# Estimation of Multi-Sinusoidal Signals: A Deadbeat Methodology

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***Abstract*— The problem of estimating the $n$ unknown amplitudes, frequencies and phases of the components of a multi-sinusoidal signal is addressed in this paper. The proposed methodology theoretically allows the exact identification of the above unknown parameters within an arbitrarily small finite time in the noise-free scenario. The measured signal is processed by a bank of Volterra integral operators with a suitably designed kernel, that yields a set of auxiliary signals which are computable on-line by causal linear filters. These auxiliary signals are in turn used to estimate the frequencies in an adaptive fashion, while the amplitudes and the phases estimates can be calculated by means of algebraic formulas. The effectiveness of the estimation technique is evaluated and compared with other existing finite-time estimators via numerical simulations.**

## I. Introduction

The parametric estimation of a signal composed by a given number of sinusoids is one of the fundamental issues arising in several areas of engineering, such as, for instance, vibration diagnostics and prognosis, power quality monitoring and periodic disturbance rejection. Several methods are available in literature for the adaptive estimation of the amplitude, frequency and phase (AFP) of a single sinusoid (see, for example, [1], [2], [3], [4], [5], [6], and the references cited therein), while the AFP problem for a multi-sinusoidal signal has recently received renewed attention. Besides the well-known Fast Fourier Transform (FFT), which is far the most common tool used for harmonic extraction, several algorithmic alternatives have been conceived, being the Phase-Locked-Loop (PLL) and Adaptive Notch Filtering (ANF) the most successful methods for their ease of implementation. Although both PLL and ANF in their original formulation only apply to single-sinusoidal signals, multiple PLLs or ANFs can be combined to address the estimation problem in the multi-sinusoidal scenario. In [7], $n$ enhanced-PLL (EPLL) units (see [8]) are deployed to extract the $n$ harmonics and inter-harmonics of a multi-sinusoidal signal. Analogously, in [9] a bank of $n$ ANF modules is used for the same task, with the advantage of being less computationally intensive than [7]. The problem becomes more challenging in case of an input with two frequencies that are close to each other. It has been shown in [10] that any two nearby frequencies can be discriminated by a couple of PLLs equipped with a “de-correlation” module. An alternative solution is given in [11], where the estimates from two identifiers are separated by enforcing a minimum frequency interval. However, such methods with de-correlation are hardly applicable for a number of sinusoids larger than two.

Another family of methodologies to track multiple frequencies relies on adaptive observers. These techniques are interesting since global or semi-global stability is ensured in most cases (see [12], [13], [14], [15] [16], [17], [18] and [19]). In particular, [16] and [17] deal with direct adaptation mechanisms for the squares of the frequencies with semi-global stability guarantees.

Despite the large number of AFP techniques, relatively few deadbeat AFP estimation techniques are available in the literature. This type of estimators are needed in scenarios where the estimates are required to converge in a neighborhood of the true values within a predetermined finite time, *independently from the unknown initial conditions*. A deadbeat AFP estimation method is firstly addressed in [20] based on the concept of algebraic derivatives. However, re-initialization may be needed due to the presence of singularities. This issue has been tackled in [21] and [22] by recursive least squares algorithms. In [23], the algebraic identification approach is further extended to address the parameter estimation of two sinusoidal signals. Moreover, a modulating function-based approach is presented in [24], which allows non-asymptotic frequency detection by processing the input with truncated periodic functions. A new tool for finite-time estimation has been recently proposed in [25], [26], where Volterra operators with a suitably designed kernel function allow to annihilate in finite-time the effect of the unknown initial conditions on the estimate. Compared with the algebraic identification method, the kernel-based one features internal stability, thus not requiring periodic re-initialization. Resorting to the said kernel-based design, a novel finite-time frequency identifier is presented in [27], where Volterra operators are paired with a robust sliding-mode adaptation law.

The present paper presents a deadbeat AFP estimator that employs Volterra operators with novel kernel functions. Compared to the previous kernels proposed by the authors in [26] and [27], yielding to Linear Time Varying (LTV) filters, the new ones admit a linear time invariant (LTI) realization. Moreover, the new kernel functions do not annihilate the initial conditions, that instead take part to the estimation as extended parameters, allowing for the retrieval of both the amplitude and the phase of the sinusoidal components.

## II. Problem statement and preliminaries

Consider the following multi-sinusoidal signal

$$y(t) = \sum_{i=1}^{n} A_i \sin(\vartheta_i(t)),\ \dot{\vartheta}_i = \omega_i,\ \vartheta_i(0) = \phi_i\,, \quad (1)$$

where $A_i \in \mathbb{R}_{>0}$ and $\omega_i \in \mathbb{R}_{>0}$ denote respectively the unknown amplitudes and the angular frequencies, verifying the inequality $\omega_i > 0, \omega_i \neq \omega_j$ for $i \neq j$, while $\phi_i$ denotes the initial phase of each sinusoid. As mentioned in the Introduction, our objective consists in estimating $A_i$, $\omega_i$ and $\phi_i$ within an arbitrarily small finite time.

The signal (1) can be thought of as being generated by the following observable autonomous marginally-stable

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dynamical system:

$$\begin{cases} \dot{\mathbf{w}}(t) &= \mathbf{A}_w \mathbf{w}(t) \\ y(t) &= \mathbf{c}_w^\top \mathbf{w}(t) \end{cases}, \quad (2)$$

where $\mathbf{w}(t) \triangleq [w_0(t) \dots w_r(t) \dots w_{2n-1}(t)]^\top \in \mathbb{R}^{2n}$,

$$\mathbf{A}_w \triangleq \begin{bmatrix} \mathbf{J}_1 & 0 & \cdots & 0 \\ 0 & \mathbf{J}_2 & \ddots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \mathbf{J}_n \end{bmatrix}, \quad \mathbf{c}_w \triangleq \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix},$$

$$\mathbf{J}_i \triangleq \begin{bmatrix} 0 & 1 \\ -\omega_i^2 & 0 \end{bmatrix}, \quad c_i^\top \triangleq [\, 1 \quad 0 \,],$$

and with initial conditions

$$w_{2i-2}(0) = A_i \sin\phi_i, \;\; w_{2i-1}(0) = A_i \omega_i \cos\phi_i, \quad \forall i \in \{1, \dots, n\}. \quad (3)$$

The associated characteristic polynomial, having purely imaginary roots occurring in complex-conjugate pairs, is given by

$$\begin{aligned} P(s) &= \prod_{i=1}^{n} (s^2 + \omega_i{}^2) \\ &= s^{2n} + \alpha_{n-1} s^{2n-2} + \cdots + \alpha_1 s^2 + \alpha_0 \end{aligned} \quad (4)$$

where $s$ is Laplace variable, $(\alpha_0, \alpha_1, \cdots, \alpha_{n-1})$ are the coefficients of the characteristic polynomial, simply determined by the unknown frequencies $\omega_i$, $i = 1, 2, \cdots, n$.

Being (2) observable, the state vector $\mathbf{w}(t)$ admits a linear transformation of coordinates $\mathbf{z}(t) = \mathbf{T}\mathbf{w}(t)$ with $\mathbf{T}$ is defined later in (7), such that the signal generator of $y(t)$ can be rewritten in an observer canonical form. Consider

$$\mathbf{z}(t) \triangleq [z_0(t) \; z_1(t) \; \dots \; z_r(t) \; \dots \; z_{2n-1}(t)]^\top \in \mathbb{R}^{2n},$$

the canonical system evolving from the unknown initial state $\mathbf{z}(0) = \mathbf{T}\mathbf{w}(0)$, is given as follows:

$$\begin{cases} \dot{\mathbf{z}}(t) = \mathbf{A}_z \mathbf{z}(t), \\ y(t) = \mathbf{c}_z^\top \mathbf{z}(t), \end{cases} \quad t \in \mathbb{R}_{\geq 0} \quad (5)$$

where $\mathbf{A}_z = \mathbf{T}\mathbf{A}_w\mathbf{T}^{-1}$, $\mathbf{c}_z^\top = \mathbf{c}_w^\top \mathbf{T}^{-1}$ are given by

$$\mathbf{A}_z = \begin{bmatrix} a_{2n-1} & 1 & 0 & \cdots & 0 \\ a_{2n-2} & 0 & 1 & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ a_1 & 0 & 0 & \cdots & 1 \\ a_0 & 0 & 0 & \cdots & 0 \end{bmatrix}, \quad \mathbf{c}_z = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}. \quad (6)$$

with $a_{2i+1} = 0$ and $a_{2i} = -\alpha_i$, $\forall i = \{0, 1, \cdots, n-1\}$. The transformation matrix $\mathbf{T}$ is determined by:

$$\mathbf{T} = \mathbf{M}\mathcal{O} \quad (7)$$

where $\mathcal{O}$ is the observability matrix of (2) and $\mathbf{M}$ is a triangular matrix with respect to $a_1, a_2, \cdots, a_{n-1}$

$$\mathbf{M} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ -a_{2n-1} & 1 & 0 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ -a_2 & -a_3 & \cdots & 1 & 0 \\ -a_1 & -a_2 & \cdots & -a_{2n-1} & 1 \end{bmatrix}.$$

In the following, a deadbeat algorithm is introduced to address the identification of the unknown system parameters $\alpha_i$ and the initial conditions $\mathbf{z}(0)$. Thereby the frequency are computed as the zeros of the characteristic polynomial $P(s)$, while the amplitudes and phases are determined by inverting (3) with the current frequency estimates.

Letting $x(t) \in \mathbb{R}$, $\forall t \geq 0$ be an $i$-th order differentiable signal, in this paper we denote by $x^{(1)}$ the $i$-th order derivative signal. Moreover, given a kernel function $K(\cdot,\cdot)$ in two variables, its $i$-th order derivative with respect to the second argument will be denoted as $K^{(i)}(t,\tau)$, $i \in \mathbb{Z}_{\geq 0}$.

Consider a Volterra integral operator (see [26] for a detailed review on the subject) with respect to a kernel function $K(\cdot,\cdot)$

$$[V_K x](t) \triangleq \int_0^t K(t,\tau) x(\tau) d\tau, \;\; t \in \mathbb{R}_{\geq 0}. \quad (8)$$

For the sake of practical implementability, it is worth to point out that the transformed signal $[V_K\, x](t)$, for $t \geq 0$, can be obtained as the output of a dynamic system described by the following scalar integro-differential equation:

$$\begin{cases} \xi^{(1)}(t) &= K(t,t)x(t) + \displaystyle\int_0^t \left( \frac{\partial}{\partial t} K(t,\tau) \right) x(\tau) d\tau \\ [V_K x](t) &= \xi(t) \end{cases} \quad (9)$$

where $\xi(0) = \int_0^0 K(0,\tau)x(\tau)\, d\tau$ and $\xi^{(1)}(0) = 0$.

The following result is useful in dealing with the application of Volterra operators to the derivatives of a signal.

*Lemma 2.1:* [26] For a given $i \geq 0$, consider a signal $x(\cdot) \in \mathcal{L}^2(\mathbb{R}_{\geq 0})$ that admits a $i$-th weak derivative in $\mathbb{R}_{\geq 0}$ and a kernel function $K(\cdot,\cdot) \in \mathcal{HS}$, admitting the $i$-th derivative (in the conventional sense) with respect to the second argument. Then, it holds that:

$$\begin{aligned} \left[ V_K x^{(i)} \right](t) &= \sum_{j=0}^{i-1} (-1)^{i-j-1} x^{(j)}(t) K^{(i-j-1)}(t,t) \\ &+ \sum_{j=0}^{i-1} (-1)^{i-j} x^{(j)}(0) K^{(i-j-1)}(t,0) + (-1)^i \left[ V_{K^{(i)}}\, x \right](t) \end{aligned} \quad (10)$$

that is, the function $\left[V_K x^{(i)}\right](\cdot)$ is non-anticipative with respect to the lower-order derivatives $x(\cdot)$, $x^{(1)}(\cdot), \dots, x^{(i-1)}(\cdot)$. □

The properties of the Volterra operator depend significantly on the shape of the kernel function. In this connection, we define a class of kernel functions that plays an important role in this framework.

*Definition 2.1:* If a kernel $K(\cdot,\cdot) \in \mathcal{HS}$ which is at least $(i-1)$-th order differentiable with respect to the second argument, verifies the condition

$$K^{(j)}(t,t) = 0, \; \forall j \in \{0, 1, \dots, i-1\} \quad (11)$$

then, it is called an $i$-th order Bivariate (strict) Causal Kernel (BC-K).

Here, we introduce a BC-K that fulfills (11):

$$K(t,\tau) = e^{-\beta(t-\tau)} \left( 1 - e^{-\beta(t-\tau)} \right)^N \quad (12)$$

with the parameter $\beta \in \mathbb{R}_{>0}$. Indeed, the condition (11) up to the $N$-th order is met by the factor $\left(1 - e^{-\beta(t-\tau)}\right)^N$.

## III. FINITE-TIME AMPLITUDE, FREQUENCY AND PHASE ESTIMATION

For the sake of further discussion, it is worth to introduce the differential-constraint model of (5):

$$\begin{cases} y^{(2n)}(t) \; = \sum\limits_{i=0}^{2n-1} a_i y^{(i)}(t), \;\; \forall t \in \mathbb{R}_{\geq 0}, \\ y^{(2i)}(0) = \quad y_0^{(2i)}, \; i \in \{0, \ldots, n-1\} \end{cases} \tag{13}$$

where $y_0^{(i)}$, $i \in \{0, \ldots, 2n-1\}$ represent the unknown initial conditions on hidden output derivatives. Notably, the state-variables of the observer canonical realization can be expressed as a linear combination of the output derivatives:

$$z_r(t) = y^{(r)}(t) - \sum_{j=0}^{r-1} a_{2n-r+j} \, y^{(j)}(t) \, , r \in \{0, 1, \cdots, 2n-1\} \tag{14}$$

where we have used the convention $\sum_{j=0}^{k} \{\cdot\} = 0, \;\; \forall k < 0$.

Assuming that $K(\cdot, \cdot)$ is a $2n$-th order Bivariate Causal kernel function satisfying the condition (11), thanks to Lemma 2.1, it is immediate to show that

$$\left[ V_K y^{(i)} \right](t) = \sum_{j=0}^{i-1} (-1)^{i-j} y^{(j)}(0) K^{(i-j-1)}(t, 0) + (-1)^i \big[ V_{K^{(i)}} \, y \big](t) \tag{15}$$

for all $i \in \{0, 1, \cdots, 2n\}$.

Consider the case $i = 1$, from (15) we have that

$$\big[ V_{K^{(1)}} \, y \big](t) = -y(0) K(t, 0) - \left[ V_K y^{(1)} \right](t) \, .$$

Moreover, performing the substitution of $y$ with $y^{(2n-1)}$ we have that also the following integral equation holds

$$\big[ V_{K^{(1)}} \, y^{(2n-1)} \big](t) = -y^{(2n-1)}(0) K(t, 0) - \left[ V_K y^{(2n)} \right](t) \, .$$

Therefore, owing to the I/O relationship (13), it holds that

$$\big[ V_{K^{(1)}} \, y^{(2n-1)} \big](t) = -y^{(2n-1)}(0) K(t, 0) - \sum_{i=0}^{2n-1} a_i \left[ V_K y^{(i)} \right](t)$$

which can be rearranged as

$$\begin{aligned} (-1)^{2n-1} \big[ V_{K^{(2n)}} \, y \big](t) = -y^{(2n-1)}(0) K(t, 0) \\ - \sum_{j=0}^{2n-2} (-1)^{2n-1-j} y^{(j)}(0) K^{(2n-j-1)}(t, 0) \\ - \sum_{i=0}^{2n-1} a_i \Bigg( \sum_{j=0}^{i-1} (-1)^{i-j} y^{(j)}(0) K^{(i-j-1)}(t, 0) \\ + (-1)^i \big[ V_{K^{(i)}} \, y \big](t) \Big) \, . \end{aligned}$$

After some cumbersome algebra, we get

$$\begin{aligned} (-1)^{2n-1} [V_{K^{(2n)}} y](t) + \sum_{i=0}^{2n-1} a_i (-1)^i [V_{K^{(i)}} y](t) \\ = - \sum_{r=0}^{2n-1} K^{(2n-r-1)} \, (-1)^{2n-r-1} \\ \times \left( y^{(r)}(0) - \sum_{j=0}^{r-1} a_{2n-r+j} \, y^{(j)}(0) \right) \end{aligned} \tag{16}$$

that, thanks to (14), can be written in a compact form

$$\big[ V_{K^{(2n)}} \, y \big](t) = \sum_{i=0}^{n-1} \alpha_i \big[ V_{K^{(2i)}} \, y \big](t) + \sum_{r=0}^{2n-1} \gamma_r(t) z_r(0) \tag{17}$$

where $\gamma_r(t) = K^{(2n-r-1)}(t, 0)(-1)^{2n-r-1}$.

Noting that the right-hand side of (17) is linear with respect to the parameters $\alpha_i$ and the initial state $z_r(0)$, it can be recast in vector form

$$\left[ V_{K^{(2n)}} \, y \right](t) = \boldsymbol{\nu}(t)^\top \boldsymbol{\theta} \tag{18}$$

where $\boldsymbol{\theta} \triangleq [\alpha_0, \, \alpha_1, \, \ldots, \, \alpha_{n-1}, \, z_0(0), \, z_1(0), \, \ldots, \, z_{2n-1}(0)]^\top$ is an extended parameter vector that contains, besides the model model parameters, also the initial conditions of output derivatives, while

$$\boldsymbol{\nu}(t) \triangleq \big[ [V_K y](t), \, [V_{K^{(2)}} y](t), \, \ldots, \, [V_{K^{(2n-2)}} y](t), \\ \gamma_0(t), \, \gamma_1(t), \, \ldots, \, \gamma_{2n-1}(t) \big]^\top$$

is a vector of known signals. For the sake of the further discussion, let us partition $\boldsymbol{\nu}(t)$ as follows: $\boldsymbol{\nu}(t) = [\mathbf{z}_e(t), \boldsymbol{\gamma}(t)]$ where $\mathbf{z}_e(t)$ contains signals obtainable by processing $y(t)$ by Volterra operators, while $\boldsymbol{\gamma}(t)$ contains known (kernel-dependent) functions of time. In the following, we show that $\mathbf{z}_e(t)$ can be obtained by processing the measurable output through a stable linear filter.

Consider a BC-K in the form of (12) with $N = 2n + 1$:

$$K(t, \tau) = e^{-\beta(t-\tau)} \left( 1 - e^{-\beta(t-\tau)} \right)^{2n+1} \tag{19}$$

with the the design parameter $\beta \in \mathbb{R}_{>0}$.

For any $i \in \{0, 1, 2, \ldots, 2n\}$, the $i$-th derivative of the designed kernel with respect to the second argument can be expressed as:

$$K^{(i)}(t, \tau) = \sum_{j=1}^{2n+2} e^{-j\beta t} f_{i,j}(\tau) \, . \tag{20}$$

Let $K_{i,j}(t, \tau) \triangleq e^{-j\beta t} f_{i,j}(\tau)$, then we have

$$\frac{\partial}{\partial t} K_{i,j}(t, \tau) = -j\beta e^{-j\beta t} f_{i,j}(\tau).$$

Moreover, by the linearity of the Volterra operator, it follows that $\left[ V_{K^{(i)}} \, y \right](t) = \sum_{j=1}^{2n+2} \left[ V_{K_{i,j}} y \right](t)$. Defining the internal state vector

$$\boldsymbol{\xi}(t) = [\xi_{0,1}(t), \, \xi_{0,2}(t), \ldots, \xi_{0,2n+2}, \, \xi_{2,1}(t), \, \ldots, \, \xi_{2n,2n+2}]^\top,$$

with $\xi_{i,j}(t) \triangleq [V_{K_{i,j}} y](t)$. Then the augmented signal vector $\mathbf{z}_a(t) \triangleq \left[ \mathbf{z}_e(t) \;\; \left[ V_{K^{(2n)}} \, y \right](t) \right]^\top$ can be computed by the following stable LTI system:

$$\begin{cases} \boldsymbol{\xi}^{(1)}(t) & = & \mathbf{G}_\xi \boldsymbol{\xi}(t) + \mathbf{E} y(t) \\ \mathbf{z}_a(t) & = & \mathbf{H} \boldsymbol{\xi}(t) \end{cases} \tag{21}$$

with $\boldsymbol{\xi}(0) = 0 \in \mathbb{R}^{(n+1)\times(2n+2)}$ and where $\mathbf{G}_\xi$ is a diagonal, time invariant and Hurwitz matrix, defined by $\mathbf{G}_\xi = \text{blockdiag}[\mathbf{G}, \ldots, \mathbf{G}]$, with $\mathbf{G} = \text{diag}(-\beta, -2\beta, \ldots, -(2n+2)\beta)$, and $\mathbf{H}$ is defined by $\mathbf{H} = \text{blockdiag}[\mathbf{1}^\top, \ldots, \mathbf{1}^\top]$, with $\mathbf{1}^\top$ denotes a row vector of ones with $2n+2$ elements. Finally, the vector $\mathbf{E} = [\mathbf{E}_0, \mathbf{E}_2, ..., \mathbf{E}_{2n}]^\top$ can be obtained as described in the following lines. Since the functions $K_i, j(t,\tau)$, evaluated for $\tau = t$,

$$K_{i,j}(t,t) = \lambda_{i,j} \triangleq (-1)^{j-1}\begin{pmatrix} 2n+1 \\ j-1 \end{pmatrix}(j\beta)^i$$

are constant, then $\mathbf{E}_i$ is given by $\mathbf{E}_i = [\lambda_{i,1},\, \lambda_{i,2},\, \ldots,\, \lambda_{i,2n+2}]^\top$. In order to form a well-posed algebraic system based on (18) conventional augmentation tools used in system's identification can be employed. The covariance filtering technique is adopted here to construct a linear algebraic system. Let us multiple $\boldsymbol{\nu}(t)$ on both sides of (18), leading to:

$$\mathbf{S}(t) = \mathbf{R}(t)\boldsymbol{\theta} \tag{22}$$

where $\mathbf{S}(t) \triangleq \boldsymbol{\nu}(t)[V_{K^{(2n)}}](t) \in \mathbb{R}^{3n\times 1}$ and $\mathbf{R}(t) \triangleq \boldsymbol{\nu}(t)\boldsymbol{\nu}^\top(t) \in \mathbb{R}^{3n\times 3n}$.

Note that rank $(\mathbf{R}(t)) = 1$, $\forall t > 0$, hence we apply to both sides of the (22) a low-pass filtering operation, obtaining

$$\begin{cases} \dot{\mathbf{S}}_f(t) = -g\mathbf{S}_f(t) + \mathbf{S}(t) \\ \dot{\mathbf{R}}_f(t) = -g\mathbf{R}_f(t) + \mathbf{R}(t) \end{cases}, \tag{23}$$

where $\mathbf{S}_f(0) = 0 \in \mathbb{R}^{3n\times 1},\ \mathbf{R}_f(0) = 0 \in \mathbb{R}^{3n\times 3n}$.

Now, let

$$\mathcal{F}_{r,j} = (-1)^{j-1}\begin{pmatrix} 2n+1 \\ j-1 \end{pmatrix}(-j\beta)^{2n-r-1}, \tag{24}$$

it is worth noting that $\gamma_r(t)$, $r = 0, 1, \ldots, 2n-1$ contained in the regressor $\boldsymbol{\nu}(t)$ can be represented as the sum of exponential functions

$$\gamma_r(t) = \sum_{j=1}^{2n+2} e^{-j\beta t}\mathcal{F}_{r,j}$$

which decay to zero as $t \to \infty$. The following technical result characterizes a specialized persistency of excitation condition (PE) on signal $\boldsymbol{\nu}(t)$ that is needed to prove the convergence of the proposed algorithm.

*Lemma 3.1:* (Finite-time persistency of excitation) Given the multi-sinusoidal measurement $y(t)$ (see (1)) and the designed kernel (19), there exist some $\epsilon \in \mathbb{R}_{>0}$, $t_\epsilon \in \mathbb{R}_{>0}$ and $T \in \mathbb{R}_{>0}$ such that

$$\int_{t-t_\epsilon}^{t} \boldsymbol{\nu}(\tau)\boldsymbol{\nu}^\top(\tau)d\tau \geq \epsilon\mathbf{I},\ \forall t \in [t_\epsilon, t_\epsilon + T]. \tag{25}$$

*Proof:* Let us split $\boldsymbol{\nu}(t)$ into two vector signals $\boldsymbol{\nu}_1(t) \in \mathbb{R}^n$ and $\boldsymbol{\nu}_2(t) \in \mathbb{R}^{2n}$, such that

$$\mathscr{L}\{\boldsymbol{\nu}_1(t)\} = \mathbf{G}_1(s)\mathscr{L}\{y(t)\},$$

and

$$\boldsymbol{\nu}_2(t) = \mathbf{G}_2\boldsymbol{\psi}_2(t)$$

where

$$\mathbf{G}_1(s) = \begin{bmatrix} \kappa_0(s) & \kappa_2(s) & \cdots & \kappa_{2n-2}(s) \end{bmatrix}^\top \in \mathbb{C}^n$$

with $\kappa_i(s) \triangleq \sum_{j=1}^{2n+2} \frac{\lambda_{i,j}}{s+j\beta}$, $i = 0, 2, \ldots, 2n-2$, and

$$\boldsymbol{\psi}_2(t) \triangleq \begin{bmatrix} e^{-\beta t} & e^{-2\beta t} & \cdots & e^{-(2n+2)\beta t} \end{bmatrix}^\top, \tag{26}$$

$$\mathbf{G}_2 = \begin{bmatrix} \mathcal{F}_{0,1} & \mathcal{F}_{0,2} & \cdots & \mathcal{F}_{0,2n+2} \\ \mathcal{F}_{1,1} & \mathcal{F}_{1,2} & \cdots & \mathcal{F}_{1,2n+2} \\ \vdots & \ddots & \ddots & \vdots \\ \mathcal{F}_{2n-1,1} & \mathcal{F}_{2n-1,2} & \cdots & \mathcal{F}_{2n-1,2n+2} \end{bmatrix} \in \mathbb{R}^{2n\times(2n+2)}.$$

Since $y(t)$ takes on the multi-sinusoidal form (1), it can be concluded that $y(t)$ is sufficient rich of order $2n$. Thanks to the linear independence of the complex vectors $\mathbf{G}_1(j\omega_1)), \cdots, \mathbf{G}_1(j\omega_n))$ on the complex space $\mathbb{C}^n$, $\boldsymbol{\nu}_1(t)$ is PE for all $t \geq 0$ ([28, Chapter 2]).

Moreover, for the signal $\boldsymbol{\psi}_2(t)$ defined in (26), there always exists a finite time interval $[\underline{t}, \bar{t}]$ with $\bar{t} > \underline{t}$ over which the elements of $\boldsymbol{\psi}_2(t)$ are linearly independent functions [29]. It also implies that for any $t > \underline{t}$, there exist a constant $\epsilon_2 \in \mathbb{R}_{>0}$, such that

$$\int_{t-\underline{t}}^{t} \boldsymbol{\psi}_2(\tau)\boldsymbol{\psi}_2^\top(\tau)d\tau \geq \epsilon_2\mathbf{I}\,.$$

Then, in view of (24), $\mathbf{G}_2$ is full row rank of $2n$. Hence, we have

$$\int_{t-\underline{t}}^{t} \boldsymbol{\nu}_2(\tau)\boldsymbol{\nu}_2^\top(\tau)d\tau = \mathbf{G}_2\int_{t-\underline{t}}^{t} \boldsymbol{\psi}_2(\tau)\boldsymbol{\psi}_2^\top(\tau)d\tau\, \mathbf{G}_2^\top \geq \underline{g}_2^2\epsilon_2\mathbf{I} \tag{27}$$

where we denote by $\underline{g}_2$ the minimum singular value of $\mathbf{G}_2$. The inequality (27) implies $\nu_2(t)$ PE over an interval $[\underline{t}, \bar{t}]$.

By using the fact that the sinusoidal functions in $\boldsymbol{\nu}_1(t)$ and the exponential functions in $\boldsymbol{\nu}_2(t)$ are linearly independent, it can be concluded that also for $\boldsymbol{\nu}(t)$ there exist some $\epsilon \in \mathbb{R}_{>0}$, $t_\epsilon \in \mathbb{R}_{>0}$ and $T \in \mathbb{R}_{>0}$, such that the finite-time PE condition (25) holds, thus ending the proof. ∎

Owing to (23) and (25), it is straightforward to show that

$$\mathbf{R}_f(t) \geq \int_{t-t_\epsilon}^{t} e^{-g(t-\tau)}\boldsymbol{\nu}(\tau)\boldsymbol{\nu}^\top(\tau)d\tau \geq e^{-gt_\epsilon}\epsilon\mathbf{I},\ t \in [t_\epsilon, t_\epsilon + T]$$

which in turn implies that, under the PE condition, the filtered auto-covariance matrix $\mathbf{R}_f(t)$ is invertible within a time interval $t_\epsilon \leq t \leq t_\epsilon + T$. In this connection, the unknown parameter vector $\boldsymbol{\theta}$ can be estimated by

$$\hat{\boldsymbol{\theta}}(t) = \begin{cases} \boldsymbol{\theta}_0, & t < t_\epsilon, \\ \mathbf{R}_f(t)^{-1}\mathbf{S}_f(t), & t_\epsilon \leq t \leq t_\epsilon + T, \\ \mathbf{R}_f(t_\epsilon + T)^{-1}\mathbf{S}_f(t_\epsilon + T), & t > t_\epsilon + T \end{cases}$$

where $\boldsymbol{\theta}_0$ is a guessed parameter vector. It is worth noting that the algorithm is switched off after $t = t_\epsilon + T$ by freezing the estimates.

Given $\hat{\boldsymbol{\theta}}$, the estimates of the $\alpha_i$, $i \in \{0, \ldots, n-1\}$ and the initial states $z_r(0)$, $r \in \{1, \ldots, 2n-1\}$ are computable. Thanks to the the characteristic polynomial (4) that is parametrized by $\alpha_i$, the frequencies $\omega_0, \ldots, \omega_{n-1}$ are computed by letting $P(s) = 0$. From (3), using the initial states $w_i(0)$, obtained by $\mathbf{w}(0) = \mathbf{T}^{-1}\mathbf{z}(0)$ and $\omega_i$, we finally get

$$(\omega_i w_{2i-2}(0))^2 + w_{2i-1}(0)^2 = A_i^2 \omega_i^2$$

which yields

$$A_i = \sqrt{\left((\bar{\omega}_i w_{2i-2}(0))^2 + w_{2i-1}(0)^2\right)/\bar{\omega}_i^2}$$

with $\bar{\omega}_i \triangleq \max(\omega_{\min}, \omega_i)$, $\omega_{\min}$ is a *known* lower bound of the input frequencies. Finally,

$$\phi_i = \tan^{-1}\left(\frac{\omega_i w_{2i-2}(0)}{w_{2i-1}(0)}\right)$$

for all $i \in \{1, \ldots, n-1\}$.

## IV. Numerical Example

In this section, a few numerical examples are carried out to examine the behavior of the proposed methodology, the performance of which is also compared with another algebraic algorithm proposed in [23].

### A. Identification of two sinusoidal signals

Let us first consider the example used in [23], assuming $y_1(t) = \sum_{i=1}^{2} A_i \sin(\omega_i t + \phi_i)$ where $A_1 = 2$, $A_2 = 5$, $\omega_1 = 1.4\pi \approx 4.4$ rad/s, $\omega_2 = 0.6\pi \approx 1.89$ rad/s, $\phi_1 = 1$ rad and $\phi_2 = 0.5$ rad.

The parameters of the algebraic algorithm [23] are set as $\epsilon = 1s$, $\zeta = 0.707$, $\omega_n = 31.4$ rad/s, while the proposed method is tuned by $\beta = 1$, $g = 1$. All the estimates are initialized by zero. According to Fig. 1, in the noise-free scenario, both methods are able to capture the sinusoidal parameters precisely in finite-time.

Instead of using a pure sinusoidal signal, let us assume the input $y_1(t)$ is corrupted by a bounded disturbance $d(t)$ with uniform distribution in the interval $[-0.1, 0.1]$. Keeping the tuning parameters unchanged, the behavior of the two methods in the presence of $d(t)$ are shown in Fig. 2. It is observed that the kernel-based method succeeds in AFP detection with fast convergence speed and slightly better noise immunity than [23]. It is worth noting that the algebraic estimator may be susceptible to numerical problems in the noisy scenario, due to its internal instability (as shown in the results at $t = 3.8$s).

### B. Identification of three sinusoidal signals

In the second example, we investigate the behavior of the proposed method in the presence of three sinusoidal components with nearby frequencies. Consider $y_2(t) = \sum_{i=1}^{3} A_i \sin(\omega_i t + \phi_i)$ where $A_1 = 2$, $A_2 = 5$, $A_3 = 4$, $\omega_1 = 2$ rad/s, $\omega_2 = 2.2$ rad/s, $\omega_3 = 5$ rad/s, $\phi_1 = 1$, $\phi_2 = 0.7$ rad and $\phi_3 = 0.2$ rad.

The estimation results in the noise-free scenario with tuning gains tuning $\beta = 1$ and $g = 1$ are reported in Fig. 3, showing the two nearby frequencies ($\omega_1$ and $\omega_2$) are precisely discriminated, while the other sinusoidal parameters are accurately identified as well. Moreover, Fig. 4 shows the estimated parameters when the measurement is perturbed by a bounded disturbance within $[-0.5, 0.5]$. Despite some degradation, the proposed methodology can provide satisfactory estimates within a short period of time.

## V. Concluding Remarks

In this paper, the problem of AFP identification for a multi-sinusoidal signal has been addressed. A novel estimator is designed to provide reliable amplitude, frequency and phase estimates in finite-time. While relying upon strong theoretical foundations referring to the algebra of integral operators, the estimator ends up with a simple linear filter that, applied to the measured signal, produces an array of auxiliary signals that are exploited to retrieve all the parameters of the sinusoidal components in one shot.

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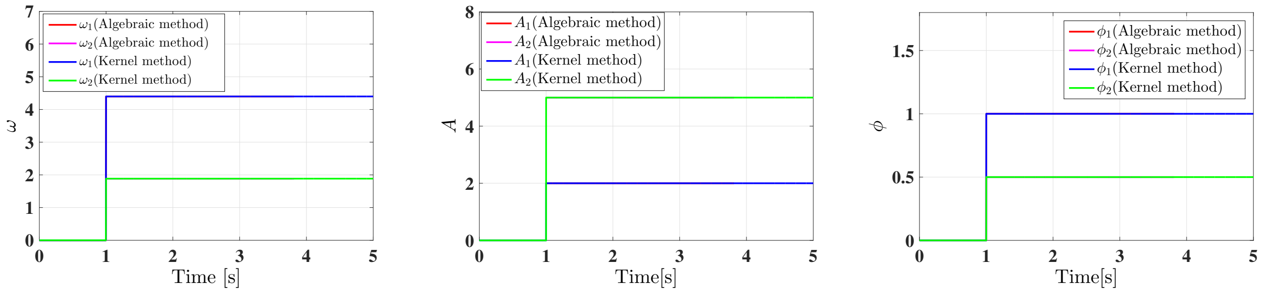


Fig. 1. Time behavior of the AFP estimates in noise-free scenario (estimates are overlapped for both methods).

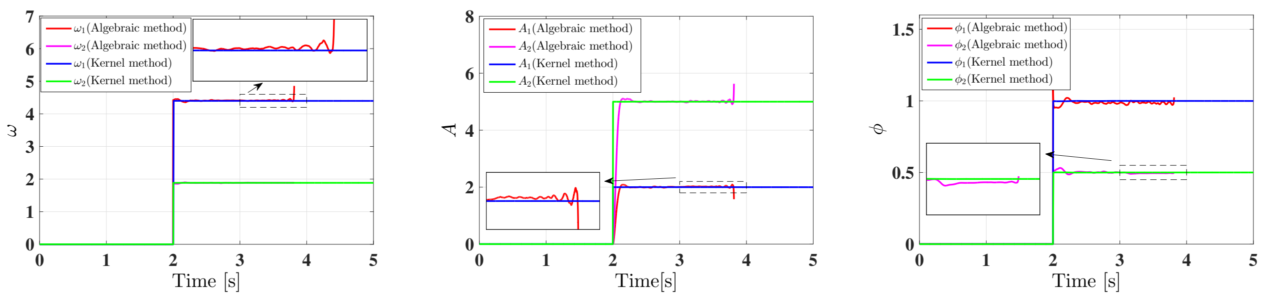


Fig. 2. Time behavior of the AFP estimates in noisy scenario.

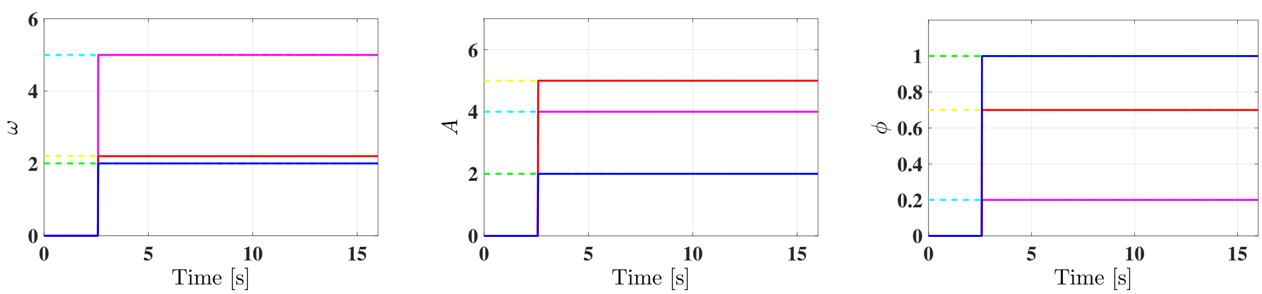


Fig. 3. Time behavior of the AFP estimates in noise-free scenario.

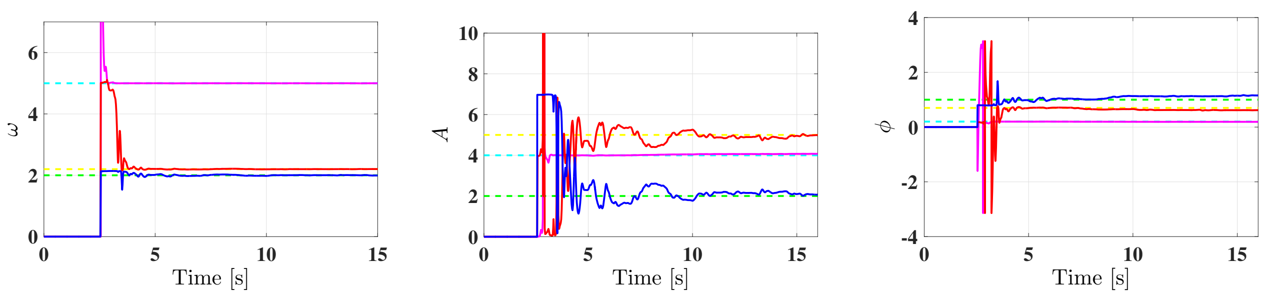


Fig. 4. Time behavior of the AFP estimates in noisy scenario.


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