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Spectral quasi-clustering estimates for certain semiregular systems

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ABSTRACT

We show a quasi-clustering result for a subclass of the class of Semiregular Metric Globally Elliptic Systems (SMGES) including certain quantum optics models (such as Jaynes-Cummings and its generalizations) which describe light-matter interaction. More precisely, we show that for the class of systems with polynomial coefficients we consider, the spectrum concentrates within the union of intervals (not necessarily disjoint, but at most intersecting in an a priori finite number) centered at a sequence determined in terms of invariants of the (total) symbol and width decreasing as the centers go to infinity.

Contents

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1. Introduction

In this paper we prove a spectral quasi-clustering for large eigenvalues of a subclass of systems belonging to the class of Semiregular Metric Globally Elliptic Systems (SMGES), introduced in [7] (see Definition 2.1 below).

By *spectral quasi-clustering* we mean the concentration of the spectrum of a positive self-adjoint ψdo within the union of certain intervals with centers at a sequence determined in terms of invariants of the symbol, and diameters decreasing as the centers go to infinity.

We speak of "quasi-clustering" when the various intervals in whose union the spectrum is lying may intersect (in an at most uniformly finite number) and speak of "clustering" when such intervals do not intersect anymore when the centers are sufficiently large.

Determining when such a clustering/quasi-clustering takes place is interesting since it actually completes the spectral asymptotic information given by the Weyl asymptotics. In fact, it gives quite a precise location of the spectrum for large eigenvalues when the centers are in a neighborhood of $+\infty$ on the real line.

Keeping as examples the Jaynes-Cummings model and its generalizations of Section 2 of [7], we consider SMGES systems whose semiprincipal symbols possess matrix invariants (i.e., the coefficients of the characteristic polynomial of the semiprincipal symbol) that are functions of the harmonic oscillator $p_{2,\alpha}$ (see Section 4) and subprincipal of its diagonalization (by some unitary symbol e_0) which is constant on the bicharacteristics of p_2 and $p_{2,\alpha}$.

To start describing more precisely our results, we give some references to problems of eigenvalue clustering in various situations, focusing only on those that are most relevant to this paper (see, e.g., the reference lists of the quoted ones for more contributions). First, we recall that Duistermaat and Guillemin in [1] gave a clustering result for the m-th root of a scalar positive elliptic self-adjoint ψdo P of order $m > 0$ on a compact smooth boundaryless manifold under the hypothesis that the bicharacteristics of $\sqrt[m]{p}$ are all periodic with the same period, where p denotes the principal symbol of P. Conversely, they showed that if that clustering occurs then the flow of $\sqrt[m]{p}$ is periodic. Next, Weinstein [15] proved also an eigenvalue clustering result for a Schrödinger operator on a compact Riemannian manifold, deepening the description of the asymptotic structure of the clusters. We will later recall the arguments used in that paper since they are relevant to our work. Later, Colin de Verdière [2] gave an even more precise result in the case of the square of a first order ψdo with zero subprincipal symbol and 2π-periodic bicharacteristics on a compact smooth manifold. He was also able to recover the multiplicities of the eigenvalues in the disjoint intervals. Taking inspiration from the ideas of Weinstein and Colin de Verdière, Helffer and Robert [4] studied *semiclassical* clustering properties for an anharmonic oscillator model, and for scalar second order globally elliptic *regular* positive self-adjoint ψdos, Helffer [3] obtained a clustering result under the hypothesis that the X-ray transform (that is, the average over a period along the bicharacteristics of the principal part) of the subprincipal symbol is identically a constant.

Clustering results for certain systems in the semiclassical case where obtained by Ivrii [6], and for *regular* 2×2 NCHOs by Parmeggiani [8–10].

In this paper, we generalize the (quasi-)clustering properties to semiregular systems by means of an idea introduced by Weinstein in the aforementioned paper [15]. In there, he studied ψdos on a compact Riemannian manifold of the form $A^2 + B$ with A a 1st-order self-adjoint, positive, elliptic ψdo, B a self-adjoint ψdo of order 0, and such that $e^{2\pi i A} = cI$ for some constant c. His approach is based on an averaging technique: the subprincipal symbol is X-ray transformed on the bicharacteristics of the principal symbol by a unitary operator conjugation and the new subprincipal term commutes with the principal one. Thus, the spectrum of the sum of the operators corresponding to the principal and subprincipal terms can be analyzed by studying that of the two terms individually and it gives the sequence at which the intervals are centered. The remainder, that is, the difference between the conjugated operator and the operator itself, is a compact operator and gives the diameter of the intervals. In fact, the compactness of the remainder leads to an energy inequality and the minimax principle completes the analysis.

The plan of the paper is the following.

In Section 2 we recall the definition the SMGES class given in [7]. We also describe the examples given by the JC-model and of an extension of it by the use of semiregular Non-Commutative Harmonic Oscillators (NCHOs, introduced by Parmeggiani and Wakayama in [11–13], see also [8–10]). In Section 3 we show that it is possible for a Fredholm operator (with non positive index and parametrix given by its adjoint) to be deformed into an isometry by adding a compact operator. This is crucial for applying the diagonalization procedure and keep control on the relation between the spectrum of the starting operator in terms of that of the diagonalized one. In Section 4 we prove that the blockwise diagonalization with scalar semiprincipal blocks of a system in our class is equal, modulo a system of order -1, to a system whose principal, semiprincipal, and subprincipal terms commute. It is here that we take inspiration from the work by Weinstein. In fact, we study the non-compact part of the operator (that is, the operator obtained by considering only the principal, semiprincipal and subprincipal parts) obtaining an explicit expression for its spectrum. Next, we recapture the spectrum of the whole operator, thanks to an energy inequality which leads to our estimate by using the minimax principle. It is at this point that what we show in Section 3 becomes crucial. In fact, the minimax principle alone is not sufficient for obtaining the result. Indeed, we need to link the spectrum of the initial operator with that of its conjugation by a suitable diagonalizing operator E. This is achieved by showing that E^* (when $\operatorname{ind} E \geq 0$), respectively E (when $\operatorname{ind} E < 0$), can be made into an isometry, that is $EE^* = I$, respectively $E^*E = I$, by adding a smoothing term. When $\operatorname{ind} E \geq 0$ we are in a good position and we may carry the spectral information of the diagonalized system onto that of the initial operator. When $\operatorname{ind} E \leq 0$, it is not enough that E can be corrected into an isometry by addition of a smoothing operator and some further analysis is needed (see Section 4).

Notation. For $X \in \mathbb{R}^n \times \mathbb{R}^n = \mathbb{R}^{2n}$ and $\langle X\rangle = (1 + |X|^2)^{1/2}$ the "Japanese bracket of X", let g be the admissible Hörmander metric $g_X = |dX|^2/\langle X\rangle^2$. We denote the (matrix-valued) Hörmander class of symbols $S(\langle X\rangle^m, g; \mathsf{M}_N)$ related to the admissible metric g on $\mathbb{R}^n \times \mathbb{R}^n$ and the g-admissible weight $\langle X\rangle$ (where $m \in \mathbb{R}$) simply by S^m, and denote by Ψ^m the corresponding class of pseudodifferential operators obtained by Weyl-quantization [5]. Finally we denote by S^m_{sreg} (and by Ψ^m_{sreg} the corresponding class of ψdos) the class of symbols $a \in S^m$ that admit an asymptotic expansion $\sum_{j\geq 0} a_{m-j}$ where the a_{m-j} are C^∞ on $\dot{\mathbb{R}}^{2n} := \mathbb{R}^{2n} \setminus \{(0, 0)\}$ and positively homogeneous of degree $m - j$, in which a_m is the principal symbol, a_{m-1} and a_{m-2} are the semiprincipal and subprincipal symbols, respectively. We put $p_2(X) = |X|^2/2$ and, for $\alpha \in \mathbb{R}^n$ with $\alpha_j > 0$, $1 \leq j \leq n$,

$$p_{2,\alpha}(X) = \sum_{j=1}^{n} \alpha_j (x_j^2 + \xi_j^2)/2.$$

The Hamilton vector field associated with a function f on the phase-space will be denoted by

$$H_f = \sum_{j=1}^{n} \Big(\frac{\partial f}{\partial \xi_j} \frac{\partial}{\partial x_j} - \frac{\partial f}{\partial x_j} \frac{\partial}{\partial \xi_j} \Big),$$

and by $(t, X) \mapsto \exp(tH_f)(X)$ its flow. We finally denote by $B^s = B^s(\mathbb{R}^n; \mathbb{C}^N)$ the Shubin-Sobolev spaces of order $s \in \mathbb{R}$ (so that, in particular, B^2 is the maximal domain for the L^2 realization of $p_2^{\mathrm{w}}(x, D)$).

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2. Setting

In this section, we recall the definition of the SMGES class and introduce, as examples of that class, the classical JC-model and its extension to systems of an $N \geq 3$ energy level atom and $n = N - 1$ cavity-modes of the electromagnetic field.

2.1. The SMGES class

The class under study, introduced in [7], is given by the systems of order m having scalar and elliptic principal symbol whose semiprincipal symbol (that is, the isotropic positively homogeneous term of degree $m-1$ in the asymptotic expansion of the symbol) can be blockwise diagonalized in scalar blocks satisfying the condition of eigenvalues separation.

Definition 2.1. We say that an M_N-valued symbol $a \in S^m_{\mathrm{sreg}}$ is a **semiregular metric globally elliptic system** (SMGES for short) of order m, when

$$a(X) = a(X)^* = q_m(X)I_N + a_{m-1}(X) + a_{m-2}(X) + S^{m-3}_{\mathrm{sreg}}, \quad X \neq 0,$$

where:

- $q_m \in C^\infty(\dot{\mathbb{R}}^{2n}; \mathbb{R})$ is positively homogeneous of degree m and such that $|X|^m \approx q_m(X)$ for all $X \neq 0$;
- $a_{m-1} = a^*_{m-1}$ is such that there exists $r \geq 1$ and $e_0 \in C^\infty(\dot{\mathbb{R}}^{2n}; \mathsf{M}_N)$ **unitary** and positively homogeneous of degree 0 such that

$$e_0(X)^* a_{m-1}(X) e_0(X) = \mathsf{diag}(\lambda_{m-1,j}(X) I_{N_j}; 1 \leq j \leq r), \quad X \neq 0$$

  where $N = N_1 + N_2 + \ldots + N_r$ and $\lambda_{m-1,j} \in C^\infty(\dot{\mathbb{R}}^{2n}; \mathbb{R})$ are positively homogeneous of degree $m-1$ and such that

$$j < k \Longrightarrow \lambda_{m-1,j}(X) < \lambda_{m-1,k}(X), \quad \forall X \neq 0.$$

2.2. JC-model by semiregular NCHOs and generalizations

We give here two examples of semiregular NCHOs in the SMGES class (due to Jaynes and Cummings [14]), relevant to Quantum Optics, that serve as a model of the class we consider in this work.

It will be convenient to use the following notation. We denote by $\boldsymbol{\sigma}_j$, $j = 0, \ldots, 3$, the Pauli-matrices, i.e.

$$\boldsymbol{\sigma}_0 = I_2, \quad \boldsymbol{\sigma}_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \boldsymbol{\sigma}_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \boldsymbol{\sigma}_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

and

$$\boldsymbol{\sigma}_\pm = \frac{1}{2}(\boldsymbol{\sigma}_1 \pm i\boldsymbol{\sigma}_2).$$

Let $\langle\cdot,\cdot\rangle$ be the canonical Hermitian product in $\mathbb{C}^N$, and $e_1,\dots,e_N$ be the canonical basis of $\mathbb{C}^N$. Let

$$E_{jk} := e_k^* \otimes e_j, \quad 1 \le j,k \le N,$$

be the basis of $\mathsf{M}_N(\mathbb{C})$, where E_{jk} acts on $\mathbb{C}^N$ as

$$E_{jk}w = \langle w, e_k\rangle e_j, \quad w \in \mathbb{C}^N.$$

Hence we have the relation

$$E_{jk}E_{hm} = \delta_{hk}E_{jm}.$$

We also let, for $X = (x,\xi) \in \mathbb{R}^n \times \mathbb{R}^n = \mathbb{R}^{2n}$,

$$\psi_j(X) := \frac{x_j + i\xi_j}{\sqrt{2}}, \quad 1 \le j \le n,$$

so that $\psi_j^{\mathrm{w}}(x,D)$ is the annihilation operator and $\psi_j^{\mathrm{w}}(x,D)^* = (\bar{\psi}_j)^{\mathrm{w}}(x,D)$ is the creation operator, with respect to the j-th variable. Hence,

$$\sum_{j=1}^{n} \psi_j^{\mathrm{w}}(x,D)^*\psi_j^{\mathrm{w}}(x,D) = p_2^{\mathrm{w}}(x,D) - \frac{n}{2}.$$

2.2.1. The JC-model by semiregular NCHOs

This is the model of a two-level atom in one cavity, given by the 2×2 system in one real variable $x \in \mathbb{R}$

$$A^{\mathrm{w}}(x,D) = p_2^{\mathrm{w}}(x,D)I_2 + \alpha\Big(\boldsymbol{\sigma}_+\psi^{\mathrm{w}}(x,D)^* + \boldsymbol{\sigma}_-\psi^{\mathrm{w}}(x,D)\Big) + \gamma\boldsymbol{\sigma}_3, \quad \alpha,\gamma \in \mathbb{R}\setminus\{0\},$$

where the atom levels are given by $\pm\gamma$.

In this case, for the principal symbol of the diagonalizer e_0 we have

$$e_0(X) = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ \psi(X)/|\psi(X)| & -\psi(X)/|\psi(X)| \end{bmatrix}, \quad X \neq 0, \tag{2.1}$$

and for the subprincipal symbol b_0 of the diagonalized operator

$$b_0(X) = -\frac{1}{2}I_2. \tag{2.2}$$

2.2.2. The JC-model for an N-level atom and $n = N-1$ cavity-modes (in the Ξ-configuration)

In this case, for $\alpha_1, \dots \alpha_{N-1} \in \mathbb{R} \setminus \{0\}$, $\gamma_1, \dots \gamma_{N-1} \in \mathbb{R}$ with $0 < \gamma_1 < \gamma_2 < \dots < \gamma_{N-1}$, we consider the $N \times N$ system in $\mathbb{R}^n$, $n = N-1$, given by

$$\begin{aligned} A^{\mathrm{w}}(x,D) &= p_2^{\mathrm{w}}(x,D) I_N \\ &\quad + \sum_{k=1}^{N-1} \alpha_k \Big(\psi_k^{\mathrm{w}}(x,D)^* E_{k,k+1} + \psi_k^{\mathrm{w}}(x,D) E_{k+1,k} \Big) + \sum_{k=1}^{N-1} \gamma_k E_{k+1,k+1}. \end{aligned}$$

In this case, the levels of the atom are given by 0 and the γ_k.

When $N = 3$, for the principal symbol of the diagonalizer e_0 we have, writing $\alpha\psi(X) = (\alpha_1\psi_1(X), \alpha_2\psi_2(X))$ so that for the relative norm one has $|\alpha\psi(X)| = p_{2,\alpha}(X)^{1/2}$,

$$e_0(X) = \begin{bmatrix} \frac{\alpha_2 \overline{\psi_2(X)}}{|\alpha\psi(X)|} & \frac{\alpha_1 \overline{\psi_1(X)}}{\sqrt{2}|\alpha\psi(X)|} & \frac{\alpha_1 \overline{\psi_1(X)}}{\sqrt{2}|\alpha\psi(X)|} \\ 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{-\alpha_1 \overline{\psi_1(X)}}{|\alpha\psi(X)|} & \frac{\alpha_2 \overline{\psi_2(X)}}{\sqrt{2}|\alpha\psi(X)|} & \frac{\alpha_2 \overline{\psi_2(X)}}{\sqrt{2}|\alpha\psi(X)|} \end{bmatrix}, \quad X \neq 0, \tag{2.3}$$

and for the subprincipal symbol b_0 of the diagonalized operator

$$\begin{aligned} b_0(X) = \mathrm{diag}\Big(& \frac{-\alpha_2^2 |\psi_2(X)|^2 + (\gamma_2 - 1)\alpha_1^2 |\psi_1(X)|^2}{|\alpha\psi(X)|}, \\ & \frac{-\alpha_1^2|\psi_1(X)|^2 + \gamma_1 |\alpha\psi(X)|^2 + (\gamma_2+1)\alpha_2^2|\psi_2(X)|^2}{2|\alpha\psi(X)|} I_2 \Big), \quad X \neq 0. \end{aligned} \tag{2.4}$$

3. The "isometrization"

Definition 3.1. Let H be a (separable) Hilbert space. We say that the linear bounded operator $U \colon H \longrightarrow H$ is **quasi-unitary** if $U^*U = I + F_1$ and $UU^* = I + F_2$, where the $F_1, F_2 \colon H \longrightarrow H$ are compact and I is the identity operator in H.

We show in Theorem 3.2 below that, given a quasi-unitary pseudodifferential system $U \in \Psi^0$ (realized as a bounded operator in $H = L^2$), when $F_j \in \Psi^{-\ell_j}$, $\ell_1, \ell_2 > 0$, we may perturb U, or its adjoint, by an operator of the same order of F_1 or F_2, to make it, or its adjoint, into an isometry.

Recall that a linear bounded operator A on a Hilbert space into itself is an isometry if $A^*A = I$.

This will be fundamental in Section 4 since in Theorem 4.2 below we will need to relate the spectrum of an SMGES to that of its diagonalization. In fact, we will see that the conjugation by an isometry changes the point spectrum of a positive ψdo by adding, at most, the eigenvalue 0. Hence, the conjugation by an isometry preserves the asymptotic properties of the spectrum of an SMGES.

Theorem 3.2. *Let $U \in \Psi^0_{\mathrm{sreg}}$ and suppose that $U^*U = I + F_1$ and $UU^* = I + F_2$, where $F_j \in \Psi^{-\ell_j}$, $\ell_1, \ell_2 > 0$. Then,*

(i) *When $\operatorname{ind} U \leq 0$, there is $K \in \Psi^{-\ell_1}$ (i.e. of the same order as F_1) such that $U + K$ is an isometry;*
(ii) *When $\operatorname{ind} U \geq 0$, the same holds for U^*, that is, there is $K \in \Psi^{-\ell_2}$ (i.e. of the same order as F_2) such that $U^* + K$ is an isometry.*

Finally, when $\operatorname{ind} U \leq 0$ and F_1 is smoothing, resp. $\operatorname{ind} U \geq 0$ and F_2 is smoothing, then K is also smoothing.

Proof. Fundamental step. We start by considering the case in which U is such that $U^*U = I + F_1$ where F_1 has order $-\ell_1 < 0$ and U is *injective*, that is,

$$-1 \notin \operatorname{Spec}(F_1).$$

We want to construct a ψdo R such that UR is an isometry, where $R = I + K_1$, with K_1 a ψdo of the same order as F_1. Namely, R must formally be the inverse of a square root of $I + F_1$. To give a precise meaning to R as a ψdo we follow the approach by Helffer in [3]. It is based on the construction of R as a bounded linear operator $L^2 \to L^2$ commuting with F. Once R is given, one then shows that it is indeed a ψdo by inverting the square root of $I + F_0$, where F_0 is the composition of F_1 with the projection onto a finite-codimensional vector subspace of L^2, such that $\|F_0\|_{L^2 \to L^2} \leq 1/2$.

As F_1 is self-adjoint and compact (since it has order $-\ell_1 < 0$), we may consider its eigenvalues $\mu_j \in \mathbb{R}$, $j \geq 0$, repeated according to multiplicity, and a corresponding orthonormal basis $(\phi_j)_{j\geq 0}$ of L^2 made of eigenfunctions of F_1, with ϕ_j belonging to μ_j. Observe that, since $-1 \notin \operatorname{Spec}(F_1)$, we have that $I + F_1 = U^*U > 0$, whence we define the linear and bounded operator $R = (I + F_1)^{-1/2} \colon L^2 \longrightarrow L^2$ by

$$R\phi_j = (1 + \mu_j)^{-1/2}\phi_j, \quad j \geq 0.$$

If now c_j is the j-th coefficient in the Taylor series of $(1+t)^{-1/2}$ at $t = 0$, we consider a ψdo G such that

$$G - \sum_{j=0}^{k} c_j F_1^j \in \Psi^{-\ell_1(k+1)}, \quad \forall k \geq 0. \tag{3.1}$$

We wish to prove that $G - R$ is smoothing, because that then yields R is a ψdo. To do that, we need to invert $\sum_{j\geq 0} c_j F_1^j$, and the problem is given by the possible eigenvalues μ_j of F_1 with $|\mu_j| \geq 1$. We therefore deform F_1 to the linear and bounded operator F_0 defined on the basis $(\phi_j)_{j\geq 0}$ by

$$F_0\phi_j := \begin{cases} 0, \text{ if } |\mu_j| > 1/2, \\ F_1\phi_j, \text{ if } |\mu_j| \leq 1/2. \end{cases}$$

Hence, in particular, $\|F_0\|_{L^2\to L^2} \leq 1/2$. Note also that there are only finitely many js such that $|\mu_j| > 1/2$. We have that F_0 is a ψdo. In fact, for all $\phi \in L^2$,

$$(F_1 - F_0)\phi = \sum_{j\geq 0;\ |\mu_j|>1/2} \mu_j(\phi, \phi_j)_0\phi_j, \tag{3.2}$$

which shows that it is smoothing, as for its Schwartz-kernel we have

$$\mathbb{R}^{2n} \ni (x, y) \longmapsto \mathsf{K}_{F_1-F_0}(x, y) = \sum_{j\geq 0;\ |\mu_j|>1/2} \mu_j\phi_j(x)\overline{{}^t\phi_j(y)} \in \mathscr{S}(\mathbb{R}^{2n}; \mathsf{M}_N).$$

The same of course applies to $F_1^j - F_0^j$ for all $j \geq 1$ (it just suffices to substitute the eigenvalue in (3.2) by its j-th power). Now, define

$$R_0 := \sum_{j\geq 0} c_jF_0^j,$$

which is therefore a bounded operator $R_0 \colon L^2 \longrightarrow L^2$, because $\|F_0\|_{L^2\to L^2} \leq 1/2$. In addition, as

$$(R - R_0)\phi = \sum_{j\geq 0;\ |\mu_j|>1/2} (1 + \mu_j)^{-1/2}(\phi, \phi_j)_0\phi_j, \quad \forall\phi \in L^2,$$

it has Schwartz kernel

$$\mathbb{R}^{2n} \ni (x, y) \longmapsto \mathsf{K}_{R-R_0}(x, y) = \sum_{j\geq 0;\ |\mu_j|>1/2} (1 + \mu_j)^{-1/2}\phi_j(x)\overline{{}^t\phi_j(y)} \in \mathscr{S}(\mathbb{R}^{2n}; \mathsf{M}_N),$$

whence it is smoothing too.

We next write for any given $k \geq 0$

$$G - R_0 = A_k + B_k - C_k,$$

where

$$A_k := G - \sum_{j=0}^{2k} c_jF_1^j, \quad B_k := \sum_{j=0}^{2k} c_j(F_1^j - F_0^j), \quad C_k := R_0 - \sum_{j=0}^{2k} c_jF_0^j.$$

We have that $A_k, B_k, C_k \colon L^2 \longrightarrow L^2$ are all bounded. Moreover, $A_k \in \Psi^{-\ell_1(2k+1)}$ by (3.1) and B_k is smoothing for all k. We hence only need to study C_k. We have

$$C_k = F_0^{k+1}\Big(\sum_{j\geq 2k+1} c_j F_0^{j-(2k+1)}\Big)F_0^k,$$

which shows that $C_k\colon B^{-\ell_1 k}(\mathbb{R}^n;\mathbb{C}^N) \longrightarrow B^{\ell_1(k+1)}(\mathbb{R}^n;\mathbb{C}^N)$ is bounded for all k, because the operator $\sum_{j\geq 2k+1} c_j F_0^{j-(2k+1)}\colon L^2 \longrightarrow L^2$ is bounded and $F_0^j \in \Psi^{-\ell_1 j}$ for all j. This shows that $G - R_0$ is smoothing and therefore that R is a ψdo.

To complete the proof in this case, let $U_1 = UR$. Then $U_1^* U_1 = R^* U^* U R = (I + F_1)R^2 = I$ since R and $I + F_1$ commute because their eigenspaces coincide. Observe that $R = I + K_1$ where $K_1 \in \Psi^{-\ell_1}$ (i.e. it has the same order of F_1). Therefore $U_1 = U(I+K_1) = U+K$ with $K \in \Psi^{-\ell_1}$. Note that when F_1 is smoothing then K is smoothing too. This concludes the proof in the case $-1 \notin \operatorname{Spec}(F_1)$.

Case (i). Consider now the case when U is not injective (i.e., when $-1 \in \operatorname{Spec}(F_1)$; otherwise we are already done by the previous construction). We show how to modify U through a smoothing operator Q so that $U_1 := U + Q$ is injective (so as to be able to apply the previous construction). Consider the set

$$Z_1 := \{j;\ \mu_j = -1\}.$$

Then Z_1 is a finite set, by the compactness of F_1. Consider next an orthonormal system $(\psi_j)_{j\geq 0} \subset \mathscr{S}$ of L^2 made of eigenfunctions of F_2 (which is also compact), and denote by μ'_j the eigenvalues of F_2 (repeated according to multiplicity). Let also

$$Z_2 := \{j;\ \mu'_j = -1\}.$$

As before, also Z_2 is a finite set, since F_2 is compact. We then have

$$\operatorname{Ker} U = \{\phi \in L^2;\ U^* U\phi = 0\} = \operatorname{Span}\{\phi_j;\ j \in Z_1\},$$

and, likewise,

$$\operatorname{Ker} U^* = \{\psi \in L^2;\ UU^*\psi = 0\} = \operatorname{Span}\{\psi_j;\ j \in Z_2\}.$$

We hence construct $Q\colon L^2 \longrightarrow L^2$ as an injective operator that does not vanish on $\operatorname{Ker} U \setminus \{0\}$ with range in $\operatorname{Ker} U^*$. As $\operatorname{ind} U \leq 0$, we have that $\operatorname{card} Z_1 \leq \operatorname{card} Z_2$, whence we have an injective map $f\colon Z_1 \longrightarrow Z_2$. Define then

$$Q\phi := \sum_{j\in Z_1} (\phi, \phi_j)_0 \psi_{f(j)}, \quad \phi \in L^2.$$

Therefore Q is smoothing, since

$$\mathbb{R}^{2n} \ni (x,y) \longmapsto \mathsf{K}_Q(x,y) = \sum_{j\in Z_1} \psi_{f(j)}(x)\overline{{}^t\phi_j(y)} \in \mathscr{S}(\mathbb{R}^{2n};\mathsf{M}_N),$$

and $U+Q$ is injective, for

$$(U+Q)\phi = 0 \Longleftrightarrow \underbrace{U\phi}_{\in \mathrm{Range}(U)} = \underbrace{-Q\phi}_{\in \mathrm{Ker}\, U^*},$$

which in turn, since $\mathrm{Range}(U) = (\mathrm{Ker}\, U^*)^\perp$, yields

$$U\phi = -Q\phi \in (\mathrm{Ker}\, U^*)^\perp \cap \mathrm{Ker}\, U^* = \{0\},$$

and hence $\phi = 0$ because $\mathrm{Ker}\, Q \cap \mathrm{Ker}\, U = \{0\}$ by construction. By the fundamental step we hence have the existence of the desired K, which is also smoothing when F_1 is smoothing. This concludes the proof of the case (i).

Case (ii). It immediately follows from the previous constructions applied to U^*.

This concludes the proof. □

Remark 3.3. If in Theorem 3.2 one has $\mathrm{ind}\, U = 0$ then the function f in the proof of case (i) is bijective, whence $U+Q$ is onto. In fact,

$$(U+Q)^*\phi = 0 \Longleftrightarrow \underbrace{U^*\phi}_{\in \mathrm{Range}(U^*)} = \underbrace{-Q^*\phi}_{\in \mathrm{Ker}\, U},$$

and therefore

$$U^*\phi = -Q^*\phi \in (\mathrm{Ker}\, U)^\perp \cap \mathrm{Ker}\, U = \{0\},$$

which means $\phi = 0$ since $\mathrm{Ker}\, Q^* \cap \mathrm{Ker}\, U^* = \{0\}$ by the bijectivity of f. Since $R^* U_1^* U_1 R = I$ with R invertible, $U_1 R$ is unitary. In fact, it is invertible (U_1 and R are invertible) and the left inverse is unique.

4. Spectral quasi-clustering theorem

In this section, we prove a quasi-clustering theorem, Theorem 4.2 below, for a class of SMGES for which:

- The principal part is the scalar harmonic oscillator p_2;
- The semiprincipal part has eigenvalues $\lambda_{1,h}$ ($h = 1, \ldots, r$, where r is the number of blocks of b_1, the diagonalization of a_1) of the form

$$\lambda_{1,h}(X) = p_{1/2}^{(h)}(p_{2,\alpha}(X)), \quad X \neq 0, \quad 1 \leq h \leq r,$$

  where $p_{1/2}$ is smooth and positively homogeneous of degree $1/2$ (off a compact set);

- The subprincipal part b_0 of the diagonalized system is *constant* on the bicharacteristics of p_2 and $p_{2,\alpha}$.

This class contains the Jaynes-Cummings model and its generalizations as of Section 2 of [7] (with $\beta_j = \beta$ for all j in the notation of [7]).

The use of Theorem 3.2 and Remark 3.3 will be crucial in the proof of the quasi-clustering theorem.

It will be convenient, given a semiregular symbol $a \in S^{\mu}_{\mathrm{sreg}}$, to write $A\colon D(A) = \{u \in L^2;\ a^{\mathrm{w}}(x,D)u \in L^2\} \subset L^2 \longrightarrow L^2$ for the L^2 maximal realization of $a^{\mathrm{w}}(x,D)$. When a_μ is elliptic, we have $D(A) = B^\mu$.

We will be using the blockwise diagonalization theorem proved in [7] (where r denotes the number of blocks), valid for the class SMGES. We hence may find $e_0 \in C^\infty(\dot{\mathbb{R}}^{2n}; \mathsf{M}_N)$ such that $e_0 e_0^* = e_0^* e_0 = I_N$ and for which $e_0^* a_1 e_0 = b_1$ is blockwise diagonal, with r blocks $N_j \times N_j$, $N_1 + \ldots + N_r = N$, that is $b_1 = \mathrm{diag}(\lambda_{1,h} I_{N_h};\ 1 \le h \le r)$. In this case the subprincipal term b_0 of the blockwise-diagonalized operator has the form (see Corollary 6.5 in [7])

$$b_0 = \mathrm{diag}(b_{0,h};\ 1 \le j \le r), \quad \text{where} \quad b_{0,h} = \pi_h\big(e_0^* a_0 e_0 - i\{e_0^*, a_2\} e_0\big)\pi_h^*,$$

π_h being the orthogonal projector from $\mathbb{C}^N$ onto the h-th block $\mathbb{C}^{N_h}$.

Now, when for the diagonalizer E we have $\mathrm{ind}\, E \ge 0$ we are in a good position and we may carry the spectral information of the diagonalized system onto that of the initial operator. When we have $\mathrm{ind}\, E \le 0$, it is not enough that E can be corrected into an isometry by addition of a smoothing operator. We need to add a suitable $N \times N$ system to our given one, and extend it to a blockwise diagonal $2N \times 2N$ system, whose diagonalizator is of the blockwise diagonal form $\mathrm{diag}(E, E^*)$. Since the latter has now index 0, we may correct it into an isometry by addition of a smoothing operator. Such an extension is chosen so that its only contribution is adding some explicitly known centers of intervals occurring in the quasi-clustering.

We next prove a lemma that gives our result when the decoupling operator has a *nonnegative index.*

Lemma 4.1. *Let $a = a^* \sim \sum_{j \ge 0} a_{2-j} \in S^2_{\mathrm{sreg}}$ be a 2nd-order SMGES with principal symbol $a_2 = p_2 I_N$, such that the corresponding unbounded operator $A > 0$ (this is no restriction, in view of the Sharp-Gårding inequality; see [9], Thm. 3.3.22). Suppose that the coefficients of the characteristic polynomial $\lambda \mapsto \det(\lambda - a_1(X))$ of the semiprincipal term a_1 are functions of $p_{2,\alpha}$ and that b_0, the subprincipal symbol of the blockwise diagonalization of A, is constant on the bicharacteristics of p_2 and $p_{2,\alpha}$. In addition, suppose there are $m_1, \ldots, m_n \in \mathbb{Z}_+ \setminus \{0\}$ coprime such that*

$$\frac{m_1}{\alpha_1} = \ldots = \frac{m_n}{\alpha_n} =: q, \tag{4.1}$$

and that for the operator E_0, associated with the principal symbol of the diagonalizer e_0 of A, we have $\operatorname{ind} E_0 \geq 0$. Then, with $b_0 = \operatorname{diag}(b_{0,h};\ 1 \leq h \leq r)$, the eigenspaces of P_2 are invariant for $P_{2,\alpha}$ and the eigenspaces of $(P_2 + P_{2,\alpha}) \otimes I_{N_h}$ are invariant for $B_{0,h}$, for all $h = 1, \ldots, r$. Moreover, for each $h = 1, \ldots, r$, there is an orthonormal basis $\{\phi^{(h)}_{\gamma,j}\}_{\gamma\in\mathbb{Z}^n_+, 1\leq j\leq N_h} \subset \mathscr{S}(\mathbb{R}^n;\mathbb{C}^{N_h})$ of $L^2(\mathbb{R}^n;\mathbb{C}^{N_h})$ such that $\operatorname{Ker}\big(P_2 \otimes I_{N_h} - (k + \frac{n}{2})\big) = \operatorname{Span}\{\phi^{(h)}_{\gamma,j};\ |\gamma| = k,\ 1 \leq j \leq N_h\}$, for all $k \in \mathbb{Z}_+$ and

$$B_{0,h}\phi^{(h)}_{\gamma,j} = \mu^{(h)}_{\gamma,j}\phi^{(h)}_{\gamma,j}, \ \textit{with}\ |\mu^{(h)}_{\gamma,j}| \leq \|B_{0,h}\|_{L^2\to L^2}\,, \quad \forall\gamma,\ \forall j = 1, \ldots, N_h,$$

and smooth functions $p^{(h)}_{1/2} \colon \mathbb{R}_+ \longrightarrow \mathbb{R}$, positively homogeneous of degree $1/2$, such that $\lambda_{1,h} = p^{(h)}_{1/2}(p_{2,\alpha})$, $1 \leq h \leq r$, and finally a constant $M > 0$ such that

$$\operatorname{Spec}(A) \subset \bigcup_{h=1}^{r} S_h(A), \tag{4.2}$$

where, for each $h = 1, \ldots, r$,

$$S_h(A) := \bigcup_{k\geq 0} \bigcup_{\substack{\gamma\in\mathbb{Z}^n_+\\ |\gamma|=k}} \bigcup_{j=1}^{N_h} \Big(k + \frac{n}{2} + p^{(h)}_{1/2}(\alpha(\gamma + 1/2)) + \mu^{(h)}_{\gamma,j}\Big) + \Big[-\frac{M}{\sqrt{k+n/2}}, \frac{M}{\sqrt{k+n/2}}\Big], \tag{4.3}$$

with $\alpha(\gamma + 1/2) := \sum_{j=1}^{n} \alpha_j(\gamma_j + 1/2)$.

Proof. The proof takes inspiration from the approach by Weinstein [15] which we adapt to semiregular systems of ψdos. The main idea is to investigate the spectrum of A by studying the spectrum of the part of its blockwise diagonalization B which has nonnegative order as a ψdo (the difference being a compact operator). Of course, it will suffice to work for a single block of B, which is parametrized by $h = 1, \ldots, r$. Hence, we may suppose that $r = 1$ and that b_2 and b_1 are scalar operators.

Let P be the self-adjoint maximal L^2 realization of $p^{\mathrm{w}} := p_2^{\mathrm{w}} + p_{1/2}(p^{\mathrm{w}}_{\alpha,2}) + b_0^{\mathrm{w}}$ with $D(P) = D(P_2) = B^2(\mathbb{R}^n;\mathbb{C}^N)$. Recall that for the semiprincipal term b_1 of B we have $b_1(X) = (p_{1/2} \circ p_{2,\alpha})(X)$ for $X \neq 0$ with $p_{1/2}$ smooth and positively homogeneous of degree $1/2$ (by virtue of the hypothesis that the characteristic polynomial of a_1 have coefficients which are smooth functions of $p_{2,\alpha}$).

The first step in the proof is to show that

$$b^{\mathrm{w}} - p^{\mathrm{w}} = k_1^{\mathrm{w}} \in \Psi^{-1}_{\mathrm{sreg}}.$$

Since

$$b^{\mathrm{w}} = p_2^{\mathrm{w}} + (p_{1/2} \circ p_{2,\alpha})^{\mathrm{w}} + b_0^{\mathrm{w}} + \Psi^{-1}_{\mathrm{sreg}},$$

and since, by Theorem 1.11.2 in [3],

$$(p_{1/2} \circ p_{2,\alpha})^{\mathrm{w}} - p_{1/2}(p_{2,\alpha}^{\mathrm{w}}) \in \Psi^{-1}_{\mathrm{sreg}},$$

we get indeed that $k_1 \in S^{-1}_{\mathrm{sreg}}$.

For a later purpose, it is convenient to notice that $e^{\pm i2\pi q P_{2,\alpha}} = \mathrm{id}$, where $2\pi q$ is the period of the bicharacteristics of $p_{2,\alpha}$. In fact, for $\phi \in \mathscr{S}$,

$$e^{\pm i2\pi q P_{2,\alpha}}\phi = \bigotimes_{k=1}^{n} e^{\pm i2\pi q\alpha_k P_{2,k}}\phi = \bigotimes_{k=1}^{n} \underbrace{e^{\pm i2\pi m_k P_{2,k}}}_{=\mathrm{id}}\phi = \phi,$$

since the $P_{2,k}$ commute over $\mathscr{S}(\mathbb{R}^n; \mathbb{C}^N)$. The fact that $2\pi q$ is an integer multiple of the period of the bicharacteristics of $p_{2,\alpha}$ follows form the fact that, being

$$H_{p_{2,\alpha}} = \sum_{j=1}^{n} \alpha_j \Big(\xi_j \partial_{x_j} - x_j \partial_{\xi_j}\Big),$$

the bicharacteristic flow is for all $t \in \mathbb{R}$ and $X \in \mathbb{R}^n_x \times \mathbb{R}^n_\xi$ given by

$$\begin{aligned}\exp(tH_{p_{2,\alpha}})(X) &= \sum_{j=1}^{n}(\cos(\alpha_j t)x_j + \sin(\alpha_j t)\xi_j) + \sum_{j=1}^{n}(-\sin(\alpha_j t)x_j + \cos(\alpha_j t)\xi_j) \\ &= \sum_{j=1}^{n} \exp(\alpha_j t H_{p_2^{(j)}})(X_j), \ \text{ where } \ X_j = (x_j, \xi_j),\ p_2^{(j)}(X_j) = |X_j|^2/2.\end{aligned}$$

We now want to show that $2\pi q$ is indeed *the* period of the bicharacteristics of $P_{2,\alpha}$. Suppose by contradiction that there is $0 < q' < q$ such that $2\pi q = 2\pi q' m'$ with $0 < m' \in \mathbb{Z}_+$ and $\exp(\pm 2\pi q' H_{p_{2,\alpha}}) = \mathrm{id}$, then we must have $\exp(\pm 2\pi q' \alpha_j H_{p_2^{(j)}}) = \mathrm{id}$ for all $j = 1, \ldots, n$. Therefore $2\pi q' \alpha_j \in 2\pi\mathbb{Z}$ which implies that m' divides m_j for all j, which is impossible. Therefore $2\pi q$ is the period of the bicharacteristics of $p_{2,\alpha}$.

We next show that the commutator $[p_{2,\alpha}^{\mathrm{w}}, b_0^{\mathrm{w}}] = 0$. Since $[p_{2,\alpha}^{\mathrm{w}}, b_0^{\mathrm{w}}]\big|_{\mathscr{S}} = [P_{2,\alpha}, B_0]\big|_{\mathscr{S}}$ and since $[p_{2,\alpha}^{\mathrm{w}}, b_0^{\mathrm{w}}] \in \Psi^0$, it follows that we may extend $[P_{2,\alpha}, B_0]\big|_{\mathscr{S}}$ as a bounded linear operator $[P_{2,\alpha}, B_0]\colon L^2 \longrightarrow L^2$. Hence, if we show that $[p_{2,\alpha}^{\mathrm{w}}, b_0^{\mathrm{w}}] = 0$ then also $[P_{2,\alpha}, B_0] = 0$.

Now, $b_0 \circ \exp(tH_{p_{2,\alpha}}) = b_0$ for all t by hypothesis. Hence

$$b_0 = \mathsf{R}_\alpha(b_0) := (2\pi q)^{-1} \int_0^{2\pi q} b_0 \circ \exp(tH_{p_{2,\alpha}})\, dt$$

(the X-ray transform of b_0 with respect to the bicharacteristics of $p_{2,\alpha}$), and on $\mathscr{S}$

$$[p_{2,\alpha}^{\mathrm{w}}, b_0^{\mathrm{w}}] = [p_{2,\alpha}^{\mathrm{w}}, \mathsf{R}_\alpha(b_0)^{\mathrm{w}}]$$

$$= \frac{-i}{2\pi q} \int_0^{2\pi q} \partial_t (e^{itP_{2,\alpha}} b_0^{\mathrm{w}} e^{-itP_{2,\alpha}})\, dt = \frac{-i}{2\pi q} [e^{itP_{2,\alpha}} b_0^{\mathrm{w}} e^{-itP_{2,\alpha}}]_0^{2\pi q} = 0.$$

In addition, also $b_0 \circ \exp(tH_{p_2}) = b_0$ for all t by hypothesis. Hence, we have also that on $\mathscr{S}$ ($\mathsf{R}(b_0)$ being the X-ray transform of b_0 with respect to the bicharacteristics of p_2)

$$[p_2^{\mathrm{w}}, b_0^{\mathrm{w}}] = [p_2^{\mathrm{w}}, \mathsf{R}(b_0)^{\mathrm{w}}] = \frac{-i}{2\pi q} \int_0^{2\pi q} \partial_t (e^{itP_2} b_0^{\mathrm{w}} e^{-itP_2})\, dt = \frac{-i}{2\pi q} [e^{itP_2} b_0^{\mathrm{w}} e^{-itP_2}]_0^{2\pi q} = 0.$$

Recall that the eigenspaces of P_2, made of Hermite functions, are invariant for $P_{2,\alpha}$ and vice versa. Therefore, the eigenspaces of $P_2 + P_{2,\alpha}$ are invariant for B_0. We may hence choose an orthonormal system $\{\phi_{\gamma,j};\ \gamma \in \mathbb{Z}_+^n,\ 1 \le j \le N\} \subset \mathscr{S}(\mathbb{R}^n; \mathbb{C}^N)$ of $L^2(\mathbb{R}^n; \mathbb{C}^N)$, made of eigenfunctions of both P and P_2, that also diagonalizes $B_0\big|_{W_k}$ on each space $W_k := \mathrm{Span}_{\mathbb{C}}\{\phi_{\gamma,j};\ |\gamma| = k,\ 1 \le j \le N\}$, $k \in \mathbb{Z}_+$. It follows that the eigenvalue of P associated with the eigenfunctions $\phi_{\gamma,j}$, for $|\gamma| = k$ and $1 \le j \le N$ is

$$k + \frac{n}{2} + p_{1/2}(\alpha(\gamma + 1/2)) + \mu_{\gamma,j}, \quad 1 \le j \le N,$$

where $\mu_{\gamma,j} \in \mathbb{R}$ is such that $B_0 \phi_{\gamma,j} = \mu_{\gamma,j} \phi_{\gamma,j}$, and $\alpha(\gamma + 1/2) = \sum_{h=1}^n \alpha_h(\gamma_h + 1/2)$.

Hence,

$$\mathrm{Spec}(P) = \bigcup_{k \ge 0} C_k,$$

where

$$C_k := \Big\{ k + \frac{n}{2} + p_{1/2}(\alpha(\gamma + 1/2)) + \mu_{\gamma,j};\ \gamma \in \mathbb{Z}_+^n, |\gamma| = k,\ 1 \le j \le N \Big\}.$$

We next wish to show that there is $M > 0$ such that

$$\mathrm{Spec}(A) \subset \bigcup_{k \ge 0} \Big(C_k + \Big[-\frac{M}{\sqrt{k + n/2}}, \frac{M}{\sqrt{k + n/2}} \Big] \Big), \tag{4.4}$$

where, for two sets $I, I' \subset \mathbb{R}$, we write $I + I' = \{a + b;\ a \in I,\ b \in I'\}$.

For that, we have to consider the diagonalizer e^{w} (and hence its L^2 bounded extension E) of a^{w} (see Theorem 3.1.3 in [7]). Then, $\mathrm{ind}\, E = \mathrm{ind}\, E_0 \ge 0$ by hypothesis since the index of an operator is invariant under compact perturbations. Thus, by the quasi-isometrization Theorem 3.2, we may assume that $E^* \colon L^2 \to L^2$ is an isometry (that is, $EE^* = I$). Letting

$$\tilde{r}^{\mathrm{w}} := (e^{\mathrm{w}})^* a^{\mathrm{w}} e^{\mathrm{w}} - p^{\mathrm{w}} = \big((e^{\mathrm{w}})^* a^{\mathrm{w}} e^{\mathrm{w}} - b^{\mathrm{w}} \big) + (b^{\mathrm{w}} - p^{\mathrm{w}}) \in \Psi_{\mathrm{sreg}}^{-1},$$

and noting that $(p_2^{\mathrm{w}})^{1/4} \in \Psi_{\mathrm{sreg}}^{1/2}$ with principal symbol $p_2^{1/4}$, we have that

$$(p_2^{\mathrm{w}})^{1/4}\tilde{r}^{\mathrm{w}}(p_2^{\mathrm{w}})^{1/4} \in \Psi^0_{\mathrm{sreg}}$$

can be extended to a bounded operator in $L^2(\mathbb{R}^n;\mathbb{C}^N)$. Hence, there is $M > 0$ such that for all $\psi \in \mathscr{S}(\mathbb{R}^n;\mathbb{C}^N)$

$$-M\|\psi\|_0^2 \leq \big((p_2^{\mathrm{w}})^{1/4}\tilde{r}^{\mathrm{w}}(p_2^{\mathrm{w}})^{1/4}\psi,\psi\big)_0 \leq M\|\psi\|_0^2, \tag{4.5}$$

that we rewrite in terms of the L^2 realizations of the ψdos involved as

$$-M\|\psi\|_0^2 \leq \big(P_2^{1/4}\tilde{R}P_2^{1/4}\psi,\psi\big)_0 \leq M\|\psi\|_0^2. \tag{4.6}$$

Now, recalling that $P_2^{1/4}\colon D(P_2^{1/4}) \subset L^2 \longrightarrow L^2$ is the self-adjoint unbounded L^2 realization of $(p_2^{\mathrm{w}})^{1/4}$, which is elliptic, we have that $D(P_2^{1/4}) = B^{1/2}(\mathbb{R}^n;\mathbb{C}^N)$ (dense in L^2) and $P_2^{1/4}$ is invertible with bounded inverse $P_2^{-1/4} : L^2 \to B^{1/2} \hookrightarrow L^2$. Therefore, by substituting $P_2^{-1/4}\phi$ for ψ in (4.6), we get that for all $\phi \in \mathscr{S}$

$$-M(P_2^{-1/2}\phi,\phi)_0 \leq (\tilde{R}\phi,\phi)_0 \leq M(P_2^{-1/2}\phi,\phi)_0.$$

Hence, for all $\phi \in B^2$,

$$\begin{aligned}\Big(\big(P - MP_2^{-1/2}\big)\phi,\phi\Big)_0 &\leq \big(\underbrace{P+\tilde{R}}_{=E^*AE}\phi,\phi\big)_0\\ &\leq \big((P + MP_2^{-1/2})\phi,\phi\big)_0,\end{aligned}$$

which leads to (4.4) for E^*AE by the minimax principle. But we also have that

$$\mathrm{Spec}(E^*AE) \setminus \{0\} = \mathrm{Spec}(A).$$

In fact,

$$\mathrm{Spec}(A) \subset \mathrm{Spec}(E^*AE),$$

since

$$A\phi_\lambda = \lambda\phi_\lambda, \quad \phi_\lambda \neq 0,$$

implies

$$(E^*AE)\,E^*\phi_\lambda = \lambda E^*\phi_\lambda,\ E^*\phi_\lambda \neq 0.$$

Moreover,

$$\mathrm{Spec}(E^*AE) \setminus \{0\} \subset \mathrm{Spec}(A),$$

since

$$E^*AE\psi_\zeta = \zeta\psi_\zeta, \quad \psi_\zeta \neq 0 \tag{4.7}$$

implies

$$AE\psi_\zeta = \zeta E\psi_\zeta.$$

Now, when $\psi_\zeta \notin \ker E$ then $\zeta \in \mathrm{Spec}(A)$. When $\psi_\zeta \in \ker E$ then $\zeta = 0$ by (4.7) while $0 \notin \mathrm{Spec}(A)$ since $A > 0$.

This concludes the proof of the lemma. □

We next generalize the previous result by removing the hypothesis on the non-negativity of the decoupling operator index.

Theorem 4.2. *Let $a = a^* \sim \sum_{j\geq 0} a_{2-j} \in S^2_{\mathrm{sreg}}$ be a 2nd-order SMGES with principal symbol $a_2 = p_2 I_N$, such that the corresponding unbounded operator $A > 0$ (this is no restriction). Suppose that the coefficients of the characteristic polynomial $\lambda \mapsto \det(\lambda - a_1(X))$ of the semiprincipal term a_1 are smooth functions of $p_{2,\alpha}$ and that there is a unitary diagonalizer e_0 of the semiprincipal symbol such that for the subprincipal symbol $b_0 = \mathrm{diag}(b_{0,h};\ 1 \leq h \leq r)$ of the resulting blockwise diagonalization of A we have*

$$b_0 \circ \exp(tH_{p_2}) = b_0, \quad b_0 \circ \exp(tH_{p_{2,\alpha}}) = b_0, \quad \forall t \in \mathbb{R}. \tag{4.8}$$

In addition, suppose there are $m_1, \ldots, m_n \in \mathbb{Z}_+ \setminus \{0\}$ coprime such that

$$\frac{m_1}{\alpha_1} = \ldots = \frac{m_n}{\alpha_n} =: q. \tag{4.9}$$

Then the eigenspaces of P_2 are invariant for $P_{2,\alpha}$ and the eigenspaces of $(P_2+P_{2,\alpha})\otimes I_{N_h}$ are invariant for $B_{0,h}$, for all $h = 1, \ldots, r$. Moreover, for each $h = 1, \ldots, r$, there is an orthonormal basis $\{\phi^{(h)}_{\gamma,j}\}_{\gamma\in\mathbb{Z}^n_+, 1\leq j\leq N_h} \subset \mathscr{S}(\mathbb{R}^n; \mathbb{C}^{N_h})$ of $L^2(\mathbb{R}^n; \mathbb{C}^{N_h})$ such that for all $k \in \mathbb{Z}_+$ we have $\mathrm{Ker}\big(P_2 \otimes I_{N_h} - (k + \frac{n}{2})\big) = \mathrm{Span}\{\phi^{(h)}_{\gamma,j};\ |\gamma| = k,\ 1 \leq j \leq N_h\}$, and

$$B_{0,h}\phi_{\gamma,j,h} = \mu^{(h)}_{\gamma,j}\phi^{(h)}_{\gamma,j}, \ \ \textit{with}\ |\mu^{(h)}_{\gamma,j}| \leq \|B_{0,h}\|_{L^2\to L^2}, \quad \forall \gamma \in \mathbb{Z}^n_+,\ \forall j = 1, \ldots, N_h,$$

and smooth function $p^{(1)}_{1/2}, \ldots, p^{(r)}_{1/2} \colon \mathbb{R}_+ \longrightarrow \mathbb{R}$, positively homogeneous of degree 1/2, such that $\lambda_{1,h} = p^{(h)}_{1/2}(p_{2,\alpha})$, $1 \leq h \leq r$, and, finally, constants $M, c > 0$ such that

$$\mathrm{Spec}(A) \subset \bigcup_{h=1}^{r+1} S_h(A), \tag{4.10}$$

where, for each $h = 1, \ldots, r+1$,

$$S_h(A) := \bigcup_{k\geq 0} \bigcup_{\substack{\gamma\in\mathbb{Z}_+^n \\ |\gamma|=k}} \bigcup_{j=1}^{N_h} (k + \frac{n}{2} + p_{1/2}^{(h)}(\alpha(\gamma+\frac{1}{2})) + \mu_{\gamma,j}^{(h)}) + \Big[-\frac{M}{\sqrt{k+n/2}}, \frac{M}{\sqrt{k+n/2}}\Big], \tag{4.11}$$

with

$$\alpha(\gamma+1/2) := \sum_{j=1}^{n} \alpha_j(\gamma_j + 1/2), \quad N_{r+1} := N, \quad p_{1/2}^{(r+1)} := 0, \quad \mu_{\gamma,j}^{(r+1)} := c,$$

for all γ and j.

Proof. The proof follows by an argument based on the construction of a system $\tilde{A}$ associated with A, having a decoupling operator $\tilde{e}_0$ with a nonnegative index. In fact, consider the blockwise diagonal system with two $N \times N$ blocks

$$\tilde{A} := \left[\begin{array}{c|c} A & 0_N \\ \hline 0_N & P_2 I_N + P_0 \end{array}\right],$$

where p_2 is the harmonic oscillator and

$$p_0 := -e_0^*(e_{-2}e_0^* p_2 + p_2 e_0 e_{-2}^* - \frac{i}{2}\left(e_0\left\{p_2, e_0^*\right\} + \left\{e_0, p_2 e_0^*\right\}\right) + e_{-1} p_2 e_{-1}^*) e_0 + cI_N,$$

where $c > 0$ is a real constant such that $P_0 > 0$ (so that also $P_2 I_N + P_0 > 0$). Hence,

$$\tilde{E}^* \tilde{A} \tilde{E} = \left[\begin{array}{c|c} B + \Psi^{-\infty} & 0_N \\ \hline 0_N & (P_2 + c) I_N + R \end{array}\right],$$

where $\tilde{e} := \left[\begin{array}{c|c} e & 0_N \\ \hline 0_N & e^* \end{array}\right]$ (e and e^* are the total symbols of E and E^*, respectively) and $R \in \Psi_{\rm sreg}^{-1}$ since by Proposition 6.1 in [7] (or by a straightforward computation, using the composition formula for matrix-valued ψdos) the subprincipal symbol of $E(P_2 I_N + P_0)E^*$ is cI_N by the definition of p_0.

Now, $\tilde{A}$ satisfies the hypotheses of Lemma 4.1. In fact, $\tilde{A} > 0$, $\operatorname{ind} \tilde{E} = \operatorname{ind} E + \operatorname{ind} E^* = 0$ and by Corollary 6.5 in [7] one has the subprincipal blocks

$$b_{0,h} = \pi_h(e_0^* a_0 e_0 - i\{e_0^*, p_2\} e_0)\pi_h^*.$$

By hypothesis $b_0 \circ \exp(tH_{p_{2,\alpha}}) = b_0$ and $b_0 \circ \exp(tH_{p_2}) = b_0$ for all t. Now, Lemma 4.1 yields (4.2) for $\tilde{A}$, that is,

$$\operatorname{Spec}(\tilde{A}) \subset \bigcup_{h=1}^{r+1} S_h(A),$$

where, for each $h = 1, \dots, r+1$,

$$S_h(A) := \bigcup_{k\geq 0} \bigcup_{\substack{\gamma\in\mathbb{Z}_+^n \\ |\gamma|=k}} \bigcup_{j=1}^{N_h} \big(k + \frac{n}{2} + p_{1/2}^{(h)}(\alpha(\gamma + \frac{1}{2})) + \mu_{\gamma,j}^{(h)}\big) + \Big[-\frac{M}{\sqrt{k+n/2}}, \frac{M}{\sqrt{k+n/2}}\Big]$$

with $N_{r+1} := N$, $p_{1/2}^{(r+1)} := 0$ and $\mu_{\gamma,j}^{(r+1)} := c$ for all γ and j. Moreover, since $\tilde{A}$ is blockwise diagonal, we have $\mathrm{Spec}(\tilde{A}) = \mathrm{Spec}\,(A) \cup \mathrm{Spec}\,(P_2 I_N + P_0)$. Hence,

$$\mathrm{Spec}(A) \subset \bigcup_{h=1}^{r+1} S_h(A),$$

which completes the proof. □

Remark 4.3. A condition on e_0 granting that (4.8) be satisfied is that there exist smooth functions $\mathbb{R} \times \dot{\mathbb{R}}_X^{2n} \ni (t, X) \longmapsto (f_t(X), g_t(X)) \in \mathsf{M_N} \times \mathsf{M_N}$, positively homogeneous of degree 1, such that

$$\{p_2, f_t\} = 0, \quad e_0 \circ \exp(tH_{p_{2,\alpha}}) = f_t e_0, \quad a_0 = f_t^* \Big(a_0 \circ \exp(tH_{p_{2,\alpha}})\Big) f_t, \tag{4.12}$$

$$\{p_2, g_t\} = 0, \quad e_0 \circ \exp(tH_{p_2}) = g_t e_0, \quad a_0 = g_t^* \Big(a_0 \circ \exp(tH_{p_2})\Big) g_t. \tag{4.13}$$

This happens, e.g., for the 2×2 JC model. It fails, though, for the 3×3 JC model (in the Ξ-configuration; $n = 2$) for which, however, the subprincipal b_0 satisfies (4.8).

In fact, for the 2×2 JC model we have, using $p_{2,\alpha} = \alpha p_2$ and (2.1),

$$e_0 \circ \exp(tH_{p_2}) = \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & e^{-it} \end{bmatrix}}_{=g_t} e_0,$$

$$e_0 \circ \exp(tH_{p_{2,\alpha}}) = \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & e^{-i\alpha t} \end{bmatrix}}_{=f_t} e_0,$$

and (since we are dealing with diagonal matrices, for $a_0 = \mathrm{diag}(\gamma, -\gamma)$)

$$a_0 \circ \exp(tH_{p_2}) = g_t^* a_0 g_t, \quad a_0 \circ \exp(tH_{p_{2,\alpha}}) = f_t^* a_0 f_t.$$

In addition, by (2.2), b_0 is constant on the bicharacteristics of p_2 and $p_{2,\alpha}$.

In case of the 3×3 JC model, we instead have, using (2.3),

$$e_0 \circ \exp(tH_{p_2}) = \underbrace{\begin{bmatrix} e^{it} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{-it} \end{bmatrix}}_{=g_t} e_0,$$

while

$$e_0 \circ \exp(tH_{p_{2,\alpha}}) = \begin{bmatrix} e^{it\alpha_2}\frac{\alpha_2^2|\psi_2|^2}{|\alpha\psi|^2} + e^{it\alpha_1}\frac{\alpha_1^2|\psi_1|^2}{|\alpha\psi|^2} & 0 & (e^{it\alpha_1} - e^{it\alpha_2})\frac{\overline{\psi_1\psi_2}}{|\alpha\psi|^2} \\ 0 & 1 & 0 \\ (e^{-it\alpha_2} - e^{-it\alpha_1})\frac{\psi_1\psi_2}{|\alpha\psi|^2} & 0 & e^{-it\alpha_1}\frac{\alpha_1^2|\psi_1|^2}{|\alpha\psi|^2} + e^{-it\alpha_2}\frac{\alpha_2^2|\psi_2|^2}{|\alpha\psi|^2} \end{bmatrix},$$

so that if $\alpha_1 \neq \alpha_2$ we cannot factor a matrix f_t. However, by (2.4) and the fact that

$$\psi_j \circ \exp(tH_{p_{2,\alpha}}) = e^{-i\alpha_j t}\psi_j, \;\; j = 1, 2,$$

we have that the subprincipal symbol of the diagonalized operator is clearly constant on the bicharacteristics of p_2 and $p_{2,\alpha}$.

As a closing observation, we wish to remark that it is exactly when the index of the diagonalizer is negative that we have to throw in the spectrum the points given by the supplementary $N \times N$ auxiliary system $P_2 I_N + P_0$. However, those points are explicitly known and they give a small perturbation to the quasi-clustering of the system A we are interested in.

Declaration of competing interest

The authors declare that there is no competing interest.

Data availability

No data was used for the research described in the article.

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