# Asymptotically conical Calabi–Yau conifolds

Alessio Di Lorenzo

A thesis presented for the degree of

**Doctor of Philosophy**

Department of Mathematics
University College London
October 2023

I, Alessio Di Lorenzo, confirm that the work presented in my thesis is my own. Where information has been derived from other sources, I confirm that this has been indicated in the thesis.

**Abstract**

In this thesis we construct Ricci-flat metrics on asymptotically conical Calabi–Yau manifolds with isolated canonical singularities. These metrics will have two end behaviours: on one side, there is a complete asymptotically conical end, and on the other side there is one or more incomplete ends with polynomial decay to the conical Calabi–Yau metrics of the model cones at the singularities. This will be done by employing analysis on weighted Hölder spaces and techniques of Gromov–Hausdorff convergence on non-compact spaces, linking metric geometry with algebraic geometry. As an application, we will employ a gluing construction to prove the existence of special Lagrangian $n$-spheres vanishing cycles for smoothings in a small enough neighbourhood in the space of versal deformations of an asymptotically conical Calabi–Yau conifold with only nodal singularities.

## Impact statement

The study of Ricci-flat metrics on Kähler manifolds dates back to the work of Eugenio Calabi who, in the paper Calabi (1957), noticed that the Kähler environment simplifies enormously the task of prescribing the Ricci curvature on a compact manifold, giving birth to the *Calabi conjecture*. The conjecture became a theorem thanks to the work of Yau, Yau (1978), who proved that any Kähler class in a compact Kähler manifold with vanishing first Chern class admits a Ricci-flat representative. In the following years, several generalisations of this result have been proven. In particular, in recent years, Conlon and Hein proved a version of the Calabi conjecture for asymptotically conical manifolds in Conlon & Hein (2013a). Moreover, in Eyssidieux et al. (2009), the authors prove a version for singular compact manifolds. While the existence for singular Calabi–Yau manifolds was proven, not much was known about the behaviour of the singular Ricci-flat metric close to the singularities of the manifold until the work Hein & Sun (2017). Describing the behaviour of the metric close to the singularities opens the door to applying other techniques, *e.g.* gluing constructions, to the singular metric. This thesis has the aim of constructing Ricci-flat metrics in the non-compact case of asymptotically conical Calabi–Yau conifolds, and describing their behaviour close to the singularities, which can be helpful in studying, as an example, versal deformations of Calabi–Yau cones and bubble-tree structure of isolated singularities of Kähler–Einstein metrics.

# Contents

# Chapter 1

# Introduction

## 1.1 Background

Prescribing the curvature of a smooth manifold $M$ means choosing a suitable tensor field and looking for a metric $g$ on $M$ (sometimes with additional properties) such that its curvature tensor is equal to the chosen one at each point.

Trying to prescribe the Ricci curvature of a Riemannian manifold $(M, g)$ is generally speaking a very complicated matter.

However, when restricting ourselves to consider only the compact Kähler case, things get a bit easier. In the paper Calabi (1957), Eugenio Calabi asked whether, given a Kähler metric on a compact complex manifold, one could find another Kähler metric cohomologous to the former such that its Ricci form is an a priori selected form belonging to the first Chern class of the manifold. He conjectured it was the case, and gave birth to the *Calabi Conjecture*.
The solution to this problem had been awaited for more than two decades when Shing-Tung Yau published the paper Yau (1978) wherein he settled the question, an effort (paired with other contributions, among which those related to the Positive Mass Conjecture of General Relativity Theory) which won him the Fields Medal in 1982.

The main point that differentiates the Kähler case from the general one is the fact that, as directly pointed out by Calabi himself, the solution of the Calabi Conjecture can be seen to be equivalent — by means of the cohomological tools the Kähler setting provides us with, among which the 1:1 correspondence between hermitian forms and real $(1, 1)$-forms is of paramount importance — to the solution of a nonlinear second-order partial differential equation, categorised as of the *complex*

*Monge–Ampère type*:

$$(\omega + dd^c\varphi)^n = e^f\omega^n.$$

The complex Monge–Ampère equation obviously carries huge similarities with the real Monge–Ampère one (the same way the latter deals with the determinant of the real hessian $D^2u$ of a function $u$, the former deals with the determinant of the complex hessian $D^2_{\mathbb{C}}u$), so that most of the arguments used in the proof of the Calabi Conjecture were actually developed to solve the real Monge–Ampère equation, and then adapted to fit in the complex case. Yau himself says, in Yau (1978), that a large amount of the work that enabled him to solve the problem had been done whilst he was working on the real Monge–Ampère equation, and directly cites the work of Pogorelov Pogorelov (1971) on the second-order estimate for the real Monge–Ampère equation, explaining how it had had direct bearing on his work.

Much more recently, in Hein & Sun (2017), H.-J. Hein and S. Sun studied smoothable compact Calabi–Yau manifolds with isolated conical singularities, with the objective to describe the behaviour of the unique singular Ricci-flat Kähler current in a given cohomology class, built via pluripotential techniques in Eyssidieux et al. (2011), near these singularities.

This amounts to employing a continuity method not unlike the one used in the celebrated solution of the Calabi conjecture due to S.-T. Yau in Yau (1978), which however presents some further difficulties. These are overcome by various techniques, among which results on Gromov–Hausdorff convergence and K-stability.

On the other hand, in the papers Conlon & Hein (2013a) and Conlon & Hein (2013b), R.J. Conlon and H.-J. Hein studied the Calabi–Yau problem for asymptotically conical Calabi–Yau manifolds. Heuristically, they are complex manifolds that admit a holomorphic $(n, 0)$-form that vanishes nowhere and such that the metric at infinity is diffeomorphic to that of a cone.

In particular, they managed to prove existence and uniqueness of Ricci-flat metrics in a Kähler class $\alpha$ for (smooth) asymptotically conical Calabi–Yau manifolds, with rates depending on the asymptotically conical structure, if one assumes a certain condition on the decay of the forms contained in the class $\alpha$. The condition ensures that the class has metrics that are "Ricci-flat at infinity" up to some decaying functions.

This is used to construct certain *reference metrics* on the manifold, which are useful objects that can be used as starting points for a continuity method to solve the following complex Monge–Ampère equation:

$$\omega^n = i^{n^2}\Omega \wedge \overline{\Omega},$$

where $\Omega$ is the holomorphic volume form of the Calabi–Yau manifold. Indeed, solutions to this equation will correspond to Ricci-flat metrics for the Calabi–Yau manifold. The equation is the same as the one above if $f$ is pluriharmonic, *i.e.* $dd^c f = 0$, up to replacing $\Omega$ with $e^{f/2}\Omega$ (which is holomorphic).

In Conlon & Hein (2013b), it is proven that if we take a compact Kähler orbifold $X$ without codimension one singularities and a suborbifold divisor $D$ such that $X^{\mathrm{sing}} \subset D$ and $-K_X = (k+1)D$ for some $k > 0$, then one can give an asymptotically conical Calabi–Yau structure to $X \backslash D$, by using the normal exponential map around $D$. Moreover, it is also proven that every Kähler class satisfies the decay condition mentioned above with rate of decay at infinity $\mu = -2$.

Here we will consider smoothings of Calabi–Yau cones. As part of the context, note that it can be proven that an affine variety $M$ with trivial canonical bundle is a smoothing of a (regular) Calabi–Yau cone if and only if $M = X \backslash D$ for a Fano manifold $X$ and a divisor $D$ that satisfies the condition above. If we consider a family of smoothings of a Calabi–Yau cone, there will be divisors on the base space that will correspond to singular varieties (partial smoothings). This is discussed in Section 2.3.1.

In Chapter 4, as an application of the main result, we will focus on constructing *Special Lagrangian submanifolds* in smoothings of asymptotically conical Calabi–Yau conifolds. Special Lagrangians are a particular type of submanifold in a Calabi–Yau manifold $(X, \omega, \Omega)$ on which both the symplectic form $\omega$ and the imaginary part $\mathrm{Im}(\Omega)$ of the holomorphic volume form vanish. This makes it so that these submanifolds possess special properties, among which being volume minimisers in their homology class.

Constructing such submanifolds is no easy task in general. We will employ a gluing construction, similar to the case in dimension two carried out in Spotti (2014). The existence of these special Lagrangians will be based on arguments similar to the ones used to prove the main result. Finally, we will discuss the only other known method to construct special Lagrangians in particular cases, namely using antiholomorphic involutions, and compare it with the gluing construction method.

## 1.2 Result

We will cover what is needed to solve the Calabi–Yau problem for asymptotically conical Calabi–Yau manifolds with isolated canonical singularities. We assume the following property for the cone at the singularities.

**Definition 1.2.1.** *A Calabi–Yau cone $C$ with smooth cross-section and with Ricci-flat Kähler cone metric $\omega_C = dd^c r^2$ is called* strongly regular *if we can write it as the blow-down of the zero section of $\frac{1}{q}K_Z$ for some Kähler–Einstein Fano manifold $Z$ and $q \in \mathbb{N}$, and $-\frac{1}{q}K_Z$ is very ample.*

**Definition 1.2.2.** *Let $(C_0, g_0, \Omega_0)$ and $(C_\infty, g_\infty, \Omega_\infty)$ be Calabi–Yau cones of dimension $n$ with Ricci-flat Kähler metrics $g_0, g_\infty$ and holomorphic volume forms $\Omega_0, \Omega_\infty$ such that*

$$\omega_0^n = i^{n^2}\Omega_0 \wedge \overline{\Omega}_0 \quad \text{and} \quad \omega_\infty^n = i^{n^2}\Omega_\infty \wedge \overline{\Omega}_\infty,$$

*where $\omega_0, \omega_\infty$ are the Kähler forms associated to $g_0, g_\infty$.*

*Consider a Kähler manifold $(X, g, \Omega)$ of dimension $n$ with a canonical singularity $x \in X$. Then it is said to be an* asymptotically conical almost Calabi–Yau conifold *if*

*i) $K_X$ is trivialised by $\Omega$;*

*ii) $(X, g, \Omega)$ has an asymptotically conical structure with respect to the cone at infinity $(C_\infty, g_\infty, \Omega_\infty)$ as in Definition 2.3.1;*

*iii) the germ $(X, x)$ is isomorphic to the germ $(C_0, o)$ of the vertex of the cone $o \in C_0$;*

*iv) the metric $g$ is polynomially asymptotic to the metric $g_0$ of the cone $(C_0, g_0, \Omega_0)$ in a holomorphic chart giving a biholomorphism of a neighbourhood of $x$ to a neighbourhood of the vertex $o \in C_0$.*

*We say it is* Calabi–Yau *if in addition the $(1,1)$-form $\omega$ associated to the metric $g$ satisfies*

$$\omega^n = i^{n^2}\Omega \wedge \overline{\Omega}$$

*away from the singularity $x$.*

*The definition clearly applies also if there are a finite number of canonical singularities.*

**Definition 1.2.3.** *An asymptotically conical almost Calabi–Yau conifold is said to be* smoothable *if there exists* $\pi\colon (\mathcal{X},\mathcal{L}) \to \Delta$ *a family of almost Calabi–Yau manifolds, where* $\mathcal{X}$ *is smooth and* $\mathcal{L}$ *is a line bundle on* $\mathcal{X}$ *such that*

*i)* $\pi^{-1}(s) =: X_s$ *is smooth for all* $s \neq 0$, $X_0 \simeq X$, *and denote* $L_s := \mathcal{L}|_{X_s}$*;*

*ii) for each* $x \in X_0 \backslash X_0^{reg}$ *the germ* $(\mathcal{X}, x)$ *is isomorphic to a neighbourhood of the vertex* $o \in C_x$ *in a smoothing of a strongly regular Calabi–Yau cone* $(C_x, \omega_{C_x})$*;*

*iii) there is a family* $\Omega_s$ *of holomorphic n-forms on* $X_s^{reg}$ *coming from restrictions of a nowhere vanishing section* $\Omega$ *of the relative canonical bundle* $K_{\mathcal{X}/\Delta}$*.*

*Remark* 1.2.4. Note that since the family is smooth, there exists a neighbourhood $\mathcal{V}$ of $x$ in $\mathcal{X}$ and a smooth Kähler manifold $(Z,\eta)$ such that there is an embedding $i\colon \mathcal{V} \to Z$, and that there exists a weak Kähler metric $\omega \in c_1(\mathcal{L})$ on $\mathcal{X}$ in the sense of Definition 2.4.4 such that $\omega|_{\mathcal{V}} \equiv i^*\eta$ on $\mathcal{V}$.

We prove the following main theorem in the case of regular cone at infinity, *c.f.* Section 2.2.

**Theorem 1.2.5.** *Let* $(X, g, \Omega)$ *be a smoothable asymptotically conical almost Calabi–Yau conifold, with regular cone at infinity* $(C_\infty, g_\infty, \Omega_\infty)$ *and each canonical singularity* $x \in X \backslash X^{reg}$*, is modelled on strongly regular cones* $(C_x, g_x, \Omega_x)$*. Suppose the Kähler class* $[\omega]$ *of the Kähler form* $\omega$ *associated to* $g$ *is* $\mu$*-almost compactly supported in the sense of Definition 2.5.1 for some* $\mu < 0$*.*

*Then there exists an asymptotically conical Calabi–Yau metric* $g_0$ *on* $X^{reg}$ *with Kähler form* $\omega_0 \in [\omega]$*, such that*

$$\omega_0^n = i^{n^2}\Omega \wedge \overline{\Omega},$$

$g_0$ *is asymptotically conical approaching* $g_\infty$ *and it has conical singularities with rate* $\lambda_x > 0$ *and tangent cone* $(C_x, \omega_{C_x})$ *at each* $x \in X^{sing}$*.*

In the regular case, the family $\pi\colon (\mathcal{X},\mathcal{L}) \to \Delta$ can be compactified fibre by fibre with a (single) divisor at infinity $D$, in the sense of Theorem 1.2.6, *i)*. The divisor $D$ will correspond to the quotient of the cone $C_\infty$ by the $\mathbb{C}^*$ action induced by the Reeb vector field defining the Sasakian structure on its link; in particular it will need to be a Kähler–Einstein Fano manifold. In this case, the asymptotically conical structure need not be given a priori, but can be naturally constructed using a normal exponential map procedure, once $D$ can be related to the canonical bundle.

Compare with the smooth case in Conlon & Hein (2013b), which is a refinement of the Tian–Yau construction (see, for instance, Tian & Yau (2019)). We can then rewrite Theorem 1.2.5 in the following way.

**Theorem 1.2.6.** *Let $(X, g, \Omega)$ be a smoothable almost Calabi–Yau conifold, with each canonical singularity $x \in X \backslash X^{reg}$ modelled on strongly regular cones $(C_x, g_x, \Omega_x)$.*

*Let $\pi\colon (\mathcal{X}, \mathcal{L}) \to \Delta$ be a family of Calabi–Yau manifolds giving a smoothing of $X \simeq X_0$. Suppose we have*

*i) a (fibre-wise) compactification $\overline{\mathcal{X}} := \mathcal{X} \cup D$ such that $\overline{\mathcal{X}}$ and $\overline{X}_s := X_s \cup D$, where $X_s := \pi^{-1}(s)$, have natural structures of complex varieties and the fibres $\overline{X}_s$ are compact; here $D \subseteq \overline{\mathcal{X}}$ is a subvariety such that $D = \cap_s \overline{X}_s$ and it is a smooth divisor in each fibre $\overline{X}_s$;*

*ii) a line bundle $\overline{\mathcal{L}}$ on $\overline{\mathcal{X}}$ such that $\overline{\mathcal{L}}|_{\mathcal{X}} \equiv \mathcal{L}$;*

*iii) $D$ admits Kähler–Einstein metrics of positive scalar curvature;*

*iv) each $\overline{X}_s$ is a (compact) Fano variety of index at least $2$;*

*v) $-K_{\overline{X}_s} \equiv (k+1)D$ as divisors in $\overline{X}_s$ for all $s \in \Delta$, for some natural number $k > 0$.*

*Then there exists an asymptotically conical Calabi–Yau metric $g$ on $X^{reg}$ with Kähler form $\omega$ such that*

$$\omega^n = i^{n^2} \Omega \wedge \overline{\Omega},$$

*$g$ is asymptotically conical approaching the conical metric $g_\infty$ of a quasi-regular cone at infinity $(C_\infty, g_\infty, \Omega_\infty)$, has conical singularities with rate $\lambda_x > 0$ and tangent cone $(C_x, \omega_{C_x})$ at each $x \in X^{sing}$.*

*Remark* 1.2.7. In what follows we will prove the theorem assuming $X_0$ has only one singular point $x \in X_0$. The presence of a greater (finite) number of singularities adds no difficulty to the proof.

*Remark* 1.2.8. The strongly regular property is used to gain polynomial convergence in a holomorphic gauge close to the singularity, as in Hein & Sun (2017), Section 3.3.

*Remark* 1.2.9. The cones at infinity will be given by the blow-downs — shrinkings of the zero sections to one point — of the total spaces $N^{-1}_{D/\overline{X}_s}$ of the duals of the normal bundles of $D$ in $\overline{X}_s$ for each $s \in \Delta$. Equivalently, it is the total space $N_{D/\overline{X}_s} \backslash 0$ of the normal bundle itself with infinity end at the zero section $0 \subset N_{D/\overline{X}_s}$. In particular, following Conlon & Hein (2013b), we will be able to construct asymptotically conical structure precisely thanks to the condition $-K_{\overline{X}_s} \equiv (k+1)D$ and thanks to the existence of a positive Kähler–Einstein metric on $D$, in a spirit similar to the construction of Ricci-flat metrics on Calabi–Yau cones done by Calabi in Calabi (1957). Moreover, the asymptotically conical structures will be uniform in $s$, in a suitable sense.

*Remark* 1.2.10. The condition on $\lambda$ is meant to be as defined in Theorem 3.4.9, in the same way as in Theorem 2.11 of Hein & Sun (2017).

*Example* 1.2.11. Consider the cone

$$C = \left\{ z_1^3 + z_2^3 + z_3^3 + z_4^3 = 0 \right\} \subset \mathbb{C}^4.$$

It can be shown that $C$ is Calabi–Yau, and a possible family of deformations is the following:

$$\left\{ \sum_i z_i^3 + \sum_i t_i z_i = \varepsilon \right\} \subset \mathbb{C}^4 \times \mathbb{C}^5.$$

For some choices, the fibers will correspond to asymptotically conical Calabi–Yau manifolds with isolated conical singularities. For instance, if we choose

$$\varepsilon = \sum_i \left(-t_i/3\right)^{3/2} + \sum_i t_i \left(-t_i/3\right)^{1/2},$$

then this has a singularity at

$$x = \left( \left(-t_1/3\right)^{1/2}, \ldots, \left(-t_4/3\right)^{1/2} \right).$$

After some computations, we can rewrite the family as

$$\sum_i 3x_i(z_i - x_i)^2 + (z_i - x_i)^3 = 0,$$

hence if we denote $y_i = x_i^{1/2}(z_i - x_i)$, up to a polynomial error of order 3, we can describe the singularity type with the cone

$$\sum_i y_i^2 = 0,$$

which is an ordinary double point.

Note that if we take the fibre-wise compactification of the family in $\mathbb{P}^4 \times \mathbb{C}^5$, all of the fibres share a divisor at infinity (namely, the projectivisation of $C$ itself).

An application of the main result is using it to construct special Lagrangian submanifolds in smoothings of certain asymptotically conical Calabi–Yau manifolds.

**Definition 1.2.12.** *A singularity $x$ in Calabi–Yau manifold $X$ is called a* nodal singularity *if there exists a neighbourhood of $x$ which is biholomorphic to a neighbourhood of the vertex of the cone $\{\sum_i z_i^2 = 0\}$. This cone is called* ordinary double point *and can be given a structure of Calabi–Yau cone with metric*

$$\eta_0 = dd^c \left( \sum_{i=0}^{n} |z_i|^2 \right)^{(n-1)/n} .$$

We will prove the following statement.

**Proposition 1.2.13.** *If a smoothing $\mathcal{X}$ is versal for each node of its central fibre $X$, where $(X, g, \Omega)$ is a smoothable asymptotically conical Calabi–Yau manifold, then each node of $X$ is the limit of vanishing special Lagrangian $n$-spheres in the nearby fibres of $\mathcal{X}$.*

# Chapter 2

# Preliminaries

In this chapter, we will focus on preliminary results that will be useful in the remainder of the work. In addition to presenting these preliminary results, we also plan to provide examples to further expand and clarify the concepts discussed in the following chapters.

## 2.1 Riemannian tangent cones

To study singularities on a manifold, a basic notion is that of tangent cone. Assume $(M_i, m_i, g_i)$ is a sequence of pointed smooth Riemannian manifolds with $\mathrm{Ric}_{M_i} \geq -(n-1)$, and assume there is a (pointed) Gromov–Hausdorff limit $(Y, y)$. Roughly speaking, one wants to describe the infinitesimal structure around a singularity, and thus considers a pointed Gromov–Hausdorff sequence $\{(Y, y, r_i^{-1}d)\}$, where $r_i \to 0$, "zooming in" on the singularity $y$. Such a sequence subconverges to some space $(Y_y, y_\infty, d_\infty)$, by Gromov's precompactness theorem.

**Definition 2.1.1.** *The limit $Y_y$ of any subsequence as above is called a tangent cone at $y$.*

*Remark* 2.1.2. If instead of considering a sequence $\{(Y, y, r_i^{-1}d)\}$, we consider a sequence $\{(Y, y, r_i d)\}$, we get a notion of tangent cone at *infinity.*

For a given $y \in Y$, the tangent cone need not be unique (see for instance Cheeger & Colding (1997), Example 8.41).

In the *noncollapsed case*, that is, when

$$\mathrm{Vol}(B_1(m_i)) \geq v > 0,$$

it is possible to show that every tangent cone is a metric cone, and in our case we have

$$(C(Z) = \mathbb{R}_{>0} \times Z, \overline{g}),$$

on some length space $(Z, g)$ of diameter less or equal than $\pi$. Note that $\overline{g} = dr^2 + r^2 g$ on the regular part of $C(Z)$. The space $Z$ is then called the *link* of $C(Z)$.

## 2.2 Sasakian geometry

Our work will deal mostly with Calabi–Yau cones, which necessarily need to be complex and Kähler. To do this, we can use the structure of *Sasakian* manifolds on their links.

Sasakian geometry can be thought of as an odd dimensional version of Kähler geometry. For a more detailed introduction on Sasakian geometry, see Boyer & Galicki (2008) and Sparks (2011).

**Definition 2.2.1.** *A compact Riemannian manifold* $(S, g)$ *is called Sasakian if its metric cone* $(C(S) = \mathbb{R}_{>0} \times S, \overline{g} = dr^2 + r^2 g)$ *is Kähler.*

*Remark* 2.2.2. Necessarily, $S$ is of odd dimension $2n - 1$. Moreover, there is a privileged vector field on the cone $C(S)$, namely $r\partial_r$, which generates homoteties.

**Definition 2.2.3.** *The vector field* $J(r\partial_r) =: \xi$ *is called the Reeb vector field. This is tangential to any slice* $\{\lambda\} \times S$ *in* $C(S)$*, in particular to* $S \simeq \{1\} \times S$*.*

This Reeb vector field $\xi$ has unit length on $(S, g)$ and in particular is nowhere zero; it is Killing, and thus its integral curves are geodesics. The corresponding foliation $\mathcal{F}_\xi$ will be called the *Reeb foliation.*

Note that $\xi$ induces the action of a torus $\mathcal{T}$ on $S$; we say that $C(S)$ is *quasi-regular* if $\xi$ induces a $U(1)$-action on $S$, which is equivalent to saying $\dim \mathcal{T} = 1$. In case the action has no fixed points, we say that $C(S)$ is *regular.*

On the cone $C(S)$ we thus have a Kähler metric

$$g_C = dr^2 + r^2 g_S.$$

We define the 1-form on $C(S)$

$$\eta = J\left(\frac{dr}{r}\right) = \frac{1}{r^2} g_C(\xi, \cdot).$$

This is the *contact form* of the Sasakian structure when pulled back to the link $S$ via the embedding $i\colon S \to C(S)$ which identifies $S \simeq \{r = 1\}$. Note that $\eta$ is homogeneous of degree zero via $r\partial_r$. By the expression above, we can write

$$\eta = Jd\log r = \frac{1}{2}d^c \log r,$$

and thus

$$d\eta = \frac{1}{2}dd^c \log r.$$

We may now write the metric $g_C$ as

$$g_C = dr^2 + r^2(\eta \otimes \eta + g_\Sigma),$$

where $g_\Sigma$ is a Kähler metric on the distribution orthogonal to the span of $r\partial_r$ and $\xi = J(r\partial_r)$. In the quasi-regular case, this is the tangent space to $\Sigma = S/\langle\xi\rangle$, which is a Kähler compact orbifold. In the regular case it is a smooth manifold. One can compute

$$\omega_\Sigma = \frac{1}{2}d\eta,$$

from which we get the expression

$$\omega_C = \frac{1}{2}d(r^2\eta) = dd^c r^2.$$

Note that in particular, while $\eta, d\eta$ are forms on the cone $C(S)$, by construction

- the 1-form $\eta$ passes to the Sasakian manifold $S$ as the connection 1-form;
- the 2-form $d\eta$ passes to the Sasakian manifold $S$ and to the quotient $\Sigma = S/\langle\xi\rangle$. Here the differentiation $d$ is taken on the cone, hence $d\eta$ need not be exact on $\Sigma$.

The statements above can be expressed in the following.

**Theorem 2.2.4** (Boyer & Galicki (2008), Theorem 7.5.1)**.** *Let $(S, g)$ be a compact quasi-regular Sasakian manifold as above, and $\Sigma$ the space of leaves as a topological space.*

*Then $\Sigma$ carries an orbifold structure $(\Sigma, \Delta)$ with an orbifold Kähler metric $\omega$ which defines an integral class $[\pi^*\omega]$ in $H^2_{orb}(Z, \mathbb{Z})$. Moreover, the projection $\pi\colon S \to \Sigma$ is an orbifold Riemannian submersion, and a principal $S^1$-orbibundle over $\Sigma$. Furthermore, $\omega$ satisfies $\pi^*\omega = d\eta$, where $\eta$ is the 1-form connection on the bundle.*

*If moreover $S$ is regular, then the orbifold structure is trivial and $\pi$ is a principal circle bundle over a smooth algebraic variety.*

We can ask ourselves the question of when such a cone is Ricci-flat. This happens to be a well studied case, and the following proposition explains the consequences.

**Proposition 2.2.5.** *Let $(S, g)$ be a Sasakian manifold of dimension $2n-1$. Then the following are equivalent:*

1. *$(S, g)$ is* Sasaki–Einstein, *with* $\mathrm{Ric}_g = 2(n-1)g$;

2. *the Kähler cone is Ricci-flat,* $\mathrm{Ric}_{\overline{g}} = 0$.

*Example* 2.2.6. The manifold $\mathbb{C}^n \backslash \{0\}$ is a Calabi–Yau cone with link $S^{2n-1}$. On $S^{2n-1}$ the (standard) Reeb vector field acts as $U(1)$, and we get $\Sigma = S^{2n-1}/U(1) = \mathbb{CP}^{n-1}$, which is a Kähler–Einstein Fano manifold. Thus $\mathbb{C}^n \backslash \{0\} = C(S^{2n-1})$ is a regular cone.

*Remark* 2.2.7. If we consider a divisor $D$ in $X$ such that $-K_X = (k+1)D$, the cone at infinity that we consider will be $(N_D^{-1})^\times$ (this is only true if $k > 1$), the blow-down at the zero section of the dual of the normal cone to $D$ in $X$. This entails that $D$ takes the role of the space $\Sigma = S/\langle \xi \rangle$ in the above, and thus if we assume it is Kähler–Einstein we have a Calabi–Yau cone at infinity.

## 2.3 Asymptotically conical structures

In this section we give the definition of asymptotically conical structure for Calabi–Yau manifolds and give the asymptotically conical structure in the case covered by Theorem 1.2.6.

**Definition 2.3.1.** *Let $(C, g_0, \Omega_0)$ be a Calabi–Yau cone with a Ricci-flat Kähler metric $g_0$ and holomorphic volume form $\Omega_0$. Let $(M, g, \Omega)$ be a Calabi–Yau manifold with metric $g$ and holomorphic volume form $\Omega$.*

*We say $(M, g, \Omega)$ is* asymptotically conical *of rate $\lambda < 0$ to $C$ if there exists a diffeomorphism $\Phi \colon C \backslash K \to M \backslash K'$ away from compacts $K, K'$ such that for all $j \in \mathbb{N}$,*

$$\left|\nabla_{g_0}^j \left(\Phi^* g - g_0\right)\right|_{g_0} + \left|\nabla_{g_0}^j \left(\Phi^* \Omega - \Omega_0\right)\right|_{g_0} = O(r^{\lambda - j}),$$

*where $r$ is the radius function of $g_0$.*

**Definition 2.3.2.** *Let* $(M, g)$ *be asymptotically conical with cone* $(C, g_0)$ *and let* $r$ *be a radius function on* $M$. *We define the norm*

$$||\varphi||_{C^{k,\gamma}_\beta} := \sum_{j\leq k} ||r^{-(\beta-j)}\nabla^j\varphi||_{L^\infty} + [\nabla^k\varphi]_{C^{0,\gamma}_{\beta-k-\gamma}},$$

*where*

$$[\nabla^k\varphi]_{C^{0,\gamma}_{\beta-k-\gamma}} := \sup_{x\neq y} \min\{(r(x))^{-(\beta-k-\gamma)}, (r(y))^{-(\beta-k-\gamma)}\}\frac{||\nabla^k\varphi(x) - \nabla^k\varphi(y)||}{d(x,y)^\gamma},$$

*and the distance between* $\nabla^k\varphi(x)$ *and* $\nabla^k\varphi(y)$ *is computed via parallel transport along the minimal geodesic from* $x$ *to* $y$.

Consider now a Fano manifold $X$ and a Kähler–Einstein divisor $D$ in $X$ such that we have $-K_X = (k+1)[D]$, where $[D]$ is the line bundle determined by $D$.

We can give an asymptotically conical structure to $X\backslash D$ via a diffeomorphism $\Phi$ of a tubular neighbourhood of $D$ (which can be thought of as a tubular neighbourhood of the zero section in the normal bundle $N_D$ of $D$ in $X$) with a Calabi–Yau cone. This diffeomorphism will be taken to be the same as in Proposition 2.1 in Conlon & Hein (2013b), *i.e.* the normal exponential map $\exp\colon (T^{1,0}D)^\perp \to X\backslash D$, due to the following.

Note that the cone is $K_D^{1/k}$ as via adjunction we can write $-K_D = kN_D$; by taking the dual bundle, a neighbourhood of the zero section in $N_D$ indeed corresponds to a neighbourhood at infinity of $K_D^{1/k}$. This is done in the following way.

Consider a tubular neighbourhood $U$ of $D$ which is diffeomorphic to a neighbourhood of the zero section of the normal bundle $N_D$ of $D$ in $X$. By Section 2.1 of Conlon & Hein (2013b), we know there is a covering map $N_D\backslash\ 0 \to K_D\backslash\ 0$ (this works also if $D$ is an orbifold). We sketch its construction here.

**Proposition 2.3.3.** *We have a covering map* $N_D\backslash\ 0 \to K_D\backslash\ 0$ *given locally in orbifold charts* $(U, \Gamma), (U', \Gamma')$ *by*

$$t\partial_s \mapsto t^{-k}\partial_s \lrcorner\, (s^{k+1}\Omega)|_D,$$

*where* $t \in \mathbb{C}^*$, $s$ *is the defining section of* $D$ *in* $U$, $\partial_s$ *is the unique trivialising section of* $N_D$ *on which* $ds|_D \equiv 1$, *and* $\Omega$ *is the* $\Gamma$*-invariant holomorphic volume form that blows up to order* $k+1$ *along* $D$.

Consider the unitary bundle (that is, the circle bundle given by vectors of norm one) in $K_D$ and denote it by $S_{K_D}$. Heuristically, the unitary bundle $S_{N_D}$ in $N_D$ is obtained via the fibre-wise map $z \mapsto z^{-k}$ (recalling the relation $-K_D = kN_D$ obtained by adjunction). Hence $S_{N_D} \simeq S_{K_D}/\mathbb{Z}_k$ with an inversion. In particular this means we have a covering map via extending this map the total spaces of the line bundles without the zero sections.

Now, via the Calabi ansatz we can construct a Ricci-flat metric on $K_D \backslash 0$ (see Conlon & Hein (2013b), 2.1.2), and the radius on $K_D$ is given[1] by $||z||^{1/n} = r^{1/n}$ which then becomes $\rho := r^{-k/n}$ on $N_D$. By LeBrun (1994), Proposition 3.1, this choice of the distance function $r$ on $N_D$ gives rise to a Calabi–Yau cone structure on $N_D \backslash\ 0$ (*c.f.* Conlon & Hein (2013b), 2.1.2).

As mentioned above, it can be shown that the normal exponential map of $D$ in $X$ (with an arbitrary smooth metric on $X$) can be used to define a diffeomorphism $\Phi$. Using this diffeomorphism, we can get explicit rates of convergence. In particular,

$$\Phi^*\Omega - \Omega_0 = O(r^{-n/k}),$$

where $\Omega_0$ and $r$ are respectively the holomorphic volume form and the radius function on $N_D$. Part of the problem we address is how to extend this diffeomorphism in families in a controlled fashion.

### 2.3.1 Affine smoothings

A natural example on which Theorem 1.2.5 can be applied is *partial smoothings* of cones. Consider as a starter the following proposition.

**Proposition 2.3.4** (Proposition 5.1, Conlon & Hein (2013a)). *An $n$-dimensional smooth affine variety $X$ with trivial canonical bundle is a smoothing of the cone $C = \left(\frac{1}{k}K_D\right)^{\times}$ if and only if $X = \overline{X}\backslash D$ for some $n$-dimensional Fano manifold $\overline{X}$ of index at least 2 containing $D$ as an anticanonical divisor such that $-K_X = (k+1)[D]$.*

We sketch the proof here. Consider a family of Fano manifolds $\overline{\mathcal{X}} \to \Delta$ with index greater or equal than 2 and a divisor $\overline{\mathcal{X}}$ on each fibre $\overline{X}_s$ such that we have $-K_{\overline{X}_s} = (k+1)[D]$. If $\overline{X}_s\backslash D_s =: X_s$ is an affine variety and a smoothing of the cone $(\frac{1}{k}K_D)^{\times}$, $X_s$ can be naturally be recompactified into $\overline{X}_s$ by passing to the completion of $X_s$ in the total space of $\mathcal{O}(1)$ over the weighted projective space $\mathbb{P}(w)$, where $w$ is the weight vector of the $\mathbb{C}^*$-action on the cone. The condition on the

[1]The hermitian metric $||\cdot||$ on the line bundle $K_D$ is induced by the Kähler–Einstein metric on $D$.

canonical bundle (if we assume that $D$ is a Kähler–Einstein Fano variety) enables us to give an asymptotically conical structure on $X_s := \overline{X}_s \backslash D$ by considering a tubular neighbourhood $U \simeq \Delta \times D$ of $D$ in $\overline{X}_s$ and defining the diffeomorphism on $\Delta$ as above.

Conversely, let us construct a smoothing of the blow-down $(N_D^*)^\times$ at the zero section of the dual of the normal cone of $D$ in $X$.

Consider the line bundle $p\colon [D] \to \overline{X}$ and $s$ a defining section of $[D]$ (*i.e.* such that $D = \{s = 0\}$). Then we can define

$$\overline{X}_t := \{v \in [D] \,|\, tv = s(p(v))\}.$$

Note that $\overline{X}_0 = [D]|_D$ (the total space of the line bundle $[D]$ restricted to the points of $D$) and $\overline{X}_t \simeq \overline{X}$ for any $t \in \mathbb{C}^*$. Moreover, $\overline{X}_s \cap \overline{X}_t = D$ in the zero section for all $t, s \in \mathbb{C}$, which is the type of family that we consider.

If now $[D] > 0$, we can compactify the total space of $[D]$ by adding a single point and then remove the zero section to construct an affine variety $V$. This is done in the following way.

Suppose at first $[D]$ is very ample. Then we have a diagram

$$\begin{tikzcd} {[D] \simeq \varphi_{[D]}^* \mathcal{O}(1)} \arrow[r, hook] \arrow[d] & \mathbb{P}^{N+1} \backslash \{p\} \simeq \mathcal{O}(1) \arrow[d] \\ \overline{X} \arrow[r, hook, "\varphi_{[D]}"'] & \mathbb{P}^N \end{tikzcd}$$

We can compactify $\mathbb{P}^{N+1} \backslash \{p\}$ to $\mathbb{P}^{N+1}$, this gives a compactification $[D]^*$ of $[D]$. Now,

$$\mathbb{C}^{N+1} \simeq \mathbb{P}^{N+1} \backslash \mathbb{P}^N \supset_{\varphi_{[D]}} [D]^* \backslash \overline{X} =: ([D]^*)^\times,$$

where $\overline{X}$ is the zero section in $[D]$. Hence $([D]^*)^\times$ is an affine variety and if we consider the construction over $D \subset \overline{X}$ we get exactly $(N_D^*)^\times =: V$.

Now the family $\overline{X}_t$ induces a subfamily in $([D]^*)^\times \simeq \mathbb{C} \times V$ such that all of the fibers are copies of $X := \overline{X} \backslash D$ (the compactification does not add points to $\overline{X}_t$ for $t \neq 0$ since they are already compact varieties, and $\overline{X}_t \cap \overline{X} = D$) and the central fibre is $(N_D^*)^\times$.

If $[D]$ is not very ample, we can use Proposition 8.8.2 in Grothendieck (1961). This roughly says that that if a line bundle $L$ is anti-ample, the zero section in the total space $L$ can be contracted to a point and the result is an affine cone. In our case, we can apply the proposition to $[D]^\vee$ (dual bundle of $[D]$), so that we get that we can compactify the "section at infinity" of $[D]$ (corresponding to the zero section in $[D]^\vee$) to a point. In the total space of $[D]^\vee$ we have already removed the zero section of $[D]$, as it would correspond to the section at infinity of $[D]^\vee$.

Note that by Proposition 2.3.4, the assumptions that $X_s$ is affine and $\overline{X}_s$ is Fano are necessary and sufficient to get a smoothing of a Calabi–Yau cone.

The smoothing above gives a family whose general fibre $\overline{X}_s$ for $s \neq 0$ has no more singularities than those of $D$. More in general, we can consider the space of *versal deformations* of a cone $C$.

**Definition 2.3.5.** *[Versal deformation, after Kas & Schlessinger (1972)] Let $C$ be an cone defined by the equations $f_i = 0$ for $i = 1, \ldots, p$, where $f_i$ are polynomial functions on $\mathbb{C}^n$. Let $M$ denote the submodule of $\mathbb{C}[z_1, \ldots, z_n]^p$ generated by*

$$\frac{\partial f}{\partial z_j} = \left( \frac{\partial f_1}{\partial z_j}, \cdots, \frac{\partial f_p}{\partial z_j} \right) \in \mathbb{C}[z_1, \ldots, z_n]^p, \; j = 1, \ldots, n,$$

*and the ideal $(f_1, \ldots, f_p)$. More practically,*

$$M = \left\{ \sum_{i=1}^{p} f_i a_i + \sum_{i=1}^{n} g_j \frac{\partial f}{\partial z_j} \,\middle|\, a_i \in \mathbb{C}[z_1, \ldots, z_n]^p, g_i \in \mathbb{C}[z_1, \ldots, z_n] \right\}.$$

*The assumption (by definition) that $C$ has an isolated singularity at the vertex is equivalent to the condition*

$$\dim \mathbb{C}[z_1, \ldots, z_n]^p / M < \infty.$$

*The* space of versal deformations *of the cone $C$ is then the analytic subspace*

$$\mathcal{C} := \left\{ f_j + \sum_{i=1}^{N} t_i P_i^{(j)} = 0 \,\middle|\, j = 1, \ldots, p \right\} \subset \mathbb{C}^N \times \mathbb{C}^n,$$

*where $P_i^{(j)}$ determine a $\mathbb{C}$-basis of the vector space $\mathbb{C}[z_1, \ldots, z_n]^p / M$.*

A cone $C$ can be realised as an algebraic variety in $\mathbb{C}^n$ in such a way that the Reeb action of $\mathbb{C}^*$ generated by $r\partial_r$, $J(r\partial_r)$ on $C$ is the restriction of a $\mathbb{C}^*$-action on $\mathbb{C}^n$ with weights $\underline{w} = (w_1, \ldots, w_n)$. The structure is constructed through a *Remmert reduction* — a contraction of the zero section — as in Grauert (1962), Section 3.2.

Not all of the fibers of a versal deformation are necessarily smooth, *c.f.* Example 1.2.11.

### 2.3.2 Classification of asymptotically conical Calabi–Yau manifolds

By recent results contained in Conlon & Hein (2022), all smooth complete Calabi–Yau manifolds asymptotic to some given Calabi–Yau cone at a polynomial rate infinity have been classified. This result is general in the type of Calabi–Yau cone at infinity; in particular, it applies to manifolds whose cone at infinity is irregular. Moreover, every asymptotically conical Calabi–Yau manifold with a quasi-regular asymptotic cone comes from the construction described in Conlon & Hein (2013b), *i.e.* removing a orbifold divisor $D$ satisfying $-K_{\overline{X}} = (k+1)[D]$ from a compact orbifold $\overline{X}$.

We discuss the classification result in this section. Let $(C, \omega_0)$ be a Calabi–Yau cone, $\xi$ its Reeb vector field.

**Definition 2.3.6.** *An affine variety $V$ is a* deformation of negative weight *of $C$ if there exists a sequence $(\xi_i)_i$ in the Lie algebra of $\mathbb{T} = \{e^{t\xi} \mid t \in \mathbb{R}\}$, and a sequence of positive real numbers $(c_i)_i$ such that*

- *$\xi_i \to \xi$ as $i \to \infty$, and $-J(c_i\xi_i)$ generates an effective algebraic $\mathbb{C}^*$-action on $C$. Note that this means that if $\xi$ is the Reeb vector field of an irregular cone, it can be approximated by Reeb vector fields of quasi-regular cones;*
- *there exists a $\mathbb{C}^*$-equivariant deformation of $V$ to $C$. A $\mathbb{C}^*$-equivariant deformation of an affine variety $V$ to the cone $C$ is a triple $(\mathcal{V}_i, p_i, \sigma_i)$ consisting of a family of varieties with the projection, and a fibrewise action, such that*
  - *$\mathcal{V}_i$ is an irriducible affine variety;*
  - *$p_i \colon \mathcal{V}_i \to \mathbb{C}$ is a regular function with $p^{-1}(0) \simeq C$ and $p^{-1}(t) \simeq V$ for $t \neq 0$;*
  - *$\sigma_i \colon \mathbb{C}^* \times \mathcal{V}_i \to \mathcal{V}_i$ is an effective $\mathbb{C}^*$-action on $\mathcal{V}_i$ such that $p_i(\sigma_i(z,x)) = z^{\mu_i} p_i(x)$ for some $\mu_i \in \mathbb{N}$, and $\sigma_i$ restricted to the central fibre is the $\mathbb{C}^*$-action generated by $-J(c_i\xi_i)$;*
  - *$\lim_{z\to 0} \sigma_i(z, o) = o$, where $o$ is the vertex of $C$;*
  - *the sequence $\lambda_i := -(k_i\mu_i)/c_i$ is uniformly bounded away from zero, where[2] $k_i = \sup_k\{(\mathcal{V}_i, p_i) \simeq_{\mathbb{C}[t]/(t^k)} (\mathbb{C} \times C, \pi_1)\}$.*

[2]By the symbol $\simeq_{\mathbb{C}[t]/(t^k)}$ we mean that $(\mathcal{V}_i, p_i)$ is isomorphic to the trivial family $(\mathbb{C} \times C, \pi_1)$ when pulled back by a base change $\mathrm{Spec}(\mathbb{C}[t]/(t^k)) \to \mathbb{C}$.

*Given $V$ as above, $\lambda(V) := \inf \limsup_{i\to\infty} \lambda_i$ is called the $\xi$-*weight *of $V$. The infimum is taken over all of the possible choices of sequences $(\xi_i), (\mathcal{V}_i, p_i, \sigma_i)$.*

*Example* 2.3.7. Consider again the cubic

$$C = \left\{ \sum_{i=1}^{4} z_i^3 = 0 \right\} \subseteq \mathbb{C}^4,$$

then its space of versal deformations as in Definition 2.3.5 is given by

$$\mathcal{C} = \left\{ t z_1 z_2 z_3 z_4 + \sum_{i=1}^{4} z_i^3 + \sum_{1\le i<j<k\le 4} t_{ijk} z_i z_j z_k + \sum_{1\le i<j\le 4} t_{ij} z_i z_j + \sum_{i=1}^{4} t_i z_i = \varepsilon \right\}.$$

We have $\mathcal{C} \subseteq \mathbb{C}^{16} \times \mathbb{C}^4$. On $C$ we have a natural $\mathbb{C}^*$ action coming from the restriction of the action on $\mathbb{C}^4$ with weights $(1,1,1,1)$ that leaves $C$ invariant.

We want to extend the action to $\mathcal{C}$, by specifying the weights on $\mathbb{C}^{16}$, to make each deformation invariant. Note that if we consider the trivial action on $\mathbb{C}^{16}$ we get $\lambda \cdot \mathcal{C}$ by the expression

$$\left\{ t\frac{\lambda^4}{\lambda^3} z_1 z_2 z_3 z_4 + \sum_{i=1}^{4} z_i^3 + \sum_{1\le i<j<k\le 4} t_{ijk} \frac{\lambda^3}{\lambda^3} z_i z_j z_k + \sum_{1\le i<j\le 4} t_{ij} \frac{\lambda^2}{\lambda^3} z_i z_j + \sum_{i=1}^{4} t_i \frac{\lambda}{\lambda^3} z_i = \varepsilon \frac{1}{\lambda^3} \right\}.$$

It is clear from the above that, to get invariance, we need to specify the action on $\mathbb{C}^{16}$ as

$$\lambda \cdot (t, t_{ijk}, t_{ij}, t_i, \varepsilon) = (\lambda^{-1} t, \lambda^0 t_{ijk}, \lambda^1 t_{ij}, \lambda^2 t_i, \lambda^3 \varepsilon).$$

We want to consider only the deformations such that if $\lambda$ approaches infinity, we get $C$ back. For this to happen, we need to choose $t = 0, t_{ijk} = 0$, *i.e.* nullify the variables with negative weights. One can check that these are the deformations of negative weight as in the definition above.

The classification is the following.

**Theorem 2.3.8** (Theorem B, Conlon & Hein (2022)). *Every asymptotically conical Calabi–Yau manifold $(M, g, J, \Omega)$ is equivalent, up to diffeomorphism, to a holomorphic crepant resolution $\pi\colon M \to V$, where $V$ is is a deformation of negative $\xi$-weight of a Calabi–Yau cone $C$ with Reeb vector field $\xi$, such that the complex manifold $M$ admits a Kähler form. In particular, $M$ is a quasi-projective algebraic manifold.*

## 2.4 Kähler geometry on normal spaces

Since we are going to deal with singular spaces, we review in this section the basic notions of complex analysis on normal analytic spaces, following Section 16.3 in Guedj & Zeriahi (2017).

In the following, $X$ will be a normal analytic space of pure dimension $n$. We can think of $X$ as a complex manifold with no singularities of codimension one. We write $X^{reg}$ for the regular part of $X$.

**Definition 2.4.1** (Plurisubharmonic functions)**.** *A* plurisubharmonic function *on $X$ is an upper semi-continuous function on $X$ with values in $\mathbb{R} \cup \{-\infty\}$, which is not locally $-\infty$, and extends to a plurisubharmonic function in some local embedding $X \to \mathbb{C}^N$.*

*We say that a function $\varphi$ is strictly plurisubharmonic (or any other regularity,* e.g. *$C^\infty$) if it extends to a strictly plurisubharmonic function in some local embedding.*

*Remark* 2.4.2. By Fornaess & Narasimhan (1980), a continuous function is plurisubharmonic if and only if its restriction to the regular part $X^{reg}$ is so. Moreover, a bounded plurisubharmonic function on $X^{reg}$ extends to $X$.

**Definition 2.4.3.** *A* plurisubharmonic potential *on $X$ is a family $(U_i, \varphi_i)_{i\in I}$, where $(U_i)_i$ is an open covering of $X$ and $\varphi_i$ is a strictly plurisubharmonic function on $U_i$ such that $\varphi_i - \varphi_j$ is pluriharmonic on $U_i \cap U_j$. If the $\varphi_i$ are all $C^\infty$, the potential will be called a* Kähler potential*.*

We define an equivalence relation on Kähler potentials by requiring that $(U_i, \varphi_i) \sim (V_j, \psi_j)$ if $\varphi_i - \psi_j$ is pluriharmonic on $U_i \cap V_j$.

**Definition 2.4.4.** *A* Kähler metric *$\omega$ on $X$ is an equivalence class of Kähler potentials.*

**Definition 2.4.5.** *A* positive current *on $X$ is an equivalence class of plurisubharmonic potentials.*

*Remark* 2.4.6. Note that such a definition can be applied to define Kähler metrics on an orbifold $X$ as well, by considering locally invariant potentials.

**Definition 2.4.7.** *Let $\omega$ be a Kähler metric on $X$ with Kähler potentials $(U_i, \varphi_i)_i$. An upper semi-continuous function $\varphi\colon X \to \mathbb{R} \cup \{-\infty\}$ is said to be (strictly) $\omega$-*plurisubharmonic *if $\varphi + \varphi_i$ is (strictly) plurisubharmonic on $U_i$ for all $i$.*

*If $\varphi$ is strictly plurisubharmonic, the family $(U_i, \varphi+\varphi_i)_i$ defines a positive current (a Kähler metric if $\varphi$ is smooth) that will be denoted by $\omega + dd^c\varphi$.*

## 2.5 On the almost compactly supported assumption

Consider a manifold or orbifold $\overline{X}$ and a Kähler–Einstein divisor $D$ in $\overline{X}$ such that $-K_{\overline{X}} = (k+1)[D]$, where $[D]$ is the line bundle determined by $D$ and such that $X^{sing} \subset D$. Let $X := \overline{X} \backslash D$. We have seen in Section 2.3 that $X$ can be endowed with a asymptotically conical structure.

**Definition 2.5.1.** *Let $\mu < 0$. We say that a Kähler class $[\omega]$ on $X$ is $\mu$-*almost compactly supported *if for any representative $\omega$ there exist a real $(1,1)$-form $\xi$ and a real function $\phi$ on $X$ such that*

$$\begin{cases} \omega &= \xi + dd^c\phi \qquad \text{on } X\backslash K', \\ \Phi^*\xi &= O(r^\mu), \end{cases}$$

*where $K' \subseteq X\backslash D$ is a (big enough) compact set, $\Phi\colon X\backslash K' \to C\backslash K''$ is the diffeomorphism of the asymptotically conical structure, and $r$ is the distance on the cone $C$.*

Note that in this definition $\xi$ is not necessarily Kähler.

Take a $\mu$-almost compactly supported class $\alpha$ in $X$. The following proposition will tell us that in our situation, $\mu \leq -2$.

**Proposition 2.5.2** (Proposition 2.5, Conlon & Hein (2013b))**.** *Let $\overline{X}$ and $D$ be as above. Then the restriction map $H^{1,1}(\overline{X}, \mathbb{R}) \to H^2(X, \mathbb{R})$ is surjective. In particular, every Kähler class on $X$ is $(-2)$-almost compactly supported.*

The proof relies on an application of a Gysin sequence and estimates on $\xi$ in terms of the cone metric, using the normal exponential map diffeomorphism (*c.f.* Proposition 2.1 in Conlon & Hein (2013b)).

Suppose now we have a Kähler form $\omega$ on $\overline{X}$. We restrict it to $X$ so as to get $\omega|_X$. Now, $\omega|_X$ belongs to a $(-2)$-almost compactly supported class, thus $\omega|_X = \xi + dd^c\varphi$ for some real form $\xi$ such that $|\xi| = O(r^\mu)$ in $X$, by Proposition 2.5.2 above.

**Proposition 2.5.3.** *Suppose we are in the situation described above. Let $U \subset \overline{X}$ be a tubular neighbourhood of $D$ in $\overline{X}$. Then:*

*i) We have an equivalence of cohomology classes*

$$[\xi]_{H^2(U\backslash D)} = [(\pi^*\omega|_D)|_{U\backslash D}]_{H^2(U\backslash D)},$$

*where $\pi$ is the projection from $U$ to $D$;*

*ii) If $\omega|_D \in c_1(D)$, then $\varphi \sim \log r + \varphi_0$, where $r$ is the distance function on the cone at infinity and $\varphi_0$ can be extended to $X$.*

*Proof.* For *i)*, consider the Gysin sequence

$$H^0(D) \to H^{1,1}(X) \to H^2(X \backslash D) \to H^1(D) = 0,$$

where the first map is the wedge product with $\pi^* c_1(D)$ and the second map is the restriction. The group $H^1(D)$ vanishes as $D$ is Fano. This is true even if $D$ is an orbifold, by the existence of orbifold Kähler metrics with positive Ricci curvature, Bochner formulas and an orbifold version of Hirzebruch–Riemann–Roch.

We claim that $[\omega|_{U\backslash D}]_{H^2(U\backslash D)} = [(\pi^*\omega|_D)|_{U\backslash D}]_{H^2(U\backslash D)}$.

Indeed, the projection $\pi$ from $U$ to $D$ and the identity on $U$ are homotopic, thus inducing a 1:1 correspondence in cohomology. Then the claim is due to the fact that $\omega_{X\backslash D}$ and $\omega_D$ come from restriction of the same metric on $U$, namely $\omega$. A similar result can be achieved if we assume that the metric $\omega$ on $X\backslash D$ and the metric $\omega'$ on $D$ share the same cohomology class on $U$; *i.e.* that there exist a 2-form $\xi$ on $U$ such that $\omega|_{U\backslash D} = \xi|_{U\backslash D} + d\eta'$ on $U\backslash D$ and $\omega' = \xi|_D + d\eta''$ on $D$, for certain 1-forms $\eta', \eta''$ respectively on $U\backslash D$ and $D$.

Now, we can also check that if $\omega|_D$ is Kähler–Einstein, then we can take $\xi$ compactly supported.

As a first thing, $U\backslash D \simeq S \times \mathbb{R}$ topologically, where $U$ is a tubular neighbourhood of $D$, and thus $H^2(S) \simeq H^2(U\backslash D)$ (where $S$ is the Sasakian manifold associated to $D$). Note that by Theorem 2.2.4, we have $\pi_S^*\omega|_D = d\eta$, where $\eta$ is the connection[3] 1-form on $S$ and $\pi_S \colon S \to D$ is the projection from $S$, hence $\pi^*\omega|_D = \pi^* d\eta$ on $U\backslash D$. Note that $||d\eta||_{g_S}$ is bounded on $S$ because $\eta$ is a smooth connection, and $g_S = g_D \oplus \eta^2$.

If $\omega|_D$ is Kähler–Einstein,

$$\begin{aligned} g_C &= dr^2 + r^2 g_S \\ &= dr^2 + r^2(g_D + \eta^2), \end{aligned}$$

where $g_C$ is the conical metric, $g_D$ is the metric on $D$ and $\eta$ the connection 1-form

[3]By what explained in Section 2.2, while forms on the cone, $\eta$ and $d\eta$ pass respectively to the Sasakian manifold $S$ and the transverse quotient $S/\mathcal{F}_\xi \simeq D$.

on the unitary bundle $S \to D$. Now we have

$$||\pi^*\omega|_D||_{g_C} = r^{-2}||\omega|_D||_{g_D} = r^{-2}||d\eta||_{g_S}.$$

Note that this is related to the result in Proposition 2.5.2, which states that every Kähler class in $X\backslash D$ coming from a Kähler class in $X$ is $(-2)$-almost compactly supported.

Suppose now $c_1(D) = [\omega|_D]$, hence

$$\mathrm{Ric}(\omega|_D) = \omega|_D + dd^c\phi_D,$$

for some function $\phi_D$ on $D$. From here, the Kähler form $\omega_C$ corresponding to the conical metric $g_C$ takes then the form

$$\omega_C = \pi^*\omega|_D + dd^c(\pi^*\phi_D) + dd^c\psi_{Cal},$$

where $\psi_{Cal}$ is a potential coming from the Calabi construction of the conical metric, see Calabi (1957). In this case, we expect the metric $\omega$ on $X\backslash D$ to be cohomologically compactly supported. Indeed, this discrepancy is given by the cohomology class of $\pi^*\omega|_D$.

Indeed, $\pi^*\omega|_D = \pi^*d\eta$ on $U\backslash D$, where $U$ is the tubular neighbourhood of $D$ and again $\eta$ is the connection form on the associated Sasakian manifold, and in this case the associated line bundle is trivial. Indeed, we have that $[d\eta]_{H^2(D)} = c_1(S)$, since $d\eta$ yields the curvature of the bundle[4]. But since $\omega|_D$ belongs to the first Chern class of $D$, this entails $c_1(S) = 0$. This comes from the fact that if $\omega|_D$ is Kähler–Einstein, and thus in particular $[\omega|_D]_{H^2(D)} = c_1(D)$, then $[\pi_S^*\omega|_D]_{H^2(S)} = 0$. Indeed, let again $U$ be a tubular neighbourhood of $D$. The claim above can be proven by use of the Gysin sequence

$$H^{k-2}(D) \to H^k(U) \to H^k(U\backslash D) \simeq H^k(S) \to H^{k-1}(D),$$

where the first map is the wedge product with $\pi^*c_1(D)$ and the second map is the restriction. Note that $H^{k-2}(D) \simeq H^k_c(U)$ by means of the pullback map $\pi^*$, as we have

$$H^{k-2}(D) \simeq H^{2n-k}(D) \simeq H^{2n-k}(U) \simeq H^k_c(U)$$

[4]Note that despite the notation, $d\eta$ is not necessarily exact on $D$.

by repeated application of the Poincaré duality. Hence we get the sequence

$$H^k_c(U) \to H^k(U) \to H^k(S) \to H^{k-1}_c(U).$$

Note that the exactness of the sequence yields the claim.

For *ii)*, suppose $\omega|_D \in c_1(D)$ and hence $[\omega_{U\setminus D}] = 0$, from which we have $\omega_{U\setminus D} = \xi + dd^c\varphi$ and $\xi \simeq \pi^*\omega|_D = d\eta$. Note that by computing the tranverse Kähler form in a Sasakian manifold (*c.f.* equations (2.9) and (2.10) in Martelli et al. (2008)), we get that the conical metric has the form

$$\omega_C = \frac{1}{2}d\left(r^2\pi_r^*\eta\right).$$

By construction as in Section 2.2, we have $\pi_r^*\eta = r^{-2}d^c\log(r)$. Here $\pi_r$ is the cone projection $\pi_r\colon C(S) \to S \simeq \{r = 1\}$. Now, we can write on $U$

$$\omega = \pi^*\omega|_D + dd^c\phi_\omega$$

for some function $\phi_\omega$ on $U$. Here $\pi\colon U \to D$ is the projection given by the construction of the tubular neighbourhood $U$. Note that by the diffeomorphism given by the normal exponential map, whose construction is mentioned in Section 2.3, $U$ can be identified with a neighbourhood at infinity of the cone $C(S)$. Moreover, by the fact that $\omega|_D \in c_1(D)$, we have

$$\pi^*\omega|_D = dd^c(\pi^*\phi_D)$$

for some function $\phi_D$ on $D$. This entails that

$$\omega = dd^c(\pi^*\phi_D + \phi_\omega)$$

on $U$. At this point, we then have

$$dd^c(\pi^*\phi_D) = \frac{1}{2}d\left(r^2\pi_r^*\eta\right) = \frac{1}{2}dd^c\log r.$$

Hence we can write $\pi^*\phi_D = \frac{1}{2}\log(r) + h$, where $h$ is a pluriharmonic function on $U$. □

*Example* 2.5.4 (Fubini–Study case).

As before, consider a compact manifold $\overline{X}$ and a Kähler–Einstein divisor $D$ in $\overline{X}$ such that $-K_{\overline{X}} = (k+1)[D]$, where $[D]$ is the line bundle determined by $D$.

Consider the embedding given by a suitable multiple of $-K_{\overline{X}}$ (or, by the equality, $[D]$)

$$\varphi_{|-nK_{\overline{X}}|} \colon \overline{X} \hookrightarrow \mathbb{CP}^N.$$

Given this embedding, we have $-nK_{\overline{X}} = \mathcal{O}_{\mathbb{CP}^N}(1)|_{\overline{X}} = -\frac{1}{N+1}K_{\mathbb{CP}^N}|_{\overline{X}}$. This entails

$$\begin{aligned} n(k+1)[D] &= -\frac{1}{N+1}K_{\mathbb{CP}^N}|_{\overline{X}} \\ n(k+1)[D]|_D &= -\frac{1}{N+1}K_{\mathbb{CP}^N}|_D \\ -n\frac{k+1}{k}K_D &= -\frac{1}{N+1}K_{\mathbb{CP}^N}|_D, \end{aligned}$$

where the last equation is true by the adjunction formula $K_D = K_{\overline{X}}|_D + [D]|_D$. This implies

$$c_1(D) = \frac{k}{n(k+1)(N+1)}c_1(-K_{\mathbb{CP}^N}|_D),$$

and thus the Fubini–Study metric restricted to $D$ belongs to a multiple of the first Chern class of $D$, which entails we can find a Kähler–Einstein metric that is cohomologous to it. Hence, in this case, $\omega_{FS}$ takes the role of the global metric $\omega$ above, and restricted to $D$ it belongs to $c_1(D)$.

Starting from this, one can build a conical metric with infinite end at $D$ in the following way.

We can find a Calabi–Yau metric $\omega_0$ on this neighbourhood starting from a Kähler–Einstein metric

$$\omega_D = \omega_{FS}|_D + dd^c\psi$$

in the Kähler class of the restriction of the Fubini–Study metric. Now, on $U$, this metric is of the form

$$\omega_0 = \pi^*\omega_D + dd^c\psi_{\mathrm{Cal}},$$

where $\pi$ is the projection of the line bundle and $\psi_{\mathrm{Cal}}$ is a smooth function on $U$ that can be constructed thanks to the Calabi construction.

In case this operation creates zones in which the metric is no longer positive, we can add suitably chosen bump $(1,1)$-forms, and thus getting a proper metric $\hat{\omega}$.

This is compactly supported because $\pi^*\omega|_D = \pi^*d\eta$ on $U\backslash D$, where $U$ is the tubular neighbourhood of $D$ and again $\eta$ is the connection form on the associated Sasakian manifold, and in this case the associated line bundle is trivial. Indeed, we have that $[d\eta] = c_1(S)$, but since $\omega|_D$ belongs to the first Chern class of $D$, this entails $c_1(S) = 0$ by the Gysin sequence.

This is an example of the construction of a reference metric that will be useful to gain a priori estimates on the continuity path. The general construction (in the case we have a non-vanishing $\xi$) will be carried out in Section 3.2.

## 2.6 Sobolev inequalities in families

We want to consider uniform Sobolev constants in families to gain uniform estimates for the complex Monge–Ampère equation. Recall that the $C^0$ estimate argument in the work of Yau (Yau (1978)) makes extensive use of this inequality to set up the Moser iteration scheme.

As a starter, consider the following proposition in the compact case, *c.f.* Section 3 in Rong & Zhang (2011), Croke (1980), Gallot (1983), P. Li (1980).

**Proposition 2.6.1.** *Suppose we have a smoothing $(X_s, g_s)_s$ of a singular compact Calabi-Yau variety $X_0$ of dimension $n > 2$ such that*

$$\mathrm{Ric}_{g_s} \equiv 0, \quad \mathrm{Vol}_{g_s}(X_s) \equiv V, \quad \mathrm{diam}_{g_s}(X_s) \leq D.$$

*Then there exist uniform constants $C_1, C_2 > 0$ independent of $s$ such that for $s \neq 0$ and any smooth function $\chi$ on $X_s$, we have*

$$||\chi||^2_{L^{4n/(2n-2)}(g_s)} \leq C_1 \left( ||d\chi||^2_{L^2(g_s)} + ||\chi||^2_{L^2(g_s)} \right),$$

*and if $\int_{X_s} \chi \, \mathrm{dvol}_{g_s} = 0$,*

$$||\chi||^2_{L^{4n/(2n-2)}(g_s)} \leq C_2 ||d\chi||^2_{L^2(g_s)}.$$

We want a similar proposition in the asymptotically conical case, dropping the diameter and volume assumption and assuming instead non-collapsing properties. Indeed, locally we have the following.

**Theorem 2.6.2** (Anderson (1992), Theorem 4.1)**.** *Let $\mathbb{B}(r)$ be a geodesic ball in a complete Riemannian manifold with the lower curvature bound $\mathrm{Ric}_M \geq -\delta^2(n-1)$. If $v(r) = \mathrm{vol}\,\mathbb{B}(r)$ and $b_\delta(r)$ is the volume of a geodesic $r$-ball in the space form of constant curvature $-\delta^2$, then for all $r \leq \min\left(\frac{1}{4}\,\mathrm{diam}_M, 1\right)$,*

$$\inf_{\Omega \Subset \mathbb{B}(r)} \frac{\mathrm{vol}\,\partial\Omega}{\mathrm{vol}\,\Omega^{\frac{n-1}{n}}} \geq c(n,\delta) \left( \frac{v(r)}{b_\delta(r)} \right)^{\frac{1}{n}}.$$

On the other hand, in the asymptotically conical case we have the following.

**Theorem 2.6.3** (Hein (2011), Corollary 1.3). *Let $(M, g)$ be an asymptotically conical manifold of real dimension $n$ with diffeomorphism $\Phi$ defining the asymptotically conical structure with a cone $(C, g_C)$. Suppose we have $c \geq 1$ such that*

$$c^{-1} g_C \leq \Phi^* g \leq c g_C.$$

*Then for all compactly supported $\chi \in C^\infty(M)$,*

$$||\chi||^2_{L^{2n/(n-2)}(g)} \leq c' \left( ||d\chi||^2_{L^2(g)} \right),$$

*for a constant $c' > 0$.*

Note that this inequality is sufficient to start a Moser iteration and gain a $C^0$ estimate, as in Section 2.3 of Conlon & Hein (2013a).

The point is now to combine Theorem 2.6.3 with Proposition 2.6.1. Once we have uniform bounds on the metrics of the family with respect to the cone metric, the result is essentially the same as in the compact case.

**Definition 2.6.4.** *A manifold $(M, g)$ of dimension $n$ is called SOB($n$) if there exist $x_0 \in M$ and $C_0 \geq 1$ such that $\mathbb{B}_s(x_0) \backslash \overline{\mathbb{B}}_t(x_0)$ is connected for all $s > t \geq C_0$, $\mathrm{Vol}(\mathbb{B}_r(x_0)) \leq C_1 r^n$ for all $r \geq C_0$, $\mathrm{Vol}(\mathbb{B}_{(1-C_0^{-1})r}(x)) \geq C_0^{-1} r^n$, $\mathrm{Ric} \geq -K$ and $\mathrm{Ric} \geq -K r^{-2}$ for $r \geq C_0$.*

*Remark* 2.6.5. Note that an asymptotically conical manifold of dimension $n$ which is asymptotic to a Ricci-flat cone is automatically SOB($n$).

**Theorem 2.6.6.** *Suppose we have a smoothing $(X_s, g_s)_s$ of an asymptotically conical singular Calabi-Yau variety $X_0$ of complex dimension $n > 2$ which are all SOB($2n$) with constants $C_0, C_1, K$ independent of $s$, such that they satisfy the non-collapsing condition*

$$\inf_{x \in M} \mathrm{Vol}_{g_s}(\mathbb{B}_1(x)) \geq \kappa,$$

*where the constant $\kappa > 0$ is independent of $s$. Then there exists an uniform constant $C_{Sob} > 0$ independent of $s$ such that for $s \neq 0$ and any smooth compactly supported function $\chi$ on $X_s$, we have*

$$||\chi||^2_{L^{4n/(2n-2)}(g_s)} \leq C_{Sob} \left( ||d\chi||^2_{L^2(g_s)} \right).$$

*Proof.* We follow Hein (2011), that deals with the case of a single asymptotically conical manifold, in particular proving Theorem 2.6.3. In the proof, we find explicit description of the constants in the Sobolev inequality and the geometric quantities on which it depends. By our hypothesis those geometric quantities will be uniformly bounded, thus implying a uniform Sobolev constant.

As a first, by Corollary 2.6 in Hein (2011), we have that for all balls $B = \mathbb{B}_r$ and for all $p \in [1, 2n)$, $\alpha \in [1, \frac{2n}{2n-p}]$ and $\chi \in C_0^\infty(B)$,

$$\left( \fint_B |\chi|^{\alpha p} \right)^{1/\alpha p} \leq C(n,p,K) r \left( \fint_B |d\chi|^p \right)^{1/p}.$$

In particular, for $p = 2$ and $\alpha = 2n/(2n-2)$ we get

$$\left( \frac{1}{\operatorname{Vol} B} \int_B |\chi|^{4n/(2n-2)} \right)^{(2n-2)/2n} \leq C(n,K) r \left( \frac{1}{\operatorname{Vol} B} \int_B |d\chi|^2 \right).$$

Now, for $r > 0$ big enough, we have

$$\left( \int_{\mathbb{B}_r} |\chi|^{4n/(2n-2)} \right)^{(2n-2)/2n} \leq C(n,K) r^{-1} \left( \int_{\mathbb{B}_r} |d\chi|^2 \right),$$

since by the asymptotically conical structure, $cr^{2n} \leq \operatorname{Vol} \mathbb{B}_r \leq Cr^{2n}$ for a big enough $r > 0$.

This is a local Sobolev inequality, substantially a rephrasing of Theorem 2.6.2, and it can be used to gain a global one. Note that if we compute the dependency with respect to the volume of the ball, thanks to the $\mathrm{SOB}(2n)$ assumptions, we note that if we consider the annulus $A := \mathbb{B}_{r+1} \backslash \overline{\mathbb{B}}_r$, we have

$$\begin{aligned}
\left( \int_A |\chi|^{4n/(2n-2)} \right)^{(2n-2)/2n} &\leq C(n,K) \operatorname{Vol}(\mathbb{B}_1)^{-1/2n} \left( \int_A |d\chi|^2 \right) \\
&\leq C(n,K) \kappa^{-1/2n} \left( \int_A |d\chi|^2 \right)
\end{aligned}$$

eventually for a different constant $C = C(n,K)$, where the second inequality is due to the definition of $\kappa = \inf_x \operatorname{Vol}(\mathbb{B}_1(x)) \leq \operatorname{Vol}(\mathbb{B}_1)$.

Following Corollary 2.8 in Hein (2011), we fix $x_0 \in X_s$ define $r_m = 1 + m/100$, $A_m := \mathbb{B}_{r_{m+1}}(x_0) \backslash \overline{\mathbb{B}}_{r_m}(x_0)$ for $m \in \mathbb{N}$.

Take $\eta_0 \in C_0^\infty(\mathbb{B}_{r_2}(x_0))$ such that $0 \leq \eta_0 \leq 1$, $\eta_0 \equiv 1$ on $\mathbb{B}_{r_1}(x_0)$, $|\nabla \eta_0| \leq 200$. Then do the same for $m \geq 1$ by taking $\eta_m \in C_0^\infty(A_{m-1} \cup \overline{A}_m \cup A_{m+1})$ such that

$0 \leq \eta_m \leq 1$ with $\eta_m \equiv 1$ on $A_m$ and $|\nabla \eta_m| \leq 200$. Then, by the above and thanks to the non-collapsing bound, for a fixed $m$ we find

$$||\eta_m \chi||^2_{4n/(n-2)} \leq C(n, K)\kappa^{-1/2n}||\nabla(\eta_m \chi)||^2_2.$$

At this point, we can sum over $m$, and we get a global Sobolev inequality. □

Note that given a lower bound on Ricci curvature, we have that if we consider a non-collapsing bound, local bounds on volume and local upper bound on diameter, there are some relationship between them. This is discussed more in detail in Proposition 2.7.1. In particular, this explains the difference of hypothesis in the statement of the compact version, Proposition 2.6.1.

## 2.7 Bounds on volume, diameter and non-collapsing condition

In this section, we will explore the interplay between a lower bound on Ricci curvature, non-collapsing conditions, local volume bounds, and local upper bounds on diameter.

**Proposition 2.7.1.** *Suppose* $(M, g)$ *is a complete Riemannian manifold with* $\mathrm{Ric}_g \geq -Kg$. *Let* $V_r := \mathrm{Vol}(\mathbb{B}_r)$, $2r = \mathrm{diam}(\mathbb{B}_r)$, $\kappa = \inf_{p \in \mathbb{B}_r} \mathrm{Vol}(\mathbb{B}_1(p))$. *Then*

*i) if* $2r \leq D$, *we get an upper bound on the volume*

$$V_r \leq C(n, K)D^n;$$

*ii) if* $V_r = V$ *and* $2r \leq D$, *we get a non-collapsing bound*

$$\kappa = C(n, K)\frac{V}{D^n};$$

*iii) if* $K \leq 0$, $\kappa > 0$ *and* $V_r \leq V$, *we get an upper bound on the diameter*

$$r \leq C(n, K)\frac{V}{\kappa}.$$

*Note that the constants* $C(n, K) > 0$, *depending only on the dimension* $n$ *and the lower bound on the Ricci curvature* $K$, *may vary from line to line.*

*Proof.* Let $\omega_K(r)$ denote the volume of the ball of radius $r$ in the constant sectional curvature space of Ricci curvature $-K$.

i) if we only have $2r \leq D$, we can use the fact that $V_r/\omega_K(r) \to 1$ for $r \to 0$ to get the inequality

$$V_r \leq \omega_K(r) \leq CD^n$$

where $C = C(n, K) > 0$ and because the volume growth is at most euclidean, by an application of the Bishop–Gromov inequality;

ii) if we have $V_r = V$ and $2r \leq D$, we get a non collapsing bound by applying Bishop–Gromov directly, as (here let $r \geq 1$)

$$\frac{V_1}{\omega_K(1)} \geq \frac{V_r}{\omega_K(r)} \geq C\frac{V}{D^n},$$

where $C = C(n, K) > 0$ and the last inequality is true again because the volume growth is at most euclidean.

Note that the same argument can be carried out on a compact manifold $(M, g)$. By letting $r \to \mathrm{diam}(M) =: D$, we have directly $\mathrm{Vol}(\mathbb{B}_D) = \mathrm{Vol}(M)$;

iii) if we have $\kappa > 0$ and $V_r \leq V$, by the Bishop–Gromov inequality we have

$$\frac{V_r}{\omega_K(r)} \geq \frac{V_{r+2}}{\omega_K(r+2)}.$$

For positive numbers $a \leq c, b \leq d$, we have

$$\frac{a}{b} \geq \frac{c}{d} \Longrightarrow \frac{a}{b} \geq \frac{c-a}{d-b}.$$

Hence we get

$$\frac{V_r}{\omega_K(r)} \geq \frac{V_{r+2} - V_r}{\omega_K(r+2) - \omega_K(r)}.$$

From this, we get

$$V_r \geq \frac{\omega_K(r)}{(\omega_K(r+2) - \omega_K(r))} \mathrm{Vol}(\mathbb{B}_{r+2} \backslash \mathbb{B}_r) \geq Cr \,\mathrm{Vol}(\mathbb{B}_1(q)) \geq C\kappa r,$$

where $q$ is a point on $\partial\mathbb{B}_{r+1}$ (hence $\mathbb{B}_1(q) \subseteq \mathbb{B}_{r+2} \backslash \mathbb{B}_r$). The inequality

$$\frac{\omega_K(r)}{(\omega_K(r+2) - \omega_K(r))} \geq Cr$$

where $C = C(n, K) > 0$ is a constant depending on the dimension and the lower bound for the Ricci curvature, is true because the volume growth of balls is at worst linear by Yau (1976), Theorem 7 and the discussion in the Appendix, point (iii). Here we use the fact that $\mathrm{Ric} \geq 0$. At this point we have

$$r \leq \frac{V}{C\kappa},$$

thus getting an upper bound on the diameter.

$\square$

In our discussion, in particular for the $C^0$ estimate in Section 3.5.2, we want a uniform bound on the diameter only given an upper bound on the volume, in order to then gain a non-collapsing bound. This is not given by the previous proposition, as in the general case it would be a circular argument.

To gain the bound we need, we adapt Theorem 2.1 of Rong & Zhang (2011) in a local case. Note that as in the application of Lemma 3.2.6, we make use of the fact that the family can be locally embedded in a smooth Kähler manifold $(Z, \eta)$.

# Chapter 3

# Main theorem

This chapter will be devoted to the proof of Theorem 1.2.5. We will need to employ a continuity path, thus proving openness and closedness in the space of solution. Choosing which space of solutions to consider will be a crucial part of the proof.

## 3.1 Framework

To prove Theorem 1.2.5, we will follow the steps of Calabi (Calabi (1958)), who proposed to consider a continuity path to solve the equation

$$(\omega + dd^c\varphi)^n = e^f\omega^n \tag{3.1}$$

on a compact Kähler manifold $X$, in $C^\infty(X)$. Since the Ricci curvature is linked to the volume form, solving this equation gives a way to solve the prescribed Ricci curvature problem.

A continuity path is a family of equations

$$(\omega + dd^c\varphi_t)^n = e^{tf}\omega^n, \tag{3.2}$$

depending on $t \in [0, 1]$. The continuity path method goes as follows. We consider the set

$$T := \{t \in [0, 1] \,|\, \varphi_t \text{ is a solution in } H\},$$

where $H$ is a certain space of functions (*e.g.* for the compact case, one can choose the Hölder space $C^{k+2,\alpha}$).

A fundamental step is proving that $0 \in T$ (or, more precisely, that $T$ is non-empty). In the compact case considered by Calabi, one can see that $\varphi_0 \equiv 0$ solves the equation for $H = C^{k+2,\alpha}(X)$.

The following step is to ensure that $T$ is both open and closed. This would entail that $T = [0,1]$, as it is non-empty, and hence we can solve the original equation with $\varphi_1$.

To follow the continuity path strategy, we need to be able to find a solution to an equation related to the one that we want to solve, and set up the continuity path starting from there. This is made in order to prove that the space $T \subseteq [0,1]$ of solutions is non-empty. To reach this goal, a direct way is to choose an already existing metric with the desired properties. In our case, we need to find metrics that have the property of having controlled geometry both at infinity and close to the singular point.

The construction of such starting points, to which we give the name *reference metrics*, will be done in the following section.

Subsequently, we will use bespoke analysis on manifolds with conical singularities to gain openness of the continuity path.

Finally, to determine closedness in the continuity path we have assumed we are in the situation in which we can find a *smoothing* of our manifold. Note that if we are considering the smoothing of a Calabi–Yau cone, we are exactly in this situation. By the hypothesis of Theorem 1.2.5 we have such a smoothing $\mathcal{X} \to \Delta$ and that the central fibre $X_0$ is isomorphic to the manifold $X$, while the other fibres $X_s$ with $s \neq 0$ are smooth asymptotically conical manifolds with the same cone at infinity. Moreover, we have a Kähler metric $\omega$ on $\mathcal{X}$ which can be restricted to Kähler metrics $\omega_s := \omega|_{X_s}$ on $X_s$.

The assumption on the existence of a smoothing is used, as in Hein & Sun (2017), to rule out the possibility of a "bubbling" effect when considering the tangent cone at the singularity when taking the limit. We address this issue in Section 3.5.5.

## 3.2 Reference metrics

As pointed out before, to compare metrics in the family (and from there obtain uniform a priori estimates) one needs to fix a reference metrics whose behaviour is shared in the family:

- In the proof of the compact case as in Yau (1978), the reference metric is only one and it is the starting smooth Kähler metric $\omega$; even if we do not have a family, it is still necessary to compare $\omega + dd^c\varphi$ with $\omega$;

- In Hein & Sun (2017), where the authors consider compact Calabi–Yau manifolds with canonical singularities, one just fixes the Fubini–Study metric given by the smoothing (which is projective);
- in Section 3 of G. Chen (2019), where the author deals with the asymptotically cylindrical case with a canonical singularity, one considers a flat family of projective varieties (a compactification) and then identifies a divisor in there with the right properties so that one can construct an asymptotically cylindrical metric on the complement of the divisor.

We want to create background asymptotically conical metrics that have controlled uniform geometry both at infinity (where they will be asymptotically conical with the usual cone) and on the "compact part" close to the singularity $x$, where they can be controlled by the metric $\omega_s$ on $\overline{X}_s$ restricted to $X_s$. Note that this metric has the property to be the restriction of a smooth Kähler metric $\eta$ in an ambient space $Z$, close to the singularity $x$.

The existence of such a family of starting points for the continuity path is not obvious and the construction will consist of several steps, which go as follows:

1. Choose a Kähler metric $\omega$ on the family $\mathcal{X}$. This gives metrics $\omega_s$ on each of the fibers and a way to compute distance between them;
2. Change the behaviour of the metric $\omega_s$ at infinity with respect to the asymptotically conical structure, to make them asymptotically conical. This change, as all of the changes that will follow, will be uniform in $s$;
3. Improve the rate of convergence of the Ricci potential $\rho_s$ of the metrics, in case it is too slow;
4. Modify the metric on the singular fibre $X_0$ close to the singularity, so that the metric is conical;
5. Choose functions $F_s$ that agree with the Ricci potential of the singular metric in $X_0$, are Ricci-flat close to the singularity, and agree with the Ricci potential of the asymptotically conical metrics that we already have.

After constructing these $F_s$, we will set up a continuity path on each $X_s$ with metrics which have as Ricci potentials in the starting points the functions $F_s$ and in the ending points the identically zero functions. These continuity paths will already have solutions on the smooth fibres thanks to Conlon & Hein (2013a).

*Remark* 3.2.1. Note that all of the modifications that we are going to make to the starting metric $\omega_s$ will only consist of adding $dd^c$-exact forms. This ensures that we are not changing the cohomology class of $\omega_s$, which will still be $c_1(L_s)$.

### 3.2.1 Prescription at infinity

In this section we will deal with points 1., 2. and 3. of the program above.

We will start with the construction on one fixed asymptotically conical manifold (eventually with isolated canonical singularities), and then extend the prescription in family later, by checking that the quantities needed for the construction can be taken to be uniform.

Suppose we have a Kähler metric $\omega$ on $X$ such that the class $[\omega|_X]$ is $\mu$-almost compactly supported as in Definition 2.5.1, with $(1,1)$-form $\xi$ such that $|\xi| \leq K' r^\mu$ for a constant $K' > 0$ outside of a fixed compact set. In case $X$ has an analytic singularity $x$, we assume the metric $\omega$ is smooth on the regular part $X^{reg}$ of $X$, with bounded Kähler potentials.

Then we want to modify it so as to find an asymptotically conical Kähler metric in the same class. In other words, we want to prescribe the behaviour at infinity.

We follow the computations carried out in Conlon & Hein (2013a). By Lemma 2.15 in Conlon & Hein (2013a), on $X$ for all $\alpha > 0$ there exists a plurisubharmonic function $h_\alpha$ which is equal to $(r \circ \Phi)^{2\alpha}$ on[1] $\{r > R\}$ for some $R \gg 0$, and is strictly plurisubharmonic there. We can also take $h_\alpha$ such that $dd^c h_\alpha = 0$ in a compact subset $K$ containing the singularity for all $\alpha$ by construction. Fix $\alpha \in (0,1)$, and consider

$$\hat{\omega} = \omega - dd^c(\zeta\phi) + C dd^c((1-\zeta_S)h_\alpha) + c dd^c h_1,$$

with $C, S$ to be determined and $c > 0$ the "scaling" constant, $\zeta$ is a cut-off function which is $\equiv 1$ on $\{r > 3R\}$, $\equiv 0$ on $\{r < 2R\}$, and $\zeta_S(x) := \zeta(x/S)$, and $\phi$ is such that outside a compact set $\omega - \xi = dd^c\phi$.

Then

- On $K \cup \{1 < r < 2R\}$, $\hat{\omega} \simeq \omega + C dd^c h_\alpha + c dd^c h_1 \geq \omega > 0$ by plurisubharmonicity of $h_\alpha$ on $X$;

- On $\{3R < r < 2SR\}$, $\hat{\omega} \simeq \xi + C dd^c h_\alpha + c dd^c h_1 > 0$ after increasing $R$ if

[1]We will write $r$ for $\Phi^* r$.

necessary to make $|\xi| = O(r^\mu)$ small enough in norm. Indeed we need

$$|Cdd^c h_\alpha + cdd^c h_1| > K' r^\mu \geq |\xi|,$$

from which we get $R > (K'^{-1}|Cdd^c h_\alpha + cdd^c h_1|)^{1/\mu}$;

- On $\{3SR < r\}$, $\hat{\omega} > 0$ for the same reason as in the previous point (note that here $Cdd^c((1-\zeta_S)h_\alpha) \equiv 0$), we get $R > (K'^{-1}|cdd^c h_1|)^{1/\mu}$;

- On $\{2R \leq r \leq 3R\}$, now $R$ is fixed and this set is compact. We can choose $S > 3/2$ so that we get $2SR > 3R$, hence $1 - \zeta_S \equiv 1$ in this region. If we choose a big enough $C > 0$, then $\hat{\omega} > 0$. We can take

$$C > \frac{|\omega - dd^c(\zeta\phi) + cdd^c h_1|_{\{2R \leq r \leq 3R\}}}{|dd^c(h_\alpha)|_{\{2R \leq r \leq 3R\}}};$$

- On $\{2SR < r < 3SR\}$, $\hat{\omega} > 0$ choosing $S > 3/2$ big enough (depending on $R, C$), as $h_\alpha$ is of lower order compared to $h_1$ (recall $\alpha \in (0,1)$). Indeed if

$$\frac{|cdd^c h_1|}{|Cdd^c h_\alpha|} \geq K'' r^{2-2\alpha},$$

  it suffices to choose $S > (K'')^{1/(2\alpha-2)}(2R)^{-1}$.

Then $\hat{\omega}$ is a genuine Kähler metric which is equal to $\xi + dd^c r^2$ at infinity. Note that modulo changing the cut-off function accordingly, this construction works for a manifold with an isolated canonical singularity as well.

For technical details that will be discussed later, on the singular manifold we would prefer to have a metric $\hat{\omega}_0$ such that close to the singularity $x$ it can be written as $\hat{\omega}_0 \simeq i^*\eta$, where $i\colon U \to Z$ is an embedding of a neighbourhood $U$ of $x$ into a smooth Kähler manifold $(Z, \eta)$. Note that $\hat{\omega}_0$ possesses such a property, by starting from $\omega_0 = \omega|_{X_0}$, obtained by restricting the metric $\omega \in H^{1,1}(\mathcal{X}, \mathbb{R})$ on the central fibre $X_0$. The equality $\omega_0 = i^*\eta$ on $U$ is true by the hypothesis of Theorem 1.2.5.

We now aim to have this construction in a family, in the quasi-regular case of Theorem 1.2.6.

Suppose that we have $\overline{\mathcal{X}}$ with a codimension two set $D \subseteq \overline{\mathcal{X}}$ such that $\mathcal{X} = \overline{\mathcal{X}} \backslash D \to \Delta$ is a family of asymptotically conical manifolds ($D$ is a Kähler–Einstein divisor for all fibres $\overline{X}_s$).

We want each fibre $X_s$ to be Kähler with Kähler metrics $\omega_s$ which are such that $\omega_s|_D \equiv \omega_{s'}|_D$ for any $s, s' \in \Delta$. Recall that we have exactly $\overline{X}_s \cap \overline{X}_{s'} = D$ if $s \neq s'$. We thus consider the Kähler metric $\omega$ on $\overline{\mathcal{X}}$. At this point, we define $\omega_s := \omega|_{\overline{X}_s}$. Note that these are smooth Kähler metrics for $s \neq 0$. Denote $\omega_D := \omega_s|_D$.

This construction enables us to define the asymptotically conical structures on $X_s$ as in Section 2.3 in a controlled fashion. Indeed, we can take for all $s$ the normal exponential map with respect to $\omega_s$, and all of these maps come from the normal exponential map of $D$ in $\overline{\mathcal{X}}$ with respect to the metric $\omega$.

For each $s \in \Delta$, we then have a $\mu_s$-almost compactly supported Kähler class $[\omega_s]$. Indeed, note that by Proposition 2.5.2 these $\mu_s$ exist and they satisfy $\mu_s \leq -2$. In particular they can be taken uniform equal to $\mu = \max_s \mu_s \leq -2$. Then, following the procedure above, we can construct background metrics $\hat{\omega}_s$ for all $s$; to be certain that all background metrics are genuine Kähler metrics, we need that for the respective $K_s$ there exists $K > 0$ such that $\sup_\Delta K_s \leq K$.

Recall that by the results of Section 2.5, if we have that $\omega_s|_{X_s}$ is $\mu$-almost compactly supported with form $\xi_s$, we have

$$[\xi_s]_{H^2(U_s \setminus D)} = [(\pi_s^* \omega_D)|_{U_s \setminus D}]_{H^2(U_s \setminus D)}.$$

Note that up to shrinking the $U_s$, we can choose a neighbourhood $\mathcal{U}$ of $D$ in $\overline{\mathcal{X}}$ such that $U_s = \mathcal{U} \cap \overline{X}_s$. This can be done as the space $\Delta$ is a precompact set. This is equivalent to giving a uniform bound on the volume with respect to $\omega_s$ of the compact sets $K_s$ in the $\mu$-almost compactly supported definition.

Moreover, we can choose the projections $\pi_s \colon U_s \to D$ as restrictions to $U_s$ of the orthogonal projection $\pi \colon \mathcal{U} \to D$ with respect to the metric $\omega$. By arguments as in the proof of *i)* in Proposition 2.5.3, since we have uniform bounds on the norms of the projections (as they come from restrictions of the same projection $\pi$) and all of the asymptotically conical structure come from the same normal exponential map with respect to $\omega$, we can choose a uniform $K > 0$.

This is equivalent to the following. Consider the diffeomorphisms $\Phi_s \colon X_s \backslash K_s \to C_\infty \backslash K'$. We define $\Psi_{0,s} \colon X_s \backslash K_s \to X_0 \backslash K_0$ as $\Psi_{0,s} := \Phi_0^{-1} \circ \Phi_s$. We can pack the $\rho_s$, radius functions at infinity in $X_s$, in a function $\rho$ on $\mathcal{X} \backslash \mathcal{K}$ by eventually defining $\rho_s := \Psi_{0,s}^{-1} \circ \rho_0 \circ \Psi_{0,s}$.

*Remark* 3.2.2. Note that by construction, given a compact $\mathcal{K} \subseteq \mathcal{X}$, we have

$$|\operatorname{Ric}(\hat{\omega}_s)|_{\hat{\omega}_s} \leq C_{\mathcal{K}}, \quad |\operatorname{Sec}(\hat{\omega}_s)|_{\hat{\omega}_s} \leq C'_{\mathcal{K}} \quad \text{on } X_s \backslash K_s,$$

where $K_s = \mathcal{K} \cap X_s$ and $C_{\mathcal{K}} > 0$ does not depend on $s$. This is consequence of the uniform bound $K$ that we found above which makes the metrics uniformly asymptotic to the conical metric $\omega_\infty$ on the Calabi–Yau cone $C_\infty$.

Indeed, $\omega_\infty$ is Ricci-flat by construction. Moreover, by writing $g_\infty = d\rho^2 \oplus \rho^2 g_S$, where $S$ is the link of the cone at infinity and $\rho$ is the radius on $C_\infty$, we find that the Riemannian curvature on the cone decays like $\rho^{-2}$, since the link $(S, g_S)$ is a smooth manifold.

A priori, the decay at infinity of the Ricci potential $F_s$ of $\hat{\omega}_s$ could be bigger than $-2$, and this can cause problems due to the presence of non-trivial harmonic functions of rate in $[0, 2]$ on the cone at infinity. Thus we prove the following lemma, which follows the idea of Lemma 2.12 in Conlon & Hein (2013a), where the authors deal with the same problem in the smooth asymptotically conical setting.

**Lemma 3.2.3.** *Consider the same data as before, and the Monge–Ampère problem*

$$\begin{cases} \omega^n = e^F i^{n^2} \Omega \wedge \overline{\Omega} & \text{on } X \backslash \{x\}, \\ |\Phi^* F|_{\omega_C} \leq C_1 \rho^\mu & \mu \in (-2, 0), \end{cases}$$

*where $F, \Omega$ are given, and $\rho$ is a radius function relative to the cone at infinity. Then the problem has a solution if a corresponding problem*

$$\begin{cases} \omega_u^n = e^{\tilde{F}} i^{n^2} \Omega \wedge \overline{\Omega} & \text{on } X \backslash \{x\}, \\ |\Phi^* \tilde{F}|_{\omega_C} \leq C_2 \rho^\lambda & \lambda < -2, \end{cases}$$

*has a solution, where $\omega_u := \omega + dd^c u$, $u$ is a fixed function and $C_2$ depends only on $C_1$. Moreover, if we fix a compact $K$, we can choose $\tilde{F}$ such that it agrees with $F$ on $K$.*

*Proof.* Consider

$$(\omega + dd^c u)^n = \omega^n + \omega^{n-1} \wedge dd^c u + O(|dd^c u|^2).$$

By the equation for $\omega$, the first two terms are equal to

$$e^F (1 + C\Delta_\omega u) i^{n^2} \Omega \wedge \overline{\Omega},$$

where $C = C(n) > 0$ is a fixed dimensional constant. By the study of the Poisson equation, we can at first solve $e^F(1 + C\Delta_\omega u) = 1$ outside the compact region $K$

that contains $x$. Then we choose $u_1$ as a solution of this equation. Note that we can solve this as $F = O(\rho^\mu)$ at infinity. Then

$$\begin{aligned}(\omega + dd^c u_1)^n &= \left(1 + \frac{O(|dd^c u_1|^2)}{i^{n^2}\Omega \wedge \overline{\Omega}}\right) i^{n^2}\Omega \wedge \overline{\Omega} \\ &= e^{F_1} i^{n^2}\Omega \wedge \overline{\Omega},\end{aligned}$$

in $X \backslash K$, where $F_1 := \log\left(1 + \frac{O(|dd^c u_1|^2)}{i^{n^2}\Omega\wedge\overline{\Omega}}\right)$.

Now, since $F = O(\rho^\mu)$ at infinity, the equation $\Delta u_1 = C^{-1}(e^{-F} - 1)$ implies $u_1 = O(\rho^{\mu+2})$ and hence $dd^c u_1 = O(\rho^\mu)$, from which $F_1 = O(|dd^c u_1|^2) = O(r^{2\mu})$. Note that up to modifying $u_1$ on an open set $\Omega \supseteq K$, for instance by multiplying by a cut-off function[2] so that $u_1 \equiv 0$ in $K$, we still have that $F_1$ agrees with $F$ in $K$. Note that a priori $\omega + dd^c u_1$ may not be a positive $(1,1)$-form on a compact region, but substituting $u_1$ with $\epsilon u_1$ for a suitable $\epsilon > 0$ keeps the form Kähler — since $\omega$ is Kähler — and is of no consequence for the argument as it consists only of finite steps. Note that we still retain the property $O(|dd^c(\epsilon u_1)|^2) = O(r^{2\mu})$.

We can repeat this process iteratively until we obtain $F_k$ such that $2^k\mu < -2$. We can then define $\tilde{F} := F_k$ and $u := \sum_i^k u_i$. □

*Remark* 3.2.4. Note that we can proceed iteratively and change $F_s$ simultaneously for each $s \in \Delta$ until $\max_s \mu_s < -2$.

### 3.2.2 Prescription at the singularity

In this section we will deal with the points 4. and 5. of the program sketched in Section 3.2.

We will call $\hat{\omega}_s$ the metrics on $X_s$ constructed in the previous section. Note that these metrics have the right behaviour at infinity, but they do not interact at all with the conical behaviour close to the singularity $x$. Hence we need to modify them still some more to get proper reference metrics to use for our continuity path. In particular, we will want metrics on the singular space $X_0$ to possess metric conical behaviour near the singularity.

Call $\mathcal{V}$ a neighbourhood of the singularity $x$ in $\mathcal{X}$, and $V_s := \mathcal{V} \cap X_s$.

Let us consider the case $s = 0$. We want to construct a continuity path of the

[2] Equivalently, solving the equation $e^F(1 + C\Delta_\omega u) = \xi e^F + (1-\xi)$ where $\xi$ is a cut-off function that is equal to 1 in $K$.

type

$$(\hat{\omega}_0 + dd^c\psi_t)^n = e^{tF} i^{n^2} \Omega_0 \wedge \overline{\Omega_0}, \tag{3.3}$$

where $F$ is such that $\hat{\omega}_0 + dd^c\psi_t$ has metric conical behaviour close to the singularity.

This is easier done if we consider, as we have done in the previous sections, a *smoothing* of the singular manifold $X_0$, so as to gain a priori estimates on smooth manifolds; once these are secured, one passes to the singular limit taking care of the fact that solutions do not degenerate at the singularity.

Hence, rather than a single continuity path, we are going to consider a family of continuity paths over $\Delta$, of the same type as Equation (3.3), using the constructed metrics $\hat{\omega}_s$ and suitable chosen functions $F_s$ on $X_s$ for each $s \in \Delta$.

We pack the construction of the $F_s$ that we are going to consider in the following proposition. Note that the construction of the functions $F_s$ and of the potentials $\psi_{t,s}$ are tightly interconnected due to the Monge–Ampère equation.

**Proposition 3.2.5.** *Suppose we have the metrics $\hat{\omega}_s$ on $X_s$ as described above. Starting from these metrics, we can build metrics $\omega_{t,s} = \hat{\omega}_s + dd^c\psi_{t,s}$ on $X_s$ for $s \neq 0$ such that they satisfy the continuity path*

$$\omega_{t,s}^n = e^{F_{t,s}} i^{n^2} \Omega_s \wedge \overline{\Omega}_s.$$

*Here $F_{t,s}$ are functions on $X_s$ such that*

*i) $F_{1,s}$ are the restriction of the same function $\mathcal{F}$ on $\mathcal{V}$, up to shrinking $\mathcal{V}$; in particular, they have uniform $C^k$ estimates in $s$ on $V_s$ depending on estimates for $\mathcal{F}$;*

*ii) are pluriharmonic on $\mathcal{V}$;*

*iii) $F_{1,0}$ is such that $\omega_{1,0}$ is the pullback of the metric on the model cone close to the singularity $x$;*

*iv) $F_{1,s}$ are the Ricci potentials of the metrics $\hat{\omega}_s$ outside $K_s$ for all $s$; in particular, they have uniform estimates in $s$ outside of $K_s$ depending on estimates for $\hat{\omega}_s$ (more precisely, $\rho_s$);*

*v) $F_{0,s} \equiv 0$ for all $s$.*

*We can also choose $F_{t,s}$ so that $F_{t,s} = tF_{1,s}$. This is not required, but in the following this property will be assumed.*

We will describe the construction claimed in the proposition in the following.

The Ricci potential $F_{1,0}$ can be taken to be pluriharmonic — and more, exactly the conical metric — near the singularity $x$ (in $V_0$, up to shrinking) by the following lemma.

**Lemma 3.2.6** (Arezzo & Spotti (2016), Proposition 2.4)**.** *Let $(X, \omega)$ be a Kähler manifold which has singularities analytically, but not metrically, modelled on Calabi–Yau cones. Assume moreover that the metric $\omega$ is a smooth Kähler metric on the regular part $X^{reg}$, and that locally near the singularities is the restriction of a smooth Kähler form in an embedding of a neighbourhood of each singular point into a smooth Kähler manifold.*

*Then there exists a Kähler metric $\omega'$ on the regular part $X^{reg}$ with metrically conical singularities which satisfies $[\omega'] = [\omega] \in H^{1,1}(X^{reg}, \mathbb{R})$.*

*Example* 3.2.7. An algebraic variety $X \subseteq \mathbb{P}^n$ with conical singularities satisfies the hypothesis of the lemma.

The idea under the proof of the lemma is that one can glue the Kähler potential of the Calabi–Yau metric of the cone in a neighbourhood $U$ of the singular point in the manifold. This can be done as $\omega|_{U\setminus\{0\}} = i^*\eta$, where $i$ is the embedding into the smooth Kähler manifold and $\eta$ can be written in coordinates as $\eta \simeq dd^c\left(|z|^2 + O(|z|^4)\right)$. Using this expression, we can use a cut-off function $\chi_\delta$ on $\mathbb{C}^n$, so that we can define the form

$$\eta_\delta = dd^c\left(|z|^2 + \chi_\delta(z)O(|z|^4)\right).$$

This is still Kähler and flat close to an arbitrary small neighbourhood of $\mathbb{C}^n$. Moreover it agrees with $\omega$ if $|z| > \delta$. Consider the map $\phi$ that gives the model of the Calabi–Yau cone close to the singularity. Then we can consider

$$\omega' := \varepsilon^2\phi_*\left(dd^c\left((1-\chi_{\delta'})r^2\right)\right) + \tilde{\eta},$$

where $\tilde{\eta}$ is another suitable modification of $\eta_\delta$ so that $\tilde{\eta} \equiv 0$ close to the singularity, with also suitable choices of $\varepsilon$ and $\delta'$ so that this metric stays Kähler.

Now, following Hein & Sun (2017), the function $F$, the Ricci potential of this new metric $\omega'$, is then the real part of a holomorphic function $F^h$ in $V_0$. This function extends to a holomorphic function $\mathcal{F}^h$ in $\mathcal{V}$ by analytic continuation.

Then we can find a continuous function $\mathcal{F}$ on $\mathcal{V}$ such that $\mathcal{F}$ is smooth away from $x$, $\mathcal{F}|_{\mathcal{V}} = \mathrm{Re}(\mathcal{F}^h)$, $\mathcal{F}_{V_0} = F$. Define $F_s := \mathcal{F}|_{V_s}$.

We need to extend these functions $F_s$ outside $V_s$.

Write $\rho_s$ for the Ricci potential of the metric $\hat{\omega}_s$ on $X_s$. We want to construct directly the potential $F_{t,s}$ by interpolating between the Ricci potential $\rho_s$ on $X_s \backslash K_s$ and the pluriharmonic potential $F_s$ in $V_s$. In particular, we want smooth function $F_{t,s}$ such that

$$\begin{aligned} F_{t,s} &\equiv tF_s \quad \text{on } V_s, \\ F_{t,s} &\equiv t\rho_s \quad \text{on } X_s \backslash K_s. \end{aligned}$$

We directly only extend $F_s$ and then choose $F_{t,s} := tF_s$.

Now, we can directly extend $\mathcal{F}$ from $\mathcal{V}$ to the whole $\mathcal{X}$. We can also pack the $\rho_s$ in a function $\rho$ on $\mathcal{X} \backslash \mathcal{K}$, where $x \in \mathcal{K}$, by Remark 3.2.4. At this point, we want to extend $\mathcal{F}$ to a continuous function that is smooth outside $\{x\}$ and such that

$$\mathcal{F} \equiv \rho \ \text{ in } \mathcal{X} \backslash \mathcal{K}, \qquad \mathcal{F}|_{X_0} \equiv F_0.$$

We do it in the following way. Let us write

$$\mathcal{X} = \mathcal{X}^{ext} \cup \mathcal{X}^{neck} \cup \mathcal{X}^{int},$$

where $\mathcal{X}^{ext} := \mathcal{X} \backslash \mathcal{K}$, $\mathcal{X}^{int} = \mathcal{V}$ and $\mathcal{X}^{neck}$ is an open set containing the remaining region and overlapping $\mathcal{X}^{ext}$ and $\mathcal{X}^{int}$. We have diffeomorphisms $\Phi_s \colon X_s \backslash V_s' \to X_0 \backslash V_0'$, where $V_s'$ are adequate shrinkings of $V_s$ for all $s$, given by orthogonal projection with respect to the Kähler metric $\omega$ on $\mathcal{X}$. In particular we can choose $V_s'$ such that $\Phi_s$ is defined in $\mathcal{X}^{neck} \cap X_s$. Clearly $\Phi_0 \equiv \mathrm{Id}$.

Let $\{\chi_{ext}, \chi_{neck}, \chi_{int}\}$ be a partition of unity subordinate to the previous open covering. We extend $F_0$ constantly on the whole $\mathcal{X}^{neck}$. More precisely, we consider the function $\overline{F} \colon \mathcal{X}^{neck} \to \mathbb{R}$ such that

$$\overline{F}(x) = F_0(\Phi_s(x)) \quad \text{for } x \in \mathcal{X}^{neck} \cap X_s.$$

Then we define our extension as

$$\overline{\mathcal{F}} := \chi_{ext}\rho + \chi_{neck}\overline{F} + \chi_{int}\mathcal{F}$$

This extension satisfies the requirements up to shrinking $\mathcal{V}$ and enlarging $\mathcal{K}$. Indeed, for all $x \in X_0$ we have

$$\begin{aligned}\overline{\mathcal{F}}(x) &= \chi_{ext}(x)\rho(x) + \chi_{neck}(x)\overline{F}(x) + \chi_{int}(x)\mathcal{F}(x) \\ &= (\chi_{ext}(x) + \chi_{neck}(x) + \chi_{int}(x))\, F_0(x) \\ &= F_0(x),\end{aligned}$$

by definition of partitions of unity.

Then we have uniform estimates for all derivatives of the functions $F_s := \overline{\mathcal{F}}|_{X_s}$, since they descend from estimates on the single smooth function $\overline{\mathcal{F}}$.

Once we have built these functions, on $X_s$ we can already find solutions $\psi_{t,s}$ with a priori estimates depending on $X_s$, $\hat{\omega}_s$ and $F_{t,s}$ thanks to the results of Conlon & Hein (2013a), Conlon & Hein (2013b). Note that these results do not imply the existence for $s \neq 0$, but the $F_{t,0}$'s are still well defined functions.

## 3.3 Continuity path

At this point we can clearly state what is the continuity path that we consider, completing the program in Section 3.2.

In the previous section, we have found a family of functions $F_{t,s}$ on $X_s$ with the properties expressed in Proposition 3.2.5. For all $s$, the continuity paths we consider are

$$\omega_{t,s}^n = c_{t,s} e^{tF_s} i^{n^2} \Omega_s \wedge \overline{\Omega}_s, \tag{3.4}$$

where $\omega_{t,s} = \hat{\omega}_s + dd^c\psi_{t,s}$, and the constant $c_{t,s} > 0$ are chosen so that

$$\mathrm{Vol}_{\omega_{t,s}}(V_s) = \mathrm{Vol}_{\omega_{1,0}}(V_0)$$

for all $t \in [0,1]$ and $s \in \Delta\backslash\{0\}$. This can be done as the right hand side is finite by construction of the metric $\omega_{1,0}$, which on $V_0$ is the pullback of the conical metric on the model cone $(C_x, \omega_{C_x})$. The reason we add this multiplicative constant is to simplify the argument in Section 3.5.2, but it is not strictly necessary.

For $s \neq 0$, since $F_s$ is pluriharmonic in $V_s$, the metric $\omega_{t,s}$ is Ricci-flat in $V_s$. By construction, the functions $\psi_{t,s}$ are Hölder continuous on $X_s$ and have uniform rates of decay $\mu + 2$ at infinity.

Hence we can consider the space

$$\mathcal{TS} := \{(t,s) \in [0,1] \times \Delta \,|\, \exists\, \psi_{t,s} \in \mathcal{U}_s\},$$

where $\mathcal{U}_s$ is the space of $C^{k,\alpha}$-regular functions on $X_s$ such that they have the prescribed behaviour at infinity; if $s = 0$, we request that the functions are $C^{k,\alpha}$-regular on the smooth part and they also have the prescribed behaviour closed to the singularity.

Note that we already know that $([0,1] \times (\Delta \backslash \{0\})) \cup \{(1,0)\} \subseteq \mathcal{TS}$, because we have constructed $\omega_{1,0}$ and we can solve the equations on smooth asymptotically conical manifolds by Conlon & Hein (2013a).

The main continuity path that we consider is the one at the singular fibre $X_0$. Hence we want the space

$$\mathcal{T} := \{t \in [0,1], | \, \exists \, \psi_{t,0} \in \mathcal{U}_0\}$$

to be open and closed. We know already that $1 \in \mathcal{T}$, by the construction of $\omega_{1,0}$.

Section 3.4 will be dedicated to the proof of the openness of $\mathcal{T}$. In Section 3.5 we will prove that the set is also closed.

## 3.4 Openness

In this section, we deal with the openness of the set $\mathcal{T}$ as in Section 3.3 and the choice of a suitable space $\mathcal{U}$ for the continuity path.

As in the proof of the Calabi–Yau theorem, the proof will rely on an application of the *Implicit Function Theorem* in suitable Banach spaces.

Solving the Monge–Ampère equation can be indeed rephrased as controlling the surjectivity of the operator

$$\varphi \mapsto \mathcal{M}(\varphi) := \log \frac{(\omega + dd^c\varphi)^n}{\omega^n},$$

on a suitable space $H$ and towards a suitable space $F$. To prove openness, it suffices to prove that this operator is locally invertible; to do this, we can use the Implicit Function Theorem for Banach spaces, *c.f.* Section 1.2 in Aubin (1998), Theorem 2.5.7 in Marsden et al. (2002), or Theorem 4.1.9. However, unlike the smooth case, we will have to keep track of some additional issues.

Firstly, to control the polynomial decay of these metrics, one has to work with *weighted Hölder spaces* both at infinity and close to the singularity. Moreover, when linearising the equation, it turns out that one has to study the invertibility of the Laplace operator $\Delta$ (which is the linearisation of the Monge–Ampère operator) in such weighted spaces. Now, focusing on the singular part, since in this case harmonic functions are not only the constants, if $f = \text{const.} + O(r^{\varepsilon})$ for some $\varepsilon > 0$, then

$$\Delta^{-1} f = h_0 + \cdots + h_I + \text{const.} \cdot r^2 + O(r^{2+\varepsilon}),$$

where each $h_i$ is harmonic and $h_i \sim r^{\mu_i}$ for some $\mu_i \in [0, 2]$. This is a priori a problem: if $\mu_i \in [0, 2)$, $dd^c h_i$ may blow up for $r \to 0$, and if $\mu_i = 2$, the term $dd^c h_i$ may force us to change the cone model along the path.

These issues will be solved through a fundamental lemma that shows that for $\mu_i \in [0, 2)$, $h_i$ is actually pluriharmonic (and hence does not change the metric), and for $\mu_i = 2$, $dd^c h_i$ corresponds to an automorphism of the cone (hence, the cone does not change).

For what concerns the part at infinity, we again have to check some property of $\Delta$, to make sure to get the right rates to be able to invert the Laplace operator. We can divide the problem into three cases, namely when the rate of convergence of $f$ at infinity $\mu$ is in the interval $(-2, 0)$, when it is in the interval $(-2n, -2)$ and when it is less than $-2n$. We can avoid taking care of the first case thanks to Lemma 3.2.3. The other two cases are dealt with in a manner similar to van Coevering (2009).

The theorem we want to prove is thus the following.

**Theorem 3.4.1** (Openness). *Let $X$ be an asymptotically conical Kähler manifold of dimension $n \geq 2$ with $x \in X$ an isolated canonical singularity and trivial canonical bundle. Assume that the germ $(X, x)$ is isomorphic to a neighbourhood of the vertex in some Calabi–Yau cone $(C_0, \omega_0)$ and that the cone model at infinity is a Calabi–Yau cone $(C_\infty, \omega_\infty)$.*

*Fix a holomorphic volume form $\Omega$ on $X$. Let $F \colon X \backslash \{x\} \to \mathbb{R}$ be a smooth function which is pluriharmonic in a neighbourhood of $x$. Assume there exists an asymptotically conical Kähler metric $\omega$ on $X$, polynomially decaying in a holomorphic gauge $P$ to the conical metric $P^*\omega_0$ in a neighbourhood of $x$, such that*

$$\omega^n = e^F i^{n^2} \Omega \wedge \overline{\Omega}.$$

*Then for some $\delta_0 > 0$ and all $\delta \in (-\delta_0, \delta_0)$ there exists a Kähler metric $\omega_\delta$ representing the same Kähler class as $\omega$ such that*

$$\omega_\delta^n = c_\delta e^{(1+\delta)F} i^{n^2} \Omega \wedge \overline{\Omega},$$

*where the constant $c_\delta$ is determined by integrating both sides over $V$, a neighbourhood* *of the singularity $x \in X$, and $\omega_\delta$ is again polynomially decaying to the conical metric $P_\delta^* \omega_0$ in a neighbourhood of $x$, eventually for a different holomorphic gauge $P_\delta$.*

*Remark* 3.4.2. Recall that by Lemma 3.2.3 we can assume without loss of generality that the Ricci potential $F$ has rate of decay $\mu < -2$ at infinity.

The theorem is proved if we show that the Monge–Ampère operator $\mathcal{M}$ is locally invertible in our context. To do this, we first need to understand what are the spaces $\mathcal{U}, \mathcal{F}$ such that

$$\mathcal{M} \colon \mathcal{U} \to \mathcal{F}$$

is adequate to solve our problem, *i.e.* such that

$$d\mathcal{M}|_{u=0} \colon T_0\mathcal{U} \to T_0\mathcal{F}$$

is an isomorphism. In other words, since $d\mathcal{M}|_{u=0} = \frac{1}{2}\Delta_\omega$, we need to consider the Poisson equation

$$\Delta_\omega \varphi = f,$$

with $\varphi \in T_0\mathcal{U}, f \in T_0\mathcal{F}$ (where $\mathcal{U}, \mathcal{F}$ are yet to be defined).

### 3.4.1 Linear analysis

We cover some preliminary definitions and results that deal with the injectivity and surjectivity of the Laplace operator $\Delta$ on the asymptotically conical Calabi–Yau manifolds that we consider.

**Definition 3.4.3** (Conifold)**.** *A* Riemannian manifold with isolated conical singularities *(or* conifold*) is a Riemannian manifold $(M, g)$ without boundary such that $M = M_0 \sqcup M_1 \sqcup \cdots \sqcup M_K$ $(K \in \mathbb{N})$ such that $M_0$ is the closure of a domain with smooth boundary and such that for all $k \in \{1, \dots, K\}$ there exists a Riemannian cone $C_k$ with radius function $r$ and a diffeomorphism $P_k \colon U_k \to M_k$, where $U_k = \{r < 1\} \subseteq C_k$, such that*

$$\left|\nabla^j \left(P_k^* g - g_{C_k}\right)\right|_{g_{C_k}} \leq c_k r^{\lambda_k - j} \qquad \forall\, j \in \mathbb{N},$$

*for some constants $c_k, \lambda_k > 0$. If $M_0$ is compact, the conifold $M$ will be called* compact*.*

Note that if $X$ is an asymptotically conical Kähler manifold with $x \in X$ an isolated canonical singularity with a metric that is polynomially asymptotic to the one of the cone close to the singularity, then it is a conifold.

**Definition 3.4.4.** *Let $M$ be a conifold of dimension $n > 3$ with $x \in M$ an isolated canonical singularity modelled on a cone $(C_0, g_0)$. We define the norm*

$$||u||_{C^{k,\alpha}_{\nu,\mu}(M)} := ||u||_{C^{k,\alpha}_{\mu}(M\setminus V)} + ||u \circ P||_{C^{k,\alpha}_{\nu}(V)},$$

*where $V$ is a neighbourhood of $x$ and the first norm takes into account the rate of decay $\mu$ at infinity (see Definition 2.3.2), while the second takes into account the rate of decay $\nu$ towards $x$.*

*We then define $C^{k,\alpha}_{\nu,\mu}(M)$ as the subset of $C^{k,\alpha}(M)$ consisting of functions $u$ with finite $||u||_{C^{k,\alpha}_{\nu,\mu}(M)}$. These are functions in $C^{k,\alpha}(M^{reg})$ that can be extended to the singular point as described in Section 2.4.*

*We will denote by $C^{k,\alpha}_{*,\mu}(M)$ (resp. $C^{k,\alpha}_{\nu,*}(M)$) the subset of functions in $C^{k,\alpha}(M)$ such that $||u||_{C^{k,\alpha}_{\mu}(M\setminus V)}$ (resp. $||u \circ P||_{C^{k,\alpha}_{\nu}(V)}$) is finite.*

**Proposition 3.4.5.** *Let $(M, g)$ be a connected asymptotically conical conifold of dimension $m \geq 3$. Then*

$$||u||_{\frac{2m}{m-2}} \leq c||\nabla u||_2$$

*for all $u \in C^1_0(M)$ with $\frac{u}{r} \in L^2$ at the singularity.*

*Proof.* This can be proven by following the exact same reasoning as in the compact case Proposition 2.5 in Hein & Sun (2017). The main point is that we can get a local Sobolev inequality near the singularity given the assumption $\frac{u}{r} \in L^2$, *c.f.* Lemma 2.3 in Hein & Sun (2017), and a Sobolev inequality on asymptotically conical manifolds, *i.e.* Theorem 2.6.3.

□

**Definition 3.4.6.** *Let $B \subset M$ a geodesic ball of radius $\rho$. Define the* rescaled *norm as*

$$||u||_{C^{k,\alpha}_{sc}(B)} := \rho^{k+\alpha} \sum [\nabla^k u]_{C^{0,\alpha}} + \sum_{j=0}^{k} \rho^j ||\nabla^j u||_{L^\infty(B)}.$$

**Theorem 3.4.7.** *Let $M$ be a connected conifold of real dimension $n \geq 3$. For all $k \in \mathbb{N}$ and $\alpha \in (0, 1)$, there exists a constant $c = c(k, \alpha)$ such that the following is true. For all $f \in L^\infty_\mu(M) \cap C^{k,\alpha}_{\nu,\mu}(M)$ with $\mu \in (-n, -2)$, $\nu > -2$, there exists a solution $u \in L^\infty_{\mu+2}(M)$ with $\Delta u = f$. Moreover,*

$$||u||_{L^\infty_{\mu+2}} \leq c||f||_{L^\infty_\mu}$$

*and*

$$||u||_{C^{k+2,\alpha}_{sc}(B/2)} \leq c\left(||u||_{L^\infty(B)} + ||f||_{C^{k,\alpha}_{sc}(B)}\right).$$

*Proof.* The second inequality follows by standard elliptic theory; one can apply the standard Schauder estimates on a ball of radius one and then argue by rescaling, since by the definition of conifold the metric close to the singularity is the same as the one on the cone up to polynomial error.

As for the existence, fix a smoothly bounded domain $\Omega \Subset M$. Then there exists a unique $v \in C^{k+2,\alpha}(\overline{\Omega})$ with $\Delta v = f$ and $v|_{\partial\Omega} \equiv 0$. Extend $v$ by zero to the whole of $M$ and let $u = v$. Supposing the first inequality is proved, we can construct a global solution by considering a sequence of domains $\Omega$ that exhaust $M$. Note that we cannot construct a harmonic function this way, as then $v$ would be identically zero for all domains $\Omega$.

Let us then prove the inequality. First, we can secure a bound

$$||u||_{L^\infty(V)} \leq c||f||_{L^\infty(V)},$$

where $V$ is a neighbourhood of the singularity $x$, via the Moser iteration carried out in Proposition 2.7 of Hein & Sun (2017). To set up this Moser iteration, we need to use the Sobolev inequality on the singular region given by Proposition 3.4.5.

Moreover, we can secure a bound

$$||u||_{L^\infty_{\mu+2}(M\setminus V)} \leq c||f||_{L^\infty_{\mu}(M\setminus V)}$$

as in Section 3 of Hein (2011), by employing yet another Moser iteration, again thanks to the Sobolev inequality of Theorem 2.6.3.

Combining the two inequalities yields the desired estimate.

□

**Proposition 3.4.8.** *Assume $M$ has only one conical singularity $x \in M$. Let $K \subset M$ be a compact set containing the singularity $x$. With the same assumptions as above, if*

- $\mu \in (-n, -2)$. *If $\Delta u = f$ on $M\setminus K$, then we have an inequality*

$$||u||_{C^{k+2,\alpha}_{\mu+2}(M\setminus K)} \leq C\left(||f||_{C^{k,\alpha}_{\mu}(M\setminus K)}\right);$$

- $\mu \in (-n-\delta, -n)$ *with $0 < \delta \ll 1$. If $\Delta u = f$ on $M\setminus K$, then on $M\setminus K$ we can write $u = A\rho^{2-n} + v$ for a constant $A$ depending on $f$ and we have an* *inequality*

$$||v||_{C^{k+2,\alpha}_{\mu+2}(M\backslash K)} \leq C\left(||f||_{C^{k,\alpha}_{\mu}(M\backslash K)}\right).$$

*Proof.* For reference in the smooth asymptotically conical case, check van Coevering (2009), Theorem 2.30.

For the first point, we have $|f(x)| \leq ||f||_{C^0_\mu}\rho(x)^\mu$, where $\rho$ is a radius function for the cone at infinity. Define

$$u(y) = \int_{M\backslash K} G(y,x)f(x)\,\mathrm{dvol}_g(x),$$

where $G$ is the Green's function on $M\backslash K$.

By van Coevering (2009), Theorem 2.29, we get

$$|u(y)| \leq C||f||_{C^0_\mu}\int_{M\backslash K} d(y,x)^{2-n}\rho(x)^\mu\,\mathrm{dvol}_g(x),$$

where $d$ is the distance function. Now, note that for $x\to\infty$, $\rho(x)$ can be approximated with $d(x,y)$. Thus since $\mu\in(-n,-2)$,

$$\int_{M\backslash K} d(y,x)^{2-n}\rho(x)^\mu\,\mathrm{dvol}_g(x) \leq C'\rho(y)^{\mu+2},$$

hence the point is proven.

For the second point, let us assume first that $\int_{M\backslash K} f\,\mathrm{dvol}_g = 0$. Then the equation $\Delta u = f$ is solved on $M\backslash K$ by means of a Green's function:

$$u = \int_{M\backslash K}\left(G(y,x) - G(y,0)\right)f(x)\,\mathrm{dvol}_g(x),$$

where $0\in M\backslash K$ is an arbitrarily chosen point. Note that

$$\int_{M\backslash K} G(y,0)f(x)\,\mathrm{dvol}_g(x) = 0$$

because $f$ has zero mean value. Then we get the estimate

$$\begin{aligned}|u(y)| &\leq C||f||_{C^0_\mu}\int_{M\backslash K} d(x,y)^{2-n-\varepsilon}d(0,x)^\varepsilon\rho(x)^\mu\,\mathrm{dvol}(x)\\ &\leq \begin{cases} C\rho(y)^{\mu+2} & \mu\in(-n-\varepsilon,-2);\\ C\rho(y)^{2-n-\varepsilon} & \mu\leq -n-\varepsilon;\end{cases}\end{aligned}$$

by reasoning similar to the first point. This entails that $u \in C^0_{\mu+2}$ if $\mu \in (-n-\varepsilon, -n)$.

In general, we have

$$\int_{M\setminus K} \left(f - A\Delta(\rho^{2-n})\right) \mathrm{dvol}_g = 0,$$

for some constant $A \in \mathbb{R}$, *c.f.* D. Joyce (2000), Theorem 8.5.1 for the ALE case, and its generalisation in the asymptotically conical case in van Coevering (2009), Theorem 2.30.

Then by what we have seen before, there exists $v \in C^{k+2,\alpha}_{\mu+2}$ such that $\Delta v = f - A\Delta(\rho^{2-n})$, and thus

$$||v||_{C^{k+2,\alpha}_{\mu+2}} \leq C\left(||f||_{C^{k,\alpha}_{\mu}} + ||A\Delta(\rho^{2-n})||_{C^{k,\alpha}_{\mu}}\right).$$

One can find that

$$|A| \leq \frac{1}{(n-2)V}\int_{M\setminus K} |f|\,\mathrm{dvol}_g,$$

where $V$ is the volume of the link of the cone at infinity. Hence

$$|A| \leq \frac{||f||_{C^0_\mu}}{(n-2)V}\int_{M\setminus K} \rho^\mu\,\mathrm{dvol}_g \leq C_1||f||_{C^0_\mu},$$

where the constant $C_1$ is finite since $\mu \leq -n$.

□

**Theorem 3.4.9.** *Let $(X,\omega)$ be an asymptotically conical Kähler conifold of dimension $n \geq 2$ with $x \in X$ an isolated conical singularity and trivial canonical bundle. Assume that the germ $(X,x)$ is isomorphic to a neighbourhood of the vertex in some Calabi–Yau cone $(C_0,\omega_0)$ and that the cone model at infinity is a Calabi–Yau cone $(C_\infty,\omega_\infty)$.*

*Let $\lambda > 0$ be such that*

$$|\nabla^j_{\omega_0}\Delta_{P^*\omega}(r^{\mu_i}\phi_i)|_{\omega_0} = O(r^{\mu_i-2+\lambda-j}),$$

*where $\{r^{\mu_i}\phi_i\}_i$ is a basis of harmonic functions on $(C_0,\omega_0)$. Then there exist a $\nu' \in (-2,\lambda]$ such that if*

$$f = \overline{f} + f_0,\ \overline{f} \in C^{k,\alpha}_{\nu,\mu}(X),\ f_0 \in \mathbb{R},$$

*then the unique bounded solution $u$ to $\Delta_\omega u = f$ satisfies*

$$u = \overline{u} + \chi\left(\frac{f_0}{2n}r^2 + h\right)\circ P^{-1} \qquad \text{if } \mu \in (-2n, -2),$$
$$u = \overline{u} + \chi\left(\frac{f_0}{2n}r^2 + h\right)\circ P^{-1} + A\chi_\infty \rho^{2-2n}\circ \Phi^{-1} \qquad \text{if } \mu \in (-2n-\delta, -2n),$$

*where $\chi$ is a cut-off function that is equal to 1 close to $x$, $h$ is a harmonic function on the model cone at the singularity of rate in $[0,2]$ and $\chi_\infty$ is a cut-off function that is equal to 1 at infinity. Moreover,*

$$||\overline{u}||_{C^{k+2,\alpha}_{\nu'+2,\mu+2}(X)} \leq C\left(||\overline{f}||_{C^{k,\alpha}_{\nu,\mu}(X)} + |f_0|\right).$$

For $\mu \in (-2n, -2)$, the solution $u$ exist by Theorem 3.4.7. We refer to Theorem 2.11 in Hein & Sun (2017) for the remainder of the proof, which consists of local arguments around the singularity. The behaviour at infinity is given by Proposition 3.4.8.

We explain the behaviour of solutions to $\Delta_\omega \varphi = f$ close to the singularity in the following. Let us suppose we work on $V = \{r < 1\} \subseteq C_0$, since the Laplace operator on the manifold near the singularity is but a perturbation of that on the cone. Write

$$\varphi(r,y) = \sum_i \varphi_i(r)\phi_i(y), \qquad f(r,y) = \sum_i f_i(r)\phi_i(y),$$

where $\{\phi_i\}_i$ is a Fourier basis consisting of eigenfunctions of the Laplacian on the link $S$ of $C_0$, with respective eigenvalues $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \dots$. These extend to homogeneous harmonic functions $r^{\mu_i^\pm}\phi_i$ on $C_0$; to determine the value of $\mu_i^\pm$, we can solve the homogeneous equation

$$\Delta(r^{\mu_i^\pm}\phi_i) = 0,$$

rewriting it as

$$(r^{\mu_i^\pm}\phi_i)'' + \frac{m-1}{r}(r^{\mu_i^\pm}\phi_i)' - \frac{\lambda_i}{r^2}(r^{\mu_i^\pm}\phi_i) = 0,$$

where by $(\,\cdot\,)'$ we mean differentiation by $r$. We thus get

$$\mu_i^\pm = -\frac{m-2}{2} \pm \sqrt{\frac{(m-2)^2}{4} + \lambda_i}.$$

We apply the same reasoning to get the equations

$$\varphi_i'' + \frac{m-1}{r}\varphi_i' - \frac{\lambda_i}{r^2}\varphi_i = f_i,$$

from which one gets, with some work,

$$\varphi_i(r) = A_i^{\pm} r^{\mu_i^{\pm}} + \overline{\varphi}_i(r),$$

where $\overline{\varphi}_i(r) \in C^{k+2,\alpha}_{\nu+2}$ if $f_i \in C^{k,\alpha}_{\nu}$.

To choose viable values of $\nu$, we can reason as follows. Clearly, we must ask for $\nu$ to be less or equal than $\lambda$, that is, the rate in Definition 3.4.3, as then the solution would satisfy the right decay. Heuristically, we have to consider how $\nu$ relates to $\mu_i^{\pm}$. Firstly, we cannot choose a solution containing a multiple of $r^{\mu_i^-}\phi_i$, as $\mu_i^- < 0$ and thus the solution would blow-up for $r \to 0$: this wouldn't have geometric meaning. Moreover, recall that we are solving the equation on $V$, thus we need to specify the behaviour of the solution on $\partial V = \{r = 1\}$.

Thus we are left with two cases:

- if the rate $\nu$ of $f$ is between 0 and $\mu_1^+ - 2$, everything is fine: we have one degree of freedom for each $i$ given by the choice of the constant $A_i$, and one constraint given by boundary condition. In this case the operator $\Delta$ is thus invertible (up to some normalisations);
- if the rate $\nu$ of $f$ is greater than $\mu_1^+ - 2$, we have a problem since $r^{\mu_1^+ - 2} \neq O(r^{\nu})$ and hence this doesn't allow the metric to be conical, as the Laplace operator with arbitrary boundary condition is surjective no longer (we cannot use the degree of freedom given by the harmonic part $A_1 r^{\mu_1^+}$).

Generally speaking, there is no reason $\mu_1^+$ must be greater than 2, so we cannot always choose $\nu$ in the first range. However, in our case, we prove that the contributions of the harmonic functions $r^{\mu_i^+}\phi_i$ with rate $\mu_i^+ \leq 2$ fit well in the geometric picture (*i.e.* they don't make the solution blow up after taking $dd^c$). For this, we have the following theorem.

**Theorem 3.4.10.** *Consider a non-flat Kähler Riemannian cone with non-negative Ricci curvature. Then*

1. *if $u$ is a real-valued $\mu$-homogeneous harmonic function with $\mu > 0$, then $\mu > 1$; if $1 < \mu < 2$, then $dd^c u = 0$. If $\mu = 2$, then $u = u_1 + u_2$ where $u_1, u_2$ are*

*2-homogeneous,* $dd^c u_1 = 0$ *and* $u_2$ *is* $\xi$*-invariant (where* $\xi$ *is the Reeb vector field of the cone);*

2. *if a real-valued* $\mu$*-homogeneous harmonic function with* $\mu > 0$ *is* $\xi$*-invariant, then* $\mu \geq 2$ *with equality if and only if* $\operatorname{Ric} \nabla u = 0$ *and* $\nabla u$ *is a holomorphic vector field;*

3. *if the cone is Ricci-flat, the space of all holomorphic vector field that commute with the dilation* $r\partial_r$ *can be written as* $\mathfrak{p} \oplus J\mathfrak{p}$ *where* $\mathfrak{p}$ *is spanned by* $r\partial_r$ *and vector field of the type* $\nabla u$ *as in the point above. All elements in* $J\mathfrak{p}$ *are moreover Killing vector fields.*

The upshot is that in the reasoning above we can ignore the rates $\mu_i^+ \leq 2$, and substantially replace $\mu_1^+$ with $\mu_1^{>2}$, that is, the first harmonic rate strictly greater than 2 (and thus one can always choose $\nu$ between 0 and $\mu_1^{>2} - 2$).

### 3.4.2 Proof of Theorem 3.4.1

In this section we prove the openness of the continuity path.

The proof is pretty similar to the one in the compact case carried out in Hein & Sun (2017), so we will only outline the passages that differ. In particular, we will focus on defining the correct function spaces required to apply the inverse function theorem.

For this, we will make use of Theorem 3.4.9, which is a restatement of Theorem 2.11 in Hein & Sun (2017) taking into account the structure at infinity. Substantially, we only need to chose the right rates at infinity, as shown in the smooth case in Hein (2011).

In more generality, by Proposition 3.4.9 we have that

$$\Delta \colon C^{k+2,\alpha}_{*,\beta+2} \to C^{k,\alpha}_{*,\beta}$$

is Frehdolm if and only if $\beta + 2 \notin \mathcal{P}_\infty$, where $\mathcal{P}_\infty$ is the set of exceptional weights given by the rates of eigenfunctions of the Laplacian on the cone $C_\infty$.

We have then two cases to consider, when the rate of decay at infinity $\mu$ is between $-2n$ and $-2$, and when $\mu$ is less than $-2n$. Clearly, the second case is contained in the first, as a function with rate at infinity $\mu$ in particular has also rate at infinity $\mu'$ for all $\mu' > \mu$; we can however be more precise in the description of the solution by adding harmonic functions on the cone at infinity. We write this description for the first harmonic function, which has the form $\sim A\rho^{2-2n}$, where $\rho$ is the distance function on the cone $C_\infty$, hence for rates in $(-2n-\delta, -2n)$ for $\delta$ small enough.

Note that by Remark 3.4.2, these are the only possible cases.

- Case $\mu \in (-2n, -2)$.

  In this case, by Theorem 3.4.9, a solution to $\Delta u = f$ exists and is unique and if

$$f = \overline{f} + f_0, \quad \overline{f} \in C^{k,\alpha}_{\nu,\mu}(X),\ f_0 \in \mathbb{R},$$

  then

$$u = \overline{u} + \chi\left(\frac{f_0}{2n} r^2\right) \circ P^{-1},$$

  where $\chi$ is a cut-off function that is equal to 1 close to $x$ and equal to 0 outside a compact subset. We also have the estimate

$$||\overline{u}||_{C^{k+2,\alpha}_{\nu+2,\mu+2}} \leq C\left(||\overline{f}||_{C^{k,\alpha}_{\nu,\mu}} + |f_0|\right).$$

  This enables us to choose the following as function spaces. Let $u, f\colon X\backslash\{x\} \to \mathbb{R}$ be of type $U, F$ respectively if

$$u = \overline{u} + \chi(p + \frac{1}{2}(r^2 \circ \Psi - r^2)) \circ P^{-1},$$

  and

$$f = \overline{f} + f_0,$$

  with $\overline{u} \in C^{k+2,\alpha}_{\nu+2,\mu+2}(X)$, $p \in \mathcal{P}$, $\Psi \in G$, and $\overline{f} \in C^{k,\alpha}_{\nu,\mu}(X)$, $f_0 \in \mathbb{R}$. Here $\mathcal{P}$ is the vector space spanned by the homogeneous pluriharmonic functions with growth rate in $[0, 2]$ on the cone $C_0$, and $G$ is the identity component of the group of homolorphic automorphisms of $C_0$ commuting with $r\partial_r$, so that $\mathrm{Lie}(G) = \mathfrak{p} \oplus J\mathfrak{p}$.

  Then define

$$\mathcal{U} := \{u\colon X^{reg} \to \mathbb{R} \,|\, u \text{ of type U}, \omega + dd^c u > 0\}$$
$$\mathcal{F} := \{f\colon X^{reg} \to \mathbb{R} \,|\, f \text{ of type F}\},$$

  and let

$$\mathcal{M}(u) := \log \frac{(\omega + dd^c u)^n}{\omega^n}$$

be the Monge–Ampère operator.

We have that $\mathcal{U}, \mathcal{F}$ have $C^1$ Banach structures and $\mathcal{M}$ is a $C^1$ map from $\mathcal{U}$ to $\mathcal{F}$. The argument is the same as in Theorem 2.19 in Hein & Sun (2017), writing $\mathcal{U}$ as the image of a certain map $S$ and making use of the rank theorem (Theorem 2.5.15 in Marsden et al. (2002)) to show that this image is $C^1$.

Then

$$\begin{aligned} T_0\mathcal{U} &= \left\{u \in C^{k+2,\alpha}_{\nu+2,\mu+2}(X) \oplus \chi(\mathcal{P} \oplus \mathbb{R}r^2 \oplus \mathcal{H}) \circ P^{-1}\right\} \\ T_0\mathcal{F} &= \left\{f \in C^{k;\alpha}_{\nu,\mu}(X) \oplus \mathbb{R}\right\}, \end{aligned}$$

where $\mathcal{H}$ is the space of all $\xi$-invariant 2-homogeneous harmonic functions on $C_0$, where $\xi$ is the Reeb vector field of $C_0$. This comes from differentiating the Lie group $G$. We need to check for which rates $\nu, \mu$, the linearised operator $\Delta\colon T_0\mathcal{U} \to T_0\mathcal{F}$ is an isomorphism. These are given by Theorem 3.4.9.

- Case $-2n - \delta < \mu < -2n$.

In this case we cannot expect $u$ to have rate of decay $\mu + 2$, but this is true up to a term $\sim A\rho^{2-2n}$, where $\rho$ is the distance function on the cone $C_\infty$. This is because we need to add harmonic functions on the cone at infinity to gain surjectivity of the operator.

Hence we need to modify the space $\mathcal{U}$ so as to include this function; this is done by changing the “type $U$” functions so as to be of the form

$$u = \overline{u} + \chi(p + \frac{1}{2}(r^2 \circ \Psi - r^2)) \circ P^{-1} + \chi_\infty \rho^{2-2n} \circ \Phi^{-1},$$

where $\Phi$ is the diffeomorphism of the asymptotically conical structure, so that the tangent space to $\mathcal{U}$ takes the form

$$T_0\mathcal{U} = \left\{u \in C^{k+2,\alpha}_{\nu+2,\mu+2}(X) \oplus \chi(\mathcal{P} \oplus \mathbb{R}r^2 \oplus \mathcal{H}) \circ P^{-1} \oplus \chi_\infty \mathbb{R}\rho^{2-2n} \circ \Phi^{-1}\right\},$$

and by the uniqueness result discussed above, the Laplacian is an isomorphism.

In this case we alter the metric. Compare the smooth case in Conlon & Hein (2013a) in which the case $\mu \in (-2n - \delta, -2n)$ for the Monge–Ampère equation has indeed, in the same way, a solution in $\mathbb{R}\rho^{2-2n} \oplus C^{k+2,\alpha}_{*,\mu+2}$.

# 3.5 Closedness

In this section, we deal with the closedness of the set $T$ as in Section 3.3. We will secure a priori estimates and describe how the "bubbling" effect doesn't take place in our situation.

## 3.5.1 Passing to the limit

As we noted before, the existence of $\psi_{t,0}$ is not guaranteed for any $t \in [0,1)$. The following theorem will take care of the existence provided a priori estimates on the potentials. The existence will be in a suitable weak sense so as to take care of the singularity $x \in X_0$.

**Theorem 3.5.1.** *Let $\mathcal{K} \subset \mathcal{X} \backslash \{x\}$ be a compact subset, and let $K_s := \mathcal{K} \cap X_s$. Suppose that we have a priori estimates*

$$||\psi_{t,s}|_{K_s}||_{C^{k,\alpha}} \leq C_{k,\alpha},$$
$$||\psi_{t,s}|_{K_s}||_{C^{k,\alpha}_\mu} \leq C^\mu_{k,\alpha},$$

*for all $\mathcal{K}$, where $C_{k,\alpha}$ and $C^\mu_{k,\alpha}$ are independent of $(t,s)$, but depend on $\mathcal{K}$. Suppose moreover we have a priori estimates*

$$||\psi_{t,s}||_{C^0} \leq C, \quad C^{-1}\hat{\omega}_s \leq \omega_{t,s}$$

*for $C > 0$ is a constant constant independent of $t, s$ and the compact $\mathcal{K}$.*

*Then for all $t_i \to t_\infty$ and $s_i \to 0$ there exist a subsequence $(t_{i_k}, s_{i_k})$, such that we have a metric $\omega_{t_\infty,0} = \hat{\omega}_0 + dd^c\psi_{t_\infty,0}$ on $X_0^{reg}$ and a smooth family of embeddings $\hat{\Phi}_s\colon X_0^{reg} \to X_s$. These embeddings are the identity for $s = 0$ and diffeomorphisms with the images in general, and $d\hat{\Phi}_s|_K^{-1} \circ J_s \circ d\hat{\Phi}_s|_K$ converges in the $C^\infty$ sense to $J_0$, where $J_s$ is the complex structure on $X_s$.*

*These data are then such that*

$$\hat{\Phi}_s|_K^* \psi_{t_{i_k}, s_{i_k}} \to \psi_{t_\infty,0}$$

*in the $C^\infty_\mu$ sense on $K$, for all compact subsets $K \subset X_0^{reg}$. Moreover, $\omega_{t_\infty,0}$ can be extended on $X_0$ in the sense of currents and*

$$\omega^n_{t_\infty,0} = e^{F_{t_\infty,0}} i^{n^2} \Omega_0 \wedge \overline{\Omega}_0.$$

*Remark* 3.5.2. While the main analytic argument in Theorem 3.5.1 will involve uniform estimates on closed subsets, we manage to get the lower bound $C$ on $\omega_{t,s}$ with respect to $\hat{\omega}_s$ as well as the uniform $C^0$ bound to be uniformly positive and independent of the closed subset $\mathcal{K}$.

The uniform $C^0$ bound is necessary to get a Kähler metric in the limit in the sense of currents, so that the potential $\psi_{t_\infty,0}$ for the singular metric is bounded on $X_0$. Note that even with uniform $C^0$ bound it is not obvious that the potential extends continuously, as already mentioned in Eyssidieux et al. (2009), Theorem A; in the case of compact irreducible Calabi–Yau manifolds, by recent results in Guedj et al. (2023), the potential is also continuous.

As for the $C^2$ bound, if such bound is not uniform from below we would not be able to extend the metric to the singular point $x$ in the sense of Definition 2.4.4. This is because on the exceptional divisor on a resolution, the potential $\psi_{t_\infty,0}$ would not be *strictly* $\hat{\omega}_s$-plurisubharmonic on $X_0$ in the sense of Definition 2.4.1.

*Proof.* We want to adapt Theorem 1.4 of Rong & Zhang (2011). The steps are similar to the cylindrical case in G. Chen (2019).

On $K_0 \cap \{\rho \leq R\}$, where $R$ is a fixed constant and $\rho$ is the distance function on the cone at infinity, one can follow the same arguments as Rong & Zhang (2011), using the $C^\infty_{loc}$ estimates we have as hypothesis. See also Proposition 3.4 of G. Chen (2019) for the cylindrical equivalent. Roughly, given the sequence of embeddings and that we have $||\psi_{t,s}|_{K_s}||_{C^{k,\alpha}} \leq C_{k,\alpha}$, via Ascoli–Arzelà we can find a limit $\Phi^*\psi_{t_k,s_k} \to \psi_{t_\infty,0}$.

For $R_0 \geq R$ one can restrict oneself to $\{R_0 \leq \rho \leq R_0 + 1\}$; here again we have a uniform $C^\infty$ bound (found above using the $C^{k,\alpha}$ estimates and bootstrapping), and hence we can find a converging subsequence there; moreover, by the $C^{k,\alpha}_\mu$ bounds at infinity, we can actually find that the convergence is in $C^\infty_\mu$.

We claim that the limit satisfies the equation.

First, consider a resolution of singularities of $X$, with $E$ the exceptional divisor over $x$. The divisor $E$ is a pluripolar set, *i.e.* we can locally write $E = \{u = -\infty\}$ for a plurisubharmonic function $u$. In this case, the function $u(z)$ can be locally given by $\log(|f(z)|^2)$, where $f$ is the holomorphic function that locally defines the divisor. By Proposition 5.24 in Demailly (2012), since $E$ is pluripolar, we can extend $\psi_{t_\infty,0}$ on $E$ by the $C^0$ estimate $||\psi_{t_\infty,0}||_0 \leq C$, getting an extension $\psi_{t_\infty,0} \in PSH(X_0, \omega_{t_\infty,0}) \cap L^\infty(X_0)$. The extension is build in the following way: without loss of generality, we can assume that the function $u$ locally defining the pluripolar set $E$ is non-positive. The extension will be locally given by $\left(\sup_\varepsilon\left(\psi_{t_\infty,0} + \varepsilon u\right)\right)^*$, where for a function $u$,

$$u^*(z) := \lim_{\epsilon \to 0} \sup_{\mathbb{B}_(z)} u$$

is the regularised upper envelope, which is always upper semicontinuous.

Then, $\omega_{t_\infty,0}$ can be extended on the resolution of $X$ so that the equation is satisfied everywhere but on the exceptional divisor $E$. Here, the pullback of $\omega_{t_\infty,0}$ can be locally written as $dd^c\phi$ with $||\phi||_0 \leq C$ by the a priori estimates. By Proposition 4.6.4 in Klimek (1991), we have that

$$\int_E (dd^c\phi)^n = 0$$

on the exceptional divisor $E$, again due to the fact that $E$ is pluripolar. Note that $(dd^c\phi)^n$ can be defined in the weak sense thanks to the work of Bedford and Taylor in Bedford & Taylor (1976).

Now, $\phi$ can be extended to the resolution by the same reasoning as above. Since the Monge–Ampère mass on the exceptional divisor $E$ is 0, we have that the exceptional divisor doesn't "interfere" with the Monge–Ampère equation when considered in the weak sense. □

Note that it is not obvious that the metrics $\omega_{t_\infty,0}$ share the same conical metric singularity at $x$; this will be a point of later discussion in Section 3.5.5.

Now we want to establish uniform a priori estimates for $\psi_{t,s}$ with $s \neq 0$. These would imply the closedness of the continuity path as we have seen above.

### 3.5.2 $C^0$ estimate

Here we take care of the $C^0$ estimate.

As in the work of Yau's, the main point to get this uniform estimate is to apply a Moser iteration technique. Following Joyce's work on ALE manifolds and the Calabi conjecture in D. Joyce (2000), it is clear that the principal ingredient that one needs to possess in order to have a chance to generalise the estimate in a non-compact setting is the Sobolev inequality

$$||\chi||^2_{L^{4n/(2n-2)}(g_s)} \leq C\left(||d\chi||^2_{L^2(g_s)}\right).$$

for a compactly supported function $\chi$. This is true on smooth asymptotically conical manifolds, as we have seen in Theorem 2.6.3.

**Proposition 3.5.3.** *There is a constant $C > 0$ independent of $t \in [0,1]$ and $s \in \Delta\backslash\{0\}$ such that*
$$||\psi_{t,s}||_{C^0(X_s)} \leq C.$$

We want to use the Sobolev inequality with the metrics $\omega_{1,s}$ as background rather than the metric $\hat{\omega}_s$ since we have control on the Ricci curvature close to the singularity $x$; recall indeed that the $\omega_{1,s}$ have been constructed to be Ricci-flat in $V_s$ for all $s$. Here we can apply Theorem 2.6.6 to get a uniform Sobolev constant for $s \in \Delta\backslash\{0\}$, after checking that the non-collapsing bound applies.

As in the construction of the reference metrics in Section 3.2.2, we can consider three regions of $\mathcal{X}$. These are defined by two compact sets. We have $\mathcal{V}$, the neighbourhood of the singularity $x$ on which the metrics $\omega_{1,s}$ are Ricci-flat, and have a compact set $\mathcal{K}$ such that the metrics $\omega_{1,s}$ are asymptotic to the cone metric on $\mathcal{X}\backslash\mathcal{K} \cap X_s = X_s\backslash K_s$.

We firstly check on the non-compact part $X_s\backslash V_s$. On $\mathcal{X}\backslash\mathcal{K}$ we have uniform asymptotically conical behaviour of the metrics, which entails a uniform non-collapsing bound. On $\mathcal{K}\backslash\overline{\mathcal{V}}$ we have a uniform non-collapsing bound by compactness.

Then we need to check on the compact part $V_s$. Note that this is the more delicate part as it is the one close to the singularity. Note that we have chosen the continuity path to be so that $\mathrm{Vol}_{\omega_{t,s}}(V_s) \equiv V > 0$ for all $s \in \Delta$. Consider now the following theorem, which adapts Theorem 2.1 in Rong & Zhang (2011) and Proposition 3.2 in Hein & Sun (2017).

**Theorem 3.5.4.** *Let $\pi\colon \mathcal{X} \to \Delta$ be a smoothing of a singular asymptotically conical Calabi–Yau manifold $X_0$ as in Theorem 1.2.5, with an isolated canonical singularity $x \in X_0$. Let $\mathcal{V}$ be a neighbourhood of $x$ in $\mathcal{X}$ such that there exist metrics $\omega_{1,s}$ as in the construction in Section 3.2. Then the diameter of $V_s$ satisfies*

$$\mathrm{diam}_{\omega_{1,s}}(V_s) \leq C\,\mathrm{Vol}_{\omega_{1,s}}(K_s) + 2,$$

*where $C > 0$ is a constant independent of $s$.*

*Proof.* Note that by the definition of the potentials $F_{1,s}$ in Proposition 3.2.5 *i)*, we have uniform bounds

$$\mathrm{diam}_{\omega_{1,s}}(K_s\backslash V_s) \leq D_0 \quad \text{and} \quad v \leq \mathrm{Vol}_{\omega_{1,s}}(K_s\backslash V_s) \leq V,$$

since $\mathcal{K}\backslash\overline{\mathcal{V}}$ is a compact region. Moreover, we have a uniform lower bound on the Ricci curvature $\mathrm{Ric}(\omega_{1,s}) \geq -K\omega_{1,s}$ on $K_s \backslash V_s$ again by compactness and the definition of the $F_{1,s}$.

Take now a point $p \in V_s$ and let $d := d_{\omega_{1,s}}(p, \partial V_s)$. Without loss of generality we can assume $d \geq 3$. Consider the geodesic $\gamma$ with respect to $\omega_{1,s}$ such that $\gamma(0) = p$, $\gamma(d) \in \partial V_s$. Denote $q := \gamma(d-1)$. Recall that on $V_s$, we have $\mathrm{Ric}(\omega_{1,s}) \equiv 0$. Then by the Bishop–Gromov inequality we get

$$\begin{aligned}\mathrm{Vol}(\mathbb{B}_{d-2}(p)) &\geq \frac{1}{\left(\frac{d}{d+2}\right)^{2n} - 1} \mathrm{Vol}(\mathbb{B}_d(p) \backslash \mathbb{B}_{d-2}(p)) \\ &\geq \frac{d-1}{C} \mathrm{Vol}(\mathbb{B}_1(q)),\end{aligned}$$

where $C > 0$ is a dimensional constant. Now, from above we know that

$$\mathrm{diam}_{\omega_{1,s}}(K_s \backslash V_s) \leq D_0,$$

hence $K_s \backslash V_s \subseteq \mathbb{B}_{D_0+1}(q)$. Again by Bishop–Gromov, we have

$$\mathrm{Vol}(\mathbb{B}_1(q)) \geq \frac{1}{C} \mathrm{Vol}(\mathbb{B}_{D_0+1}(q)) \geq \frac{1}{C} \mathrm{Vol}_{\omega_{1,s}}(K_s \backslash V_s) \geq \frac{v}{C},$$

where now $C = C(K, D_0) > 0$ also depends on the uniform lower Ricci bound and the diameter bound on $K_s \backslash V_s$. From here we then get

$$C \,\mathrm{Vol}(\mathbb{B}_{d-2}(p)) + 1 \geq d.$$

Now, consider the geodesic $\gamma$ in $V_s$ of length equal to the diameter of $V_s$. The geodesic $\gamma$ joins two points $p_0, p_1$ on $\partial V_s$; if this was not true, the geodesic could be extended in $V_s$ and thus the the diameter would be bigger. Consider the point $p$ on $\gamma$ that is equally distant to $p_0$ and $p_1$. Then $d = d_{\omega_{1,s}}(p, \partial V_s) = \frac{1}{2} \mathrm{diam}_{\omega_{1,s}}(V_s)$. Then

$$\mathrm{diam}_{\omega_{1,s}}(V_s) \leq C \,\mathrm{Vol}(\mathbb{B}_{d-2}(p)) + 2 \leq C \,\mathrm{Vol}(K_s) + 2,$$

since $\mathbb{B}_{d-2}(p) \subseteq K_s$. □

By the theorem we thus get

$$\mathrm{diam}_{\omega_{1,s}}(V_s) \leq C \,\mathrm{Vol}(K_s) + 2 =: D,$$

where $D$ is independent of $s$, since $\mathrm{Vol}(K_s)$ is uniformly bounded by compactness of $\mathcal{K}\backslash\mathcal{V}$ and the choice of volume $\mathrm{Vol}(V_s) \equiv V$ in Section 3.3. Hence we can use Proposition 2.7.1 *ii)* to get a non-collapsing bound $\kappa > 0$ on the region $V_s$.

By choosing the minimum among these numbers we have a uniform non-collapsing bound, hence a uniform Sobolev constant by Theorem 2.6.6. Once we have this uniform Sobolev inequality, we can follow D. Joyce (2000) to secure the estimate via Moser iteration. The procedure is the same as the one in the estimate for the compact smooth case, except that we need to carefully check that the boundary terms when applying Stokes' theorem vanish.

To gain independence of $t$, apply the above using $\omega_{1,s} = \hat{\omega}_s + dd^c\psi_{1,s}$ as background metric, and $\psi_{t,s} - \psi_{1,s}$ as potential. Then the relative Kähler potential $\psi_{t,s} - \psi_{1,s}$ has a $C^0$ bound which is uniform in $t$ and $s$, by the same argument as above and by construction of the density function (Ricci potential) $F_{t,s} := tF_{1,s}$. Now, $\psi_{1,s}$ is uniformly $C^0$-bounded in $s$ by the above. From here it follows that we get a uniform $C^0$ bound for $\psi_{t,s}$. Note that this step is not different in spirit from the one to gain a priori $C^0$ estimates in Yau's original proof in Yau (1978), with the only difference that we shift the background metrics to $\omega_{1,s}$ because we found a uniform Sobolev constant in $s$ for those metrics, *c.f.* Lemma 3.1 in Hein & Sun (2017) and Proposition 3.2 in G. Chen (2019).

### 3.5.3 $C^2$ estimate

This section will take care of the $C^2$ estimate. The argument we follow is substantially the same as in the compact case of Yau Yau (1978), with attention to the uniformity of the estimate, *c.f.* Rong & Zhang (2011) and G. Chen (2019).

The $C^2$ estimate follows by the Chern–Lu inequality argument (see Proposition 7.1 of Rubinstein (2014)), applied to the compact part and the part at infinity, which are both with controlled geometry.

**Proposition 3.5.5** (Chern–Lu inequality)**.** *Let $f\colon (M,\omega) \to (N,\eta)$ be a holomorphic map between Kähler manifolds. Then*

$$\begin{aligned}|\partial f|^2\Delta_\omega \log|\partial f|^2 &= ((\mathrm{Ric}\,\omega)^\sharp \otimes \eta)(\partial f, \overline{\partial} f) - (\omega^\sharp \otimes \omega^\sharp \otimes R_\eta)(\partial f, \overline{\partial} f, \partial f, \overline{\partial} f) + e(f)\\ &\geq ((\mathrm{Ric}\,\omega)^\sharp \otimes \eta)(\partial f, \overline{\partial} f) - (\omega^\sharp \otimes \omega^\sharp \otimes R_\eta)(\partial f, \overline{\partial} f, \partial f, \overline{\partial} f),\end{aligned}$$

*where $e(f) = |\nabla\partial f|^2 - |\partial f|^2|\partial \log|\partial f|^2|^2$ and $R_\eta$ is the full Riemannian curvature tensor of $\eta$.*

The estimate now goes as follows.

**Proposition 3.5.6.** *For any closed subset $\mathcal{K} \subset \mathcal{X} \backslash \{x\}$, there exists a constant $C_K > 0$ independent of $s$ such that on $\mathcal{K} \cap X_s$ we have*

$$C\hat{\omega}_s \leq \omega_{t,s} \leq C_K \hat{\omega}_s,$$

*where $C > 0$ is a constant independent of $s$ and $\mathcal{K}$.*

By Chern–Lu applied to the identity map

$$\xi_{t,s} \colon (X_s, \omega_{t,s}) \to (X_s, \hat{\omega}_s),$$

which is holomorphic and in particular harmonic, we have

$$\Delta_{\omega_{t,s}} \log |\partial \xi_{t,s}|^2 \geq \frac{\operatorname{Ric}(\omega_{t,s})(\partial \xi_{t,s}, \overline{\partial \xi_{t,s}})}{|\partial \xi_{t,s}|^2} - \frac{\operatorname{Sec}(\hat{\omega}_s)(\partial \xi_{t,s}, \overline{\partial \xi_{t,s}}, \partial \xi_{t,s}, \overline{\partial \xi_{t,s}})}{|\partial \xi_{t,s}|^2}.$$

Note that

$$|\partial \xi_{t,s}|^2 = 2 \operatorname{tr}_{\omega_{t,s}} \hat{\omega}_s = \operatorname{tr}_{\omega_{t,s}} (\omega_{t,s} - dd^c \psi_{t,s}) = 2n - \Delta_{\omega_{t,s}} \psi_{t,s},$$

hence finding uniform bounds on $\operatorname{tr}_{\omega_{t,s}} \hat{\omega}_s$ is equivalent to finding uniform bounds for $\Delta_{\omega_{t,s}} \psi_{t,s}$, that is, $C^2$ estimates.

Now, consider $U_s := X_s \backslash K_s$, a neighbourhood at infinity in $X_s$. The sectional curvature $\operatorname{Sec}_{\hat{\omega}_s}$ is uniformly bounded from above on $U_s$ by Remark 3.2.2. The upper bound is uniform since at infinity the metric is the same for all $s \in \Delta \backslash \{0\}$.

Moreover the Ricci form of the metric $\omega_{t,s}$ is bounded by $C \operatorname{tr}_{\omega_{t,s}} \hat{\omega}_s$ since $dd^c F_{t,s}$ is bounded using the $\hat{\omega}_s$-norm by Proposition 3.2.5. Indeed, $\hat{\omega}_s$ and $\omega_{t,s}$ share the same behaviour at infinity, up to polynomial error. Moreover, $\operatorname{Ric}(\hat{\omega}_s)$ is bounded with respect to the $\hat{\omega}_s$-norm on $U_s$, by definition of the reference metric $\hat{\omega}_s$, see Remark 3.2.2.

From this we have

$$|\operatorname{Ric}(\omega_{t,s})|_{\omega_{t,s}} \leq C \operatorname{tr}_{\omega_{t,s}} \hat{\omega}_s,$$

and

$$-\frac{\operatorname{Sec}(\hat{\omega}_s)(\partial \xi_{t,s}, \overline{\partial \xi_{t,s}}, \partial \xi_{t,s}, \overline{\partial \xi_{t,s}})}{|\partial \xi_{t,s}|^2} \geq -C |\partial \xi_{t,s}|^2 = -2C \operatorname{tr}_{\omega_{t,s}} \hat{\omega}_s.$$

Then,

$$\Delta_{\omega_{t,s}} \log \operatorname{tr}_{\omega_{t,s}} \hat{\omega}_s \geq -C \operatorname{tr}_{\omega_{t,s}} \hat{\omega}_s \tag{3.5}$$

on $U_s$, where $C > 0$ is independent of $s \in \Delta \backslash \{0\}$.

On the compact part $V_s$, we have no control on the sectional curvature of $\hat{\omega}_s$. However, we know $\hat{\omega}_s$ is the fixed Kähler metric $\omega_s = \omega|_{X_s}$ from its construction, and $\omega$ is the restriction of the smooth metric $\eta$ on the ambient Kähler manifold $Z$. Thus here we can use yet again the Chern–Lu inequality applied to the inclusion map $i\colon (V_s, \omega_{t,s}) \to (Z, \eta)$. Note that $\hat{\omega}_s = \omega_s = i^*\eta$ on $V_s$. Then we have

$$\Delta_{\omega_{t,s}} \log \operatorname{tr}_{\omega_{t,s}} \eta_s \geq |\operatorname{Ric}(\omega_{t,s})|_{\omega_{t,s}} - |\operatorname{Sec}(\eta)|_\eta \operatorname{tr}_{\omega_{t,s}} \eta_s,$$

where $\eta_s := \eta|_{i(V_s)} = i^*\eta$. Again the Ricci curvature is bounded by $\operatorname{tr}_{\omega_{t,s}} \hat{\omega}_s$, and $\operatorname{Sec}(\eta)$ is bounded since $\eta$ is a smooth Kähler metric. Thus Equation 3.5 is true on $X_s$ as a whole. Now we get

$$\Delta_{\omega_{t,s}}(\log \operatorname{tr}_{\omega_{t,s}} \hat{\omega}_s - C\psi_{t,s}) \geq C \operatorname{tr}_{\omega_{t,s}} \hat{\omega}_s - 2Cn.$$

Now, since the potentials $\psi_{t,s}$ go to zero as we approach infinity, then outside of a big enough region, eventually depending on $t, s$, we have

$$\log \operatorname{tr}_{\omega_{t,s}} \hat{\omega}_s - C\psi_{t,s} \leq C''$$

for a constant $C''$ independent of $t, s$. This is true because $\omega_{t,s}$ and $\hat{\omega}_s$ have the exact same behaviour at infinity, hence $\operatorname{tr}_{\omega_{t,s}} \hat{\omega}_s$ will approach $2n$ from below outside a big enough region.

Now, either we have a bound on $\operatorname{tr}_{\omega_{t,s}} \hat{\omega}_s$ independently of $t, s$ (recall that we already have a uniform $C^0$ bound for $\psi_{t,s}$), in case $\operatorname{tr}_{\omega_{t,s}} \hat{\omega}_s$ is increasing when approaching infinity, or the expression has a maximum at a point $x \in X_s$. In the latter case, at $x$ we have by the maximum principle

$$0 \geq C \operatorname{tr}_{\omega_{t,s}} \hat{\omega}_s - 2Cn,$$

hence we get again a uniform bound. This upper bound then implies that there exist a $C > 0$ independent of $t, s$ such that

$$C\hat{\omega}_s \leq \omega_{t,s}.$$

For the lower bound, note that for every compact $\mathcal{K} \subset \mathcal{X}\backslash\{x\}$ we can find a $C_{\mathcal{K}} > 0$ such that

$$\omega_{t,s} \leq C_{\mathcal{K}}\hat{\omega}_s$$

on $K_s = \mathcal{K} \cup X_s$. Indeed, by compactness of $\mathcal{K}$ and the uniform bounds for $F_{t,s}$, we have that there exists $c_{\mathcal{K}} > 0$ such that

$$c_{\mathcal{K}} \omega_{t,s}^n \leq \hat{\omega}_s^n \qquad \text{on } K_s.$$

We now use the following mixed Monge–Ampère inequality.

**Lemma 3.5.7** (Theorem 1.3 in Dinew (2009), Proposition 1.11 in Boucksom et al. (2010))**.** *Suppose that we have a collection* $\{\delta_i\}_{i=1,\dots,n}$ *of closed positive* $(1,1)$*-forms such that*

$$\delta_d^n \geq f_d \mu,$$

*where* $\mu$ *is a positive measure and the* $f_d$ *are non-negative. Then*

$$\delta_1 \wedge \cdots \wedge \delta_n \geq f_1^{1/n} \cdots f_n^{1/n} \mu.$$

Applying the lemma with $\delta_1 = \hat{\omega}_s$, $\delta_i = \omega_{t,s}$ for $i = 2, \dots, n$ and $\mu = \omega_{t,s}^n$ gives

$$(\operatorname{tr}_{\omega_{t,s}} \hat{\omega}_s) \omega_{t,s}^n = n \hat{\omega}_s \wedge \omega_{t,s}^{n-1} \geq n c_{\mathcal{K}}^{1/n} \omega_{t,s}^n,$$

hence the lower bound $\operatorname{tr}_{\omega_{t,s}} \hat{\omega}_s \geq n c_{\mathcal{K}}^{1/n}$ on $K_s$, which translates to $\omega_{t,s} \leq C_{\mathcal{K}} \hat{\omega}_s$.

If $\mathcal{K}$ is not compact, the same result applies since, as pointed out above, $\operatorname{tr}_{\omega_{t,s}} \hat{\omega}_s$ approaches $2n$ from below when going to infinity. Hence there will be a non-compact region $\mathcal{K}' \subseteq \mathcal{K}$ such that $\operatorname{tr}_{\omega_{t,s}} \hat{\omega}_s \geq n$ in $\mathcal{K}'$. Since the closure of the remaining region $\mathcal{K} \backslash \mathcal{K}'$ is compact, we find the bound on $\mathcal{K}$ by the previous case.

### 3.5.4 Final estimates

Recall that the $C^{2,\alpha}$ estimate in the proof of the Calabi Conjecture (see Błocki (2003)) is a completely local argument and thus can be carried out in our case with no modification; it depends only on the previous estimates, that we have already found to be uniform in $(t, s)$.

The usual bootstrapping techinque assures that we can then find a priori $C^{k,\alpha}$ estimates for any $k \in \mathbb{N}, \alpha \in (0, 1)$.

Let us now focus on the $C^0_\mu$ estimate, for $\mu \in (-2n, -2)$. The idea is to use a Moser iteration on $\psi_{t,s} \rho_s^{-\mu}$, where $\rho_s$ is a radius function on $X_s$. Note that if $\mu$ is the rate of $F_{t,s}$, then necessarily the rate $\gamma$ of $\psi_{t,s}$ is greater or equal than $\mu + 2$ (however, its $C^2_\gamma$ norm doesn't have a uniform bound, a priori). It is important here that $\mu$ needs to be uniform (independent of $s$).

For a fixed $(s_i, t_i)$, $s_i \neq 0$, the arguments carried out in the ALE case in D. Joyce (2000) pass unchanged to this case, as they only rely on dimension and volume growth. In particular we have the following.

**Proposition 3.5.8.** *Let* $(X, \omega)$ *be a (smooth) asymptotically conical Calabi–Yau manifold of complex dimension* $n > 2$. *Let* $\mu, \gamma, \delta$ *satisfy* $\mu + 2 \leq \gamma < \delta < 0$. *Then there exists a constant* $C = C(||dd^c\varphi||_{C^0}, ||f||_{C^0_\mu(X)}) > 0$ *such that if* $f \in C^0_\mu(X)$ *and* $\varphi \in C^2_\gamma(X)$ *satisfy*

$$(\omega + dd^c\varphi)^n = e^f \omega^n,$$

*then* $\varphi \in C^0_\delta(X)$ *and* $||\varphi||_{C^0_\delta(X)} \leq C$.

*Remark* 3.5.9. As pointed out above, while we require that $\varphi \in C^2_\gamma(X)$, we don't have a uniform bound on the norm in $C^2_\gamma(X)$, while the bound on the $C^0_\delta(X)$ norm is a priori (hence uniform in $t$, for instance).

From here, again following D. Joyce (2000), we can get higher order weighted estimates.

**Proposition 3.5.10.** *Let* $(X, \omega)$ *as above. Let* $\alpha \in (0, 1)$, $\mu + 2 \leq \delta < 0$. *Let* $f \in C^{3,\alpha}_\mu(X)$ *and* $\varphi \in C^0_\delta(X) \cap C^{5,\alpha}(X)$ *satisfy*

$$(\omega + dd^c\varphi)^n = e^f \omega^n.$$

*Then there exists* $C = C\left(||f||_{C^{3,\alpha}_\mu(X)}, ||\varphi||_{C^0_\delta(X)}, ||\varphi||_{C^{5,\alpha}(X)}\right) > 0$ *such that* $||\varphi||_{C^{5,\alpha}_\delta} \leq C$.

*Furthermore, if* $f \in C^{k,\alpha}_\mu(X)$ *and* $\varphi \in C^{k+2,\alpha}(X)$ *for some integer* $k \geq 3$, *then* $\varphi \in C^{k+2,\alpha}_\delta(X)$.

It remains to check whether we can choose $\mu$ to be uniform along the sequence $(s_i, t_i)$ (by Joyce's work, we only need to check the $s_i$-part, that is, in which way $\mu$ depends on the goemetry of $X_{s_i}$).

Ultimately, all of the estimates depend on a uniform bound on $||F_{1,s}||_{C^{k,\alpha}_{*,\mu}(X_s)}$ for some integer $k \geq 3$, hence it is enough find such a bound, and the bound exists by construction of $F_{1,s}$.

### 3.5.5 Gromov–Hausdorff limit and polynomial decay

In the previous section we have shown that we can find a limit in $C^\infty_\mu$ that satisfies the Monge–Ampère equation, away from the singularity. The remaining part is now to show that this tangent cone at $t = t_\infty$ is indeed the model cone $C_x$ at the singularity, and from there that we have a local biholomorphism in which the decay to the conical metric is polynomial. In particular, for the first part we want to prove the following.

**Proposition 3.5.11.** *The tangent cone $C(Y)$ at $x$ for the metric $\omega_{t_\infty,0}$ is isomorphic to the model cone $C_x$ as a Ricci-flat Kähler cone.*

To do this, we need an algebraic description of the tangent cone $C(Y)$. The relation between Kähler geometry and the corresponding algebraic geometry has been extensively studied in Donaldson & Sun (2014) and Donaldson & Sun (2017).

The main point here is that a priori we can find two sequences $a_i < b_i$ of positive reals, converging to $\infty$, such that $(X, a_i^2\omega_{t_i,0}, x)$ converges to $C(Y)$ and $(X, b_i^2\omega_{t_i,0}, x)$ converges to $C_x$ in the pointed Gromov–Hausdorff topology. The idea is to decide whether it is the convergence $\omega_{t_i,0} \to \omega_{t_\infty,0}$ — which gives the tangent cone $C(Y)$ — or the convergence to the metric cone to be the "quickest" in this limit of rescalings.

After passing to the algebraic side, an isomorphism can be proven to exist by showing that there is a test configuration from the model cone $C_x$ to the — potentially — new cone at $t_\infty$ in the sense of K-stability, by employing some results described in Donaldson & Sun (2017). This will be enough to infer that the cones are isomorphic, combining a result of on K-polystability of $\mathbb{Q}$-Fano varieties Berman (2016) and a uniqueness result on test configurations of K-polystable varietes. This is contained in Section 3.2 of Hein & Sun (2017), and can be followed verbatim as it consists of local arguments.

Following the approach of Hein & Sun (2017), we will thus prove the following lemma that links the Kähler and algebraic geometry of our manifold.

**Lemma 3.5.12.** *Given a sequence $(t_i, s_i) \to (t_\infty, 0)$, any subsequential Gromov–Hausdorff limit of $(X_s, \omega_{t_i,s_i})$ is naturally isomorphic to $X$ as a quasi-projective variety and is isometric to the metric completion of $(X\backslash\{x\}, \omega_{t_\infty,0})$.*

One possible proof of the lemma relies on the natural compactifications $\overline{X}_s$ of the spaces $X_s$. This follows the path of Hein & Sun (2017) and G. Chen (2019). The former argument uses directly the results of Donaldson & Sun (2014), Donaldson & Sun (2017) and Arezzo & Spotti (2016). The latter, since it deals with a non-compact environment, cannot do the same. Instead, to circumnavigate the problem, the author exploits the natural compactification of the space he considers in a projective space. Then, by modifying the metric in this compactification, he then uses the Donaldson–Sun theory on the compact spaces instead.

Recalling we consider only the regular case, we can follow the steps of G. Chen (2019) and take the Gromov–Hausdorff limit for compactifications of our spaces.

*Proof of Lemma 3.5.12.* For all $s_i$, choose a point $x_i \in V_{s_i}$ such that $x_i$ converges[3] to $x \in V_0$. We want to show that $\{(X_{s_{i_k}}, \omega_{t_{i_k}, s_{i_k}}, x_i)\}_k$ is precompact in the Gromov–Hausdorff pointed sense. This implies it admits a Gromov–Hausdorff limit $(X, d_X, x)$, and that $(X, d_X)$ is isometric to the completion of $\tilde{\omega}$. The following reasoning is the same as in the asymptotically cylindrical case in G. Chen (2019), which we report for completeness.

Due to the uniform $C^\infty_{loc}$ convergence we have found above, for any point in $q \in V_{s_i}$ such that $d_{\omega_{t_i,s_i}}(q, \partial V_{s_i}) = d$, we have a uniform lower bound $B(q, d) \geq \varepsilon$.

Let now $p \in V_{s_i}$ be such that $D = d_{\omega_{t_i,s_i}}(p, \partial V_{s_i})$. If $D > 3d$, we can choose $q$ on the minimal geodesic connecting $p$ to the boundary of $V_{s_i}$ such that $d(p, q) = D - d$. Exploiting the Ricci-flatness of $\omega_{t_i,s_i}$ in $V_{s_i}$, we can use the Bishop–Gromov inequality to get

$$\mathrm{Vol}\,(B(p, D - 2d)) \geq \frac{\mathrm{Vol}\,(B(p, D) \backslash B(p, D - 2d))}{\left(\frac{D}{D-2d}\right)^{2n} - 1} \geq \frac{C D\,\mathrm{Vol}(B(p, d))}{d}.$$

Since the volume of $V_{s_i}$ is uniformly bounded, so is $D$. Hence we can use the following proposition.

**Proposition 3.5.13.** *A subset $\mathcal{X}$ of metric spaces modulo isometry is precompact with respect to the pointed Gromov–Hausdorff topology if and only if there exist $\varepsilon > 0$ and $N(\varepsilon) < \infty$ such that for all $[W] \in \mathcal{X}$ there exists a $\varepsilon$-dense set in $W$ with at most $N(\varepsilon)$ elements.*

*Remark* 3.5.14. The statement above usually appears in the literature with a reference to compact metric spaces (for instance in Proposition 0.24 of Cheeger (2001)). In the non-compact setting, see Corollary 1.10 of Petersen (2006), we can say that a collection $\mathcal{X}$ is precompact if and only if for any $R > 0$ the collection $\mathcal{X}_R = \{B(x, R) \,|\, B(x, R) \subset (X, x) \in \mathcal{X}\}$ is precompact. Thus we can apply the same result on collections of balls and it passes unchanged.

By the bound found above, we can find a Gromov–Hausdorff limit for the sequence $\{(X_{s_{i_k}}, \omega_{t_{i_k}, s_{i_k}})\}_k$, that we are going to denote by $(X, d_X)$. In the compact case, thanks to the results of Donaldson–Sun (Theorem 1.2 in Donaldson & Sun

[3] If we consider the smooth family of embeddings $\Phi_s$ in Proposition 3.5.1, this means $x_i \to \Phi_{s_i}(x)$ with respect to the metrics $\omega_{t_i,s_i}$, or better, $d_{\omega_{t_i,s_i}}(\Phi_{s_i}(x), x_i) \to 0$ for $i \to \infty$.

(2014)) we can directly affirm that the limit $X$ is a projective variety and it is isometric to the metric completion of $(X\backslash\{x\}, \omega_{t_\infty})$.

In our case, the space $(X, d_X)$ is however non-compact, so we cannot apply the results of Donaldson–Sun directly.

In this limit, the only problem we need to take care of is the behaviour at the singularities. These are contained in the "compact part" of the manifold, *i.e.* a big enough ball centred at the singularity $x$. Thus our objective is to modify the metric at infinity to get a compact metric. We can apply the results of Donaldson–Sun to this modified metric, and the behaviour on the compact part will be the same as that of the original metric.

To do this, we consider the $(1,1)$-forms

$$\omega|_{X_s} + dd^c\psi_{t,s},$$

where $\omega$ is the smooth Kähler metric on $X_s$ used in the construction of the reference metrics $\hat{\omega}_s$ in Section 3.2. We need to check if these forms define metrics, modulo modifying them appropriately.

- On $\{r \geq R_0\}$:

  here we know that

  $$\hat{\omega}_s + dd^c\psi_{t,s} > 0.$$

  Let $U$ be a neighbourhood of $D$ which is isomorphic to a neighbourhood of the zero section in the normal bundle to $D$, and denote by $\rho$ the distance from the zero section. Recall from the construction of the covering map, $\rho = r^{-k/n}$, where $r$ is the radius on the cone. Let $\hat{g}_s$ and $g|_{X_s}$ be the Riemannian metric associated respectively to the Kähler metrics $\hat{\omega}_s$ and $\omega|_{X_s}$. Then we can write locally

  $$\begin{aligned} \hat{g}_s &= \rho^{-2\left(\frac{n}{k}+1\right)}d\rho^2 + \rho^{-2\frac{n}{k}}g_S + o(1); \\ g|_{X_s} &= d\rho^2 + h_\rho, \end{aligned}$$

  for a suitable smooth form $h_\rho$ and metric $g_S$ pulled back from the Sasakian manifold $S$ such that $D \simeq S/\langle Jr\partial_r\rangle$. The writing of the asymptotics for $\hat{g}_s$ come from the fact that it is an asymptotically conical metric (thus asymptotically of the form $dr^2 + r^2 g_S$) and that $r \sim \rho^{-n/k}$ at infinity, while the asymptotics for $g|_{X_s}$ can be deduced from a normal version of the Gauss lemma.

At this point, we prove that for any $\varepsilon > 0$ we can find a bound on $\rho$ such that

$$\rho^{2+2\frac{n}{k}+\varepsilon}\hat{g}_s - g|_{X_s} < 0.$$

Indeed, we can rewrite the expression as

$$\left(\frac{\rho^{2+2\frac{n}{k}+\varepsilon}}{\rho^{2+2\frac{n}{k}}} - 1\right) d\rho^2 + \frac{\rho^{2+2\frac{n}{k}+\varepsilon}}{\rho^{2\frac{n}{k}}} g_S - h_\rho.$$

Then, since $g$ is smooth, we can find $C_s > 0$ such that $-h_\rho \leq -C_s\rho^2 g_S$. Choosing $\rho$ such that $\rho^\varepsilon < 1$ and $\rho^\varepsilon < C_s$ gives the inequality. Since all of the $X_s$ have the same geometry at infinity, we can substitute $C_s$ with $C := \max C_s$, which is then a finite number.

Now, we can prove that if

$$\hat{\omega}_s + dd^c\psi_{t,s} > 0,$$

and $dd^c\psi_{t,s} \sim r^\mu = \rho^{-\frac{n\mu}{k}}$, then $dd^c\psi_{t,s}$ is small enough in $\omega$-norm for a suitable neighbourhood of $D$ so that we also have

$$\omega|_{X_s} + dd^c\psi_{t,s} > 0,$$

which is what we want to prove. Indeed, by the inequality (and by the fact that we are considering 2-covariant tensors) we get

$$||\rho^{-2-2\frac{n}{k}-\varepsilon} dd^c\psi_{t,s}||_{\hat{g}_s} > ||dd^c\psi_{t,s}||_g.$$

Note that the norm with respect to $\hat{g}_s$ is not weighted, and thus the left hand side might be infinite. Here the norms are considered only on the neighbourhood of $D$. We know that

$$||dd^c\psi_{t,s}||_{\hat{g}_s,\mu} := ||r^{-\mu} dd^c\psi_{t,s}||_{\hat{g}_s} = ||\rho^{\frac{n\mu}{k}} dd^c\psi_{t,s}||_{\hat{g}_s} < \infty.$$

Then the inequality we found above can be used to infer, in the case all of the considered quantities are finite, that

$$\begin{aligned}
||dd^c\psi_{t,s}||_g &< ||\rho^{-2-2\frac{n}{k}-\varepsilon} dd^c\psi_{t,s}||_{\hat{g}_s} \\
&= ||\rho^{-2-2\frac{n}{k}-\varepsilon}\rho^{-\frac{n\mu}{k}} dd^c\psi_{t,s}||_{\hat{g}_s,\mu} \\
&\leq \max_\rho \rho^{-\frac{n}{k}(2+\mu)-2-\varepsilon}||dd^c\psi_{t,s}||_{\hat{g}_s,\mu+2}.
\end{aligned}$$

We now need $-\frac{n}{k}(2+\mu) - 2 - \varepsilon > 0$, which implies

$$\varepsilon < -\frac{n}{k}(2+\mu) - 2.$$

Note that via the bootstrapping technique of Section 3.2.2 we can choose a more negative $\mu$, with minimum represented by $\mu = -2n$. This means we are allowed to choose, without loss of generality,

$$\varepsilon < -\frac{n}{k}(2-2n) - 2 = 2\frac{n^2}{k} - 2\frac{n}{k} - 2.$$

The right hand side is now bigger than 0, as it is equivalent to

$$n^2 - n > k,$$

and since $k$ (the Fano index) is at most $n$, then this is true for $n > 2$.

In particular this means that we can take $\rho$ small such that the open condition $\omega|_{X_s} + dd^c\psi_{t,s} > 0$ is satisfied.

- On $\{r \leq R_0\}$:

  On this compact part we have

$$\omega + \beta \geq \hat{\omega}_s$$

  for some bump $(1,1)$-form $\beta$, by construction of the reference metrics in Section 3.2. Thus the form $\omega + dd^c\psi_{t,s} + \beta$ is positive.

This means that, after having modified the metric near infinity, we have found a metric which is compact on $X_s$ and such that the diameter, volume and Ricci curvature have two-sided bounds. At infinity it is true because the metric $\omega$ can be taken to be the same for all $X_s$ and hence has bounds independent of $s$, and on the compact part we have only added bump forms which again are uniformly bounded as shown in Section 3.2. □

There is hope of repeating the argument of G. Chen (2019) in the case the cone at infinity is quasi-regular. This is because in the quasi-regular case we have a natural compactification of all the manifolds $X_s = \overline{X}_s \backslash D$ we consider, which is just $\overline{X}_s$.

Note that however the argument works only if the $\overline{X}_s$ are themselves smooth manifolds. In case $D$ is an orbifold (*i.e.* the quasi-regular case), we cannot apply the results of Donaldson–Sun directly, hence an extension of those results to orbifolds is needed.

We will now sketch a possible way of tackling the quasi-regular and irregular cases. Note that the exposition will only be a sketch, and more work is needed to prove the results in full completeness. To try to get around the problem stated above we reason as follows, similarly to the remarks in Section 3.1 of Székelyhidi (2020).

Suppose we have a sequence $(M_i, L_i, \omega_i, p_i)$ of complete pointed $n$-dimensional Kähler manifolds with line bundles $L_i \to M_i$ equipped with Hermitian metrics of curvature $-i\omega_i$. Moreover, $\mathrm{Ric}(\omega_i) = \lambda_i \omega_i$ with $|\lambda_i| \leq 1$ and $\mathrm{Vol}(B(p_i, 1)) > \kappa > 0$ for all $i$, given a fixed $\kappa > 0$. These are the conditions in the class $\mathcal{K}(n, \kappa)$ in Donaldson & Sun (2017), apart from compactness.

Let us call $(Z, p)$ the pointed Gromov–Hausdorff limit of the sequence $(M_i, \omega_i, p_i)$. Then we can claim that Theorems 1.1, 1.2 and 1.3 of Donaldson & Sun (2017) hold in this context. In particular, these imply Lemma 3.5.12.

Indeed, the basic construction in Donaldson & Sun (2014) involves grafting a holomorphic function from a tangent cone to $Z$ onto $M_i$ for large $i$, and then using the Hörmander $L^2$-estimate to perturb the resulting approximately holomorphic section of a power of $L_i$ to a holomorphic section $s$. This is a local construction and the Hörmander estimate holds on complete Kähler manifolds. Moreover, thanks to the non-collapsing condition and Ricci curvature bound we can control the Sobolev constant on geodesic balls by a theorem of Anderson in Anderson (1992).

Let us explain more carefully this procedure a bit more carefully.

Let us suppose $(Z, p)$ is the non-collapsed pointed Gromov–Hausdorff limit of a sequence of complete polarised Kähler–Einstein manifolds $(M_i, g_i, p_i)$ with the same assumptions as in Donaldson & Sun (2017):

- $\mathrm{Ric}(g_i) = \lambda_i g_i$ with $|\lambda_i| \leq 1$;
- $\mathrm{Vol}(\mathbb{B}_r(p_i)) \geq \kappa r^{2n}$ for all $r \in (0, 1]$;
- the Kähler form is given by the curvature of a line bundle $L_i$.

Via the results of Donaldson & Sun (2014) and Donaldson & Sun (2017), $Z$ is a normal complex variety with a singular Kähler–Einstein metric $\omega_Z$ and the metric tangent cone $Z_p$ at $p$ is homeomorphic to a normal affine variety uniquely determined by $(Z, p)$. The tangent cone admits a singular Ricci flat cone metric $\omega_{Z_p}$. Note that Donaldson & Sun (2014) and Donaldson & Sun (2017) deal with compact complex manifolds; however, as already pointed out in Székelyhidi (2020), Section 3.1, the results can be extended to non-compact complete complex manifolds. Indeed, following Donaldson & Sun (2014), we take the following steps:

- graft holomorphic functions from a tangent cone to $Z$ onto $M_i$, through cut-offs functions and approximations. This is contained in Section 3.2.1 of Donaldson & Sun (2014) and is part of the proof of Theorem 3.2, which aims to give a uniform lower bound on the "density of states" function $\rho_{k,M_i}$, also known as Bergman function, defined by

  $$\rho_{k,M_i} = \sum |s_\alpha|^2,$$

  where $(s_\alpha)_\alpha$ is any orthonormal basis of $H^0(M_i, L_i^k)$. This function is tightly linked to how one can embed $M_i$ in a projective space using sections of $L_i^k$ as in the Kodaira Embedding Theorem. Indeed, part of the Kodaira Embedding Theorem asserts that for each fixed $X$ we have $\rho_{k,X} > 0$ for big enough $k$. To present $Z$ as an affine variety one will indeed need to embed all of the $M_i$ in the same affine space, and a uniform lower bound on the Bergman function would enable such an uniform embedding.

  Now, all of the arguments dealing with the tangent cone $C(Y)$ at the base point $p$ pass unchanged to the non-compact case, as they only rely on properties of the link $Y$. Moreover, the completion of the proof of the theorem contained in Section 3.2.3 of Donaldson & Sun (2014) only deals with choosing suitable embeddings of open subsets $\tilde{U}$, which have been defined previously on the tangent cone as modifications of certain annuli $\{\delta < |z| < R\}$, in the regular part of $Z$. Thus the completion of the proof is once again a local argument;

- use Hörmander $L^2$-estimate to perturb an almost holomorphic section of $L_i^k$ to an actual holomorphic section $s$, for some $k \in \mathbb{N}$. This is contained in Proposition 2.3 of Donaldson & Sun (2014). The main ingredients are Hodge Theory for the $\overline{\partial}$ operator and indeed the Hörmander estimate. Results from Hodge Theory pass to the non-compact case if we consider $L^2$-sections (*c.f.*

section VI of Demailly (2012)), and the estimate works on complete non-compact Kähler manifolds (*c.f.* Theorem 4.5 in Demailly (2012), section VIII);

- employ Moser's iteration and Bochner–Weitzenbock formulas to bound the $L^\infty$ norm of $s$ and $\nabla s$ in terms of the $L^2$ norm of $s$, which still work in the non-compact setting. These sections have also other separating properties, *c.f.* Donaldson & Sun (2014) Proposition 4.6 and 4.7. The proposition that we consider is the following.

  **Proposition 3.5.15** (*c.f.* Donaldson & Sun (2014), Proposition 2.1). *There are constants $K_0, K_1$, depending only on $n, \kappa, k$, such that if $s$ is a holomorphic section of $L_i^k$ for any $k > 0$, we have*

  $$||s||_{L^\infty} \leq K_0 ||s||_{L^2}, \quad ||\nabla s||_{L^\infty} \leq K_1 ||s||_{L^2}.$$

  These estimates can be proved using the Sobolev inequality, given a bound on the Sobolev constant on geodesic balls. Thanks to Theorem 2.6.2, we have such a bound depending on the Ricci bound and the non-collapsing bound $\kappa$.

At this point, we would like to use these sections $s$, thereby creating a map from $H^0(M_i, L_i^k)$ to some vector space of dimension bounded by the constants $n, \kappa$. In this way, we could embed all of the $M_i$ in the same $\mathbb{CP}^N$ as affine varieties $W_i$. After this, by following the same arguments as in Section 2 of Donaldson & Sun (2017), we could construct an analytic set $W$ as limit of analytic sets $W_i$ obtained by embedding the $M_i$, thus endowing $(Z, \mathcal{O}_Z)$ with the structure of a normal complex space. However, some problems arise and can be object of further work. Contrary to the compact case, is not clear a priori which sections we should take to contruct the embeddings as the dimension of $H^0(M_i, L_i^k)$ is not necessarily finite. Moreover, while in the compact case we can take limits in the Hilbert scheme, while it is not obvious how to take limits in the affine case.

By all of the above, we can the state the following version of the result of Donaldson–Sun as in Hein & Sun (2017) (see also Theorem 3.1 in Spotti et al. (2016)) as a conjecture.

**Conjecture 3.5.16** (*c.f.* Theorem 1.2, Donaldson & Sun (2014)). *Given constants $n, \kappa$, there is a fixed constant $k$ and an integer $N$ with the following effect:*

- *Consider a complete Kähler manifold $M$, polarised with a line bundle $L$, such as above, with $\operatorname{Vol} B_r \geq \kappa r^{2n}$ for $r \leq 1$. Then $M$ can be embedded as a quasi-projective variety in a linear subspace of $\mathbb{CP}^N$ by sections of $L^k$;*

- *Let $(M_i, p_i)$ be a pointed sequence of complete Kähler manifolds as above, with pointed Gromov–Hausdorff limit $(Z, p)$. Then $Z$ is homeomorphic to a normal quasi-projective variety $W$ in $\mathbb{CP}^N$. After passing to a subsequence and taking a suitable sequence of analytic transformations, we can suppose that the varieties $M_i \simeq W_i \subset \mathbb{CP}^N$ converge as analytic varieties to $W$.*

Applying the results of this conjecture in our case, in particular the second point, would complete the proof in the irregular case. Note that the uniform non-collapsing bound holds by the discussion in Section 3.5.2.

Now, having proved Lemma 3.5.12 enables us to prove Proposition 3.5.11. We then follow exactly Section 3.3 of Hein & Sun (2017) to prove the polynomial decay, as it consists of local arguments. In particular we have

**Theorem 3.5.17.** *There is a complex analytic isomorphism $P\colon U \to V_0$ between open neighbourhoods $U$ of the vertex $o$ in $C_x$ and $V_0$ of $x$ in $X_0$ such that*

$$\sup_{\partial\mathbb{B}_r(o)} |\nabla^j_{\omega_{C_x}}(P^*\omega - \omega_C)|_{\omega_C} \leq C_j r^{d-j}$$

*for some $d > 0$ and all $j \in \mathbb{N}$.*

The proof roughly consists of first finding a "broken holomorphic gauge" — the local biholomorphism $P$ — in which the limit metric $\omega_{t_\infty,0}$ converges uniformly to the metric on the cone $\omega_{C_x}$ for $r \to 0$, on $\partial\mathbb{B}_r(o)$, by employing some arguments similar to those in Donaldson & Sun (2017). Subsequently, one improves this holomorphic gauge to one such that the convergence is polynomial. This is done by considering the linearised Ricci-flat equation on $(1,1)$-forms and applying Schauder estimates in there, after checking that certain estimates on annuli work.

# Chapter 4

# Special Lagrangian vanishing cycles

Special Lagrangian submanifolds are a class of geometric objects in differential geometry and symplectic geometry. They are a special type of submanifolds in a Calabi-Yau manifolds that are both Lagrangian and calibrated by the real part of the holomorphic volume form of the Calabi-Yau manifold. This "special" condition ensures that the submanifold has minimal volume among all submanifolds in its homology class.

Special Lagrangian submanifolds play an important role in string theory, where they arise as solutions to certain supersymmetric equations of motion. They also have important applications in mirror symmetry. All of this is related to the *SYZ conjecture*, exposed in Strominger et al. (1996), which deals with special Lagrangian fibrations in six dimensional Calabi–Yau manifolds.

For references about the subject, refer to D. D. Joyce (2007), Hitchin (2001), Y. Li (2022).

## 4.1 Background

Constructing special Lagrangian submanifolds is not an easy task. There is substantially only one known method, by considering fixed loci of antiholomorphic involutions. Constructing Calabi–Yau metrics on manifolds with conical singularities can help in the endeavour of counting special Lagrangian submanifolds in smoothings of such Calabi–Yau's, thus giving another method to find these submanifolds. When referring to the compact case, Appendix A in Hein & Sun (2017) gives a general picture of the situation and contains the following statement:

**Proposition 4.1.1.** *Let $X$ be a compact Calabi–Yau manifold with at worst nodal* *singularities as in Definition 1.2.12. Let $\mathcal{X}$ be a smoothing of $X$ that is versal for every node of $X$. Then every node of $X$ is the limit of vanishing special Lagrangian $n$-spheres in the nearby fibers of $\mathcal{X}$.*

In Hein & Sun (2017), the proof is only sketched, and in fact a technical difficulty appears to have been overlooked. In work of Chan in Chan (2005), a version of the result above is covered in Theorem 4.31 and Theorem 6.1. However, Chan's construction of a smoothing is abstract and the connection between the given family $\mathcal{X}$ and the constructed smoothing is not a priori clear. Moreover, Chan's work covers only the compact 3-fold case.

We aim to discuss similar statements in the non-compact case, and as a byproduct fill in the details of Hein and Sun's sketch.

We will prove it by essentially using an $n$-dimensional version of the gluing construction contained in Spotti (2014), as suggested in Hein & Sun (2017), which deals with Del Pezzo surfaces, after having proved a suitable polynomial convergence of the Calabi–Yau metric in $X$ to the Calabi–Yau metric on the tangent cone to the singularity.

The idea is to smooth out the singularities by gluing in scaled copies of Stenzel's asymptotically conical Calabi–Yau metric on $T^*S^n$. It is a known fact that the zero section in $T^*S^n$ is a special Lagrangian with respect to the Stenzel metric.

Firstly, let us introduce some background material on special Lagrangians.

**Definition 4.1.2.** *The quadruple $((M, g), \omega, \Omega, J)$ is an* almost Calabi–Yau manifold *if*

- *$(M, g)$ is a Riemannian manifold of real dimension $2n$;*
- *$J$ is integrable, $\omega = g(J\,\cdot\,,\,\cdot\,)$ and $d\omega = 0$;*
- *$\Omega$ is a complex form of bidegree $(\dim_{\mathbb{C}} M, 0)$, nowhere vanishing and holomorphic.*

*If the Monge–Ampère equation*

$$\omega^n = i^{n^2}\Omega \wedge \overline{\Omega}$$

*is also satisfied, then $M$ is Calabi–Yau.*

*Remark* 4.1.3. We can recover the complex structure $J$ from $\Omega$ by

$$\Lambda^{1,0}_J M := \mathrm{Ker}\left[\Lambda^1_{\mathbb{C}} M \ni \alpha \mapsto \Omega \wedge \alpha \in \Lambda^{n+1}_{\mathbb{C}} M\right],$$

which can be seen as locally writing

$$\Omega \simeq_{loc} dz^1 \wedge \cdots \wedge dz^n.$$

We will generally only consider the Calabi–Yau case, but most of the following can be applied to the almost Calabi–Yau case.

*Remark* 4.1.4. Note that the substitution $\Omega \mapsto e^{i\theta}\Omega$ preserves the volume form for all $\theta$:

$$e^{i\theta}\Omega \wedge \overline{e^{i\theta}\Omega} = e^{i(\theta-\theta)}\Omega \wedge \overline{\Omega} = \Omega \wedge \overline{\Omega}.$$

**Definition 4.1.5.** *Let $L \subset M$ be a real submanifold of dimension $n$. Then $L$ is:*

- Lagrangian *if $\omega|_L \equiv 0$;*
- Special Lagrangian *if it is Lagrangian and there exists a real number $\theta \in [-\pi, \pi]$ such that*
$$\mathrm{Im}(e^{i\theta}\Omega)|_L \equiv 0.$$

By the last property, one can show

$$\mathrm{vol}_L = (i^{n^2}\Omega \wedge \overline{\Omega})|_L = \mathrm{Re}(e^{i\theta}\Omega)|_L.$$

Now, if $(\omega, \Omega)$ is Calabi–Yau, we have

$$-\mathrm{vol}_\Pi \leq \mathrm{Re}(e^{i\theta}\Omega)|_\Pi \leq \mathrm{vol}_\Pi \qquad \forall\, \Pi \subset T_xM,\ \dim \Pi = n,$$

where $\Pi$ is a linear subspace of $T_xM$. This presents $\mathrm{Re}(e^{i\theta}\Omega)$ as a *calibration* whenever $M$ is Calabi–Yau. In particular, this entails that a compact special Lagrangian without boundary $L$ minimises the volume functional in its homology class. To see this, let $L'$ be such that $[L] = [L'] \in H_n(M, \mathbb{Z})$. Then

$$\begin{aligned}\mathrm{vol}(L) = \int_L \mathrm{vol}_L &= \int_L \mathrm{Re}(e^{i\theta}\Omega)|_L \\ &= \int_{L'} \mathrm{Re}(e^{i\theta}\Omega)|_{L'} \leq \int_{L'} \mathrm{vol}_{L'} = \mathrm{vol}(L').\end{aligned}$$

Note that the equality between the two lines is true by Stokes' Theorem because $\mathrm{Re}(e^{i\theta}\Omega)$ is a closed form and $L$ and $L'$ are homologous.

The main way of constructing such special Lagrangian submanifold is by considering fixed loci of antiholomorphic involutions.

**Definition 4.1.6.** *Let $(X, \omega, \Omega)$ be a Calabi–Yau manifold of complex dimension $n$. An* antiholomorphic involution $i \colon X \to X$ *is a smooth map such that*

*i)* $i^*\omega = -\omega$;

*ii)* $i^*\Omega = \overline{\Omega}$.

*Remark* 4.1.7. If $L \subset X$ has real dimension $n$ and $L \subseteq \mathrm{Fix}(i)$, the fixed locus of $i$, then $L$ is special Lagrangian.

**Thomas–Yau conjecture**

There is a similarity between the problem of existence of special Lagrangian submanifolds and the existence of constant scalar curvature Kähler metrics. Indeed, given a submanifold $L$ of dimension $n$, we can define its *phase* $e^{i\theta(s)}$ as the function from $L$ to $S^1$ such that

$$\mathrm{Im}(e^{i\theta(s)}\Omega)|_L \equiv 0.$$

The special Lagrangian condition is then substantially the PDE problem

$$e^{i\theta(s)} \equiv e^{i\theta},$$

where $\theta$ is a constant.

This has inspired the following conjecture by Thomas–Yau (*c.f.* Thomas (2001) and Thomas & Yau (2002)):

$$\exists\, L \text{ sLag in a given holomogy class } [L] \Longleftrightarrow \text{the class } [L] \text{ is stable},$$

where stability is in some algebraic sense. One of the main hardships in studying this conjecture is related to finding the precise statement of this stability condition. A prediction by Joyce involves defining stability using the framework of Bridgeland stability conditions on the derived Fukaya category of the almost Calabi–Yau manifold, *c.f.* Y. Li (2022).

### 4.1.1 Stenzel metric

Consider the affine smoothing of the ordinary double point (ODP) given by $V := \{\sum_{i=0}^n z_i^2 = 1\}$. This can be identified with the cotangent bundle of the $n$-sphere $T^*S^n = \{(x, \xi) \in \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \,|\, |x| = 1, \langle x, \xi\rangle = 0\}$, identified with the tangent bundle, using the diffeomorphism

$$(x,\xi) \mapsto z = x\cosh(|\xi|) + i\frac{\sinh(|\xi|)}{|\xi|}\xi.$$

Following Stenzel (1993), we can find a Ricci-flat metric on $V$, called *Stenzel metric*, given by $\eta_1 = dd^c u(r^2)$, where $r^2 = |z|^2$ and $u$ satisfies the differential equation

$$\frac{d}{dw}\left(\left(\frac{d}{dw}u(\cosh w)\right)^n\right) = cn(\sinh w)^{n-1},$$

for some constant $c > 0$. If $n = 2$, then $u(r^2) = \sqrt{r^2+1}$ satisfies the equation; for higher dimensions, $u$ can be expressed in integral form, and in particular one can check that $u(r^2) \sim r^{2(n-1)/n}$ for large enough $r$.

On the ODP itself $C := \{\sum_{i=0}^n z_i^2 = 0\}$, the Stenzel metric takes the form

$$\eta_0 = dd^c\left(\sum_{i=0}^n |z_i|^2\right)^{(n-1)/n},$$

as we can identify $C$ with $T^*S^n \backslash S^n$, where $S^n$ is the zero section of the bundle.

On $V = \{\sum_{i=0}^n z_i^2 = 1\} \simeq T^*S^n$, the zero section $L \simeq S^n$ plays an important role. It corresponds to the real trace

$$\left\{ z_i \in \mathbb{R} \,\middle|\, \sum_{i=0}^n z_i^2 = 1 \right\} \subset V,$$

and is thus the fixed locus of the antiholomorphic involution $i(z) = \bar{z}$. As noted in Remark 4.1.7, $L$ is then a special Lagrangian submanifold with respect to the Stenzel Calabi–Yau metric $\eta_1$ described above.

### 4.1.2 Deformations of special Lagrangians

We give the proof of the following standard theorem about deformations of special Lagrangian submanifolds, *c.f.* Theorem 8.4.7 in D. D. Joyce (2007).

**Theorem 4.1.8.** *Let $(\omega_t, \Omega_t)$, $t \in (-\delta_0, \delta_0)$ be a smooth one-parameter family of Calabi–Yau structures on a smooth manifold $M$. Suppose $L$ is a compact special Lagrangian of phase $e^{i\theta_0}$. If $[\omega_t|_L] = 0$ for all $t$, then there exist $\delta \in (0, \delta_0)$ and a one-parameter family of special Lagrangians $L_t$ of phases $e^{i\theta_t}$ in $(M, \omega_t, \Omega_t)$ for $t \in (-\delta, \delta)$ with $L_0 = L$. Moreover, if $\left[\mathrm{Im}(e^{-i\theta_0}\Omega_t)|_L\right] = 0$, then $\theta_t \equiv \theta_0$.*

As a fundamental tool in the following proof, and subsequent ones, we will employ the following quantitative version of the Implicit Function Theorem in Banach Spaces.

**Theorem 4.1.9** (Implicit Function Theorem)**.** *Let $E\colon X \to Y$ be a differentiable map between Banach spaces and let $R(x) := E(x) - E(0) - D_0E(x)$ be the non-linear part of $E$. Assume there exist $L, r_0, C > 0$ such that*

- *$|R(x) - R(y)|_Y \leq L(|x-y|_X)(|x|_X + |y|_X)$, for all $x, y$ in $\mathbb{B}_{r_0}(0) \subset X$;*
- *$D_0E$ is invertible with norm of the inverse bounded by $C$.*

*If for $r < \min\left\{r_0, \frac{1}{2LC}\right\}$ we have $|E(0)|_Y \leq \frac{r}{2C}$, then there exists a unique solution of the equation $E(x) = 0$ in $\mathbb{B}_r(0) \subset X$.*

*Proof of Theorem 4.1.8.* Consider sections $\sigma_t \in C^\infty(L_0, N_{L_0|M}) \simeq C^\infty(L_0, T^*L_0)$ of the normal bundle of $L_0$ in $M$ (or equivalently 1-forms $\gamma_t$ on $L_0$), where the isomorphism is given by the Lagrangian Neighbourhood Theorem. These sections define submanifolds $L_t \subset M$ by considering the graph of $\sigma_t$ in $M$ via the normal exponential map with respect to $\omega_0$.

For $L_t$ to be a special Lagrangian submanifold, the following equations need to be satisfied:

$$\begin{cases} i_{L_t}^*\omega_t & = 0, \\ \operatorname{Im}\left(i_{L_t}^*(e^{-i\theta_t}\Omega_t)\right) & = 0. \end{cases}$$

For the first equation, we can write

$$\begin{aligned} i_{L_t}^*\omega_t &= i_{L_0}^*\omega_0 + t\frac{d}{dt}(i_{L_t}^*\omega_t)|_{t=0} + Q_1(t) && (4.1) \\ &= t\frac{d}{dt}(i_{L_t}^*\omega_t)|_{t=0} + Q_1(t), && (4.2) \end{aligned}$$

where $Q_1(t)$ is the nonlinear part. Analysing the derivative, we find

$$\begin{aligned} \frac{d}{dt}(i_{L_t}^*\omega_t)|_{t=0} &= i_{L_0}\dot{\omega}_0 + L_v\omega_0 \\ &= i_{L_0}\dot{\omega}_0 + d\gamma_0, \end{aligned}$$

where $L_v$ is the Lie derivative with respect to $v$ and $v \in TL_0$ is such that $\sigma_t(x) = \exp^\perp(tv)$.

For the second equation,

$$\begin{aligned}\operatorname{Im}\left(i_{L_t}^*(e^{-i\theta_t}\Omega_t)\right) &= \operatorname{Im}\left(i_{L_0}^*(e^{-i\theta_0}\Omega_0)\right) + t\frac{d}{dt}\left(\operatorname{Im}\left(i_{L_t}^*(e^{-i\theta_t}\Omega_t)\right)\right)|_{t=0} + Q_2(t) \\ &= t\frac{d}{dt}\left(\operatorname{Im}\left(i_{L_t}^*(e^{-i\theta_t}\Omega_t)\right)\right)|_{t=0} + Q_2(t),\end{aligned}$$

where $Q_2(t)$ is the nonlinear part. Again, analysing the derivative, we find

$$\begin{aligned}\frac{d}{dt}\left(\operatorname{Im}\left(i_{L_t}^*(e^{-i\theta_t}\Omega_t)\right)\right)|_{t=0} &= \operatorname{Im}\left(\frac{d}{dt}\left(i_{L_t}^*(e^{-i\theta_t}\Omega_t)\right)|_{t=0}\right) \\ &= \operatorname{Im}\left(i_{L_0}^*\frac{d}{dt}(e^{-i\theta_t}\Omega_t) + L_v(e^{-i\theta_0}\Omega_0)\right) \\ &= \operatorname{Im}\left(i_{L_0}^*\left(-i\dot{\theta}_0 e^{-i\theta_0}\Omega_0 + e^{-i\theta_0}\dot{\Omega}_0\right) + e^{-i\theta_0}L_v\Omega_0\right).\end{aligned}$$

Without loss of generality, we can assume $\theta_0 = 0$. Then

$$\begin{aligned}\frac{d}{dt}\left(\operatorname{Im}\left(i_{L_t}^*(e^{-i\theta_t}\Omega_t)\right)\right)|_{t=0} &= -\dot{\theta}_0 i_{L_0}^*\Omega_0 + \operatorname{Im}\left(i_{L_0}^*\dot{\Omega}_0 + L_v\Omega_0\right) \\ &= -\dot{\theta}_0 i_{L_0}^*\Omega_0 + \operatorname{Im}\left(i_{L_0}^*\dot{\Omega}_0\right) + d*\gamma_0,\end{aligned}$$

where the last equality comes from the equality $L_v\Omega_0 = d{*}\gamma_0$, *c.f.* D. D. Joyce (2007), Theorem 8.4.5 (recall that $\Omega_0$ is the volume form on $L$ and then $*\gamma_0 = i_v\Omega_0$).

At this point, we can rewrite the problem as the invertibility of the linear operator $d \oplus d^*\colon \Omega^1 \to \Omega^2 \oplus \Omega^0$. Indeed,

$$L_t \text{ is sLag} \iff \begin{cases} d\gamma &= -i_{L_0}^*\dot{\omega}_0 + t^{-1}Q_1(\gamma_0) \\ d^*\gamma &= *\left(\dot{\theta}_0 i_{L_0}^*\Omega_0 - \operatorname{Im}\left(i_{L_0}^*\dot{\Omega}_0\right) + t^{-1}Q_2(e^{i\dot{\theta}_0},\gamma_0)\right).\end{cases}$$

Let us check injectivity. Suppose we have $(d\gamma_1, d^*\gamma_1) = (d\gamma_2, d^*\gamma_2)$. Then $d(\gamma_1 - \gamma_2) = 0$ and $d^*(\gamma_1 - \gamma_2) = 0$, hence $\gamma_1 - \gamma_2$ is harmonic. Then the operator is invertible on the image up to harmonic forms, thus we consider the operator on 1-forms modulo harmonic forms (that is, on the $L^2$-orthogonal complement of harmonic forms).

Let us check that the right hand sides are in the range of the operator $d \oplus d^*$. By hypothesis, $i_{L_t}^*\omega_t$ is an exact form on $L_0$ for all $t$, hence so is $Q_1(\gamma_0)$, by Equation (4.2). Moreover, we need to look for the phase $e^{i\theta_t}$ so that it satisfies the relation

$$\int_{L_0} i_{L_t}^*\Omega_t \in e^{i\theta_t}\mathbb{R}.$$

By differentiating with respect to $t$, and restricting to $t = 0$, this implies

$$\int_{L_0} \left( \dot{\theta}_0 i^*_{L_0} \Omega_0 - \mathrm{Im} \left( i^*_{L_0} \dot{\Omega}_0 \right) \right) = \int_{L_0} d * \gamma_0 = 0,$$

hence the integrand on the left hand side must be coexact up to harmonic forms, and thus so must be $Q_2$. Then we can conclude by the Implicit Function Theorem, Theorem 4.1.9.

If moreover $[\mathrm{Im}(e^{-i\theta_0}\Omega_t)|_{L_0}] = 0$ for all $t$, then we can choose directly $e^{i\theta_t} \equiv e^{i\theta_0}$. □

## 4.2 Gluing construction

Consider a family $\mathcal{X} \to \Delta$ as in Theorem 1.2.5, and assume we only have nodal singularities in the sense of Definition 1.2.12.

**Proposition 4.2.1.** *If the smoothing $\mathcal{X}$ is versal for each node of its central fibre $X$, then each node of $X$ is the limit of a vanishing special Lagrangian $n$-sphere in the nearby fibres of $\mathcal{X}$.*

This result is obtained through a *gluing construction*, and a *perturbation* result. The steps are the following:

1. Consider the singular Calabi–Yau metric $\omega_0$ on $X_0$, constructed in Theorem 1.2.5. At each node, the metric $\omega_0$ is polynomially asymptotic to the Stenzel metric $\omega_C$ on $\{\sum_i z_i^2 = 0\}$. We can moreover prove that $\omega_0 = \omega_C + dd^c u$ for some $u$ that decays with rate $2 + \lambda$, *c.f.* Lemma A.1 in Hein & Sun (2017);

2. We can glue (appropriately scaled) copies of the Stenzel metric on $\{\sum_i z_i^2 = t\}$ to appropriately constructed asymptotically conical metrics $\omega_t$, by employing an argument similar to the one of Spotti (2014), that we will expose below;

3. In $\{\sum_i z_i^2 = 1\} \simeq T^*S^n$ there exists a special Lagrangian $L$ with respect to the Stenzel metric, namely the zero section;

4. The metrics $\omega_t$, which are not a priori Ricci-flat, will be starting points for an application of the Implicit Function Theorem in appropriate Banach spaces. In this way, we will find asymptotically conical Calabi–Yau metrics $\omega_t^{CY}$ with the crucial property that suitable rescalings converge smoothly to the Stenzel metric on $T^*S^n$;

5. As a byproduct of this construction, thanks to the deformation result on special Lagrangians of Theorem 4.1.8, we can construct special Lagrangians $L_t^{CY}$ with respect to $\omega_t^{CY}$, starting from the special Lagrangian $n$-sphere $L$ on the manifold $\{\sum_i z_i^2 = 1\}$ and using an appropriately rescaled path of metrics.

The gluing construction goes as follows, and follows Section 2 of Spotti (2014).

We will assume that the smoothing is *versal* at each node, or *generic* in the sense of Spotti (2014). This means that if $t$ is the parameter of the base $\Delta$ of the smoothing and $\{\sum_i z_i^2 = s\}$ is the family of versal deformation of the node, then

$$s = s(t) = Ct + O(t^2)$$

with $C \neq 0$. This is used in the construction to avoid blowing up of the Ricci curvature of the glued metric.

Throughout this section, we will assume there is only one singularity $x \in \mathcal{X}$. We want to construct on $X_t$ a metric $\omega_t$ that coincides with the Stenzel metric close to the singularity $x \in \mathcal{X}$. A general deformation of the node is given by $\{\sum_i z_i^2 = t\}$. Up to a change of variables, we can assume $t$ is real and positive. By the genericity assumption, we can identify $\{\sum_i z_i^2 = t\} \cap \{|z| \leq C\}$ with a subset $V_t \subset X_t$ for some $C > 0$. By adapting Lemma 2.1 in Spotti (2014), we get the following.

**Proposition 4.2.2.** *There exists a diffeomorphism*

$$F_t \colon X_0 \backslash N_0 \to X_t \backslash N_t,$$

*such that for* $|z|^2 \leq 4$ *we have*

$$F_t(z_0, \ldots, z_n) = w, \quad w_i = z_i + \frac{t}{2|z|^2}\overline{z}_i,$$

*where* $N_0 = \{|z|^2 \leq t\}$ *and* $N_t = \{|w|^2 \leq 3t/2\}$.

*Sketch of the proof.* We can find a diffeomorphism between $V_0 \backslash N_0$ and $V_t \backslash N_t$ defined as above. For the non compact part, we can interpolate this diffeomorphism with the one given by the asymptotically conical structures on $X_0$ and $X_t$, as they are asymptotic to the same cone. These asymptotically conical structures can be given in the same way as described in Section 2.5. □

Now, we can identify $V_1 := \{\sum_{i=0}^n z_i^2 = 1\}$ with the cotangent bundle of the $n$-sphere $T^*S^n$ and we can consider the Stenzel metric on $V_1$ given by $\eta_1 = dd^c u(r^2)$

as described in Section 4.1.1.

Pulling back by the holomorphic map $j_t\colon z_i \mapsto w_i = \sqrt{t}z_i$ and scaling the metric, we have a family of Ricci-flat metrics $\eta_{\delta,t} = \delta^2 j_t^* \eta_1 =: dd^c u_t$ on $V_t = \{\sum_{i=0}^n w_i^2 = t\}$ such that $\mathrm{diam}_{\eta_{\delta,t}}(L_t) = \delta D$, where $D = \mathrm{diam}_{\eta_1}(L_1)$. From now on, we set the bijective relation between $\delta$ and $t$ as

$$\delta^2 = t^{(n-1)/n}.$$

Note that this choice is natural given that the radius function on the Calabi–Yau cone $\{\sum_i z_i^2 = 0\}$ is given by

$$r(z) = |z|^{(n-1)/n}.$$

Consider now the singular Calabi–Yau metric $\omega_0$ on $X_0$.

**Lemma 4.2.3** (Lemma A.1, Hein & Sun (2017)). *Locally close to the singularity $x$,*

$$P^*\omega_0 = \omega_C + dd^c\phi$$

*for some function $\phi \in C^\infty_{2+\gamma}$, where $\gamma > 0$ and $P$ is a biholomorphism between a neighbourhood of $x$ and a neighbourhood of the vertex in the cone.*

*Proof.* By Theorem 1.2.5, we know that $P^*\omega_0 - \omega_C = dd^c\phi$ for a smooth function $\phi$ such that $dd^c\phi \in C^\infty_\gamma$, where the rate $\gamma > 0$ is intended at the singularity. In particular, $\Delta\phi \in C^\infty_\gamma$. By Proposition 2.9 in Hein & Sun (2017), $\phi = \overline{\phi} + h$, where $h$ is a finite sum of homogeneous harmonic functions with rates in $[0, 2+\gamma)$ and $\overline{\phi} \in C^\infty_{2+\gamma}$. Reducing $\gamma$ if necessary we can assume there are no indicial roots between 2 and $2+\gamma$, and therefore, using point *2)* of Lemma 3.4.10 as in Section 3.4.2, we can modify $P$ so that $h$ is a sum of homogenous harmonic functions of rate in $[0, 2)$. These are all pluriharmonic by point *1)* of Lemma 3.4.10 and therefore we can take $\phi = \overline{\phi}$. □

From the above, we can write

$$\omega_0 = dd^c\varphi_0^{\mathrm{ext}}$$

on $V_0$, where

$$\varphi_0^{\mathrm{ext}}(z) = |z|^{2(n-1)/n} + O(|z|^{(2+\gamma)(n-1)/n}).$$

Starting from this metric, we can define the first piece of our gluing, that will correspond to the external part away from the singularity:

$$\omega_t^{\mathrm{ext}} := dd^c \left( (F_t)_* \varphi_0^{\mathrm{ext}} \right)$$

on $V_t$.

Now we want to glue the Stenzel metric $\eta_t$ to $\omega_t^{\mathrm{ext}}$. Recall $\delta^2 = t^{(n-1)/n}$, and consider the gluing region $\{\delta^\alpha \leq |w| \leq 2\delta^\alpha\} \cap V_t$, for $\alpha \in [0, 2)$.

Write[1] $\varphi_t^{\mathrm{int}}(w) := \delta^2 j_t^* u(w)$, that is, the potential of the Stenzel metric.

We have the following estimates.

- $|\nabla^k_{\eta_t}(\varphi_t^{\mathrm{int}} - |w|^{2(n-1)/n})|_{\eta_t} \leq C\delta^{2n/(n-1)}|w|^{-\varepsilon-k(n-1)/n}$, for some $\varepsilon > 0$, where $\varepsilon$ is such that $|u(z) - |z|^{2(n-1)/n}| \leq C|z|^{-\varepsilon}$;

- $\left|\nabla^k_{\eta_t}\left(\varphi_t^{\mathrm{ext}} - \left(\sqrt{\frac{|w|^2+\sqrt{|w|^4-\delta^{4n/(n-1)}}}{2}}\right)^{2(n-1)/n}\right)\right|_{\eta_t} \leq C|w|^{(2+\gamma)(n-1)/n-k(n-1)/n}$.
  This comes from the fact that $|w| = \sqrt{|z|^2 + \frac{t^2}{4|z|^2}}$ under the diffeomorphism $F_t$, and the expression in parenthesis is its inverse, that is, $|z|$ written in $w$-coordinates.

At this point we can compute

$$\begin{aligned}|\nabla^k_{\eta_t}(\varphi_t^{\mathrm{ext}} - \varphi_t^{\mathrm{int}})|_{\eta_t} \leq C &\left( |w|^{(2+\gamma)(n-1)/n-k(n-1)/n} + \delta^{2n/(n-1)}|w|^{-\varepsilon-k(n-1)/n} \right) \\ &+ \left|\nabla^k_{\eta_t}\left(|w|^{2(n-1)/n} - \left(\sqrt{\frac{|w|^2+\sqrt{|w|^4-\delta^{4n/(n-1)}}}{2}}\right)^{2(n-1)/n}\right)\right|_{\eta_t},\end{aligned}$$

hence

$$\begin{aligned}|\nabla^k_{\eta_t}(\varphi_t^{\mathrm{ext}} - \varphi_t^{\mathrm{int}})|_{\eta_t} \leq C(&|w|^{(2+\gamma)(n-1)/n-k(n-1)/n} + \delta^{2n/(n-1)}|w|^{-\varepsilon-k(n-1)/n} \\ &+ \delta^{4n/(n-1)}|w|^{-4+2(n-1)/n-k(n-1)/n}),\end{aligned}$$

where the last inequality is due to the following. Suppose $k = 0$, the case $k \geq 1$ is analogous. Write $\beta = 2(n-1)/n$, and use the diffeomorphism $F_t$ to write the

[1]For $n = 2$, $\varphi_t^{\mathrm{int}}(w) = \sqrt{|w|^2 + \delta^4}$.

expression in terms of the coordinate $z$. We get

$$\begin{aligned}\left|\left(|z|^2+\frac{t^2}{4}|z|^{-2}\right)^{\beta/2}-|z|^\beta\right|_{\eta_t} &= \left||z|^\beta\left(1+\frac{t^2}{4}|z|^{-4}\right)^{\beta/2}-|z|^\beta\right|_{\eta_t} \\ &= \left||z|^\beta\left(1+\frac{\beta}{2}\left(\frac{t^2}{4}|z|^{-4}\right)+O(|z|^{-8})\right)-|z|^\beta\right|_{\eta_t} \\ &\leq Ct^2|z|^{-4+\beta}.\end{aligned}$$

From the above, on the annulus[2] $\delta^\alpha \leq |w| \leq 2\delta^\alpha$,

$$\begin{aligned}|\nabla^k_{\eta_t}(\varphi^{\text{ext}}_t-\varphi^{\text{int}}_t)|_{\eta_t} &= O(\delta^{\alpha(2+\gamma)(n-1)/n-k\alpha(n-1)/n})+O(\delta^{2n/(n-1)-\varepsilon\alpha-k\alpha(n-1)/n}) \\ &\quad +O(\delta^{4n/(n-1)-4\alpha+2\alpha(n-1)/n-k\alpha(n-1)/n}),\end{aligned}$$

and the error is minimised for $\alpha^* := \frac{2n}{n-1}\left(\frac{(2+\gamma)(n-1)}{n}+\varepsilon\right)^{-1}$. By Stenzel (1993), *c.f.* Lemma 5.14 in Conlon & Hein (2013a), we have $\varepsilon = 1$ when $n = 2$, $\varepsilon = 8/3-\nu$ for all $\nu > 0$ when $n = 3$, and $\varepsilon = 2+2/n$ when $n \geq 4$. Hence, if $\gamma$ is the rate of Lemma 4.2.3,

- $\alpha^* = 4/3$ when $n = 2$, as one can take $\gamma = 2$ in this case, see Spotti (2014), Lemma 2.2;
- $\alpha^* = 9/(12+2\gamma-3\nu)$ for all $\nu > 0$ when $n = 3$;
- $\alpha^* = n^2(n-1)^{-1}((2+\gamma/2)n-\gamma/2)^{-1}$ when $n \geq 4$.

**Lemma 4.2.4.** *For $|w|_{|X_t} \leq 2$, define on $X_t$ the form*

$$\omega^1_t := i\partial_t\overline{\partial}_t(\chi_t\varphi^{\text{ext}}_t+(1-\chi_t)\varphi^{\text{int}}_t),$$

*where $\chi_t := \chi\left(\delta^{-\alpha^*}|w|\right)$ is a cut-off function supported in $|w| \geq \delta^{\alpha^*}$ and identically one in $|w| \geq 2\delta^{\alpha^*}$. As above, here $\delta^2 = t^{(n-1)/n}$. For $\delta$ sufficiently small,*

- $||\nabla^k_{\eta_t}(\omega^1_t-\eta_t)||_{\eta_t} = O(\delta^{\alpha^*-\alpha^*k\frac{n-1}{n}})$;
- $\omega^1_t > 0$.

*Proof.* The first point comes from the estimates above by observing that $||\nabla^k_{\eta_t}\chi_t||_{\eta_t} = O(\delta^{-\alpha^*k\frac{n-1}{n}})$ when $\delta^{\alpha^*} \leq |w| \leq 2\delta^{\alpha^*}$. Then the positivity of $\omega^1_t$ is derived by the positivity of $\eta_t$. □

[2]Suppose $\delta^{2\alpha} \leq |z|^2 \leq 4\delta^{2\alpha}$, and recall $|z|^2 \geq t = \delta^{2n/(n-1)}$. Then $\delta^{2\alpha} \leq |w|^2 \leq 4\delta^{2\alpha} + \delta^{2n/(n-1)} \leq 5\delta^{\min(2\alpha,2n/(n-1))} = 5\delta^{2\alpha}$. One can check that the reverse implication holds as well.

At this point, we have a candidate metric $\omega_t^1$ close to the singularity in the family that coincides with Stenzel's metric in $X_0$, and that outside of a compact set coincides with the metric $\omega_0$.

We now want to construct an approximate Calabi–Yau candidate metric $\omega_t^2$ away from the singularities, on the non-compact part. To do this, again we start by considering the singular Calabi–Yau metric $\omega_0$ on $X_0$ given by Theorem 1.2.5. By construction, we can write

$$\omega_0 = \beta_0 + i\partial_0\overline{\partial}_0\phi_0,$$

where $\beta$ is a Kähler form on the family $\mathcal{X}$ and $\beta_t := \beta|_{X_t}$; by definition, $\beta_t \in c_1(L_t)$. We can then define

$$\omega_t^2 := \beta_t + i\partial_t\overline{\partial}_t(F_t)_*\phi_0.$$

The goal is to glue the metrics $\omega_t^1$ and $\omega_t^2$ to a metric $\omega_t$. Indeed, this metric would have the property of converging to the Calabi–Yau metric on the singular manifold $X_0$ away from the singularity and being exactly the Stenzel metric close to the singularity.

Recall we denoted $V_t$ the subset of $X_t$ that can be identified with $\{\sum_i z_i^2 = t\}$. We have

**Proposition 4.2.5.** *There exist functions $p_t, \psi_t$ on $V_t$ such that*

$$\omega_t := \begin{cases} \omega_t^1 & \text{on } |w||_{X_t} \leq 1; \\ i\partial_t\overline{\partial}_t\left(\chi_t(\varphi_t^{\text{ext}} - p_t) + (1-\chi_t)\left(\psi_t + (F_t)_*\phi_0\right)\right) & \text{on } 1 \leq |w||_{X_t} \leq 2; \\ \omega_t^2 & \text{elsewhere} \end{cases}$$

*is a Kähler metric in $[\beta_t]_{H^{1,1}(X_t)} = c_1(L_t)$, where $\chi_t$ is a cut-off function supported in $|w| \leq 2$ and equal to $1$ for $|w| \leq 1$.*

*Proof.* On $V_0$, we can write

$$\omega_0 = \beta_0 + i\partial\overline{\partial}\phi_0.$$

Fix a never vanishing section $\sigma$ on $V_t \cap \{|w| \leq 2\}$ of the line bundle $\mathcal{L}$ restricted to $\bigcup_{t\in\Delta} V_t \cap \{|w| \leq 2\}$, for small enough $t$. This exists because $\bigcup_{t\in\Delta} V_t \cap \{|w| \leq 2\}$ is contractible. Define sections $\sigma_t := \sigma|_{V_t}$. Define $s_t := |\sigma_t|^2_{\beta_t}$, then we can write $\beta_t = -dd^c \log s_t$ on $\{|w| \leq 2\}$.

Consider the function

$$p_0 := \varphi_0^{\text{ext}} + \log|\sigma_0|^2_{\omega_0}.$$

Note that the norm is taken with respect to $\omega_0$. This function is pluriharmonic as $dd^c p_0 = \omega_0 - \omega_0 = 0$. We can write it as the real part of a holomorphic function $h$ on $V_0$, which can be extended to a holomorphic function $H$ on $\bigcup_{t\in\Delta} V_t \cap \{|w| \leq 2\}$. Thus we can define the function $p_t := \mathrm{Re}(H|_{V_t})$ which is pluriharmonic on $V_t \cap \{|w| \leq 2\}$.

Define $\psi_t := -\log s_t$. In this way, on the annulus $1 \leq |w||_{X_t} \leq 2$, we have $\omega_t^1 = dd^c \varphi_t^{\mathrm{ext}} = dd^c(\varphi_t^{\mathrm{ext}} - p_t)$ and $\omega_t^2 = dd^c(\psi_t + (F_t)_*\phi_0)$.

Define $a_t := \varphi_t^{\mathrm{ext}} - p_t$ and $b_t := \psi_t + (F_t)_*\phi_0$. At this point, it is clear that $\omega_t$ is a Kähler metric for small enough $t$ if

$$|a_t - b_t| + ||d(a_t - b_t)||_{\omega_t^2} \to 0 \quad \text{for } t \to 0.$$

Indeed, if these are small enough, the potential offsetting given by the cut-off functions $\chi_t$ (which could make $\omega_t$ more negative) is also smaller, hence the metric stays positive.

Now, by definitions and estimates for $F_t$,

$$\begin{aligned} |a_t - b_t| &= |(F_t)_*\varphi_0^{\mathrm{ext}} - p_t + \log s_t - (F_t)_*\phi_0| \\ &= |\left((F_t)_*p_0 - p_t\right) + \left(\log s_t - (F_t)_* \log s_0\right)| = O(t), \end{aligned}$$

using the fact that $\log|\sigma_0|^2_{\omega_0} = \log s_0 - \phi_0$. We have the same estimate for the other derivatives. □

**Existence of special Lagrangian vanishing cycles assuming convergence to Stenzel's metric**

At this point, for suitable $t$, we have constructed a metric $\omega_t$ that coincides with the Stenzel metric close to the singularity (in the family $\mathcal{X}$) and coincides — up to the diffeomorphism $F_t$ — with the Calabi–Yau metric on $X_0$ outside a neighbourhood of the singularity.

We can thus find special Lagrangians $L_t$ with respect to $(\omega_t, \Omega_t)$ that correspond to the zero sections with respect to the identifications of $\{\sum_i z_i^2 = t\}$ with $T^*S^n$.

By carefully checking errors and applying Theorem 4.1.9, we will be able to produce Calabi–Yau metrics $\omega_t^{CY}$ and control their rescaled convergence to the Stenzel metric $\eta_1$. We will then use Theorem 4.1.8, which enables us to perturb the special Lagrangian submanifold $L = L_1$ with respect to the Stenzel metric $\eta_1$ to special Lagrangians with respect to $\omega_t^{CY}$.

On $V_1 = \{\sum_{i=0}^n z_i^2 = 1\} \cap \{|z| \leq C\}$, consider the rescaled metric $\delta^{-2}(j_t)_*\omega_t$, where as above $j_t\colon V_t \to V_1$, $j_t(z) = \sqrt{t}z$. By definition, $\delta^{-2}(j_t)_*\omega_t \to \eta_1$ in $C^\infty(V_1)$ for $t \to 0$, where $\eta_1$ is the Stenzel metric.

We want to apply Theorem 4.1.8 on the path of (still to be constructed) asymptotically conical Calabi–Yau metrics $\omega_t^{CY}$. Assuming that we have also have a "rescaled" convergence $\delta^{-2}(j_t)_*\omega_t^{CY} \to \eta_1$ in $C^\infty(V_1)$ for $t \to 0$, we have the following existence result.

**Lemma 4.2.6.** *Suppose $\delta^{-2}(j_t)_*\omega_t^{CY}|_{V_1} \to \eta_1$ in $C^\infty(V_1)$. Then, for $t$ sufficiently small, we can find special Lagrangians $L_t^{CY}$ with respect to $\omega_t^{CY}$.*

*Proof.* Since we know $\delta^{-2}(j_t)_*\omega_t^{CY}|_{V_1} \to \eta_1$, we have a path $[\delta^{-2}(j_t)_*\omega_t^{CY}]_t$ of metrics on $V_1$ such that

- for $t = 0$ we have $\eta_1$;
- $[\delta^{-2}(j_t)_*\omega_t^{CY}|_L] \equiv 0$ for all $t$ on $L$, which is the standard special Lagrangian $n$-sphere on $V_1$ with respect to $\eta_1$. This will come from the construction of $\omega_t^{CY}$, which will be in the same cohomology class of $\omega_t$.

In particular we can apply Theorem 4.1.8 and get existence of special Lagrangian $n$-spheres $\overline{L}_t$ with respect to $\delta^{-2}(j_t)_*\omega_t^{CY}$, hence special Lagrangians $L_t^{CY} := \delta^2(j_t)^{-1}(\overline{L}_t)$ with respect to $\omega_t^{CY}$. □

The lemma above follows *provided* that the convergence $\delta^{-2}(j_t)_*\omega_t^{CY}|_{V_1} \to \eta_1$ in $C^\infty(V_1)$ holds. The rest of the argument is therefore devoted to proving such convergence.

Consider the following weight function $\rho_t\colon X_t \to \mathbb{R}_{>0}$, such that

- $\rho_t(|w|_{|X_t}) \equiv \delta$, if $|w|_{|X_t} \leq 2\delta^{n/(n-1)}$;
- $\rho_t(|w|_{|X_t}) \equiv |w|_{|X_t}^{(n-1)/n}$, if $3\delta^{n/(n-1)} \leq |w|_{|X_t} \leq 1/2$;
- $\rho_t(|w|_{|X_t}) \equiv 1$, if $1 \leq |w|_{|X_t} \leq 2$.

Define then the weighted norm as

$$||\varphi||_{C^{k,\gamma}_{\rho_t,\beta,\lambda}(X_t)} := \sum_{j\leq k} ||\rho_t^{-(\beta-j)}\nabla^j\varphi||_{L^\infty(X_t)} + [\varphi]_{C^{k,\gamma}_{\rho_t,\beta}(X_t)} + ||\varphi||_{C^{k,\gamma}_{\lambda}(X_t\setminus K_t)},$$

where

$$[\varphi]_{C^{k,\gamma}_{\rho_t,\beta}} := \sup_{x\neq y} \min\{(\rho_t(x))^{-(\beta-k-\gamma)}, (\rho_t(y))^{-(\beta-k-\gamma)}\} \frac{||\nabla_t^k\varphi(x) - \nabla_t^k\varphi(y)||_t}{d_t(x,y)^\gamma},$$

$||\varphi||_{C^{k,\gamma}_{\lambda}(X_t \setminus K_t)}$ is the asymptotically conical norm defined in Definition 2.3.2 and $K_t$ is the compact set of the asymptotically conical structure.

Define $C^{k,\gamma}_{\rho_t,\beta,\lambda}(X_t)$ as the completion of $C^\infty(X_t)$ with respect to $||\cdot||_{C^{k,\gamma}_{\rho_t,\beta,\lambda}}$.

By construction, we will write $\omega_t^{CY} = \omega_t + dd^c\varphi_t^{CY}$ for some potential $\varphi_t^{CY}$.

**Lemma 4.2.7.** *Suppose*

$$||dd^c\varphi_t^{CY}||_{C^{0,\gamma}_{\rho_t,\beta-2,\lambda-2}} = O(\delta^\mu),$$

*for some $\lambda, \beta \in \mathbb{R}$ and $\mu > 2-\beta$. Then $\delta^{-2}(j_t)_*\omega_t^{CY}|_{V_1} \to \eta_1$ in $L^\infty_{\eta_1}(V_1)$.*

*Proof.* By the bound and by construction of the weighted norm we can infer that the unweighted norm behaves as

$$||dd^c\varphi_t^{CY}||_{L^\infty_{\omega_t}} \leq O(\delta^{\mu-2+\beta}).$$

Since $\mu > 2-\beta$, we have $||dd^c\varphi_t^{CY}||_{L^\infty_{\omega_t}} \to 0$ for $\delta \to 0$. Then

$$\begin{aligned} ||\delta^{-2}(j_t)_*\left(\omega_t - \omega_t^{CY}\right)|_{V_1}||_{L^\infty_{\eta_1}} &= ||\delta^{-2}(j_t)_*\left(dd^c\varphi_t^{CY}\right)|_{V_1}||_{L^\infty_{\eta_1}} \\ &= ||dd^c\varphi_t^{CY}|_{V_t}||_{L^\infty_{\omega_t}} \to 0, \end{aligned}$$

and since by construction $\delta^{-2}(j_t)_*\omega_t|_{V_1} \to \eta_1$, we get $\delta^{-2}(j_t)_*\omega_t^{CY}|_{V_1} \to \eta_1$ in $L^\infty_{\eta_1}(V_1)$. □

**Lemma 4.2.8.** *Under the hypothesis of Lemma 4.2.7, we have $\delta^{-2}(j_t)_*\omega_t^{CY}|_{V_1} \to \eta_1$ in $C^\infty(V_1)$.*

*Sketch of the proof.* We have a one parameter family of Riemannian manifolds with boundary $(V_1, \delta^{-2}(j_t)_*\omega_t^{CY}|_{V_1})_\delta$, all of whose metrics are Ricci-flat (in particular, Einstein). We can follow the same argument as in Theorem 10.25 in Cheeger (2001), which deals with the closed case.

The main point here is that we can find a *harmonic* coordinate system with domain a collection of balls $\{\mathbb{B}_r(p_i)\}_i$. In harmonic coordinates, the equation $\mathrm{Ric}(g) \equiv 0$ is a quasi-linear elliptic system of equations for the metric $g$. By standard elliptic estimates, plus convergence in $L^\infty$, the convergence takes place in $C^\infty$. □

We know already that the asymptotically conical Calabi–Yau metric $\omega_t^{CY}$ on $X_t$ is the one constructed in Conlon & Hein (2013a); the existence result of Conlon & Hein (2013a) does not however yield estimates depending on the smoothing parameter $t$.

To gain such estimates, we can solve again the Monge–Ampère equation

$$(\omega_t + dd^c\varphi_t^{CY})^n = i^{n^2}\Omega_t \wedge \overline{\Omega}_t \quad (= (\omega_t^{CY})^n), \tag{4.3}$$

by employing the Implicit Function Theorem (Theorem 4.1.9) on the Banach space $C^{k,\gamma}_{\rho_t,\alpha}(X_t)$ and checking a priori estimates. This would automatically give estimates for $||\varphi_t^{CY}||_{C^{k,\gamma}_{\rho_t,\alpha}}$ depending on $\delta$.

Note that this is not fundamentally different from the procedure we employed in Chapter 3, where we concentrated on solving the Monge–Ampère equation by keeping track of both the rate of decay at infinity and the rate of decay at the singularities. This time we have to keep track of estimates depending on the shrinking parameter $\delta$.

**Proposition 4.2.9.** *There exists a solution $\varphi_t^{CY}$ of Equation 4.3 such that*

$$||dd^c\varphi_t^{CY}||_{C^{0,\gamma}_{\rho_t,\beta-2,\lambda-2}} = O(\delta^\mu),$$

*with $\mu = \alpha^* - \alpha^*(\beta-2)\frac{n-1}{n}$, where $\alpha^*$ is the optimal exponent obtained in the construction of the glued metric $\omega_t$ and $\lambda$ is the rate of decay at infinity of the asymptotically conical metric on $X_0$.*

The proof of this estimate consists of three steps.

i) Securing estimates for the Ricci potential $f_t = \log\big((\omega_t + dd^c\varphi_t^{CY})^n/\omega_t^n\big)$, which is the initial error in Theorem 4.1.9, and analysing their dependence on $\delta$;

ii) checking that the linearisation of the Monge–Ampère operator, *i.e.* the Laplacian

$$\Delta_{\omega_t} \colon C^{2,\gamma}_{\rho_t,\beta,\lambda}(X_t) \to C^{0,\gamma}_{\rho_t,\beta-2,\lambda-2}(X_t),$$

is invertible with norm of the inverse independent of $\delta$, for a small enough $\delta$; for $\beta > 0$, the Laplacian $\Delta_{\omega_0}$ is not invertible, its kernel consisting of the constant and harmonic functions on the cone of rate less than $\beta$. However, we have the following results about harmonic functions on asymptotically conical manifolds:

**Lemma 4.2.10** (Conlon & Hein (2013a), Corollary 3.9)**.** *Any harmonic function of rate strictly less than 2 on an asymptotically conical Kähler manifold must already be pluriharmonic.*

Note that this is strictly related (in fact, a consequence) of Theorem 3.4.10. This allows us to consider functions in $C^{2,\gamma}_{\rho_t,\beta,\lambda}(X_t)$ modulo constants and homogeneous harmonic functions of rate less than 2 on the cone. Note that this is similar to the procedure employed in the openness part in Section 3.4.2, where the main point is using Theorem 3.4.9;

iii) apply the Implicit Function Theorem, *i.e.* Theorem 4.1.9.

Let us start with point i).

We divide the analysis in different regions (as we have different estimates given by the gluing construction).

- On $\{|w|_{|X_t} \leq \delta^{\alpha^*}\}$, $\omega_t$ is Ricci-flat, so $f_t \equiv 0$;
- on the gluing region $\delta^{\alpha^*} \leq |w|_{|X_t} \leq 2\delta^{\alpha^*}$, we have

$$\omega_t = i\partial_t\overline{\partial}_t(\chi_\delta\varphi_t^{\text{ext}} + (1-\chi_t)\varphi_t^{\text{int}}),$$

  where $\varphi_t^{\text{ext}} = (F_t)_*\varphi_0$ and $dd^c\varphi_t^{\text{int}} = \eta_t$. Hence here it suffices to use the estimate

$$||\nabla^k_{\eta_t}(\omega_t^1 - \eta_t)||_{\eta_t} = O(\delta^{\alpha^* - \alpha^* k\frac{n-1}{n}})$$

  found in Lemma 4.2.4, so that

$$||\nabla^k_{\omega_t} f_t||_{\omega_t} = O(\delta^{\alpha^* - \alpha^* k\frac{n-1}{n}}).$$

- on $\{2\delta^{\alpha^*} \leq |w|_{|X_t} \leq 1\}$, we have $\omega_t \equiv dd^c((F_t)_*\varphi_0)$, where $\varphi_0$ is the Kähler potential of $\omega_0$. This implies smooth convergence of $\omega_t$ to $\omega_0$, hence uniform estimates for a small enough $t$ by compactness.

  This entails that we only need to check the difference between the complex structures under the diffeomorphism, getting

$$||\nabla^k_{\omega_t}(J_t - (F_t)_*J_0)||_{\omega_t} = O(t) = O(\delta^{\frac{2n}{n-1}}),$$

  for all $k \in \mathbb{N}$, hence

$$||\nabla^k_{\omega_t} f_t||_{\omega_t} = O(\delta^{\frac{2n}{n-1}}).$$

- on $X_t \backslash \{|w|_{|X_t} \leq 1\}$, again we have $\omega_t \equiv dd^c((F_t)_*\varphi_0)$. Here the we need to consider the right rate $\lambda$ at infinity, so that the complex structure on $X_t$ decays to the one on the cone. Thus the rest is as above,

$$||\nabla^k_{\omega_t} f_t||_{\omega_t} = O(\delta^{\frac{2n}{n-1}}).$$

Note that the error $f_t$ is maximised in the gluing region, where it is $O(\delta^{\alpha^*})$.

For point ii), the harmonic functions on the smoothings of the ordinary double point are the main obstructions to the invertibility of the Laplace operator. The following lemma fixes the family of harmonic functions on $X_t$ that we want to consider.

**Lemma 4.2.11.** *Let $r^{\mu_j}\phi_j$ be a homogeneous harmonic function on the cone $\{\sum_i z_i^2 = 0\}$ with rate $\mu_j$ in $[0, 2)$. In particular, let $\mu_0 = 0$ so that the constant function 1 is the homogeneous harmonic function of rate 0.*

*For $t \neq 0$, there exist a harmonic function $h_j^t$ on $(\{\sum_i z_i^2 = t\}, \eta_t)$ asymptotic to $r^{\mu_j}\phi_j$ with respect to the asymptotically conical structure given by the diffeomorphism $F_t$ defined by*

$$F_t(z_0, \dots, z_n) = w, \quad w_i = z_i + \frac{t}{2|z|^2}\overline{z}_i.$$

*Proof.* First, we will define $h_j^1$ on $\{\sum_i z_i^2 = 1\}$. Let $\{|z| \leq 1\} \subset \{\sum_i z_i^2 = 1\}$ the compact subset where $F_1$ is not defined. Let $\chi$ be a cutoff function such that $\chi \equiv 0$ on $\{|z| < 1\}$ and $\chi \equiv 1$ on $\{|z| > 2\}$.

Consider the solution $h_j^{(r)}$ of the problem

$$\begin{cases} \Delta_{\eta_1} h_j^{(r)} &= -\Delta_{\eta_1}(\chi(F_1)_*(r^{\mu_j}\phi_j)), \\ h_j^{(r)}|_{\partial\mathbb{B}_r} &\equiv 0, \end{cases}$$

on $\mathbb{B}_r$ for $r > 2$.

Then $\overline{h}_j^{(r)} := h_j^{(r)} + \chi(F_1)_*(r^{\mu_j}\phi_j)$ is harmonic on $\mathbb{B}_r$, and moreover $|\overline{h}_j^{(r)}||_{\partial\mathbb{B}_r} \leq Cr^{\mu_j}\max_{\partial\mathbb{B}_r}|(F_1)_*\phi_j|$. Note that since $r^{\mu_j}\phi_j$ is harmonic on the cone,

$$||\Delta(\chi(F_1)_*(r^{\mu_j}\phi_j))||_{L^\infty_{\mu-2}} \leq C$$

for a $C > 0$ independent of $r$, where $\mu - 2 < \mu_j - 2$ is the rate of decay of $\Delta(\chi(F_1)_*(r^{\mu_j}\phi_j))$. This is given by the rate of decay of the asymptotically conical structure and that of the the complex structure, which are less than $-2$ (*c.f.* Proposition 5.9 in Conlon & Hein (2013a)).

By Theorem 3.4.7, we have

$$||h_j^{(r)}||_{L^\infty_\mu} \leq C||\Delta(\chi(F_1)_*(r^{\mu_j}\phi_j))||_{L^\infty_{\mu-2}} \leq C,$$

where $C > 0$ is independent of $r$.

Then, by letting $r \to \infty$, we get a function $\overline{h}_j^{(\infty)}$ on $\{\sum_i z_i^2 = 1\}$ such that $\Delta\overline{h}_j^{(\infty)} = 0$ and $||h_j^{(\infty)}||_{L^\infty_\mu} \leq C$. Moreover, since $\Delta(\chi(F_1)_*(r^{\mu_j}\phi_j))$ is smooth, the convergence $\overline{h}_j^{(r)} \to \overline{h}_j^{(\infty)}$ is locally smooth. We define $h_j^1 := \overline{h}_j^{(\infty)}$.

Then on $\{\sum_i z_i^2 = t\}$ we define

$$h_j^t := \delta^{\mu_j}(j_t)^* h_j^1$$

by rescaling in the appropriate way, since $\eta_t = \delta^2(j_t)^*\eta_1$. □

**Lemma 4.2.12.** *Let $V_t \simeq \{\sum_i z_i^2 = t,\ |z| < 1\} \subset X_t$ and $\chi_t$ be a cutoff function such that $\chi_t \equiv 1$ on $V_t$ and $\chi_t \equiv 0$ on $X_t \backslash \tilde{V}_t$, where $\tilde{V}_t := \{\sum_i z_i^2 = t,\ |z| < 2\} \supset V_t$, then*

$$||i\partial_t\overline{\partial}_t(\chi_t h_j^t)||_{L^\infty(\omega_t)} \leq C,\ ||i\partial_t\overline{\partial}_t(\chi_t h_j^t)||_{C^{0,\gamma}_{\rho_t,\mu_j-2,\lambda-2}} \leq C,$$

*where $C > 0$ is independent of $t$. In particular,*

$$||\Delta_{\omega_t}(\chi_t h_j^t)||_{L^\infty(\omega_t)} \leq C,\ ||\Delta_{\omega_t}(\chi_t h_j^t)||_{C^{0,\gamma}_{\rho_t,\mu_j-2,\lambda-2}} \leq C.$$

*Proof.* By Lemma 4.2.10, $h_j^t$ is pluriharmonic on $\{\sum_i z_i^2 = t\}$. This implies

$$||i\partial_t\overline{\partial}_t(\chi_t h_j^t)||_{L^\infty(\omega_t)} \leq C,\ ||i\partial_t\overline{\partial}_t(\chi_t h_j^t)||_{C^{0,\gamma}_{\rho_t,\mu_j-2,\lambda-2}} \leq C,$$

which follow by noting that the estimates on $\chi_t, \partial\chi_t, \overline{\partial}\chi_t, i\partial\overline{\partial}\chi_t$ with respect to $\omega_t$ are independent of $t$, because $d\chi_t$ has support outside of the gluing region, where we have $\omega_t \equiv dd^c((F_t)_*\varphi_0)$, thus $\omega_t$ smoothly converges to $\omega_0$, yielding uniform estimates for small enough $t$ by compactness.

Moreover, $h_j^t$ is harmonic with respect to $\omega_t$ as well, hence

$$||\Delta_{\omega_t}(\chi_t h_j^t)||_{L^\infty(\omega_t)} \leq C,\ ||\Delta_{\omega_t}(\chi_t h_j^t)||_{C^{0,\gamma}_{\rho_t,\mu_j-2,\lambda-2}} \leq C,$$

again because we are in the region where $\omega_t \equiv dd^c((F_t)_*\varphi_0)$. □

**Lemma 4.2.13.** *Let $t_i \to 0$, and as before let $\delta_i^2 = t_i^{(n-1)/n}$. Then*

$$\delta_i^{-1}(j_{t_i})_*\rho_{t_i} \to \rho_1$$

*on $\{|z|_{\eta_1} \leq 3\} \subset \{\sum_i z_i^2 = 1\}$.*

*Proof.* On $\{|z| \leq 2\}$ we have $\rho_1 \equiv 1$ and $(j_{t_i})_*\rho_{t_i} \equiv \delta_i$. Outside, we have

$$\begin{aligned}
\rho_1(z) = |z|_{\eta_1}^{(n-1)/n} &= \lim_i |z|^{(n-1)/n}_{\delta_i^{-2}(j_{t_i})_*\omega_{t_i}} \\
&= \lim_i \delta_i^{-1}|z/\sqrt{t_i}|^{(n-1)/n}_{\omega_{t_i}} \\
&= \lim_i \delta_i^{-1}\rho_{t_i}(z/\sqrt{t_i}) \\
&= \lim_i \delta_i^{-1}(j_{t_i})_*\rho_{t_i}(z).
\end{aligned}$$

□

The pairing below will fix the orthogonal space to the constants and the harmonic functions on $X_t$ in $C^{2,\gamma}_{\beta,\lambda}(X_t)$.

**Definition 4.2.14.** *Define the pairing on $X_t$ given by*

$$(u, v) \mapsto \langle u, v\rangle_{\rho_t,\beta} := \int_{X_t} (uv)\rho_t^{-2\beta-2n}\omega_t^n,$$

*for all pairs $(u, v)$ of functions such that the integral is finite.*

*Remark* 4.2.15. When applying the pairing in the following, one of the functions will always be compactly supported.

The following definition introduces the space we are going to use in the application of the Implicit Function Theorem.

**Definition 4.2.16.** *Let $\{r^{\mu_i}\phi_i\}_i$ be a basis for homogeneous harmonic functions on the ODP with rates in $(0, \beta)$, $\{h_i^t\}_i$ the associated harmonic functions as in Lemma 4.2.11. Let $\varphi \in C^{2,\gamma}_{\rho_t,\beta,\lambda}(X_t)$. Then we can write*

$$\varphi = \overline{\varphi} + \chi_t\left(\sum_j \lambda_j h_j^t\right),$$

*where,*

$$\langle\overline{\varphi}, \chi_t\rangle_{\rho_t,\beta} = 0,\ \langle\overline{\varphi}, \chi_t h_j^t\rangle_{\rho_t,\beta} = 0,$$

*for all $j$.*

*Define $K_t := \mathrm{Span}_{\mathbb{R}}\{\chi_t, \{\chi_t h_j^t\}_j\}$, which is a finite dimensional vector space. Let $K_t^{\perp} \subset C^{2,\gamma}_{\rho_t,\beta,\lambda}(X_t)$ be the subspace of functions $\overline{\varphi}$ such that the conditions above occur. Then $C^{2,\gamma}_{\rho_t,\beta,\lambda}(X_t) = K_t \oplus K_t^{\perp}$. Define the following norm on $C^{2,\gamma}_{\rho_t,\beta,\lambda}(X_t)$:*

$$||\varphi||_{K_t \oplus K_t^{\perp}} := ||\overline{\varphi}||_{C^{2,\gamma}_{\rho_t,\beta,\lambda}(X_t)} + \left(\sum_j |\lambda_j|\right).$$

The following weighted Schauder estimates will be independent of the smoothing parameter $\delta$. This makes them crucial ingredients to estimate the norm of the inverse of the Laplace operator independently of $\delta$.

**Lemma 4.2.17.** *If $\delta$ is sufficiently small, then*

$$||\varphi||_{C^{2,\gamma}_{\rho_t,\beta,\lambda}} \leq C\left(||\varphi||_{L^{\infty}_{\rho_t,\beta,\lambda}} + ||\Delta_{\omega_t}\varphi||_{C^{0,\gamma}_{\rho_t,\beta-2,\lambda-2}}\right)$$

*holds for some $C > 0$ independent of $\delta$ and for all $\beta, \lambda$.*

*Proof.* Every point $p \in X_t$ has a neighbourhood of diameter approximately $\rho_t(p)$ that, after rescaling to unit size, has bounded geometry. Now, the weighted $C^{2,\gamma}_{\rho_t,\beta,\lambda}$ norm of $\varphi_i$ in this neighbourhood is equivalent to the standard unweighted $C^{2,\gamma}$ norm of $\rho_t^{-\beta}\varphi_i$ after rescaling. Then the estimate follows by local Schauder estimates in these neighbourhoods, which have bounded geometry. See Lemma 3.1 in Spotti (2014) for more details.

□

**Lemma 4.2.18.** *If $\delta$ is sufficiently small, then*

$$||\overline{\varphi}||_{C^{2,\gamma}_{\rho_t,\beta,\lambda}} \leq C\left(||\overline{\varphi}||_{L^{\infty}_{\rho_t,\beta,\lambda}} + ||\Delta_{\omega_t}\varphi||_{C^{0,\gamma}_{\rho_t,\beta-2,\lambda-2}} + |\lambda_0| + \sum_j |\lambda_j|\right)$$

*holds for some $C > 0$ independent of $\delta$ and for all $\beta, \lambda$.*

*Proof.* Applying Lemma 4.2.17 to $\overline{\varphi}$, we get

$$\begin{aligned}
||\overline{\varphi}||_{C^{2,\gamma}_{\rho_t,\beta,\lambda}} &\leq C\left(||\overline{\varphi}||_{L^{\infty}_{\rho_t,\beta,\lambda}} + ||\Delta_{\omega_t}\overline{\varphi}||_{C^{0,\gamma}_{\rho_t,\beta-2,\lambda-2}}\right) \\
&\leq C\left(||\overline{\varphi}||_{L^{\infty}_{\rho_t,\beta,\lambda}} + ||\Delta_{\omega_t}\varphi||_{C^{0,\gamma}_{\rho_t,\beta-2,\lambda-2}} + \sum_j |\lambda_j| ||\Delta(\chi_t h_j^t)||_{C^{0,\gamma}_{\rho_t,\beta-2,\lambda-2}}\right).
\end{aligned}$$

By the estimates in Lemma 4.2.11, the claim follows. □

**Lemma 4.2.19.** *Write* $\varphi \in C^{2,\gamma}_{\rho_t,\beta,\lambda}(X_t)$ *as*

$$\varphi = \overline{\varphi} + \chi_t \left( \sum_j \lambda_j h_j^t \right),$$

*as described above.*

*If* $\delta$ *is sufficiently small and* $\beta \in (0,2) \cap \mathcal{P}^{\complement}$*, where* $\mathcal{P}$ *is the set of exceptional weights of the Laplace operator on the ODP, and* $\lambda \in (2-2n, 0)$*, then*

$$||\varphi||_{K_t \oplus K_t^{\perp}} = ||\overline{\varphi}||_{C^{2,\gamma}_{\rho_t,\beta,\lambda}} + \left( \sum_j |\lambda_j| \right) \leq C \left( ||\Delta_{\omega_t} \varphi||_{C^{0,\gamma}_{\rho_t,\beta-2,\lambda-2}} \right)$$

*for some constant* $C > 0$ *independent of* $\delta$*.*

*Proof.* Note that for $t = 0$, this is essentially is Theorem 3.4.9.

We can argue by contradiction. Assume the estimate does not hold. Then there exist a sequence $t_i \to 0$ such that there exist smooth functions $\varphi_i$ on $X_{t_i}$ with $|\varphi_i|_K \equiv 1$ and $||\Delta_{\omega_t} \varphi_i||_{C^{0,\gamma}_{\rho_t,\beta-2,\lambda-2}} \to 0$.

Now by assumption, $|\lambda_j^i| \leq 1$, thus we have (sub-)convergence $\lambda_j^i \to \lambda_j^0$. Suppose there exists $j$ such that $\lambda_j^0 \neq 0$. Then by Lemma 4.2.18, in the limit when $t_i \to 0$ we have

$$\Delta_0 \varphi_0 = 0,$$

which means the limit is given by a non-zero harmonic function $\varphi_0$. This function must also be bounded as its rate of vanishing at the singularity is $\beta > 0$. But since $\varphi_0$ also approaches 0 at infinity for the choice of $\lambda$, such an harmonic function does not exist on the singular manifold $X_0$, see Theorem 3.4.9. Moreover, $\varphi_0$ and the harmonic functions $r^{\mu_j} \phi_j$ are linearly independent functions, because they have different rates of vanishing at the singular point, so we reach a contradiction.

Thus $\lambda_j^0 = 0$ for all $j$, and by Lemma 4.2.18, $||\overline{\varphi}_i||_{L^{\infty}_{\rho_t,\beta,\lambda}} \geq c > 0$, where $c$ is independent of $i$.

Now, note that by the results of Section 3.4, in particular Theorem 3.4.7, for $\lambda \in (2-2n, 0)$ the estimate is satisfied outside a compact set $V_{t_i} \subset X_{t_i}$ for all $i$.

Hence we can assume there exists a sequence $p_i \in V_{t_i}$ of points such that

$$\rho_{t_i}^{-\beta}(p_i) |\overline{\varphi}_i(p_i)| \geq c > 0.$$

Let us divide into cases.

- Suppose $\rho_{t_i}(p_i) \geq C > 0$ for all $i$, then $p_i \to p_0 \in V_0^{reg}$. By Ascoli–Arzelà, we can assume $\varphi_i \to \varphi_0$ in $C^{2,\gamma-\varepsilon}$ and $\varphi_0(p_0) > 0$. Moreover, $\varphi_0 = \overline{\varphi}_0$ by the argument above and
$$\Delta_0 \overline{\varphi}_0 = 0$$
in the weak sense on $X_0$ so long as $\beta > 2 - 2n$. The choice of $\beta$, as noted above, makes it so that $\varphi_0$ is less singular than the Green's function at $x$, and the choice of $\lambda$ makes it so that $\varphi_0$ goes to zero at infinity. Hence, by Theorem 3.4.9 we get $\overline{\varphi}_0 \equiv 0$, which contradicts the bound $||\overline{\varphi}_i||_{L^\infty_{\rho_t,\beta,\lambda}} \geq c$.

- Suppose $\rho_{t_i}(p_i) \to 0$ and $\delta_i^{-1} \rho_{t_i}(p_i) \to C > 0$, where $\delta_i^2 = t_i^{(n-1)/n}$.

As in the proof of Lemma 4.2.17, we transport everything on $\{\sum_i z_i^2 = 1\}$ by using the map $j_t$. We are going to rescale the metric, so we can work in the compact region where $\chi_t \equiv 1$. We define
$$\overline{\Phi}_i := \delta_i^{-\beta} (j_t)_* \overline{\varphi}_i.$$
Consider the scaled metric $\delta_i^{-2}(j_t)_* \omega_t$. Then we have $||\overline{\Phi}_i||_{C^{2,\gamma}} \leq C$ on compact sets of $\{\sum_i z_i^2 = 1\}$.

Moreover, $|\overline{\Phi}_i(z)| \leq C(1 + |z|^{-\beta(n-1)/n})^{-1}$ (the estimate comes from rescaling the $L^\infty$ bound, recalling that $\delta^2 = t^{(n-1)/n}$), and $\overline{\Phi}_i(p_i) = c > 0$ for all $i$. By Ascoli–Arzelà, $\overline{\Phi}_i \to \overline{\Phi}_0$ in $C^{2,\gamma-\varepsilon}$ for all compact subsets, $\lim_{|z|\to\infty} \overline{\Phi}_0(z) = 0$ and $|\overline{\Phi}_0(p_0)| = c > 0$. By $||\Delta_{\omega_t} \varphi_i||_{C^{0,\gamma}_{\rho_t,\beta-2,\lambda-2}} \to 0$, and the fact that $\delta_i^{-2}(j_t)_* \omega_t \to \eta_1$, we get
$$\Delta_{\eta_1} \overline{\Phi}_0 \equiv 0,$$
since the $h_j^{t_i}$ are pluriharmonic functions on $\{\sum_i z_i^2 = t_i\}$, hence harmonic with respect to $\omega_{t_i}$ where $\chi_{t_i} \equiv 1$. The rate of decay at infinity of $\overline{\Phi}_0$ on $\{\sum_i z_i^2 = 1\}$ is $\beta$ by construction.

Now, the only harmonic functions on $\{\sum_i z_i^2 = 1\}$ with respect to $\eta_1$ of rate $\beta$ are the functions $h_j^1$. Hence, to reach a contradiction, we need to prove that $\overline{\Phi}_0$ is orthogonal to these harmonic functions, *i.e.*
$$\int_{\{\sum_i z_i^2 = 1\}} (\overline{\Phi}_0 \cdot h_j^1) \rho_1^{-2n-2\beta} \eta_1^n = 0$$

for all $j$. By definition, we have

$$\langle \overline{\varphi}_i, \chi_t h_j^{t_i} \rangle_{\rho_t,\beta} \equiv 0$$

for all $i$. We can thus write

$$\begin{aligned}
0 &= \delta_i^{\beta-\mu_j} \langle \overline{\varphi}_i, \chi_t h_j^{t_i} \rangle_{\rho_t,\beta} \\
&= \delta_i^{\beta-\mu_j} \int_{X_{t_i}} (\overline{\varphi}_i \cdot \chi_{t_i} h_j^{t_i}) \rho_{t_i}^{-2n-2\beta} \omega_{t_i}^n \\
&= \delta_i^{\beta-\mu_j} \int_{\{\sum_i z_i^2 = t_i\}} (\overline{\varphi}_i \cdot \chi_{t_i} h_j^{t_i}) \rho_{t_i}^{-2n-2\beta} \omega_{t_i}^n \\
&= \delta_i^{\beta-\mu_j} \int_{\{\sum_i z_i^2 = 1\}} ((j_{t_i})_* \overline{\varphi}_i \cdot ((j_{t_i})_* \chi_{t_i} (j_{t_i})_* h_j^{t_i}) ((j_{t_i})_* \rho_{t_i})^{-2n-2\beta} ((j_{t_i})_* \omega_{t_i})^n \\
&= \int_{\{\sum_i z_i^2 = 1\}} (\delta_i^{-\beta} (j_{t_i})_* \overline{\varphi}_i \cdot \delta_i^{-\mu_j} ((j_{t_i})_* \chi_{t_i} (j_{t_i})_* h_j^{t_i}) (\delta_i^{-1} (j_{t_i})_* \rho_{t_i})^{-2n-2\beta} (\delta_i^{-2} (j_{t_i})_* \omega_{t_i})^n
\end{aligned}$$

for all $i$. Remember that $h_j^1 = \delta_i^{-\mu_j} (j_{t_i})_* h_j^{t_i}$ by definition, and note that

$$|\delta_i^{-\beta} (j_{t_i})_* \overline{\varphi}_i|_{\eta_1} \leq C \rho_1^{\beta},$$
$$|\delta_i^{-\mu_j} (j_{t_i})_* h_j^{t_i}|_{\eta_1} \leq C \rho_1^{\mu_j}$$

for all $i$ and a fixed constant $C > 0$. Moreover, note that $(j_{t_i})_* \chi_{t_i}$ converges to the identity function for $t_i \to 0$. The integrand then is dominated by $C\rho_1^{-2n-(\beta-\mu_j)}$, which is integrable since $\beta > \mu_j$. Then by dominated convergence and by Lemma 4.2.13 the integral converges to

$$\int_{\{\sum_i z_i^2 = 1\}} (\overline{\Phi}_0 \cdot h_j^1) \rho_1^{-2n-2\beta} \eta_1^n,$$

which then vanishes. Thus $\overline{\Phi}_0$ is orthogonal to the harmonic functions, hence $\Phi_0 \equiv 0$ and we get a contradiction.

- Suppose $\rho_{t_i}(p_i) \to 0$ and $\delta_i^{-1} \rho_{t_i}(p_i) \to \infty$.

  This case is similar to the previous one, but we have to "blow-up the metric more", since when transported in $\{\sum_i z_i^2 = 1\}$ the points $p_i$ go to infinity, that is, $|p_i| =: r_i^2 \to \infty$, with $r_i \delta_i \to 0$.

  Here we thus need to consider $(r_i\delta_i)^{-2} (j_{t_i r_i^{-2n/(n-1)}})_* (F_{t_i}^*)\, \omega_{t_i}$, which converges to $\eta_0$ on the cone $\{\sum_i z_i^2 = 0\}$. Here we pull-back the metric on the cone, then rescale it appropriately so that the points $p_i$ have norm 1. We rescale the functions $\overline{\varphi}_i$ in the same way, defining

$$\overline{\Xi}_i := (r_i\delta_i)^{-\beta}(j_{t_i r_i^{-2n/(n-1)}})_*(F_{t_i}^*)\,\overline{\varphi}_i,$$

which converges to $\overline{\Xi}_0$. Here $\Delta_{\eta_0}\overline{\Xi}_0 \equiv 0$, and $|\overline{\Xi}_0(p_0)| = c > 0$, where $|p_0| = 1$ by the definition of the blow-up. Moreover,

$$|\overline{\Xi}_0|_{\eta_0} \leq Cr^{\beta}.$$

Since $\beta \notin \mathcal{P}$, there is no such harmonic function on the cone, hence $\overline{\Xi}_0 \equiv 0$, which is again a contradiction.

Note that we avoid considering the rates $\beta \in \mathcal{P}$ to ensure that the operator remains Fredholm.

□

For point iii), first we show that

$$||f_\delta||_{C^{0,\gamma}_{\rho_t,\beta-2,\lambda-2}} = O(\delta^{\alpha^*-\alpha^*(\beta-2)\frac{n-1}{n}}).$$

On $\{\delta^{\alpha^*} \leq |w|_{|X_t} \leq 1\}$ we have $f_\delta \equiv 0$, hence there is nothing to prove. The error is maximised on the gluing region $\{\delta^{\alpha^*} \leq |w|_{|X_t} \leq 2\delta^{\alpha^*}\}$, where we have

$$\begin{aligned} ||f_\delta||_{L^\infty_{\rho_t,\beta-2,\lambda^*-2}} &\leq C\sup\{\rho^{-(\beta-2)}|f_\delta|\} \\ &\leq C\delta^{\alpha^*}\sup\{|w|^{-(\beta-2)\frac{n-1}{n}}\} \\ &\leq O(\delta^{\alpha^*-\alpha^*(\beta-2)\frac{n-1}{n}}), \end{aligned}$$

by the estimate found above. On the other regions, the error is less than the already found $O(\delta^{\alpha^*-\alpha^*(\beta-2)\frac{n-1}{n}})$.

We get the same estimate for the Hölder seminorm $[f_\delta]_{C^{0,\gamma}_{\rho_t,\beta-2,\lambda^*-2}}$ noticing that $[f_\delta]_{C^{0,\gamma}} = O(\delta^{\alpha^*-\alpha^*\gamma\frac{n-1}{n}})$ as described in point i). Hence we find

$$\mu = \alpha^* - \alpha^*(\beta-2)\frac{n-1}{n}.$$

Now, for the argument of Lemma 4.2.7 to work, we need to choose $\beta$ so that the fundamental bound

$$\mu > (2-\beta)$$

holds. We will need a stronger inequality, namely

$$\mu > (n-1)(2-\beta),$$

for the non-linear estimates to work.

**Lemma 4.2.20.** *The inequality*

$$\mu = \alpha^* - \alpha^*(\beta-2)\frac{n-1}{n} > (n-1)(2-\beta)$$

*is satisfied*

- *for $n = 2$, if $\beta \in (-2, 2)$;*
- *for $n = 3$, if $\beta \in (2 - \frac{9}{2(9+2\gamma-3\nu)}, 2)$, where $\nu > 0$ is such that $\alpha^* = \frac{9}{12+2\gamma-3\nu}$;*
- *for $n \geq 4$, if $\beta \in (2 - \frac{n^2}{(n-1)((2+\gamma/2)n^2-(3+\gamma)n+\gamma/2)}, 2)$,*

*where $\gamma > 0$ is the rate in Lemma 4.2.3.*

As a final part of the estimates, let us deal with the non-linear part.

**Lemma 4.2.21.** *The operator $i\partial_t\overline{\partial}_t\colon K \oplus K^\perp \to C^{0,\gamma}_{\rho_t,\beta-2,\lambda-2}(X_t)$ is bounded uniformly in $t$,* i.e.

$$||i\partial_t\overline{\partial}_t\varphi||_{C^{0,\gamma}_{\rho_t,\beta-2,\lambda-2}} \leq C||\varphi||_{K\oplus K^\perp}$$

*for a $C > 0$ independent of $t$.*

*Proof.* For the $K$ part, this follows directly by the estimates on the harmonic functions $h^t_j$ in Lemma 4.2.12. For the $K^\perp$ part, it follows by the definition of the $C^{2,\gamma}_{\rho_t,\beta,\lambda}$ norm. □

We can write the non-linear part $R$ of the Monge–Ampère operator as

$$R(\varphi) = \frac{1}{\omega^n}\left(\sum_{k\geq 2}\binom{n}{k}(i\partial\overline{\partial}\varphi)^k \wedge \omega^{n-k}\right) - e^f.$$

From here we can see that

$$||R(\varphi_1)-R(\varphi_2)||_{C^{0,\gamma}_{\rho_t,\beta-2,\lambda-2}} \leq C\delta^{(n-1)(\beta-2)}||\varphi_1-\varphi_2||_{K_t\oplus K_t^\perp}\left(||\varphi_1||_{K_t\oplus K_t^\perp} + ||\varphi_2||_{K_t\oplus K_t^\perp}\right).$$

Indeed, for any $k \geq 2$ we have

$$\left|\left|\left(\frac{(i\partial\overline{\partial}\varphi_1)^k \wedge \omega^{n-k}}{\omega^n}\right) - \left(\frac{(i\partial\overline{\partial}\varphi_2)^k \wedge \omega^{n-k}}{\omega^n}\right)\right|\right|_{C^{0,\gamma}_{\rho_t,\beta-2,\lambda-2}}$$
$$\leq C\delta^{(k-1)(\beta-2)}||\varphi_1 - \varphi_2||_{K_t\oplus K_t^{\perp}} \left(||\varphi_1||^{k-1}_{K_t\oplus K_t^{\perp}} + ||\varphi_2||^{k-1}_{K_t\oplus K_t^{\perp}}\right)$$

due to Lemma 4.2.21. In particular,

$$\left|\left|\left(\frac{(i\lambda_j^1\partial\overline{\partial}\chi_t h_j^t)^k \wedge \omega^{n-k}}{\omega^n}\right) - \left(\frac{(i\lambda_j^2\partial\overline{\partial}\chi_t h_j^t)^k \wedge \omega^{n-k}}{\omega^n}\right)\right|\right|_{C^{0,\gamma}_{\rho_t,\beta-2,\lambda-2}}$$
$$= |(\lambda_j^1)^k - (\lambda_j^2)^k| \left|\left|\left(\frac{(i\partial\overline{\partial}\chi_t h_j^t)^k \wedge \omega^{n-k}}{\omega^n}\right)\right|\right|_{C^{0,\gamma}_{\rho_t,\beta-2,\lambda-2}}$$
$$\leq C|\lambda_j^1 - \lambda_j^2| \left(|\lambda_j^1|^{k-1} + |\lambda_j^2|^{k-1}\right) \leq C\delta^{(k-1)(\beta-2)}|\lambda_j^1 - \lambda_j^2| \left(|\lambda_j^1|^{k-1} + |\lambda_j^2|^{k-1}\right),$$

and the other estimates follow similarly.

For small $\delta$ we have $\delta^{(n-1)(\beta-2)} > \delta^{(k-1)(\beta-2)}$, and we can take $\varphi$ in $\mathbb{B}_1(0) \subset K_t \oplus K_t^{\perp}$, so that

$$||\varphi||_{K_t\oplus K_t^{\perp}} > ||\varphi||^{k-1}_{K_t\oplus K_t^{\perp}}.$$

Applying Theorem 4.1.9 with $L = C\delta^{(n-1)(\beta-2)}$, and since the initial error is much smaller than $L^{-1}(\delta)$, that is, by Lemma 4.2.20,

$$\delta^{\mu} \ll \delta^{(n-1)(2-\beta)},$$

we thus have

**Proposition 4.2.22.** *Suppose $\delta$ is sufficiently small, $\lambda \in (2-2n, 0)$, $\beta$ is in the intervals of Lemma 4.2.20 and $\beta \notin \mathcal{P}$, where $\mathcal{P}$ is the set of exceptional weights of the Laplace operator on the ODP, then the equation*

$$\operatorname{Ric}\left(\omega_t + dd^c\varphi_t^{CY}\right) \equiv 0$$

*admits a solution with $||dd^c\varphi_t^{CY}||_{C^{0,\gamma}_{\rho_t,\beta-2,\lambda-2}} = O(\delta^{\mu})$, where $\mu = \alpha^* - \alpha^*(\beta-2)\frac{n-1}{n}$.*

*Proof.* By applying Theorem 4.1.9 as described above, we get a solution $\varphi_t^{CY}$ such that

$$||\varphi_t^{CY}||_{K_t\oplus K_t^{\perp}} = O(\delta^{\mu}),$$

and by applying Lemma 4.2.21, we get

$$||dd^c\varphi_t^{CY}||_{C^{0,\gamma}_{\rho_t,\beta-2,\lambda-2}} = O(\delta^\mu).$$

□

## 4.3 Explicit examples

In this section, we will apply the gluing construction to particular examples of smoothings of asymptotically conical Calabi–Yau conifolds. Moreover, we will compare the gluing construction method with the only other known method to construct special Lagrangian submanifolds, namely by using antiholomorphic involutions.

### 4.3.1 Special Lagrangians through fibred products

Consider cones of the type

$$C_d = \left\{x^2 + y^2 + z^d + w^d = 0\right\} \subseteq \mathbb{C}^4,$$

for some $d \in \mathbb{N}$, $d > 2$. A singularity modelled on the vertex of this cone is a particular case of a family of singularities called *Brieskorn–Pham* singularities. Note that by Collins & Székelyhidi (2019), Theorem 8.1, all of the Brieskorn–Pham cones of weights $(d, d, 2, 2)$ as above admit Calabi–Yau cone metrics with respect to a natural Reeb vector field.

Indeed, as discussed in Section 2.2, checking whether a cone admits a Calabi–Yau metric is equivalent to checking whether its link admits a Sasaki–Einstein metric of positive curvature, hence — in the quasi-regular case — whether the Fano orbifold obtained by considering the orbits of the Reeb vector field admits a Kähler–Einstein metric of positive curvature. In the compact smooth case, by the Yau–Tian–Donaldson conjecture proved in X. Chen et al. (2015a), X. Chen et al. (2015c), X. Chen et al. (2015b), the existence of a Kähler–Einstein metric of positive curvature is equivalent to an algebraic notion of stability called *K-stability*.

In the paper Collins & Székelyhidi (2019), the link between Sasaki–Einstein metrics and K-stability is studied extensively. In particular we have the following.

**Theorem 4.3.1** (Collins & Székelyhidi (2019), Theorem 1.2, 8.1)**.** *Consider the*

*Brieskorn–Pham singularity*

$$Z_{p,q} = \{x^2 + y^2 + z^p + w^q = 0\} \subseteq \mathbb{C}^4.$$

*For a suitable choice of Reeb vector field $\xi$, the pair $(Z_{p,q}, \xi)$ admits a Ricci-flat Kähler cone metric if and only if $2p > q$ and $2q > p$. In particular, $(Z_{p,p}, \xi)$ admits a Ricci-flat cone metric.*

The theorem is proven essentially by exploiting the particular symmetry of $Z_{p,q}$, which admits a 2-torus action.

### Application of gluing construction, Theorem 4.2.1

**Proposition 4.3.2.** *Consider a Calabi–Yau 3-fold of the type*

$$C = \{xy - c = p_d(z, w)\} \subseteq \mathbb{C}^4,$$

*where $p_d(z, w) = \prod_{i=1}^{d} (\langle \alpha_i, (z, w)\rangle + \beta_i)$, $\alpha_i \in \mathbb{C}^2, \beta_i \in \mathbb{C}$, is a polynomial of degree $d$ in $z, w$ which is the product of $d$ polynomials of degree 1. Suppose $\alpha_i, \beta_i$ are so that the intersection of any three lines of the type $\{\langle \alpha_i, (z, w)\rangle + \beta_i = 0\}$ is empty.*

*Then, if $|c| < \varepsilon$ for a small enough $\varepsilon$, we can find $\delta := \binom{d}{2}$ special Lagrangian submanifolds of $C$ obtained by the gluing construction of Proposition 4.2.1.*

*Remark* 4.3.3. We can write

$$z^p - w^p = (z - w)(z - \zeta w)(z - \zeta^2 w) \cdots (z - \zeta^{p-1} w),$$

where $\zeta^p = 1$ is a primitive root of unity. Hence polynomials of the type above are particular deformations of the Brieskorn–Pham singularity[3] $C_d$.

*Remark* 4.3.4. Note that the number of homology classes all represented by Lagrangian cycles is given exactly by the Milnor number $\mu = (d - 1)^2$, as in Milnor (1968), Chapter 9. The gluing construction gives a way to check that $\delta$ of these can be represented by special Lagrangian cycles.

If we consider the remaining $\mu - \delta = \binom{d-1}{2}$ possible special Lagrangians, these can be constructed using antiholomorphic involutions for some choices of $\beta_i$, $c$.

[3] This ceases to be true in dimension $n > 2$; for instance, $x^3 + y^3 + z^3$ cannot be written as the product of three polynomials of degree one.

*Proof.* Apply Proposition 4.2.1 noticing that if $c = 0$ then $C$ has exactly $\delta$ nodes corresponding to the joining of the singularity at $(0, 0) \in \{xy = 0\}$ and the $\delta$ intersection points of the $d$ lines corresponding to $\{p_d(z, w) = 0\}$. $\square$

These explicit examples have quite rich geometry, and we will describe how these geometric properties can be used to say something about Lagrangian and special Lagrangian submanifolds. For simplicity we analyse the case $d = 3$. Consider the cone

$$C = \{xy - zw(z - w) = 0\} \subseteq \mathbb{C}^4.$$

This is a Brieskorn–Pham-type singularity of weights $(3, 3, 2, 2)$. We now want to consider $C$ as the double fibred product $S_1 \times_{\mathbb{C}} S_2$ of the following manifolds:

$$S_1 = \{xy = t\} \subseteq \mathbb{C}^2 \times \mathbb{C},$$
$$S_2 = \{zw(z - w + \beta) = t\} \subseteq \mathbb{C}^2 \times \mathbb{C}.$$

Among all of the possible versal deformations, we can focus on the ones that preserve this fibred product structure, by deforming $S_1$ and $S_2$ separately. In this case, since $S_1$ is a ODP, these are all of the versal deformations. This leads to the possible deformation space given by

$$\{xy - c = zw(z - w + \beta)\} \subseteq \mathbb{C}^2 \times \mathbb{C}^4,$$

where $S_1 = \{xy - c = t\}$ and $S_2 = \{zw(z - w + \beta) = t\}$. It is clear that we have a singular fibre for $S_1$ for $t = c$, which has one node at the point $(x, y) = (0, 0)$. By some computations, the singular fibres of $S_2$ are over the points $t = 0$ and $t = (-\beta/3)^3$. The former has three nodes given by the points $(z, w) = (0, 0), (-\beta, 0), (0, \beta)$, while the latter has one node at the point $(z, w) = (-\beta/3, \beta/3)$.

With this picture in mind, we can see that a singular deformation of $C$ is realised when singular fibres of $S_1$ and $S_2$ are over the same point. For instance, this happens if $c = 0$, since both fibers would be singular over $t = 0$.

The idea, similarly to the study in Section 5 of Smith & Thomas (2003), is now to describe Lagrangians spheres as torus fibrations over a path conjoining a singular fibre of $S_1$ with a singular fibre of $S_2$ on the base of the fibration. For instance, $S_1$ has a singular fibre at $t = c$, and consider the singular fibre of $S_2$ over $t = (-\beta/3)^3$. We can choose the path $\gamma(t) = c + ((-\beta/3)^3 - c)t$.

The torus is constructed by considering the cycles that shrink to the nodes at the

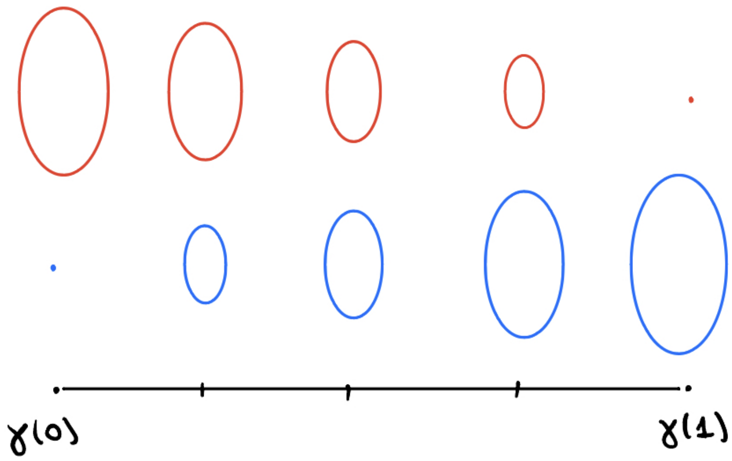


Figure 4.1: Two shrinking $S^1$ cycles forming a $S^3$.

extremes on the path $\gamma(0) = c$ and $\gamma(1) = (-\beta/3)^3$, so that in the general fibre we have a torus $S^1 \times S^1$, while at the singular points $\gamma(0)$ and $\gamma(1)$ we have respectively $\{(0,0)\} \times S^1$ and $S^1 \times \{(-\beta/3, \beta/3)\}$. This construction yields an $S^3$ Lagrangian submanifold of the fibred product $S_1 \times_{\mathbb{C}} S_2$.

*Remark* 4.3.5. Consider the general situation in which the degree $d$ of $p_d$ can be different from 3. Then we have $d$ lines with interception at the origin if $\beta_i = 0$ for all $i$. Suppose we are in the general situation in which all $\alpha_i$ are different from one another.

Then, by deforming the $d$ lines by changing the $\beta_i$ to be non-zero, we get $d$ lines on the plane that intersect generically in $\delta = \binom{d}{2}$ points. These correspond to nodes on the singular fibre of the surface $S_2$ of the type $zw = 0$, and thus to cycles on the generic fibres that shrink at $t = 0$.

On the other hand, the $d$ lines divide the plane in $\binom{d-1}{2} = \mu - \delta$ compact portions (plus the non-compact parts)[4]. These correspond to cycles that shrink at $t = t^* \neq 0$, a different $t^*$ for each cycle, depending on $\alpha_i, \beta_i$. In the case $d = 3$ above, there is only $\binom{2}{2} = 1$ cycle and the $t^*$ at which it shrinks is $t^* = (-\beta/3)^3$.

### Analysis of the Lagrangian phases

Now, the point of Proposition 4.3.2 is that it guarantees that, at least in the limit of small $c$, these $\delta$ Lagrangian classes admit a special Lagrangian representative.

[4]This is again only relative to dimension 2; in dimension $n > 2$ we would have $\mu = (d-1)^n \neq \binom{d}{n} + \binom{d-1}{n}$.

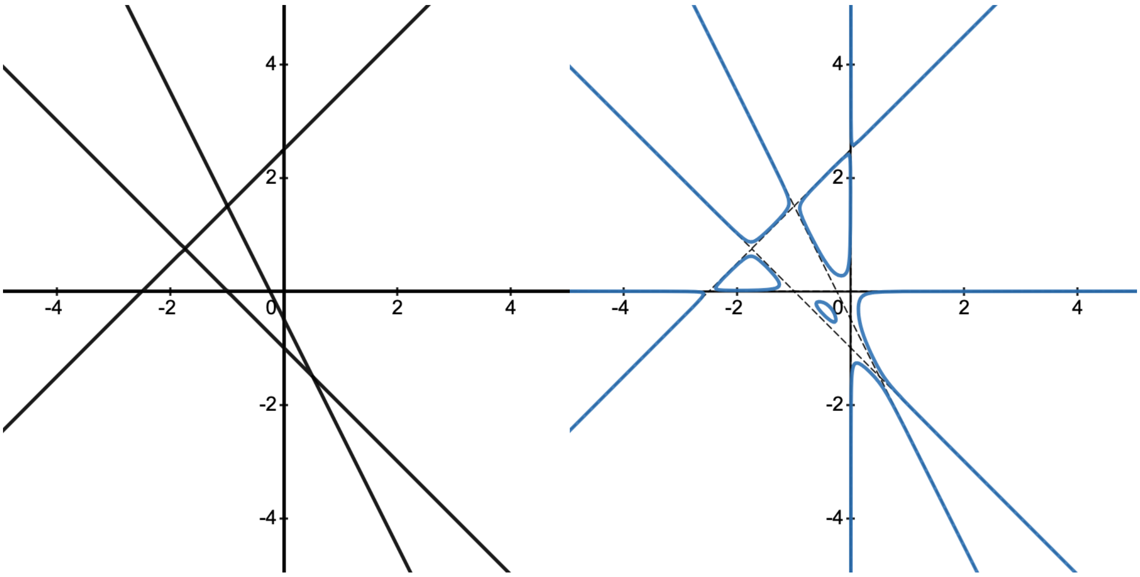


Figure 4.2: Case $d = 5$, $\binom{5}{2} = 10$ intersection points and $\binom{4}{2} = 6$ compact areas, for $t = 0$ and $t \neq 0$.

We want to compute the phases of these special Lagrangians, so as to subsequently compare with the phases of the special Lagrangians found by antiholomorphic involutions.

To compute the phases, we can proceed as follows. The form $\Omega$ must satisfy

$$d(f_1 - f_2) \wedge \Omega = dx \wedge dy \wedge dz \wedge dw, \tag{4.4}$$

where $f_1(x, y) = xy - c$ and $f_2(z, w) = zw(z - w + d)$.

Consider $\{xy - c = t\}$ first. Here we consider

$$\omega_1 = (dt \wedge dx \wedge dy)\,|_{xy-c=t},$$

which yields

$$\omega_1 = \begin{cases} dt \wedge dx/x & \text{if } \partial_y f_1 = x \neq 0; \\ -dt \wedge dy/y & \text{if } \partial_x f_1 = y \neq 0. \end{cases}$$

One can check that the expressions glue on $\{f_1 = t\}$ by the relation $df_1 = 0$. More generally we can define $\lambda_1$ so that $\omega_1 = dt \wedge \lambda_1$, hence

$$\lambda_1 = \begin{cases} dx/x & \text{if } \partial_y f_1 = x \neq 0; \\ -dy/y & \text{if } \partial_x f_1 = y \neq 0. \end{cases}$$

Similarly, on $\{zw(z-w+\beta)=t\}$ we get

$$\lambda_2 = \begin{cases} dz/(2zw - w^2 + \beta w) & \text{if } \partial_z f_2 \neq 0; \\ -dw/(z^2 - 2zw + \beta z) & \text{if } \partial_w f_2 \neq 0. \end{cases}$$

At this point, we can see that we can write

$$\Omega = dt \wedge \lambda_1 \wedge \lambda_2,$$

and by definition this satisfies Equation (4.4) on the fibred product.

To compute the phase of a special Lagrangian $L$ in the given homology class, it suffices to integrate this form on $L$, as the total phase of a closed Lagrangian is a topological quantity. Denoting $\Lambda_1$ and[5] $\Lambda_2$ the periods of $\lambda_1$ and $\lambda_2$ on the shrinking cycles $\gamma_1$ and $\gamma_2$, we get

$$\int_L \Omega = \Lambda_1 \Lambda_2 \left( \int_{z_1}^{z_2} dt \right) = 2\pi i \Lambda_2 (z_2 - z_1),$$

where $z_1$ and $z_2$ are the initial and final points of the path connecting the singular points. Hence the phase depends directly on the angle of $(z_2 - z_1)$.

### Antiholomorphic involutions

Finally, we compare the existence results that can be obtained via the gluing procedure of Proposition 4.3.2 with the method of antiholomorphic involutions, that we defined in Definition 4.1.6.

Now, as anticipated in Remark 4.3.4, we show that for some choices of deformation parameters we can find special Lagrangian representatives for all homology classes containing a Lagrangian submanifold.

**Proposition 4.3.6.** *For the deformation of $C$ given by*

$$\{xy = zw(z-w+\beta) + c\},$$

*with $\beta \in \mathbb{R}_{>0}$ and $c \in (0, (-\beta/3)^3)$, we can find $\mu = (3-1)^2 = 4$ special Lagrangian submanifolds obtained via antiholomorphic involutions.*

*Remark* 4.3.7. Proposition 4.3.2 enables us to find $\delta$ special Lagrangian submanifolds of whichever phase so long as $c$ is close to 0 (the phase is given by $e^{i \operatorname{Arg} c} = c/|c|$).

[5] Note that $\Lambda_1 = \int_\gamma \lambda_1 = \int_\gamma \frac{dx}{x} = \int_0^{2\pi} \frac{ire^{i\theta} d\theta}{re^{i\theta}} = 2\pi i$.

On the other hand, Proposition 4.3.6 enables us to find all $\mu$ special Lagrangians, $\delta$ of which have phase 1 and the remaining have phase $-1$ (or viceversa).

*Remark* 4.3.8. If we consider polynomials of the type

$$p_3(z,w) = \left(\langle \alpha_1, (z,w)\rangle + \beta_1\right)\left(\langle \alpha_2, (z,w)\rangle + \beta_2\right)\left(\langle \alpha_3, (z,w)\rangle + \beta_3\right),$$

then (for a general choice of $\alpha_i, \beta_i$) we can always change coordinates so that

$$p_3(z,w) = zw(z - w + \beta)$$

with $\beta \in \mathbb{R}_{>0}$. In particular we can choose real $\alpha_i, \beta_i$. Note that this is not true anymore if $d > 3$.

*Proof.* Describing the shrinking cycles amounts to choosing a antiholomorphic involution on $S_1$ and $S_2$, so that the cycles correspond to the fixed locus. On $\{xy = t\}$, in the case $t$ is real positive, we can choose the involution $(x,y) \mapsto (\overline{y}, \overline{x})$. This yields

$$\begin{cases} x = \sqrt{t}e^{-i\theta}, \\ y = \sqrt{t}e^{i\theta}, \end{cases}$$

hence $\theta \mapsto \sqrt{t}(e^{-i\theta}, e^{i\theta})$ gives the $S^1$-cycle. This can be seen as the zero section in the identification of $\{xy = t\}$ with $T^*S^1$. Note that we can always reduce to the case in which $t$ is real positive by eventually using the change of coordinates $(x,y) \mapsto e^{-i\psi/2}(x,y)$, where $\psi = \mathrm{Arg}(t)$.

Without loss of generality, let us temporarily fix $\beta = 3$ to simplify computations, so that the singular point is at $(t,z,w) = (1,1,-1)$. On $\{zw(z - w - 3) = t\}$, we choose the involution $(z,w) \mapsto (\overline{z}, \overline{w})$. Here again $t$ is real and positive. To describe the cycle, we solve the equation

$$zw^2 - z(z-3)w + t = 0$$

with respect to $w$.

- If $t = 0$, the discriminant of this equation is $\Delta = z^2(z-3)^2$. Hence there are always two solutions for $w$ apart for the double points at $z = 0, z = 3$. The cycle is given by the solutions $(z, w(z))$ for $z \in [0,3]$;

- if $t = 1$, the discriminant of this equation is $\Delta = z(z-1)^2(z-4)$. Here the cycle is shrunken at the double point $z = 1$, corresponding to the singularity $(z, w) = (1, -1)$;

- if $t \in (0, 1)$, the discriminant of this equation is $\Delta = z^2(z-3)^2 - 4tz$. The derivative of the discriminant $\partial_t \Delta$ with respect to $t$ is $-4z < 0$. This implies that for $t < 1$ there exists a $S^1$-cycle when $z$ varies between the two "central" solutions of $\Delta = 0$. The cycle can be explicitly expressed as $(z, w(z))$, where

$$w(z) = \frac{1}{2}\left[(z-3) \pm \sqrt{(z-3)^2 - 4tz^{-1}}\right].$$

Note that $\Delta = (4 - 4t) > 0$ at $z = 1$. Moreover, we can see three curves defining solutions of the equations, two that approach $\pm\infty$ for $z \to 0$ and are asymptotic to $z = 0$, $z = w - 3$ for $z \to -\infty$, and another asymptotic to $z = w - 3$ and $z = 0$ for $z \to \infty$.

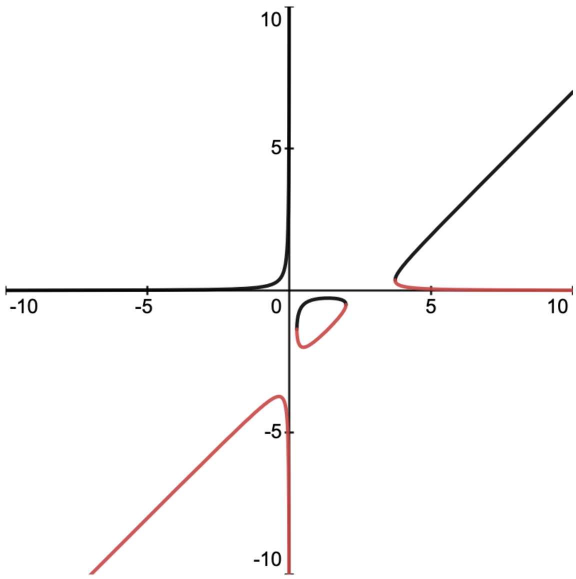


Figure 4.3: Cycle related to the involution $(z, w) \mapsto (\overline{z}, \overline{w})$.

We thus have described the $S^3$ Lagrangian sphere that connects the singular point of $S_1$ at $t = 0$ with the singular point of $S_2$ at $t = 1$. This can be seen to be special Lagrangian as it contained in the fixed locus of the involution $(x, y, z, w) \mapsto (\overline{y}, \overline{x}, \overline{z}, \overline{w})$, which is antiholomorphic and antisymplectic. It remains to study how this special Lagrangian behaves with respect to the three other Lagrangian spheres obtained by perturbing the other three singularities in the space of versal deformations of $S_1$.

To do this, we will describe the other cycles related to the singularities at $t = 0$, *i.e.* $(z, w) = (0, 0), (-\beta, 0), (0, \beta)$.

- Cycle related to $(z, w) = (0, 0)$.

  Fix $t \in (0, (-\beta/3)^3)$. Note that close to the singularity we can describe the surface approximately as
$$\beta z w = t,$$
  by discarding higher order terms. Since $t$ and $\beta$ have different signs, we can choose the involution $(z, w) \mapsto (-\overline{w}, -\overline{z})$. Locally it would give
$$|z|^2 = -t/\beta,$$
  hence
$$\theta \mapsto \begin{cases} z = \sqrt{-t/\beta}\, e^{i\theta}, \\ w = \sqrt{-t/\beta}\, e^{-i\theta}. \end{cases}$$
  Which is effectively an $S^1$ cycle. The following computations show that the cycle persists if we put back the higher order terms.

  We have $z = -\overline{w}$ and $w = -\overline{z}$. Write $z = re^{i\theta}$, thus
$$\begin{aligned} zw(z - w + \beta) &= -r^2(z + \overline{z} + \beta) \\ &= -2r^3 \cos\theta - r^2\beta \end{aligned}$$
  Thus the fixed locus is described by the equation
$$2r^3 \cos\theta + r^2\beta = -t.$$
  For any given $\theta$, and for $t \in (0, (-\beta/3)^3)$, the equation has a real positive solution $r(\theta) \in (r_-, r_+)$, where $r_-$ is the positive solution to $-2r^3 + \beta r^2 = -t$ and $r_+$ is the second largest positive solution to $2r^3 + \beta r^2 = -t$.

  Thus we find an $S^1$ cycle given by
$$\theta \mapsto \begin{cases} z = r(\theta)e^{i\theta}, \\ w = -r(\theta)e^{-i\theta}. \end{cases}$$

- Cycle related to $(z, w) = (0, \beta)$.

  Let us make the variable change $(z, w) = (z, v + \beta)$, from which we get the equation
$$z(v + \beta)(z - v) = t.$$

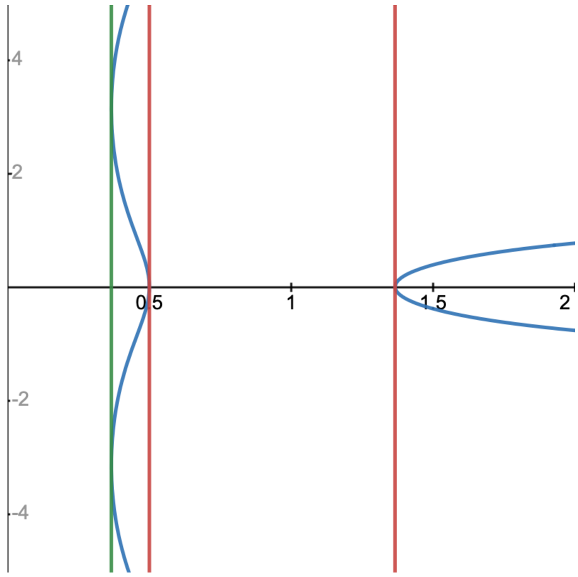


Figure 4.4: Cycle related to the singularity $(z, w) = (0, 0)$, in polar coordinates.

Locally close to the singularity we can describe this equation approximately with

$$\beta z(z - v) = t.$$

Denote $\xi = z - v$. Then since $t$ and $\beta$ have different signs, we consider the involution $(z, \xi) \mapsto (-\overline{\xi}, -\overline{z})$, hence $(z, v) \mapsto (-\overline{z} + \overline{v}, \overline{v})$.

To compute the fixed locus, we see that $v \in \mathbb{R}$ and $z + \overline{z} = v$, which implies $z = v/2 + is$ for some $s \in \mathbb{R}$. The equation thus becomes

$$\begin{aligned} z(v + \beta)(z - v) &= (v/2 + is)(v + \beta)(-v/2 + is) \\ &= -(v^2/4 + s^2)(v + \beta) = t, \end{aligned}$$

from which $(v, s)$ are solutions of the polynomial equation

$$v^3 + \beta v^2 + 4s^2 v + 4(s^2\beta + t) = 0.$$

We thus find an $S^1$ cycle for this involution as well.

- Cycle related to $(z, w) = (-\beta, 0)$.

  Similarly, we change variables with $(z, w) = (u - \beta, w)$, from which we get

$$(u - \beta)w(u - w) = t,$$

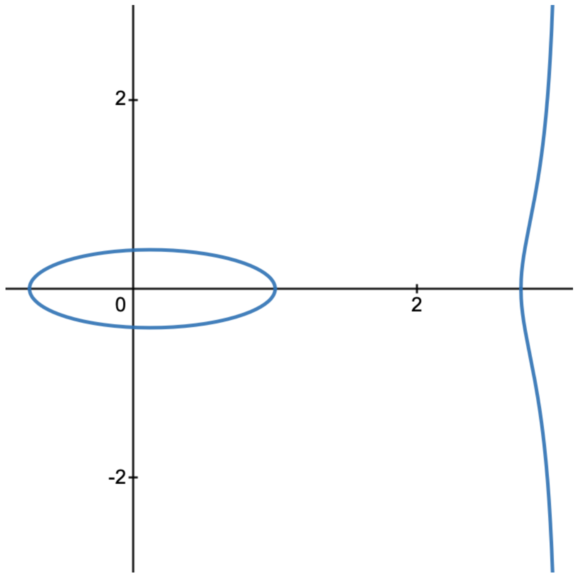


Figure 4.5: Cycle related to the singularity $(z, w) = (0, \beta)$.

we call $u - w = \zeta$ and consider the involution $(w, \zeta) \mapsto (\overline{\zeta}, \overline{w})$, hence $(u, w) \mapsto (\overline{u}, \overline{u} - \overline{w})$.

The fixed locus is thus given by $u = v \in \mathbb{R}$ and $w = v/2 + is$ for some $s \in \mathbb{R}$, from which the polynomial equation

$$v^3 - \beta v^2 + 4s^2 v - 4(s^2\beta + t) = 0.$$

Note that this equation is the same as the one in the previous case after the variable change $v \mapsto -v$.

$\square$

### Higher dimension

As pointed out in the previous section, we cannot directly adapt the arguments of the three dimensional case to higher dimensions. We can however reduce the analysis to the three dimensional case.

Firstly, consider the Brieskorn–Pham singularities of the type

$$C_d^n = \left\{ \sum_{i=1}^{n} z_i^2 = z_{n+1}^d + z_{n+2}^d \right\} \subseteq \mathbb{C}^{n+2},$$

for some $d > 2$ such that the cone admits a Calabi–Yau metric. It is clear in this case that we can again write $C_d^n = S_1 \times_{\mathbb{C}} S_2$ with

$$S_1 = \left\{ \sum_{i=1}^{n} z_i^2 = t \right\}, \quad S_2 = \left\{ z_{n+1}^d + z_{n+2}^d = t \right\}.$$

We can follow the same reasoning as above and consider deformations of the type

$$\left\{ \sum_{i=1}^{n} z_i^2 - c = p_d(z_{n+1}, z_{n+2}) \right\}$$

for $p_d(z,w) = \prod_{i=1}^{d} \left( \langle \alpha_i, (z,w) \rangle + \beta_i \right)$.

In the same way as we considered products of shrinking cycles on a path, we can more generally note that we can write

$$S^n = \bigcup_{t \in [0,1]} \left( t S^{n-k} \right) \times \left( (1-t) S^k \right)$$

for any $k \in \{0, \dots, n\}$. We then get corollaries of Proposition 4.3.2 and Proposition 4.3.6:

**Corollary 4.3.9.** *Consider a Calabi–Yau $(n+1)$-fold of the type*

$$C_d^n = \left\{ \sum_{i=1}^{n} z_i^2 - c = p_d(z_{n+1}, z_{n+2}) \right\} \subseteq \mathbb{C}^{n+2},$$

*where $p_d(z,w) = \prod_{i=1}^{d} \left( \langle \alpha_i, (z,w) \rangle + \beta_i \right)$. Suppose $\alpha_i, \beta_i$ are so that the intersection of any three lines of the type $\{ \langle \alpha_i, (z,w) \rangle + \beta_i = 0 \}$ is empty.*

*Then, if $|c| < \varepsilon$ for a small enough $\varepsilon$, we can find $\delta := \binom{d}{2}$ special Lagrangian submanifolds of $C_d^n$ obtained by the gluing construction of Proposition 4.2.1.*

**Corollary 4.3.10.** *For the deformation of $C_3^n$ given by*

$$\left\{ \sum_{i=1}^{n} z_i^2 = z_{n+1} z_{n+2} (z_{n+1} - z_{n+2} + \beta) + c \right\},$$

*with $\beta \in \mathbb{R}_{>0}$ and $c \in (0, (-\beta/3)^3)$, we can find $\mu = (3-1)^2 = 4$ special Lagrangian submanifolds obtained via antiholomorphic involutions.*

The proofs of these corollaries can be carried out exactly as in the three dimensional case, by focusing on the higher degree part and recalling the existence of a special Lagrangian sphere $S^{n-1} \subseteq \{ \sum_{i=1}^{n} z_i^2 = t \}$ that shrinks to the node at $t = 0$.

# References


Anderson, M. (1992). The $L^2$-Structure of Moduli spaces of Einstein Metrics on 4-Manifolds. *Geometric and functional analysis*, *2*(1), 29-89. Retrieved from `http://eudml.org/doc/58134`

Arezzo, C., & Spotti, C. (2016). On cscK resolutions of conically singular cscK varieties. *J. Funct. Anal.*, *271*(2), 474–494. Retrieved from `https://doi.org/10.1016/j.jfa.2016.04.025` doi: 10.1016/j.jfa.2016.04.025

Aubin, T. (1998). *Some nonlinear problems in riemannian geometry.* Springer Monographs in Mathematics.

Bedford, E., & Taylor, B. (1976). The Dirichlet Problem for a Complex Monge–Ampère Equation. *Inventiones mathematicae*, *37*, 1–44.

Berman, R. J. (2016). K-polystability of $\mathbb{Q}$-Fano varieties admitting Kähler-Einstein metrics. *Invent. Math.*, *203*(3), 973–1025. Retrieved from `https://doi.org/10.1007/s00222-015-0607-7` doi: 10.1007/s00222-015-0607-7

Błocki, Z. (2003). Regularity of the degenerate Monge-Ampère equation on compact Kähler manifolds. *Mathematische Zeitschrift*(244), 153-161.

Boucksom, S., Eyssidieux, P., Guedj, V., & Zeriahi, A. (2010). Monge-Ampère equations in big cohomology classes. *Acta Math.*, *205*(2), 199–262. Retrieved from `https://doi.org/10.1007/s11511-010-0054-7` doi: 10.1007/s11511-010-0054-7

Boyer, C. P., & Galicki, K. (2008). *Sasakian geometry.* Oxford University Press, Oxford.

Calabi, E. (1957). On Kähler manifolds with vanishing canonical class. In *Algebraic geometry and topology. A symposium in honor of S. Lefschetz* (pp. 78–89). Princeton Univ. Press, Princeton, NJ.

Calabi, E. (1958). Improper affine hypersurfaces and a generalization of a theorem of K. Jörgens. *Michigan Math. J.*, *5*, 105-126.

Chan, Y.-M. (2005). *Calabi–Yau and Special Lagrangian 3-folds with conical singularities and their desingularisations* (Unpublished doctoral dissertation). University of Oxford.

Cheeger, J. (2001). *Degeneration of riemannian metrics under ricci curvature bounds.*

Cheeger, J., & Colding, T. (1997, January 1). On the structure of spaces with Ricci curvature bounded below, I. *Journal of Differential Geometry*, *46*(3), 406–480. doi: 10.4310/jdg/1214459974

Chen, G. (2019, Sep). $G_2$ manifolds with nodal singularities along circles. *The Journal of Geometric Analysis*, *31*(2), 1360–1414. Retrieved from `http://dx.doi.org/10.1007/s12220-019-00283-3` doi: 10.1007/s12220-019-00283-3

Chen, X., Donaldson, S., & Sun, S. (2015a). Kähler-Einstein metrics on Fano manifolds. I: Approximation of metrics with cone singularities. *J. Amer. Math. Soc.*, *28*(1), 183–197. Retrieved from `https://doi.org/10.1090/S0894-0347-2014-00799-2` doi: 10.1090/S0894-0347-2014-00799-2

Chen, X., Donaldson, S., & Sun, S. (2015b). Kähler-Einstein metrics on Fano manifolds. III: Limits as cone angle approaches $2\pi$ and completion of the main proof. *J. Amer. Math. Soc.*, *28*(1), 235–278. Retrieved from `https://doi.org/10.1090/S0894-0347-2014-00801-8` doi: 10.1090/S0894-0347-2014-00801-8

Chen, X., Donaldson, S., & Sun, S. (2015c). Kähler-Einstein metrics on Fano manifolds. II: Limits with cone angle less than $2\pi$. *J. Amer. Math. Soc.*, *28*(1), 199–234. Retrieved from `https://doi.org/10.1090/S0894-0347-2014-00800-6` doi: 10.1090/S0894-0347-2014-00800-6

Collins, T., & Székelyhidi, G. (2019, may). Sasaki–Einstein metrics and K–stability. *Geometry &amp Topology*, *23*(3), 1339–1413. Retrieved from `https://doi.org/10.2140%2Fgt.2019.23.1339` doi: 10.2140/gt.2019.23.1339

Conlon, R. J., & Hein, H.-J. (2013a, Dec). Asymptotically conical Calabi–Yau manifolds, I. *Duke Mathematical Journal*, *162*(15). Retrieved from `http://dx.doi.org/10.1215/00127094-2382452` doi: 10.1215/00127094-2382452

Conlon, R. J., & Hein, H.-J. (2013b). Asymptotically conical Calabi–Yau metrics on quasi-projective varieties. *Geometric and Functional Analysis*, *25*, 517-552.

Conlon, R. J., & Hein, H.-J. (2022). Classification of asymptotically conical Calabi-Yau manifolds. Retrieved from `https://arxiv.org/abs/2201.00870` doi: 10.48550/ARXIV.2201.00870

Croke, C. B. (1980). Some isoperimetric inequalities and eigenvalue estimates. *Ann. Sci. École Norm. Sup. (4)*, *13*(4), 419–435. Retrieved from `http://www.numdam.org/item?id=ASENS_1980_4_13_4_419_0`

Demailly, J.-P. (2012). *Complex analytic and differential geometry.*

Dinew, S. (2009). An inequality for Mixed Monge–Ampère measures. *Math Z.*.

Donaldson, S., & Sun, S. (2014). Gromov-Hausdorff limits of Kähler manifolds and algebraic geometry. *Acta Math.*, *213*(1), 63–106. Retrieved from `https://doi.org/10.1007/s11511-014-0116-3` doi: 10.1007/s11511-014-0116-3

Donaldson, S., & Sun, S. (2017). Gromov-Hausdorff limits of Kähler manifolds and algebraic geometry, II. *J. Differential Geom.*, *107*(2), 327–371. Retrieved from `https://doi.org/10.4310/jdg/1506650422` doi: 10.4310/jdg/1506650422

Eyssidieux, P., Guedj, V., & Zeriahi, A. (2009). Singular Kähler-Einstein metrics. *J. Amer. Math. Soc.*, *22*(3), 607–639. Retrieved from `https://doi.org/10.1090/S0894-0347-09-00629-8` doi: 10.1090/S0894-0347-09-00629-8

Eyssidieux, P., Guedj, V., & Zeriahi, A. (2011, Mar). Viscosity solutions to degenerate complex Monge–Ampère equations. *Communications on Pure and Applied Mathematics*, *64*(8), 1059–1094. Retrieved from `http://dx.doi.org/10.1002/cpa.20364` doi: 10.1002/cpa.20364

Fornaess, J.-E., & Narasimhan, R. (1980). The Levi problem on complex space with singularities. *Math. Ann.*.

Gallot, S. (1983). A sobolev inequality and some geometric applications. *Spectra of Riemannian manifolds*, 45-55. Retrieved from `https://cir.nii.ac.jp/crid/1573387449125807488`

Grauert, H. (1962). Über Modifikationen und exzeptionelle analytische Mengen. *Math. Ann.*, *146*, 331–368. Retrieved from `https://doi.org/10.1007/BF01441136` doi: 10.1007/BF01441136

Grothendieck, A. (1961). Éléments de géométrie algébrique. II. Étude globale élémentaire de quelques classes de morphismes. *Inst. Hautes Études Sci. Publ. Math.*(8), 222. Retrieved from `http://www.numdam.org/item?id=PMIHES_1961__8__222_0`

Guedj, V., Guenancia, H., & Zeriahi, A. (2023). Continuity of singular Kähler-Einstein potentials. *Int. Math. Res. Not. IMRN*(2), 1355–1377. Retrieved from `https://doi.org/10.1093/imrn/rnab294` doi: 10.1093/imrn/rnab294

Guedj, V., & Zeriahi, A. (2017). *Degenerate complex monge-ampère equations*. European Mathematical Society.

Hein, H.-J. (2011). Weighted Sobolev inequalities under lower Ricci curvature bounds. *Proc. Amer. Math. Soc.*, *139*(8), 2943–2955. Retrieved from `https://doi.org/10.1090/S0002-9939-2011-10799-8` doi: 10.1090/S0002-9939-2011-10799-8

Hein, H.-J., & Sun, S. (2017). Calabi-Yau manifolds with isolated conical singularities. *Publ. Math. Inst. Hautes Études Sci.*, *126*, 73–130. Retrieved from `https://doi.org/10.1007/s10240-017-0092-1` doi: 10.1007/s10240-017-0092-1

Hitchin, N. (2001). Lectures on special Lagrangian submanifolds. In *Winter School on Mirror Symmetry, Vector Bundles and Lagrangian Submanifolds (Cambridge, MA, 1999)* (Vol. 23, pp. 151–182). Amer. Math. Soc., Providence, RI. Retrieved from `https://doi.org/10.1090/amsip/023/06` doi: 10.1090/amsip/023/06

Joyce, D. (2000). *Compact manifolds with special holonomy.* Oxford Mathematical Monographs.

Joyce, D. D. (2007). *Riemannian holonomy groups and calibrated geometry* (Vol. 12). Oxford University Press, Oxford.

Kas, A., & Schlessinger, M. (1972). On the Versal Deformation of a Complex Space with an Isolated Singularity. *Math. Ann.*, *196*, 23-29.

Klimek, M. (1991). *Pluripotential theory.* London Mathematical Society.

LeBrun, C. (1994). Fano Manifolds, Contact Structures, and Quaternionic Geometry. Retrieved from `https://arxiv.org/abs/dg-ga/9409001` doi: 10.48550/ARXIV.DG-GA/9409001

Li, P. (1980). On the Sobolev constant and the $p$-spectrum of a compact Riemannian manifold. *Ann. Sci. École Norm. Sup. (4)*, *13*(4), 451–468. Retrieved from http://www.numdam.org/item?id=ASENS_1980_4_13_4_451_0

Li, Y. (2022). Thomas-Yau conjecture and holomorphic curves.

Marsden, J.-E., Ratiu, T., & Abraham, R. (2002). *Manifolds, Tensor Analysis and Applications.*

Martelli, D., Sparks, J., & Yau, S.-T. (2008, Apr). Sasaki–Einstein Manifolds and Volume Minimisation. *Communications in Mathematical Physics*, *280*(3), 611–673. Retrieved from http://dx.doi.org/10.1007/s00220-008-0479-4 doi: 10.1007/s00220-008-0479-4

Milnor, J. (1968). *Singular points of complex hypersurfaces.* Princeton University Press, Princeton, N.J.; University of Tokyo Press, Tokyo.

Petersen, P. (2006). *Riemannian geometry.* Springer New York. Retrieved from https://books.google.co.uk/books?id=9cekXdo52hEC

Pogorelov, A. (1971). On the regularity of generalized solutions of the equation $\det(\partial^2 u/\partial x_i \partial x_j) = \varphi(x_1, x_2, \ldots, x_n) > 0$. *Doklady Akademii Nauk SSSR*, *200*, 534-537.

Rong, X., & Zhang, Y. (2011). Continuity of extremal transitions and flops for Calabi-Yau manifolds. *J. Differential Geom.*, *89*(2), 233–269. Retrieved from http://projecteuclid.org/euclid.jdg/1324477411 (Appendix B by Mark Gross)

Rubinstein, Y. A. (2014, Dec). Geometric and spectral analysis. *Contemporary Mathematics*. Retrieved from http://dx.doi.org/10.1090/conm/630 doi: 10.1090/conm/630

Smith, I., & Thomas, R. (2003). Symplectic surgeries from singularities. *Turkish J. Math.*, *27*(1), 231–250.

Sparks, J. (2011). Sasaki-Einstein manifolds. In *Surveys in differential geometry. Volume XVI. Geometry of special holonomy and related topics* (Vol. 16, pp. 265–324). Int. Press, Somerville, MA. Retrieved from https://doi.org/10.4310/SDG.2011.v16.n1.a6 doi: 10.4310/SDG.2011.v16.n1.a6

Spotti, C. (2014). Deformations of nodal Kähler–Einstein Del Pezzo surfaces with discrete automorphism groups. *Journal of the London Mathematical Society*, *89*(2).

Spotti, C., Sun, S., & Yao, C. (2016, 11). Existence and deformations of Kähler–Einstein metrics on smoothable $\mathbb{Q}$ -Fano varieties. *Duke Math. J.*, *165*(16), 3043–3083. Retrieved from https://doi.org/10.1215/00127094-3645330 doi: 10.1215/00127094-3645330

Stenzel, M. B. (1993). Ricci-flat metrics on the complexification of a compact rank one symmetric space. *Manuscripta Math.*, *80*(2), 151–163. Retrieved from https://doi.org/10.1007/BF03026543 doi: 10.1007/BF03026543

Strominger, A., Yau, S.-T., & Zaslow, E. (1996). Mirror symmetry is $T$-duality. *Nuclear Phys. B*, *479*(1-2), 243–259. Retrieved from https://doi.org/10.1016/0550-3213(96)00434-8 doi: 10.1016/0550-3213(96)00434-8

Székelyhidi, G. (2020). Uniqueness of some Calabi-Yau metrics on $\mathbf{C}^n$. *Geom. Funct. Anal.*, *30*(4), 1152–1182. Retrieved from https://doi.org/10.1007/s00039-020-00543-3 doi: 10.1007/s00039-020-00543-3

Thomas, R. P. (2001). Moment maps, monodromy and mirror manifolds. In *Symplectic geometry and mirror symmetry (Seoul, 2000)* (pp. 467–498). World Sci. Publ., River Edge, NJ. Retrieved from https://doi.org/10.1142/9789812799821_0013 doi: 10.1142/9789812799821_0013

Thomas, R. P., & Yau, S.-T. (2002). Special Lagrangians, stable bundles and mean curvature flow. *Comm. Anal. Geom.*, *10*(5), 1075–1113. Retrieved from https://doi.org/10.4310/CAG.2002.v10.n5.a8 doi: 10.4310/CAG.2002.v10.n5.a8

Tian, G., & Yau, S. T. (2019). Complete Kähler manifolds with zero Ricci curvature. I. In *Selected works of Shing-Tung Yau. Part 1. 1971–1991. Vol. 5. Kähler geometry II* (pp. 281–311). Int. Press, Boston, MA. (Reprint of [1040196])

van Coevering, C. (2009). Regularity of asymptotically conical Ricci-flat Kähler metrics. Retrieved from https://arxiv.org/abs/0912.3946 doi: 10.48550/ARXIV.0912.3946

Yau, S.-T. (1976). Some function-theoretic properties of complete Riemannian manifold and their applications to geometry. *Indiana Univ. Math. J.*.

Yau, S.-T. (1978). On the Ricci curvature of a compact Kähler manifold and the complex Monge-Ampère equation. *Communications in Pure and Applied Mathematics*, 339-411.