Robust Parametric Estimation of Biased Sinusoidal Signals: a Parallel Pre-filtering Approach

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Abstract— In this paper, a parallel pre-filtering scheme is presented to address the problem of estimating the parameters of a sinusoidal signal from biased and noisy measurements. Extending some recent result on pre-filtering-based frequency estimators, a parallel pre-filtering scheme is proposed to deal with the unknown offset and bounded measurement perturbations, which are typically present in several practical applications. A simple frequency estimator, having parallel second-order pre-filters, is introduced. The behaviour of the proposed algorithm with respect to bounded additive disturbances is characterized by Input-to-State Stability arguments. Numerical examples shows the effectiveness of the proposed technique.

I. INTRODUCTION

Methodologies for the on-line identification of a sinusoidal signal from uncertain measurements are widely employed in many engineering applications such as active noise cancellation, vibrations monitoring in mechanical system and periodic disturbance rejection, to mention a few. A variety of techniques has been presented in the literature for estimating the unknown sinusoids in terms of estimating the amplitude, the frequency and the phase (AFP). For a deeper insight on the topic, the reader is referred to [1], [2], [3], [4], [5], [6], [7], [8], [9], [10], and the references cited therein.

Recent research attention has devoted to the AFP estimation in presence of a constant bias, which occurs in most practical applications, such as the offsets in physical transducers and A/D converters. In [11], the estimation problem of a biased sinusoidal signal is addressed by an adaptive quasi-notch-filtering scheme, while a novel PLL-based method can be found in [12], where the conventional PLL algorithm is expanded by an additional adaptation law for the offset term. The framework in [13] proposes an improved fourth-order estimator with suitable filtering technique to handle the presence of bias. By adopting a hybrid switching scheme with respect to the adaptive gains, the effect of high-frequency band noise is attenuated leading to accurate estimates in steady state. Moreover, in [14], a recursive method, relying on second-order generalized integrators is presented to reconstruct the unbiased sinusoid from a biased measurement. Another recursive frequency estimator based on the modulating functions is introduced in [15], in which the use of modulating functions allows to transform a differential expression to annihilate the bias. Inspired by the nonlinear estimator proposed in [16], a refined identifier is introduced in [17] to cope with the structured "polynomial-like" perturbations (including bias and drift phenomena as special cases).

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Techniques based on adaptive observers have also been proposed in recently literature. In this context, the frequency of the periodic signal is often modelled as an unknown parameter to be identified through an adaptive observer algorithm (see, for instance, [18], [19], [20] and [21], and the references cited therein). A notable feature of adaptive observer schemes is the possibility of carrying out multi-sinusoidal estimation by expanding the dynamic model with a suitable system transformation (see, for example, [22], [23] and [24]).

In [19] it has been shown that an adaptive observer method can achieve good noise immunity by exploiting a pre-filtering action. In [16] and [17] the authors have show that the signals generated by suitable pre-filters can be directly used to estimate the unknown frequency and the amplitude of the measured noisy sinusoid. In order to further improve the method presented in [17], the pre-filter was conceived as a cascade of first-order low-pass (LP) filters, in this work, we propose to use a parallel pre-filtering scheme in which two LP filters with different pole locations are used to generate auxiliary signals that are directly used to estimate the parameters of the sinusoid. This enhanced structure allows to simplify the adaptation law with respect to [17] and [19], while maintaining the robustness properties with respect to bounded measurement perturbations. Compared with the interesting algorithms proposed in [13], [25], [26] and based on a second-order pre-filter, the comparative simulations reported in this paper show that the use of two parallel pre-filters may lead to improved steady-state and transient performances. As in the case of the seminal papers [13] and [25], we characterize the behaviour of the estimator in presence of a bounded additive measurement disturbance. While in [20] and [26] a leakage correction term is added to the adaptation law to prevent the parameter drift in case of external perturbation, we prove that the proposed method is inherently ISS with respect to the exogenous disturbance. An important by-product of the ISS analysis is a set of useful tuning guidelines, since the dissipation rate and the ISS-asymptotic-gain are both expressed in terms of the estimator's parameters.

II. A PARALLEL PRE-FILTERING SCHEME

Consider an unknown sinusoidal signal corrupted by a norm-bounded time-varying disturbance $d(t) : |d(t)| < \bar{d}$

$$y(t) = v(t) + d(t) \qquad (1)$$

in which

$$v(t) = a_0 + a \sin(\vartheta(t)), \;\; \dot{\vartheta}(t) = \omega^*, \;\; \vartheta_0 = \phi \qquad (2)$$

where a_0 represents a constant offset.

Let us denote by $\hat{\mathbf{x}}_k(t) = [\hat{x}_{k,1}(t) \cdots \hat{x}_{k,n}(t)]^\top \in \mathbb{R}^n, k \in \{1, 2\}$ the state vector of the following pair of

filters, driven by the noisy measurement $y(t)$ and evolving from arbitrary initial conditions:

$$\begin{cases} \dot{\hat{\mathbf{x}}}_k(t) = \mathbf{F}_k\hat{\mathbf{x}}_k(t) + \mathbf{g}_k y(t), \ \ \forall k = 1,2, \ \ t \in \mathbb{R}_{\geq 0} \\ \hat{\mathbf{x}}_k(0) = \overline{\mathbf{x}}_k, \end{cases}$$

$$(3)$$

where $\mathbf{F}_k \in \mathbb{R}^{n \times n}, \forall k \in \{1,2\}$ is given by

$$\mathbf{F}_k = \begin{bmatrix} -\lambda_k & 0 & \cdots & \cdots & 0 \\ \beta_k\lambda_k & -\lambda_k & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \beta_k\lambda_k & -\lambda_k \end{bmatrix},$$

and

$$\mathbf{g}_k = \begin{bmatrix} \beta_k\lambda_k & 0 & \cdots & 0 \end{bmatrix}^\top, \ \forall k \in \{1,2\}.$$

The parameters $\lambda_1, \lambda_2 \in \mathbb{R}_{>0}$, with $\lambda_1 \neq \lambda_2$ represent the cut-off frequencies of the low-pass filters, while $\beta_1, \beta_2 \in (0,1]$ act as damping gains. The order n of the pre-filters is selected by the designer depending on the complexity of the frequency estimation algorithm and on the performance requirements. First, we describe a simple adaptive scheme characterized by $n = 2$. Then, we show that, at the cost of increasing of the order of the pre-filter, the algorithm can be modified to get enhanced transient and stationary performances.

Now, let $\hat{\mathbf{x}}(t) = [\hat{\mathbf{x}}_1(t) \ \hat{\mathbf{x}}_2(t)]^\top$ be the combined state vector whose dynamics is described by

$$\begin{cases} \dot{\hat{\mathbf{x}}}(t) = \mathbf{A}_{\lambda,\beta}\hat{\mathbf{x}}(t) + \mathbf{b}_{\lambda,\beta}y(t), \ \ t \in \mathbb{R}_{\geq 0} \\ \hat{\mathbf{x}}(0) = \overline{\mathbf{x}}, \end{cases}$$

$$(4)$$

where $\overline{\mathbf{x}} = [\overline{\mathbf{x}}_1^\top, \overline{\mathbf{x}}_2^\top]^\top$ and

$$\mathbf{A}_{\lambda,\beta} = \begin{bmatrix} \mathbf{F}_1 & 0 \\ 0 & \mathbf{F}_2 \end{bmatrix}, \quad \text{and } \mathbf{b}_{\lambda,\beta} = \begin{bmatrix} \mathbf{g}_1 \\ \mathbf{g}_2 \end{bmatrix},$$

Let us analyze the behaviour of the filters in the noise-free scenario, describing how the state vectors of the filter can be used to obtain an estimate of the frequency of the biased sinusoid $v(t)$. To this end, we denote by $\mathbf{x}_k(t), k \in \{1,2\}$ the virtual state vectors obeying the dynamics (3) driven by the disturbance-free signal $v(t)$ (not measurable) in place of $y(t)$. Moreover, let $\mathbf{x}(t) = [\mathbf{x}_1(t) \ \mathbf{x}_2(t)]^\top$ be the combined virtual state vector.

In view of the realization of the two filters, we have

$$\mathbf{c}^\top \mathbf{F}_k^q \mathbf{g}_k = 0, \ \ \forall q \in \{0,\ldots,n-2\}, \tag{5}$$

where $\mathbf{c} = \begin{bmatrix} 0 & \cdots & 0 & 1 \end{bmatrix}^\top$. In view of (5), the derivatives of the n-th state variables $x_{1,n}(t)$ and $x_{2,n}(t)$ of the two filters can be expressed as:

$$\frac{d^q}{dt^q}x_{k,n}(t) = \mathbf{c}^\top \mathbf{F}_k^q \mathbf{x}_k(t), \ q \in \{1,\ldots,n-1\},$$
$$\frac{d^n}{dt^n}x_{k,n}(t) = \mathbf{c}^\top \mathbf{F}_k^{n-1}\left(\mathbf{F}_k\mathbf{x}_k(t) + \mathbf{g}_k v(t)\right).$$

Denoting by $G_{k,n}(s)$, $k \in \{1,2\}$ the transfer functions of the filters in the Laplace domain from the input signal to the n-th state variables: $\mathcal{L}[x_{k,n}](s) = G_{k,n}(s)\mathcal{L}[v](s)$, $k \in$

$\{1,2\}$, we have that

$$G_{k,n}(s) = \frac{\lambda_k^n \beta_k^n}{(s+\lambda_k)^n}.$$

Neglecting the initial conditions of the internal filter's states, the Laplace transform of the n-th state variables can be expressed as:

$$\mathcal{L}[x_{k,n}](s) = G_{k,n}(s)a\frac{s\sin(\phi) + \omega^*\cos(\phi)}{s^2 + \omega^{*2}} + G_{k,n}(s)\frac{a_0}{s}.$$

and then the transform of the q-th derivative of $x_{k,n}$ is

$$\mathcal{L}\left[\frac{d^q x_{k,n}}{dt^q}\right](s) = G_{k,n}(s)a\frac{s\sin(\phi) + \omega^*\cos(\phi)}{s^2 + \omega^{*2}}s^q$$
$$+ G_{k,n}(s)a_0 s^{q-1}.$$

Thus, it is readily seen that in the time-domain, the derivatives $\frac{d^q}{dt^q}x_{k,n}(t), \forall 1 \leq q \leq n$ own the asymptotic sinusoidal steady-state behavior as $t \to \infty$. For the sake of further analysis, let us consider the stationary sinusoidal conditions

$$\frac{d^q}{dt^q}x_{k,n}(t) \approx a_{k,q}\sin\vartheta_{k,q}(t), \forall t >> 0 \ k \in \{1,2\} \tag{6}$$

where

$$\begin{aligned} a_{k,q} &= a\,\omega^{*q}|G_{k,n}(j\omega^*)|, \\ \vartheta_{k,q}(t) &= \vartheta(t) + \angle G_{k,n}(j\omega^*) + \frac{\pi}{2}q. \end{aligned} \tag{7}$$

III. ORDER $2+2$ FREQUENCY ESTIMATOR

In the noise-free scenario, let us fix the dimension of each pre-filter to $n = 2$, and enforce $\beta_1 = \beta_2 = \beta$, then consider the auxiliary signals $z_1(t) = x_{1,2}(t) - x_{2,2}(t)$ and $z_2(t) = \ddot{x}_{1,2}(t) - \ddot{x}_{2,2}(t)$, which tend asymptotically to a sinusoidal stationary equilibrium:

$$\begin{aligned} z_1(t) &\xrightarrow{t\to\infty} \bar{z}_1(t) = a_1\sin(\vartheta_1(t)) - a_2\sin(\vartheta_2(t)) \\ z_2(t) &\xrightarrow{t\to\infty} \bar{z}_2(t) = -\Omega^*\bar{z}_1(t) \end{aligned} \tag{8}$$

where $\Omega^* = \omega^{*2}$ denotes the true unknown squared-frequency ,

$$\begin{aligned} a_k &= a\,|G_{k,2}(j\omega^*)|, \\ \vartheta_k(t) &= \vartheta(t) + \angle G_{k,2}(j\omega^*), k \in \{1,2\}, \end{aligned} \tag{9}$$

and

$$G_{k,2}(s) = \frac{\beta^2\lambda_k^2}{s^2 + 2\lambda_k s + \lambda_k^2}.$$

In view of (8), it is worth noting that $\bar{z}_1(t)$ is a single sinusoidal signal with amplitude a_{z_1}, which will be instrumental for the stability analysis in the upcoming section. After some algebra, we obtain

$$a_{z_1} = \sqrt{\zeta_1^2 + \zeta_2^2} \tag{10}$$

in which

$$\begin{aligned} \zeta_1 &= a_1\cos\angle G_{1,2}(j\omega^*) - a_2\cos\angle G_{2,2}(j\omega^*) \\ \zeta_2 &= a_1\sin\angle G_{1,2}(j\omega^*) - a_2\sin\angle G_{2,2}(j\omega^*) \end{aligned}$$

Moreover, $\bar{z}_1(t), \bar{z}_2(t)$ defined in (8) satisfy $\Omega^*\bar{z}_1(t) + \bar{z}_2(t) = 0$ Accordingly, a recursive algorithm based on the gradient method is proposed:

$$\dot{\hat{\Omega}}(t) = -\mu_1\bar{z}_1(t)\left(\hat{\Omega}(t)\bar{z}_1(t) + \bar{z}_2(t)\right) \tag{11}$$

where $\mu_1 > 0$ is the adaptation gain. In practice, due to the presence of noise and to the initial transient of the filter, the stationary sinusoids $\bar{z}_1(t)$ and $\bar{z}_2(t)$ are not available. Hence, a modified-realizable adaptation law is proposed in the following section.

IV. STABILITY ANALYSIS OF THE ADAPTIVE SCHEME

The following realizable adaptation law, exploiting the available perturbed auxiliary signals $\hat{z}_1(t)$, $\hat{z}_2(t)$, is proposed:

$$\dot{\hat{\Omega}}(t) = -\mu_1 \hat{z}_1(t) \left(\hat{\Omega}(t)\hat{z}_1(t) + \hat{z}_2(t) \right). \qquad (12)$$

For simplicity, let us consider the combined vector $\hat{z}(t) = [\hat{z}_1(t) \ \hat{z}_2(t)]^\top$ that can be expressed in a compact form as

$$\hat{z}(t) = \Lambda_1 \left[\ y(t)^\top \quad \hat{x}_1(t)^\top \quad \hat{x}_2(t)^\top \ \right]^\top, \qquad (13)$$

with

$$\Lambda_1 = \begin{bmatrix} 0 & c^\top & -c^\top \\ c^\top(F_1 g_1 - F_2 g_2) & c^\top F_1^2 & -c^\top F_2^2 \end{bmatrix}.$$

In order to address the stability of the estimator, note that there exists an (unknown) initial filter's state $x(0) = \overline{x}_0$ giving rise to a filtered state trajectory $\overline{x}(t)$ whose projection on the subspace containing $z(t)$ matches the stationary sinusoidal behavior since the very beginning, when driven by the unperturbed sinusoid $v(t)$. Specifically:

$$\overline{x}(t) : \Lambda_1 \left[v(t)^\top, \overline{x}(t)^\top \right]^\top = \overline{z}(t) \triangleq \begin{bmatrix} \bar{z}_1(t) \\ \bar{z}_2(t) \end{bmatrix}, \forall t \in \mathbb{R}_{\geq 0}. \qquad (14)$$

The dynamics of the error vector $\tilde{x}(t) = \hat{x}(t) - \bar{x}(t)$ is governed by

$$\begin{cases} \dot{\tilde{x}}(t) = A_{\lambda,\beta}\tilde{x}(t) + b_{\lambda,\beta} d(t), \quad t \in \mathbb{R}_{\geq 0} \\ \tilde{x}(0) = \hat{x}_0 - \bar{x}_0. \end{cases} \qquad (15)$$

Since $A_{\lambda,\beta}$ is Hurwitz, it is easy to prove that the error dynamics ISS with respect to $\bar{d}$. Denote by $\gamma_x(s)$ the corresponding ISS asymptotic gain. Hence, for any arbitrary $\nu \in \mathbb{R}_{>0}$ and for any finite-norm initial error $\tilde{x}_0$, the error vector $\tilde{x}(t)$ enters in a closed ball of radius $\gamma_x(\bar{d}) + \nu$ in finite time.

Furthermore, let $\hat{z}(t) = [\hat{z}_1(t) \ \hat{z}_2(t)]^\top$ and $\tilde{z}(t) = \hat{z}(t) - \bar{z}(t)$ denote the perturbed auxiliary signals and the corresponding error dynamics. The ISS of $\tilde{x}(t)$ implies that $\tilde{z}(t)$ is ISS and the trajectory enters in a closed ball of radius $\gamma_z(\bar{d}) + \delta$ centered at the origin in finite-time:

$$\gamma_z(s) = \bar{\lambda}_1(\gamma_x(s) + s), \forall s \in \mathbb{R}_{\geq 0}, \ \delta = \bar{\lambda}_1 \nu \qquad (16)$$

where $\bar{\lambda}_1 = ||\Lambda_1||$.

Defining the frequency estimation error $\tilde{\Omega}(t) \triangleq \hat{\Omega}(t) - \Omega^*$, and applying the identity $\bar{z}_2(t) = -\Omega^* \bar{z}_1(t)$, let us rewrite (12) in terms of error signals

$$\begin{aligned} \dot{\tilde{\Omega}}(t) &= -\mu_1\big(\bar{z}_1(t) + \tilde{z}_1(t)\big) \\ &\quad \times \big(\hat{\Omega}(t)(\bar{z}_1(t) + \tilde{z}_1(t)) - \Omega^* \bar{z}_1(t) + \tilde{z}_2(t) \big) \\ &= -\mu_1 \tilde{\Omega}\bar{z}_1(t)^2 + \mu_1 \tilde{\Omega} f_1(t,\tilde{z}) + \mu_1 f_2(t,\tilde{z}) \end{aligned} \qquad (17)$$

where

$$f_1(t,\tilde{z}) = -2\bar{z}_1(t)\tilde{z}_1(t) - \tilde{z}_1(t)^2$$
$$f_2(t,\tilde{z}) = \Omega^*(f_1(t,\tilde{z}) + \tilde{z}_1(t)\tilde{z}_1(t)) - (\bar{z}_1(t) + \tilde{z}_1(t))\tilde{z}_2(t).$$

Note that the functions $f_1(t,\tilde{z})$ and $f_2(t,\tilde{z})$ verify $f_1(t,0) = 0$, $f_2(t,0) = 0$ for all $t \in \mathbb{R}_{\geq 0}$. Moreover, owing to the boundedness of $z(t)$, there exist two $\mathcal{K}_\infty$-functions $\sigma_1(\cdot)$ and $\sigma_2(\cdot)$ such that

$$|f_1(t,\tilde{z})| \leq \sigma_1(|\tilde{z}|), \quad |f_2(t,\tilde{z})| \leq \sigma_2(|\tilde{z}|). \qquad (18)$$

In the following, we use a conservative assumption that $\sigma_2(s), \forall s \in \mathbb{R}_{\geq 0}$ is such that the ratio $\sigma_2(s)/\sigma_1(s)$ is, in turn, a $\mathcal{K}$-function.

Theorem 4.1 (ISS of the adaptive frequency identifier): Given the sinusoidal signal $v(t)$ and the perturbed measurement model (1), if the bound on the measurement disturbance d verifies the inequality:

$$||d||_\infty < \bar{d} < \gamma_z^{-1} \left(\sigma_1^{-1} \left(\frac{\omega^* \kappa a_{z_1}^2 \ln\left((1-\kappa_\Delta)/\kappa\right)}{2\mu_1 \pi \kappa a_{z_1}^2 - 4\omega^* \ln\left(1/\kappa\right)} \right) \right) \qquad (19)$$

where a_{z_1} is given by (10), $\kappa : \kappa \in (0,1)$ is chosen arbitrarily, $\kappa_\Delta \in \mathbb{R}_{>0}$ such that $1 - \kappa_\Delta > \kappa$ and μ_1 is chosen large enough to verify the two inequalities

$$\omega^* \ln\left(1/\kappa\right)/(\mu_1 \kappa a_{z_1}^2) < \pi/2 \qquad (20)$$

and

$$\cos^2\left(\frac{\omega^* \ln\left(1/\kappa\right)}{\mu_1 \kappa a_{z_1}^2} \right) - \kappa \geq \frac{\omega^* \kappa \ln\left((1-\kappa_\Delta)/\kappa\right)}{2\mu_1 \pi \kappa a_{z_1}^2 - 4\omega^* \ln\left(1/\kappa\right)} \qquad (21)$$

then the frequency estimation system given by the two filters (4) with $n = 2$ and by (12) and (13) is ISS with respect to $d(t)$.

Proof: Consider the following candidate Lyapunov function $V(\tilde{\Omega}) = \frac{1}{2}\tilde{\Omega}^2$. In view of (17) and (18), the time-derivative of $V(t)$ along the system's trajectory subsumes the following inequality:

$$\begin{aligned} \frac{\partial V}{\partial \tilde{\Omega}}\dot{\tilde{\Omega}}(t) &= \tilde{\Omega}(t)\dot{\tilde{\Omega}}(t) \\ &\leq -\mu_1\big(\bar{z}_1(t)^2 - \sigma_1(|\tilde{z}(t)|)\big)\tilde{\Omega}(t)^2 \\ &\quad + \mu_1\sigma_2(|\tilde{z}(t)|)|\tilde{\Omega}(t)| \end{aligned} \qquad (22)$$

Note that in inequality (22) the stationary sinusoidal signal $\bar{z}_1(t)$ appears explicitly. At this point, under the assumption of $\bar{d}$ given by (19), then the period of the squared sinusoid $\bar{z}_1^2(t)$ can be partitioned in three intervals: P_2, in which it holds that $\big(\bar{z}_1(t)^2 - \sigma_1(\gamma_z(\bar{d}) + \delta)\big) > \kappa a_{z_1}^2$ and P_1, P_3, in which this inequality is not guaranteed. In the following, we will denote by t_0, t_1 and t_2 the transition instants between the aforementioned modes of behavior, as described in Fig.1. In this respect, we have that if the interval P_2 lasts for more than a specified $\overline{T}_\epsilon$, then the discrete-time Lyapunov function obtained by sampling the continuous-time Lyapunov function at the end of the two phases is a discrete-ISS Lyapunov function. During P_2, for a time interval of length T_ϵ we have:

$$\dot{V}(t) \leq -\mu_1 \kappa a_{z_1}^2 \tilde{\Omega}(t)^2 + \mu_1 \sigma_2(|\tilde{z}(t)|)|\tilde{\Omega}(t)| \qquad (23)$$

Now, we complete the squares, getting to

$$\begin{aligned} \dot{V}(t) &\leq -\mu_1\frac{\kappa a_{z_1}^2}{2}\tilde{\Omega}(t)^2 + \frac{\mu_1}{2\kappa a_{z_1}^2}\sigma_2(|\tilde{z}(t)|)^2 \\ &\leq -\mu_1\frac{\kappa a_{z_1}^2}{2}(V(t) - \sigma_e(\bar{d})) \end{aligned} \qquad (24)$$

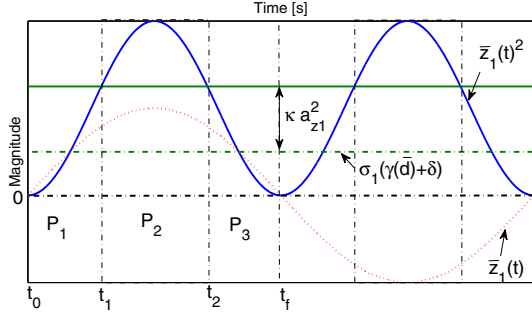Fig. 1. An example plot of the excitation signal $\overline{z}_1^2(t)$ (blue line) induced by the stationary sinusoidal signal $\overline{z}_1(t)$ with amplitude a_{z_1} (dotted red line), as well as two horizontal thresholds $\sigma_1(\gamma_z(\bar{d}) + \delta)$ (dotted green line) and $\sigma_1(\gamma_z(\bar{d}) + \delta) + \kappa a_{z_1}^2$ (green line)

with $\sigma_e(\bar{d}) = (\kappa^2 a_{z_1}^4)^{-1}\sigma_2(|\tilde{\mathbf{z}}(t)|)^2$. Analogously, during P_1 and P_3, for a time T_d, with $T_d = \frac{\pi}{2\omega^*} - \frac{T_\epsilon}{2}$, we obtain a upper bound for $\dot{V}(t)$ by letting $\sigma_d(\bar{d}) = \frac{\sigma_2(|\tilde{\mathbf{z}}(t)|)^2}{8\sigma_1(|\tilde{\mathbf{z}}(t)|)^2}$

$$\dot{V}(t) \leq 2\mu_1\sigma_1(|\tilde{\mathbf{z}}(t)|)\left(V(t) + \sigma_d(\bar{d})\right) \tag{25}$$

Applying the Gronwall-Bellman Lemma to (24), the value of the Lyapunov function during P_2 can be bounded as follows:

$$
\begin{aligned}
V(t) &\leq V(t_1) + (1 - e^{-\mu_1\kappa a_{z_1}^2(t-t_1)/2}) \\
&\quad \times (\sigma_e(\bar{d}) - V(t_1)) \\
&= e^{-\mu_1\kappa a_{z_1}^2(t-t_1)/2}V(t_1) + \sigma_e(\bar{d}) \\
&\quad \times (1 - e^{-\mu_1\kappa a_{z_1}^2(t-t_1)/2}), \; \forall t \in [t_1, t_2)
\end{aligned}
\tag{26}
$$

Following the same step as above, we obtain the further bound of $V(t)$ in dis-excited intervals P_1 and P_3:

$$
\begin{aligned}
V(t) &\leq V(t_p)e^{2\mu_1\sigma_1(|\tilde{\mathbf{z}}(t)|)(t-t_p)} \\
&\quad + (e^{2\mu_1\sigma_1(|\tilde{\mathbf{z}}(t)|)(t-t_p)} - 1)\sigma_d(\bar{d}), \\
&\forall t \in [t_0, t_1), \text{ if } p = 0 \text{ or } \forall t \in [t_2, t_0 + \frac{\pi}{\omega^*}), \text{ if } p = 2
\end{aligned}
\tag{27}
$$

Due to the poor excitation during P_1 and P_3, at the end of these intervals we can establish a conservative bound

$$
V(t_e) \leq e^{2\mu_1\sigma_1(|\tilde{\mathbf{z}}(t)|)T_d}\left(V(t_p) + \sigma_d(\bar{d})\right)
$$
$$
t_e = t_1, \text{ if } p = 0 \text{ or } t_e = t_0 + \frac{\pi}{\omega^*}, \text{ if } p = 2 \tag{28}
$$

Then, in view of (26), we get the inequality:

$$
\begin{aligned}
V(t_2) &\leq e^{-\mu_1\kappa a_{z_1}^2 T_\epsilon/2}\left[e^{2\mu_1\sigma_1(|\tilde{\mathbf{z}}(t)|)T_d}\left(V(t_0) + \sigma_d(\bar{d})\right)\right] \\
&\quad + \sigma_e(\bar{d})(1 - e^{-\mu_1\kappa a_{z_1}^2 T_\epsilon/2}) \tag{29}
\end{aligned}
$$

Finally, defining $V_k \triangleq V\left(t_0 + k\frac{\pi}{\omega^*}\right)$, and considering that t_0 is arbitrary within the set $t_0 \in \{t : \overline{z}_1^2(t) = 0\}$, from inequalities (28) and (29), we can readily derive the

following indexed expression

$$
\begin{aligned}
V_{k+1} &\leq e^{4\mu_1\sigma_1(|\tilde{\mathbf{z}}(t)|)T_d - \mu_1\kappa a_{z_1}^2 T_\epsilon/2}V_k \\
&\quad + \sigma_e(\bar{d})(1 - e^{-\mu_1\kappa a_{z_1}^2 T_\epsilon/2})e^{2\mu_1\sigma_1(|\tilde{\mathbf{z}}(t)|)T_d} \\
&\quad + (e^{4\mu_1\sigma_1(|\tilde{\mathbf{z}}(t)|)T_d - \mu_1\kappa a_{z_1}^2 T_\epsilon/2} + e^{2\mu_1\sigma_1(|\tilde{\mathbf{z}}(t)|)T_d})\sigma_d(\bar{d}) \tag{30}
\end{aligned}
$$

In the following lines, we will show that, a minimum time-duration of phase P_2 denoted by $\overline{T}_\epsilon$ is ensured as long as the bound of $\bar{d}$ is verified. In view of (19), we have that

$$
\begin{aligned}
\sigma_1(\gamma_z(\bar{d}) + \delta) &< \frac{\omega^*\kappa a_{z_1}^2 \ln\left((1 - \kappa_\Delta)/\kappa\right)}{2\mu_1\pi\kappa a_{z_1}^2 - 4\omega^* \ln\left(1/\kappa\right)} \\
&= \frac{\ln\left((1 - \kappa_\Delta)/\kappa\right)}{4\mu_1(\pi/2\omega^* - \ln\left(1/\kappa\right)/\mu_1\kappa a_{z_1}^2)} \tag{31}
\end{aligned}
$$

Then, in view of (21), there exists a positive constant $\overline{T}_d$ that bounds the length of the dis-excitation interval $(T_d \leq \overline{T}_d)$:

$$
\overline{T}_d \triangleq \min\left\{t \leq \frac{\pi}{\omega^*} : \right.
$$
$$
\left. \bar{z}_1(t_0 + t)^2 \geq a_{z_1}^2 \cos^2\left(\frac{\omega^*\ln\left(1/\kappa\right)}{\mu_1\kappa a_{z_1}^2}\right)\right\}
$$

Owing to the fact that $\bar{z}_1(t_0 + \overline{T}_d) = a_{z_1}^2\sin^2(\omega^*(t_0 + \overline{T}_d))$, then $\overline{T}_d$ can be computed as the minimum positive solution of the equation:

$$
a_{z_1}^2\sin^2\left(\omega^*\overline{T}_d\right) = a_{z_1}^2\cos^2\left(\frac{\omega^*\ln\left(1/\kappa\right)}{\mu_1\kappa a_{z_1}^2}\right)
$$

In view of (20),

$$
\overline{T}_d = \frac{\pi}{2\omega^*} - \frac{\ln\left(1/\kappa\right)}{\mu_1\kappa a_{z_1}^2}
$$

Hence,

$$
\begin{aligned}
T_\epsilon &\geq \frac{\pi}{\omega^*} - 2\overline{T}_d \\
&\geq \frac{2\ln\left(1/\kappa\right)}{\mu_1\kappa a_{z_1}^2} = \overline{T}_\epsilon
\end{aligned}
$$

Combining $\overline{T}_d$ and (31), we also have that

$$
e^{4\mu_1\sigma_1(|\tilde{\mathbf{z}}(t)|)T_d} < e^{4\mu_1\sigma_1(|\tilde{\mathbf{z}}(t)|)\overline{T}_d} < \frac{(1 - \kappa_\Delta)}{\kappa}
$$

Next, we will prove that the discrete-time system emerging from sampling the Lyapunov function at the transition time instants is ISS with respect to the measurement disturbance. Since $t_2 - t_1 = T_\epsilon \geq \frac{2\ln\left(1/\kappa\right)}{\mu_1\kappa a_{z_1}^2}$, picking an $\epsilon \in \mathbb{R}_{>0}$ such that $1 - \kappa_\Delta < \epsilon < 1$, then we can guarantee the following difference bound on the discrete (sampled) Lyapunov function sequence:

$$
V_{k+1} \leq \epsilon V_k + \sigma_e(\bar{d})(1 - \kappa)\frac{1}{\sqrt{\kappa}} + (1 + \frac{1}{\sqrt{\kappa}})\sigma_d(\bar{d})
$$

which leads to the following compact form:

$$
V_{k+1} - V_k \leq -(1 - \epsilon)V_k + \sigma(\bar{d})
$$

where the function $\sigma(\cdot)$ is $\mathcal{K}$-function defined as

$$
\sigma(s) = \sigma_e(s)(1 - \kappa)\frac{1}{\sqrt{\kappa}} + (1 + \frac{1}{\sqrt{\kappa}})\sigma_d(s), \; \forall s \in \mathbb{R}_{\geq 0}
$$

Now, we can conclude that the discrete dynamics induced by sampling the frequency estimator in correspondence of the transitions is Input-to-State stable (ISS).

Finally, we will recover the ISS for the continuous-time system by using the continuity of $V(t)$ and the boundedness of its time-derivative in the inter-sampling. Thanks to the periodicity of the excitation signal $\bar{z}_1(t)$, let us denote by $t_0(k)$, $t_1(k)$, $t_2(k)$ the transition time-instants of the k-th period of $\bar{z}_1(t)$, and $k(t)$ the index of the current period: $k(t) = k : t \in \left[t_0(k), t_0(k+1)\right)$. Between two samples, the Lyapunov function can be bounded by,

$$
\begin{aligned}
V(t) \leq & \frac{1}{\sqrt{\kappa}} \left[V(t_0) + \sigma_d(\bar{d}) \right] \\
& + \left[\sqrt{\kappa} \left(V(t_0) + \sigma_d(\bar{d}) \right) + \sigma_e(\bar{d}) \right] \\
& + \left[V(t_0) + \frac{1}{\sqrt{\kappa}} \sigma_e(\bar{d}) + (1 + \frac{1}{\sqrt{\kappa}}) \sigma_d(\bar{d}) \right] \\
= & \frac{1 + \sqrt{\kappa} + \kappa}{\sqrt{\kappa}} V_{k(t)} + (1 + \frac{1}{\sqrt{\kappa}}) \sigma_e(\bar{d}) + (2 + \frac{2}{\sqrt{\kappa}}) \sigma_d(\bar{d})
\end{aligned}
\tag{32}
$$

Since $k(t) \xrightarrow[t \to \infty]{} \infty$ (i.e., an infinite number of excited phases with length T_ϵ occurs asymptotically), the estimation error in the inter-sampling times converges to a region whose radius depends only on the assumed disturbance bound. ∎

V. AMPLITUDE AND PHASE ESTIMATION

At this stage, in view of (7) and using the estimated frequency $\hat{\omega}$ of the input sinusoid obtained with the method described in the previous section, the amplitude and phase $(a_{k,q}, \vartheta_{k,q})$ of the auxiliary derivatives can be recovered by

$$
\hat{a}_{k,q}(t) = \sqrt{\left(\hat{\Omega} \left(\frac{d^q}{dt^q} x_{k,n}(t) \right)^2 + \left(\frac{d^{q+1}}{dt^{q+1}} x_{k,n}(t) \right)^2 \right) / \hat{\Omega}}
$$
$$
\hat{\vartheta}_{k,q}(t) = \angle \frac{d^{q+1}}{dt^{q+1}} x_{k,n}(t) + j\hat{\omega} \frac{d^q}{dt^q} x_{k,n}(t), \quad \forall n \geq q + 1
\tag{33}
$$

Owing to the asymptotic sinusoidal behavior of the filtered signals (see (6) and (7)), we finally get:

$$
\hat{a}(t) = \frac{\hat{a}_{k,q}}{\hat{\omega}^q} \left[\sqrt{\lambda_k^2 + \hat{\omega}^2} / (\beta_k \lambda_k) \right]^n
$$
$$
\hat{\vartheta}(t) = \hat{\vartheta}_{k,1} + n \arctan(\hat{\omega}/\lambda_k) - \frac{\pi}{2} q
\tag{34}
$$

To achieve smoother magnitude estimate, the direct formulas for the amplitude described in (33) and (34) can be amended by an adaptive mechanism:

$$
\dot{\hat{a}}_{k,q}(t) = -\rho \hat{\omega}(t) \left[\hat{\omega}(t) \hat{a}_{k,q}(t) \right.
$$
$$
\left. - \sqrt{\left(\hat{\Omega}(t) \left(\frac{d^q}{dt^q} x_{k,n}(t) \right)^2 + \left(\frac{d^{q+1}}{dt^{q+1}} x_{k,n}(t) \right)^2 \right)} \right]
\tag{35}
$$

and

$$
\dot{\hat{a}}(t) = -\rho \hat{\omega}^q \left(\hat{a}(t) \hat{\omega}^q - \hat{a}_{k,q} \left[\sqrt{\lambda_k^2 + \hat{\omega}^2} / (\beta_k \lambda_k) \right]^n \right)
\tag{36}
$$

in which $\rho \in \mathbb{R}_{>0}$ is the tuning gain fixed by the designers.

VI. NUMERICAL RESULTS

In this section, the behavior of the proposed method is analyzed and compared with three recent AFP techniques proposed in [12], [14] and [25] respectively. The algorithms considered in this section have been tuned to have approximately the same response time when fed by a unitary amplitude-sinusoid of frequency $1/(2\pi)$ and initialized with zero initial conditions (indeed, the initial transient of the frequency-estimates shown in Fig.2 puts in evidence that the considered methods share approximately the same rise-time). The comparison is carried out for the case of a biased sinusoidal signal with both frequency and offset steps.

Let us consider a sinusoidal measurement that is corrupted by a bounded disturbance:

$$
\hat{y}(t) = \sigma(t) + 3\sin(\omega(t) t + \pi/4) + d(t), \tag{37}
$$

where $d(t)$ is a random noise with uniform distribution in the interval [-0.5, 0.5], $\omega(t) = 4, \forall t \in [0,10)$, $\omega(t) = 8, \forall t \in [10,35)$, $\omega(t) = 2, \forall t \in [35,50)$ and $\sigma(t) = 1, \forall t \in [0,20)$, $\sigma(t) = 3, \forall t \in [20,50)$. All the methods are initialized with the same initial condition $\hat{\omega}(0) = 1$. Method [12] is tuned with: $\mu_0 = 1.5$, $\mu_1 = 3$, $\mu_2 = 3$, $\mu_3 = 0.8$, while method [14] is tuned with: $K_s = 1$, $\lambda_1 = 1$, $\omega_s = 4$, $Q_0 = (1/\lambda)\mathbf{I}$. For method [25], we set $\gamma_0 = \lambda_0^2 = 8$, $\gamma_1 = 6$, $k = 0.18$. Finally, the tuning parameters of the proposed method are selected by: $\lambda_1 = 9$, $\lambda_2 = 2$, $\beta_1 = \beta_2 = 0.6$, $\mu_1 = 4$. The simulation results are given in Fig.2.
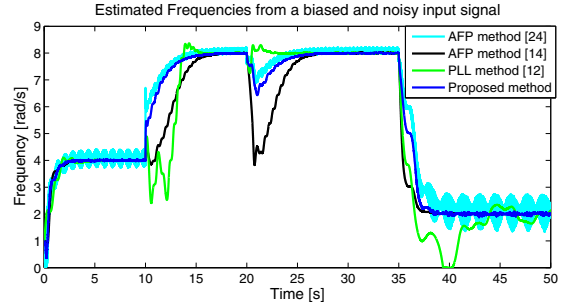

Fig. 2. Time-behavior of the estimated frequency by using the proposed AFP method (blue line) compared with the time behaviors of the estimated frequency by the AFP methods [14] (black line), [12] (green line) and [25] (cyan line).

All the methods are capable to track the initial frequency satisfactorily with the similar response time. However, the PLL method [12] suffers from worse transient for new frequencies, especially for the final frequency which is 75% less than the intermediate one. The AFP method [14] is more sensitive to a bias change, however, it performs the best noise attenuation at the cost of slow reaction to a significant frequency change. Thanks to the parallel structure of the pre-filtering scheme, the proposed method can handle considerable frequency variations with relatively faster response time and comparable robustness against noise. In the meantime, in comparison with the single pre-filter-based method [25] (note that the switching dynamic consists in this algorithm is switched off in order to consider the time-varying frequency), the proposed method that relies on two parallel pre-filters improves the steady state behavior significantly without a significant loss in the transient performance.
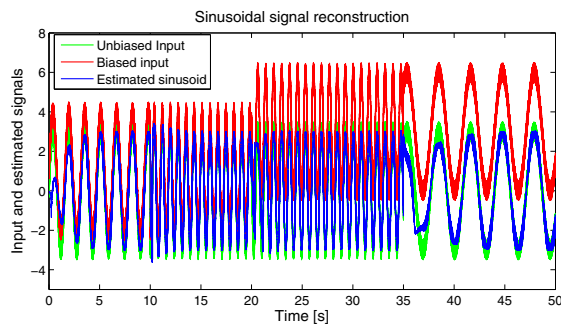
Fig. 3. Estimated sinusoidal signal by the proposed AFP method (blue line). To appreciate the time-behavior of the estimated signal, the biased noisy input is depicted (red line), as well as the same signal without the time-varying bias term (green line).

The sinusoidal signal reconstructed by the proposed method with $\rho = 0.2$ is depicted in 3, in which the unbiased signal is recovered successfully in a smooth manner even in the presence of noise.

The importance of the proposed amplitude adaptation scheme (35)-(36) emerges clear by examining the amplitude estimates. According to Fig.4, in which we compare the behavior of the adaptive algorithm with the direct formula in (33)-(34), both the transient and stationary performance is indeed improved by the recursive algorithm.
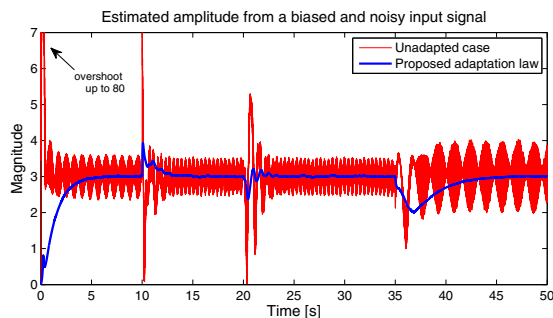

Fig. 4. Comparison of the behaviors in terms of amplitude estimation of the proposed AFP method with adaptive amplitude estimation algorithm (35)-(36) (blue line) and direct formula (33)-(34) (red line).

VII. CONCLUSIONS

In this paper, a novel parallel pre-filtering scheme has been proposed to be embedded in a sinusoid estimator to address unknown off-sets in the measurements and guaranteeing, at the same time, robustness against measurement noise. ISS stability has been proven and extensive simulation results have been reported showing the effectiveness of the proposed technique.

Future research efforts will be devoted to extend the proposed method to the case of multi-sinusoidal estimation.

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