A Nonlinear Adaptive Observer with Excitation-based Switching

Gilberto Pin, Boli Chen and Thomas Parisini

***Abstract*— This paper presents a MIMO nonlinear adaptive observer, which is characterized by a robust excitation-based switching strategy. The proposed switching algorithm allows to address the scenario of poor excitation, while a conservative minimum duration of excitation interval for ensuring a progressive improvement is determined. The robustness of the devised method with respect to the bounded unstructured perturbation is studied by a input-to-state stability analysis. Simple simulation results show the effectiveness of the proposed technique.**

I. Introduction

The design of adaptive observers for uncertain nonlinear systems has received increased attention in the past few years.

Since the seminal paper [1], in which an adaptive observer has been developed for SISO nonlinear systems that can be transformed into an observable canonical form by a suitable state-affine parametrization, several works have focused on state transformations as the main tool to design stable state observers. The adaptive observers of [2] and [3] employ filtered state-space transformation to guarantee the asymptotic convergence of the state estimate under persistency of excitation. According to [4] , the adaptive observers based on filtered transforms guarantee bounded state and parameter estimation error under persistency of excitation, but even a small bounded disturbance may lead the parameters to diverge indefinitely when the excitation level weakens. To overcome this problem, a projection operator was introduced to guarantee the boundedness of the parameter estimate.

Two major issues can be evidenced with regard to the aforementioned adaptive observers. First, to guarantee the convergence of the estimated parameters to the true values in nominal conditions, the boundedness of the parameter estimation error in presence of disturbances, the condition of persistent excitation must be satisfied , which is not realistic for many applications. Second, the system has to be converted in the so called canonical observer-form. This second issue is addressed in [5], that considers multiple-input multiple-output (MIMO) nonlinear system with unmodelled dynamics, nonlinear parametrization and external bounded perturbations. The presence of unmodelled dynamics is accounted in [5] by the introduction of bounding terms that dominate the uncertainty. An alternative method used to design observers for system that cannot be recast in the canonical observer form by trivial state transformations consists in augmenting the dynamics of the observer with a filter, that takes as input the parameter-affine nonlinear function (see e.g. [6], [7] and [8]). We refer in particular to those systems in which the parameter-affine term does not only depend on measurable inputs and outputs but also on hidden (although observable) state variables (see e.g. [9], [7]). The augmentation filter is used for instance in [10] to design of an observer for linear time-varying MIMO systems. More recently, this tool has been also employed to estimate unknown parameters affecting both the state and the output equations [11].

G. Pin is with Electrolux Professional S.p.A., Italy (gilberto.pin@electrolux.it); B. Chen is with Imperial College London, UK (boli.chen10@imperial.ac.uk); T. Parisini is with Imperial College London, UK and also with University of Trieste, Italy (t.parisini@gmail.com).

In this framework, we propose an observer for a class of nonlinear systems with state-affine linear parametrization capable to address the presence of measurement disturbances and transient loss of excitation by means of a suitably designed switching strategy. *Compared to traditional convergence proofs of adaptive control and observers theory, that rely on an integral-type persistency of excitation condition (see, e.g. [12], [4], [13]), our methodology allows to check the excitation level in real-time and to instantaneously take suitable provisions to prevent the divergence of the state and parameter estimates.*

In our setup, the convergence of the parameters toward the true values is guaranteed provided that the excitation phases (each of finite duration) last more than a given time-length. During the poorly excited phases, the parameter update law is stopped and only the state estimation is carried out in a specific kind of "dual-mode" adaptive observation.

II. Problem Formulation with Assumptions

Consider a MIMO nonlinear system having the following linear-in-the-parameters structure

$$\begin{cases} \dot{x}(t) = \mathbf{A}_x x(t) + \mathbf{G}_x(x(t), u(t))\theta^* + f_x(y(t), u(t)) \\ y(t) = \mathbf{C}_x x(t) + f_y(y(t), u(t)) \end{cases} \tag{1}$$

where the initial condition $x(0) = x_0$ is unknown. Moreover $x(t) \in \mathbb{R}^{n_x}$, $u(t) \in \mathbb{R}^{n_u}$, $y(t) \in \mathbb{R}^{n_y}$ denote the state, input and output, respectively, $f_x(\cdot,\cdot) : \mathbb{R}^{n_y} \times \mathbb{R}^{n_u} \to \mathbb{R}^{n_x}$, $f_y(\cdot,\cdot) : \mathbb{R}^{n_y} \times \mathbb{R}^{n_u} \to \mathbb{R}^{n_y}$ and $\mathbf{G}_x(\cdot,\cdot) : \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} \to \mathbb{R}^{n_x \times n_\theta}$ are known functions, θ^* is the unknown parameter vector assumed constant unless otherwise specified.

While the input of the system is accessible, the output measurement is subjected to an additive bounded perturbation:

$$\breve{y}(t) = y(t) + d_y(t)\,,$$

where $d_y(t)$ is an additive measurement uncertainty bounded by a constant $\bar{d}_y > 0 : |d_y(t)| \le \bar{d}_y, \forall t \in \mathbb{R}_{\ge 0}$.

Although system (1) presents a linear parameterization, it includes most of the models considered in the literature concerned with the design of adaptive observers ([7], [13], [4]).

Let us introduce following assumptions on system (1).

Assumption 1: The pair $(\mathbf{A}_x, \mathbf{C}_x)$ is known and fully observable.

Assumption 2: A (possibly conservative) bound $\bar{\theta}^*$ on the norm of the parameter vector is known, that is, there exists $\bar{\theta}^* > 0$ such that $|\theta^*| \le \bar{\theta}^*$.

As will be seen later on, the constraints considered in Assumption 2 will be actively enforced by the estimation scheme through an adaptation law based on derivative projection.

The following further assumption will also bee needed.

Assumption 3: The observed (true) state is bounded by a compact set, i.e. $x \in X$ where $X \subset \mathbb{R}^{n_x}$ denotes a hypersphere of radius $\bar{x}$. Thus, $|x(t)| \le \bar{x}, \forall t \in \mathbb{R}$.

Assumption 4: The elements of $\mathbf{G}_x(\cdot)$ are locally Lipschitz continuous functions in their argument, with Lipschitz constant $\bar{\gamma}$:

$$||\mathbf{G}_x(x', u) - \mathbf{G}_x(x'', u)|| \le \bar{\gamma}|x' - x''|, \forall x', x'' \in X$$

uniformly in $u, \forall u \in \mathbb{R}^{n_u}$. Moreover

$$\sup_{u\in\mathbb{R}^{n_u}} ||\mathbf{G}_x(0,u)|| < +\infty\,.$$

Finally, the functions $f_x(y(t),u(t))$ and $f_y(y(t),u(t))$ are Lipschitz continuous with respect to the first argument:

$$\begin{array}{l} ||f_x(y',u)-f_x(y'',u)|| \le \bar{f}_x|y'-y''|, \forall y',\, y'' \in \mathbb{R}^{n_y} \\ ||f_y(y',u)-f_y(y'',u)|| \le \bar{f}_y|y'-y''|, \forall y',\, y'' \in \mathbb{R}^{n_y} \end{array}$$

uniformly in $u, \forall u \in \mathbb{R}^{n_u}$.

It is worth noting that since x is confined to the bounded set X, we are able to extend $\mathbf{G}_x(x,u)$ into a global Lipschitz function by Lipschitz extension (see [7] and the references therein). Define a saturation function $\sigma : \mathbb{R}^{n_x} \to X$, we immediately have $\mathbf{G}_x(\sigma(x),u)$ a global Lipschitz function that coincides with $\mathbf{G}_x(x,u)$ on X. Thus, it is makes no difference that we consider $\mathbf{G}_x(x,u)$ as a global Lipschitz function for following deduction.

Furthermore, Assumption 3 is needed for the subsequent stability analysis, but it is not used in the actual implementation of the observer. The conditions on $\mathbf{G}_x(\cdot)$ and the bound $\bar{x}$ allow to establish the following further bound

$$\|\mathbf{G}_x(x(t),u(t))\| \le \bar{\bar{\gamma}}, \forall t \in \mathbb{R}\,,$$

where $\bar{\bar{\gamma}} \triangleq \bar{\gamma}\bar{x} + \sup_{u\in\mathbb{R}^{n_u}} ||\mathbf{G}_x(\mathbf{0}_{n_x\times 1},u)||$.

Without loss of generality, let us consider the case

$$n_\theta > n_y \ge \mathrm{rank}(\mathbf{C}_x^\top \mathbf{C}_x)\,,$$

and let us "augment" the system by the following synthetic output filter, driven by the noisy measurement vector:

$$\dot{\breve{y}}_e(t) = \mathbf{A}_e\breve{y}_e(t) + \mathbf{B}_e\breve{y}(t) \tag{2}$$

where $\breve{y}_e(t) \in \mathbb{R}^{n_e}$ with arbitrary initial conditions $\breve{y}_{e_0}$. The dimension n_e satisfies

$$n_e \ge n_\theta - \mathrm{rank}(\mathbf{C}_x^\top \mathbf{C}_x)\,.$$

For the sake of the forthcoming analysis, it is convenient to split the extended filtered output vector in two components:

$$\breve{y}_e(t) = y_e(t) + d_e(t)$$

where $y_e(t)$ is produced by a virtual filter driven by the unperturbed output:

$$\dot{y}_e(t) = \mathbf{A}_e y_e(t) + \mathbf{B}_e y(t) \tag{3}$$

with $y_e(0) = \mathbf{0}_{n_e\times 1}$ and $d_e(t)$ is produced by the virtual disturbance filter

$$\dot{d}_e(t) = \mathbf{A}_e d_e(t) + \mathbf{B}_e d_y(t) \tag{4}$$

with $d_e(0) = \mathbf{0}_{n_e\times 1}$. Matrices $\mathbf{A}_e$ and $\mathbf{B}_e$ are chosen such that $\mathbf{A}_e$ is Hurwitz and the pair $(\mathbf{A}_e, \mathbf{B}_e)$ is controllable.

In view of (1), (3) and (4), the overall augmented system dynamics with the extended perturbed output measurement equation can be written as follows:

$$\left\{ \begin{array}{l} \dot{z}(t) = \mathbf{A}_z z(t) + \mathbf{G}_z(z(t),u(t))\theta^* + f_z(\eta(t),u(t)) \\ \eta(t) = \mathbf{C}_z z(t) + f_\eta(\eta(t),u(t)) \\ \breve{\eta}(t) = \eta(t) + d_\eta(t) \end{array} \right. \tag{5}$$

with $z(0) = z_0$ and where we let

$$\begin{array}{ll} z(t) \triangleq [x^\top(t)\ x_e^\top(t)]^\top, & z_0 \triangleq [x_0^\top(t)\ \breve{y}_{e0}^\top]^\top, \\ \eta(t) \triangleq [y^\top(t)\ y_e^\top(t)]^\top, & \breve{\eta}(t) \triangleq [\breve{y}^\top(t)\ \breve{y}_e^\top(t)]^\top, \\ d_\eta(t) \triangleq [d_y^\top(t)\ d_e^\top(t)]^\top, & \end{array}$$

$$\mathbf{A}_z \triangleq \left[\begin{array}{cc} \mathbf{A}_x & \mathbf{0}_{n_x\times n_e} \\ \mathbf{B}_e\mathbf{C}_x & \mathbf{A}_e \end{array} \right], \quad \mathbf{C}_z \triangleq \left[\begin{array}{cc} \mathbf{C}_x & \mathbf{0}_{n_y\times n_e} \\ \mathbf{0}_{n_e\times n_x} & \mathbf{I}_{n_e} \end{array} \right].$$

Moreover, by letting

$$\mathbf{T}_{zx} \triangleq [\mathbf{I}_{n_x}\ \mathbf{0}_{n_x\times n_e}], \quad \mathbf{T}_{\eta y} \triangleq [\mathbf{I}_{n_y}\ \mathbf{0}_{n_y\times n_e}],$$

we have defined:

$$\mathbf{G}_z(z(t),u(t)) \triangleq \left[\begin{array}{c} \mathbf{G}_x(\mathbf{T}_{zx}z(t),u(t)) \\ \mathbf{0}_{n_e\times n_\theta} \end{array} \right],$$

$$f_z(\eta(t),u(t)) \triangleq \left[\begin{array}{c} f_x(\mathbf{T}_{\eta y}\eta(t),u(t)) \\ \mathbf{B}_e f_y(\mathbf{T}_{\eta y}\eta(t),u(t)) \end{array} \right],$$

and

$$f_\eta(\eta(t),u(t)) \triangleq \left[\begin{array}{c} f_y(\mathbf{T}_{\eta y}\eta(t),u(t)) \\ \mathbf{0}_{n_e\times 1} \end{array} \right].$$

Remark 2.1: A few observations are now in place.

1) If a state estimate $\hat{z}(t)$ for the augmented system (5) is available, then an estimate of the the state of the original system (1) can be simply computed as
$$\hat{x}(t) = \mathbf{T}_{zx}\hat{z}(t)\,.$$
2) There exist a constant $\bar{d}_\eta$ such that
$$\bar{d}_\eta > 0 : |d_\eta(t)| \le \bar{d}_\eta, \forall t \in \mathbb{R}_{\ge 0}\,.$$
3) if Assumption 4 holds, $\mathbf{G}_z(\cdot,\cdot)$ is also Lipschitz with respect to the first argument, uniformly in the second one, with the same Lipschitz constant $\bar{\gamma}$ as $\mathbf{G}_x(\cdot,\cdot)$. Analogously, both $f_z(\cdot,\cdot)$ and $f_\eta(\cdot,\cdot)$ are Lipschitz continuous, with Lipschitz constants: $\bar{f}_z \triangleq \bar{f}_x + ||\mathbf{B}_e||\bar{f}_y$ and $\bar{f}_\eta = \bar{f}_y$, respectively.

Now, we introduce the structure of the adaptive observer, consisting of the measured output filter (2) and the dynamic components (6), (7) and (8) given in the following:

1) Augmented state estimator:

$$\begin{aligned} \dot{\hat{z}}(t) &= (\mathbf{A}_z - \mathbf{L}\mathbf{C}_z)\hat{z}(t) + \mathbf{L}(\breve{\eta}(t) - f_\eta(\breve{\eta}(t),u(t))) \\ &\quad + \mathbf{G}_z(\hat{z}(t),u(t))\hat{\theta}(t) + f_z(\breve{\eta}(t),u(t)) + \Xi(t)\,\dot{\hat{\theta}}(t) \end{aligned} \tag{6}$$

with arbitrary initial conditions $\hat{z}(0) = \hat{z}_0$. The gain matrix $\mathbf{L}$ is given by

$$\mathbf{L} \triangleq \left[\begin{array}{cc} \mathbf{L}_x & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{array} \right]$$

where $\mathbf{L}_x$ is a suitable gain matrix such that $\mathbf{A}_x - \mathbf{L}_x\mathbf{C}_x$ is Hurwitz (which is always possible thanks to Assumption 1).

2) Parameter-affine state-dependent matrix filter:

$$\dot{\Xi}(t) = (\mathbf{A}_z - \mathbf{L}\mathbf{C}_z)\Xi(t) + \mathbf{G}_z(\hat{z}(t),u(t)) \tag{7}$$

with $\Xi(0) = \mathbf{0}_{n_z\times n_\theta}$ and where $\Xi(t) \in \mathbb{R}^{n_z\times n_\theta}$ is an auxiliary time-varying matrix whose elements are obtained by filtering the state-dependent parameter-affine matrix $\mathbf{G}_z(\hat{z}(t),u(t))$.

3) Parameters' adaptation law:

$$\dot{\hat{\theta}}(t) = \Psi(t)\mathrm{Proj}\left[\dot{\hat{\theta}}_{pre}(t)\right]_{|\hat{\theta}|\le\bar{\theta}^*} \tag{8}$$

with $\hat{\theta}(0) = \hat{\theta}_0$. $\Psi(t)$ is a switching signal defined later on and

$$\begin{aligned} \dot{\hat{\theta}}_{pre}(t) \triangleq \quad & -\mu\big(\Xi^\top(t)\Xi(t) + \rho^2\mathbf{I}\big)^{-1}\Xi^\top(t)\Lambda(\mathbf{C}_z\hat{z}(t) \\ & -\breve{\eta}(t) + f_\eta(\breve{\eta}(t),u(t))), \end{aligned}$$

where μ is a positive constant and $\Lambda \in \mathbb{R}^{n_z\times(n_e+n_y)}$ is a suitable chosen constant matrix, such that[1]:

$$\Lambda C_z \ge 0\,, \quad \mathrm{rank}\,(\Lambda C_z) \ge n_\theta\,.$$

[1] A feasible choice for Λ is $\Lambda = C_z^\top$.

The parameters' derivative projection operator is defined as:

$$\mathrm{Proj}\left[\dot{\hat{\theta}}_{pre}(t)\right]_{|\hat{\theta}|\le\bar{\theta}^*} \triangleq \begin{cases} \mathrm{nsp}(\hat{\theta}^\top(t))\Big(\mathrm{nsp}(\hat{\theta}^\top(t))\Big)^\top \dot{\hat{\theta}}_{pre}(t), \\ \qquad \text{if } |\hat{\theta}| = \bar{\theta}^* \text{ and } \hat{\theta}^\top(t)\dot{\hat{\theta}}_{pre}(t) > 0 \\ \dot{\hat{\theta}}_{pre}(t), \quad \text{otherwise} \end{cases}$$

in which $\mathrm{nsp}(\cdot)$ denotes the null-space of a row vector. The observer components (6) and (8) are equipped with a binary signal $\Psi(t)$: $\mathbb{N}_{\ge 0} \to \{0,1\}$, in charge of activating/suppressing the parameter adaptation. The signal $\Psi(t)$ is generated by the following hysteretic switching dynamics:

$$\Psi(t) = \begin{cases} 1, & \text{if } \min \mathrm{eig}\,(\Phi(\Xi(t))) \ge 2\bar{\delta} \\ 0, & \text{if } \min \mathrm{eig}\,(\Phi(\Xi(t))) < 2\underline{\delta} \\ \Psi(t^-), & \text{if } 2\underline{\delta} \le \min \mathrm{eig}\,(\Phi(\Xi(t))) < 2\bar{\delta} \end{cases} \tag{9}$$

where

$$\Phi(\Xi(t)) = \left(\Xi^\top(t)\Xi(t) + \rho^2\mathbf{I}\right)^{-1}\Xi^\top(t)\Lambda\mathbf{C}_z\Xi(t)$$

represents the excitation matrix. We point out that transitions are driven by an excitation-detection logic. The transition thresholds $\underline{\delta}$, $\bar{\delta}$ are fixed by the designer such that $0 < 2\underline{\delta} < 2\bar{\delta} < 1$. The introduction of an hysteresis is motivated by the need to ensure that the time between transitions has a minimum finite duration.

Remark 2.2: The condition $\mathrm{rank}\,(\Lambda C_z) \ge n_\theta$ ensures that there exist $(t, \Xi(t))$ such that $\min \mathrm{eig}\,(\Phi(\Xi(t))) > 0$.

Remark 2.3: Note that, instead of computing the eigenvalues, it is possible to reformulate the switching on the basis of a simpler (though more conservative) criterion. Owing to the fact that

$$\mathbf{X}(t) \triangleq (\Xi^\top(t)\Xi(t) + \rho^2\mathbf{I})^{-1}\Xi^\top(t)\Lambda\mathbf{C}_z\Xi(t) \ge 0\,,$$

the following inequalities hold:

$$\max\ \mathrm{eig}(\mathbf{X}(t)) \le \sum \mathrm{eig}(\mathbf{X}(t)) = \mathrm{tr}(\mathbf{X}(t))$$

and

$$\begin{aligned} \min\ \mathrm{eig}(\mathbf{X}(t)) &\ge \frac{\prod \mathrm{eig}(\mathbf{X}(t))}{(\max\ \mathrm{eig}(\mathbf{X}(t)))^{n_\theta - 1}} \\ &= \frac{\det(\mathbf{X}(t))}{(\max\ \mathrm{eig}(\mathbf{X}(t)))^{n_\theta - 1}} \\ &\ge \frac{\det(\mathbf{X}(t))}{\mathrm{tr}(\mathbf{X}(t)))^{n_\theta - 1}} \end{aligned}$$

If the above inequalities are used to generate $\Psi(t)$ in (9) some conservatism is introduced, as a smaller set of state trajectories allows to reach the activation level.

III. Stability analysis

In order to address the stability of the adaptive observer, let us define the augmented state-estimation error vector:

$$\tilde{z}(t) \triangleq \hat{z}(t) - z(t),$$

the parameter estimation error

$$\tilde{\theta}(t) \triangleq \hat{\theta}(t) - \theta^*,$$

and their linear time-varying combination

$$\tilde{\varphi}(t) \triangleq \Xi(t)\tilde{\theta}(t) - \tilde{z}(t)\,.$$

As $(\mathbf{A}_z - \mathbf{L}\mathbf{C}_z)$ is Hurwitz, it follows that, for any positive definite matrix $\mathbf{Q}$, there exists a positive definite matrix $\mathbf{P}$ which solves the Lyapunov Equation

$$(\mathbf{A}_z - \mathbf{L}\mathbf{C}_z)^\top\mathbf{P} + \mathbf{P}(\mathbf{A}_z - \mathbf{L}\mathbf{C}_z) = -2\mathbf{Q}\,. \tag{10}$$

Let us introduce the candidate Lyapunov function

$$V(t) \triangleq \frac{1}{2}\left(\tilde{z}^\top(t)\mathbf{P}\tilde{z}(t) + \tilde{\theta}^\top(t)\tilde{\theta}(t) + g\tilde{\varphi}^\top(t)\mathbf{P}\tilde{\varphi}(t)\right) \tag{11}$$

where g is a positive constant.

In the following analysis, the excitation-based switching scheme depicted in Fig. 1 will be exploited. More specifically, let $k_{0\to1}(t)$ denote a counter for the transitions to excitation, described by the jump dynamics given below:

$$k_{0\to1}(t) = \begin{cases} k_{0\to1}(t^-) + 1, & \text{if } \Psi(t^-) = 0 \text{ and } \Psi(t) = 1 \\ k_{0\to1}(t^-), & \text{if } \Psi(t^-) = 1 \text{ and } \Psi(t) = 0 \end{cases}$$

with $k_{0\to1}(0) = 0$. Analogously, let $k_{1\to0}(t)$ be a counter for the transition from excitation to dis-excitation

$$k_{1\to0}(t) = \begin{cases} k_{1\to0}(t^-) + 1, & \text{if } \Psi(t^-) = 1 \text{ and } \Psi(t) = 0 \\ k_{1\to0}(t^-), & \text{if } \Psi(t^-) = 0 \text{ and } \Psi(t) = 1 \end{cases}$$

with $k_{1\to0}(0) = 1$. Moreover, let $t_{0\to1}(k)$ and $t_{1\to0}(k)$ denote the transition time-instants:

$$t_{0\to1}(k) \triangleq \inf(t \ge 0 : k_{0\to1}(t) = k)\,,$$
$$t_{1\to0}(k) \triangleq \inf(t \ge 0 : k_{1\to0}(t) = k)\,.$$

Without loss of generality and taking into account that the system starts from zero-excitement, then

$$t_{0\to1}(k) > t_{1\to0}(k), \forall k \in \mathbb{N}\,.$$

Hence, the integer k always identifies a two-phase time-window made of a dis-excitation interval followed by an active estimation interval (see Fig. 1).

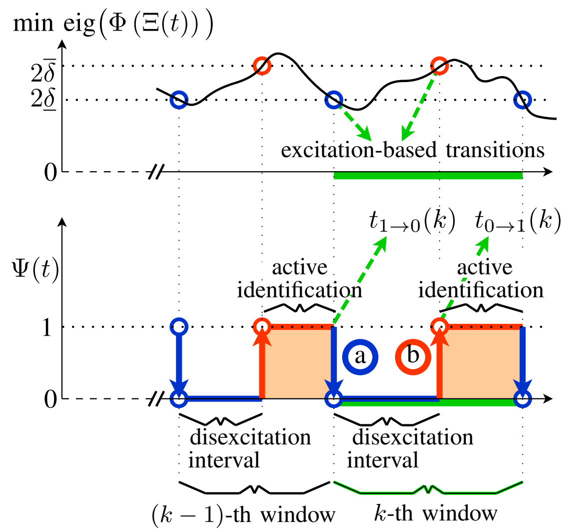

Fig. 1. Scheme of the excitation-based switching scheme for enabling/disabling the parameter adaptation. The transitions to dis-excitation (a) and to active identification phases (b) have been evidenced.

In the next section, the ISS stability properties of the proposed adaptive estimation scheme will be addressed. To this end, two intermediate results are provided in which we show that the derivative of the candidate Lyapunov function can be suitably bounded from above in both modes of behaviour (dis-excitation and excitation).

More precisely, we first establish an upper bound of the Lyapunov-function time-derivative for an arbitrary time-window k with poor excitation, in which $\Psi(t) = 0, \forall t \in \left[t_{1\to0}(k), t_{0\to1}(k)\right)$. Subsequently, the time-behaviour of the Lyapunov function in a k-th time window in which $\Psi(t) = 1, \forall t \in \left[t_{0\to1}(k), t_{1\to0}(k+1)\right)$ will be characterized.

Lemma 3.1 (Dis-excitation phase): Assume that

$$\Psi(t) = 0,\, \forall t \in \mathbb{R}_{\geq 0}\,.$$

Then under assumption 1-3, the time-derivative $\dot{V}$ of the candidate Lyapunov function V given in (11) for the adaptive observer and parametric estimator given by (2), (6), (7) and (8) verifies the inequality

$$\dot{V}(t) \leq \beta_0\big(L_0 V(t_{1\to 0}(k)) + \sigma_0(|d_\eta(t)|) - V(t)\big)$$

for suitable positive scalars β_0 and L_0, and for a suitable function σ_0. □

Proof: As $\Psi(t) = 0, \forall t \in \big[t_{1\to 0}(k), t_{0\to 1}(k)\big)$, we have

$$\dot{\hat{\theta}}(t) = \mathbf{0}_{n_\theta \times 1} \forall t \in \big[t_{1\to 0}(k), t_{0\to 1}(k)\big)$$

and hence $\tilde{\theta}(t) = \tilde{\theta}\big(t_{1\to 0}(k)\big)$. In this phase, $\tilde{z}(t)$ evolves according to the differential equation

$$\begin{aligned}\dot{\hat{z}}(t) = \;& (\mathbf{A}_z - \mathbf{L}\mathbf{C}_z)\hat{z}(t) \\ & + \mathbf{L}\big(\eta(t) + d_\eta(t) - f_\eta(\breve{\eta}(t), \eta(t), u(t))\big) \\ & + \mathbf{G}_z(\hat{z}(t), u(t))\hat{\theta}(t) - \mathbf{A}_z z(t) - \mathbf{G}_z(z(t), u(t))\theta^* \\ & + f_z(\breve{\eta}(t), u(t)) - f_z(\eta(t), u(t))\end{aligned}$$

After some algebra, we obtain

$$\begin{aligned}\dot{\tilde{z}}(t) = (\mathbf{A}_z - \mathbf{L}\mathbf{C}_z)\tilde{z}(t) + \mathbf{L}d_\eta(t) + \tilde{\mathbf{G}}_z(\hat{z}(t), z(t), u(t))\tilde{\theta}(t) \\ + \mathbf{G}_z(z(t), u(t))\tilde{\theta}(t) + \tilde{\mathbf{G}}_z(\hat{z}(t), z(t), u(t))\theta^* \\ + \tilde{f}_z(\breve{\eta}(t), \eta(t), u(t)) - \mathbf{L}\tilde{f}_\eta(\breve{\eta}(t), \eta(t), u(t))\end{aligned}$$

where

$$\begin{aligned}\tilde{\mathbf{G}}_z(\hat{z}(t), z(t), u(t)) &\triangleq \mathbf{G}_z(\hat{z}(t), u(t)) - \mathbf{G}_z(z(t), u(t)) \\ \tilde{f}_z(\breve{\eta}(t), \eta(t), u(t)) &\triangleq f_z(\breve{\eta}(t), u(t)) - f_z(\eta(t), u(t)) \\ \tilde{f}_\eta(\breve{\eta}(t), \eta(t), u(t)) &\triangleq f_\eta(\breve{\eta}(t), u(t)) - f_\eta(\eta(t), u(t))\end{aligned}$$

Analogously, $\tilde{\varphi}(t)$ evolves according to

$$\begin{aligned}\dot{\hat{\varphi}}(t) &= \dot{\Xi}(t)\tilde{\theta}(t) + \Xi(t)\dot{\hat{\theta}}(t) - \dot{\hat{z}}(t) \\ &= (\mathbf{A}_z - \mathbf{L}\mathbf{C}_z)\Xi(t)\tilde{\theta}(t) + \mathbf{G}_z(\hat{z}(t), u(t))\tilde{\theta}(t) \\ &\quad -(\mathbf{A}_z - \mathbf{L}\mathbf{C}_z)\tilde{z}(t) - \mathbf{G}_z(\hat{z}(t), u(t))\tilde{\theta}(t) \\ &\quad -\tilde{\mathbf{G}}_z(\hat{z}(t), z(t), u(t))\theta^* - \tilde{f}_z(\breve{\eta}(t), \eta(t), u(t)) \\ &\quad +\mathbf{L}\tilde{f}_\eta(\breve{\eta}(t), \eta(t), u(t)) - \mathbf{L}d_\eta(t)\end{aligned}$$

and hence

$$\begin{aligned}\dot{\tilde{\varphi}}(t) &= (\mathbf{A}_z - \mathbf{L}\mathbf{C}_z)\tilde{\varphi}(t) - \tilde{\mathbf{G}}_z(\hat{z}(t), z(t), u(t))\theta^* \\ &\quad -\tilde{f}_z(\breve{\eta}(t), \eta(t), u(t)) \\ &\quad +\mathbf{L}\tilde{f}_\eta(\breve{\eta}(t), \eta(t), u(t)) - \mathbf{L}d_\eta(t)\end{aligned}$$

Now, we let:

$$\bar{l} \triangleq \|\mathbf{L}\|\,, \quad \underline{q} \triangleq \min \operatorname{eig}(\mathbf{Q})\,.$$

and

$$\bar{p} \triangleq \max \operatorname{eig}(\mathbf{P}) \tag{12}$$

The time-derivative of the Lyapunov function is given by:

$$\begin{aligned}\dot{V}(t) = \;& \frac{1}{2}\big(\tilde{z}^\top(t)\mathbf{P}\dot{\tilde{z}}(t) + (\dot{\tilde{z}})^\top(t)\mathbf{P}\tilde{z}(t)\big) \\ & + \frac{g}{2}\big(\tilde{\varphi}^\top(t)\mathbf{P}\dot{\tilde{\varphi}}(t) + (\dot{\tilde{\varphi}})^\top(t)\mathbf{P}\tilde{\varphi}(t)\big)\end{aligned}$$

In view of the inequality $|\tilde{\theta}(t)| < 2\bar{\theta}^*$ and after some algebra $\dot{V}(t)$ can be bounded as follows:

$$\begin{aligned}\dot{V}(t) \leq & -\frac{\underline{q} - 3\bar{p}\bar{\gamma}\bar{\theta}^*}{2}|\tilde{z}(t)|^2 + \bar{p}\bar{\bar{\gamma}}|\tilde{z}(t)||\tilde{\theta}(t_{1\to 0}(k))| \\ & -\frac{\underline{q} - 3\bar{p}\bar{\gamma}\bar{\theta}^*}{2}|\tilde{z}(t)|^2 + \bar{p}(\bar{l} + \bar{f}_z + \bar{l}\bar{f}_\eta)|\tilde{z}(t)||d_\eta(t)| \\ & -\frac{g\underline{q}}{2}|\tilde{\varphi}(t)|^2 + g\bar{p}\bar{\gamma}\bar{\theta}^*|\tilde{z}(t)||\tilde{\varphi}(t)| - \frac{g\underline{q}}{2}|\tilde{\varphi}(t)|^2 \\ & + g\bar{p}(\bar{l} + \bar{f}_z + \bar{l}\bar{f}_\eta)|\tilde{\varphi}(t)||d_\eta(t)|\,.\end{aligned}$$

By completing squares, we obtain the following upper bound for $\dot{V}(t)$:

$$\begin{aligned}\dot{V}(t) \leq & -\Big(\frac{\underline{q} - 3\bar{p}\bar{\gamma}\bar{\theta}^*}{2} - \frac{g}{\underline{q}}\big(\bar{p}\bar{\gamma}\bar{\theta}^*\big)^2\Big)|\tilde{z}(t)|^2 \\ & -\frac{g\underline{q}}{2}|\tilde{\varphi}(t)|^2 + \frac{\bar{p}^2\bar{\bar{\gamma}}^2}{\underline{q} - 3\bar{p}\bar{\gamma}\bar{\theta}^*}|\tilde{\theta}(t_{1\to 0}(k))|^2 \\ & + \Big(\frac{\bar{p}^2\big(\bar{l} + \bar{f}_z + \bar{l}\bar{f}_\eta\big)^2}{\underline{q} - 3\bar{p}\bar{\gamma}\bar{\theta}^*} + \frac{g}{\underline{q}}\bar{p}^2\big(\bar{l} + \bar{f}_z + \bar{l}\bar{f}_\eta\big)^2\Big)|d_\eta(t)|^2\end{aligned}$$

By observing that

$$|\tilde{\theta}(t_{1\to 0}(k))|^2 \leq 2V(t_{1\to 0}(k))\,,$$

we get:

$$\begin{aligned}\dot{V}(t) \leq & -\Big(\frac{\underline{q} - 3\bar{p}\bar{\gamma}\bar{\theta}^*}{2} - \frac{g}{\underline{q}}\big(\bar{p}\bar{\gamma}\bar{\theta}^*\big)^2\Big)|\tilde{z}(t)|^2 - \frac{g\underline{q}}{2}|\tilde{\varphi}(t)|^2 \\ & -\frac{1}{2}|\tilde{\theta}(t)|^2 + 2\Big(\frac{\bar{p}^2\bar{\bar{\gamma}}^2}{\underline{q} - 3\bar{p}\bar{\gamma}\bar{\theta}^*} + \frac{1}{2}\Big)V(t_{1\to 0}(k)) \\ & + \Big(\frac{\bar{p}^2\big(\bar{l} + \bar{f}_z + \bar{l}\bar{f}_\eta\big)^2}{\underline{q} - 3\bar{p}\bar{\gamma}\bar{\theta}^*} + \frac{g}{\underline{q}}\bar{p}^2\big(\bar{l} + \bar{f}_z + \bar{l}\bar{f}_\eta\big)^2\Big)|d_\eta(t)|^2\end{aligned}$$

and hence, after some algebra, it follows that

$$\dot{V}(t) \leq \beta_0\big(L_0 V(t_{1\to 0}(k)) + \sigma_0(|d_\eta(t)|) - V(t)\big)$$

where

$$\beta_0 \triangleq 2\min\left\{\frac{\underline{q} - 3\bar{p}\bar{\gamma}\bar{\theta}^*}{2} - \frac{g}{\underline{q}}\big(\bar{p}\bar{\gamma}\bar{\theta}^*\big)^2, \frac{g\underline{q}}{2}, \frac{1}{2}\right\}\,,$$

$$L_0 \triangleq \frac{2}{\beta_0}\left\{\frac{\bar{p}^2\bar{\bar{\gamma}}^2}{\underline{q} - 3\bar{p}\bar{\gamma}\bar{\theta}^*} + \frac{1}{2}\right\}\,,$$

and

$$\sigma_0(s) \triangleq \frac{1}{\beta_0}\left(\frac{\bar{p}^2\big(\bar{l} + \bar{f}_z + \bar{l}\bar{f}_\eta\big)^2}{\underline{q} - 3\bar{p}\bar{\gamma}\bar{\theta}^*} + \frac{g}{\underline{q}}\bar{p}^2\big(\bar{l} + \bar{f}_z + \bar{l}\bar{f}_\eta\big)^2\right)s^2, \quad \forall s \in \mathbb{R}_{\geq 0}\,.$$

Hence, for $\beta_0 > 0$ (which is possible through a suitable design, as will be shown later on in the paper), the proof is concluded. ■

It is worth noting in the above proof that, without adaptation, in presence of parameter mismatch, the Lyapunov function is not guaranteed to converge toward zero. However, for $\beta_0 > 0$, it admits a bound that depends on the noise level and on the initial value of the Lyapunov function itself.

Now, consider the active identification phase of the k-th time-window, in which $\Psi(t) = 1, \forall t \in \big[t_{0\to 1}(k), t_{1\to 0}(k+1)\big)$.

Lemma 3.2 (Excitation phase): Assume that

$$\Psi(t) = 1,\, \forall t \in \mathbb{R}_{\geq 0}\,.$$

Then under assumption 1-3, the time-derivative $\dot{V}$ of the candidate Lyapunov function V given in (11) for the adaptive observer and parametric estimator given by (2), (6), (7) and (8) verifies the inequality

$$\dot{V}(t) \leq -\beta_1 \left[V(t) - \sigma_1(\bar{d}_\eta) \right],$$

for a suitable positive scalar β_1 and for a suitable function σ_1, with respect to any bounded additive measurement perturbation $d_\eta(t) \leq \bar{d}_\eta$. □

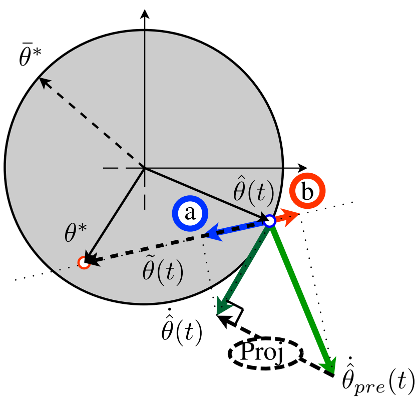

Fig. 2. 2D pictorial representation of the projection-based adaptation. When $|\hat{\theta}(t)| = \bar{\theta}^*$ and $\dot{\hat{\theta}}_{pre}(t) = \theta^*$ points out of the feasible region, then the derivative of the parameter vector is obtained by projecting $\dot{\hat{\theta}}_{pre}(t)$ to the tangential hyperplane. To visually compare the values of the scalar products $-\tilde{\theta}^\top \dot{\hat{\theta}}_{pre}(t)$ and $-\tilde{\theta}^\top \dot{\hat{\theta}}(t)$, consider the projected vectors (a) and (b) respectively.

Proof: The state estimation vector evolves according to the differential equation

$$\begin{aligned} \dot{\tilde{z}}(t) \;=\; & (\mathbf{A}_z - \mathbf{L}\mathbf{C}_z)\tilde{z}(t) + \mathbf{L}d_\eta(t) \\ & + \tilde{\mathbf{G}}_z(\hat{z}(t), z(t), u(t))\tilde{\theta}(t) + \mathbf{G}_z(z(t), u(t))\tilde{\theta}(t) \\ & + \tilde{\mathbf{G}}_z(\hat{z}(t), z(t), u(t))\theta^* + \tilde{f}_z(\breve{\eta}(t), \eta(t), u(t)) \\ & - \mathbf{L}\tilde{f}_\eta(\breve{\eta}(t), \eta(t), u(t)) + \Xi(t)\dot{\hat{\theta}}(t) \end{aligned}$$

Moreover

$$\begin{aligned} \dot{\hat{\theta}}(t) = \text{Proj}\, \{ & -\mu(\Xi^\top(t)\Xi(t) + \rho^2\mathbf{I})^{-1}\Xi^\top(t)\Lambda \\ & \times (\mathbf{C}_z\hat{z}(t) - \breve{\eta}(t) + f_\eta(\breve{\eta}(t), u(t)))\}_{|\hat{\theta}| \leq \bar{\theta}^*} \end{aligned}$$

Hence, after some algebra, we get

$$\begin{aligned} \dot{\hat{\theta}}(t) = \text{Proj} \quad \{ & -\mu(\Xi^\top(t)\Xi(t) + \rho^2\mathbf{I})^{-1}\Xi^\top(t) \\ & \times \Lambda \mathbf{C}_z \Xi(t)\tilde{\theta}(t) \\ & + \mu(\Xi^\top(t)\Xi(t) + \rho^2\mathbf{I})^{-1}\Xi^\top(t) \\ & \times \Lambda \big(d_\eta(t) - \tilde{f}_\eta(\breve{\eta}(t), \eta(t), u(t))\big) \\ & + \mu(\Xi^\top(t)\Xi(t) + \rho^2\mathbf{I})^{-1}\Xi^\top(t) \\ & \times \Lambda \mathbf{C}_z \tilde{\varphi}(t)\}_{|\hat{\theta}| \leq \bar{\theta}^*} \end{aligned}$$

To analyze the influence of the derivative projection on the parameters' estimates, consider now the vectors depicted in Fig. 2. Owing to the convexity of the admissible set, we can bound the scalar product $-\tilde{\theta}^\top(t)\dot{\hat{\theta}}(t)$ as follows:

$$\begin{aligned} -\tilde{\theta}^\top(t)\dot{\hat{\theta}}(t) \;&=\; -\tilde{\theta}^\top(t)\text{Proj}\left\{ \dot{\hat{\theta}}_{pre}(t) \right\}_{|\hat{\theta}| \leq \bar{\theta}^*} \\ &\geq\; -\tilde{\theta}^\top(t)\dot{\hat{\theta}}_{pre}(t) \end{aligned}$$

and hence

$$\begin{aligned} & \tilde{\theta}^\top(t)\dot{\hat{\theta}}(t) \leq \\ & \quad -\mu\tilde{\theta}^\top(t)(\Xi^\top(t)\Xi(t) + \rho^2\mathbf{I})^{-1}\Xi^\top(t)\Lambda\mathbf{C}_z\Xi(t)\tilde{\theta}(t) \\ & \qquad + \mu\tilde{\theta}^\top(t)(\Xi^\top(t)\Xi(t) + \rho^2\mathbf{I})^{-1}\Xi^\top(t)\Lambda \\ & \qquad \times \big(d_\eta(t) - \tilde{f}_\eta(\breve{\eta}(t), \eta(t), u(t))\big) \\ & \qquad + \mu\tilde{\theta}^\top(t)(\Xi^\top(t)\Xi(t) + \rho^2\mathbf{I})^{-1}\Xi^\top(t)\Lambda\mathbf{C}_z\tilde{\varphi}(t). \end{aligned}$$

By letting $\bar{c} \triangleq \|\mathbf{C}_z\|$ and $\bar{\lambda} \triangleq \|\Lambda\|$, the time-derivative of the Lyapunov function can be bounded as follows:

$$\begin{aligned} \dot{V}(t) \leq & -\underline{q}|\tilde{z}(t)|^2 - g\underline{q}|\tilde{\varphi}(t)|^2 - \mu\underline{\delta}|\tilde{\theta}(t)|^2 \\ & + \bar{p}(\bar{l} + \bar{f}_z + \bar{l}\bar{f}_\eta)|\tilde{z}(t)||d_\eta(t)| + \bar{p}\bar{\gamma}|\tilde{z}(t)|^2|\tilde{\theta}(t)| \\ & + \bar{p}\bar{\bar{\gamma}}|\tilde{z}(t)||\tilde{\theta}(t)| + \bar{p}\bar{\gamma}\bar{\theta}^*|\tilde{z}(t)|^2 + \mu\bar{p}\bar{c}\bar{\lambda}|\tilde{z}(t)|^2 \\ & + \mu\bar{p}\bar{\lambda}(1 + \bar{f}_\eta)|\tilde{z}(t)||d_\eta(t)| + \mu\frac{\bar{\lambda}(1 + \bar{f}_\eta)}{2\rho}|\tilde{\theta}(t)||d_\eta(t)| \\ & + \mu\frac{\bar{c}\bar{\lambda}}{2\rho}|\tilde{\theta}(t)||\tilde{\varphi}(t)| + g\bar{p}(\bar{l} + \bar{f}_z + \bar{l}\bar{f}_\eta)|\tilde{\varphi}(t)||d_\eta(t)| \\ & + g\bar{p}\bar{\gamma}\bar{\theta}^*|\tilde{z}(t)||\tilde{\varphi}(t)|, \end{aligned}$$

and thus, after some algebra, we obtain

$$\begin{aligned} \dot{V}(t) \leq & -(\underline{q} - \mu\bar{p}\bar{c}\bar{\lambda} - 3\bar{p}\bar{\gamma}\bar{\theta}^*)|\tilde{z}(t)|^2 - g\underline{q}|\tilde{\varphi}(t)|^2 - \mu\underline{\delta}|\tilde{\theta}(t)|^2 \\ & + \bar{p}\big(\bar{l} + \bar{f}_z + \bar{l}\bar{f}_\eta + \mu\bar{\lambda}(1 + \bar{f}_\eta)\big)|\tilde{z}(t)||d_\eta(t)| \\ & + \mu\frac{\bar{c}\bar{\lambda}}{2\rho}|\tilde{\theta}(t)||\tilde{\varphi}(t)| + g\bar{p}\bar{\gamma}\bar{\theta}^*|\tilde{z}(t)||\tilde{\varphi}(t)| + \bar{p}\bar{\bar{\gamma}}|\tilde{z}(t)||\tilde{\theta}(t)| \\ & + \mu\frac{\bar{\lambda}(1 + \bar{f}_\eta)}{2\rho}|\tilde{\theta}(t)||d_\eta(t)| + g\bar{p}(\bar{l} + \bar{f}_z + \bar{l}\bar{f}_\eta)|\tilde{\varphi}(t)||d_\eta(t)|. \end{aligned}$$

By re-arranging the above inequality to put in evidence the square monomial and the binomial terms, we have:

$$\begin{aligned} \dot{V}(t) \leq \quad & -(\underline{q} - \mu\bar{p}\bar{c}\bar{\lambda} - 3\bar{p}\bar{\gamma}\bar{\theta}^*)|\tilde{z}(t)|^2 \\ & + \bar{p}\big(\bar{l} + \bar{f}_z + \bar{l}\bar{f}_\eta + \mu\bar{\lambda}(1 + \bar{f}_\eta)\big)|\tilde{z}(t)||d_\eta(t)| \\ & - \frac{g\underline{q}}{2}|\tilde{\varphi}(t)|^2 + g\bar{p}\bar{\gamma}\bar{\theta}^*|\tilde{z}(t)||\tilde{\varphi}(t)| \\ & - \frac{g\underline{q}}{2}|\tilde{\varphi}(t)|^2 + g\bar{p}(\bar{l} + \bar{f}_z + \bar{l}\bar{f}_\eta)|\tilde{\varphi}(t)||d_\eta(t)| \\ & - \frac{\mu\underline{\delta}}{3}|\tilde{\theta}(t)|^2 + \bar{p}\bar{\bar{\gamma}}|\tilde{z}(t)||\tilde{\theta}(t)| \\ & - \frac{\mu\underline{\delta}}{3}|\tilde{\theta}(t)|^2 + \mu\frac{\bar{\lambda}(1 + \bar{f}_\eta)}{2\rho}|\tilde{\theta}(t)||d_\eta(t)| \\ & - \frac{\mu\underline{\delta}}{3}|\tilde{\theta}(t)|^2 + \mu\frac{\bar{c}\bar{\lambda}}{2\rho}|\tilde{\theta}(t)||\tilde{\varphi}(t)|. \end{aligned}$$

Now, we complete the squares, getting to

$$\begin{aligned} \dot{V}(t) \quad \leq & -\frac{(\underline{q} - \mu\bar{p}\bar{c}\bar{\lambda} - 3\bar{p}\bar{\gamma}\bar{\theta}^*)}{2}|\tilde{z}(t)|^2 \\ & + \frac{\bar{p}^2\big(\bar{l} + \bar{f}_z + \bar{l}\bar{f}_\eta + \mu\bar{\lambda}(1 + \bar{f}_\eta)\big)^2}{2(\underline{q} + \mu\bar{p}\bar{c}\bar{\lambda} - 3\bar{p}\bar{\gamma}\bar{\theta}^*)}|d_\eta(t)|^2 \\ & - \frac{g\underline{q}}{2}|\tilde{\varphi}(t)|^2 + \frac{g}{\underline{q}}(\bar{p}\bar{\gamma}\bar{\theta}^*)^2|\tilde{z}(t)|^2 - \frac{\mu\underline{\delta}}{2}|\tilde{\theta}(t)|^2 \\ & + \frac{g}{\underline{q}}\bar{p}^2(\bar{l} + \bar{f}_z + \bar{l}\bar{f}_\eta)^2|d_\eta(t)|^2 + \frac{3(\bar{p}\bar{\bar{\gamma}})^2}{2\mu\underline{\delta}}|\tilde{z}(t)|^2 \\ & + \frac{3\left(\mu\frac{\bar{\lambda}(1 + \bar{f}_\eta)}{2\rho}\right)^2}{2\mu\underline{\delta}}|d_\eta(t)|^2 + \frac{3\left(\mu\frac{\bar{c}\bar{\lambda}}{2\rho}\right)^2}{2\mu\underline{\delta}}|\tilde{\varphi}(t)|^2 \end{aligned}$$

and hence

$$\begin{aligned}\dot V(t) \quad \le & -\left[\frac{(\underline{q}-\mu\bar p\bar c\bar\lambda-3\bar p\bar\gamma\bar\theta^*)}{2}-\frac{g}{\underline{q}}(\bar p\bar\gamma\bar\theta^*)^2 \right. \\ & \left. -\frac{3(\bar p\bar{\bar\gamma})^2}{2\mu\underline{\delta}}\right]|\tilde z(t)|^2 \\ & -\frac{\mu\underline{\delta}}{2}|\tilde\theta(t)|^2-\left(\frac{g\underline{q}}{2}-\frac{3\mu\bar c^2\bar\lambda^2}{8\rho^2\underline{\delta}}\right)|\tilde\varphi(t)|^2 \\ & +\left[\frac{\bar p^2\big(\bar l+\bar f_z+\bar l\bar f_\eta+\mu\bar\lambda(1+\bar f_\eta)\big)^2}{2(\underline{q}+\mu\bar p\bar c\bar\lambda-3\bar p\bar\gamma\bar\theta^*)}\right. \\ & \left. +\frac{3\mu\big(\bar\lambda(1+\bar f_\eta)\big)^2}{8\rho^2\underline{\delta}}+\frac{g}{\underline{q}}\bar p^2(\bar l+\bar f_z+\bar l\bar f_\eta)^2\right]|d_\eta(t)|^2\end{aligned}$$

Summing up, for the case $\Psi(t)=1$ (excitation), the following inequality can be established:

$$\dot V(t) \le -\beta_1\left[V(t)-\sigma_1(\bar d_\eta)\right],$$

where

$$\begin{aligned}&\beta_1 \triangleq \\ &2\min\left\{\frac{(\underline{q}-\mu\bar p\bar c\bar\lambda-3\bar p\bar\gamma\bar\theta^*)}{2}-\frac{g}{\underline{q}}(\bar p\bar\gamma\bar\theta^*)^2-\frac{3(\bar p\bar{\bar\gamma})^2}{2\mu\underline{\delta}},\right. \\ &\qquad\left.\frac{\mu\underline{\delta}}{2},\ \frac{g\underline{q}}{2}-\frac{3\mu\bar c^2\bar\lambda^2}{8\rho^2\underline{\delta}}\right\} \qquad (13)\end{aligned}$$

and (for $\beta_1>0$)

$$\begin{aligned}\sigma_1(s) \triangleq \frac{1}{\beta_1} & \left[\frac{\bar p^2\big(\bar l+\bar f_z+\bar l\bar f_\eta+\mu\bar\lambda(1+\bar f_\eta)\big)^2}{2(\underline{q}+\mu\bar p\bar c\bar\lambda-3\bar p\bar\gamma\bar\theta^*)}\right. \\ & \left. +\frac{3\mu\big(\bar\lambda(1+\bar f_\eta)\big)^2}{8\rho^2\underline{\delta}}+\frac{g}{\underline{q}}\bar p^2(\bar l+\bar f_z+\bar l\bar f_\eta)^2\right]s^2. \\ & \forall s\in\mathbb{R}_{\ge 0}\end{aligned}$$

Hence, for $\beta_1>0$ (which is possible through a suitable design, as will be shown later on in the paper), the proof is concluded. ■

The bounds on the derivative of the Lyapunov function in the two phases of dis-excitation and excitation (assuming that these phases go on indefinitely) have been proven by claiming the existence of a design procedure that guarantees that a positive value of β_0 in Lemma 3.1 and of β_1 in Lemma 3.2 can be constructed, respectively. As $\beta_1>0$ implies $\beta_0>0$ ($L_0>0$ as well), we are now only proceeding to analyze the conditions making β_1 strictly positive. More specifically, we prove the existence of a parameter set which ensures the monotonic decrease of the Lyapunov function for any prescribed value of the excitation threshold $\underline{\delta}$.

First, we set the excitation threshold $\underline{\delta}$ and the $\mathbf{Q}$ matrix arbitrarily, determining $\underline{q}$. Then, letting

$$\bar p \ge \mu, \qquad (14)$$

we determine a sufficiently small value of μ such that the first term in (13) is positive:

$$\frac{(\underline{q}-\mu^2\bar c\bar\lambda-3\mu\bar\gamma\bar\theta^*)}{2}-\mu^2\frac{g}{\underline{q}}(\bar\gamma\bar\theta^*)^2-\mu\frac{3\bar{\bar\gamma}^2}{2\underline{\delta}}>0.$$

Being the Lyapunov parameter $g>0$ arbitrary, let us fix $g=1$ for simplicity. Now, we can always determine a sufficiently small value of μ such that the inequality holds true. Next, we compute the output-injection gain $\mathbf{L}$ in order to let inequality (14) hold true (this task can be achieved by suitably allocating the poles of $\mathbf{A}_z-\mathbf{L}\mathbf{C}_z$, such that the solution $\mathbf{P}$ of (10), whose eigenvalues are bounded by (12), allows to fulfill (14)). Finally, to render β_1 strict-positive, we choose a regularization parameter ρ such that

$$\frac{g\underline{q}}{2}-\frac{3\mu\bar c^2\bar\lambda^2}{8\rho^2\underline{\delta}}>0.$$

Remark 3.1: Note that low values of $\bar p$, that enforce the positiveness of β_1, correspond to high-gain output injections through $\mathbf{L}$. This may result in high values of $\bar l$. As a consequence, also σ_1, that is, the worst-case sensitivity to bounded noises, is increased. Two possible ways to reduce the noise sensitivity are: to increase the regularization parameter ρ and to increase the threshold $\underline{\delta}$. The latter however, reduces the estimator switch-on times and should be avoided if possible.

Remark 3.2: The *semi-global* attribute, referred to the stability of the observer, arises from the assumption that the observed system state is bounded by a constant $\bar x$. However, for any finite $\bar x$ arbitrarily large, there exists a suitable choice for the design parameters which allows to enforce the ISS property.

IV. Robustness under Alternate excitation and Active identification phases

By linking the available Lyapunov bounds established in the previous section in correspondence to the transitions, we can study the worst-case behaviour of the system under alternate switching. It is also possible to determine the sufficient conditions (namely, a minimum time-duration of the active identification phases) that guarantee the asymptotic ISS property of the discrete dynamics induced by sampling the adaptive observer in correspondence of the transitions. This important result is given in the following theorem.

Theorem 4.1: Under the same assumptions of Lemma 3.1 and 3.2, consider the adaptive observer (2), (6), (7), (8) equipped with the excitation level-based switching strategy defined in (9). Moreover, assume that any excitation phase ΔT_e lasts longer than $\beta_1^{-1}\ln(L_0)$. Then, the discrete dynamics induced by sampling the adaptive observer in correspondence of the switching transitions has the asymptotic ISS property. □

Proof: By the Gronwall-Bellman Lemma, the value of the Lyapunov function within the dis-excitation intervals can be bounded as follows:

$$\begin{aligned}V(t)\le\ & V(t_{1\to 0}(k))+\left(1-e^{-\beta_0(t-t_{1\to 0}(k))}\right) \\ & \times\left(L_0V(t_{1\to 0}(k))+\sigma_0(\bar d_\eta)-V(t_{1\to 0}(k))\right) \\ & \forall t\in[t_{1\to 0}(k),t_{0\to 1}(k)),\forall k\in\mathbb{Z}\ge 1\end{aligned}$$

Instead, during the excitation phases, the Lyapunov function can be bounded as

$$\begin{aligned}V(t)\le\ & V(t_{0\to 1}(k))+\left(1-e^{-\beta_1(t-t_{0\to 1}(k))}\right) \\ & \times\left(\sigma_1(\bar d_\eta)-V(t_{0\to 1}(k))\right) \\ & \forall t\in[t_{0\to 1}(k),t_{1\to 0}(k+1)),\forall k\in\mathbb{Z}\ge 1\end{aligned}$$

In order to link the two modes of behaviour, let us denote by $V_k=V(t_{1\to 0}(k))$ the value of the Lyapunov function sampled at the k-th transition to dis-excitement, occurring at time $t_{1\to 0}(k)$ (or equivalently, at the end of the $(k-1)$-th active identification phase).

Due to the poor excitation during the interval $[t_{1\to 0}(k),t_{0\to 1}(k))$, at the transition time $t_{0\to 1}(k)$ we can establish the (conservative) bound

$$V(t_{0\to 1}(k))\le L_0V_k+\sigma_0(\bar d_\eta).$$

Such a bound holds for any duration the disexcitation phase. For any subsequent active identification time $t=t_{0\to 1}(k)+$

Δt with $\Delta t < t_{1\to 0}(k+1) - t_{0\to 1}(k)$, we get the inequality:

$$\begin{aligned} V(t) &\leq V(t_{0\to 1}(k)) + \left(1 - e^{-\beta_1 (t - t_{0\to 1}(k))}\right) \\ &\quad \times \left(\sigma_1(\bar{d}_\eta) - V(t_{0\to 1}(k))\right) \\ &= \sigma_1(\bar{d}_\eta) - e^{-\beta_1(t-t_{0\to 1}(k))}\sigma_1(\bar{d}_\eta) \\ &\quad + e^{-\beta_1(t-t_{0\to 1}(k))} V(t_{0\to 1}(k)) \\ &\leq \sigma_1(\bar{d}_\eta) - e^{-\beta_1(t-t_{0\to 1}(k))}\sigma_1(\bar{d}_\eta) \\ &\quad + e^{-\beta_1(t-t_{0\to 1}(k))}\left(\sigma_0(\bar{d}_\eta) + L_0 V_k\right) \\ &= \sigma_1(\bar{d}_\eta) + e^{-\beta_1 \Delta t}\left(\sigma_0(\bar{d}_\eta) - \sigma_1(\bar{d}_\eta) + L_0 V_k\right) \end{aligned}$$

Now, let us arbitrarily set $\bar{\omega} < 1$ and set $\Delta T_e = -\beta_1^{-1} \ln\left(L_0^{-1}\bar{\omega}\right)$. If the active identification phase is long enough to verify the inequality $t_{1\to 0}(k+1) - t_{0\to 1}(k) > \Delta T_e$, then we can guarantee the following difference bound on the discrete (sampled) Lyapunov function sequence:

$$V_{k+1} \leq \sigma_1(\bar{d}_\eta) + \frac{\bar{\omega}}{L_0}\left(\sigma_0(\bar{d}_\eta) - \sigma_1(\bar{d}_\eta)\right) + \bar{\omega} V_k$$

which can be rearranged in the following compact form:

$$V_{k+1} - V_k \leq -(1-\bar{\omega})V_k + \sigma(\bar{d}_\eta)$$

where $\sigma(s) = \sigma_1(s) + \bar{\omega} L_0^{-1}\left(\sigma_0(s) - \sigma_1(s)\right), \forall s \geq 0$.

We can conclude that V_k is a discrete ISS Lyapunov function for the sampled sequence, with samples taken at the end of the excitation phases assumed always to last longer than $\beta_1^{-1}\ln(L_0)$.

In order to analyze the inter-sampling behaviour, for any time t , let $k(t)$ denote the index of the current time-window: $k(t) = k : t \in \left[t_{1\to 0}(k), t_{1\to 0}(k+1)\right)$. Between two samples, the Lyapunov function can be bounded as follows:

$$\begin{aligned} V(t) \leq &\max_{t\in\left[t_{1\to 0}(k(t)), t_{0\to 1}(k(t))\right)} \Big\{V_{k(t)} \\ &+ \left(1 - e^{-\beta_0(t - t_{1\to 0}(k(t)))}\right)\left(L_0 V_{k(t)} + \sigma_0(\bar{d}_\eta) - V_{k(t)}\right)\Big\} \\ + &\max_{t\in\left[t_{0\to 1}(k(t)), t_{1\to 0}(k(t)+1)\right)} \Big\{\sigma_1(\bar{d}_\eta) \\ &- e^{-\beta_1(t-t_{0\to 1}(k(t)))}\left(\sigma_1(\bar{d}_\eta) - \sigma_0(\bar{d}_\eta) - L_0 V_{k(t)}\right)\Big\} \end{aligned}$$

thus leading to

$$\begin{aligned} V(t) \leq &\max_{\Delta t_1 \in \mathbb{R}_{\geq 0}} \\ &\Big\{V_{k(t)} + \left(1 - e^{-\beta_0 \Delta t_1}\right)\left(L_0 V_{k(t)} + \sigma_0(\bar{d}_\eta) - V_{k(t)}\right)\Big\} \\ + &\max_{\Delta t_2 \in \mathbb{R}_{\geq 0}} \Big\{\sigma_1(\bar{d}_\eta) - e^{-\beta_1 \Delta t_2}\left(\sigma_1(\bar{d}_\eta) - \sigma_0(\bar{d}_\eta) - L_0 V_{k(t)}\right)\Big\} \\ \leq &\left[(1+L_0)V_{k(t)} + \sigma_0(\bar{d}_\eta)\right] + \left[\sigma_1(\bar{d}_\eta) + \sigma_0(\bar{d}_\eta) + L_0 V_{k(t)}\right] \\ &= (1+2L_0)V_{k(t)} + \sigma_1(\bar{d}_\eta) + 2\sigma_0(\bar{d}_\eta) \end{aligned} \tag{15}$$

If we let $k(t) \xrightarrow[t\to\infty]{} \infty$ (i.e., an infinite number of active identification phases occurs asymptotically), then the estimation errors in the inter-sampling times converge to a region whose radius depends only on the assumed disturbance bound.[2] ∎

V. Simulation Results

Consider

$$\begin{cases} \dot{x}_1(t) = 2x_2 + \theta_1^* \sin x_2 \\ \dot{x}_2(t) = -4x_1 - 8x_2 + \theta_2^* x_1^3 + u(t) \\ y(t) = x_1(t) + d_y(t) \end{cases} \tag{16}$$

with $\theta_1^* = -1$, $\theta_2^* = -2$. $d_y(t)$ denotes a bounded measurement noise with uniform distribution in the interval

[2]Note that this reasoning leads also allows to recover the ISS result for a single excitation phase lasting indefinitely, since it can be thought as made up of an infinite number of excitation phases of finite duration separated by dis-excitation phases of arbitrarily short length

$[-0.05, 0.05]$. The external input is given by $u(t) = 1 + \alpha(t)\sin t$, $t \geq 0$, where $\alpha(t) = 1$, $t \in [0, 10)$, $\alpha(t) = 20$, $t \in [10, 70)$, and $\alpha(t) = 1$, $t \in [70, 100)$. In view of (5), the choice $A_e = -1$, $B_e = 1$ leads to augment (16) as

$$\begin{cases} \dot{z}(t) = \mathbf{A}_z z(t) + \mathbf{G}_z(z(t), u(t))\theta^* + B_z u(t) \\ \breve{\eta}(t) = \mathbf{C}_z z(t) + d_\eta(t) \end{cases} \tag{17}$$

with $z(0) = z_0$ and where

$$\mathbf{A}_z = \begin{bmatrix} 0 & 2 & 0 \\ -4 & -8 & 0 \\ 1 & 0 & -1 \end{bmatrix}, \mathbf{B}_z = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \mathbf{C}_z = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Moreover

$$\mathbf{G}_z(z(t), u(t)) = \begin{bmatrix} \sin x_2 & 0 \\ 0 & x_1^3 \\ 0 & 0 \end{bmatrix}, \ \mathbf{L} = \begin{bmatrix} -4 & 0 \\ 13.5 & 0 \\ 0 & 0 \end{bmatrix},$$

where $\mathbf{L}$ denotes the observer gain matrix that can be constructed using the gain matrix $\mathbf{L}_x$ which, in turn, can be easily calculated by placing the eigenvalues at $(-1, -3)$.

The tuning parameters are: $\rho = 0.5$, $\mu = 15$, $\bar{\delta} = 0.075$, $\underline{\delta} = 0.05$, $\Lambda = C_z^\top$. In the mean time, the initial values are $\hat{z}(0) = [0,\ 0,\ 0]^\top$, $\hat{\theta}_1(0) = 0.3$, $\hat{\theta}_2(0) = 0$.

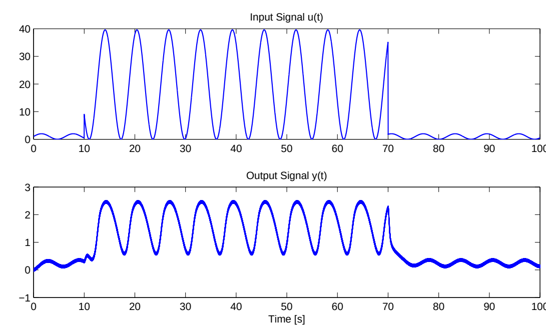

Fig. 3. Behaviour of the input signal $u(t)$ and the output signal $y(t)$.

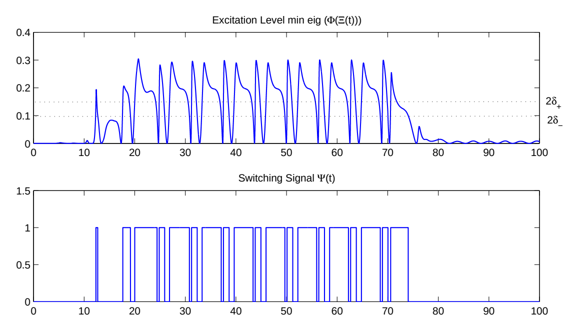

Fig. 4. Behaviour of the excitation level and of the switching signal $\Psi(t)$.

Fig. 3 show the input $u(t)$ and output $y(t)$, respectively, while the switching signal $\Psi(t)$ and the excitation level $\min \operatorname{eig}(\Phi(\Xi(t)))$ are plotted in Fig. 4, in which on-off switching level signal based on the detected excitation can be observed. In Fig.5 we compare the estimated parameters of both algorithms with and without switching strategy. Thanks to the switching algorithm, the estimates improve progressively and reach the true values after a finite number of active identification intervals. In the meantime, the estimated parameters are 'frozen' at the previous estimates in poorly

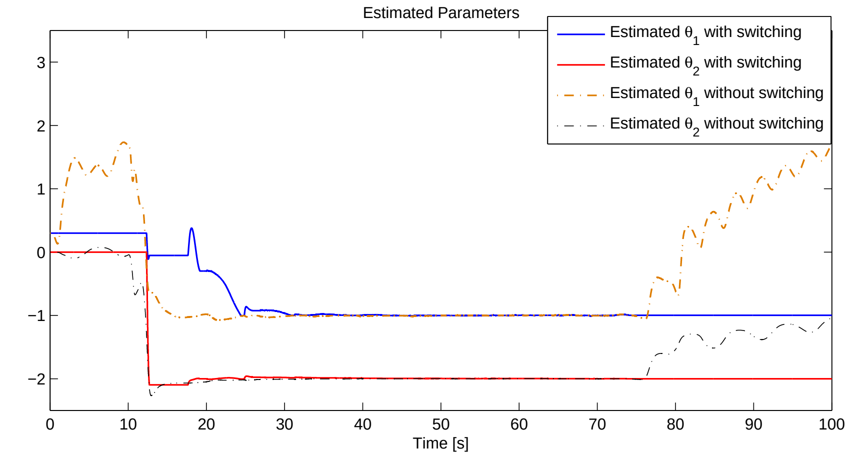

Fig. 5. Estimated parameters $\hat{\theta}_1$ and $\hat{\theta}_2$ with switching algorithm (solid line) and without switching algorithm (dashed line)

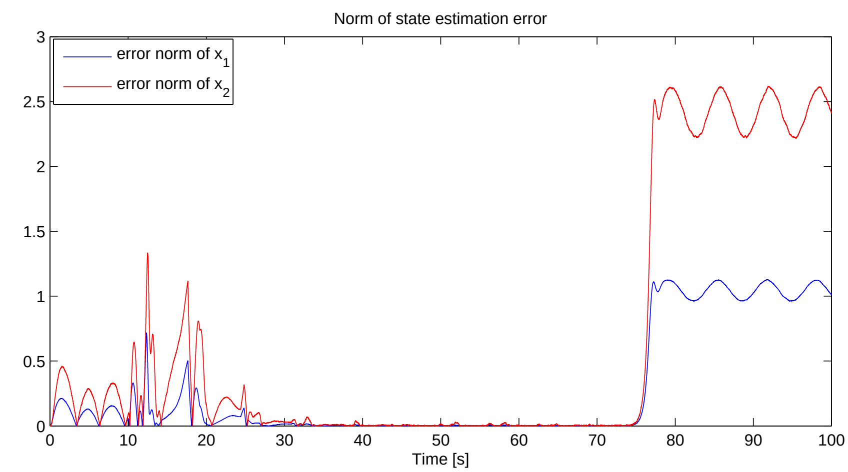

Fig. 6. Norm of state estimation errors $\tilde{x}_1$ and $\tilde{x}_2$ with switching algorithm (solid line) and without switching algorithm (dashed line)

excited conditions; conversely, the estimates diverge in the dis-excitation phase when the switching criterion is not used.

Finally, Fig. 6 shows that the state estimation errors are bounded, which is consistent with Theorem 4.1 (see (15) in the proof showing the boundedness of the Lyapunov function for the inter-sampling behaviour).

VI. Concluding Remarks

A MIMO adaptive observer for uncertain linearly-parametrized nonlinear systems has been proposed. The adaptive observer is equipped with a robust excitation-based switching strategy allowing to address poor excitation scenarios. The robustness of the proposed algorithm is shown and an ISS stability analysis is provided. The early simulation results show promising performances of the proposed technique.

Future research efforts will be devoted to extend the class of parametrized uncertain systems dealt with in this paper and to apply the algorithm to some specific estimation contexts like, for example, the estimation of a finite number of frequencies of multiple sinusoidal signals even in presence of bounded additive disturbances.

References

[1] G. Bastin and M. Gevers, “Stable adaptive observers for nonlinear time-varying systems,” *IEEE Trans. Automatic Control*, vol. 33, no. 7, pp. 650–658, 1988.

[2] Y. Cho and R. Rajamani, “A systematic approach to adaptive observer synthesis for nonlinear systems,” *IEEE Trans. Automatic Control*, vol. 42, no. 4, pp. 534–537, 1997.

[3] R. Marino and P.Tomei, “Global adaptive observers for nonlinear systems via filtered transformations,” *IEEE Trans. Automatic Control*, vol. 37, no. 8, pp. 1239–1245, 1992.

[4] R. Marino, G. Santosuosso, and P. Tomei, “Robust adaptive observers for nonlinear systems with bounded disturbances,” *IEEE Trans. Automatic Control*, vol. 46, no. 6, pp. 967–972, 2001.

[5] Y. Liu, “Robust adaptive observer for nonlinear systems with unmodeled dynamics,” *Automatica*, vol. 45, no. 8, pp. 1891–1895, 2009.

[6] G.Besanon, J. Len-Morales, and O.Huerta-Guevara, “On adaptive observers for state affine systems,” *Int. J. Control*, vol. 79, no. 6, pp. 581–591, 2006.

[7] M.Farza, M.M’Saad, T.Maatoug, and M.Kamoun, “Adaptive observers for nonlinearly parametrized class of nonlinear systems,” *Automatica*, vol. 45, pp. 2292–2299, 2009.

[8] X. Liang, Q. Zhang, and X.Xia, “Adaptive synchronization for generalized lorentz systems,” *IEEE Trans. Automatic Control*, vol. 53, no. 7, pp. 1740–1746, 2002.

[9] H. Hammouri and M. .Farza, “Nonlinear observers for locally uniformly observable systems,” *ESAIM:Control, Optimization and Calculus of Variations*, vol. 9, pp. 353–370, 2003.

[10] Q. Zhang, “Adaptive observer for multi-input-multi-output (mimo) linear time-varying systems,” *IEEE Trans. Automatic Control*, vol. 47, no. 3, pp. 525–529, 2002.

[11] X.Li, Q. Zhang, and H. Su, “An adaptive observer for joint estimation of states and parameters in both state and output equations,” *Int J. Adapt. Control Signal Processing*, vol. 25, pp. 831–842, 2011.

[12] O. Stamnes, O. Aamo, and G.-O. Kaasa, “Redesign of adaptive observers for improved parameter identification in nonlinear systems,” *Automatica*, vol. 47, no. 2, pp. 403–410, 2011.

[13] G.Besanon, “Remarks on nonlinear adaptive observers design,” *Systems and Control Letters*, vol. 41, no. 4, pp. 271–280, 2000.