An Adaptive Observer-based Estimator for Multi-sinusoidal Signals

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***Abstract*— This paper presents a robust estimation methodology that is capable to identify the amplitudes, frequencies and phases (AFP) of the components of a biased multi-sinusoidal signal in presence of a bounded disturbance on the measurement. The proposed method is based on an adaptive observer, whose parameter adaptation law is equipped by an excitation-based switching logic. The stability analysis proves the existence of a tuning parameter set for which the estimator's dynamics is ISS with respect to bounded measurement noise. The effectiveness of the algorithm is illustrated by simple simulation examples.**

I. INTRODUCTION

The problem of estimating the amplitude, frequency and phase of a sinusoidal signal arises in a variety of practical applications. In particular, the identification of frequency is a fundamental issue receiving extensive research efforts in many engineering fields such as power monitoring, vibration control and periodic disturbance rejection. While the Fast Fourier Transform (FFT) is usually preferred when discrete-time samples are available over finite-length intervals under the assumption that the frequency content is constant within the said time-window, several other methods have been conceived to track time-varying amplitude/frequencies. Among them, it is worth to recall the adaptive notch-filtering method (ANF) (see [1], [2]) and Phase-Locked-Loop (PLL) (see [3], [4], [5] and [6]) for their simple practical implementation. Nevertheless, the switching algorithm has to be reset if the nominal frequency changes. Moreover, estimators that incorporate multiple PLL-based techniques in parallel with a decorrelator factor which endows the methods with the ability to discriminate two nearby frequencies have been conceived to realize detection of two arbitrary sinusoids simultaneously (see [7], [8]).

Recently, increasing attention has been paid to the use of the adaptive observers for the sinusoidal estimation problem (see [9] and [10]). By means of an adaptive observer, globally or semi-globally (in case of a noisy measurement) convergence are readily obtained (see [11], [12], [13]). The adaptive observer technique makes it possible to achieve multi-sinusoidal estimation by expanding the dynamic model with proper system transformation. However, due to reparametrization, the estimated frequencies are usually not directly adapted. Instead, the parameter adaptation laws regard a set of parameters related nonlinearly to the frequency such as the coefficients of the characteristic polynomial of the autonomous signal-generatror system (see [14], [15], [16], [17], [18] and [19]). Among these methods, [15], [17], [18] are capable to handle a biased multi-sinusoidal signal, while [19] has been applied in a nonlinear plant for disturbance cancellation. Apart from the aforementioned adaptive observer-based technique, an asymptotically convergent estimator for n-frequencies using contraction theory is proposed in [20].

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Motivated by the adaptive observer proposed in [21] and its extension to the single sinusoidal case [13], the presented paper deals with a new methodology that is capable to offer reliable estimates of amplitudes, frequencies, phases and offset from a biased signal comprising n sinusoids. *In contrast with other methods that adapt the coefficients of the characteristic polynomial, a direct adaptation law for the squares of the frequencies is provided.* The stability analysis indicates that the robustness is guaranteed even if the measurements are corrupted by unstructured bounded disturbances, which is likely to appear in real-world applications. More specifically, once the tuning parameters are suitably fixed, ISS property with respect to the additive measurement noise is ensured.

The paper is organized as follows: the problem is formulated in Section II. In Section III, the adaptive observer-based estimator is proposed. Then, the stability analysis of the presented technique is dealt with in Section IV. Finally, simulation results showing the effectiveness of the algorithm dealt with in the paper are given in Section V.

II. PROBLEM FORMULATION AND PRELIMINARIES

Consider a biased multi-sinusoidal signal

$$y(t) = A_0 + \sum_{i=1}^{n} A_i \sin(\omega_i t + \phi_i) \tag{1}$$

where the amplitudes verify the inequality $A_i \geq 0$, A_0 is an unknown constant bias, the frequencies parameters are strict-positive and unique: $\omega_i > 0$, $\omega_i \neq \omega_j$ for $i \neq j$ and ϕ_i is the unknown initial phase of each sinusoid.

The signal y(t) is assumed to be generated by the following observable autonomous marginally stable dynamical system:

$$\begin{cases} \dot{x}(t) &= \mathbf{A}_x x(t) + \sum_{i=1}^{n} \mathbf{A}_i x(t)\theta_i^* \\ x(0) &= x_0 \end{cases} \qquad y(t) = \mathbf{C}_x x(t) \tag{2}$$

with $x(t) \in \mathbb{R}^{(2n+1)}$ and where x_0 represents the unknown initial condition which leads the output to match the stationary sinusoidal behavior since the very beginning, and $\theta_i^* = a_i + \Omega_i$ with $\Omega_i = \omega_i^2$, $\forall i \in \{1,..,n\}$

$$\mathbf{A}_x = \begin{bmatrix} \mathbf{S}_1 & \mathbf{0}_{2\times2} & \cdots & \mathbf{0}_{2\times2} & 0 \\ \mathbf{0}_{2\times2} & \mathbf{S}_2 & \ddots & \mathbf{0}_{2\times2} & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \mathbf{0}_{2\times2} & \ddots & \ddots & \mathbf{S}_n & 0 \\ 0 & \cdots & \cdots & 0 & 0 \end{bmatrix}, \quad \mathbf{C}_x^\top = \begin{bmatrix} \mathbf{c}_1 \\ \mathbf{c}_2 \\ \vdots \\ \mathbf{c}_n \\ 1 \end{bmatrix},$$

and

$$\mathbf{S}_i = \begin{bmatrix} 0 & 1 \\ a_i & 0 \end{bmatrix}, \quad \mathbf{c}_i = \begin{bmatrix} 1 & 0 \end{bmatrix},$$

$a_1, a_2 \cdots a_n$ are non-zero constants can be selected arbitrarely with the only requirements to satisfy $a_i \in \mathbb{R}$, $a_i \neq a_j$ for $i \neq j$, and $\mathbf{A}_i$ is a matrix with $(2i, 2i-1)$th entry -1

and 0 for the all the others, for instance:

$$\mathbf{A}_1 = \begin{bmatrix} \mathbf{S}_0 & \mathbf{0}_{2\times(2n-1)} \\ \mathbf{0}_{(2n-1)\times 2} & \mathbf{0}_{(2n-1)\times(2n-1)} \end{bmatrix},$$

$$\mathbf{A}_2 = \begin{bmatrix} \mathbf{0}_{2\times 2} & \mathbf{0}_{2\times 2} & \mathbf{0}_{2\times(2n-3)} \\ \mathbf{0}_{2\times 2} & \mathbf{S}_0 & \mathbf{0}_{2\times(2n-3)} \\ \mathbf{0}_{(2n-3)\times 2} & \mathbf{0}_{(2n-3)\times 2} & \mathbf{0}_{(2n-3)\times(2n-3)} \end{bmatrix}$$

in which

$$\mathbf{S}_0 = \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix}$$

In order consider the measurement uncertainty, let us assume that $y(t)$ is corrupted by an additive disturbance $d(t)$, bounded by a constant $\bar{d} > 0 : |d(t)| < \bar{d}, \forall t \in \mathbb{R}_{\geq 0}$. Then, the perturbed signal that is available from the measurement can be written as

$$\hat{y}(t) = A_0 + \sum_{i=1}^{n} A_i \sin(\omega_i t + \phi_i) + d(t) \tag{3}$$

Thanks to (2), the signal $\hat{y}(t)$ can be thought as generated by the observable system

$$\begin{cases} \dot{x}(t) = \mathbf{A}_x x(t) + \mathbf{G}_x(x(t))\theta^* \\ x(0) = x_0 \end{cases} \\ \hat{y}(t) = \mathbf{C}_x x(t) + d(t) \tag{4}$$

in which

$$\mathbf{G}_x(x(t)) = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ -x_1 & 0 & \ddots & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & -x_3 & \ddots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & \cdots & -x_{2n-1} \\ 0 & 0 & \cdots & 0 \end{bmatrix},$$

and θ^* denotes the true parameter vector $[\theta_1^* \; \theta_2^* \; \cdots \; \theta_n^*]^\top$.

Remark 2.1: The elements of $\mathbf{G}_x(\cdot)$ are globally Lipschitz continuous functions, that follows:

$$||\mathbf{G}_x(x') - \mathbf{G}_x(x'')|| \leq |x' - x''|, \forall x', x'' \in \mathbb{R}^{n_x}$$

Moreover, the true state $x(t)$ is norm-bounded for any initial condition, i.e. $|x(t)| \leq \bar{x}, \forall t \in \mathbb{R}$. Both the Lipschitz condition on $\mathbf{G}_x(\cdot)$ and the bound $\bar{x}$ allows to establish the following further bound

$$\|\mathbf{G}_x(x(t))\| \leq \bar{x}, \forall t \in \mathbb{R},$$

Now, assuming that the estimates $\hat{x}(t)$ and $\hat{\theta}(t)$ are available, then the full AFP estimates are obtained by

$$\hat{\Omega}_i = \hat{\theta}_i - a_i, \quad \hat{\omega}_i = \sqrt{\hat{\theta}_i - a_i}, \tag{5}$$

$$\hat{A}_i = \sqrt{\left(\hat{\Omega}_i \hat{x}_{2i-1}^2 + \hat{x}_{2i}^2\right) / \hat{\Omega}_i}, \;\; i = 1, 2, \cdots, n, \tag{6}$$

and

$$\hat{\varphi}_i = \angle\left(\hat{x}_{2i} + j\hat{\omega}_i \, \hat{x}_{2i-1}\right), \;\; i = 1, 2, \cdots, n. \tag{7}$$

In addition, the offset is evaluated directly by $\hat{A}_0 = \hat{x}_{2n+1}$.

In order to proceed with the further analysis, the following assumption is needed.

Assumption 1: The frequencies of the sinusoids are bounded by a positive constant $\bar{\omega}$, such that $\omega_i < \bar{\omega}$, $\forall i \in \{1, \cdots, n\}$.

According to assumption 1, there exists a known positive constant $\bar{\theta}^*$, such that $|\theta^*| \leq \bar{\theta}^*$. More specifically, in the remaining parts of the paper we consider $\theta^* \in \Theta^*$, where $\Theta^* \subset \mathbb{R}^n$ is a hypersphere of radius $\bar{\theta}^*$. The constraint on θ^* is instrumental for proving the stability of the parameter adaptation law introduced in the next section.

III. Filtered-Augmentation-based adaptive observer

Now, we are going to introduce the adaptive observer that is based on the model (4). In order to validate the estimation scheme, $\hat{y}(t)$ is augmented by the output of a synthetic filter driven by the noisy measurement vector:

$$\dot{\hat{y}}_e(t) = \mathbf{A}_e \hat{y}_e(t) + \mathbf{B}_e \hat{y}(t) \tag{8}$$

where $\mathbf{A}_e$ and $\mathbf{B}_e$ are fixed by the designer such that $\mathbf{A}_e$ is Hurwitz and the pair $(\mathbf{A}_e, \mathbf{B}_e)$ is controllable. $\hat{y}_e(t) \in \mathbb{R}^{n_e}$ denotes the accessible state vector and with arbitrary initial condition $\hat{y}_{e_0}$. According to [13], the dimension n_e of the augmented dynamics must verify $n_e \geq n - 1$.

For the sake of the forthcoming analysis, it is convenient to split the expanded filtered output vector in two components:

$$\hat{y}_e(t) = y_e(t) + d_e(t)$$

where $y_e(t)$ and $d_e(t)$ are produced by two virtual filters driven by the unperturbed output and by the measurement disturbance respectively:

$$\dot{y}_e(t) = \mathbf{A}_e y_e(t) + \mathbf{B}_e y(t) \tag{9}$$

and

$$\dot{d}_e(t) = \mathbf{A}_e d_e(t) + \mathbf{B}_e d(t) \tag{10}$$

Consequently, in view of (4), (9) and (10), the overall augmented system dynamics with the extended perturbed output measurement equation can be written as follows:

$$\begin{cases} \dot{z}(t) = \mathbf{A}_z z(t) + \mathbf{G}_z(z(t))\theta^* \\ \eta(t) = \mathbf{C}_z z(t) \\ \hat{\eta}(t) = \eta(t) + d_\eta(t) \end{cases} \tag{11}$$

with $z(0) = z_0 \in \mathbb{R}^{n_z}$, $n_z = 2n + 1 + n_e$, and $z(t) \triangleq [x^\top(t) \; x_e^\top(t)]^\top, z_0 \triangleq [x_0^\top(t) \; \hat{y}_{e_0}^\top]^\top, \hat{\eta}(t) \triangleq [\hat{y}^\top(t) \; \hat{y}_e^\top(t)]^\top, d_\eta(t) \triangleq [d^\top(t) \; d_e^\top(t)]^\top$

$$\mathbf{A}_z \triangleq \begin{bmatrix} \mathbf{A}_x & \mathbf{0}_{(2n+1)\times n_e} \\ \mathbf{B}_e \mathbf{C}_x & \mathbf{A}_e \end{bmatrix},$$

$$\mathbf{C}_z \triangleq \begin{bmatrix} \mathbf{C}_x & \mathbf{0}_{1\times n_e} \\ \mathbf{0}_{n_e\times(2n+1)} & \mathbf{I}_{n_e} \end{bmatrix}.$$

and

$$\mathbf{G}_z(z(t)) \triangleq \begin{bmatrix} \mathbf{G}_x(\mathbf{T}_{zx} z(t)) \\ \mathbf{0}_{n_e \times n} \end{bmatrix},$$

with the transformation matrix given by $\mathbf{T}_{zx} \triangleq [\mathbf{I}_{2n+1} \; \mathbf{0}_{(2n+1)\times n_e}]$. It is worth noting that $\mathbf{G}_z(z(t))$ is also Lipschitz with the same Lipschitz constant as $\mathbf{G}_x(x(t))$, and norm-bounded by $\bar{x}$. Moreover, the assumed norm-bound $\bar{d}$ on the output noise implies the existence of $\bar{d}_\eta$ such that $\bar{d}_\eta > 0 : |d_\eta(t)| \leq \bar{d}_\eta, \forall t \in \mathbb{R}_{\geq 0}$.

Now, we introduce the structure of the adaptive observer, consisting of the measured output filter (8) and the dynamic components (12), (13) and (14) described below:

1) Augmented state estimator:

$$\dot{\hat{z}}(t) = (\mathbf{A}_z - \mathbf{L}\mathbf{C}_z)\hat{z}(t) + \mathbf{L}\hat{\eta}(t) + \mathbf{G}_z(\hat{z}(t))\hat{\theta}(t) + \Xi(t)\,\dot{\hat{\theta}}(t) \quad (12)$$

with $\hat{z}(0) = \hat{z}_0$. The gain matrix $\mathbf{L}$ is given by

$$\mathbf{L} \triangleq \begin{bmatrix} \mathbf{L}_x & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$$

where $\mathbf{L}_x$ is a suitable gain matrix such that $\mathbf{A}_x - \mathbf{L}_x\mathbf{C}_x$ is Hurwitz.

2) Parameter-affine state-dependent matrix filter:

$$\dot{\Xi}(t) = (\mathbf{A}_z - \mathbf{L}\mathbf{C}_z)\Xi(t) + \mathbf{G}_z(\hat{z}(t)) \quad (13)$$

with $\Xi(0) = \mathbf{0}_{n_z \times n}$ and where $\Xi(t) \in \mathbb{R}^{n_z \times n}$ is an auxiliary time-varying matrix whose elements are driven by the state-dependent parameter-affine matrix $\mathbf{G}_z(\hat{z}(t))$.

3) Parameters' adaptation law:

Herein, an projection operator $\mathcal{P}$ is utilized to confine the estimated parameter $\hat{\theta}$ to the predefined convex region Θ^*

$$\dot{\hat{\theta}}(t) = \Psi(t)\mathcal{P}\left[\dot{\hat{\theta}}_{pre}(t)\right]_{|\hat{\theta}| \leq \bar{\theta}^*} \quad (14)$$

with $\hat{\theta}(0) = \hat{\theta}_0$ set arbitrarily. $\Psi(t)$ represents an on-off switching signal that will be specified later on.

$$\dot{\hat{\theta}}_{pre}(t) \triangleq - \mu\big(\Xi^\top(t)\Xi(t) + \rho^2\mathbf{I}\big)^{-1}\Xi^\top(t)\mathbf{C}_z^\top\left(\mathbf{C}_z\hat{z}(t) - \hat{\eta}(t)\right)$$

where μ is a positive constant, and $\rho \in \mathbb{R}$ is another constant fixed by the designer. The parameters' derivative projection operator is defined as:

$$\mathcal{P}\left[\dot{\hat{\theta}}_{pre}(t)\right]_{|\hat{\theta}| \leq \bar{\theta}^*} \triangleq \begin{cases} \mathrm{nsp}(\hat{\theta}^\top(t))\Big(\mathrm{nsp}\big(\hat{\theta}^\top(t)\big)\Big)^\top\dot{\hat{\theta}}_{pre}(t), \\ \qquad\qquad \text{if } |\hat{\theta}| = \bar{\theta}^* \text{ and } \hat{\theta}^\top(t)\dot{\hat{\theta}}_{pre}(t) > 0 \\ \dot{\hat{\theta}}_{pre}(t), \quad \text{otherwise} \end{cases}$$

in which $\mathrm{nsp}(\cdot)$ denotes the null-space of a row vector. In a compact form, the parameter adaptation law can be expressed as

$$\dot{\hat{\theta}}(t) = \Psi(t)\left[\dot{\hat{\theta}}_{pre}(t) - \mathcal{I}(\theta)\frac{\hat{\theta}(t)\hat{\theta}^\top(t)}{\bar{\theta}^{*2}}\dot{\hat{\theta}}_{pre}(t)\right]$$

where $\mathcal{I}(\theta)$ denote the indicator function given by

$$\mathcal{I}(\theta) \triangleq \begin{cases} 1, & \text{if } |\hat{\theta}(t)| = \bar{\theta}^* \text{ and } \hat{\theta}^\top(t)\dot{\hat{\theta}}_{pre}(t) > 0 \\ 0, & \text{otherwise} \end{cases}$$

The activation/suppression of the parameter adaptation is determined by the binary switching signal $\Psi(t)$, which possesses the following hysteretic property:

$$\Psi(t) = \begin{cases} 1, & \text{if } \min\mathrm{eig}\left(\Phi(\Xi(t))\right) \geq 2\bar{\delta} \\ 0, & \text{if } \min\mathrm{eig}\left(\Phi(\Xi(t))\right) < 2\underline{\delta} \\ \Psi(t^-), & \text{if } 2\underline{\delta} \leq \min\mathrm{eig}\left(\Phi(\Xi(t))\right) < 2\bar{\delta} \end{cases} \quad (15)$$

where

$$\Phi(\Xi(t)) = \big(\Xi^\top(t)\Xi(t) + \rho^2\mathbf{I}\big)^{-1}\Xi^\top(t)\mathbf{C}_z^\top\mathbf{C}_z\Xi(t)$$

represents the excitation matrix. The transition thresholds $\underline{\delta}$, $\bar{\delta}$ are fixed by the designer such that $0 < 2\underline{\delta} < 2\bar{\delta} < 1$. The introduction of the hysteresis is inspired by the need to ensure a minimum finite duration between transitions.

IV. Stability analysis

A. Excitation Phase

Consider an arbitrary active identification phase, in which $\Psi(t) = 1$ and $\Phi(\Xi(t)) \geq 2\underline{\delta}$. In order to address the stability of the adaptive observer, let us define the augmented state-estimation error vector: $\tilde{z}(t) \triangleq \hat{z}(t) - z(t)$, the parameter estimation error $\tilde{\theta}(t) \triangleq \hat{\theta}(t) - \theta^*$, and their linear time-varying combination $\tilde{\varphi}(t) \triangleq \Xi(t)\tilde{\theta}(t) - \tilde{z}(t)$. Then, the state-estimation error evolves according to the differential equation

$$\begin{aligned} \dot{\tilde{z}}(t) &= (\mathbf{A}_z - \mathbf{L}\mathbf{C}_z)\tilde{z}(t) + \mathbf{L}d_\eta(t) + \mathbf{G}_z(\tilde{z}(t))\tilde{\theta}(t) \\ &\quad + \mathbf{G}_z(z(t))\tilde{\theta}(t) + \mathbf{G}_z(\tilde{z}(t))\theta^* + \Xi(t)\dot{\hat{\theta}}(t) \end{aligned} \quad (16)$$

where $\mathbf{G}_z(\tilde{z}(t)) \triangleq \mathbf{G}_z(\hat{z}(t)) - \mathbf{G}_z(z(t))$. Meanwhile, the dynamic of the auxiliary variable $\tilde{\varphi}(t)$ evolves according to

$$\dot{\tilde{\varphi}}(t) = (\mathbf{A}_z - \mathbf{L}\mathbf{C}_z)\tilde{\varphi}(t) - \mathbf{L}d_\eta(t) - \mathbf{G}_z(\tilde{z}(t))\theta^* \quad (17)$$

The upcoming analysis is carried out in order to demonstrate benefit of using the derivative projection on the parameters' estimates. To this end, in the following lines we will only focus on the scenario that the projection operator is activated, since $\dot{\hat{\theta}}(t) = \dot{\hat{\theta}}_{pre}(t)$ for all the other conditions. For convenience, let us assume that $\frac{\hat{\theta}(t)\hat{\theta}^\top(t)}{\bar{\theta}^{*2}}\dot{\hat{\theta}}_{pre}(t) = \sigma(t)\hat{\theta}(t)$, where $\sigma(t)$ is a variable depending on $\dot{\hat{\theta}}_{pre}(t)$. Owing to the fact that $\mathcal{I}_\theta = 1$ and $|\hat{\theta}(t)| = \bar{\theta}^*$, we have that

$$\begin{aligned} \tilde{\theta}^\top(t)\dot{\hat{\theta}}(t) &= \tilde{\theta}^\top(t)\left[\dot{\hat{\theta}}_{pre}(t) - \sigma\hat{\theta}(t)\right] \\ &= \tilde{\theta}^\top(t)\dot{\hat{\theta}}_{pre}(t) - \left[\Big\langle\hat{\theta}^\top(t),\, \sigma\hat{\theta}(t)\Big\rangle - \Big\langle\theta^{*\top},\, \sigma\hat{\theta}(t)\Big\rangle\right] \end{aligned}$$

in which $\langle\cdot,\cdot\rangle$ denotes the inner product. In virtue of

$$\Big\langle\hat{\theta}^\top(t),\, \sigma\hat{\theta}(t)\Big\rangle = \sigma\bar{\theta}^{*2} \geq \sigma\bar{\theta}^*|\theta^*|\cos\phi_\theta = \Big\langle\theta^{*\top},\, \sigma\hat{\theta}(t)\Big\rangle$$

we can finally bound the scalar product $\tilde{\theta}^\top(t)\dot{\hat{\theta}}(t)$ by:

$$\tilde{\theta}^\top(t)\dot{\hat{\theta}}(t) \leq \tilde{\theta}^\top(t)\dot{\hat{\theta}}_{pre}(t)$$

where $\dot{\hat{\theta}}_{pre}(t)$ is expanded as follows:

$$\begin{aligned} \dot{\hat{\theta}}_{pre}(t) &= -\mu\big(\Xi^\top(t)\Xi(t) + \rho^2\mathbf{I}\big)^{-1}\Xi^\top(t)\mathbf{C}_z^\top\mathbf{C}_z\Xi(t)\tilde{\theta}(t) \\ &\quad + \mu\big(\Xi^\top(t)\Xi(t) + \rho^2\mathbf{I}\big)^{-1}\Xi^\top(t)\mathbf{C}_z^\top d_\eta(t) \\ &\quad + \mu\big(\Xi^\top(t)\Xi(t) + \rho^2\mathbf{I}\big)^{-1}\Xi^\top(t)\mathbf{C}_z^\top\mathbf{C}_z\tilde{\varphi}(t) \end{aligned} \quad (18)$$

Theorem 4.1 (ISS of the dynamic estimator): If assumption 1 and the excitation condition hold, then given the sinusoidal signal $y(t)$ defined in (1) and the perturbed measurement (3), there exist suitable choices of μ and ρ such that the adaptive observer-based estimator given by (8), (12), (13) and (14) is ISS with respect to any bounded disturbance d_η and in turn ISS with respect to bounded measurement disturbance $|d(t)| \leq \bar{d}$.

Proof: Since $(\mathbf{A}_z - \mathbf{L}\mathbf{C}_z)$ is Hurwitz, for any positive definite matrix $\mathbf{Q}$, there exist a positive definite matrix $\mathbf{P}$ that solves the linear Lyapunov equation

$$(\mathbf{A}_z - \mathbf{L}\mathbf{C}_z)^\top\mathbf{P} + \mathbf{P}(\mathbf{A}_z - \mathbf{L}\mathbf{C}_z) = -2\mathbf{Q}\,.$$

Now, let us introduce the candidate Lyapunov function

$$V(t) \triangleq \frac{1}{2}\big(\tilde{z}^\top(t)\mathbf{P}\tilde{z}(t) + \tilde{\theta}^\top(t)\tilde{\theta}(t) + g\tilde{\varphi}^\top(t)\mathbf{P}\tilde{\varphi}(t)\big) \quad (19)$$

where g is a positive constant. By letting:

$$\bar{c} \triangleq \|\mathbf{C}_z\|,\ \bar{l} \triangleq \|\mathbf{L}\|,\ \underline{q} \triangleq \min \text{eig}(\mathbf{Q}),\ \bar{p} \triangleq \max \text{eig}(\mathbf{P})$$

The time-derivative of the Lyapunov function can be bounded as follows:

$$\begin{aligned}\dot{V}(t) \leq & -\underline{q}|\tilde{z}(t)|^2 - g\underline{q}|\tilde{\varphi}(t)|^2 - \mu\underline{\delta}|\tilde{\theta}(t)|^2 + \bar{p}\bar{l}|\tilde{z}(t)||d_\eta(t)| \\ & + \bar{p}|\tilde{z}(t)|^2|\tilde{\theta}(t)| + \bar{p}\bar{x}|\tilde{z}(t)||\tilde{\theta}(t)| + \bar{p}\bar{\theta}^*|\tilde{z}(t)|^2 \\ & + \mu\bar{p}\bar{c}^2|\tilde{z}(t)|^2 + \mu\bar{p}\bar{c}|\tilde{z}(t)||d_\eta(t)| + \mu\frac{\bar{c}}{2\rho}|\tilde{\theta}(t)||d_\eta(t)| \\ & + \mu\frac{\bar{c}^2}{2\rho}|\tilde{\theta}(t)||\tilde{\varphi}(t)| + g\bar{p}\bar{l}|\tilde{\varphi}(t)||d_\eta(t)| + g\bar{p}\bar{\theta}^*|\tilde{z}(t)||\tilde{\varphi}(t)|,\end{aligned}$$

thus, after some algebra and by re-arranging the above inequality to put in evidence the square monomial and the binomial terms, we have:

$$\begin{aligned}\dot{V}(t) \leq & -(\underline{q} - \mu\bar{p}\bar{c}^2 - 3\bar{p}\bar{\theta}^*)|\tilde{z}(t)|^2 + \bar{p}\big(\bar{l} + \mu\bar{c}\big)|\tilde{z}(t)||d_\eta(t)| \\ & - \frac{g\underline{q}}{2}|\tilde{\varphi}(t)|^2 + g\bar{p}\bar{\theta}^*|\tilde{z}(t)||\tilde{\varphi}(t)| - \frac{g\underline{q}}{2}|\tilde{\varphi}(t)|^2 \\ & + g\bar{p}\bar{l}|\tilde{\varphi}(t)||d_\eta(t)| - \frac{\mu\underline{\delta}}{3}|\tilde{\theta}(t)|^2 + \bar{p}\bar{x}|\tilde{z}(t)||\tilde{\theta}(t)| \\ & - \frac{\mu\underline{\delta}}{3}|\tilde{\theta}(t)|^2 + \mu\frac{\bar{c}}{2\rho}|\tilde{\theta}(t)||d_\eta(t)| - \frac{\mu\underline{\delta}}{3}|\tilde{\theta}(t)|^2 + \mu\frac{\bar{c}^2}{2\rho}|\tilde{\theta}(t)||\tilde{\varphi}(t)|\,.\end{aligned}$$

Now, we complete the squares, getting to

$$\begin{aligned}\dot{V}(t) \leq & \\ & -\left[\frac{(\underline{q} - \mu\bar{p}\bar{c}^2 - 3\bar{p}\bar{\theta}^*)}{2} - \frac{g}{\underline{q}}\bar{p}^2\left(\bar{\theta}^*\right)^2 - \frac{3(\bar{p}\bar{x})^2}{2\mu\underline{\delta}}\right]|\tilde{z}(t)|^2 \\ & - \frac{\mu\underline{\delta}}{2}|\tilde{\theta}(t)|^2 - \left(\frac{g\underline{q}}{2} - \frac{3\mu\bar{c}^4}{8\rho^2\underline{\delta}}\right)|\tilde{\varphi}(t)|^2 \\ & + \left[\frac{\bar{p}^2\left(\bar{l} + \mu\bar{c}\right)^2}{2(\underline{q} + \mu\bar{p}\bar{c}^2 - 3\bar{p}\bar{\theta}^*)} + \frac{3\mu\bar{c}^2}{8\rho^2\underline{\delta}} + \frac{g}{\underline{q}}\bar{p}^2\bar{l}^2\right]|d_\eta(t)|^2\end{aligned}$$

Finally, the following inequality can be established:

$$\dot{V}(t) \leq -\beta_1\left[V(t) - \sigma_1(\bar{d}_\eta)\right],$$

where

$$\begin{aligned}\beta_1 \triangleq & \\ & 2\min\left\{\frac{(\underline{q} - \mu\bar{p}\bar{c}^2 - 3\bar{p}\bar{\theta}^*)}{2} - \frac{g}{\underline{q}}\bar{p}^2\left(\bar{\theta}^*\right)^2 - \frac{3(\bar{p}\bar{x})^2}{2\mu\underline{\delta}},\right. \\ & \left.\frac{\mu\underline{\delta}}{2},\ \frac{g\underline{q}}{2} - \frac{3\mu\bar{c}^4}{8\rho^2\underline{\delta}}\right\} \quad (20)\end{aligned}$$

and

$$\sigma_1(s) \triangleq \frac{1}{\beta_1}\left[\frac{\bar{p}^2\left(\bar{l} + \mu\bar{c}\right)^2}{2(\underline{q} + \mu\bar{p}\bar{c}^2 - 3\bar{p}\bar{\theta}^*)} + \frac{3\mu\bar{c}^2}{8\rho^2\underline{\delta}} + \frac{g}{\underline{q}}\bar{p}^2\bar{l}^2\right]s^2\,. \quad \forall s \in \mathbb{R}_{\geq 0}$$

Hence, the proof is concluded, iff

$$\beta_1 > 0 \quad (21)$$

In view of (21), all the components involved in (20) should be positive, wherein $\frac{\mu\underline{\delta}}{2} > 0$ can be immediately verified by choosing a positive μ. Now, we set the excitation threshold $\underline{\delta}$ and the $\mathbf{Q}$ matrix arbitrarily, determining $\underline{q}$. Then, letting $\bar{p} \leq \mu$, we determine a sufficient condition to ensure the positiveness of the first term in (20):

$$\frac{(\underline{q} - \mu^2\bar{c}^2 - 3\mu\bar{\theta}^*)}{2} - \mu^2\frac{g}{\underline{q}}\left(\bar{\theta}^*\right)^2 - \mu\frac{3\bar{x}^2}{2\underline{\delta}} > 0\,.$$

Being the Lyapunov parameter $g > 0$ arbitrary, let us fix $g = 1$ for simplicity. At this point, we can always determine a sufficiently small value of μ for which the inequality holds true. Next, by suitably allocating the poles, we compute the output-injection gain $\mathbf{L}$ that realizes the needed $\bar{p}$. Finally, to render β_1 strict-positive, we choose a regularization parameter ρ such that

$$\frac{g\underline{q}}{2} - \frac{3\mu\bar{c}^4}{8\rho^2\underline{\delta}} > 0\,.$$

■

Remark 4.1: To avoid the increase of the worst-case sensitivity to bounded noises, instead of using a low value of $\bar{p}$ that leads high-gain output injection through $\mathbf{L}$, and high values of $\bar{l}$ and σ_1 correspondingly, we can set $\bar{p} = \mu$ and increase the regularization parameter ρ.

B. Dis-Excitation Phase

Lemma 4.1 (Boundedness in dis-excitation phase): [21] When $\Psi(t) = 0$, the Lyapunov function admits a bound that depends on the noise level and on the initial value of the Lyapunov function itself before switching off:

$$\dot{V}(t) \leq \beta_0\big(L_0 V(t^-) + \sigma_0(|d_\eta(t)|) - V(t)\big)$$

where

$$\beta_0 \triangleq 2\min\left\{\frac{\underline{q} - 3\bar{p}\bar{\theta}^*}{2} - \frac{g}{\underline{q}}\left(\bar{p}\bar{\theta}^*\right)^2, \frac{g\underline{q}}{2}, \frac{1}{2}\right\},$$

$$L_0 \triangleq \frac{2}{\beta_0}\left\{\frac{\bar{p}^2\bar{x}^2}{\underline{q} - 3\bar{p}\bar{\theta}^*} + \frac{1}{2}\right\},$$

and

$$\sigma_0(s) \triangleq \frac{1}{\beta_0}\left(\frac{\bar{p}^2\bar{l}^2}{\underline{q} - 3\bar{p}\bar{\theta}^*} + \frac{g}{\underline{q}}\bar{p}^2\bar{l}^2\right)s^2, \forall s \in \mathbb{R}_{\geq 0}\,.$$

Proof: The present lemma can be proven by following the same line of reasoning adopted for the case $\Psi(t) = 1$. ■

C. Robustness Under Alternate Switching

At this stage, the stability of the adaptive observer under alternate switching is characterized by linking the results obtained for the two excitation phases.

Theorem 4.2: [21] Under the same assumptions of theorem 4.1, consider the adaptive observer (8), (12), (13), (14) equipped with the excitation-based switching strategy defined in (15). Then, the discrete dynamics induced by sampling the adaptive observer in correspondence of the switching transitions has the asymptotic ISS property if the excitation phases last longer than $\beta^{-1}\ln(L_0)$. Letting k denote a counter for the active identification phases, and V_k the value of the Lyapunov function (19) at then at the end of the k-th phase, then the following bound can be established

$$V_k < e^{-\beta_1\Delta_k}\left(\sigma_0(\bar{d}_\eta) - \sigma_1(\bar{d}_\eta) + L_0 V_{k-1}\right) + \sigma_1(\bar{d}_\eta),$$

where Δ_k is the duration of the k-th phase.

The complete proof is omitted and can be found in [21]. In view of Theorem 4.2, if an infinite number of active identification phases occurs asymptotically ($k \to \infty$), or if a single excitation phase lasts indefinitely, then the estimation error in the inter-sampling times converges to a region whose radius depends only on the assumed disturbance bound.

V. Illustrative Examples

In the following section, the devised method is compared with the adaptive-observer-based method presented in [16] which offers estimated frequency and amplitude simultaneously. The dynamic estimators are discretized by forward-Euler method with fixed sampling period $T_s = 1 \times 10^{-4}$s. All the measured inputs are affected by a bounded noise denoted by $d(t)$ which is subject to uniform distribution in the interval $[-0.25, 0.25]$.

Example 1: Consider the following sum-of-sinusoids signal, perturbed by additive noise

$$\hat{y}(t) = 2\sin 3t + \sin 2t + d(t)$$

For the sake of observer design, let $\mathbf{A}_e = -2$, $\mathbf{B}_e = 1$, while the observer poles are placed at $[-0.4, -0.5, -0.8, -1]$. The other tuning parameters are: $\mu = 10$, $\rho = 1$, $a_1 = 0$, $a_2 = -1, \overline{\delta} = 1 \times 10^{-6}$, $\underline{\delta} = 5 \times 10^{-7}$. The parameters for the other methods are chosen such that the response time of the frequency estimate is similar to the one provided by the method discussed in this paper. To this end, for method [16], we set $\gamma = 1200$, $\gamma_1 = 0.01$, $\gamma_2 = 0.005$, $l_1 = 10$, $l_2 = 50$ and the coefficients λ_i are chosen: $\lambda_0 = 4, \lambda_1 = 6, \lambda_2 = 6, \lambda_3 = 4$.

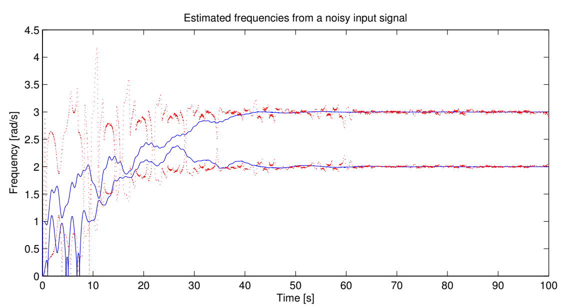

Fig. 1. Time-behavior of the estimated frequencies by using the proposed AFP method (blue line) compared with the time behaviors of the estimated frequencies by the AFP method [16] (red dotted line)

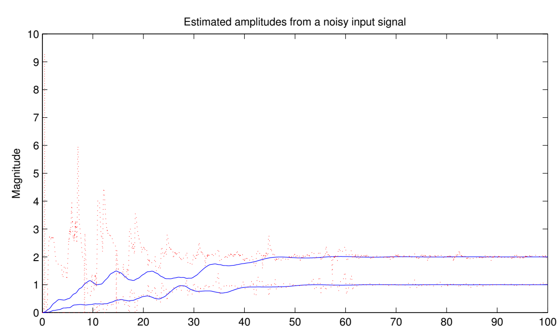

Fig. 2. Time-behavior of the estimated amplitudes by using the proposed AFP method (blue line) compared with the time behaviors of the estimated amplitudes by the AFP methods [16] (red dotted line)

It is worth noting from Fig.1 and 2 that both the estimators succeeded in detecting the frequencies and amplitudes in presence of bounded disturbance. However, the proposed method is more accurate in presence of additive noise and show a smoother transient behavior.

Example 2: Consider now a measurement signal composed of three sinusoids

$$\hat{y}(t) = 2\sin 5t + \sin 4t + 3\sin 3t + d(t)$$

The parameters of the synthetic filter are fixed as follows

$$\mathbf{A}_e = \begin{bmatrix} -2 & 0 \\ 2 & -2 \end{bmatrix} \quad \mathbf{B}_e = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Then, let us set $a_1 = 0$, $a_2 = -1$, $a_3 = -3$, $\mu = 60, \rho = 0.5, \overline{\delta} = 1 \times 10^{-7}$, $\underline{\delta} = 5 \times 10^{-8}$ respectively and place the poles at $[-0.2, -0.4, -0.5, -0.6, -0.8, -10]$. The initial values of the state variables of the observer have been all set to zero, while the parameter vector has been initialized to $\hat{\theta}(0) = [12\ 10\ 15]^\top$. As shown in Fig.3 and 4, all the three

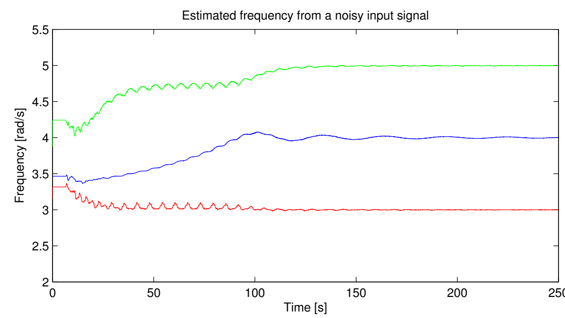

Fig. 3. Estimated frequencies obtained by using the proposed method

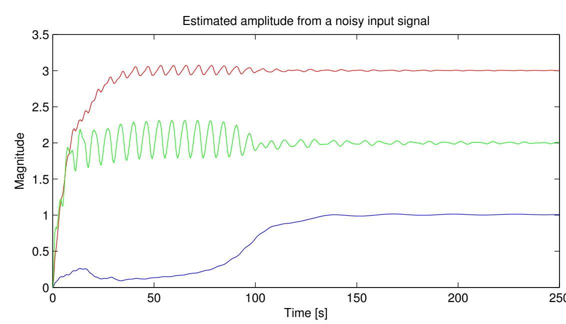

Fig. 4. Estimated amplitudes obtained by using the proposed method

frequencies are identified individually, at the same time the true amplitudes are also successfully captured.

Example 3: In order to evaluate the performance of the method in presence of measurement bias, the following signal is considered in the present example

$$\hat{y}(t) = 1 + 4\sin 3t + 2\sin 2t + d(t)$$

The parameters of the proposed method are chosen as follows: $\mathbf{A}_e = -2$, $\mathbf{B}_e = 1$, $a_1 = -2$, $a_2 = -1$, $\mu = 50$, $\rho = 0.2, \overline{\delta} = 1 \times 10^{-6}$, $\underline{\delta} = 5 \times 10^{-7}$ and the poles' location $[-0.2, -0.4, -0.5, -2, -10]$. In Fig. 5, the behaviour of the excitation level and of the switching signal $\Psi(t)$ are shown to enhance the fact that the proposed methodology allows to check in real-time the excitation level thus allowing to possibly stop the parameter-updating in case of low-exciting signals. The estimates are reported in Fig.6 and 7, where it can be observed that all the parameters including the offset are successfully estimated.

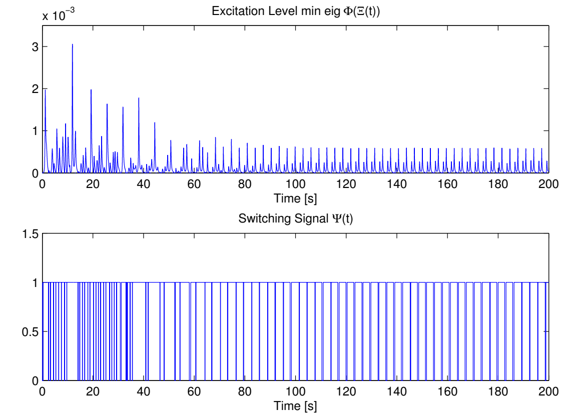

Fig. 5. Behaviour of the excitation level and of the switching signal $\Psi(t)$

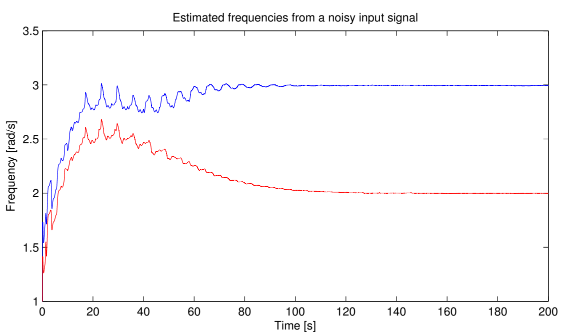

Fig. 6. Estimated frequencies obtained by using the proposed method

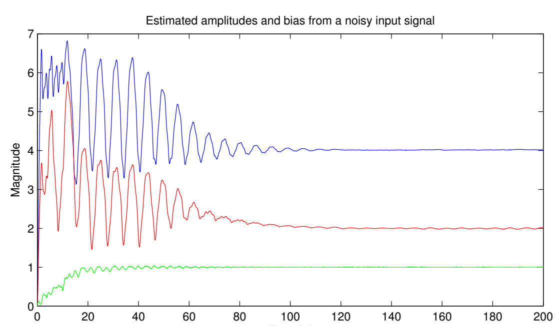

Fig. 7. Estimated amplitudes (red, blue) and bias (green) obtained by using the proposed method

VI. Conclusions

In this paper, the problem of estimating the unknown frequencies of a biased signal consisting of multiple sinusoids is addressed. To solve this task, an AFP estimator based on the filter augmented adaptive observer is presented. Compared to other methods that estimate the characteristic polynomial of the signal-generator system, the proposed algorithm allows for the direct adaptation of the squared-frequencies of the components. The proposed estimator is proven to be ISS with respect to the bounded disturbance by incorporating a switching criterion that freezes the estimates in case of poor excitement. The tuning rules of the adaptation parameters of the estimator are obtained analytically as a result of the ISS based analysis. The numerical simulation examples justify the applicability of the given estimator for on-line identification of unknown frequencies.

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