Adaptive Observer-based Sinusoid Identification: Structured and Bounded Unstructured Measurement Disturbances

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Abstract— **The paper deals with an adaptive observer methodology for estimating the parameters of an unknown sinusoidal signal from a measurement perturbed by structured and unstructured uncertainties. The proposed technique makes it possible to handle measurement signals affected by structured uncertainties like, for example, bias and drifts which are typically present in applications. The stability of the estimator with respect to bounded additive disturbances is addressed by Input-to-State Stability arguments. The effectiveness of the proposed technique is shown through numerical simulations where comparisons with some recently proposed algorithms are also provided.**

I. Introduction

The problem of estimating the unknown parameters of a sinusoidal signal from noisy measurements arises in many engineering applications such as active noise cancellation, vibration control (see [1] and the references therein) and periodic disturbance rejection (see [2], [3], [4], [5], [6]). To account for the disturbances affecting the measurements in practical applications such as offsets in physical transducers and A/D converters or drifts in sensing devices influenced by temperature variations; recent research has focused on the robust sinusoid estimation problem in presence of both structured and unstructured uncertainties (see, for example, [7], [8], [9], [10], [11] and [12] and the references cited therein).

Among the techniques proposed in the literature for estimating the Amplitude, the Frequency and the Phase (AFP) of an unknown sinusoid from uncertain measurements, the adaptive notch-filtering method is one of the most popular approaches owing to its simple practical implementation. It consists in filtering a signal with a very sharp notch whose center frequency is adaptively adjusted (see [13], [14]). Although the aforementioned method natively applies only to unbiased sinusoids, an extension of the adaptive notch-filtering scheme to the biased sinusoid estimation problem has been proposed recently in [10].

Besides adaptive notch-filters, the Phase-Locked-Loop (PLL) filter topology is another popular structure used for developing nonlinear estimation methods capable to provide robust estimates in a noisy environment (see [15], [16], [6], [17] and [18]). A recent PLL-based method conceived to retrieve the parameters of an unknown sinusoid from a biased measurement can be found in [9]. An improved fourth-order frequency estimator that can cope with bias has been also proposed in [8]. By adopting an original switching strategy, this algorithm is able to remove the effect of high-frequency band noise and to provide accurate estimates.

Recent research efforts have been devoted to incorporate the sinusoidal signal generator into an adaptive observer model, such that the parameters of interest can be identified via adaptation (see [19] and [20]). An extension of the method proposed in [20] capable to address the presence of bias has been presented in [11].

In this context, the paper deals with a new methodology which combines the AFP approach presented in [21] based on suitable pre-filtering with the design of an adaptive observer. The proposed technique allows to address the AFP estimation problem when measurements are corrupted by structured uncertainties modeled as finite-order time-polynomial functions (for example bias and drifts) and by unstructured bounded disturbances. A complete stability analysis is carried out and extensive simulation trials are provided in which the proposed AFP estimation algorithm is compared with some recently proposed tools.

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II. Notation and Basic Definitions

Let $\mathbb{R}$, $\mathbb{R}_{\geq 0}$ and $\mathbb{R}_{>0}$ denote the real, the non-negative real and the strict positive real sets of numbers, respectively. Given a vector $\mathbf{x} \in \mathbb{R}^n$, we will denote as $|\mathbf{x}|$ the Euclidean norm of $\mathbf{x}$. Given a time-varying vector $\mathbf{x}(t) \in \mathbb{R}^n$, $t \in \mathbb{R}_{\geq 0}$ we will denote as $\|\mathbf{x}\|_\infty$ the quantity $\|\mathbf{x}\|_\infty = \sup_{t\geq 0}|\mathbf{x}(t)|$. Given a matrix $\mathbf{A} \in \mathbb{R}^{n\times n}$, then $||\mathbf{A}||$ will denote $\max_{\mathbf{x}\in\mathbb{R}^n\setminus 0}\{|\mathbf{A}\mathbf{x}|/|\mathbf{x}|\}$.

The notions of functions of class $\mathcal{K}$, class $\mathcal{K}_\infty$, and class $\mathcal{KL}$ are used to characterize stability properties. A function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ belongs to the class $\mathcal{K}$ if it is continuous, strictly increasing and $\alpha(0) = 0$. If, in addition $\lim_{s\to\infty} = \infty$ then it belongs to the class $\mathcal{K}_\infty$. A continuous function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ belongs to the class $\mathcal{KL}$ if, for any fixed $t \in \mathbb{R}_{\geq 0}$, the function $\beta(\cdot, t)$ is a $\mathcal{K}$-function with respect to the first argument and if, for any fixed $s \in \mathbb{R}_{\geq 0}$, the function $\beta(s, t)$ is monotonically decreasing with respect to t and $\lim_{t\to\infty} \beta(s, t) = 0$. Given an i-times differentiable vector of signals $\mathbf{u}(t) \in \mathbb{R}^n, \forall t \in \mathbb{R}_{\geq 0}$, we denote by $\frac{d^i}{dt^i}\mathbf{u}(t)$ the vector of i-th derivative signals.

Consider the following dynamical system

$$\dot{\mathbf{x}} = f(\mathbf{x}, \mathbf{u}) \tag{1}$$

with $\mathbf{x} \in \mathbb{R}^n$, $\mathbf{u} \in \mathbb{R}^m$, $f(0, 0) = 0$ and $f(\mathbf{x}, \mathbf{u})$ locally Lipschitz in $\mathbb{R}^n \times \mathbb{R}^m$.

Definition 2.1 (ISS): The system (1) is ISS (Input-to-State Stable) if there exist a $\mathcal{KL}$-function $\beta(\cdot, \cdot)$ and a class $\mathcal{K}$-function such that, for any input $\mathbf{u} \in \mathbb{R}^m$ and any initial condition $\mathbf{x}_0 \in \mathbb{R}^n$, the trajectory of the system verifies

$$|\mathbf{x}(t)| \leq \beta(|\mathbf{x}_0|, t) + \gamma(\|\mathbf{u}\|_\infty) \tag{2}$$

Definition 2.2 (ISS-Lyapunov Function): A function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ of class $\mathcal{C}^1$ is an ISS-Lyapunov function for (1) if there exist three $\mathcal{K}_\infty$-functions $\underline{\alpha}(\cdot)$, $\overline{\alpha}(\cdot)$, $\alpha(\cdot)$ and a $\mathcal{K}$-function $\mathcal{X}(\cdot)$ such that

$$\underline{\alpha}(|\mathbf{x}|) \leq V(\mathbf{x}) \leq \overline{\alpha}(|\mathbf{x}|), \ \ \forall \mathbf{x} \in \mathbb{R}^n \tag{3}$$

and

$$|\mathbf{x}| \geq \mathcal{X}(|\mathbf{u}|) \Rightarrow \frac{\partial V}{\partial \mathbf{x}} f(\mathbf{x}, u) \leq -\alpha(|\mathbf{x}|), \ \forall \mathbf{x}\in\mathbb{R}^n, \ \forall \mathbf{u}\in\mathbb{R}^m \tag{4}$$

Theorem 2.1 ([22]): The system (1) is ISS if and only if it admits an ISS-Lyapunov function. □

III. Problem statement and Preliminaries

Our objective consists in estimating the amplitude, the frequency and the phase of a sinusoidal signal

$$s(t) = A\sin(\vartheta(t)) \;\; t \in \mathbb{R}_{\geq 0}$$
$$\begin{cases} \dot{\vartheta}(t) &= \omega^*, \;\; t \in \mathbb{R}_{\geq 0} \\ \vartheta_0 &= \phi, \end{cases} \tag{5}$$

given the perturbed measurement:

$$\hat{y}(t) = s(t) + \sum_{k=1}^{n_d} b_k t^{k-1} + d(t), \;\; t \in \mathbb{R}_{\geq 0} \tag{6}$$

where the term $\sum_{k=1}^{n_d} b_k t^{k-1}$ represents a time-polynomial structured perturbation, with b_k unknown for any $k \in \{1, \ldots, n_d\}$, and where $d(t)$ is a bounded additive unstructured disturbance with $\|d\|_\infty \leq \overline{d}$, $\overline{d} \in \mathbb{R}_{\geq 0}$. In the sequel, we will refer to $d(t)$ as *measurement noise*. We assume that the frequency of the sinusoid is bounded by a constant $\bar{\omega}$, known conservatively: $\omega^* \leq \bar{\omega}$.

In order to cope with structured and unstructured uncertainty without differentiation, we are going to extend the pre-filtering strategy proposed in [21] (see also the GPI observer approach [23] for a possible alternative way to recover the time-derivatives). To this end, let us consider the AFP problem for the noise-free signal

$$y(t) = s(t) + \sum_{k=1}^{n_d} b_k t^{k-1}, \;\; t \in \mathbb{R}_{\geq 0}. \tag{7}$$

In such a simplified setting, the pre-filtering method consists in computing n_d auxiliary filtered signals $x_1(t)$, $x_2(t)$,..., $x_{1+n_d}(t)$ obtained as follows

$$\begin{cases} \dot{x}_1(t) = \lambda\left(\beta y(t) - x_1(t)\right), \;\; t \in \mathbb{R}_{\geq 0} \\ x_1(0) = x_{1_0}. \end{cases}$$

$$\begin{cases} \dot{x}_k(t) = \lambda\left(\beta x_{k-1}(t) - x_k(t)\right), \;\; t \in \mathbb{R}_{\geq 0} \\ x_k(0) = x_{k_0}. \end{cases}$$

for $k \in \{2, \ldots, 1+n_d\}$, where $\lambda \in \mathbb{R}_{>0}$ is an arbitrary positive constant and $\beta \in (0, 1]$ is a damping coefficient. Defining

$$\mathbf{x}(t) \triangleq [\, x_1(t), \ldots, x_{1+n_d}(t)]^\top,$$

a state-space realization of the filter producing the signal $x_{1+n_d}(t)$ is

$$\begin{cases} \dot{\mathbf{x}} = \mathbf{A}_{\lambda,\beta}\mathbf{x}(t) + \boldsymbol{b}_{\lambda,\beta}\, y(t), \;\; t \in \mathbb{R}_{\geq 0} \\ \mathbf{x}(0) = \mathbf{x}_0, \end{cases} \tag{8}$$
$$x_{1+n_d}(t) = \mathbf{c}^\top \mathbf{x}(t), \;\; t \in \mathbb{R}_{\geq 0}$$

with arbitrary initial conditions $\mathbf{x}_0 \in \mathbb{R}^{1+n_d}$ and where

$$\mathbf{A}_{\lambda,\beta} = \begin{bmatrix} -\lambda & 0 & \cdots & \cdots & 0 \\ \beta\lambda & -\lambda & \ddots & & \vdots \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \beta\lambda & -\lambda \end{bmatrix},$$

$$\mathbf{b}_{\lambda,\beta} = [\; \beta\lambda \;\; 0 \;\; \cdots \;\; 0 \;]^\top,$$

$$\mathbf{c}^\top = [\; 0 \;\; \cdots \;\; 0 \;\; 1 \;].$$

In view of the proposed filter's structure, it holds that

$$\mathbf{c}^\top \mathbf{A}_{\lambda,\beta}^k \mathbf{b}_{\lambda,\beta} = 0, \;\; \forall k \in \{0, \ldots, n_d - 1\},$$

then, $\dot{x}_{1+n_d}(t)$, $\ddot{x}_{1+n_d}(t)$,..., $\frac{d^{1+n_d}}{dt^{1+n_d}} x_{1+n_d}(t)$ are all available.

$$\begin{aligned} \dot{x}_{1+n_d}(t) &= \mathbf{c}^\top \mathbf{A}_{\lambda,\beta}\mathbf{x}(t), \\ &\;\;\vdots \\ \frac{d^k}{dt^k} x_{1+n_d}(t) &= \mathbf{c}^\top \mathbf{A}_{\lambda,\beta}^k \mathbf{x}(t), \\ &\;\;\vdots \\ \frac{d^{1+n_d}}{dt^{1+n_d}} x_{1+n_d}(t) &= \mathbf{c}^\top \mathbf{A}_{\lambda,\beta}^{n_d}\left(\mathbf{A}_{\lambda,\beta}\mathbf{x}(t) + \mathbf{b}_{\lambda,\beta} y(t)\right). \end{aligned} \tag{9}$$

Let us denote by $H_k(s)$ the transfer function

$$H_k(s) = \frac{\lambda^k \beta^k}{(\lambda + s)^k}, \tag{10}$$

such that

$$\mathcal{L}[x_k](s) = H_k(s)\mathcal{L}[y](s). \tag{11}$$

At this point, noting that the Laplace transform of the measured signal can be expressed as

$$\mathcal{L}[y](s) = A\frac{s\sin(\phi) + \omega^*\cos(\phi)}{s^2 + \omega^{*2}} + \sum_{k=1}^{n_d} b_k (k-1)! \frac{1}{s^k},$$

then, neglecting the initial conditions of the internal filter's states, we have that:

$$\begin{aligned} \mathcal{L}[x_{1+n_d}](s) &= H_{1+n_d}(s) A \frac{s\sin(\phi) + \omega^*\cos(\phi)}{s^2 + \omega^{*2}} \\ &\quad + H_{1+n_d}(s) \sum_{k=1}^{n_d} b_k (k-1)! \frac{1}{s^k}. \end{aligned}$$

The transform of the n_d-th derivative of x_{1+n_d} writes

$$\begin{aligned} \mathcal{L}\left[\frac{d^{n_d} x_{1+n_d}}{dt^{n_d}}\right](s) &= H_{1+n_d}(s) A \frac{s\sin(\phi) + \omega^*\cos(\phi)}{s^2 + \omega^{*2}} s^{n_d} \\ &\quad + H_{1+n_d}(s) \sum_{k=1}^{n_d} b_k (k-1)! s^{n_d - k} \end{aligned}$$

which, in the time-domain, leads to the following asymptotic sinusoidal steady-state behavior:

$$\frac{d^{n_d}}{dt^{n_d}} x_{1+n_d}(t) \xrightarrow{t\to\infty} \frac{d^{n_d}}{dt^{n_d}} \bar{x}_{1+n_d}(t) = A_z \sin(\vartheta_z(t)) \tag{12}$$

where

$$\begin{aligned} A_z &= A\,\omega^{*n_d}\, |H_{1+n_d}(j\omega^*)|, \\ \vartheta_z(t) &= \vartheta(t) + \angle H_{1+n_d}(j\omega^*) + \frac{\pi}{2} n_d. \end{aligned} \tag{13}$$

Consider the vector of auxiliary derivatives

$$\mathbf{z}(t) = \begin{pmatrix} z_1(t) \\ z_2(t) \end{pmatrix} \triangleq \begin{pmatrix} \frac{d^{n_d}}{dt^{n_d}} x_{1+n_d}(t) \\ \frac{d^{1+n_d}}{dt^{1+n_d}} x_{1+n_d}(t) \end{pmatrix}$$

such that $\mathbf{z}(t)$ tends asymptotically to sinusoidal stationary equilibrium as well

$$z_1(t) \xrightarrow{t\to\infty} \bar{z}_1(t) = \frac{d^{n_d}}{dt^{n_d}}\bar{x}_{1+n_d}(t) = A_z \sin(\vartheta_z(t))$$
$$z_2(t) \xrightarrow{t\to\infty} \bar{z}_2(t) = A_z\omega^* \cos(\vartheta_z(t))$$

For the sake of the further discussion, let us introduce the matrix

$$\mathbf{\Lambda} = \begin{bmatrix} 0 & \mathbf{c}^\top \mathbf{A}_{\lambda,\beta}^{n_d} \\ \mathbf{c}^\top \mathbf{A}_{\lambda,\beta}^{n_d}\mathbf{b}_{\lambda,\beta} & \mathbf{c}^\top \mathbf{A}_{\lambda,\beta}^{1+n_d} \end{bmatrix}.$$

In view of (9), we have that the vector of auxiliary derivatives $\mathbf{z}(t)$ can be expressed in compact form by

$$\mathbf{z}(t) = \mathbf{\Lambda}\left[y(t)^\top \;\; \mathbf{x}(t)^\top\right]^\top. \tag{14}$$

Now, it is worth to point out that there exists an initial filter's state $\mathbf{x}(0) = \bar{\mathbf{x}}_0$ which leads to a filtered state trajectory $\bar{\mathbf{x}}(t)$ whose projection on the $\mathbf{z}$ subspace ($\mathbf{z}(t) = \mathbf{\Lambda}\left[\; y(t)^\top \;\; \bar{\mathbf{x}}(t)^\top \;\right]^\top$) matches the stationary sinusoidal behavior since the very beginning, that is:

$$\bar{\mathbf{x}}(t), t\in\mathbb{R}_{\geq 0}: \;\; \mathbf{z}(t) = \mathbf{\Lambda}\left[y(t)^\top \;\; \bar{\mathbf{x}}(t)^\top\right]^\top, \; \forall t \in \mathbb{R}_{\geq 0}. \tag{15}$$

Although $\bar{\mathbf{x}}_0$ is not known, it will be instrumental to study the stability properties of the estimation system that will be introducing in the next section.

It is easy to show that the vector of stationary sinusoidal derivatives $\bar{\mathbf{z}}(t)$ satisfies the following differential equation

$$\begin{cases} \dot{\bar{\mathbf{z}}}(t) = \mathbf{A}\bar{\mathbf{z}}(t) + \Omega^*\mathbf{A}_1\bar{\mathbf{z}}(t) \\ \bar{z}_1(t) = \mathbf{C}\bar{\mathbf{z}}(t) \end{cases} \tag{16}$$

where $\Omega^* = \omega^{*2}$, $\mathbf{A} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $\mathbf{A}_1 = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}$, and $\mathbf{C} = \begin{pmatrix} 1 & 0 \end{pmatrix}$.

Now, assuming that an estimate of the parameters of the auxiliary signal $x_{1+n_d}(t)$ is available ($\hat{A}_z, \hat{\omega}, \hat{\theta}_z$), then, in view of the pre-filter structure, the original parameters can obtained by:

$$\hat{A}(t) = \frac{\hat{A}_z(t)}{\hat{\omega}^{n_d}}\left(\frac{1}{|H_{1+n_d}(j\hat{\omega})|}\right), \tag{17}$$

$$\hat{\vartheta}(t) = \hat{\vartheta}_z(t) - \angle H_{1+n_d}(j\hat{\omega}) - n_d\frac{\pi}{2}. \tag{18}$$

Moreover, considering that

$$|H_{1+n_d}(j\hat{\omega})| = \left(\frac{\lambda\beta}{(\lambda^2+\hat{\omega}^2)^{\frac{1}{2}}}\right)^{1+n_d},$$

$$\angle H_{1+n_d}(j\hat{\omega}) = (1+n_d)\arctan\left(\frac{-\hat{\omega}}{\lambda}\right)$$

we can rearrange (17) and (18) in the following form:

$$\hat{A}(t) = \frac{\hat{A}_z(t)}{\hat{\omega}^{n_d}}\left(\frac{(\lambda^2+\hat{\omega}^2)^{\frac{1}{2}}}{\lambda\beta}\right)^{1+n_d}, \tag{19}$$

$$\hat{\vartheta}(t) = \hat{\vartheta}_z(t) + (1+n_d)\arctan\left(\frac{\hat{\omega}}{\lambda}\right) - n_d\frac{\pi}{2}. \tag{20}$$

IV. The Adaptive Observer

Now, we are going to address the original problem of estimating the parameters of a sinusoid from measurements corrupted simultaneously by both structured perturbations and bounded noises.

Let us denote by $\hat{\mathbf{x}}(t)$ the state vector of the following filter, driven by the noisy signal $\hat{y}(t)$ (see (6)) and evolving from an arbitrary initial condition $\hat{\mathbf{x}}_0$:

$$\begin{cases} \dot{\hat{\mathbf{x}}} = \mathbf{A}_{\lambda,\beta}\hat{\mathbf{x}}(\mathbf{t}) + \mathbf{b}_{\lambda,\beta}\hat{\mathbf{y}}(\mathbf{t}), \;\; \mathbf{t}\in\mathbb{R}_{\geq\mathbf{0}} \\ \hat{\mathbf{x}}(0) = \hat{\mathbf{x}}_0, \end{cases}$$
$$\hat{x}_{1+n_d}(t) = \mathbf{c}^\top\hat{\mathbf{x}}(t), \;\; t\in\mathbb{R}_{\geq 0}, \tag{21}$$

and let $\hat{\bar{\mathbf{z}}}(t) \triangleq \left[\hat{\bar{z}}_1(t), \hat{\bar{z}}_2(t)\right]^\top$ be the vector of the real computable perturbed derivative signals obtained by:

$$\hat{\bar{\mathbf{z}}}(t) = \mathbf{\Lambda}[\hat{y}(t)^\top \;\; \hat{\mathbf{x}}(t)^\top]^\top. \tag{22}$$

By introducing the estimated state $\hat{z}(t)$ and the estimated squared-frequency $\hat{\Omega}(t) = (\hat{\omega}(t))^2$, the following adaptive observer is proposed:

$$\begin{cases} \dot{\hat{\mathbf{z}}}(\mathbf{t}) &=& (\mathbf{A}-\mathbf{LC})\hat{\mathbf{z}}(t) + \mathbf{LC}\hat{\bar{\mathbf{z}}}(t) + \mathbf{A}_1\hat{\bar{\mathbf{z}}}(t)\hat{\Omega}(t) \\ && -\mu\xi(t)\xi(t)^\top(\hat{\mathbf{z}}(t) - \hat{\bar{\mathbf{z}}}(t)) \\ \dot{\xi}(t) &=& (\mathbf{A}-\mathbf{LC})\xi(t) + \mathbf{A}_1\hat{\bar{\mathbf{z}}}(t) \\ \dot{\hat{\Omega}}(t) &=& -\mu\xi(t)^\top(\hat{\mathbf{z}}(t) - \hat{\bar{\mathbf{z}}}(t)) \end{cases} \tag{23}$$

where $\mu\in\mathbb{R}_{\geq 0}$ is an arbitrary positive constant and $\mathbf{L}$ is the observer gain, obtained by assigning the poles of the observer such that $(\mathbf{A}-\mathbf{LC}) < 0$.

V. ISS Property of the Adaptation Scheme

In order to characterize the stability properties of the frequency estimation and adaptive observer system (21), (22) and (23), let us first analyze the stability of the filter dynamics. Introducing the error vector with respect to $\bar{\mathbf{x}}(t)$: $\tilde{\mathbf{x}}(t) \triangleq \hat{\mathbf{x}}(t) - \bar{\mathbf{x}}(t)$, and considering that $d(t) = \hat{y}(t) - y(t)$, the dynamics of $\tilde{\mathbf{x}}(t)$ can be written as:

$$\begin{cases} \dot{\tilde{\mathbf{x}}}(t) = \mathbf{A}_{\lambda,\beta}\tilde{\mathbf{x}}(t) + \mathbf{b}_{\lambda,\beta}\, d(t), \;\; t\in\mathbb{R}_{\geq 0} \\ \tilde{\mathbf{x}}(0) = \hat{\mathbf{x}}_0 - \bar{\mathbf{x}}_0. \end{cases} \tag{24}$$

Being $\mathbf{A}_{\lambda,\beta}$ Hurwitz, there exists a positive definite matrix $\mathbf{P}$ that solves the linear Lyapunov's equation: $\mathbf{P}\mathbf{A}_{\lambda,\beta} + \mathbf{A}_{\lambda,\beta}^\top\mathbf{P} = -\mathbf{I}$. Let $W(\tilde{\mathbf{x}}) \triangleq \tilde{\mathbf{x}}^\top\mathbf{P}\tilde{\mathbf{x}}$, then there exist two positive scalars $a_1, a_2 \in\mathbb{R}_{>0}$ such that

$$a_1|\tilde{\mathbf{x}}|^2 \leq W(\tilde{\mathbf{x}}) \leq a_2|\tilde{\mathbf{x}}|^2, \quad \forall\tilde{\mathbf{x}}.$$

The derivative of W along the system's state trajectory satisfies the inequality

$$\frac{\partial W}{\partial\tilde{\mathbf{x}}}(\mathbf{A}_{\lambda,\beta}\tilde{\mathbf{x}} + \mathbf{b}_{\lambda,\beta}d) \quad\leq\quad -|\tilde{\mathbf{x}}|^2 + 2\,\|\mathbf{P}\|\,|\mathbf{b}_{\lambda,\beta}|\,|d|\,|\tilde{\mathbf{x}}|.$$

For any $0<\epsilon<1$, let

$$\mathcal{X}(s) = \frac{2\,\|\mathbf{P}\|\,|\mathbf{b}_{\lambda,\beta}|}{1-\epsilon}s$$

with $s\in\mathbb{R}_{\geq 0}$. It is easy to show that

$$|\tilde{\mathbf{x}}| \geq \mathcal{X}(|d|) \;\;\Rightarrow\;\; \frac{\partial W}{\partial\tilde{\mathbf{x}}}(\mathbf{A}_{\lambda,\beta}\tilde{\mathbf{x}} + \mathbf{b}_{\lambda,\beta}d) \leq -|\tilde{\mathbf{x}}|^2,$$

and that the system is ISS with asymptotic gain

$$\gamma_x(s) = a_1^{-1}a_2\,\mathcal{X}(s).$$

In view of the ISS property of the linear auxiliary filter (24), for any arbitrary $\nu\in\mathbb{R}_{>0}$ and for any finite-norm initial error $\tilde{\mathbf{x}}_0$, the error vector $\tilde{\mathbf{x}}(t)$ will enter in a closed ball of radius $\gamma_x(\|d\|_\infty)+\nu \leq \gamma_x(\bar{d})+\nu$ in finite time $T_{\tilde{\mathbf{x}}_0,\nu}$. Thanks to (22), the vector $\tilde{\bar{\mathbf{z}}}(t) \triangleq \hat{\bar{\mathbf{z}}}(t) - \bar{\mathbf{z}}(t)$ will enter in finite-time $T_\delta = T_{\tilde{\mathbf{x}}_0,\nu}$ (for the sake of simplifying the notation, we have dropped the dependence of the reach-time T_δ on initial conditions) in a closed ball of radius $\gamma_z(\bar{d}) + \delta$ centered at the origin, with

$$\delta = \bar{\lambda}\nu, \quad \gamma_z(s) = \bar{\lambda}\left(\gamma_x(s) + s\right), \forall s \in \mathbb{R}_{\geq 0}, \tag{25}$$

where $\overline{\lambda} = ||\Lambda||$.

In view of (23), we have the expression of error dynamic as follows by defining some instrumental error variables: $\tilde{\mathbf{z}}(t) = \hat{\mathbf{z}}(t) - \bar{\mathbf{z}}(t)$, $\tilde{\Omega}(t) = \hat{\Omega}(t) - \Omega^*$, $\psi(t) = \tilde{\mathbf{z}}(t) - \xi(t)\tilde{\Omega}(t)$

$$\begin{aligned} \dot{\tilde{\mathbf{z}}}(\mathbf{t}) &= (\mathbf{A} - \mathbf{LC})\tilde{\mathbf{z}}(t) + (\mathbf{LC} + \Omega^* \mathbf{A}_1)\tilde{\bar{\mathbf{z}}}(t) \\ &\quad + \tilde{\Omega}(t)\mathbf{A}_1\hat{\bar{\mathbf{z}}}(t) - \xi(t)\mu\xi(t)^\top(\hat{\mathbf{z}}(t) - \hat{\bar{\mathbf{z}}}(t)) \end{aligned} \tag{26}$$

$$\dot{\tilde{\Omega}}(t) = -\mu\xi(t)^\top\xi(t)\tilde{\Omega}(t) + \mu\xi(t)^\top\left(\tilde{\bar{\mathbf{z}}}(t) - \psi(t)\right) \tag{27}$$

and

$$\dot{\psi}(t) = (\mathbf{A} - \mathbf{LC})\psi(t) + (\mathbf{LC} + \Omega^* \mathbf{A}_1)\tilde{\bar{\mathbf{z}}}(t) \tag{28}$$

In order to prove the convergence of the estimation error, the following assumption is needed.

Assumption 1: The solution $\xi(t)$ of $\dot{\xi}(t) = (\mathbf{A} - \mathbf{LC})\xi(t) + \mathbf{A}_1\hat{\bar{z}}(t)$ is persistently exciting in the sense that there exist $\sum_{k=1}^{2} g_k^2 A_z^2 \sin^2(\vartheta_z + \phi_{G_k}) > \epsilon > 0$ such that

$$\xi(t)^\top\xi(t) > \epsilon, \; \forall t > 0. \tag{29}$$

In a further Remark 5.1, we will discuss how the poles of the observer can be chosen to guarantee that the excitation condition (29) is always verified in nominal conditions, for a sinusoid with non-zero amplitude.

Theorem 5.1 (ISS of the adaptive observer system): If assumption 1 holds, then given the sinusoidal signal $s(t)$ generated by (5) and the perturbed measurement model (6), the adaptive observer as well as the frequency estimator given by (21), (22) and (23) are ISS with respect to any bounded additive measurement perturbation $|d(t)| \leq \overline{d}$. □

Proof: Let us introduce a Lyapunov function $V_\psi = \psi(t)^\top\mathbf{Q}\psi(t)$, where $\mathbf{Q}$ is a positive definite matrix that solves the linear Lyapunov's equation: $\mathbf{Q}(\mathbf{A} - \mathbf{LC}) + (\mathbf{A} - \mathbf{LC})^\top\mathbf{Q} = -\mathbf{I}$. In view of the $\psi(t)$ dynamics (28), the derivative of the Lyapunov function verifies the inequality

$$\frac{\partial V_\psi}{\partial \psi}\dot{\psi}(t) \leq -|\psi(t)|^2 + 2||Q||\,||\mathbf{LC} + \Omega^*\mathbf{A}_1||\,|\tilde{\bar{\mathbf{z}}}(t)|\,|\psi(t)|. \tag{30}$$

Hence, V_ψ is an ISS-Lyapunov function for error dynamic $\psi(t)$ with respect to the $\tilde{\bar{z}}(t)$. Moreover, the dynamics of $\tilde{\bar{z}}(t)$ is ISS with respect to disturbance $d(t)$, so that V_ψ is, in turn, ISS with respect to $d(t)$. Now let $V_{\tilde{\Omega}} = \frac{1}{2}\tilde{\Omega}(t)^2$ be a candidate ISS-Lyapunov function for the frequency-estimation subsystem. Then the derivative of $V_{\tilde{\Omega}}$ verifies the inequality

$$\frac{\partial V_{\tilde{\Omega}}}{\partial \tilde{\Omega}}\dot{\tilde{\Omega}}(t) \leq -\mu|\xi(t)|^2|\tilde{\Omega}(t)|^2 + \mu|\xi(t)||\tilde{\bar{\mathbf{z}}}(t) - \psi(t)|\,|\tilde{\Omega}(t)| \tag{31}$$

In view of (31), assumption 1, and considering that $|\xi(t)|$ is bounded (it is immediate to show that the dynamics of $\xi(t)$ is ISS with respect to the bounded input $\hat{\bar{z}}$), we have that $\tilde{\Omega}(t)$ ISS with respect to $\psi(t)$ and $\tilde{\bar{\mathbf{z}}}(t)$, which are all proven to be ISS with respect to the disturbance $d(t)$.

Finally, the identity $\tilde{\mathbf{z}}(t) = \psi(t) + \xi(t)\tilde{\Omega}(t)$, $\tilde{\mathbf{z}}(t)$ and the boundedness of $|\xi(t)|$together imply that also the state-estimation error $\tilde{\mathbf{z}}(t)$ is ISS with respect to $d(t)$. ■

Remark 5.1 (Observer Poles and Excitation): We remark that a central assumption for establishing the ISS property of the estimation error dynamics is that the excitation condition (29) is verified for any $t > 0$. We will show that this requirement can be fulfilled, in nominal conditions, by choosing accurately the observer poles. The dynamic equation of $\xi(t)$, in absence of noise and in stationary conditions can be written as follows

$$\dot{\xi}(t) = (\mathbf{A} - \mathbf{LC})\xi(t) + \mathbf{B}_\xi\bar{z}_1(t)$$

where $\mathbf{B}_\xi = \begin{pmatrix} 0 & -1 \end{pmatrix}^\top$. Then, in Laplace domain

$$\xi_k(s) = G_k(s)\bar{z}_1(s), \; k \in \{1, 2\}$$

in which $G_k(s) = \mathbf{e}_i^\top(s\mathbf{I} - \mathbf{A} + \mathbf{LC})^{-1}\mathbf{B}_\xi$ and $\mathbf{e}_i$ denote the i-th unit vector.

Assume that the poles of $\mathbf{A} - \mathbf{LC}$ are assigned to (p_1, p_2), which are either both on the negative real axis or complex conjugate with strictly negative real part. For the complex conjugate poles having the format $p_1, p_2 = a \pm jb$ with $a \in \mathbb{R}_{<0}$, $b \in \mathbb{R}$, it holds that:

$$\mathbf{L} = \begin{pmatrix} -(p_1 + p_2) \\ p_1 p_2 \end{pmatrix}, \quad \mathbf{A} - \mathbf{LC} = \begin{pmatrix} p_1 + p_2 & 1 \\ -p_1 p_2 & 0 \end{pmatrix}.$$

In the following lines, we will show that, by choosing the observer poles such that

$$a^2 > b^2, \; a \in \mathbb{R}_{<0}, \; b \in \mathbb{R}, \tag{32}$$

then the excitation condition (29) is verified, in nominal conditions, by any sinusoid of non-zero amplitude. Being $\bar{z}_1(t)$ sinusoidal with frequency ω, $\xi_k(t)$ is sinusoidal as well. Now, defining $p \triangleq p_1 + p_2$ and $q \triangleq p_1 p_2$, by simple algebra we obtain

$$G_1(s) = -\frac{1}{s^2 - ps + q}, \quad G_2(s) = -\frac{s - p}{s^2 - ps + q},$$

and

$$\phi_{G_1} = \arctan\frac{p\omega}{q - \omega^2}, \quad \phi_{G_2} = \arctan\frac{p^2\omega - q\omega + \omega^3}{pq}.$$

Owing to the structure of G_1 and G_2, the following inequality holds

$$\xi^\top(t)\xi(t) \geq \sum_{k=1}^{2} g_k^2 A_z^2 \sin^2(\vartheta_z + \phi_{G_k}),$$

where $g_k = |G_k(j\bar{\omega})|$ and ϕ_{G_k} represents the phase shift of $G_k(s)$ at the frequency of the sinusoid. In the following we will show, by contradiction, that $\phi_{G_1}(\omega) \neq \phi_{G_2}(\omega)$, $\forall\omega \in \mathbb{R}$. Let us make the hypothesis that there exists $\omega > 0$, such that $\phi_{G_1} = \phi_{G_2}$. The hypothesis is validated if and only if

$$\frac{p}{q - \omega^2} = \frac{p^2 - q + \omega^2}{pq},$$

which can be rearranged as

$$\omega^4 + (p^2 - 2q)\omega^2 + q^2 = 0. \tag{33}$$

In view of (32), equation (33) does not admit positive roots in the variable ω (since $p^2 - 2q = p_1^2 + p_2^2 > 0$ and $q^2 > 0$). Therefore, we can conclude that $\phi_{G_1} \neq \phi_{G_2}$, $\forall\omega > 0$.

Finally, due to the phase separation property, the following inequality is verified for all $t > 0$:

$$\xi^\top(t)\xi(t) \geq \sum_{k=1}^{2} g_k^2 A_z^2 \sin^2(\vartheta_z + \phi_{G_k}) > 0.$$

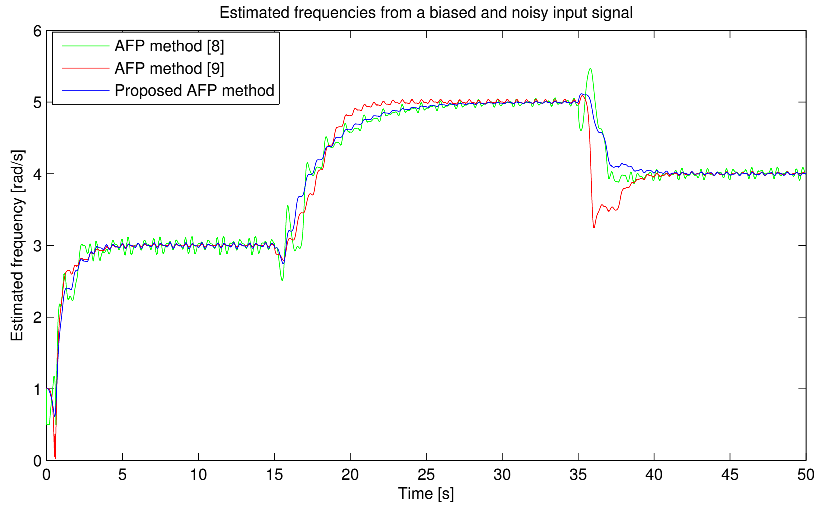

Fig. 1. Estimated frequency obtained by using the proposed AFP method (blue) compared with the frequency estimates provided by the AFP methods [9] (red line) and [8] (green)

It is worth to point out that, in case the amplitude of the sinusoid is small compared to disturbances, this condition may not be guaranteed for all $t > 0$. In order to ensure the stability of the adaptive observer, a robust (conservative) provision consists in setting $\mu = 0$, that is in disabling the adaptation, whenever poor excitation is detected.

Remark 5.2: In practical applications, the performance of the technique is indeed affected by the choice of sampling time. There exist estimation bias caused by discretization, and it increases as sampling time increases. Nevertheless, the discretization error can be compensated by proper calculation, that will not be clarified here.

VI. Simulation Results

In this section, the proposed AFP technique is simulated and compared with two recent AFP algorithms presented in [8] and [9].

A. Identification of a biased sinusoidal signal

The first evaluation deals with estimating the sinusoidal signal from the following biased and noisy measurement corrupt by high-order harmonic and random noise :

$$\hat{y}(t) = 1 + 5\sin(\omega(t)t + \pi/4) + d(t),$$

with a step-wise frequency change at time $t = 15$:

$$\omega(t) = \begin{cases} 3, & 0 \le t < 15, \\ 5, & 15 \le t < 35, \\ 4, & t \ge 35; \end{cases}$$

where $d(t)$ is a composite disturbance of bounded random noise with uniform distribution in the interval $[-0.5, 0.5]$ and a high-order harmonic denoted by $0.5\sin 15t$. The proposed AFP algorithm has been discretized in time by a forward-Euler method with sampling period $T_s = 0.001s$. It is worth noting that filter parameters λ, β are selected for pre-filtering with attenuation to the auxiliary signals subject to low-order filtering. μ is directly relating to the convergence speed of $\hat{\Omega}$, which is also affected by the poles' location. Roughly speaking, larger poles lead to a slow frequency adaptation but smoother estimates; the opposite for smaller poles.

For comparison, the parameters for all methods are chosen to ensure the estimated frequencies have similar response-time at the beginning. Then the stationary behaviour and the response to a step-wise frequency variation will be compared. All the adaptation parameters are shown below.

Algorithms	Tuning Parameters
Proposed	$\lambda = 2$, $\beta = 0.7$, $\mu = 5$, $p_1 = -4$, $p_2 = -6$
[9]	$K_s = 1, \lambda = 1, \omega_s = 2.5$
[8]	$\lambda = 3, k = 1$

With the identical initial condition $\hat{\omega}(0) = 1$, the simulation results are depicted in Figs. 1 and 2. According to Fig.1, all the three methods succeeded in tracking the sudden frequency change, yet the proposed method reveals relatively better robustness property than [8]. Note that the robustness of AFP method [8] can be improved by activating the equipped noise attenuation action of adaptively reducing the tuning parameters at the expense of longer transients for the next frequencies. Moreover, the proposed estimator is capable to provide nearly monotonic responses during transient periods.

Fig.2 shows the time-behavior of the reconstructed sinusoidal signal by the proposed AFP technique. As can be noticed, the sinusoidal signal is estimated successfully despite the presence of noise and a step-wise change in the frequency.

Remark 6.1: In the case that a harmonic disturbance exists, there is a new equilibrium point that appears in the system. It is not difficult to notice that attraction region of each equilibrium point is proportional to its amplitude. Thus harmonic with sufficient small magnitude will be considered as the noise, such that reliable estimates still can be guaranteed.

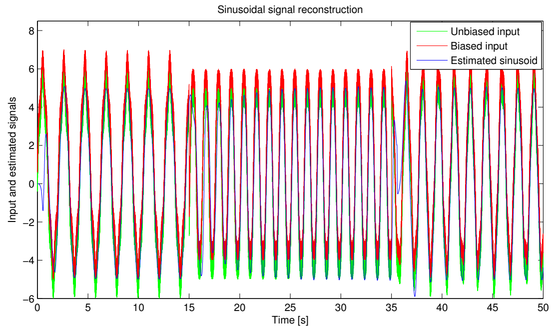

Fig. 2. Estimated sinusoidal signal by the proposed AFP method (blue). Both the pure sinusoidal signal to be estimated (green) and the biased noisy measurement (red) are shown.

B. Identification of a drifted sinusoidal signal

Let us consider a measured signal affected by time-polynomial structured perturbation:

$$\hat{y}(t) = 5\sin(3t + \pi/4) + 1 + 0.5t + d(t)\,,$$

where $d(t)$ is a random noise with the same characteristics as in the previous example. The tuning coefficients are set: $\lambda = 2.5, \beta = 0.6$ and $\mu = 10$, while the selected poles and initial condition are the same as given in previous example.

The results of the simulation are shown in Figs. 3 and 4, which show that the proposed technique is able to detect the sinusoidal signal even in presence of bounded noise and of an unknown drift term.

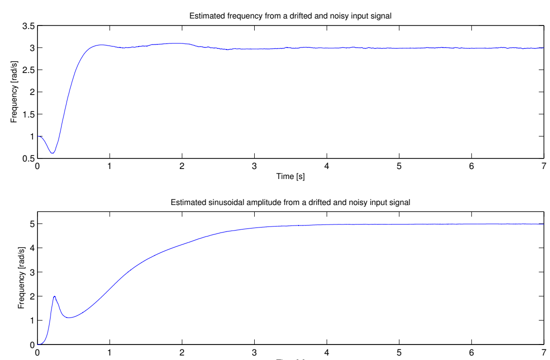

Fig. 3. Top figure: estimated frequency. Bottom figure: estimated amplitude of the sinusoid. Proposed AFP method (blue line) and the AFP method (red line). Apparently, the estimates convergence after 7s.

VII. Conclusions

In the this paper, the problem of sinusoid identification from uncertain measurements has been addressed. The proposed technique is based on the combination of a suitably designed adaptive observer with some recently proposed pre-filtering algorithms, while the dynamic order of the estimator is equal to $6 + n_d$. The stability properties have been established and comparisons with existing techniques have been provided showing the effectiveness of the proposed algorithm. Future research efforts will be devoted to the extension of the methodology to the case of multiple-frequencies estimation and to a larger class of structured measurement uncertainties.

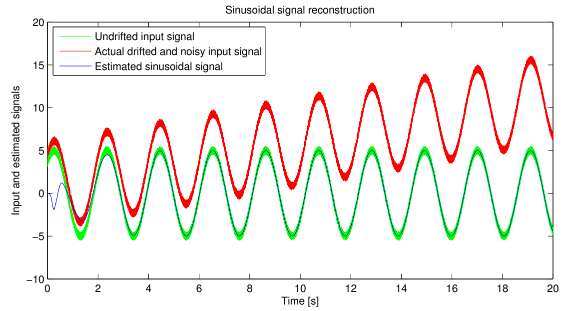

Fig. 4. Estimated sinusoidal signal by the proposed AFP method.

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