widehat{\boldsymbol{F}}(z)\Big) = -\operatorname{sgn}\mathbb{E}^{0,\boldsymbol{\pi}^{\widehat{\boldsymbol{F}}}}[H_1^{\widehat{F}}] = -1.$$

Similarly,

$$\operatorname{sgn}\lim_{z\to 0,\operatorname{Re} z=0} \frac{1}{z}\det \boldsymbol{G}(-z) = -\operatorname{sgn}\lim_{z\to 0,\operatorname{Re} z=0} \frac{1}{z}\det \boldsymbol{G}(z) = 1.$$

Thus,

$$\begin{aligned}
-\operatorname{sgn}(\det \boldsymbol{F}(0)) &= \operatorname{sgn} \lim_{z\to 0, \operatorname{Re} z=0} \frac{1}{z} \det\Big(\boldsymbol{\Delta}_{\boldsymbol{\pi}}^{-1}\widehat{\boldsymbol{F}}(z)^{\top}\boldsymbol{\Delta}_{\boldsymbol{\pi}}\boldsymbol{F}(-z)\Big) \\
&= \operatorname{sgn} \lim_{z\to 0, \operatorname{Re} z=0} \frac{1}{z} \det\Big(\boldsymbol{\Delta}_{\boldsymbol{\pi}}^{-1}\widehat{\boldsymbol{G}}(z)^{\top}\boldsymbol{\Delta}_{\boldsymbol{\pi}}\boldsymbol{G}(-z)\Big) \qquad (5.8) \\
&= \operatorname{sgn}(\det \widehat{\boldsymbol{G}}(0)).
\end{aligned}$$

Depending on whether $-\widehat{\boldsymbol{G}}(0)$ is a generator matrix or not we have $\operatorname{sgn}(\det \widehat{\boldsymbol{G}}(0)) \in \{0,1\}$ and since $\boldsymbol{F}(0)$ is invertible by assumption, $\operatorname{sgn}(\det \boldsymbol{F}(0)) = 1$. To see that this is true, note that the real parts of the eigenvalues of $\boldsymbol{Q}^{\boldsymbol{F}}$ are non-positive and hence the real parts of the eigenvalues of $-\boldsymbol{F}(0) = \boldsymbol{Q}^{\boldsymbol{F}} - \Delta_{\dagger^{\boldsymbol{F}}}$, where $\boldsymbol{\dagger}^{\boldsymbol{F}}$ has non-negative entries and is not equal to the zero vector by assumption, are strictly negative, see e.g. Proposition 1.3 in [46]. If an eigenvalue λ_i of the real matrix $-\boldsymbol{F}(0)$ is of multiplicity m and has non-trivial imaginary part, then $\overline{\lambda_i}$ is also an eigenvalue of multiplicity m and the product of these $2m$ eigenvalues is strictly positive. Thus,

$$\det \boldsymbol{F}(0) = (-1)^n \prod_{i=1}^{n} \lambda_i = \prod_{i=1}^{n} (-\lambda_i) > 0.$$

The argument for the determinant of $\widehat{\boldsymbol{G}}(0)$ is analogous. Hence, (5.8) yields a contradiction. Similarly, it follows that the case $\widehat{\boldsymbol{G}}(0) \in \mathrm{GL}_n(\mathbb{C})$ and $\widehat{\boldsymbol{F}}(0) \notin \mathrm{GL}_n(\mathbb{C})$ cannot occur.

It follows that $\lim_{z\to 0, \operatorname{Re} z\geq 0} z\boldsymbol{H}(z) = \lim_{z\to 0, \operatorname{Re} z\leq 0} z\boldsymbol{H}(z) = \mathbf{0}_{n\times n}$. Riemann's theorem therefore implies that $\boldsymbol{H}$ can be extended to a holomorphic function on $\mathbb{C}$.

We proceed by showing that $\boldsymbol{H}$ has component wise sublinear growth. If ϕ_i^G does not diverge at ∞, by our assumption that $\boldsymbol{G} \in \mathcal{A}_1$, we have $\boldsymbol{\Pi}_{i,j}^G \ll \mathrm{Leb}$ for all $j \in [n]$ and $H^{G,(i)}$ is compound Poisson. Thus, in this case it follows from the Riemann–Lebesgue lemma for Laplace transforms of absolutely continuous measures supported on $(0,\infty)$ that $\lim_{|z|\to\infty, \operatorname{Re} z\geq 0} \mathscr{L}(q_{i,j}F_{i,j}^G)(z) = 0$ for all $j \neq i$ and $\lim_{|z|\to\infty, \operatorname{Re} z\geq 0} \phi_i^G(z) = \lambda_i^G$, where λ_i^G is the jump intensity of $H^{G,(i)}$. This demonstrates that for any $i,j \in [n], i \neq j$,

$$\lim_{|z|\to\infty, \operatorname{Re} z\geq 0} \frac{\boldsymbol{G}_{i,j}(z)}{\boldsymbol{G}_{i,i}(z)} = 0.$$

Hence, for given $\varepsilon > 0$ there exists $M > 0$ such that

$$\big\|\boldsymbol{D}^{\boldsymbol{G}}(z)^{-1}(\boldsymbol{G}(z) - \boldsymbol{D}^{\boldsymbol{G}}(z))\big\| \leq \varepsilon, \quad |z| \geq M, \operatorname{Re} z \geq 0.$$

Hence, for such z

$$\begin{aligned}
\big\|\big(\mathbb{I} + \boldsymbol{D}^{\boldsymbol{G}}(z)^{-1}(\boldsymbol{G}(z) - \boldsymbol{D}^{\boldsymbol{G}}(z))\big)^{-1} - \mathbb{I}\big\| &= \Big\|\sum_{n=1}^{\infty}\Big(-\boldsymbol{D}^{\boldsymbol{G}}(z)^{-1}(\boldsymbol{G}(z) - \boldsymbol{D}^{\boldsymbol{G}}(z))\Big)^n\Big\| \\
&\leq \sum_{n=1}^{\infty}\big\|\boldsymbol{D}^{\boldsymbol{G}}(z)^{-1}(\boldsymbol{G}(z) - \boldsymbol{D}^{\boldsymbol{G}}(z))\big\|^n \\
&\leq \frac{\varepsilon}{1-\varepsilon},
\end{aligned}$$

and therefore

$$\lim_{|z|\to\infty \operatorname{Re} z\geq 0} \big(\mathbb{I} + \boldsymbol{D}^{\boldsymbol{G}}(z)^{-1}(\boldsymbol{G}(z) - \boldsymbol{D}^{\boldsymbol{G}}(z))\big)^{-1} = \mathbb{I}.$$

Moreover, since $\boldsymbol{G} \in \mathcal{A}_1$ we have

$$\alpha_i^G := \lim_{|z|\to\infty, \operatorname{Re} z \geq 0} (\boldsymbol{D}^{\boldsymbol{G}}(z))^{-1}_{i,i} = \begin{cases} \frac{1}{\lambda_i^G + \dagger_i^G - q_{i,i}^G}, & \text{if } H^{G,(i)} \text{ is compound Poisson,} \\ 0, & \text{if } H^{G,(i)} \text{ is not compound Poisson.} \end{cases}$$

Thus, using

$$\boldsymbol{G}(z) = \boldsymbol{D}^{\boldsymbol{G}}(z)\big(\mathbb{I} + \boldsymbol{D}^{\boldsymbol{G}}(z)^{-1}(\boldsymbol{G}(z) - \boldsymbol{D}^{\boldsymbol{G}}(z))\big).$$

we find

$$\lim_{|z|\to\infty, \operatorname{Re} z \geq 0} \boldsymbol{G}(z)^{-1} = \operatorname{diag}((\alpha_i^G)_{i\in[n]}),$$

Together with

$$\lim_{|z|\to\infty, \operatorname{Re} z \geq 0} \frac{1}{z}\boldsymbol{F}(z) = \operatorname{diag}((d_i^F)_{i\in[n]}),$$

we therefore obtain

$$\lim_{|z|\to\infty, \operatorname{Re} z \geq 0} \frac{1}{z}\boldsymbol{F}(z)\boldsymbol{G}(z)^{-1} = \operatorname{diag}((\alpha_i^G d_i^F)_{i\in[n]}). \tag{5.9}$$

We now argue by contradiction that $d_i^F > 0$ implies that $H^{G,(i)}$ is not compound Poisson. If $d_i^F > 0$, since by assumption either $H^{\widehat{F},(i)}$ is compound Poisson with absolutely continuous Lévy density or $\lim_{|\theta|\to\infty}|\phi_{i,i}^{\widehat{F}}(\mathrm{i}\theta)| = \infty$ and, moreover, it always holds $\lim_{\theta\to\infty}\frac{1}{\mathrm{i}\theta}\boldsymbol{F}_{i,i}(-\mathrm{i}\theta) = -d_i^F$, it follows from the Riemann–Lebesgue lemma that $\lim_{\theta\to\infty}\frac{1}{\mathrm{i}\theta}(\boldsymbol{\Delta}_{\boldsymbol{\pi}}^{-1}\widehat{\boldsymbol{F}}(\mathrm{i}\theta)^\top\boldsymbol{\Delta}_{\boldsymbol{\pi}}\boldsymbol{F}(-\mathrm{i}\theta))_{i,i}$ diverges if $H^{\widehat{F},(i)}$ is not compound Poisson or else converges to a strictly negative limit. On the other hand, if $H^{G,(i)}$ is compound Poisson, then by definition of $\mathcal{A}_1$, necessarily $\Pi^{G,(i)} \ll \mathrm{Leb}$, such that again by the Riemann–Lebesgue lemma

$$\lim_{\theta\to\infty}\frac{1}{\mathrm{i}\theta}(\boldsymbol{\Delta}_{\boldsymbol{\pi}}^{-1}\widehat{\boldsymbol{G}}(\mathrm{i}\theta)^\top\boldsymbol{\Delta}_{\boldsymbol{\pi}}\boldsymbol{G}(-\mathrm{i}\theta))_{i,i} = \begin{cases} 0, & \text{if } d_i^{\widehat{G}} = 0, \\ d_i^{\widehat{G}}(-q_{i,i}^G + \dagger_i^G + \lambda_i^G) > 0, & \text{if } d_i^{\widehat{G}} > 0. \end{cases}$$

This yields a contradiction and therefore proves that $d_i^F > 0$ implies $\alpha_i^G = 0$. Hence, by (5.9),

$$\lim_{|z|\to\infty, \operatorname{Re} z \geq 0} \frac{1}{z}\boldsymbol{F}(z)\boldsymbol{G}(z)^{-1} = \mathbf{0}_{n\times n}.$$

In the same way, we can prove that

$$\lim_{|z|\to\infty, \operatorname{Re} z \geq 0} \frac{1}{z}\boldsymbol{\Delta}_{\boldsymbol{\pi}}^{-1}\big(\widehat{\boldsymbol{G}}(z)\widehat{\boldsymbol{F}}(z)^{-1}\big)^\top\boldsymbol{\Delta}_{\boldsymbol{\pi}} = \mathbf{0}_{n\times n}.$$

Thus, under the given assumptions on the MAP Wiener–Hopf factorisation, we have shown that

$$\lim_{|z|\to\infty} \frac{1}{z}\boldsymbol{H}(z) = \mathbf{0}_{n\times n}.$$

Since we have demonstrated that $\boldsymbol{H}$ has a unique holomorphic extension to $\mathbb{C}$, the extended Liouville theorem now allows us to conclude that $\boldsymbol{H} \equiv \boldsymbol{C}$ for some constant matrix $\boldsymbol{C} \in \mathbb{C}^{n\times n}$. By considering $\boldsymbol{H}$ on $\mathbb{R}$ we see that $\boldsymbol{C}$ must be real. This gives us the desired conclusion by construction of $\boldsymbol{H}$. □

From [16, Lemma 3.20] we immediately obtain the following result.

Lemma 5.5. *Let $\boldsymbol{\Psi}$ be a MAP exponent. For any $i \in [n]$ such that ψ_i has non-trivial Gaussian part, it holds that $\lim_{|z|\to\infty, \operatorname{Re} z\geq 0}|\phi_i^{\pm}(z)| = \infty$.*

Absolute continuity of the Lévy measure matrix on $(0, \infty)$ of a bonding MAP together with information on regularity at 0 of its Lévy components also gives useful properties to determine uniqueness of its Wiener–Hopf factorisation.

Lemma 5.6. *If the Lévy measure matrix is absolutely continuous on* $(0, \infty)$ *(resp.* $(-\infty, 0)$*), then the same is true for the Lévy measure matrix of the ascending (resp. descending) ladder height MAP. In this case, if* $X^{(i)}$ *is upward (resp. downward) regular at* 0*, then the Laplace exponent* ϕ_i^+ *(resp.* ϕ_i^-*) diverges at* ∞*. Otherwise,* $H^{+,(i)}$ *(resp.* $H^{-,(i)}$*) is compound Poisson.*

Proof. We only deal with the statements on (H^+, J^+), the statement for (H^-, J^-) follows from symmetric arguments. From the construction of the ascending ladder height process (H^+, J^+) it is immediate that $H^{+,(i)}$ is strictly increasing iff $X^{(i)}$ is upward regular. Equivalently, $H^{+,(i)}$ is not compound Poisson iff $X^{(i)}$ is upward regular. Since the Lévy measure matrix $\mathbf{\Pi}$ is absolutely continuous on $(0, \infty)$, it follows from [16, Theorem 4.3] that $\mathbf{\Pi}^+$ is absolutely continuous on $(0, \infty)$. In particular, $\Pi_i^+ \ll \mathrm{Leb}$ for all $i \in [n]$ and hence $\lim_{|z|\to\infty, \mathrm{Re}\, z \geq 0} |\phi_i^+(z)| = \infty$ by Lemma A.4 if $X^{(i)}$ is upward regular. □

The uniqueness results finally allow us to give the following probabilistic interpretation of $\boldsymbol{\pi}$-friendships.

Theorem 5.7. *Let* (H^+, J^+) *and* (H^-, J^-) *be irreducible* $\boldsymbol{\pi}$*-friends with matrix Laplace exponents* $\mathbf{\Phi}^{\pm}$ *such that one of the following sets of conditions holds:*

- *(i)* *(a) the bonding MAP is irreducible and killed,*
  - *(b) the ascending and descending ladder height processes of the bonding MAP are irreducible, and at least one of them has no compound Poisson or trivial Lévy components, and*
  - *(c)* $\boldsymbol{\pi}$ *is invariant for the bonding MAP;*
- *(ii)* *(a) the bonding MAP is irreducible and unkilled,*
  - *(b)* $\mathbf{\Phi}^{\pm} \in \mathcal{A}_0 \cap \mathcal{A}_\infty$*, and*
  - *(c) the MAP exponents of the ascending and descending ladder height processes of the bonding MAP belong to* $\mathcal{A}_0 \cap \mathcal{A}_\infty$*.*

Then (H^+, J^+) *and* (H^-, J^-) *have the same distribution as the ascending and descending ladder height processes of the bonding MAP, for an appropriate scaling of local times.*

Proof.
- (i) Let $\boldsymbol{G}^{\pm}$ be the matrix Laplace exponents of the ascending/descending ladder height processes and assume, wlog, that $\boldsymbol{G}^+$ has non compound Poisson and non-trivial Lévy components. Theorem 2.1 ensures that the bonding MAP has a Wiener–Hopf factorisation (3.1) in terms of its ladder height processes. The result for killed MAPs follows by uniqueness of the Wiener–Hopf factorisation provided by Theorem 5.1 and the fact that by Lemma A.5, any matrix $\boldsymbol{C}$ such that $\boldsymbol{C}\boldsymbol{G}^+ = \mathbf{\Phi}^+$ must be diagonal with strictly positive diagonal entries.
- (ii) If J is unkilled, Lemma 3.12 implies that $\boldsymbol{\pi}$ is invariant for the modulator J of the bonding MAP. The Wiener–Hopf factorisation (3.1) for the bonding MAP holds again by Theorem 2.1. The result for unkilled MAPs then follows from the uniqueness result Theorem 5.4 and Lemma A.5.

□

Remark 5.8. Sufficient conditions for irreducibility of the ladder height modulators and necessary and sufficient conditions for finiteness of the ordinators' mean needed for the ladder height MAPs to belong to $\mathcal{A}_0$ can be found in [16, Proposition 3.5] and [12, Theorem 35], respectively. Lemma 5.5 and Lemma 5.6 give criteria that allow to check whether ladder height MAPs belong to $\mathcal{A}_1$. A characterisation of the lifetime of the bonding MAP in terms of the characteristics of $(H^\pm, J^\pm)$ is given in Lemma 3.13.

We return to our previous examples of friendship. In Example 4.10, we found that the bonding MAP had positive Gaussian part in every component, which by Lemma 5.5 implies that its ladder height processes are in $\mathcal{A}_1$. Furthermore, examining the form of the Lévy measure matrix $\mathbf{\Pi}$ of the bonding MAP, we note that assumption (i) in the example implies that $\mathbb{E}^{0,i}[|\xi_1|] < \infty$.

Since both the $\boldsymbol{\pi}$-friends in the example are unkilled, we obtain, by differentiating in (1.3) and using [18, Propositions 2.13 and 2.15], that the bonding MAP ξ oscillates. In order to check the finiteness of the mean of the ladder height process, we can look at condition (TO) in [12], which amounts to showing that $\int^\infty x\overline{\mathbf{\Pi}}(x)\,\mathrm{d}x < \infty$. We note that $\int^\infty x\overline{\mathbf{\Pi}}_{i,1}(x)\,\mathrm{d}x$ can be expressed in terms of an integral over μ_1^+. Conditions (i) and (iii) together imply that $\int_{(0,1)} y^{-4}\,\mu_1^+(\mathrm{d}y) < \infty$, which in turn yields that $\int^\infty x\overline{\mathbf{\Pi}}_{i,1}(x)\,\mathrm{d}x < \infty$ for $i = 1, 2$. Symmetrical considerations apply to $\int^\infty x\overline{\mathbf{\Pi}}_{i,2}(x)\,\mathrm{d}x$ for $i = 1, 2$, and hence condition (TO) of [12] is satisfied. It follows that the ladder height process has finite mean.

Finally, [16, Proposition 5.10] gives that the ladder height processes are irreducible, which shows that they are in $\mathcal{A}_0$. Therefore, the Wiener–Hopf factorisation in this example consists of the identified pair of $\boldsymbol{\pi}$-friends. The same argument applies, with little variation, to Example 4.14.

Appendix A. Some technical lemmas

Lemma A.1. *A matrix-valued function $\mathbf{\Psi}\colon \mathbb{R} \to \mathbb{C}^{n\times n}$ is the characteristic exponent of some $\mathbb{R}\times[n]$-valued MAP if, and only if, all of the following conditions are satisfied:*

(i) for all $i \in [n]$, $\mathbf{\Psi}_{i,i}$ is the characteristic exponent of a killed Lévy process;
(ii) for all $i, j \in [n]$ with $i \neq j$ there exists some finite measure $\rho_{i,j}$ such that $\mathbf{\Psi}_{i,j} = \mathscr{F}\rho_{i,j}$;
(iii) the vector $-\mathbf{\Psi}(0)\mathbf{1}$ is nonnegative.

Proof. Necessity is obvious by definition of a MAP exponent, so assume that (i)-(iii) hold. Let (a_i, σ_i^2, Π_i) and $\widetilde{\dagger}_i$ be the Lévy triplet and killing rate, resp., associated to $\mathbf{\Psi}_{i,i}$. Let $q_{i,j} = \rho_{i,j}(\mathbb{R})$. Then, if $q_{i,j} > 0$, $F_{i,j} = \rho_{i,j}/q_{i,j}$ is a probability measure. For $q_{i,j} = 0$ let $F_{i,j} = \delta_0$. Moreover, for

$$q_{i,i} \coloneqq -\sum_{j\neq i} q_{i,j},$$

we have

$$\dagger_i \coloneqq \widetilde{\dagger}_i + q_{i,i} = -\sum_{j=1}^n \mathbf{\Psi}_{i,j}(0) \geq 0,$$

by assumption and $\boldsymbol{Q} = (q_{i,j})_{i,j=1,\dots,n}$ is a generator matrix. Thus, if we let $\boldsymbol{G}(\theta) = (\{\mathscr{F}F_{i,j}\}(\theta))_{i,j=1,\dots n}$ and ψ_i be the Lévy–Khintchine exponent corresponding to the triplet (a_i, σ_i^2, Π_i) and killing rate $\dagger_i$, then

$$\boldsymbol{\Psi}(\theta) = \operatorname{diag}\big((\psi_i(\theta))_{i\in[n]}\big) + \boldsymbol{Q} \odot \boldsymbol{G}(\theta), \quad \theta \in \mathbb{R},$$

is a characteristic MAP exponent. □

Lemma A.2. *Let μ be a signed measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. Then, μ induces a tempered distribution with $\mu' = \nu$ for some finite signed measure ν if, and only if, μ is absolutely continuous with density $\underline{\nu} + c$, where $c \in \mathbb{R}$ and $\underline{\nu}(x) = \nu((-\infty, x])$, $x \in \mathbb{R}$.*

Proof. Suppose first that $\mu = \underline{\nu} + c$. Since ν is finite, we have $|\nu((-\infty, x])| \leq |\nu|(\mathbb{R}) < \infty$ and hence μ induces a tempered distribution. Then, for any $\phi \in \mathcal{S}(\mathbb{R})$ we have with an integration by parts

$$\langle \mu', \phi\rangle = -\int \phi'(x)\,\mu(\mathrm{d}x) = -\int \phi'(x)\nu((-\infty, x])\,\mathrm{d}x = \int \phi(x)\,\nu(\mathrm{d}x),$$

which shows $\mu' = \nu$. Conversely, if $\mu' = \nu$, it follows that $\mu = \underline{\nu} + c$ in the sense of distributions for some constant $c \in \mathbb{R}$ since $\underline{\nu}' = \nu$. □

Lemma A.3. [47, Lemme 1.5.5] *Suppose that X has absolutely continuous Lévy measure.*

(i) If X is not compound Poisson, then $\lim_{|\theta|\to\infty} \operatorname{Re}\psi(\theta) = -\infty$.
(ii) If X is compound Poisson, then $\lim_{|\theta|\to\infty} \psi(\theta) = -\Pi(\mathbb{R}) - \dagger$.

The same idea used to obtain Lemma A.3 allows us to prove the following result.

Lemma A.4. *Let X be a Lévy subordinator that is not compound Poisson with Laplace exponent ϕ. Then*

$$\lim_{|z|\to\infty, \operatorname{Re} z\geq 0} |\phi(z)| = \infty \iff \text{for some } q > 0, \lim_{|z|\to\infty, \operatorname{Re} z\geq 0} \mathscr{L}U_q^{\mathrm{sing}}(z) = 0,$$

where U_q^{sing} denotes the continuous singular part of the resolvent measure U_q of X.

Proof. Since X is not compound Poisson, U_q is a continuous measure [6, Proposition I.15] and hence $U_q = U_q^{\mathrm{cont}} + U_q^{\mathrm{sing}}$, where $U_q^{\mathrm{cont}}(\mathrm{d}x) = u_q(x)\,\mathrm{d}x$ for some L^1-density u_q. Since $\operatorname{Re}\phi(z) \geq 0$ it follows that

$$\mathscr{L}U_q^{\mathrm{cont}}(z) + \mathscr{L}U_q^{\mathrm{sing}}(z) = \mathscr{L}U_q(z) = \int_0^\infty \mathrm{e}^{(-q-\phi(z))t}\,\mathrm{d}t = \frac{1}{q+\phi(z)}, \quad z \in \mathbb{C}_+.$$

By the Riemann–Lebesgue lemma for Laplace transforms of finite, absolutely continuous measures supported on $(0, \infty)$, we have $\lim_{|z|\to\infty, \operatorname{Re} z\geq 0} \mathscr{L}U_q^{\mathrm{cont}}(z) = 0$, which implies that

$$\lim_{|z|\to\infty, \operatorname{Re} z\geq 0} \mathscr{L}U_q^{\mathrm{sing}}(z) = 0$$

if, and only if,

$$\lim_{|z|\to\infty, \operatorname{Re} z\geq 0} |\phi(z)| = \infty.$$

□

A natural criterion for divergence of $|\phi|$ at ∞ is therefore absolute continuity of U_q, which is guaranteed whenever X has strictly positive drift or $\mathbb{P}_{X_t} \ll \mathrm{Leb}$ for all $t > 0$. A convenient sufficient criterion for the latter to hold is $\Pi \ll \mathrm{Leb}$, see [44, Theorem 27.7], as already indicated by Lemma A.3.

Lemma A.5. *Let $\boldsymbol{\Psi}$ be the characteristic exponent of a MAP subordinator (ξ, J) and let $\boldsymbol{C} \in \mathbb{R}^{n\times n}$ be a matrix such that $\boldsymbol{C}\boldsymbol{\Psi}$ is again a characteristic exponent of a MAP subordinator. Then, for any $j \in [n]$ such that ψ_j is non-trivial and not compound Poisson, it holds $c_{i,j} = 0$ for any $i \neq j$ and $c_{j,j} \geq 0$.*

Proof. Let $\psi_j^{\Psi} \equiv \psi_j$ be non-trivial and not compound Poisson and suppose that there exist $i \neq j$ such that $c_{i,j} \neq 0$. Letting $\widetilde{\boldsymbol{\Psi}} = \boldsymbol{C}\boldsymbol{\Psi}$ it follows that

$$q_{j,j}^{\Psi} + \psi_j^{\Psi}(\theta) = \frac{1}{c_{i,j}}\Big(q_{i,j}^{\widetilde{\Psi}}\mathscr{F}\{\Delta_{i,j}^{\widetilde{\Psi}}\}(\theta) - \sum_{k\neq j} c_{i,k}q_{k,j}^{\Psi}\mathscr{F}\{\Delta_{k,j}^{\Psi}\}(\theta)\Big), \quad \theta \in \mathbb{R}.$$

This yields a contradiction since the function on the left is unbounded [6, Corollary 3], whereas the function on the right is bounded. In particular, $c_{j,j}(q_{j,j}^{\Psi} + \psi_j^{\Psi}) = q_{j,j}^{\widetilde{\Psi}} + \psi_j^{\widetilde{\Psi}}$, showing that $c_{j,j} \geq 0$ since $\psi_j^{\Psi}, \psi_j^{\widetilde{\Psi}}$ are subordinator exponents and ψ_j^{Ψ} is non-trivial by assumption. □

Appendix B. General Wiener–Hopf factorisation

This section is dedicated to a proof of the Wiener–Hopf factorisation of MAPs. It extends the results of [12], which dealt with situations where every component of the MAP was killed at the same (possibly zero) rate, and [19], where every component of the MAP was killed at strictly positive rate. The main idea in the proof is to use the result of [19] and take limits carefully to allow for some components to be unkilled. In what follows, we adopt Ivanovs' notation, whereby, when T is a (possibly random) time, we write $\boldsymbol{T} = \Big(\int_0^T \mathbf{1}_{\{J_t=i\}}\,\mathrm{d}t : i \in [n]\Big)$, and for a functional F_T, declare $\mathbb{E}[F_T; J_T]$ to be the matrix whose (i,j)-th entry is $\mathbb{E}^{0,i}[F_T; J_T = j]$.

Assume that J is irreducible with stationary distribution $\boldsymbol{\pi}$. Write $\mathbb{P}_*$ for the probabilities associated with an unkilled version of (ξ, J); that is, a version whose exponent is $\theta \mapsto \boldsymbol{\Psi}(\theta) + \boldsymbol{\Delta}_{\dagger}$. We define $\boldsymbol{\kappa}$ as the matrix exponent of the ascending ladder process under $\mathbb{P}_*$, in the sense that $\mathbb{E}_*[\mathrm{e}^{-\langle \boldsymbol{\gamma}, \boldsymbol{L}_t^{-1}\rangle - \alpha H_t^+}; J^+] = \mathrm{e}^{-\boldsymbol{\kappa}(\boldsymbol{\gamma},\alpha)}$; note the different convention in comparison with $\boldsymbol{\Psi}$. We can also define the descending ladder process and its matrix exponent $\widehat{\boldsymbol{\kappa}}$ by considering the dual MAP $(\widehat{\xi}, \widehat{J})$. However, as alluded to in section 2, for this purpose a slightly different choice of local time is needed, partly in case of compound Poisson components and partly due to the effects of state changes; this is done carefully by [19].

Theorem B.1. *Let $\boldsymbol{\beta}$ be a vector with non-negative entries. There exists some vector $\boldsymbol{c}$ with positive entries such that*

$$-(\boldsymbol{\Psi}(\theta) - \boldsymbol{\Delta}_{\boldsymbol{\beta}}) = \boldsymbol{\Delta}_{\boldsymbol{\pi}}^{-1}\widehat{\boldsymbol{\kappa}}(\boldsymbol{\dagger} + \boldsymbol{\beta}, \mathrm{i}\theta)^{\top}\boldsymbol{\Delta}_{\boldsymbol{\pi}}\boldsymbol{\Delta}_{\boldsymbol{c}}\boldsymbol{\kappa}(\boldsymbol{\dagger} + \boldsymbol{\beta}, -\mathrm{i}\theta).$$

Proof. We will need to initially consider MAPs with positive killing rate in every state. For this reason, let us start by replacing the rate $\dagger_i$ with $\dagger_i^{\epsilon} \coloneqq \dagger_i + \epsilon$; that is, we consider the MAP with exponent $\theta \mapsto \boldsymbol{\Psi}(\theta) - \boldsymbol{\Delta}_{\epsilon}$. Following [19], let

$$\begin{aligned} \bar{\xi}_t &= \sup\{\xi_s : s \leq t\}, \\ \overline{G}(t) &= \inf\{s \leq t : \xi_s \vee \xi_{s-} = \bar{\xi}_t\}, \text{ and} \\ \bar{J}_t &= J_{\overline{G}(t)-}\mathbf{1}_{\{\xi_{\overline{G}(t)-} = \bar{\xi}_t\}} + J_{\overline{G}(t)}\mathbf{1}_{\{\xi_{\overline{G}(t)-} < \bar{\xi}_t\}}. \end{aligned}$$

When $\bar{J}_t = j$ for some state j in which $\xi^{(j)}$ is irregular for $(0,\infty)$ and regular for $(-\infty,0]$, $\bar{\xi}_t = \xi_{\overline{G}(t)}$; otherwise, $\bar{\xi}_t = \xi_{\overline{G}(t)-}$. Assume initially that we are in the latter case, and consider the following calculation from excursion theory, in which a_j is the drift of L^{-1} in state j, and n_j is the excursion measure of (ξ, J) away from its maximum when starting in state j:

$$
\begin{aligned}
&\mathbb{E}^{0,i}\big[\mathrm{e}^{\mathrm{i}\theta\bar{\xi}_{\zeta-}-\langle\boldsymbol{\beta},\overline{\boldsymbol{G}}(\zeta-)\rangle}; \bar{J}_\zeta = j\big] \\
&= \mathbb{E}^{0,i}_*\Big[\int_0^\infty \Big(\sum\nolimits_{k\in[n]} \dagger^\epsilon_k \mathbf{1}_{\{J_t=k\}}\Big)\mathrm{e}^{-\langle\dagger^\epsilon,\boldsymbol{t}\rangle+\mathrm{i}\theta\xi_{\overline{G}(t)-}-\langle\boldsymbol{\beta},\overline{\boldsymbol{G}}(t)\rangle}\mathbf{1}_{\{J_{\overline{G}(t)-}=j\}}\,\mathrm{d}t\Big] \\
&= \mathbb{E}^{0,i}_*\Big[\int_0^\infty \mathbf{1}_{\{\bar{\xi}_t=\xi_t\}}\dagger^\epsilon_j\mathrm{e}^{-\langle\dagger^\epsilon+\boldsymbol{\beta},\boldsymbol{t}\rangle+\mathrm{i}\theta\xi_t}\mathbf{1}_{\{J_t=j\}}\,\mathrm{d}t\Big] \\
&\quad + \mathbb{E}^{0,i}_*\Big[\sum\nolimits_g \int_g^d \Big(\sum\nolimits_k \dagger^\epsilon_k\mathbf{1}_{\{J_t=k\}}\Big)\mathrm{e}^{-\langle\dagger^\epsilon,\boldsymbol{t}\rangle+\mathrm{i}\theta\xi_{g-}-\langle\boldsymbol{\beta},\boldsymbol{g}\rangle}\mathbf{1}_{\{J_{g-}=j\}}\,\mathrm{d}t\Big] \\
&= \mathbb{E}^{0,i}_*\Big[\int_0^\infty a_j\dagger^\epsilon_j\mathrm{e}^{-\langle\dagger^\epsilon+\boldsymbol{\beta},\boldsymbol{t}\rangle+\mathrm{i}\theta\xi_t}\mathbf{1}_{\{J_t=j\}}\,\mathrm{d}L_t\Big] \\
&\quad + \mathbb{E}^{0,i}_*\Big[\int_0^\infty \mathrm{e}^{-\langle\dagger^\epsilon+\boldsymbol{\beta},\boldsymbol{t}\rangle+\mathrm{i}\theta\xi_{t-}}\mathbf{1}_{\{J_{t-}=j\}}\,\mathrm{d}L_t\Big] n_j\Big(\int_0^{\bar{\zeta}}\Big(\sum\nolimits_k \dagger^\epsilon_k\mathbf{1}_{\{J_t=k\}}\Big)\mathrm{e}^{-\langle\dagger^\epsilon,\boldsymbol{t}\rangle}\,\mathrm{d}t\Big) \\
&= \mathbb{E}^{0,i}_*\Big[\int_0^\infty \mathrm{e}^{-\langle\dagger^\epsilon+\boldsymbol{\beta},\boldsymbol{L}^{-1}_t\rangle+\mathrm{i}\theta H^+_t}\mathbf{1}_{\{J^+_t=j\}}\,\mathrm{d}t\Big]\Big(a_j\dagger^\epsilon_j + n_j\big(1-\mathrm{e}^{-\langle\dagger^\epsilon,\bar{\boldsymbol{\zeta}}\rangle}\big)\Big) \\
&= \boldsymbol{\kappa}(\dagger^\epsilon+\boldsymbol{\beta},-\mathrm{i}\theta)^{-1}_{i,j}(\boldsymbol{\kappa}(\dagger^\epsilon,0)\mathbf{1})_j
\end{aligned}
$$

where in the third line the sum is over excursion intervals (g,d), and in the fourth line $\bar{\zeta}$ is the lifetime of the excursion. In other words,

$$
\mathbb{E}\big[\mathrm{e}^{\mathrm{i}\theta\bar{\xi}_{\zeta-}-\langle\boldsymbol{\beta},\overline{G}(\zeta-)\rangle}; \bar{J}_\zeta\big] = \boldsymbol{\kappa}(\dagger^\epsilon+\boldsymbol{\beta},-\mathrm{i}\theta)^{-1}\boldsymbol{\Delta}_{\boldsymbol{\kappa}(\dagger^\epsilon,0)\mathbf{1}}. \tag{B.1}
$$

We now consider the case where the state j above is one for which $\xi^{(j)}$ is irregular for $(0,\infty)$ and regular for $(-\infty,0]$. In this situation, the point 0 may be either a holding point for $\xi^{(j)}$ reflected in its supremum (if $\xi^{(j)}$ is compound Poisson) or an irregular point (otherwise.) Either way, let us define $T_0 = 0$, $S_m = \inf\{t \geq T_{m-1} : \xi_t < \bar{\xi}_t; J_t = j\}$, and $T_m = \inf\{t \geq S_m : \xi_t = \bar{\xi}_t\}$, for $n \geq 1$. This implies that $\{(S_m, T_m) : m \geq 1\}$ is the set of excursions away from the maximum that start from state j, and importantly, every S_m and T_m is a stopping time. In the case of an irregular point, $T_{m-1} = S_m$. Then, we can compute as follows, using the Markov property:

$$
\begin{aligned}
&\mathbb{E}^{0,i}\big[\mathrm{e}^{\mathrm{i}\theta\bar{\xi}_{\zeta-}-\langle\boldsymbol{\beta},\overline{\boldsymbol{G}}(\zeta-)\rangle}; \bar{J}_\zeta = j\big] \\
&= \mathbb{E}^{0,i}\Big[\sum\nolimits_{m\geq 1}\mathrm{e}^{\mathrm{i}\theta\xi_{S_m}-\langle\boldsymbol{\beta},\boldsymbol{S_m}\rangle}\mathbf{1}_{\{J_{S_m}=j,S_m<\zeta\leq T_m\}}\Big] \\
&= \mathbb{E}^{0,i}_*\Big[\sum\nolimits_{m\geq 1}\mathrm{e}^{\mathrm{i}\theta\xi_{S_m}-\langle\boldsymbol{\beta},\boldsymbol{S_m}\rangle}\mathbf{1}_{\{J_{S_m}=j\}}\int_{S_m}^{T_m}\Big(\sum\nolimits_{k\in[n]}\dagger^\epsilon_k\mathbf{1}_{\{J_t=k\}}\Big)\mathrm{e}^{-\langle\dagger^\epsilon,\boldsymbol{t}\rangle}\,\mathrm{d}t\Big] \\
&= \mathbb{E}^{0,i}_*\Big[\sum\nolimits_{m\geq 1}\mathrm{e}^{\mathrm{i}\theta\xi_{S_m}-\langle\boldsymbol{\beta}+\dagger^\epsilon,\boldsymbol{S_m}\rangle}\mathbf{1}_{\{J_{S_m}=j\}}\Big]\mathbb{E}^{0,j}_*\int_0^{T_1}\Big(\sum\nolimits_k\dagger^\epsilon_k\mathbf{1}_{\{J_t=k\}}\Big)\mathrm{e}^{-\langle\dagger^\epsilon,\boldsymbol{t}\rangle}\,\mathrm{d}t \\
&= \mathbb{E}^{0,i}_*\Big[\int_0^\infty \mathrm{e}^{-\langle\boldsymbol{\beta}+\dagger^\epsilon,\boldsymbol{L}^{-1}_t\rangle+\mathrm{i}\theta H^+_t}; J^+_t = j\Big]\mathbb{E}^{0,j}_*\big[1-\mathrm{e}^{-\langle\dagger^\epsilon,\boldsymbol{T}_1\rangle}\big] \\
&= \boldsymbol{\kappa}(\boldsymbol{\beta}+\dagger^\epsilon,-\mathrm{i}\theta)^{-1}_{i,j}(\boldsymbol{\kappa}(\dagger^\epsilon,0)\mathbf{1})_j,
\end{aligned}
$$

which implies that (B.1) holds in all cases.

Naturally, the same applies to the supremum of the dual. Using [19, Corollary 5.1], we obtain

$$\begin{aligned} &-(\boldsymbol{\Psi}(\theta)-\boldsymbol{\Delta}_{\boldsymbol{\beta}}-\boldsymbol{\Delta}_{\epsilon})^{-1}\boldsymbol{\Delta}_{\dagger^{\epsilon}} \\ &=\boldsymbol{\kappa}(\dagger^{\epsilon}+\boldsymbol{\beta},-\mathrm{i}\theta)^{-1}\boldsymbol{\Delta}_{\boldsymbol{\kappa}(\dagger^{\epsilon},0)\mathbf{1}}\boldsymbol{\Delta}_{\underline{\boldsymbol{c}}^{\epsilon}}^{-1}\boldsymbol{\Delta}_{\widehat{\boldsymbol{\kappa}}(\dagger^{\epsilon},0)\mathbf{1}}(\widehat{\boldsymbol{\kappa}}(\dagger^{\epsilon}+\boldsymbol{\beta},\mathrm{i}\theta)^{-1})^{\top}\boldsymbol{\Delta}_{\boldsymbol{\pi}}\boldsymbol{\Delta}_{\dagger^{\epsilon}}, \end{aligned} \tag{B.2}$$

where $\underline{\boldsymbol{c}}^{\epsilon}=\boldsymbol{\Delta}_{\boldsymbol{\kappa}(\boldsymbol{q}^{\epsilon},0)\mathbf{1}}(\boldsymbol{\kappa}(\boldsymbol{q}^{\epsilon},0)^{-1})^{\top}\boldsymbol{\Delta}_{\boldsymbol{\pi}}\dagger^{\epsilon}$. Simplifying this yields

$$-(\boldsymbol{\Psi}(\theta)-\boldsymbol{\Delta}_{\boldsymbol{\beta}}-\boldsymbol{\Delta}_{\epsilon})=\boldsymbol{\Delta}_{\boldsymbol{\pi}}^{-1}\widehat{\boldsymbol{\kappa}}(\dagger^{\epsilon}+\boldsymbol{\beta},\mathrm{i}\theta)^{\top}\boldsymbol{\Delta}_{\widehat{\boldsymbol{\kappa}}(\dagger^{\epsilon},0)\mathbf{1}}^{-1}\boldsymbol{\Delta}_{\underline{\boldsymbol{c}}^{\epsilon}}\boldsymbol{\Delta}_{\boldsymbol{\kappa}(\dagger^{\epsilon},0)\mathbf{1}}^{-1}\boldsymbol{\kappa}(\dagger^{\epsilon}+\boldsymbol{\beta},-\mathrm{i}\theta). \tag{B.3}$$

Now,

$$b_i^{\epsilon}:=\big(\boldsymbol{\Delta}_{\widehat{\boldsymbol{\kappa}}(\dagger^{\epsilon},0)\mathbf{1}}^{-1}\boldsymbol{\Delta}_{\underline{\boldsymbol{c}}^{\epsilon}}\boldsymbol{\Delta}_{\boldsymbol{\kappa}(\dagger^{\epsilon},0)\mathbf{1}}^{-1}\big)_{i,i}=\frac{\sum_{k\in[n]}\boldsymbol{\kappa}(\dagger^{\epsilon},0)_{k,i}^{-1}\pi(k)\dagger_k^{\epsilon}}{(\widehat{\boldsymbol{\kappa}}(\dagger^{\epsilon},0)\mathbf{1})_i},$$

and we note that $b_i^{\epsilon}>0$ for all i and all $\epsilon>0$. Assume for the moment that $\beta_i>0$ for all i. Element (i,i) of (B.2) at $\theta=0$ provides that

$$-(\boldsymbol{\Psi}(0)-\boldsymbol{\Delta}_{\boldsymbol{\beta}}-\boldsymbol{\Delta}_{\epsilon})_{i,i}^{-1}=\sum_{k\in[n]}\boldsymbol{\kappa}(\dagger^{\epsilon}+\boldsymbol{\beta},0)_{i,k}^{-1}\widehat{\boldsymbol{\kappa}}(\dagger^{\epsilon}+\boldsymbol{\beta},0)_{i,k}^{-1}\frac{\pi(i)}{b_k^{\epsilon}}.$$

The left-hand side is an element of the resolvent matrix of the irreducible Markov process J with additional killing, and therefore converges, as $\epsilon\to 0$, to a positive limit. On the right-hand side, we note that for every $i,k\in[n]$, $\lim_{\epsilon\to 0}\boldsymbol{\kappa}(\dagger^{\epsilon}+\boldsymbol{\beta},0)_{i,k}^{-1}=\boldsymbol{\kappa}(\dagger+\boldsymbol{\beta},0)_{i,k}^{-1}=\int \mathrm{e}^{-\langle\dagger+\boldsymbol{\beta},\boldsymbol{x}\rangle}\,U_{i,k}(\mathrm{d}\boldsymbol{x})\geq 0$, where U is the potential measure of $\boldsymbol{L}^{-1}$ under $\mathbb{P}_*$, and likewise for $\widehat{\boldsymbol{\kappa}}$. Specifically, when $k=i$,

$$\lim_{\epsilon\to 0}\boldsymbol{\kappa}(\dagger^{\epsilon}+\boldsymbol{\beta},0)_{i,i}^{-1}=\int \mathrm{e}^{-\langle\dagger+\boldsymbol{\beta},\boldsymbol{x}\rangle}\,U_{i,i}(\mathrm{d}\boldsymbol{x})>0,$$

and likewise $\lim_{\epsilon\to 0}\widehat{\boldsymbol{\kappa}}(\dagger^{\epsilon}+\boldsymbol{\beta},0)_{i,i}^{-1}>0$. (The reader is correct to be suspicious here: despite the irreducibility of J, a problem seems to occur when $\xi^{(i)}$ is a strictly decreasing Lévy process and transitional jumps into state i are negative, since (ξ,J) can never achieve a maximum in state i, and this indeed implies that $U_{j,i}=0$ for all $j\neq i$. However, because of the way the local time is defined for states such as i in which 0 is irregular for the process reflected in its maximum, J_t^{+} actually spends positive time in state i under $\mathbb{P}_*^{0,i}$, and so $U_{i,i}\neq 0$.) From these considerations, we see that there exists $b_i:=\lim_{\epsilon\to 0}b_i^{\epsilon}\in(0,\infty)$, for every i, and moreover that b_i does not depend on $\boldsymbol{\beta}$, so we may drop our assumption that $\boldsymbol{\beta}$ has positive entries.

Finally, we let $c_i=b_i/\pi_i$, and (B.3) can be rewritten as

$$-(\boldsymbol{\Psi}(\theta)-\boldsymbol{\Delta}_{\boldsymbol{\beta}})=\boldsymbol{\Delta}_{\boldsymbol{\pi}}^{-1}\widehat{\boldsymbol{\kappa}}(\dagger+\boldsymbol{\beta},\mathrm{i}\theta)^{\top}\boldsymbol{\Delta}_{\boldsymbol{\pi}}\boldsymbol{\Delta}_{\boldsymbol{c}}\boldsymbol{\kappa}(\dagger+\boldsymbol{\beta},-\mathrm{i}\theta),$$

which completes the proof. □

We note that the constant $\boldsymbol{c}$ depends on the killing rate $\dagger$ (as well as, of course, the law of ξ under $\mathbb{P}_*$). For Lévy processes, the dependence on the killing rate is known [39, equation (A.3)] but, lacking a Fristedt formula for MAPs, we leave it in the implicit form appearing in the proof.

Proof of Theorem 2.1. Changing the normalisation of the local times amounts to multiplying the exponents $\boldsymbol{\kappa}$ and $\widehat{\boldsymbol{\kappa}}$ on the left by diagonal matrices containing positive entries, so setting $\boldsymbol{\beta} = \mathbf{0}$ and choosing the normalisation appropriately in the preceding theorem leads to the equation

$$-\boldsymbol{\Psi}(\theta) = \boldsymbol{\Delta}_{\boldsymbol{\pi}}^{-1}\widehat{\boldsymbol{\kappa}}(\dagger, \mathrm{i}\theta)^{\top}\boldsymbol{\Delta}_{\boldsymbol{\pi}}\boldsymbol{\kappa}(\dagger, -\mathrm{i}\theta).$$

Finally, we must identify the matrix exponents appearing here, which can be done as follows:

$$\mathrm{e}^{-t\boldsymbol{\kappa}(\dagger,-\mathrm{i}\theta)} = \mathbb{E}_*\big[\mathrm{e}^{\mathrm{i}\theta H_t^+ - \langle \dagger, \boldsymbol{L}_t^{-1}\rangle}; J_t^+\big] = \mathbb{E}\big[\mathrm{e}^{\mathrm{i}\theta\xi_{L_t^{-1}}}\mathbf{1}_{\{\zeta > L_t^{-1}\}}; J_{L_t^{-1}}\big] = \mathrm{e}^{t\boldsymbol{\Psi}^+(\theta)}.$$

The exponent $\widehat{\boldsymbol{\kappa}}$ can be identified similarly, and this completes the proof. □

References

1. Larbi Alili, Loïc Chaumont, Piotr Graczyk, and Tomasz Żak, *Inversion, duality and Doob h-transforms for self-similar Markov processes*, Electron. J. Probab. **22** (2017), Paper No. 20, 18. MR 3622890
2. E. Arjas and T. P. Speed, *Symmetric Wiener-Hopf factorisations in Markov additive processes*, Z. Wahrscheinlichkeitstheorie und Verw. Gebiete **26** (1973), 105–118. MR 331515
3. Søren Asmussen, *Applied probability and queues*, 2 ed., Applications of Mathematics (New York), vol. 51, Springer-Verlag, New York, 2003, Stochastic Modelling and Applied Probability. MR 1978607
4. Søren Asmussen and Hansjörg Albrecher, *Ruin probabilities*, second ed., Advanced Series on Statistical Science & Applied Probability, vol. 14, World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2010. MR 2766220
5. M. T. Barlow, L. C. G. Rogers, and David Williams, *Wiener-Hopf factorization for matrices*, Seminar on Probability, XIV (Paris, 1978/1979) (French), Lecture Notes in Math., vol. 784, Springer, Berlin, 1980, pp. 324–331. MR 580138
6. Jean Bertoin, *Lévy processes*, Cambridge Tracts in Mathematics, vol. 121, Cambridge University Press, Cambridge, 1996. MR 1406564
7. Timothy Budd, *The peeling process on random planar maps coupled to an $O(n)$ loop model (with an appendix by Linxiao Chen)*, 2018.
8. M. E. Caballero and L. Chaumont, *Conditioned stable Lévy processes and the Lamperti representation*, J. Appl. Probab. **43** (2006), no. 4, 967–983. MR 2274630
9. M. E. Caballero, J. C. Pardo, and J. L. Pérez, *Explicit identities for Lévy processes associated to symmetric stable processes*, Bernoulli **17** (2011), no. 1, 34–59.
10. Loïc Chaumont, Henry Pantí, and Víctor Rivero, *The Lamperti representation of real-valued self-similar Markov processes*, Bernoulli **19** (2013), no. 5B, 2494–2523. MR 3160562
11. Oleksandr Chybiryakov, *The Lamperti correspondence extended to Lévy processes and semi-stable Markov processes in locally compact groups*, Stochastic Process. Appl. **116** (2006), no. 5, 857–872.
12. Steffen Dereich, Leif Döring, and Andreas E. Kyprianou, *Real self-similar processes started from the origin*, Ann. Probab. **45** (2017), no. 3, 1952–2003. MR 3650419
13. R. A. Doney and A. E. Kyprianou, *Overshoots and undershoots of Lévy processes*, Ann. Appl. Probab. **16** (2006), no. 1, 91–106. MR 2209337
14. Ronald A. Doney, *Fluctuation theory for Lévy processes*, Lecture Notes in Mathematics, vol. 1897, Springer, Berlin, 2007, Lectures from the 35th Summer School on Probability Theory held in Saint-Flour, July 6–23, 2005, Edited and with a foreword by Jean Picard. MR 2320889
15. Leif Döring, Mladen Savov, Lukas Trottner, and Alexander R. Watson, *The uniqueness of the Wiener–Hopf factorisation of Lévy processes and random walks*, Bull. Lond. Math. Soc. (to appear).
16. Leif Döring and Lukas Trottner, *Stability of overshoots of Markov additive processes*, Ann. Appl. Probab. **33** (2023), no. 6B, 5413–5458. MR 4677737
17. Roger A. Horn and Charles R. Johnson, *Matrix analysis*, 2 ed., Cambridge University Press, Cambridge, 2013. MR 2978290
18. Jevgenijs Ivanovs, *One-sided Markov Additive Processes and Related Exit Problems*, Ph.D. thesis, University of Amsterdam, 2007.
19. Jevgenijs Ivanovs, *Splitting and time reversal for Markov additive processes*, Stochastic Process. Appl. **127** (2017), no. 8, 2699–2724. MR 3660888

20. Jevgenijs Ivanovs, Onno Boxma, and Michel Mandjes, *Singularities of the matrix exponent of a Markov additive process with one-sided jumps*, Stochastic Process. Appl. **120** (2010), no. 9, 1776–1794. MR 2673974
21. H. Kaspi, *On the symmetric Wiener-Hopf factorization for Markov additive processes*, Z. Wahrsch. Verw. Gebiete **59** (1982), no. 2, 179–196. MR 650610
22. Przemysł aw Klusik and Zbigniew Palmowski, *A note on Wiener-Hopf factorization for Markov additive processes*, J. Theoret. Probab. **27** (2014), no. 1, 202–219. MR 3174223
23. S. G. Kou and Hui Wang, *First passage times of a jump diffusion process*, Adv. in Appl. Probab. **35** (2003), no. 2, 504–531. MR 1970485
24. Herbert C. Kranzer, *Asymptotic factorization in nondissipative Wiener-Hopf problems*, J. Math. Mech. **17** (1967), 577–600. MR 0220020
25. A. Kuznetsov, A. E. Kyprianou, and J. C. Pardo, *Meromorphic Lévy processes and their fluctuation identities*, Ann. Appl. Probab. **22** (2012), no. 3, 1101–1135. MR 2977987
26. A. Kuznetsov and J. C. Pardo, *Fluctuations of stable processes and exponential functionals of hypergeometric Lévy processes*, Acta Appl. Math. **123** (2013), 113–139. MR 3010227
27. A. E. Kuznetsov, *An analytical proof of the Pecherskiĭ-Rogozin identity and the Wiener-Hopf factorization*, Teor. Veroyatn. Primen. **55** (2010), no. 3, 417–431. MR 2768530
28. A. E. Kyprianou, J. C. Pardo, and V. Rivero, *Exact and asymptotic n-tuple laws at first and last passage*, Ann. Appl. Probab. **20** (2010), no. 2, 522–564.
29. A. E. Kyprianou, J. C. Pardo, and A. R. Watson, *The extended hypergeometric class of Lévy processes*, J. Appl. Probab. **51A** (2014), no. Celebrating 50 Years of The Applied Probability Trust, 391–408. MR 3317371
30. Andreas E. Kyprianou, *Fluctuations of Lévy processes with applications*, 2 ed., Universitext, Springer, Heidelberg, 2014, Introductory lectures. MR 3155252
31. ———, *Deep factorisation of the stable process*, Electron. J. Probab. **21** (2016), Paper No. 23, 28. MR 3485365
32. Andreas E. Kyprianou and Juan Carlos Pardo, *Stable Lévy processes via Lamperti-type representations*, Institute of Mathematical Statistics (IMS) Monographs, vol. 7, Cambridge University Press, Cambridge, 2022. MR 4692990
33. Andreas E. Kyprianou, Juan Carlos Pardo, and Matija Vidmar, *Double hypergeometric Lévy processes and self-similarity*, J. Appl. Probab. **58** (2021), no. 1, 254–273. MR 4222428
34. Andreas E. Kyprianou, Juan Carlos Pardo, and Alexander R. Watson, *Hitting distributions of α-stable processes via path censoring and self-similarity*, Ann. Probab. **42** (2014), no. 1, 398–430. MR 3161489
35. Andreas E. Kyprianou, Victor Rivero, and Batı¸Sengül, *Deep factorisation of the stable process II: Potentials and applications*, Ann. Inst. Henri Poincaré Probab. Stat. **54** (2018), no. 1, 343–362. MR 3765892
36. Andreas E. Kyprianou, Victor Rivero, and Weerapat Satitkanitkul, *Deep factorisation of the stable process III: the view from radial excursion theory and the point of closest reach*, Potential Anal. **53** (2020), no. 4, 1347–1375. MR 4159383
37. John Lamperti, *Semi-stable Markov processes. I*, Z. Wahrscheinlichkeitstheorie und Verw. Gebiete **22** (1972), 205–225. MR 307358
38. Eugene Lukacs, *On certain periodic characteristic functions*, Compositio Math. **13** (1956), 76–80. MR 83838
39. Pierre Patie and Mladen Savov, *Bernstein-gamma functions and exponential functionals of Lévy processes*, Electron. J. Probab. **23** (2018), Paper No. 75, 101. MR 3835481
40. N. U. Prabhu, *Stochastic storage processes*, second ed., Applications of Mathematics (New York), vol. 15, Springer-Verlag, New York, 1998, Queues, insurance risk, dams, and data communication. MR 1492990
41. È. L. Presman, *Factorization methods, and a boundary value problem for sums of random variables given on a Markov chain*, Izv. Akad. Nauk SSSR Ser. Mat. **33** (1969), 861–900. MR 256467
42. L. C. G. Rogers, *Fluid models in queueing theory and Wiener-Hopf factorization of Markov chains*, Ann. Appl. Probab. **4** (1994), no. 2, 390–413. MR 1272732
43. B. A. Rogozin, *On distributions of functionals related to boundary problems for processes with independent increments*, Theory of Probability & Its Applications **11** (1966), no. 4, 580–591.
44. Ken Sato, *Lévy processes and infinitely divisible distributions*, 2 ed., Cambridge Studies in Advanced Mathematics, vol. 68, Cambridge University Press, Cambridge, 2013. MR 3185174

45. René L. Schilling, Renming Song, and Zoran Vondraček, *Bernstein functions*, 2 ed., De Gruyter Studies in Mathematics, vol. 37, Walter de Gruyter & Co., Berlin, 2012, Theory and applications. MR 2978140
46. Robin Stephenson, *On the exponential functional of Markov additive processes, and applications to multi-type self-similar fragmentation processes and trees*, ALEA Lat. Am. J. Probab. Math. Stat. **15** (2018), no. 2, 1257–1292. MR 3867206
47. Vincent Vigon, *Simplifiez vos Lévy en titillant la factorisation de Wiener–Hopf*, Ph.D. thesis, INSA de Rouen, 2002.
48. ———, *Votre Lévy rampe-t-il?*, J. London Math. Soc. (2) **65** (2002), no. 1, 243–256. MR 1875147
49. J. L. Walsh, *The Cauchy–Goursat Theorem for Rectifiable Jordan Curves*, Proc. Natl. Acad. Sci. USA **19** (1933), no. 5, 540.
50. Aurel Wintner, *On a Class of Fourier Transforms*, Amer. J. Math. **58** (1936), no. 1, 45–90. MR 1507134

University of Mannheim, Institute of Mathematics, B6 26, 68159 Mannheim, Germany
Email address: `doering@uni-mannheim.de`

Aarhus University, Department of Mathematics, Ny Munkegade 118, 8000 Aarhus C, Denmark
Email address: `trottner@math.au.dk`

University College London, UK
Email address: `alexander.watson@ucl.ac.uk`