Discourses of mathematical reasoning: analysis of three reform frameworks

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Curricula in many countries include mathematical reasoning as an aim, a competence or proficiency that students should acquire. This inclusion has been supported by wide dissemination of frameworks advocating reform that have arisen from the research community. We present the first part of a project aiming to investigate how ideas about reasoning originating in these frameworks are recontextualised in curricula, textbooks and classrooms. We analyse discourses about reasoning in three such frameworks, identifying how each characterises the nature of mathematical reasoning and the ways students are expected to relate to it. We also examine the extent to which reasoning is construed as a goal of mathematics education or as a means to achieving other goals. In this paper, we explain the methods used for analysing reasoning discourse and identify key findings from the analysis.

Keywords: mathematical reasoning; discourse analysis; curriculum reform frameworks

Introduction

Mathematical reasoning is widely recognised to be important within mathematics education and features within many curricula worldwide. Yet reviews of its use in research (Hjelte et al., 2020), curriculum documents (Reid, 2022) and among teachers (Herbert et al., 2015) suggest that there are diverse meanings ascribed to the term. Moreover, while developing competence in mathematical reasoning may be seen as a goal in itself, it also has a role as a means to developing other aspects of mathematical knowledge. Our project *The dual role of process goals when implementing the written curriculum in Sweden and England* seeks to understand how this dual role as goal and means is manifested in curricula and to track how discourses about mathematical reasoning are recontextualised as they move through the curriculum chain from policy into practice. By comparing the curriculum chains in Sweden and in England, we hope to contribute to a general understanding of processes of recontextualisation. In this paper we present the first part of the project in which we have analysed key documents arising from the mathematics education research community that have played significant roles in influencing curriculum reforms in many countries. This analysis will provide a starting point for investigating curriculum documents, textbooks and classroom practices.

Background

Reviews of the literature have attempted to characterise the range of conceptualisations of mathematical reasoning used by researchers. Hjelte et al. (2020) focus on the definitions and theories used in empirical research, while noting that almost 20% of articles surveyed were not explicit about the meaning of mathematical reasoning. They distinguish between domain-general and domain-specific definitions. Domain-general definitions are characterised as focusing on giving reasons for a mathematical standpoint, solution, or conclusion, regardless of the specific topic. Domain-specific definitions describe mathematical reasoning as approaches to mathematical tasks in a specific mathematical domain, for example, algebraic or proportional reasoning. Such definitions include making sense of specific content.

Also based on a review of research literature, Jeannotte and Kieran (2017) synthesise the different meanings for mathematical reasoning to create a model that organises converging features of these meanings into a coherent frame. In Hjelte et al.'s (2020) terms, this model involves only domain-general types of definition. Jeannotte and Kieran (2017) define mathematical reasoning in general as "a process of communication with others or with oneself that allows for inferring mathematical utterances from other mathematical utterances" (p. 7). Their model distinguishes two broad types of mathematical reasoning, focusing respectively on searching for similarities and differences (including processes of generalising, conjecturing, identifying patterns, comparing and classifying) and validating (including validating, justifying, proving, and formal proving). Having used Jeannotte and Kieran's model while analysing curriculum documents, Reid (2022) identifies a need to add explaining as a further process.

In an empirical study of primary school teachers' perceptions of mathematical reasoning, Herbert et al. (2015) also identify a variety of meanings, categorised as: thinking, communicating thinking, problem solving, validating thinking, conjecturing, validating conjectures, and connecting aspects of mathematics. The inclusion of thinking and communicating thinking does not match with Hjelte et al.'s (2020) characterisation of domain-general reasoning as giving reasons nor Jeannotte and Kieran's (2017) definition of reasoning as involving inferring. It might, however, be seen as similar to the broader idea of making sense, included in Hjelte et al.'s characterisation of domain-specific reasoning.

While previous research has tended to focus within a single field (e.g., research literature, curriculum or classrooms) our study aims to track the evolution of discourses about reasoning as they move from the research field into curriculum, teaching resources and classroom practice. As well as using discourse analytic techniques to capture more detailed nuances of how reasoning is conceptualised, we will address the question of whether reasoning is seen as a curriculum goal in its own right or as a means to achieving other goals.

Methodology

In this first stage of the project, we identified three sources, arising from the mathematics education research community, that have provided frameworks for curriculum reform. Initially developed in the USA (*Adding it up* (AiU), (Kilpatrick et al., 2001); *Principles and Standards* (NCTM), (National Council of Teachers of Mathematics, 2000)) and Denmark *(Competences and mathematics learning* (KOM), (Niss & Højgaard, 2011)), these have had significant influence on curriculum development internationally. Our aim is to characterise discourses about mathematical reasoning within each of these sources, addressing three main questions:

- How is reasoning characterised?
- How are students expected to engage with reasoning?
- Is reasoning construed as a goal in itself and/ or as a means to other curricular goals?

It is important to note that the three sources differ in how they position mathematical reasoning in relation to other aspects of the curriculum. While KOM seeks to define the goal of mathematics education as the development of competences such as reasoning, that are “developed and practiced through the use of the content areas” (Niss & Højgaard, 2011, p. 30), AiU characterises its goal of developing mathematical proficiency as a set of “interwoven and interdependent” strands (Kilpatrick et al., 2001, p. 5), inextricably linking reasoning to the development of content aims and problem solving. Meanwhile, the NCTM standards define content and process dimensions separately. These differences may help to explain differences in the ways reasoning itself is construed in the three sources.

In order to construct a dataset, the text of each source document was searched for passages including the word *reasoning* or its cognates. Those passages not involving *mathematical* reasoning were omitted from the analysis. The analytic methods and key findings for each of the three main questions are presented in the following sections.

How is reasoning characterised?

Characterising reasoning involves both identifying what type of object or process is involved in reasoning and also identifying any properties that are ascribed to it. There are thus two sub-questions addressed in this section:

- What processes or objects are identified as types of reasoning?
- What properties are associated with reasoning?

Within the passages in the data set, each statement was coded with the process or object identified as a type of reasoning or as the product of reasoning. Where the type of reasoning was qualified by adjectives or adverbs (or adjectival/ adverbial phrases), these terms were added as property codes. The codes consisted of the terms actually used in the source or cognate terms. An example the coding is shown in Table 1.

Table 1: Examples of coding processes/ objects and properties

statement	process/object codes	property codes
On the other hand, it consists of the ability to devise and carry out informal and formal arguments (on the basis of intuition) and hereby transform heuristic reasoning to actual (valid) proofs. (KOM)	argument heuristic reasoning proof	informal formal valid

A total of 39 distinct process/ object codes were generated. However, only five of these types of reasoning were common to all three sources: *argument*, *(logical) chain*, *explanation*, *justification* and *proof*. The types of reasoning occurring most frequently are shown in Table 2, allowing us to observe some similarities and differences between the discourses of the three sources:

- All three sources include multiple references to *justification* and *proof* but with differing degrees of emphasis. While AiU prioritises *justification*, KOM prioritises *proof*.
- *Conjecture* has a strong presence in NCTM but is absent from other sources.
- In terms of Jeannotte and Kieran’s (2017) categorisation of types of reasoning found in the mathematics education research literature, NCTM and AiU involve both Type 1 (Search for similarities and differences) and Type 2 (Validating), whereas KOM includes only Type 2.

- The presence of *explanation* in all three sources resonates with Reid's (2022) addition to Jeannotte & Kieran's model.

Table 2: Types of reasoning occurring at least three times in one or more source

	AiU	**NCTM**	**KOM**
argument	1	17	9
(logical) chain	6	5	4
conjecture		23	
explanation	7	5	1
generalisation	4		
justification	16	11	5
pattern	2	3	
procedure	3		2
proof	4	13	11
refutation	1	4	
relationships	5		
strategies	3	1	
thinking	5		

Table 3 shows the most frequently occurring properties of reasoning. All three sources include both *formal* and *informal* reasoning. The properties *logical* and *deductive* occur frequently in AiU and NCTM but are surprisingly absent from KOM; it may be that these properties are assumed implicitly in KOM's emphasis on proof. NCTM notes that reasoning occurs in both *real world* and *symbolic* contexts; this may relate to its emphasis on conjecturing and recognising patterns. Interestingly, whereas NCTM highlights being *systematic* and *rigorous* – properties of the **process** of constructing an argument or proof, KOM highlights being *correct* and *valid* – properties of the **product** of reasoning.

Table 3: Properties of reasoning occurring at least three times in one or more source

	AiU	NCTM	KOM
deductive	7	4	
informal	4	4	2
logical	6	3	
formal	2	3	2
by counterexample	1	3	2
intuitive	1		5
correct	1		3
real world		3	
rigorous		3	
symbolic		3	
(un)systematic		3	
using knowledge		3	
using properties	3		
valid			3

How are students expected to engage with reasoning?

In considering how students are expected to engage with reasoning we draw on Bernstein's (2000) theory of pedagogic discourse. To be successful in acquiring a specialised discourse such as mathematics, a student needs to acquire both recognition rules, knowing how to distinguish between what is and what is not to be considered mathematical, and realisation rules, knowing how to produce legitimate mathematical texts. Bernstein also argues that acquisition is more accessible if a student is aware of the evaluation criteria for legitimating mathematical text. We are thus interested in whether student engagement with reasoning is in the form of *recognition*, *realisation* or *evaluation*. Statements in our dataset that involved student activity were coded as shown in Table 4. Whereas statements about engagement in KOM are balanced between the three forms (15 recognition, 13 realisation, 11 evaluation), production of communicable reasoning objects (i.e. realisation) is prioritised in both AiU (35 of 62 statements) and NCTM (50 of 84 statements). Closer examination also revealed differences in expectations about student engagement in evaluation: AiU includes only evaluation of the correctness of procedures and strategies; NCTM focuses mainly on the investigation and evaluation of conjectures; KOM expects students to evaluate the validity of arguments and proofs.

Table 4: Operationalisation of student engagement in reasoning

form of engagement	**operational definition**	**example**
recognition	processes of observing, understanding	Questions such as "Why do you think this is true?" and "Does anyone think the answer is different, and why do you think so?" help students see that statements need to be supported or refuted by evidence (NCTM)
realisation	constructive processes by which the student produces something that is communicable to others	Students need to be able to justify and explain ideas in order to make their reasoning clear (AiU)
evaluation	processes of judging, investigating, comparing	Here it is, among other things, an important task for the teacher to help the students understand and take a stance about when a proof suggestion is correct and complete according to the given criteria. (KOM)

Is reasoning construed as a goal and/ or as a means?

The dataset was further examined to distinguish between statements construing reasoning as a **goal** of mathematics education and statements connecting mathematical reasoning to another outcome, that is, construing reasoning as a **means** to something else. Codes were developed to describe the outcomes of reasoning and these were consolidated into four categories:

- affect, including appreciation of the nature of mathematics;
- learning; developing new ideas; sense making;
- problem solving: successful action in mathematics or in the world;
- communication.

It is notable that KOM contained only one instance of reasoning construed as a means; this is consistent with the overall project of KOM to define the aim of mathematics education in terms of acquisition of competences. In contrast, AiU contained 20 statements construing reasoning as a means to learning or developing understanding of mathematical content and 12 statements construing reasoning as a support for problem solving. Again, this is consistent with AiU’s overall model of mathematical proficiency as consisting of interwoven strands, relating reasoning to conceptual understanding, procedural fluency and strategic competence as both goal and means. NCTM was distinctive in ascribing value to reasoning for its general role as essential to appreciating the nature of mathematics.

Next steps

By analysing three mathematics education reform frameworks, we have identified and characterised distinct discourses of reasoning. These characterisations provide us with analytical tools to investigate how reasoning is construed in texts originating at other points in the curriculum chain: in official curriculum documents, in textbooks and other curricular resources, and in teachers' practices. We seek to understand how the discourses of curriculum, resources and teaching select from, supplement and transform the discourses originating in the mathematics education research community. The major challenge we are currently addressing is to develop principles for analysing teaching materials and classroom teaching. While we have means of distinguishing ways of talking about reasoning, we wish also to distinguish how the discourses may be operationalised in explanations, examples and tasks presented to students by textbooks and teachers.

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