SIAM J. Discrete Math.
Vol. 38, No. 2, pp. 1239–1249

THE RAINBOW SATURATION NUMBER IS LINEAR*

NATALIE BEHAGUE†, TOM JOHNSTON‡, SHOHAM LETZTER§,
NATASHA MORRISON¶, AND SHANNON OGDEN¶

Abstract. Given a graph H, we say that an edge-colored graph G is H-rainbow saturated if it does not contain a rainbow copy of H, but the addition of any nonedge in any color creates a rainbow copy of H. The rainbow saturation number $\mathrm{rsat}(n, H)$ is the minimum number of edges among all H-rainbow saturated edge-colored graphs on n vertices. We prove that for any nonempty graph H, the rainbow saturation number is linear in n, thus proving a conjecture of Girão, Lewis, and Popielarz. In addition, we give an improved upper bound on the rainbow saturation number of the complete graph, disproving a second conjecture of Girão, Lewis, and Popielarz.

Key words. saturation, rainbow, edge-coloring

MSC code. 05C35

DOI. 10.1137/23M1566881

1. Introduction.

For a fixed graph H, we say that a graph G is *H-saturated* if it does not contain H as a subgraph, but adding any extra edge creates a copy of H as a subgraph. The *saturation number* of H, denoted by $\mathrm{sat}(n, H)$, is the minimum number of edges in an H-saturated graph G on n vertices. This can be thought of as a dual to the classical Turán extremal number $\mathrm{ex}(n, H)$ that counts the maximum number of edges in an H-free graph on n vertices. Since a maximal H-free graph is H-saturated, $\mathrm{ex}(n, H)$ equivalently counts the maximum number of edges in an H-saturated graph on n vertices. The saturation number in contrast counts the minimum number of edges among such graphs.

The saturation number was first studied independently by Zykov [18] and Erdős, Hajnal, and Moon [5], who proved that $\mathrm{sat}(n, K_r) = (r-2)(n-1) - \binom{r-2}{2}$, where as usual K_r denotes the complete r-vertex graph. Later, Kászonyi and Tuza [10] proved that the saturation number of every graph H is linear in n, that is, $\mathrm{sat}(n, H) = O_H(n)$. For more information and many other results related to the saturation number see the survey of Faudree, Faudree, and Schmitt [6]. In this paper we provide analogous results to these for the rainbow saturation number: a variant of the saturation number for edge-colored graphs.

The generalization of saturation to edge-colored graphs was first considered by Hanson and Toft [9]. Following this, Barrus et al. [2] considered the particular case

*Received by the editors April 19, 2023; accepted for publication (in revised form) December 21, 2023; published electronically April 8, 2024.
https://doi.org/10.1137/23M1566881
Funding: The first author's research was supported by a PIMS postdoctoral fellowship. The third author's research was supported by the Royal Society. The fourth author's research was supported by NSERC Discovery Grant RGPIN-2021-02511 and NSERC Early Career Supplement DGECR-2021-00047. The fifth author was supported by NSERC CGS M.
†Mathematics Institute, University of Warwick, Coventry, CV4 7AL, UK (natalie.behague@warwick.ac.uk).
‡School of Mathematics, University of Bristol, Bristol, BS8 1UG, UK, and Heilbronn Institute for Mathematical Research, Bristol, BS8 1UG, UK (tom.johnston@bristol.ac.uk).
§Department of Mathematics, University College London, Gower Street, London, WC1E 6BT, UK (s.letzter@ucl.ac.uk).
¶Department of Mathematics and Statistics, University of Victoria, Victoria V8W 2Y2, BC, Canada (nmorrison@uvic.ca, sogden@uvic.ca).

of t-rainbow saturation, where there are exactly t colors available. Girão, Lewis, and Popielarz [8] initiated the study of the natural generalization to the case where the palette of colors available is unlimited, rather than bounded by t. This is the focus of our paper.

A *t-edge-colored* graph is an ordered pair (G,c), where c is a (not necessarily proper) edge-coloring of the graph G using colors from $[t]=\{1,2,\dots,t\}$. An edge-coloring of a graph is said to be *rainbow* if every edge is assigned a distinct color. We say that a t-edge-colored graph (G,c) is *(H,t)-rainbow saturated* if (G,c) does not contain a rainbow copy of H, but the addition of any nonedge in any color from $[t]$ creates a rainbow copy of H in G. Note that this requires $t \geq |E(H)|$. The *t-rainbow saturation number* of H, denoted by $\mathrm{rsat}_t(n,H)$, is the minimum number of edges in an (H,t)-rainbow-saturated graph on n vertices.

When the number of possible edge colors is infinite (say, the set of colors is $\mathbb{N}$), an edge-colored graph (G,c) is *H-rainbow saturated* if (G,c) does not contain a rainbow copy of H, but the addition of any nonedge in any color from $\mathbb{N}$ creates a rainbow copy of H. The *rainbow saturation number* of H, denoted by $\mathrm{rsat}(n,H)$, is the minimum number of edges in an H-rainbow saturated graph on n vertices.

In [8], Girão, Lewis, and Popielarz conjectured that, like ordinary saturation numbers, the rainbow saturation number of any nonempty graph H is at most linear in n, and they proved this for graphs with some particular properties.

THEOREM 1.1 (Girão, Lewis, and Popielarz; Theorem 2 (4) in [8]). *Let H be a graph with a nonpendant edge that is not contained in a triangle. Then* $\mathrm{rsat}(n,H)=O(n)$.

In fact, Theorem 1.1 is a consequence of a stronger result that proves $\mathrm{rsat}_t(n,H)=O(n)$ for any such H and any $t\geq e(H)$.

Our main result proves that the rainbow saturation number of every graph is linear, thus confirming the conjecture of Girão, Lewis, and Popielarz [8].

THEOREM 1.2. *Every nonempty graph H satisfies* $\mathrm{rsat}(n,H)=O(n)$.

Interestingly, when t is fixed the t-rainbow saturation number is not in general linear in n. Barrus et al. [2] proved in particular that for all $t\geq |E(H)|$, $\mathrm{rsat}_t(n,H)\geq c_1\frac{n\log n}{\log\log n}$ whenever H is a complete graph, and $\mathrm{rsat}_t(n,H)\geq c_2 n^2$ if H is a star, where c_1 and c_2 are constants depending only on t and the size of H.

The lower bound on the t-rainbow saturation number for complete graphs was improved independently by Ferrara et al. [7], by Korándi [11], and by Girão, Lewis, and Popielarz [8] to $\mathrm{rsat}_t(n,K_r)\geq c_1 n\log n$, which is tight up to a constant factor.

Girão, Lewis, and Popielarz also conjectured that for any complete graph K_r where $r\geq 3$, the rainbow saturation number satisfies $\mathrm{rsat}(n,K_r)=2(r-2)n+O(1)$ when $n\geq 2(r-2)$. We prove that $\mathrm{rsat}(n,K_r)$ is in fact significantly less than this conjectured value.

THEOREM 1.3. *Let $r\geq 3$. Then* $\mathrm{rsat}(n,K_r)\leq (r+2\sqrt{2r})n+c_r$*, where c_r is a constant depending only on r.*

In section 2, we establish some useful basic results regarding rainbow saturation, including Propositions 2.2 and 2.3, which allow us to assume that any counterexample to Theorem 1.2 is connected and has no pendant edges. We describe the construction used to prove Theorem 1.2 in section 3: given a graph H, we construct an edge-colored graph on n vertices with $O(n)$ edges such that the addition of (almost) any nonedge creates a rainbow copy of H. In section 4, we show that if e is an edge contained in a triangle, then the graph produced by this construction does not contain a rainbow copy of H. Together with Theorem 1.1, this completes the proof of Theorem 1.2. Finally, we prove Theorem 1.3 in section 5.

2. Preliminaries.

In this section we provide preliminary results that deal with some easy classes of graphs. We begin by establishing that the existence of an edge-colored graph G on n vertices with $O(n)$ edges that is "almost" H-rainbow-saturated (save for at most linearly many problematic nonedges) is enough to prove that $\mathrm{rsat}(n,H)=O(n)$. For a color c, we say that a nonedge $e\in E(\overline{G})$ is *c-bad* for G if the addition of e in color c to G does not create a rainbow copy of H. We say that a nonedge is *bad* if there exists a color c such that e is c-bad for G.

PROPOSITION 2.1. *Let H be a graph, and let G be an n-vertex edge-colored graph. Suppose that G has m bad nonedges and does not contain a rainbow copy of H. Then* $\mathrm{rsat}(n,H)\le |E(G)|+m$.

Proof. Let $\{e_1,\dots,e_m\}$ be the set of bad nonedges of G. We construct an n-vertex graph G_m and coloring χ of $E(G_m)$ as follows: set $G_0:=G$, where each edge of G_0 is colored as it is in G. Consider each $i\in[m]$ in turn. If there is a color c such that e_i is c-bad for G_{i-1}, then we define G_i to be $G_{i-1}\cup\{e_i\}$, set $\chi(e_i)=c$ (that is, e_i receives color c), and keep the colors of the other edges as in G_{i-1}. Note that there may be multiple such c, in which case we pick one arbitrarily. Otherwise, if e_i is not bad for G_{i-1}, set $G_i:=G_{i-1}$. Observe that G_m is H-rainbow-saturated, and since we added at most m edges during this process, G_m has at most $|E(G)|+m$ edges. Hence, $\mathrm{rsat}(n,H)\le|E(G)|+m$ as required. ∎

A direct result of Proposition 2.1 shows that in order to prove Theorem 1.2 it suffices to consider connected graphs H.

PROPOSITION 2.2. *Let H be a disconnected graph and let H' be a component of H that has the most vertices and, subject to this, the most edges. If* $\mathrm{rsat}(n,H')=O(n)$, *then* $\mathrm{rsat}(n,H)=O(n)$.

Proof. Let s be the number of components of H which are isomorphic to H'. As $\mathrm{rsat}(n,H')=O(n)$, we have $\mathrm{rsat}(n-m,H')=O(n)$, where m is given by $m:=2\left(|V(H)|-|V(H')|\right)-(s-1)$. Let (G,χ) be an H'-rainbow-saturated graph on $n-m$ vertices with as few edges as possible.

Define a graph G^* on n vertices as follows. Let H_2 be obtained from H' by gluing two disjoint copies of H' together at an arbitrary vertex. Let G^* be the disjoint union of G with $s-1$ disjoint copies of H_2 and two disjoint copies of each component of H that is not isomorphic to H'. See Figure 1 for an example. Let χ^* be an edge-coloring of $E(G^*)$ where $\chi^*(e)=\chi(e)$ for every edge $e\in G$, and $G^*\setminus V(G)$ is rainbow with colors not used in χ.

Observe that G^* contains no rainbow copy of H (by our choice of H', there are at most $s-1$ disjoint copies of H' in $G^*\setminus G$ and G contains no rainbow copy of H'). Moreover, the addition of any nonedge (in any color) within G creates a rainbow copy of H in G^*, since $G^*\setminus V(G)$ contains a rainbow copy of $H\setminus V(H')$ avoiding any given color. As there are at most $m(n-m)+\binom{m}{2}=O(n)$ other nonedges in G^*, by Proposition 2.1 we have $\mathrm{rsat}(n,H)=O(n)$, as required. ∎

Therefore, in what follows we may (and will) assume that H is a connected graph. Our next result shows that if H contains a vertex of degree 1, then $\mathrm{rsat}(n,H)=O(n)$.

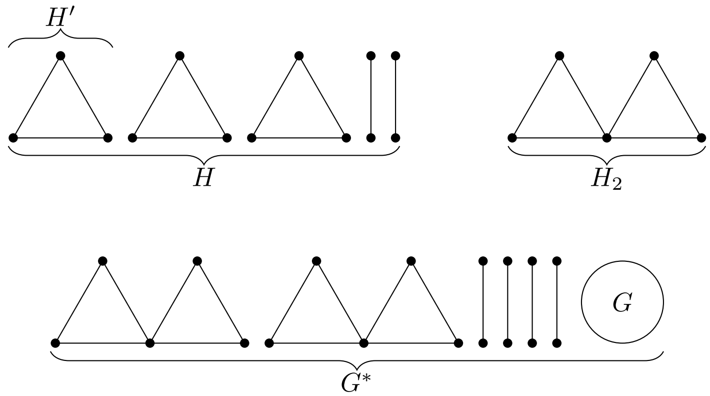

FIG. 1. *An example of the graph G^* created via the construction described in Proposition* 2.2. *Here $s = 3$.*

PROPOSITION 2.3. *Let H be a connected graph with $\delta(H) = 1$. Then* $\mathrm{rsat}(n, H) = O(n)$.

Proof. Write $k = |V(H)|$ and $n = q(k-1)+r$, where $0 \le r < k-1$. Define an edge-colored graph G on n vertices as follows: take q disjoint copies of K_{k-1} and let χ be a rainbow coloring of them. Finally, add r isolated vertices. Note that G has n vertices and $q\binom{k-1}{2} = (n-r)(\frac{k-2}{2}) = O(n)$ edges. Clearly G does not contain a rainbow copy of H. Moreover, since $\delta(H) = 1$, if any nonedge in any color is added between two distinct copies of K_{k-1} in G, then a rainbow copy of H is created. Therefore, since G has at most $q(k-1)r + \binom{r}{2} = (n-r)r + \binom{r}{2} = O(n)$ other nonedges, Proposition 2.1 implies that $\mathrm{rsat}(n, H) = O(n)$. ∎

3. The construction. In this section, we give the construction that lies at the heart of our proof of Theorem 1.2. Given a graph H and an edge $e \in E(H)$, we create an edge-colored graph on n vertices with $O(n)$ edges such that the addition of (almost) any edge creates a rainbow copy of H. The main work in the proof of Theorem 1.2 will be to show that this graph does not contain a rainbow copy of H, which will be done in section 4.

Construction 3.1. Let H be a connected graph on at least three vertices, and let $xy \in E(H)$. Write $S := N_H(x) \cap N_H(y)$ and let T be the set of edges in H with one endpoint in S and the other in $\{x, y\}$. Define H' to be the graph obtained from H by contracting the edge xy and replacing any multiedges by single edges. We label the vertices of H' as in H, with the single vertex obtained from contracting xy labeled by $x|y$. Let T' be the set of edges in H' between $x|y$ and S, and write $H'' := H' \setminus T'$.

For an integer $m \ge 2$, define $F = F(m)$ to be the graph obtained from H' by replacing the vertex $x|y$ with m duplicates of itself, denoted $v_1, v_2, \dots, v_m$. Write $M := \{v_1, v_2, \dots, v_m\}$ and label the vertices in M by $x|y$. Label the remaining vertices as in H. This means F has m vertices labeled $x|y$ and one vertex labeled v for every $v \in V(H) \setminus \{x, y\}$.

Given any integers $m \ge 2$ and $r \ge \max\{|E(H'')| + 1, 2\}$, define the graph $G := G^H_{xy}(m, r)$ as follows. Start with $|E(H'')|$ disjoint copies of $F(m)$, indexed by the edges in H'' as $F_{e_1}, \dots, F_{e_{|E(H'')|}}$. To these we add $r - |E(H'')|$ copies of $F(m)$, which we index as $F_1, \dots, F_{r-|E(H'')|}$. Finally, we obtain G by identifying the vertices corresponding to v_i for each i in turn. That is, for each $i = 1, \dots, m$, we replace the

r vertices $u_1, \ldots, u_r$ corresponding to v_i with one vertex and connect this vertex to the vertices in $\bigcup_{j=1}^{r} N(u_j)$. Note that the graph G has r copies of each vertex in $V(H') \setminus \{x|y\}$ and m copies of the vertex $x|y$, and that the labeling of the vertices naturally defines a labeling of the edges, where an edge in G whose vertices are labeled u and v is labeled uv. By construction, uv is an edge in H'.

We now define a coloring χ of G, which we will later show does not contain a rainbow copy of H. Let χ_0 be a rainbow coloring of H'' and fix a color that is not in χ_0, say, black. We extend χ_0 to a coloring of H' by coloring the edges in T' with a color $\star$. Let an edge uv in G be colored by $\chi_0(e)$, where e is the color of the edge in H' matching the label of uv. For each edge e in H'', recolor the edge corresponding to e in F_e with the color black, and finally recolor each edge colored with $\star$ with unique colors that are not used elsewhere in G.

Construction 3.1 was inspired by the construction used in [8] to prove Theorem 1.1. In fact, when the edge xy is not contained in a triangle, $N_H(x) \cap N_H(y) = \emptyset$, and the graph described in Construction 3.1 is equivalent to the construction in [8].

See Figure 2 for an example of how Construction 3.1 works. In what follows, we will use $G_{xy}^H(n)$ to refer to the edge-colored graph $(G_{xy}^H(m,r), \chi)$, where $m = n - r(|V(H)| - 2)$ and $r = \max\{|E(H'')| + 1, 2\}$; so $G_{xy}^H(n)$ has n vertices. Observe that the addition of any nonedge in any color within M creates a rainbow copy of H.

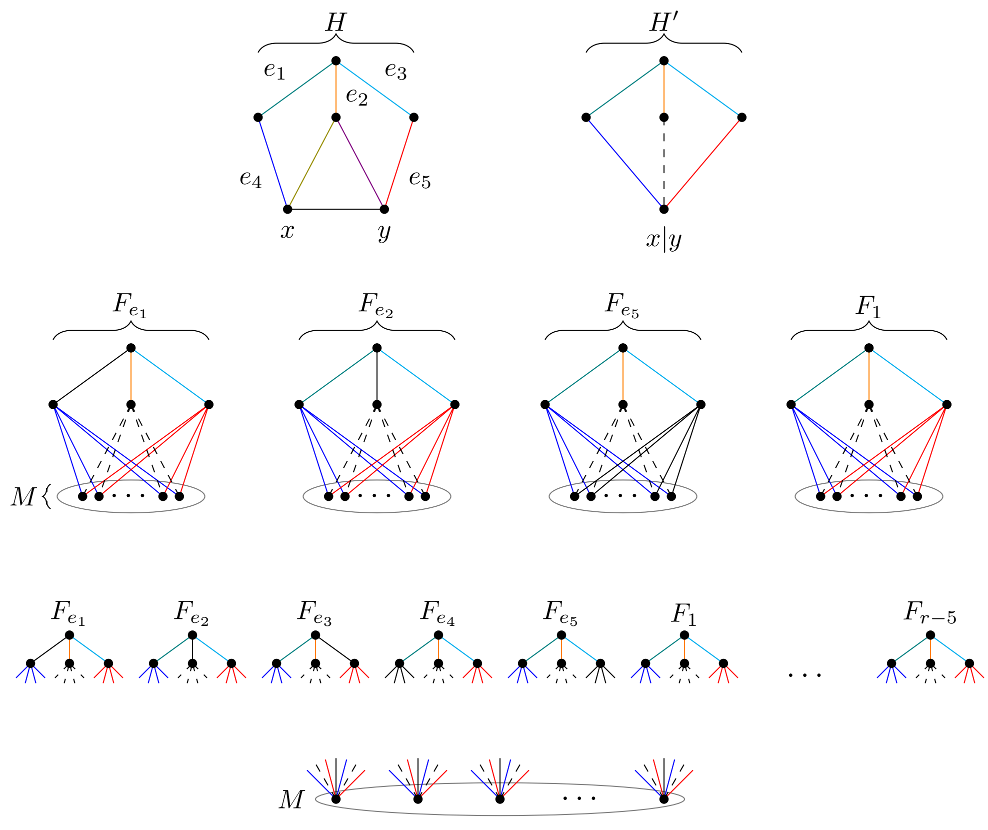

FIG. 2. *An example of the construction of the graph $G_{xy}^H(m,r)$. The dashed edge represents edges which are originally colored $\star$ and will be colored with unique colors by χ. The color black can be seen replacing other colors according to the index of the copy of F.*

Since $r = O(1)$ and $m = n - O(1)$, there are at most $O(n)$ bad nonedges in $G_{xy}^H(n)$. Therefore, if $G_{xy}^H(n)$ contains no rainbow copy of H, then Proposition 2.1 will yield $\mathrm{rsat}(n,H) = O(n)$, as required.

4. Completing the proof of Theorem 1.2.

The goal of this section is to show that G_{xy}^H contains no rainbow copy of H.

PROPOSITION 4.1. *Let H be a nonempty connected graph. For any edge $xy \in E(H)$ that is contained in a triangle and all integers m and r for which it is defined, the graph $G_{xy}^H(m,r)$ does not contain a rainbow copy of H.*

Before proving this, we show why it implies Theorem 1.2 (restated here for convenience).

THEOREM 1.2. *Every nonempty graph H satisfies* $\mathrm{rsat}(n,H) = O(n)$.

Proof. First, note that by Proposition 2.2, it suffices to prove that every connected graph H satisfies $\mathrm{rsat}(n,H) = O(n)$. Let H be a nonempty connected graph. If H has a triangle, let xy be an edge contained in a triangle, and write $G := G_{xy}^H(n)$. Then $|E(G)| = O(n)$, and G has no rainbow copies of H by Proposition 4.1. Moreover, as described at the end of section 3, for all but at most $O(n)$ nonedges e, the addition of e in any color creates a rainbow copy of H. In particular, $\mathrm{rsat}(n,H) = O(n)$, as required. Otherwise, if H is triangle-free, then $\mathrm{rsat}(n,H) = O(n)$, using either Proposition 2.3 if H has a pendant edge or Theorem 1.1 if H has a nonpendant edge. ∎

Proof of Proposition 4.1. Suppose for contradiction that there exists a connected graph H and an edge $xy \in E(H)$ contained in a triangle such that $G_{xy}^H(m,r)$ contains a rainbow copy $H_{\mathfrak{R}}$ of H for some integers m, r with $m \geq 2$ and $r \geq \max\{|E(H'')| + 1, 2\}$. Take H to have the minimal number of vertices with respect to these properties, and fix appropriate m and r such that $G := G_{xy}^H(m,r)$ has a rainbow copy of H. Recall that the vertices of G are labeled by vertices in H' (the graph obtained from H by contracting xy), M is the set of vertices labeled $x|y$, and the edges of G are labeled by edges in H'.

Since $H_{\mathfrak{R}}$ is connected it must contain at least one vertex in M, else it would be contained entirely within a copy of $H \setminus \{x,y\}$, which is impossible as $|V(H_{\mathfrak{R}})| = |V(H)|$.

CLAIM 4.2. *For every $v \in V(H) \setminus \{x,y\}$, the rainbow copy $H_{\mathfrak{R}}$ uses at least one vertex labeled v.*

Proof of claim. Suppose for contradiction that v is a vertex in $V(H) \setminus \{x,y\}$ such that $H_{\mathfrak{R}}$ uses no vertex labeled v.

Suppose first that v is the unique common neighbor of x and y in H. Then, since $H_{\mathfrak{R}}$ uses no vertex labeled v, it uses at most $|E(H)| - 2$ colors: at most $|E(H'')| = |E(H)| - |T| - 1$ colors used by χ_0, and possibly also black. This is a contradiction, as a rainbow copy of H requires $|E(H)|$ colors.

Next, suppose that the graph $H^v := H \setminus \{v\}$ is connected. By the previous paragraph, we may (and will) assume that v is not the unique common neighbor of x and y. Consider the edge-colored graph G^* obtained from $G = G_{xy}^H(m,r)$ by the deletion of every vertex labeled v. As $H_{\mathfrak{R}}$ contains no vertex labeled v, it is present in G^*, and, in particular, G^* contains a rainbow copy of H^v. Observe that $G^* = (G_{xy}^{H^v}(m,r), \chi_0')$, where χ_0' is the restriction of χ_0 to $(H^v)''$ and $r \geq \max\{|E(H^v)| + 1, 2\}$. Since v is not the unique common neighbor of x and y, the edge xy is in a triangle in H^v. Therefore, since H^v is connected, by minimality of H we know that $G_{xy}^{H^v}(m,r)$ contains no rainbow copy of H^v. Hence, we obtain a contradiction.

Therefore, we may (and will) assume that, for every non-cut vertex u in H, the rainbow copy $H_{\mathfrak{R}}$ contains a vertex labeled u. In particular, v is a cut vertex in H. Note that x and y are in the same component of H^v since $xy \in E(H)$ and $v \in V(H) \setminus \{x,y\}$. Let C be a component of H^v not containing $\{x,y\}$. Take $u \in C$ of maximum distance from v. Note that u is not a cut vertex of H, and thus, by assumption, there is a vertex u' in $H_{\mathfrak{R}}$ labeled u. Since $H_{\mathfrak{R}}$ is connected and contains at least one vertex in M, there is a path in $H_{\mathfrak{R}}$ from M to u'. However, by our choice of C, any such path passes through a vertex labeled v, which is a contradiction. ◻

Recall that T is the set of edges in H with one endpoint in $S = N_H(x) \cap N_H(y)$ and the other in $\{x,y\}$, and that T' is the set of edges in H' between $x|y$ and S. Note that $|S| = |T'| = \frac{|T|}{2} \geq 1$ since xy is contained in a triangle. By Claim 4.2, we see that $|H_{\mathfrak{R}} \cap M| \leq 2$. We will obtain a contradiction by counting the edges in $H_{\mathfrak{R}}$ (via two cases).

First, consider the case where $|H_{\mathfrak{R}} \cap M| = 2$. By Claim 4.2, $H_{\mathfrak{R}}$ contains exactly one vertex labeled v for each $v \in V(H) \setminus \{x,y\}$. Hence there are at most $2|T'|$ edges in $H_{\mathfrak{R}}$ with a label in T'. Now, consider an edge $uv \in E(H') \setminus T'$, where $u \neq x|y$ without loss of generality. Since there is exactly one vertex labeled u, all edges labeled uv belong to the same copy of F, and thus have the same color. It follows that the edges of $H_{\mathfrak{R}}$ are colored using at most $|E(H')| - |T'| + 2|T'| \leq |E(H)| - 1$ colors, which contradicts the assumption that $H_{\mathfrak{R}}$ is a rainbow copy of H.

Now, consider the case where $|H_{\mathfrak{R}} \cap M| = 1$. In this case, by Claim 4.2 there is exactly one $v \in V(H) \setminus \{x,y\}$ that labels two vertices in $H_{\mathfrak{R}}$, and every other vertex in $V(H')$ appears once as a label. We claim that for every edge $e \in E(H')$, except possibly for $vx|y$ (if this is an edge), all edges labeled e have the same color. Indeed, if e does not contain v, then the claim clearly follows, as there is at most one edge labeled e. Otherwise, $e = uv$ with $u \neq x|y$, and thus, since $e \in E(H') \setminus T'$ and all edges labeled e are in the same copy of F, they all receive the same color. Note that if $vx|y$ is an edge in H', then there are at most two edges in $H_{\mathfrak{R}}$ with this label. Altogether, the edges of $H_{\mathfrak{R}}$ see at most $|E(H')| + 1 = |E(H)| - |T'| < |E(H)|$ colors, which contradicts $H_{\mathfrak{R}}$ being a rainbow copy of H. ◻

5. Rainbow saturation for cliques. The purpose of this section is to prove Theorem 1.3, restated here for convenience.

THEOREM 1.3. *Let $r \geq 3$. Then $\mathrm{rsat}(n, K_r) \leq (r + 2\sqrt{2r})n + c_r$, where c_r is a constant depending only on r.*

We begin by proving the following lemma.

LEMMA 5.1. *For every $s \geq 3$, there is an edge-colored graph (H, χ_H) on $\binom{s}{2} + s$ vertices with the following properties:*

(1) *the largest rainbow clique in H has $\binom{s}{2} + 1$ vertices,*
(2) *for every $v \in V(H)$, there is a rainbow clique in $H \setminus \{v\}$ on $\binom{s}{2} + 1$ vertices,*
(3) *for every color c, there is a rainbow clique with $\binom{s}{2} + 1$ vertices containing no edge colored c.*

Proof. Let S be a set of size s. Let H be a complete graph on vertices $S \cup S^{(2)}$, where $S^{(2)} = \{\{x,y\} \subseteq S : x \neq y\}$ is the set of pairs of distinct elements from S. For each pair $\{x,y\} \in S^{(2)}$, color the edges $(x, \{x,y\})$ and $(y, \{x,y\})$ with the same color, picking a different color for each pair. The remaining edges receive a rainbow coloring, with colors distinct from those used so far.

We first show that the largest rainbow clique in H has size $\binom{s}{2}+1$. Suppose that T is a rainbow clique, and write $t := |T \cap S|$. For every two distinct vertices $x,y \in T \cap S$, the pair $\{x,y\}$ is not in T, since the edges $(x,\{x,y\})$ and $(y,\{x,y\})$ have the same color. It follows that $|T| \leq t + \binom{s}{2} - \binom{t}{2} \leq \binom{s}{2}+1$, using the fact that $\binom{t}{2} \geq t-1$ for $t \geq 0$. This shows that H has no rainbow cliques of size at least $\binom{s}{2}+2$. Given distinct vertices $x,y \in S$, let $T(x,y) := \{x,y\} \cup (S^{(2)} \setminus \{\{x,y\}\})$. It is not difficult to see that $T(x,y)$ is a rainbow clique of size $\binom{s}{2}+1$, and hence a maximum rainbow clique. This proves (1).

For (2), let $v \in V(H)$. If $v \in S$, then for any $x,y \in S \setminus \{v\}$, the clique $T(x,y)$ avoids v (using $s \geq 3$). If $v \in S^{(2)}$, then $v = \{x,y\}$ for some $x \neq y \in S$, and thus the clique $T(x,y)$ avoids v. This proves (2).

For (3), let c be a color that is used on H (if c does not appear on H, then (3) holds trivially). Since every color class is either an edge or a path of length 2, there is a vertex v which is incident with all edges of color c. By (2), there is a maximum rainbow clique avoiding v and thus c. This proves (3). ∎

We now complete the proof of Theorem 1.3.

Proof of Theorem 1.3. Given $r \geq 10$, let s be the largest integer such that $r \geq \binom{s}{2}+1$ (so $r \leq \binom{s+1}{2}$). Write $t := r - \binom{s}{2} - 1$. Notice that we may take n to be sufficiently large.

Let (H,χ_H) be an edge colored graph as in Lemma 5.1 (applied with s), and let T_1, T_2, and U be pairwise disjoint sets of sizes t, t, and $n-2t-|V(H)|$, respectively, which are disjoint from $V(H)$. Let G be a graph on vertex set $V(H) \cup U \cup T_1 \cup T_2$ with edge set $E(G) := \{uv : \{u,v\} \not\subseteq U \text{ and } uv \notin T_1 \times T_2\}$. We define a coloring χ on $E(G)$ such that $\chi(e) = \chi_H(e)$ for each $e \in E(H)$, and the edges of $E(G) \setminus E(H)$ receive a rainbow coloring with colors distinct from those used by χ_H. Note that the vertices of each T_i induce a rainbow clique of size t.

We claim that G contains no rainbow copy of K_{r+2}. Indeed, suppose W is a rainbow clique in G. Then W contains at most one vertex from U (as U is an independent set), at most t vertices from $T_1 \cup T_2$ (as no edges exist between T_1 and T_2), and at most $\binom{s}{2}+1$ vertices from H (by the choice of χ_H). Thus, $|W| \leq \binom{s}{2}+1+t+1 = r+1$, as claimed.

Next, we show that the addition of any nonedge xy in U in any color creates a rainbow copy of K_{r+2}. Suppose that xy is colored red, and denote by G' the graph obtained from G by adding xy in red. Then exactly one of the following holds: no edge in $G[V(H) \cup \{x,y\}]$ is red; exactly one of the edges from $\{x,y\}$ to $V(H)$ is red; red is used in H. Hence, by the properties of H given by Lemma 5.1 and by definition of G, there is a maximum rainbow clique W in H such that $G[W \cup \{x,y\}]$ does not contain any red edges. Thus $W \cup \{x,y\}$ is a rainbow clique of size $\binom{s}{2}+3 = r-t+2$ in G'. Since either T_1 or T_2 is not incident to any red edge, there is $i \in \{1,2\}$ for which $W \cup T_i \cup \{x,y\}$ is a rainbow clique of size $r+2$, as required.

Now, as $r \leq \binom{s+1}{2}$, we have $t \leq \binom{s+1}{2} - \binom{s}{2} - 1 = s-1$. Also, since $r \geq \binom{s}{2}$, we have $s-1 \leq \sqrt{2r}$. Thus, thinking of r as fixed, we have

$$\begin{aligned}
|E(G)| &= (|V(H)|+2t)(n-|V(H)|-2t) + |E(H)| + 2\binom{t}{2} + 2t|V(H)| \\
&= (|V(H)|+2t)(n-|V(H)|-2t) + O(1) \\
&= \left(\binom{s}{2} + s + 2t\right) \cdot n + O(1)
\end{aligned}$$

$$= (r+s+t-1)\cdot n + O(1)$$
$$\leq (r+2s-2)\cdot n + O(1)$$
$$\leq (r+2\sqrt{2r})\cdot n + O(1).$$

Observe that G has at most $t^2 = O(1)$ bad nonedges, namely the edges uv such that $u \in T_1$ and $v \in T_2$, so applying Proposition 2.1 yields that $\mathrm{rsat}(n, K_{r+2}) \leq (r+2\sqrt{2r})n + O(1)$. ∎

6. Conclusion and open problems. Our main theorem shows that for any graph H the rainbow saturation number $\mathrm{rsat}(n,H)$ is linear in n. Given this, it is natural to ask the following.

QUESTION 6.1. *For every graph H, does there exist a constant $c = c(H)$ such that*

$$\mathrm{rsat}(n,H) = (c(H) + o(1))n?$$

Analogous questions have been considered for saturation and the related notion of weak saturation. The limit $\lim_{n\to\infty} \frac{\mathrm{sat}(n,H)}{n}$ was conjectured to exist by Tuza; see [16, 17]. Although some progress has been made toward Tuza's conjecture (see [13, 15]), it remains open.

Another natural direction would be to generalize the notion of weak saturation to edge-colored graphs. A subgraph G of F is said to be *weakly (F,H)-saturated* if the edges of $E(F) \setminus E(G)$ can be added to G, one edge at a time, in such a way that every added edge creates a new copy of H. The minimum number of edges in a weakly (F,H)-saturated graph is known as the *weak saturation number* and is denoted $\mathrm{wsat}(F,H)$. When $F = K_n$, we write $\mathrm{wsat}(n,H)$ instead of $\mathrm{wsat}(K_n,H)$.

We say an edge-colored subgraph (G,χ) of F is *weakly H-rainbow saturated* if there exists an ordering $e_1,\dots,e_m$ of $E(F)\setminus E(G)$ such that, for any list $c_1,\dots,c_m$ of distinct colors from $\mathbb{N}$, the nonedges e_i in color c_i can be added to G, one at a time, so that every added edge creates a new rainbow copy of H. The *weak rainbow saturation number* of H, denoted by $\mathrm{rwsat}(F,H)$, is the minimum number of edges in a weakly H-rainbow saturated graph. When $F = K_n$, we write $\mathrm{rwsat}(n,H)$ instead of $\mathrm{rwsat}(K_n,H)$.

Note that we require the collection of added edges to receive distinct colors. In particular, we wish to exclude the possibility that all added edges have the same color, in which case the previously added edges do not contribute to making new rainbow copies and the problem reduces to the standard rainbow saturation number.

The study of weak saturation numbers was introduced by Bollobás [3] in 1968, where he proved that $\mathrm{sat}(n,K_m) = \mathrm{wsat}(n,K_m)$ for all $3 \leq m \leq 7$ and conjectured that this equality holds for all m. This conjecture was first proved by Lovász [12] using a beautiful generalization of the Bollobás two families theorem. It would be interesting to determine whether a similar phenomenon holds for the rainbow saturation number.

QUESTION 6.2. *Let $r \geq 3$. Is* $\mathrm{rwsat}(n,K_r) = \mathrm{rsat}(n,K_r)$?

It would be interesting to see if upper bounds on $\mathrm{rwsat}(n,K_r)$ can be found that are much stronger than those provided by Theorem 1.3 (note that $\mathrm{rsat}(n,K_r) \geq \mathrm{rwsat}(n,K_r)$).

QUESTION 6.3. *Let $r \geq 3$. Is it the case that* $\mathrm{rwsat}(n,K_r) \leq rn + O(1)$ *for n sufficiently large?*

Alon [1] proved in 1985 that $\mathrm{wsat}(n,H) = (c(H)+o(1))n$ for all graphs H. The natural generalization of this to hypergraphs was recently proved by Shapira and

Tyomkin [14]. An analogous question can be considered regarding the weak rainbow saturation number.

Question 6.4. *For every graph H, does there exist a constant $c = c(H)$ such that*

$$\mathrm{rwsat}(n, H) = (c(H) + o(1))n?$$

In the above definition of weak rainbow saturation we do not require (G, χ) to be rainbow H-free. Note, however, that one could alternatively consider the minimum number of edges in a weakly H-rainbow saturated, rainbow H-free graph. Unlike with weak saturation numbers, it is not clear whether this gives the same value as the weak rainbow saturation number of H. Indeed, a graph may contain a rainbow copy of H and still be a minimal (with respect to edge removal) weakly H-rainbow saturated graph. For example, a rainbow K_n is weakly K_n-saturated but removing any edge does not leave a weakly K_n-rainbow saturated graph, due to the stipulation that adding the edge back in any color must create a rainbow K_n.

6.1. Proper rainbow saturation. As our construction does not give a proper coloring, it is natural to ask what happens if we restrict our attention to properly edge-colored graphs. We make the following definition: A properly edge-colored graph (G, c) is *properly H-rainbow saturated* if (G, c) does not contain a rainbow copy of H, but the addition of any nonedge, in any color from $\mathbb{N}$ which preserves the proper coloring, creates a rainbow copy of H. The *proper rainbow saturation number* of H, denoted by $\mathrm{prsat}(n, H)$, is the minimum number of edges in a properly H-rainbow saturated edge-colored graph on n vertices.

Question 6.5. *Is* $\mathrm{prsat}(n, H) \le \mathrm{rsat}(n, H)$ *for all H?*

It is worth noting that the phrase "rainbow saturation" has appeared in the literature in a different context. Recently, Bushaw, Johnston, and Rombach [4] defined a different form of rainbow saturation which also requires the coloring to be proper. We will refer to this concept as *BJR proper rainbow saturation* to distinguish it from the definition above.

A graph G is *BJR properly H-rainbow saturated* if there is a proper coloring of G that does not contain a rainbow copy of H, but if any nonedge is added to G, then any proper coloring contains a rainbow copy of H. The *BJR proper rainbow saturation number* of H, denoted by $\mathrm{prsat}'(n, H)$, is the minimum number of edges in a properly H-rainbow saturated graph on n vertices.

Note that the two definitions are subtly different. Any graph G that is BJR properly H-rainbow saturated gives rise to a properly H-rainbow saturated edge-colored graph (G, c) by taking c to be the proper coloring of G that does not contain a rainbow copy of H. This observation tells us that $\mathrm{prsat}(n, H) \le \mathrm{prsat}'(n, H)$ for all n and H. However, the converse might not hold: if (G, c) is properly H-rainbow saturated, there is no guarantee that every recoloring of G plus a nonedge contains a rainbow H.

Bushaw, Johnston, and Rombach [4] proved that, for any graph H that does not include an induced even cycle, the BJR proper rainbow saturation number of H is linear in n, that is, $\mathrm{prsat}'(n, H) = O(n)$. They also showed that $\mathrm{prsat}'(n, C_4)$ is again linear in n and conjecture that, analogously to the classical saturation numbers, the proper rainbow saturation number is linear in n for every H. Private correspondence with Bushaw, Johnston, and Rombach and independently with Barnabás Janzer informs us that this can be shown by a straightforward application of a result of Kázsonyi and Tuza [10]. It follows that $\mathrm{prsat}(n, H)$ must also be linear in n for all H. It would be interesting to know if there are any graphs H where $\mathrm{prsat}(n, H)$ and $\mathrm{prsat}'(n, H)$ differ considerably.

Acknowledgment. We would like to thank the anonymous referees for their helpful comments.

REFERENCES

[1] N. Alon, *An extremal problem for sets with applications to graph theory*, J. Combin. Theory Ser. A, 40 (1985), pp. 82–89.

[2] M. D. Barrus, M. Ferrara, J. Vandenbussche, and P. S. Wenger, *Colored saturation parameters for rainbow subgraphs*, J. Graph Theory, 86 (2017), pp. 375–386, https://doi.org/10.1002/jgt.22132.

[3] B. Bollobás, *Weakly k-saturated graphs*, in Beiträge zur Graphentheorie (Kolloquium, Manebach, 1967), Teubner, Leipzig, 1968, pp. 25–31.

[4] N. Bushaw, D. Johnston, and P. Rombach, *Rainbow saturation*, Graphs Combin., 38 (2022), 166, https://doi.org/10.1007/s00373-022-02566-z.

[5] P. Erdős, A. Hajnal, and J. W. Moon, *A problem in graph theory*, Amer. Math. Monthly, 71 (1964), pp. 1107–1110, https://doi.org/10.2307/2311408.

[6] J. R. Faudree, R. J. Faudree, and J. R. Schmitt, *A survey of minimum saturated graphs*, Electron. J. Combin., DS19 (2011), 36.

[7] M. Ferrara, D. Johnston, S. Loeb, F. Pfender, A. Schulte, H. C. Smith, E. Sullivan, M. Tait, and C. Tompkins, *On edge-colored saturation problems*, J. Comb., 11 (2020), pp. 639–655, https://doi.org/10.4310/JOC.2020.v11.n4.a4.

[8] A. Girão, D. Lewis, and K. Popielarz, *Rainbow saturation of graphs*, J. Graph Theory, 94 (2020), pp. 421–444, https://doi.org/10.1002/jgt.22532.

[9] D. Hanson and B. Toft, *Edge-colored saturated graphs*, J. Graph Theory, 11 (1987), pp. 191–196, https://doi.org/10.1002/jgt.3190110209.

[10] L. Kászonyi and Z. Tuza, *Saturated graphs with minimal number of edges*, J. Graph Theory, 10 (1986), pp. 203–210, https://doi.org/10.1002/jgt.3190100209.

[11] D. Korándi, *Rainbow saturation and graph capacities*, SIAM J. Discrete Math., 32 (2018), pp. 1261–1264, https://doi.org/10.1137/17M1155429.

[12] L. Lovász, *Flats in matroids and geometric graphs*, in Combinatorial Surveys, Academic Press, London, 1977, pp. 45–86.

[13] O. Pikhurko, *Results and open problems on minimum saturated hypergraphs*, Ars Combin., 72 (2004), pp. 111–127.

[14] A. Shapira and M. Tyomkyn, *Weakly saturated hypergraphs and a conjecture of Tuza*, Proc. Amer. Math. Soc., 151 (2023), pp. 2795–2805, https://doi.org/10.1090/proc/16197.

[15] M. Truszczyński and Z. Tuza, *Asymptotic results on saturated graphs*, Discrete Math., 87 (1991), pp. 309–314.

[16] Z. Tuza, *A generalization of saturated graphs for finite languages*, in Proceedings of the 4th International Meeting of Young Computer Scientists, Vol. 185, 1986, pp. 287–293.

[17] Z. Tuza, *Extremal problems on saturated graphs and hypergraphs*, Ars Combin., 25 (1988), pp. 105–113.

[18] A. A. Zykov, *On some properties of linear complexes*, Mat. Sb. N.S., 24 (1949), pp. 163–188 (in Russian).