Received: 14 March 2024 | Revised: 29 May 2024 | Accepted: 29 May 2024

DOI: 10.1112/blms.13112

RESEARCH ARTICLE

Bulletin of the London Mathematical Society

The uniqueness of the Wiener–Hopf factorisation of Lévy processes and random walks

Leif Döring[1] | **Mladen Savov**[2,3] | **Lukas Trottner**[4] | **Alexander R. Watson**[5]

[1]Institute of Mathematics, University of Mannheim, Mannheim, Germany

[2]Faculty of Mathematics and Informatics, Sofia University, Sofia, Bulgaria

[3]Institute of Mathematics and Informatics, Bulgarian Academy of Sciences, Sofia, Bulgaria

[4]Department of Mathematics, Aarhus University, Aarhus, Denmark

[5]Department of Statistical Science, University College London, London, UK

Correspondence
Alexander R. Watson, Department of Statistical Science, University College London, London, UK.
Email: alexander.watson@ucl.ac.uk

Funding information
Carlsberg Foundation Young Researcher Fellowship, Grant/Award Number: CF20-0640; European Union - NextGenerationEU via National Recovery and Resilience Plan of the Republic of Bulgaria, Grant/Award Number: BG-RRP-2.004-0008

Abstract

We prove that the spatial Wiener–Hopf factorisation of a Lévy process or random walk without killing is unique.

MSC 2020
60G50, 60G51 (primary), 60E07 (secondary)

1 | INTRODUCTION

Let X be a Lévy process without killing, and define $\psi : \mathbb{R} \to \mathbb{C}$ to be its characteristic exponent: $\mathbb{E}e^{\mathrm{i}zX_t} = e^{-t\psi(z)}$. The spatial Wiener–Hopf factorisation of ψ [10, Theorem 6.15(iv)] consists of the identity

$$\psi(z) = a\kappa_+(z)\kappa_-(-z), \qquad z \in \mathbb{R}, \tag{1}$$

where κ_+ and κ_- are the characteristic exponents of certain subordinators, known, respectively, as the ascending and descending ladder height processes. The constant $a > 0$ is determined by the normalisation of certain local times, and we take $a = 1$ throughout.

In this work, we address the question of the uniqueness of factorisations of the form (1). That is, suppose that

$$\psi(z) = \kappa_+(z)\kappa_-(-z) = \kappa'_+(z)\kappa'_-(-z), \qquad z \in \mathbb{R}, \tag{2}$$

for some functions $\kappa_+, \kappa_-, \kappa'_+, \kappa'_-$ defined on $\mathbb{C}_u = \{z \in \mathbb{C} : \operatorname{Im} z \geqslant 0\}$ which are the characteristic exponents of certain subordinators (i.e. $\kappa_\pm(\mathrm{i}\cdot)$ and $\kappa'_\pm(\mathrm{i}\cdot)$ are Bernstein functions). We will prove the following result.

Theorem 1. *There exists some $c > 0$ such that $\kappa_+(z) = c\kappa'_+(z)$ and $\kappa'_-(z) = c\kappa_-(z)$ for all $z \in \mathbb{C}_u$.*

This result has an important probabilistic consequence for the *theory of friendship* of Lévy processes developed in Vigon [18]. If $\kappa_\pm$ are the characteristic exponents of two subordinators, and there exists a Lévy process X with characteristic exponent ψ satisfying (1), then these subordinators are called *friends*. In this case, we refer to X as the *bonding Lévy process*. In Vigon [18], necessary and sufficient conditions are given for friendship in terms of the characteristics of the subordinators. The construction of Lévy processes by these means has been the subject of intense research over the past decade [5, 6, 9, 11]. However, if one wants to use the ladder height processes of a process constructed via friendship, for example, to describe its hitting distributions [12] or its scale functions [10, §9], then it is essential that the friendship has a probabilistic meaning. This can be deduced from our result.

Corollary 2. *Two friends H^+ and H^- are equal in law to the ascending and descending ladder height processes (for some scaling of local time) of their bonding Lévy process.*

We emphasise that the difficulty in the results above is that we consider only the spatial Wiener–Hopf factorisation, and focus on processes without killing. If one has access to the spatio-temporal Wiener–Hopf factorisation (i.e. if one considers the bivariate ladder processes of X), then, as shown by Chaumont and Doney [3] and Kwaśnicki [8], even knowing just the ascending factor is enough to uniquely specify the distribution of X, and thereby also to determine the descending factor.

Likewise, when X is killed — or equivalently, when we consider factorisations of $q + \psi(z)$ for some $q > 0$ — the uniqueness of the factorisation is well known. The traditional proof proceeds via Liouville's theorem, and can be found, for example, in [7, Theorem 1(f)]. One first uses the ratios $\kappa_+(z)/\kappa'_+(z)$ and $\kappa'_-(z)/\kappa_-(z)$ to define a non-zero entire function F. Taking a continuous version of the logarithm and using asymptotic properties of the characteristic exponent of a subordinator, one observes that $\log F$ is sublinear, and therefore constant, which completes the proof. However, this argument requires a lower bound for each characteristic exponent; when X is not killed, it may be the case that one of these exponents κ has the property that $\liminf_{|z|\to\infty, z\in\mathbb{R}} \kappa(z) = 0$. Examples of such subordinators are given in [17, Example 41.23], and we remark that this condition is related to the *weak non-lattice* property defined in [14, §2.2.2].

There is a probabilistic proof of uniqueness, described in [16, pp. 583–4], but as this involves factorising the value of X at its lifetime, it too is restricted to the case where X is killed. The situation is no better for random walks. For instance, the uniqueness result found in Theorem 12.1.1 or 12.1.2 of Borovkov [2] is restricted to the setting where killing is present, and when the killing is removed (putting $|z| = 1$ in the notation of [2]), a similar issue appears involving lower bounds on the factors.

Despite many attempts, we were unable to find a satisfactory proof of uniqueness when X is not killed using either Liouville's theorem or a probabilistic argument. Instead, we approach the question using the theory of tempered distributions. This idea has precedent: Grzywny and Kwaśnicki [4] have recently made use of distribution theory to prove a generalised Liouville theorem for Lévy operators.

Our result for Lévy processes immediately resolves the same problem for random walks. Let X_1 be the step of a one-dimensional random walk $X = (X_n : n \geqslant 0)$ started from $X_0 = 0$, and assume that X_1 is not identically zero. Define

$$\tau^+ = \inf\{n \geqslant 1 : X_n > 0\}, \quad \tau^{-,w} = \inf\{n \geqslant 1 : X_n \leqslant 0\}.$$

We will now define defective random variables H^+ and $H^{-,w}$. Adjoin a 'cemetery' state Δ, and extend any function $f : \mathbb{R} \to \mathbb{C}$ by setting $f(\Delta) = 0$. Let H^+ take the value X_{τ^+} on the event $\{\tau^+ < \infty\}$ and the value Δ on its complement. Similarly, let $H^{-,w}$ take the value $X_{\tau^{-,w}}$ on the event $\{\tau^{-,w} < \infty\}$. These are the first steps of the strict ascending and weak descending ladder height processes, respectively. Following [2, Corollary 12.2.2], the spatial Wiener–Hopf factorisation of this random walk can be expressed by the identity

$$1 - \mathbb{E}\left[e^{\mathrm{i}zX_1}\right] = \left(1 - \mathbb{E}\left[e^{\mathrm{i}zH^+}\right]\right)\left(1 - \mathbb{E}\left[e^{-\mathrm{i}zH^{-,w}}\right]\right), \quad z \in \mathbb{R}.$$

Now, we can show the following result.

Theorem 3. *If V^+ is a positive defective random variable and V^- is a non-negative defective random variable satisfying*

$$1 - \mathbb{E}\left[e^{\mathrm{i}zX_1}\right] = \left(1 - \mathbb{E}\left[e^{\mathrm{i}zV^+}\right]\right)\left(1 - \mathbb{E}\left[e^{-\mathrm{i}zV^-}\right]\right), \quad z \in \mathbb{R},$$

then $V^+ \overset{d}{=} H^+$ and $V^- \overset{d}{=} H^{-,w}$.

Naturally, an analogue of Corollary 2 holds for random walks as well.

2 | PROOFS

We begin with some notation. If κ is the characteristic exponent of a subordinator, then it has the representation

$$-\kappa(z) = -q + \mathrm{i}dz + \int_{(0,\infty)} (e^{\mathrm{i}xz} - 1)\,\mu(\mathrm{d}x), \qquad \operatorname{Im} z \geqslant 0,$$

where $q \geqslant 0$ is the killing rate, $d \geqslant 0$ the drift and μ the Lévy measure, satisfying $\int_{(0,\infty)} (1 \wedge x)\,\mu(\mathrm{d}x) < \infty$. We adopt similar notation for the characteristics of the other subordinators in question.

We will use distribution theory, and refer to [19] for background. We define $\mathcal{D}$ to be the set of complex-valued smooth functions with compact support, $\mathcal{S}$ to be the set of Schwartz functions (complex-valued smooth functions with rapidly decaying derivatives) and $\mathcal{S}'$ the set of tempered distributions (continuous linear functionals from $\mathcal{S}$ to $\mathbb{C}$). The action of $h \in \mathcal{S}'$ on $\phi \in \mathcal{S}$ is written $\langle h, \phi \rangle$, and it will often be convenient to write both the distribution and the action with a dummy variable, that is, $h(x)$ and $\langle h(x), \phi(x) \rangle$. In particular, when $h \in \mathcal{S}'$ is a measure, we will often still write $h(x)$ in places where, as probabilists, we would usually use the infinitesimal notation $h(\mathrm{d}x)$.

There are a number of useful operations on distributions. The distributional derivative of h is written Dh. The reflection of $h(x)$ is written $h(-x)$, or $h(-\cdot)$ if we want to omit the dummy variable, and has the meaning $\langle h(-x), \phi(x) \rangle = \langle h(x), \phi(-x) \rangle$. Denote by $\mathcal{F}\phi(z) = \int_{\mathbb{R}} e^{\mathrm{i}xz} \phi(x)\mathrm{d}x$ the Fourier transform of $\phi \in \mathcal{S}$, and extend this to $h \in \mathcal{S}'$ by the identity $\langle \mathcal{F}h, \phi \rangle = \langle h, \mathcal{F}\phi \rangle$ [19, §6.2]. We also recall the definition of the support of a distribution from [19, §1.5], as follows. We say that $h \in \mathcal{S}'$ vanishes on an open set $G \subset \mathbb{R}$ if, for all $\phi \in \mathcal{D}$ with support in G, $\langle h, \phi \rangle = 0$. The support of h is defined as $\operatorname{supp} h = \mathbb{R} \setminus \bigcup \{G \subset \mathbb{R} : G \text{ open}, \ h \text{ vanishes on } G\}$.

When μ is the Lévy measure of some subordinator, we define a distribution $\Gamma\mu$ by

$$\langle \Gamma\mu, \phi \rangle = \int_{(0,\infty)} \big(\phi(x) - \phi(0)\big)\mu(\mathrm{d}x), \qquad \phi \in \mathcal{S}.$$

The properties of a Lévy measure imply that $\Gamma\mu \in \mathcal{S}'$. Let G_+ be the tempered distribution

$$G_+ = -q_+\delta - d_+ D\delta + \Gamma\mu_+,$$

where δ is the Dirac mass at zero. This has the property that $\mathcal{F}G_+ = -\kappa_+$. Let $U_+ = \int_0^\infty \mathbb{P}(H_t \in \cdot)\,\mathrm{d}t$ be the 0-resolvent measure of the subordinator H with characteristic exponent κ_+, and observe that for all $\epsilon > 0$, we have $\mathcal{F}(e^{-\epsilon \cdot} U_+)(z) = -\frac{1}{\kappa_+(z+\mathrm{i}\epsilon)}$ for $\operatorname{Im} z > -\epsilon$. Analogous quantities are defined for the other subordinators involved.

The argument required depends on the support of the Lévy process X. We say that X has *lattice support* if for some (and then any) $t > 0$, the support of the random variable X_t is contained in some lattice strictly contained in $\mathbb{R}$. When X has lattice support, we denote by $\eta > 0$ the minimal span of the support of X_t; that is, X_t is supported on $\eta\mathbb{Z}$ but on no strict sublattice thereof. When X does not have lattice support, let $\eta = \infty$.

Since ψ has zeroes exactly at the points of $\frac{2\pi}{\eta}\mathbb{Z}$ (understanding this set to be $\{0\}$ when $\eta = \infty$), we can sensibly define $F : \mathbb{C} \setminus \frac{2\pi}{\eta}\mathbb{Z} \to \mathbb{C}$ as the holomorphic function given by

$$F(z) = \begin{cases} \dfrac{\kappa_+(z)}{\kappa_+'(z)}, & \operatorname{Im} z \geqslant 0, \\[2ex] \dfrac{\kappa_-'(-z)}{\kappa_-(-z)}, & \operatorname{Im} z \leqslant 0. \end{cases} \tag{3}$$

Theorem 1 therefore amounts to the assertion that $F \equiv c$ for some constant $c > 0$.

We are now in a position to sketch our argument. Formally, it appears that, for $z \in \mathbb{R} \setminus \frac{2\pi}{\eta}\mathbb{Z}$,

$$\mathcal{F}(G_+ * U'_+)(z) = F(z) = \mathcal{F}\big((G'_- * U_-)(-\cdot)\big)(z), \tag{4}$$

where '$*$' is convolution. An optimistic approach is to say that $G_+ * U'_+(x)$ has support contained in $[0,\infty)$ and $G'_- * U_-(-x)$ has support contained in $(-\infty,0]$, and that (4) implies that these distributions are equal. $G_+ * U'_+(x)$ must therefore have support $\{0\}$, and hence be equal to $c\delta(x)$, where δ is the Dirac mass at 0, and this gives that $F = c$. This is close to the argument that we will make, but it is not quite valid: the hitch is that (4) holds only on the domain of F. However, this does tell us that $G_+ * U'_+(x)$ and $G'_- * U_-(-x)$ differ only by a polynomial (in the non-lattice case) or a series of polynomials weighted by $e^{\mathrm{i}ax}$ for $a \in \frac{2\pi}{\eta}\mathbb{Z}$ (in the lattice case). Showing that this perturbation is actually zero requires some bounds on the growth rates of the distributions in question, which are quite delicate in the lattice case.

This proof outline sounds relatively simple, especially in the non-lattice case, but the devil is in the details, and in particular, we need to find a rigourous interpretation of Equation (4). Addressing these difficulties adds some complexity to the proof, but the essential idea remains the same.

The proof now begins in earnest. Define the measure

$$h = (\overline{\mu}_+ + d_+\delta + q_+\mathbb{1}_{\mathbb{R}_+}) * U'_+,$$

where $\overline{\mu}_+(x) = \mathbb{1}_{\{x>0\}}\mu_+(x,\infty)$, δ is the Dirac mass at 0 and $\mathbb{1}_{\mathbb{R}_+}$ is the indicator function of $\mathbb{R}_+ = [0,\infty)$. The intuition is that $-h$ is a primitive of $G_+ * U'_+$, though we will neither prove nor use this assertion. The key ingredient is the following lemma.

Lemma 4. *For any* $\epsilon > 0$, $\int_{[1,\infty)} x^{-(2+\epsilon)} h(\mathrm{d}x) < \infty$.

Proof. We consider each part of h separately.

(i) The first part is $\delta * U'_+ = U'_+$. The following calculation provides slightly more than is required for the integrability in question, and will be used again for the remaining parts. When κ'_+ corresponds to a subordinator (without killing),

$$\begin{aligned}\int_{[1,\infty)} x^{-(1+\epsilon)} U'_+(\mathrm{d}x) &\leqslant \sum_{n=1}^{\infty} n^{-(1+\epsilon)} U'_+[n,n+1)\\ &\leqslant \sum_{n=1}^{N-1} n^{-(1+\epsilon)} U'_+[n,n+1) + \left(\frac{1}{m'_+} + \rho\right) \sum_{n=N}^{\infty} n^{-(1+\epsilon)} < \infty,\end{aligned}$$

where $\rho > 0$ is arbitrary and $m'_+ \in (0,\infty]$, and N satisfying the inequality in question exists by the renewal theorem [15, Theorem 5.3.1]. When κ'_+ corresponds to a killed subordinator, U'_+ is a finite measure, and the integrability is immediate.

(ii) Next, we observe that $\mathbb{1}_{\mathbb{R}_+} * U'_+(x) = U'_+[0,x]$, and again by the renewal theorem, $U'_+[0,x]$ grows at most linearly in x as $x \to \infty$, which establishes that $\int_1^\infty x^{-(2+\epsilon)} U'_+[0,x]\mathrm{d}x < \infty$.

(iii) We consider now the calculation

$$\int_1^\infty x^{-(2+\epsilon)}\overline{\mu}_+ * U'_+(x)\mathrm{d}x = \int_1^\infty x^{-(2+\epsilon)}\int_{[0,\infty)} \overline{\mu}_+(x-y)\,U'_+(\mathrm{d}y)\mathrm{d}x$$
$$= \int_{[0,\infty)}\int_0^\infty \overline{\mu}_+(u)(u+y)^{-(2+\epsilon)}\mathbb{1}_{\{u+y\geqslant 1\}}\mathrm{d}u\,U'_+(\mathrm{d}y) = I_1 + I_2 + I_3,$$

where the integrals I_1, I_2 and I_3 are obtained by the restrictions to $\{y \leqslant 1\}$, $\{y > 1, u \leqslant 1\}$ and $\{y > 1, u > 1\}$, respectively.

These terms can be estimated as follows:

$$I_1 = \int_{[0,1]}\int_0^\infty \overline{\mu}_+(u)(u+y)^{-(2+\epsilon)}\mathbb{1}_{\{u+y\geqslant 1\}}\,\mathrm{d}u\,U'_+(\mathrm{d}y)$$
$$\leqslant \int_{[0,1]}\int_1^\infty \overline{\mu}_+(u)u^{-(2+\epsilon)}\,\mathrm{d}u\,U'_+(\mathrm{d}y) + \int_{[0,1]}\int_0^1 \overline{\mu}_+(u)\,\mathrm{d}u\,U'_+(\mathrm{d}y)$$
$$\leqslant U'_+[0,1]\left(\overline{\mu}_+(1)\int_1^\infty u^{-(2+\epsilon)}\,\mathrm{d}u + \int_0^1 \overline{\mu}_+(u)\,\mathrm{d}u\right) < \infty.$$

Looking at I_2, we have

$$I_2 = \int_{(1,\infty)}\int_0^1 \overline{\mu}_+(u)(u+y)^{-(2+\epsilon)}\mathbb{1}_{\{u+y\geqslant 1\}}\,\mathrm{d}u\,U'_+(\mathrm{d}y)$$
$$\leqslant \int_{(1,\infty)} y^{-(2+\epsilon)}\,U'_+(\mathrm{d}y)\int_0^1 \overline{\mu}_+(u)\,\mathrm{d}u,$$

and this is finite by the argument in part (i). Finally, we turn to I_3.

$$I_3 = \int_{(1,\infty)}\int_1^\infty \overline{\mu}_+(u)(u+y)^{-(2+\epsilon)}\mathbb{1}_{\{u+y\geqslant 1\}}\,\mathrm{d}u\,U'_+(\mathrm{d}y)$$
$$\leqslant \overline{\mu}_+(1)\int_{(1,\infty)}\int_1^\infty (u+y)^{-(2+\epsilon)}\,\mathrm{d}u\,U'_+(\mathrm{d}y)$$
$$= \frac{\overline{\mu}_+(1)}{2+\epsilon}\int_{(1,\infty)} (y+1)^{-(1+\epsilon)}\,U'_+(\mathrm{d}y),$$

which again is finite by part (i), and this completes the proof. □

An important consequence is that h is a tempered distribution:

Corollary 5. $h \in \mathcal{S}'$.

Proof. A short calculation, similar to the one for I_1 in the proof above, reveals that h is finite on compact sets. The fact that $\int |\phi(x)|\mathbb{1}_{\{|x|\geqslant 1\}}\,h(\mathrm{d}x) < \infty$, for $\phi \in \mathcal{S}$, follows from the preceding lemma. These two observations imply that $\phi \mapsto \int \phi(x)\,h(\mathrm{d}x) < \infty$ is a map from $\mathcal{S}$ to $\mathbb{C}$, and the fact that it is a continuous operation can be proved using the dominated convergence theorem and the lemma. □

With this preparation, we can now give the proof of Theorem 1.

Proof of Theorem 1 (non-lattice support). Let $u(z) = \frac{F(z)}{\mathrm{i}z}$ for $z = \mathbb{C} \setminus \{0\}$, meaning $u(z) = \frac{\kappa_+(z)}{\mathrm{i}z\kappa_+'(z)}$ where $\operatorname{Im} z \geqslant 0$; note that we do not know whether u is a tempered distribution. Our next goal is to show that '$\mathcal{F}h = u$ away from zero', in a sense that we will make precise.

Let $\epsilon > 0$ and define $h_\epsilon(x) = e^{-\epsilon x}h(x) = \big(e^{-\epsilon \cdot}(\overline{\mu}_+ + d_+\delta + q_+\mathbb{1}_{\mathbb{R}_+})\big) * (e^{-\epsilon \cdot}U_+')(x)$. This is a convolution of finite measures, so we can compute its Fourier transform directly as an integral.

When $\operatorname{Im} z > -\epsilon$, we have (using Fubini's theorem for the integral):

$$
\begin{aligned}
&\int_{(0,\infty)} e^{\mathrm{i}xz-\epsilon x}(\overline{\mu}_+(x) + d_+\delta(x) + q_+\mathbb{1}_{\mathbb{R}_+}(x))\mathrm{d}x \\
&\quad = -\frac{q_+}{\mathrm{i}(z+\mathrm{i}\epsilon)} + d_+ + \int_{(0,\infty)}\int_0^x e^{\mathrm{i}x(z+\mathrm{i}\epsilon)}\mathrm{d}x\,\mu_+(\mathrm{d}y) \\
&\quad = \frac{1}{\mathrm{i}(z+\mathrm{i}\epsilon)}\left(-q_+ + \mathrm{i}d_+(z+\mathrm{i}\epsilon) + \int_{(0,\infty)}(e^{\mathrm{i}x(z+\mathrm{i}\epsilon)} - 1)\,\mu_+(\mathrm{d}y)\right) \\
&\quad = -\frac{\kappa_+(z+\mathrm{i}\epsilon)}{\mathrm{i}(z+\mathrm{i}\epsilon)}.
\end{aligned}
$$

As already mentioned, for such z, $\mathcal{F}(e^{-\epsilon\cdot}U_+')(z) = -\frac{1}{\kappa_+'(z+\mathrm{i}\epsilon)}$. Using the convolution theorem for the Fourier transform of finite measures, we obtain that $\mathcal{F}h_\epsilon(z) = u(z+\mathrm{i}\epsilon)$ for $\operatorname{Im} z > -\epsilon$.

Let $\phi \in \mathcal{S}$, and recall that $\mathcal{F}\phi \in \mathcal{S}$ [19, Lemma, p. 107]. Corollary 5 implies that $\mathcal{F}\phi(x)h(x)$ is a finite (complex) measure, and so, the following calculation can proceed via dominated convergence:

$$
\begin{aligned}
\lim_{\epsilon\to 0}\langle \mathcal{F}h_\epsilon, \phi\rangle = \lim_{\epsilon\to 0}\langle h_\epsilon, \mathcal{F}\phi\rangle &= \lim_{\epsilon\to 0}\int e^{-\epsilon x}h(x)\mathcal{F}\phi(x)\,\mathrm{d}x \\
&= \int h(x)\mathcal{F}\phi(x)\,\mathrm{d}x = \langle h, \mathcal{F}\phi\rangle = \langle \mathcal{F}h, \phi\rangle.
\end{aligned}
$$

Restricting ourselves now just to $\phi \in \mathcal{D}$ such that $0 \notin \operatorname{supp}\phi$, we obtain:

$$
\begin{aligned}
\langle \mathcal{F}h, \phi\rangle = \lim_{\epsilon\to 0}\langle \mathcal{F}h_\epsilon, \phi\rangle &= \lim_{\epsilon\to 0}\int \frac{\kappa_+(z+\mathrm{i}\epsilon)}{\mathrm{i}(z+\mathrm{i}\epsilon)\kappa_+'(z+\mathrm{i}\epsilon)}\phi(z)\,\mathrm{d}z \\
&= \int u(z)\phi(z)\,\mathrm{d}z,
\end{aligned}
$$

where for the last line, we used bounded convergence, since the domain of integration is compact and does not contain 0, and the non-lattice condition for the Lévy process implies that κ_+' has no zeroes except possibly at 0.

Using the properties of Fourier transforms [19, §6.3], this implies that for such ϕ,

$$\langle -\mathcal{F}(Dh)(z), \phi(z)\rangle = \langle \mathcal{F}h(z), \mathrm{i}z\phi(z)\rangle = \int \mathrm{i}zu(z)\phi(z)\,\mathrm{d}z = \int F(z)\phi(z)\,\mathrm{d}z. \tag{5}$$

Written informally, we have found that '$-\mathcal{F}(Dh) = F$ away from zero'.

Next, we turn to the other half plane, and define

$$\widetilde{h}(x) = -(\overline{\mu}'_- + d'_-\delta + q'_-\mathbb{1}_{\mathbb{R}_+}) * U_-(-x),$$

which by the same argument as in Corollary 5 is a tempered distribution. Carrying out the same steps as above, we find that

$$\langle -\mathcal{F}(D\widetilde{h})(z), \phi(z)\rangle = \int F(z)\phi(z)\,\mathrm{d}z, \tag{6}$$

for $\phi \in \mathcal{D}$ with support not containing 0. Putting together (5) and (6), we have that for such ϕ,

$$\langle \mathcal{F}(Dh), \phi\rangle = \langle \mathcal{F}(D\widetilde{h}), \phi\rangle.$$

In other words, $\operatorname{supp}\mathcal{F}(Dh - D\widetilde{h}) \subset \{0\}$. By [19, §2.6], there exist $N \geqslant 0$ (finite by [19, §5.2, Corollary 1]) and coefficients $(a_n)_{0\leqslant n\leqslant N} \subset \mathbb{C}$ such that

$$\mathcal{F}(Dh - D\widetilde{h}) = \sum_{n=0}^{N} a_n D^n\delta,$$

which, in turn, implies that

$$(Dh - D\widetilde{h})(x) = \sum_{n=0}^{N} \frac{a_n}{2\pi}(-\mathrm{i}x)^n$$

and

$$(h - \widetilde{h})(x) = a_{-1} + \sum_{n=0}^{N} \frac{a_n}{2\pi}\frac{1}{n+1}(-i)^n x^{n+1},$$

for some $a_{-1} \in \mathbb{C}$. Lemma 4 tells us that $a_n = 0$ for all $n \geqslant 1$, meaning that

$$Dh - D\widetilde{h} = \frac{a_0}{2\pi}.$$

Recall that $\operatorname{supp} Dh \subset [0, \infty)$ and $\operatorname{supp} D\widetilde{h} \subset (-\infty, 0]$. To make use of this property, we first split the constant:

$$Dh - \frac{a_0}{2\pi}\mathbb{1}_{\mathbb{R}_+} = D\widetilde{h} + \frac{a_0}{2\pi}\mathbb{1}_{(-\infty,0)}.$$

Consider a function $\phi \in \mathcal{D}$ with $0 \notin \operatorname{supp}\phi$. We have:

$$\left\langle Dh - \tfrac{a_0}{2\pi}\mathbb{1}_{\mathbb{R}_+}, \phi \right\rangle = \left\langle Dh - \tfrac{a_0}{2\pi}\mathbb{1}_{\mathbb{R}_+}, \phi\mathbb{1}_{\mathbb{R}_+} \right\rangle = \left\langle D\widetilde{h} + \tfrac{a_0}{2\pi}\mathbb{1}_{(-\infty,0)}, \phi\mathbb{1}_{\mathbb{R}_+} \right\rangle = 0.$$

It follows that $\operatorname{supp}\left(Dh - \tfrac{a_0}{2\pi}\mathbb{1}_{\mathbb{R}_+}\right) \subset \{0\}$, and thence that there exist $M \geqslant 0$ finite and $(b_m)_{0\leqslant m\leqslant M} \subset \mathbb{C}$ such that

$$\frac{a_0}{2\pi}\mathbb{1}_{\mathbb{R}_+} - Dh = \sum_{m=0}^{M} b_m D^m \delta.$$

But since h is a measure, this simplifies to

$$Dh = \frac{a_0}{2\pi}\mathbb{1}_{\mathbb{R}_+} - b_0\delta - b_1 D\delta. \tag{7}$$

Taking primitives in the above equation [19, §2.2, Theorem], we obtain

$$h = f - b_1\delta,$$

where f is a distribution coming from an absolutely continuous complex measure. Therefore, recalling that h is represented by a (positive) measure, $b_1 = -h(\{0\}) \leqslant 0$. On the other hand, by the equality of distributions established above,

$$D\widetilde{h} + \frac{a_0}{2\pi}\mathbb{1}_{(-\infty,0)} = Dh - \frac{a_0}{2\pi}\mathbb{1}_{\mathbb{R}_+} = -b_0\delta - b_1 D\delta$$

also so,

$$\widetilde{h} = \widetilde{f} - b_1\delta,$$

where again $\widetilde{f}$ is a distribution arising from an absolutely continuous complex measure. Thus, since $-\widetilde{h}$ is represented by a positive measure, $b_1 = -\widetilde{h}(\{0\}) \geqslant 0$. Since $b_1 \geqslant 0$ and $b_1 \leqslant 0$, we obtain that $b_1 = 0$.

We now take the Fourier transform of (7) with $b_1 = 0$, using [19, equation (6.12)] for the $\mathbb{1}_{\mathbb{R}_+}$ term, to obtain

$$-\mathcal{F}(Dh)(z) = \frac{a_0}{2\pi \mathrm{i} z} - \frac{a_0}{2}\delta(z) + b_0,$$

where the distribution $1/z$ is to be understood in the sense of principal value. In particular, by taking $\phi \in \mathcal{D}$ with $0 \notin \operatorname{supp}\phi$ and using (5),

$$\int F(z)\phi(z)\,\mathrm{d}z = \langle -\mathcal{F}(Dh), \phi\rangle = \int \left(\frac{a_0}{2\pi \mathrm{i} z} + b_0\right)\phi(z)\,\mathrm{d}z.$$

From this, it follows

$$F(z) = \frac{a_0}{2\pi \mathrm{i} z} + b_0, \qquad z \in \mathbb{C} \setminus \{0\}, \tag{8}$$

where we used the identity theorem to extend the equality to $z \notin \mathbb{R}$. Taking reciprocals in the definition (3) of F, we can repeat the proof thus far to obtain

$$\frac{1}{F(z)} = \frac{a_0^*}{2\pi \mathrm{i} z} + b_0^*, \qquad z \in \mathbb{C} \setminus \{0\}. \tag{9}$$

Multiplying (8) and (9) we arrive at

$$1 = \left(\frac{a_0^*}{2\pi \mathrm{i} z} + b_0^*\right)\left(\frac{a_0}{2\pi \mathrm{i} z} + b_0\right), \qquad z \in \mathbb{C} \setminus \{0\}.$$

Clearly, then $a_0 = a_0^* = 0$. Therefore, $F(z) = b_0$ and hence we have shown that $\kappa_+(z) = b_0 \kappa'_+(z)$ and $\kappa'_-(z) = b_0 \kappa_-(z)$ for $z \in \mathbb{C}_u \setminus \{0\}$, and the equation extends to 0 by taking limits, since all functions involved are continuous there. The fact that $b_0 > 0$ follows from the positivity of these functions at $\mathrm{i}x$ with $x > 0$. This completes the proof in the non-lattice case. □

We turn now to the case where X has lattice support, and we will begin by collecting a few facts. Let $\eta > 0$. The support of X_t is contained in $\eta\mathbb{Z}$ for some $t > 0$ if and only if it is contained in $\eta\mathbb{Z}$ for all $t > 0$ [17, Proposition 24.17], and in this case, we say that X has support contained in $\eta\mathbb{Z}$. X has support contained in $\eta\mathbb{Z}$ if and only if $\psi(2\pi/\eta) = 0$ [13, Theorem 2.4], and if this is the case, then, in fact, ψ is $2\pi/\eta$-periodic, so that $\psi(2k\pi/\eta) = 0$ for every $k \in \mathbb{Z}$. Moreover, by considering the Lévy–Itô decomposition, we can see that this holds if and only if X is a compound Poisson process whose (finite) Lévy measure has support contained in $\eta\mathbb{Z}$ [17, Corollary 24.6]. We also recall that a Lévy process X is a compound Poisson process if and only if ψ is bounded on $\mathbb{R}$ [1, Corollary 3].

From now on, and without loss of generality, we assume that X has support that is contained in $\mathbb{Z}$ and in no sublattice thereof; that is, we assume that $\eta = 1$ in the discussion above is maximal.

The key information about the structure of the functions h and $\widetilde{h}$ is captured in the following lemma. When working with lattice support, it will be convenient to use the definition $\overline{\mu}_+(x) = \mu_+(x,\infty)\mathbb{1}_{\{x \geqslant 0\}}$ so that, in particular, $\overline{\mu}_+(0)$ is equal to the total mass of the measure μ_+, which as we will shortly see is finite; we adopt the same convention for the other Lévy measure tails involved. In comparison with the non-lattice case, this does not change the definition of h as a distribution, so we are free to use the results established previously, in particular Lemma 4 and Corollary 5.

Lemma 6. *The tempered distributions h and $\widetilde{h}$ are represented by the functions*

$$h(x) = (\overline{\mu}_+ + q_+ \mathbb{1}_{\mathbb{R}_+}) * U'_+(x), \qquad \widetilde{h}(x) = -(\overline{\mu}'_- + q'_- \mathbb{1}_{\mathbb{R}_+}) * U_-(-x),$$

and their distributional derivatives are given by

$$Dh = \sum_{k \geqslant 0} \big(h(k) - h(k-1)\big)\delta_k, \qquad D\widetilde{h} = \sum_{k \leqslant 0} \big(\widetilde{h}(k) - \widetilde{h}(k+1)\big)\delta_k,$$

which are series converging in $\mathcal{S}'$. Moreover,

$$\lim_{k \to \infty} \big(h(k) - h(k-1)\big) = 0 = \lim_{k \to -\infty} \big(\widetilde{h}(k) - \widetilde{h}(k+1)\big). \tag{10}$$

Proof. Since the Lévy measure Π of X is concentrated on $\mathbb{Z} \setminus \{0\}$, the characteristic exponent ψ satisfies

$$-\psi(z) = \int_{-\infty}^{\infty} (e^{\mathrm{i}zx} - 1)\,\Pi(\mathrm{d}x) = \sum_{k\in\mathbb{Z}} a_k(e^{\mathrm{i}zk} - 1), \tag{11}$$

where $a_k = \Pi(\{k\})$ for $k \in \mathbb{Z} \setminus \{0\}$, $a_0 = 0$ and $A = \Pi(\mathbb{R}) = \sum_{k\in\mathbb{Z}} a_k < \infty$. Equation (11) can be reformulated as

$$-\psi(z) = \mathcal{F}\left(\sum_{k\in\mathbb{Z}} a_k\delta_k - A\delta\right),$$

where δ_x is a Dirac mass at $x \in \mathbb{R}$, $\delta = \delta_0$ and we note that $\sum_{k\in\mathbb{Z}} a_k\delta_k - A\delta \in \mathcal{S}'$.

We may assume that $\kappa_\pm$ are the characteristic exponents of ascending and descending ladder height processes of X (for some choice of normalisation of the local times of X at its supremum and infimum). Since the ladder height processes also live on $\mathbb{Z}$, it follows immediately that $d_\pm = 0$. In addition, $q_+q_- = 0$, since the ladder height processes cannot be killed simultaneously. Hence, we have that

$$\begin{aligned}-\kappa_\pm(z) = -q_\pm + \int_{(0,\infty)} (e^{\mathrm{i}zx} - 1)\,\mu_\pm(\mathrm{d}x) &= \mathcal{F}\left(-q_\pm\delta + \sum_{k\geqslant 1} a_k^\pm\delta_k - A^\pm\delta\right)\\ &= \mathcal{F}\left(-(q_\pm + A^\pm)\delta + \sum_{k\geqslant 1} a_k^\pm\delta_k\right),\end{aligned}$$

where $a_k^\pm = \mu_\pm(\{k\})$ for $k \geqslant 1$, $A^\pm = \sum_{k\geqslant 1} a_k^\pm < \infty$, and $\sum_{k\geqslant 1} a_k^\pm\delta_k - (q_\pm + A^\pm)\delta \in \mathcal{S}'$. The second factorisation of (2) yields that

$$0 = \psi(2\pi) = \kappa'_+(2\pi)\kappa'_-(-2\pi),$$

and therefore, at least one of the two subordinators defined by $\kappa'_\pm$ corresponds to a subordinator without killing, living on $\mathbb{Z}_+ = \mathbb{Z} \cap [0,\infty)$. Without loss of generality, assume that this applies to κ'_+, and denote the potential measure of its corresponding subordinator by $U'_+ = \sum_{k\geqslant 0} u'_k\delta_k$.

From the équations amicales inversées, see [18, Section 5.1], we have $\mu'_-(-\cdot) = \Pi * U'_+$ on $(-\infty, 0)$, which establishes that μ'_- is supported by $\mathbb{Z}_+$ as well. Next, if $d'_- > 0$ were to hold, then we would have $|\kappa'_-(z)| \sim d'_-|z|$ as $|z| \to \infty$. Since κ'_+ is 2π-periodic and is not identically zero by the previous assumption, this contradicts the fact that $\psi(z) = \kappa'_+(z)\kappa'_-(-z)$ is bounded over $z \in \mathbb{R}$. Thus, $d'_- = 0$, and we can therefore write

$$-\kappa'_-(z) = -q'_- + \int_{(0,\infty)} (e^{\mathrm{i}zx} - 1)\,\mu'_-(\mathrm{d}x) = \mathcal{F}\left(\sum_{k\geqslant 1} b_k^-\delta_k - (q'_- + B^-)\delta\right), \tag{12}$$

where $b_k^- = \mu'_-(\{k\})$ for $k \geqslant 1$ and $B^- = \sum_{k\geqslant 1} b_k^- < \infty$.

We have shown that the subordinators pertaining to $\kappa_\pm$ and $\kappa'_\pm$ have lattice support. Letting $U_- = \sum_{k\geqslant 0} u_k \delta_k$, we therefore arrive at

$$h(x) = (\overline{\mu}_+ + q_+ \mathbb{1}_{\mathbb{R}_+}) * U'_+(x) = \sum_{0\leqslant k\leqslant x} u'_k(q_+ + \overline{\mu}_+(x-k))\mathbb{1}_{\{x\geqslant 0\}}, \tag{13}$$

and

$$\widetilde{h}(x) = -(\overline{\mu}'_- + q'_- \mathbb{1}_{\mathbb{R}_+}) * U_-(-x) = -\sum_{0\leqslant k\leqslant -x} u_k\Big(q'_- + \overline{\mu}'_-(-x-k)\Big)\mathbb{1}_{\{x\leqslant 0\}}.$$

This proves the representations of $h, \widetilde{h}$ given in the statement of the result. Lemma 4 and Corollary 5 remain valid for lattice-valued processes, so $h, \widetilde{h} \in \mathcal{S}'$. Next, we observe that since both μ_+, μ'_- are supported on $\mathbb{Z}_+$, the functions $x \mapsto q_+ + \overline{\mu}_+(x)$ and $x \mapsto q'_- + \overline{\mu}'_-(x)$ are constant on $(l, l+1]$ for any $l \in \mathbb{Z}_+$. This shows that the distributional derivatives $Dh, D\widetilde{h}$ are supported by $\mathbb{Z}$. In particular, it is easy to check that

$$Dh = \sum_{k\geqslant 0}(h(k) - h(k-1))\delta_k, \quad D\widetilde{h} = \sum_{k\leqslant 0}(\widetilde{h}(k) - \widetilde{h}(k+1))\delta_k,$$

noting that $h(-1) = 0 = \widetilde{h}(1)$. We observe that the series above are convergent in $\mathcal{S}'$ thanks to Lemma 4 applied to $h, \widetilde{h}$.

Next, we show (10). We furnish the proof only for the limit to $+\infty$, the other case being analogous using (12). Firstly note that if $q'_+ > 0$, the measure U'_+ is finite and hence $h(k) - h(k-1) \to 0$ as $k \to \infty$ is obtained immediately from (13) by the dominated convergence theorem. Assume therefore $q'_+ = 0$. From (13), we get that

$$h(k) - h(k-1) = \sum_{0\leqslant l\leqslant k-1} u'_l\big(\overline{\mu}_+(k-l) - \overline{\mu}_+(k-1-l)\big) + u'_k(q_+ + \overline{\mu}_+(0)), \quad k \geqslant 0.$$

Noting that $\overline{\mu}_+(k-l) - \overline{\mu}_+(k-1-l) = -a^+_{k-l}$, we can simplify this to

$$h(k) - h(k-1) = -\sum_{0\leqslant l\leqslant k-1} u'_l a^+_{k-l} + u'_k(q_+ + \overline{\mu}_+(0)).$$

By the renewal theorem [15, Theorem 5.3.1], it holds that $\lim_{k\to\infty} u'_k = 1/m'_+$, where $m'_+ \in (0, \infty]$ is the expectation of the subordinator (without killing) pertaining to κ'_+ [15, Proposition 5.3.4 and Theorem 5.3.8]. Since $\sum_{l\geqslant 1} a^+_l < \infty$ we get, using the renewal theorem [15, Proposition 5.3.3] that

$$\lim_{k\to\infty}\big(h(k) - h(k-1)\big) = -\frac{1}{m'_+}\sum_{l\geqslant 0} a^+_l + \frac{1}{m'_+}(q_+ + \overline{\mu}_+(0)) = \frac{q_+}{m'_+}.$$

To conclude, we need to verify that, if $q'_+ = 0$ and $m'_+ < \infty$, then $q_+ = 0$. Assume the opposite, that is, that $q'_+ = 0$, $m'_+ < \infty$ and $q_+ > 0$. Then, $\lim_{z\to 0} \kappa'_+(z)/z = -\mathrm{i}m'_+$, so that (2) implies the existence of the limit

$$\mathrm{i}\lim_{z\to 0} \psi(z)/z = m'_+\kappa'_-(0) = m'_+ q'_- \in [0, \infty). \tag{14}$$

Using this and once more (2), $q_+ > 0$ implies that $q_- = 0$ and that $\lim_{z\to 0} \kappa_-(z)/z$ is finite. The latter is therefore equal to $-\mathrm{i}m_-$, where $m_- \in (0,\infty)$ is the expectation of the subordinator (without killing) pertaining to κ_-. We arrive at:

$$\mathrm{i}\lim_{z\to 0}\psi(z)/z = \mathrm{i}q_+\lim_{z\to 0}\kappa_-(-z)/z = -q_+m_- < 0,$$

which contradicts (14). This establishes $q_+ = 0$ in the remaining case, which concludes the proof of (10). □

With this lemma in place, we can tackle the theorem in the case of lattice support.

Proof of Theorem 1 (lattice support). As in the non-lattice case, we represent $\mathcal{F}Dh - \mathcal{F}D\widetilde{h}$ in two different ways.

On the one hand, Lemma 6 directly implies that

$$\mathcal{F}Dh(z) = \sum_{k\geqslant 0}(h(k) - h(k-1))e^{\mathrm{i}zk}, \quad \mathcal{F}D\widetilde{h}(z) = \sum_{k\leqslant 0}(\widetilde{h}(k) - \widetilde{h}(k+1))e^{\mathrm{i}zk}. \tag{15}$$

For brevity, let us write $v_k = h(k) - h(k-1)$ for $k \geqslant 1$, $v_k = \widetilde{h}(k) - \widetilde{h}(k+1)$ for $k \leqslant -1$ and $v_0 = h(0) - \widetilde{h}(0)$. Thus,

$$\mathcal{F}Dh(z) - \mathcal{F}D\widetilde{h}(z) = \sum_{k\in\mathbb{Z}} v_k e^{\mathrm{i}kz}. \tag{16}$$

On the other hand, emulating the argument in the non-lattice case, we see that, for any $\phi \in \mathcal{D}$ whose support is disjoint from $2\pi\mathbb{Z}$,

$$\langle -\mathcal{F}(Dh)(z), \phi(z)\rangle = \int F(z)\phi(z)\,\mathrm{d}z \tag{17}$$

and

$$\langle \mathcal{F}(Dh), \phi\rangle = \langle \mathcal{F}(D\widetilde{h}), \phi\rangle.$$

The latter implies that the support of $\mathcal{F}Dh - \mathcal{F}D\widetilde{h}$ is contained in $2\pi\mathbb{Z}$. Applying [19, §2.6 and §5.2, Corollary 1], we obtain that

$$\mathcal{F}Dh - \mathcal{F}D\widetilde{h} = \sum_{k\in\mathbb{Z}}\sum_{n=0}^{N_k} w_{k,n}D^n\delta_{2\pi k},$$

for some $N_k \geqslant 0$ and weights $w_{k,n}$, and our first task is to show that N_k is bounded in k. To do this, we appeal to [19, §5.4, Corollary]: there exists N such that, for all $\epsilon \in (0, 1/2)$, there exists a function $g \in \mathcal{S}$ supported on $2\pi\mathbb{Z} + (-\epsilon,\epsilon)$ such that

$$\mathcal{F}Dh - \mathcal{F}D\widetilde{h} = D^N g.$$

Suppose now that $N_k > N$ for some k and $w_{k,N_k} \neq 0$. Take some bump function $m \in \mathcal{S}$ with the property that $m(z) = 1$ on $[2\pi k - \epsilon, 2\pi k + \epsilon]$ and $m(z) = 0$ on $\mathbb{R}\setminus[2\pi k - r, 2\pi k + r]$, with $r \in$

$(\epsilon, 1/2)$. Then,

$$(\mathcal{F}Dh - \mathcal{F}D\widetilde{h})m = \sum_{n=0}^{N_k} w_{k,n} D^n \delta_{2\pi k} = (D^N g)m,$$

and, in particular,

$$\sum_{n=N+1}^{N_k} w_{k,n} D^n \delta_{2\pi k} = D^N \widetilde{g},$$

where $\widetilde{g} \in \mathcal{S}$ is supported on $(2\pi k - \epsilon, 2\pi k + \epsilon)$. Taking primitives N_k times, this yields

$$w_{k,N_k} \delta_{2\pi k} = \widetilde{g}_1 + P,$$

where $\widetilde{g}_1$ is the $(N_k - N)$th primitive of $\widetilde{g}$, and P is a polynomial of power at most N_k. Since $w_{k,N_k} \neq 0$, this is a contradiction. Therefore, $N_k \leqslant N$ for every $k \in \mathbb{Z}$, and so, we obtain

$$\mathcal{F}Dh - \mathcal{F}D\widetilde{h} = \sum_{k\in\mathbb{Z}} \sum_{n=0}^{N} w_{k,n} D^n \delta_{2\pi k}, \tag{18}$$

Our goal now is to take our two representations (16) and (18) and use them to show that $Dh - D\widetilde{h} = 0$. To this end, fix $l \in \mathbb{Z}$; we aim to show that $w_{l,n} = 0$ for $0 \leqslant n \leqslant N$. Rearranging (16) and (18) gives us that (as distributions)

$$\sum_{k\in\mathbb{Z}} v_k e^{\mathrm{i}kz} = \sum_{k\in\mathbb{Z}} \sum_{n=0}^{N} w_{k,n} D^n \delta_{2\pi k}(z) = \sum_{n=0}^{N} w_{l,n} D^n \delta_{2\pi l}(z) + \Upsilon(z), \tag{19}$$

where $\Upsilon \in \mathcal{S}'$ and its support does not intersect with $(2\pi(l-1), 2\pi(l+1))$. Moreover, Lemma 6 tells us that

$$\lim_{k\to\pm\infty} v_k = 0. \tag{20}$$

Assume that $w_{l,n} \neq 0$ for some n and choose n_0 maximally among these. Suppose first that n_0 is even. Then, choose $\phi_\sigma(x) = (\sqrt{2\pi}\sigma)^{-1} e^{-(x-2\pi l)^2/2\sigma^2} \in \mathcal{S}$. Taking into account that

$$\mathcal{F}\phi_\sigma(z) = e^{\mathrm{i}z2\pi l} e^{-\frac{\sigma^2 z^2}{2}}, \tag{21}$$

from (19), we derive

$$\sum_{k\in\mathbb{Z}} v_k e^{-\frac{k^2\sigma^2}{2}} = \sum_{k\in\mathbb{Z}} v_k e^{\mathrm{i}k2\pi l} e^{-\frac{k^2\sigma^2}{2}} = \sum_{n=0}^{n_0} (-1)^n w_{l,n} \phi_\sigma^{(n)}(2\pi l) + \langle \Upsilon, \phi_\sigma \rangle.$$

It is easy to check that

$$\phi_\sigma^{(n)}(2\pi l) = \frac{C_n}{\sigma^{n+1}}, \quad |C_n| > 0,$$

when n is even and $\phi_\sigma^{(n)}(l) = 0$ when n is odd. Therefore, we get

$$\sigma^{n_0+1} \sum_{k\in\mathbb{Z}} v_k e^{-\frac{k^2\sigma^2}{2}} = C_{n_0} w_{l,n_0} + \langle \Upsilon, \sigma^{n_0+1}\phi_\sigma \rangle + o(\sigma), \text{ as } \sigma \to 0. \tag{22}$$

Let us show that

$$\lim_{\sigma\to 0} \sigma \sum_{k\in\mathbb{Z}} v_k e^{-\frac{\sigma^2 k^2}{2}} = 0. \tag{23}$$

Fix $\epsilon > 0$ and, by (20), choose $K \geqslant 1$ large enough such that $|v_k| < \epsilon$ when $|k| \geqslant K$. Then,

$$\begin{aligned}\limsup_{\sigma\to 0} \sigma \left| \sum_{k\in\mathbb{Z}} v_k e^{-\frac{\sigma^2 k^2}{2}} \right| &\leqslant \epsilon \limsup_{\sigma\to 0} \sigma \sum_{|k|\geqslant K} e^{-\frac{\sigma^2 k^2}{2}} \\ &\leqslant \epsilon \limsup_{\sigma\to 0} \sigma \int_{-\infty}^{\infty} e^{-\frac{\sigma^2 x^2}{2}} \mathbb{1}_{\{|x|\geqslant K-1\}} \mathrm{d}x \leqslant \sqrt{2\pi}\epsilon.\end{aligned}$$

Given that ϵ is arbitrary, this proves (23). This shows that

$$0 = w_{l,n_0} + \lim_{\sigma\to 0} \langle \Upsilon, \sigma^{n_0+1}\phi_\sigma \rangle. \tag{24}$$

Take an infinitely differentiable function ρ such that $\rho(x) = 1$ for $|x - 2\pi l| > 1/2$ and $\rho(x) = 0$ for $|x - 2\pi l| < 1/4$. Then, from the fact that the support of Υ does not intersect with $(2\pi l - 1, 2\pi l + 1)$, we get that

$$\langle \Upsilon, \sigma^{n_0+1}\phi_\sigma \rangle = \langle \Upsilon, \sigma^{n_0+1}\phi_\sigma \rho \rangle$$

with $\sigma^{n_0+1}\phi_\sigma\rho \in \mathcal{S}$ for any $\sigma > 0$. Clearly, we have that $\lim_{\sigma\to 0} \sigma^{n_0+1}\phi_\sigma\rho = 0$ in $\mathcal{S}$, and we deduce from (24) that $w_{l,n_0} = 0$.

Now, if n_0 is instead odd, we define $\gamma_\sigma(x) = -\sigma^2\phi'_\sigma(x) = (\sqrt{2\pi}\sigma)^{-1}(x - 2\pi l)e^{-(x-2\pi l)^2/2\sigma^2} \in \mathcal{S}$, and from (21), we evaluate

$$\mathcal{F}\gamma_\sigma(z) = \mathrm{i}z\sigma^2\mathcal{F}\phi_\sigma(z) = \mathrm{i}\sigma^2 z e^{\mathrm{i}z2\pi l} e^{-\frac{\sigma^2 z^2}{2}}.$$

Also, we get easily that $\gamma_\sigma^{(n)}(2\pi l) = -\sigma^2\phi_\sigma^{(n+1)}(2\pi l) = -C_{n+1}/\sigma^n$ with $|C_{n+1}| > 0$ if n is odd, and $\gamma_\sigma^{(n)}(2\pi l) = 0$ if n is even. We use these expressions as in (22) to get

$$\sigma^{n_0+2} \sum_k v_k \mathrm{i}k e^{-\frac{k^2\sigma^2}{2}} = C_{n_0+1} w_{l,n_0} + \langle \Upsilon, \sigma^{n_0}\gamma_\sigma \rangle + o(\sigma), \text{ as } \sigma \to 0.$$

Let us check that

$$\lim_{\sigma\to 0} \sigma^2 \sum_{k\in\mathbb{Z}} v_k \mathrm{i}k e^{-\frac{\sigma^2 k^2}{2}} = 0. \tag{25}$$

Fix $\epsilon > 0$, and from (20), choose $K \geqslant 2$ large enough that $|v_k| < \epsilon$ for $k \geqslant K$. By (25), we get that

$$\limsup_{\sigma\to 0} \sigma^2 \left| \sum_{k\in\mathbb{Z}} v_k ike^{-\frac{\sigma^2 k^2}{2}} \right| \leqslant \epsilon \limsup_{\sigma\to 0} \sigma^2 \sum_{|k|\geqslant K} |k| e^{-\frac{\sigma^2 k^2}{2}}$$

$$\leqslant \frac{2K\epsilon}{K-1} \limsup_{\sigma\to 0} \sigma^2 \int_{K-1}^{\infty} xe^{-\frac{\sigma^2 x^2}{2}} \mathrm{d}x \leqslant 4\epsilon \int_0^{\infty} xe^{-\frac{x^2}{2}} \mathrm{d}x.$$

Given that ϵ is arbitrary, this verifies (25). Since necessarily $n_0 \geqslant 1$ and $\gamma_\sigma(2\pi l) = 0$, we obtain, as in the case where n_0 is even, that

$$\lim_{\sigma\to 0} \langle \Upsilon, \sigma^{n_0} \gamma_\sigma \rangle = \lim_{\sigma\to 0} \langle \Upsilon, \sigma^{n_0} \gamma_\sigma \rho \rangle = 0,$$

and thus,

$$\lim_{\sigma\to 0} \sigma^{n_0+2} \sum_{k\in\mathbb{Z}} v_k ike^{-\frac{k^2\sigma^2}{2}} = w_{l,n_0}. \tag{26}$$

From (25) and (26), it holds that $w_{l,n_0} = 0$, and we conclude that $w_{l,n} = 0$ for all $0 \leqslant n \leqslant N$ and $l \in \mathbb{Z}$. So, from (16) and (18),

$$0 = \mathcal{F}Dh(z) - \mathcal{F}D\widetilde{h}(z) = \sum_k v_k e^{\mathrm{i}kz},$$

and we conclude that $v_k = 0$ for all $\mathbb{Z}$. Given the definition of v_k, this says that

$$h(k) = h(k-1), \quad k \geqslant 1 \qquad \text{and} \qquad \widetilde{h}(k) = \widetilde{h}(k+1), \quad k \leqslant -1.$$

Thus, from (15), we arrive at

$$\mathcal{F}Dh = h(0).$$

Finally, (17) implies that F is constant on $\mathbb{C} \setminus 2\pi\mathbb{Z}$. This concludes the proof. □

Finally, we derive the theorem for random walks from the main result.

Proof of Theorem 3. Let $p_{X_1} = \mathbb{P}(X_1 \neq 0) > 0$, and define

$$\psi(z) = 1 - \mathbb{E}[e^{\mathrm{i}zX_1}] = p_{X_1}\big(1 - \mathbb{E}[e^{\mathrm{i}zX_1} \mid X_1 \neq 0]\big),$$

which is the characteristic exponent of a compound Poisson process with rate p_{X_1} and Lévy measure $\mathbb{P}(X_1 \in \cdot \mid X_1 \neq 0)$. Similarly and with analogous notation, let

$$\kappa_+(z) = 1 - \mathbb{E}[e^{\mathrm{i}zH^+}]$$

and

$$\kappa_-(z) = \begin{cases} p_{H^{-,w}}\big(1 - \mathbb{E}[e^{-\mathrm{i}zH^{-,w}} \mid H^{-,w} \neq 0]\big), & p_{H^{-,w}} > 0, \\ \mathbb{P}(H^{-,w} = \Delta), & p_{H^{-,w}} = 0, \end{cases}$$

which are both characteristic exponents of (possibly killed) compound Poisson subordinators; note that $p_{H^{-,w}} = \mathbb{P}(H^{-,w} \neq 0) = \mathbb{P}(H^{-,w} \in (0,\infty) \cup \{\Delta\})$. Using this notation, the equation

$$\psi(z) = \kappa_+(z)\kappa_-(-z)$$

represents the Wiener–Hopf factorisation of a Lévy process. On the other hand, we can define

$$\kappa'_+(z) = 1 - \mathbb{E}[e^{izV^+}],$$

and

$$\kappa'_-(z) = \begin{cases} p_{V^-}\big(1 - \mathbb{E}[e^{-izV^-} \mid V^- \neq 0]\big), & p_{V^-} > 0, \\ \mathbb{P}(V^- = \Delta), & p_{V^-} = 0, \end{cases}$$

which gives us two more characteristic exponents of compound Poisson subordinators, again possibly killed, with the property that

$$\psi(z) = \kappa'_+(z)\kappa'_-(-z).$$

Theorem 1 states that there exists $c > 0$ such that

$$\kappa'_+(z) = c\kappa_+(z), \quad \kappa_-(z) = c\kappa'_-(z). \tag{27}$$

Taking the first of these equalities and rearranging it, we obtain

$$\mathbb{E}[e^{izV^+}] = 1 - c + c\mathbb{E}[e^{izH^+}] = \mathcal{F}\big((1-c)\delta + c\mathbb{P}(H^+ \in \cdot; H^+ \neq \Delta)\big)(z).$$

Since neither V^+ nor H^+ have an atom at zero, it follows that $1 - c = 0$, that is, $c = 1$. In turn, this implies that $H^+ \overset{d}{=} V^+$. Turning now to the second equality in (27), we first see that $p_{H^{-,w}} = 0$ if and only if $p_{V^-} = 0$, in which case $H^{-,w} \overset{d}{=} V^-$. If this is not the case, then we can write

$$\mathbb{E}[e^{izH^{-,w}} \mid H^{-,w} \neq 0] = 1 - \frac{p_{V^-}}{p_{H^{-,w}}} + \frac{p_{V^-}}{p_{H^{-,w}}}\mathbb{E}[e^{izV^-} \mid V^- \neq 0].$$

By the same logic as above, we obtain that $p_{V^-} = p_{H^{-,w}}$ and that V^- conditioned on $\{V^- \neq 0\}$ has the same distribution as $H^{-,w}$ conditioned on $\{H^{-,w} \neq 0\}$. Combining these two observations yields that $H^{-,w} \overset{d}{=} V^-$, which completes the proof. □

ACKNOWLEDGEMENTS

The authors wish to thank Vincent Vigon for fruitful discussions and valuable insights during early stages of the project. LT gratefully acknowledges financial support of Carlsberg Foundation Young Researcher Fellowship grant CF20-0640. M. Savov acknowledges that this study is financed by the European Union - NextGenerationEU, through the National Recovery and Resilience Plan of the Republic of Bulgaria, project No. BG-RRP-2.004-0008. All authors thank the referee for their careful reading and comments.

JOURNAL INFORMATION

The *Bulletin of the London Mathematical Society* is wholly owned and managed by the London Mathematical Society, a not-for-profit Charity registered with the UK Charity Commission. All surplus income from its publishing programme is used to support mathematicians and mathematics research in the form of research grants, conference grants, prizes, initiatives for early career researchers and the promotion of mathematics.

ORCID

Alexander R. Watson https://orcid.org/0000-0003-0421-0091

REFERENCES

1. J. Bertoin, *Lévy processes*, Cambridge Tracts in Mathematics, vol. 121, Cambridge University Press, Cambridge, 1996, pp. x+265. ISBN: 0-521-56243-0.
2. A. A. Borovkov, *Probability theory*, 5th ed., Universitext, Springer, London, 2013, pp. xxviii+733. ISBN: 978-1-4471-5200-2; 978-1-4471-5201-9. DOI: 10.1007/978-1-4471-5201-9
3. L. Chaumont and R. Doney, *On distributions determined by their upward, space-time Wiener-Hopf factor*, J. Theoret. Probab. **33** (2020), no. 2, 1011–1033. ISSN: 0894-9840,1572-9230. DOI: 10.1007/s10959-019-00889-x
4. T. Grzywny and M. Kwaśnicki, *Liouville's theorems for Lévy operators*, arXiv: 2301.08540 [math.AP], 2023.
5. A. Kuznetsov, A. E. Kyprianou, and J. C. Pardo, *Meromorphic Lévy processes and their fluctuation identities*, Ann. Appl. Probab. **22** (2012), no. 3, 1101–1135. ISSN: 1050-5164. DOI: 10.1214/11-AAP787
6. A. Kuznetsov and J. C. Pardo, *Fluctuations of stable processes and exponential functionals of hypergeometric Lévy processes*, Acta Appl. Math. **123** (2013), 113–139. ISSN: 0167-8019. DOI: 10.1007/s10440-012-9718-y
7. A. E. Kuznetsov, *An analytical proof of the Pecherskiĭ-Rogozin identity and the Wiener-Hopf factorization*, Teor. Veroyatn. Primen. **55** (2010), no. 3, 417–431. ISSN: 0040-361X,2305-3151. DOI: 10.1137/S0040585X97984929
8. M. Kwaśnicki, *Random walks are determined by their trace on the positive half-line*, Ann. H. Lebesgue **3** (2020), 1389–1397. ISSN: 2644-9463. DOI: 10.5802/ahl.64
9. A. E. Kyprianou, J. C. Pardo, and A. R. Watson, *The extended hypergeometric class of Lévy processes*, J. Appl. Probab. **51** (2014), no. A, 391–408. ISSN: 0021-9002. DOI: 10.1239/jap/1417528488
10. A. E. Kyprianou, *Fluctuations of Lévy processes with applications*, 2nd ed., Universitext. Introductory lectures, Springer, Heidelberg, 2014, pp. xviii+455. ISBN: 978-3-642-37631-3; 978-3-642-37632-0. DOI: 10.1007/978-3-642-37632-0
11. A. E. Kyprianou, J. C. Pardo, and M. Vidmar, *Double hypergeometric Lévy processes and self-similarity*, J. Appl. Probab. **58** (2021), no. 1, 254–273. ISSN: 0021-9002,1475-6072. DOI: 10.1017/jpr.2020.86
12. A. E. Kyprianou, J. C. Pardo, and A. R. Watson, *Hitting distributions of α-stable processes via path censoring and self-similarity*, Ann. Probab. **42** (2014), no. 1, 398–430. ISSN: 0091-1798,2168-894X. DOI: 10.1214/12-AOP790
13. E. Lukacs, *Characteristic functions*, 2nd ed., Hafner Publishing Co., New York, 1970, pp. x+350.
14. P. Patie and M. Savov, *Bernstein-gamma functions and exponential functionals of Lévy processes*, Electron. J. Probab. **23** (2018), Paper No. 75, 101. DOI: 10.1214/18-EJP202
15. D. Revuz, *Markov chains*, 2nd ed., North-Holland Mathematical Library, vol. 11, North-Holland Publishing Co., Amsterdam, 1984, pp. xi+374. ISBN: 0-444-86400-8.
16. B. A. Rogozin, *On distributions of functionals related to boundary problems for processes with independent increments*, Theory Probab. Appl. **11** (1966), no. 4, 580–591.
17. K. Sato, *Lévy processes and infinitely divisible distributions*, 2nd ed., Cambridge Studies in Advanced Mathematics, vol. 68, Cambridge University Press, Cambridge, 2013, pp. xiv+521. ISBN: 978-1-107-65649-9.
18. V. Vigon, *Simplifiez vos Lévy en titillant la factorisation de Wiener–Hopf*, Ph.D. thesis, INSA de Rouen, 2002.
19. V. S. Vladimirov, *Generalized functions in mathematical physics*, Translated from the second Russian edition by G. Yankovskiĭ, "Mir", Moscow, 1979, pp. xii+362.