PROBLEM SOLVING WITH TECHNOLOGY: MULTIPLE PERSPECTIVES ON MATHEMATICAL CONJECTURING

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Research on technology and mathematics education has been a longstanding interest of the PME community. In this paper we revisit the interplay between technology and conjecturing within the process of problem-solving with an intention to capture different aspects of the processes in which students make and explore mathematical conjectures, and roles that both technology and teachers can play in this process. The focus is two-fold: first, to discuss different interpretations of conjectures and conjecturing within mathematics education, as reflected selected current works in mathematics education research; and second, to offer a discussion on progress in the implementation of these ideas with considerations of developments in technology, and our wider understandings of the role of the teachers.

INTRODUCTION

> "I have dared to undertake a dangerous journey on the basis of a slight supposition and already see the foothills of new lands. Those who have the courage to pursue the exploration, will step onto those lands…" **Immanuel Kant** (1755, I:222)

> "All positive knowledge must be reached if at all by an operation that begins with conjecture" **Charles S. Peirce** (1910, p. 283)

Prologue

> "… the intellectual progress of mankind in mathematical and scientific domains depends on our being able to make and explore conjectures…how might we change the way we educate people so as to help them make and explore conjectures…?" (Schwartz, 1992, p. 167).

These 30-year old thoughts by Judah Schwartz still hold as a relevant issue in mathematics education nowadays. In the following, we revisit what conjectures are and We propose why conjectures can become powerful springboards for 'intellectual progress' (i.e. learning both mathematical content and learning to do mathematics). We also suggest and exemplify some implications to support students in engaging in the practice of making productive conjectures.

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Conjectures - What?

- "A mathematical conjecture is a proposition about a previously unsuspected relationship thought to hold among mathematical objects." (Schwartz, 1997, p. 95) [1]
- "Conjectures are ideas formed by a person (the learner) in experience which satisfy the following properties: the idea is conscious (though not necessarily explicitly stated), uncertain and the conjecturer is concerned about its validity." (Norton, 2000, p. 290) [2]
- "Inference formed without proof or sufficient evidence. A conclusion deduced by surmise or guesswork. A proposition (as in mathematics) before it has been proved or disproved." (Merriam-Webster Dictionary) [3]
- "(Verb) Infer, predict, form (an opinion or notion) upon probabilities or slight evidence" (Online Etymology Dictionary) [4]
- "A statement strictly connected with an argumentation and a set of conceptions where the statement is potentially true because some conceptions allow the construction of an argumentation that justifies it." (Pedemonte, 2001, p. 34) [5]

This eclectic selection of definitions sheds light on the multifarious nature of both conjectures and conjecturing. If we choose the one word that is central in any definition, we find at least two characterizations: a proposition, a statement (grammar), and an idea, an inference, a prediction (semantics). What is such proposition/idea/prediction about? A "previously unsuspected relationship" [1], "a conclusion" [3], a connection to "a set of conceptions". How does a conjecture arise? From the learners "experience" [2], deduced by "guesswork" [3], based on "probabilities or slight evidence" and generated by "some conceptions" [5]. What about the conjecturers? They are "conscious" of the idea even if not explicitly, "uncertain and concerned" about the validity of the conjecture, and possibly able to undertake "the construction of an argumentation that justifies it" [5].

Conjectures - Why?

The mottos (by Kant and Pierce) heading this paper provide a first general justification for engaging in conjecturing to support learning: it may lead us to explore "a new land", generating new knowledge along the way.

More specifically, conjecturing nudges the conjecturers (a) to harness perception, previous knowledge and explicit reasoning in order to be clearer about how they envision the situation they are working on; (b) to become the 'owners' of a conjecture, and thus more (cognitively and affectively) committed to it, and (c) to become expectant and motivated for acting in pursue of evidence to prove or disprove the conjecture (Arcavi, 2000).

In the pursuit to confirm or discard a conjecture, it may happen that unexpected or counter-intuitive results emerge creating a clear disparity between outcomes and explicitly stated predictions. Students working on such activities are described, for example, in Hadas and Hershkowitz (1999). Such a disparity can be the trigger for nurturing the students' own need for re-inspection of their knowledge and assumptions, even without the teacher prompting to do so, creating propitious opportunities for meaningful learning.

Conjectures - How?

Conjecturing, as many other habits of mind (in the sense of Cuoco, Goldenberg & Mark, 1996), requires curriculum materials with activities that lend themselves well to stimulate students to conjecture. Moreover, it requires the creation of a secure environment and enactment of appropriate classroom norms that legitimize and support conjecturing while containing students' negative affective reactions, such as refusal, frustration, lack of confidence and fear of incorrect predictions, which may expose their lack of knowledge. In the implications of his extensive research on students conjecturing, Norton (2004, p. 367) mentions "interaction with peers" and "agreeable dispositions" as the "positive factors in generating constructive conjectural activity". Moreover, he states that if "teachers encourage students to verbalize their conjectures … other students might build from those assertions. But it is equally important for teachers to encourage students to be skeptical about such assertions, attempting to explain why they are viable or not (ibid.)." However, for that to happen, the conjecturer should feel that it is safe to take risks and to have the courage to persevere (Kant), because conjecturing may be necessary to attain "positive knowledge" but it may not be sufficient (Pierce).

Beyond the creation of appropriate learning materials and instituting a suitable classroom culture, students need not only learn to conjecture upon the request of the tasks or the teacher, but they should also take the agency to generate conjectures on their own. Autonomously asking what-if or what-if-not questions, experimenting with variations and invariants, observing patterns, attempting to generalize and searching for justifications place conjecturing as a promising companion to problem posing, as in Judah Schwartz's vision.

Technological environments provide a most proper arena for such a vision, since it allows students to experiment by themselves, and "to appreciate the ease of getting many examples . . . , to look for extreme cases, negative examples and non stereotypic evidence . . ." (Yerushalmy, 1993, p. 82). Such environments can provide immediate feedback as "dry" consequences of the student actions which, in some cases, may be more effective than teacher reactions, not only because of the affective implications (lack of value judgment), but also because it may engage the motivation to restate a conjecture, revise it, experiment again and even induce a need for proof.

Research on technology and mathematics education has been a longstanding interest of the PME community (e.g. Ferrara et al, 2006; Laborde et al, 2006;

Sinclair & Yerushalmy, 2016). As mathematics educators, a common goal has been to use technology to support inquiry-based practices in mathematics classrooms. In this paper, we focus on the process of mathematical conjecturing as part of technology-enhanced problem-solving activity. Our focus on the conjecturing process provides a lens through which to chart the evolution of technology designs and implementations across different mathematical content domains and phases of education.

We rely on a set of quotes underlining a set of sub themes around which we structure this paper. These quotes (Schwartz, 1992) serve as a connecting thread for the ideas that will be presented. In addition, we extend the focus to include more explicit attention to the role of the teacher, a dimension that was not foregrounded in Schwartz' original writing. Specifically, we use the following four themes: The role of mathematical conjecture in the construction of knowledge; The relations between Problem-Solving and Problem-Posing; Reflecting on the consequences of users' actions; and The impact of Digital environments' on learners' mathematical knowledge and the role of the teacher. For each of the subthemes, we will introduce a main idea, and then expand and exemplify it through recent technology and research, discussing the progress and development of these ideas over the years.

THE ROLE OF MATHEMATICAL CONJECTURE IN THE CONSTRUCTION OF KNOWLEDGE

On the role of conjecture in constructing knowledge, Schwartz states that "We need to have succeeding generations ask naturally and spontaneously about everything they see in the world around them, "What is this a case of ?"" (Schwartz, 1992). One well known natural human activity is gameplay, which can be employed to create forms of backward reasoning, which allows students to produce conjectures and reasoning in a more 'natural' way thanks to technology.

In different research studies (Soldano et al., 2019; Arzarello & Soldano, 2019; Albano et al., 2021) it has been shown how the so-called Logic of Inquiry (Hintikka, 1999; LI in what follows) can facilitate the structuring of different types of games, which allow students to produce conjectures and reasoning in a more 'natural' way thanks to technology. In fact, such games can foster a process of mathematical concepts learning, which is cognitively resonant with students' attitudes, slowing down the difficulties with and misuse of terms or logic in conjectures since early exploration stages of inquiry (Luz & Yerushalmy, 2023), but at the same time is epistemologically sound from the mathematical standpoint.

The above 'naturality' promoted by LI has deep cognitive and epistemological roots, respectively due to the rich productions of abductions by students in such activities and to the status of the mathematical truths within the game-theoretical approach. On the one hand, they allow to properly explain how the gap between the conjecturing and the proving processes (Arzarello & Soldano, 2019, p. 222) is reduced because of the processes triggered by the games proposed through a smart use of technology and of careful interventions of the teacher; on the other hand, these results suggest promising ways of deepening further the research and the design of teaching/learning situations with digital tools.

Consequently, in this section we first base on previous researches to give a short picture of the LI landscape for approaching proof in the classroom through a problem-solving approach, then we sketch a possible development of the research, which can be the starting point for a discussion in this RF.

LI has been introduced from a typical logic context (Hintikka & Sandu, 1997) into mathematics education in order to face and possibly overcome the seemingly unbridgeable 'basic double gap' between students' argumentative and proving processes underlined by most of the literature about the teaching of proof (Boero et al., 1996; Selden & Selden 2003; Pedemonte, 2007; Boero et al., 2010; Stylianides et al. 2017). The gap has both epistemological and cognitive features: it has been so described by Soldano & Arzarello (2016, p. 10):

On the one hand, there is an epistemological gap between what is empirically perceived as true and what is logically valid within a suitable theoretical framework […]. On the other hand, there is a cognitive gap between a first arguing phase in students' productions (when they are asked to explore a situation and make conjectures) and the proving phase (when they are asked to prove their conjectures in a more formal way).

LI allows to face this double gap through the design of mathematical games played by the students, who must develop strategic rules in order to win. Specifically, students, in couple, have to interpret mathematical statements as debates between two players: one assumes the role of falsifier F, who tries through her/his actions to disprove it, and the other member of the couple assumes the role of verifier V, who tries to prove it. If we consider, for example, a sentence like "$\forall x \exists y P(x,y)$" the dialectic between F and V will develop in this way: F starts by showing to V a particular individual a, chosen in the most unfavorable situation. If V finds an individual b, such that P(a,b) is true, V wins that hand, otherwise F wins it. In this way, the process to describe the truth of a sentence becomes a dialectic process: each action hides a questioning-answering dynamic, ruled both by definitory rules (the rules of inference) and by strategic rules (the rules of well reasoning in the game). According to Hintikka (1999), any kind of activity directed to the reach of an aim can be conceptualized as a game between two players, and this is true in particular for mathematical theorems. Such games, when treating geometric arguments, can be transposed into DGS environments, appropriately exploiting the robust and soft constructions.

For a simple but emblematic example (Figure 1), one builds a generic quadrilateral ABCD and only the middle points F of AC, and E of BD and the intersection point G between AC and BD are built in a robust way (Figure 1). The Verifier's aim is to make E, F G, to coincide, while the Falsifier has to make it impossible. The Verifiers moves C and the Falsifier moves D. After some hands, for the students it is clear that the Verifier always wins and that this result occurs because (s)he can always move D so to make E, F, G to coincide; in such a way the Verifier always constructs a parallelogram (DC parallel to AB and AD parallel to BC). At this point, suitable questions of the teacher can guide students to argue that ∀ D ∃ C (G = E = F) and why it is so; afterwards, the teacher can guide students to elaborate arguments and possibly proofs that "in a quadrilateral ABCD the diagonal AC and BD bisect each other if and only if the quadrilateral is a parallelogram".

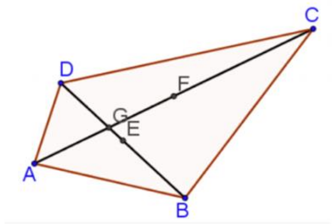

Figure 1: Initial state of the quadrilateral presented in the game

The truth of the sentence that students ascertain is based on a 'backward approach' rooted in the impossibility of building counterexamples to it: this strategy has an important epistemological and cognitive value.

The former has been discussed by Hintikka (Hintikka & Kulas, 1983), who elaborated a logical definition of semantic games, in which he reversed the standard definition of truth, given by Tarski (1933) and used in all textbooks of logic. In fact, Tarski's definition starts from the condition of truth of the simplest (atomic) sentences and proceeds recursively to the complex ones: for example, to say if A&B is true one refers to the truth of A and B. The definition in LI is in the opposite direction: it starts with complex sentences and goes inside them, according to a top-down procedure, which is in accordance with the approach sketched out in the example.

The latter is discussed in Arzarello & Soldano (2019) and one of its main points (not the only one) consists in observing a phenomenon in students' actions, productions and communications, which can generate a reduction of the above basic gap. In fact, semantic games extend the example spaces of students (Antonini, 2006) through their introduction of non-prototypical examples. Such non-prototypical examples are produced by students, typically by Falsifiers, with the aim to create difficulties to the Verifier. In the long run they produce reasoning to discover whether or not the Verifier can always win, since, to ascertain that, both players must check whether the geometric properties are still preserved. Consequently, the discussion of the new entries in their example space moves the attention from the figural to the conceptual aspects of the geometric figures (Fischbein, 1993), and activate their critical thinking (Abrami et al. 2015; Toulmin et al. 1984). We can therefore observe empirical evidence for a reduction of the basic gap in these discussions. The gap is also reduced since in semantic games two types of rules are used by students in a 'natural' way: the definitory rules, which are the possible and acceptable moves according to the game (deductive moves), and the strategic rules, which correspond to questions and answers for investigating which moves are the most convenient for a player in a specific situation (argumentative moves). There is so a deep interaction between the logic of justification and that of proving: the games are the cognitive and epistemic pivot, which allows this link between these two forms of reasoning to be made palpable. The mathematical theorems, transformed into games, embody a form of cognitive and epistemic continuity between the usual deductive logic of justification and the more argumentative logic of inquiry. This continuity constitutes a real added value to the educational games from the point of view of mathematics education.

We have based on LI for designing also a different type of games aimed at fostering students' learning of mathematical concepts in more formal contexts, e.g. in algebra, in a way which is cognitively resonant with their attitudes and epistemologically sound from the mathematical standpoint. We have called such games digital inquiry games: also in this case technology is crucial for their implementation. Space does not allow to enter into details and we defer to Albano et al. (2021) for their description.

As hinted above, the interplay between the logic of justification and that of inquiry is often characterized by the production of abductions (Peirce, 1878, p. 472 = CP 2.623; Magnani, 2001, 2009, 2015, 2023). The term refers to forms of reasoning that explain, and also sometimes discover some (possibly new) phenomenon or observation. Abduction is the process of reasoning in which explanatory hypotheses are generated and justified: in this latter sense, abduction is also often called 'Inference to the Best Explanation' (Douven, 2021).

Sometimes abductions regulate the actions made by the players in order to win and are produced within forms of backward reasoning (Gómez Chacón 1992; Shachter and Heckerman 1987; Beaney, 2021): in game-theory this way of reasoning is also called backward induction (it corresponds to what in chess is called retrograde analysis). This method is implemented, for example, in automated theorem provers, and its logical features were put forward by Hintikka (1998) and constitute the logical core of LI.

A careful analysis of abductions produced in technological games with respect to the used tools can suggest fuel for further research and for a discussion in RF, as we now shortly argument.

Magnani (2015) distinguishes between theoretical (or sentential) and manipulative abductions. Roughly speaking, theoretical abductions can be characterized at a logical level as those situations in which a hypothesis is formed and evaluated relying to the sentential aspects of natural or artificial languages. Theoretical abductions can be rendered in the Peircean syllogistic framework as transformation of a syllogism in 'barbara' (From a Rule, like 'all the A are B', and a Case, like 'x is an A', to a Result, like 'x is a B') into a fallacious syllogism (abduction), in which from knowing-selecting the Rule, and observing the Result, the Case is inferred. Such kinds of abductions, common in scientific reasonings, can be found in the students who play our games. However, many times, their abductions show also something more than a purely sentential production, which corresponds to forms of manipulative abductions. According to Magnani (2009), manipulative abductions are processes in which a hypothesis is formed and evaluated resorting to basically extra-theoretical behaviors, for example, manipulating diagrams in geometric reasonings. In our case, this happens within technological tools: the game creates a kind of an 'epistemic negotiation' between the internal framework of the student and the external reality of the diagrams built with the digital tool because of the proposed game. As claimed by Magnani (2009, p. 46), who in this relates to some of Peirce's observations (CP, 5.221): "manipulative abduction happens when we are thinking through doing and not only, in a pragmatic sense, about doing". This is an exact picture of what happens in our games: students' actions, e.g. when in the play with a parallelogram produce non-prototypical figures, have also an epistemic and not a merely performative role, which is relevant for abductive reasoning.

THE RELATIONS BETWEEN PROBLEM-SOLVING AND PROBLEM-POSING

The intellectual progress in the domains of mathematics and science depends on mathematicians and scientists "being able to make and explore conjectures. i.e. problems that we pose for ourselves" (Schwartz, 1992). In this section we demonstrate the relations between Problem-Solving and Problem-Posing through problem posing of various stakeholders in the learning process (students and teachers), and suggest possible perspectives connecting them to creativity and aesthetics.

Sinclair (2001) reports on the aesthetic dimension of problem posing, showing how students engaged in spontaneous problem posing as they explored and interacted with digital, colour-based representations of the decimal expansion of fractions. In early work, Sinclair (2001) explored the way in which an interactive, visual digital environment called *The Colour Calculator* (CC) could provoke grade 8 students' problem posing. The work drew on the role that aesthetics plays in the problem posing of mathematicians (see Sinclair, 2004), especially in terms of how mathematicians drawn to certain patterns or objects as they experiment with mathematical objects. For example, the images in Figure 2 show different patterns produced by the fraction 1/7, shown in different table widths.

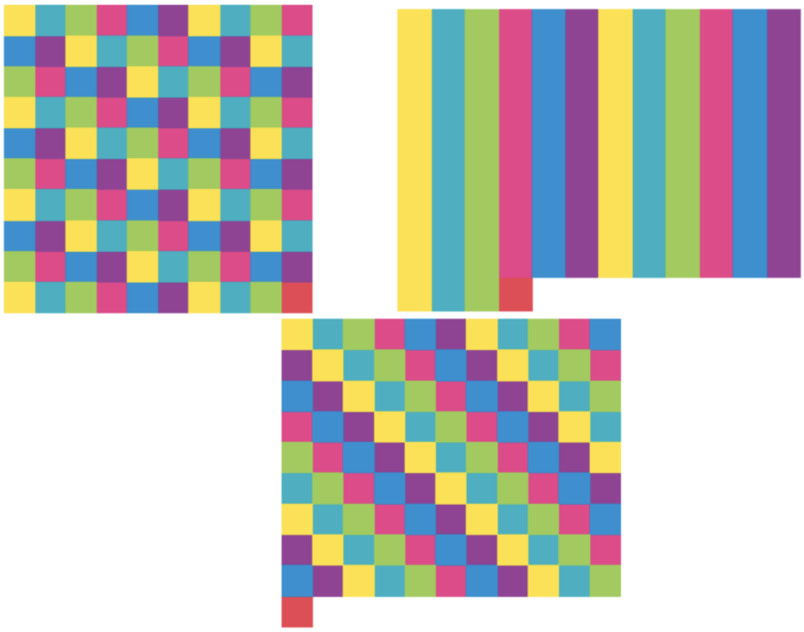

Figure 2: Patterns produced for the one seventh fraction

The students did indeed pose many problems that were mathematically interesting and that were provoked by the colour patterns and interactive options available in CC, such as: How to make a fraction that produces a table that is entirely red? I wonder how to make the diagonals go in the other direction? Is there a fraction that will not have a pattern? This finding, which corroborated Papert's (1980) view on the potential for non-experts to have aesthetic responses while engaging with mathematics. The aesthetic was operationalized less in terms of objective criteria (such as simplicity or generalizability or elegance) and more in terms of the visual appeal, thereby mobilising aesthetics more in relation to sensory knowing, such as the sensibility to visual patterns and colours.

Similar work on the aesthetic aspect of children's problem posing can be found in Eberle (2014) and Jasien et al. (2022), though in contexts involving symmetry and tessellations using non-digital technologies. One significant affordance of the digital environment of CC is the fact that it provides both symbolic and visual display and feedback, thereby enabling students to pose problems in the register of visual patterns yet connect them to numerical (in this case, fractions) registers. In their extension of the CC into sounds, which was developed for blind learners, Fernandes et al. (2011) report on similar aesthetic engagement, this time focused more on sound patterns and notes.

More recently, Sinclair and Ferrara (2021) draw on Whitehead's notion of prehension, which are subconscious aesthetic feelings, to study the way in which aesthetics is involved in student problem solving when they use *TouchCounts* (TC), a multitouch app for number sense. This work broadens the sensory scope to include the haptic/gestural experience of interacting with TC, as students use their fingers and gestures to create and manipulate cardinal quantities, which are then shown visually, aurally and symbolically. As with CC, it is therefore possible for students to draw either on the visual, oral, symbolic or haptic aspects of their mathematical interactions.

These multiple sensory presentations can be seen as way to aestheticise mathematics—by which I mean, make it more accessible to the senses—and, in so doing, to potentially draw on different ways of sensing and making sense. Other studies that were not specifically focused on problem posing did reveal some of the ways in which aestheticization provoked problem posing. For example, in Smythe et al. (2017), kindergarten children pose problems such as: “How can we make 100?”, after having made 33, “What other numbers have the same digits?”, “Are there different ways of making 10?”, and “Can I make a number that’s bigger than the screen?”

Significantly, in both CC and TC (as well as the environment studied by Eberle and Jasien et al.), there are no fixed tasks or given problem—the pedagogical goal of these environments is to engage students in explorations and experimentation, through which they might notice patterns or surprises that lead to the formulation of questions. While learners can generate such questions, it is another matter to determine whether the questions are themselves mathematically interesting, or even of aesthetic interest. For example, the question mentioned above, “Can I make a number that’s bigger than the screen?” does not have much mathematical interest in the sense that it doesn’t lead to any patterns or generalisations. Part of the process of mathematics enculturation could involve enabling students to become aware of what counts as mathematically interesting, something which teachers sometimes do in indirect ways (see Sinclair, 2008), and which Lehrer et al. (2013) have explored in more explicit ways.

Related to this, Crespo and Sinclair (2008) conducted research with elementary pre-service teachers in order to explore their ideas of what makes a mathematical problem interesting, as well as to inquire into how to promote their own problem posing. With respect to the latter, they found that having time to explore (with a set of tangram pieces) enabled pre-service teachers to pose more interesting problems than if they had to pose them without exploration, which confirmed Hawkins’ (2000) suggestion that it takes time for learners to get to know an environment enough in order to be able to notice what might be interesting (unusual, surprising, compelling) to them. Here “more interesting” related to more focused on reasoning than on facts.

Tymoczko (1993) argued that mathematics needed to engage more in explicit aesthetic criticism in order to draw attention to certain features of a work of mathematics (such as a proof or a technique or a problem), which may enhance our appreciation of that work. Sinclair (2022) suggests that the same could be true in the context of mathematics education, where engaging students in questions of what is interesting or appealing might both enable students to appreciate the values that tend to dominate school mathematics (efficiency, unity, symbolism) and perhaps open up mathematical activity to other relevant cultural or individual values.

Turning to teachers, Problem posing-through-investigations (PPI), for example, is another mathematical activity in which investigations and conjecturing in a dynamic geometry environment (DGE) lead to posing problems and solving the posed problems (Leikin 2014; Leikin & Elgrably, 2020, 2022; Elgrably & Leikin, 2021). PPI tasks start from a proof problem (either introduced by instructors or researchers, or chosen by solvers). PPI tasks require: (a) Investigating a geometrical figure (from a proof problem) in a DGE (experimenting, conjecturing and testing) to find several new properties of the given figure and related figures that are obtained using auxiliary constructions. (b) Formulating multiple new proof problems. A PPI task is completed only when all the posed problems are solved by the participants.

Figure 3 demonstrates an example of problems posed by Rasha (who studied for a teaching certificate after completion of a B.Sc. in mathematics) who chose the Midline-in-a-Triangle Theorem (see Figure 3, EP is a midline in triangle ABC). To pose new problems she first performed construction of a parallelogram, $EDCB$, used when proving the theorem, and connected different points in the figure.

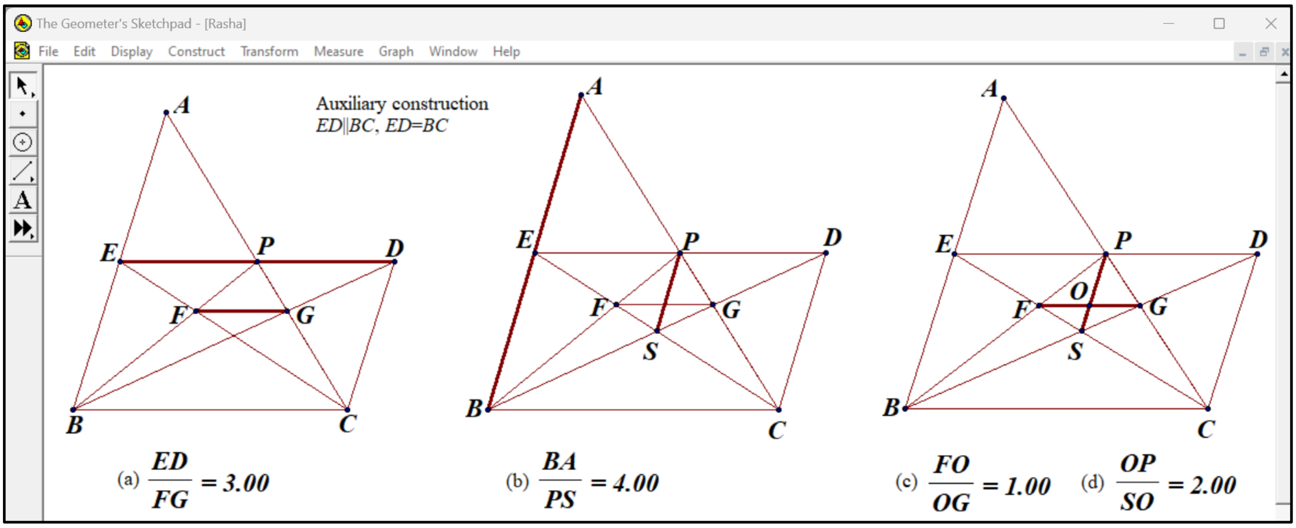

Figure 3: Examples of posed problems

Rasha discovered 4 different properties (depicted in Figure 3): (a) $\frac{ED}{FG} = 3$, (b) $\frac{BA}{PS} = 4$, (c) $\frac{FO}{OG} = 1$, and (d) $\frac{PO}{OS} = 1$, and formulated corresponding proof problems. All the problems were new for Rasha and her peers. These problems did not appear in the textbooks or in the instructional materials available to her.

Like other problem-posing tasks, PPI tasks are open (Pehkonen, 1995; Haylock, 1987; Solver, 1995) since solvers can implement different investigation strategies related to changing/extending the conditions of a given problem through performing auxiliary constructions in DGE, searching for new properties and posing different problems based on the discovered properties. Additionally, PPI openness is related to the differences in the collections of problems posed by different individuals, which include differences in terms of the number, types and complexity of the posed problems. The PPI's openness is also associated with solving a new (posed) problem in that solvers are free to choose how to prove any discovered property. The openness of PPI allows these tasks to be used as a didactical and research tool aimed at the development and evaluation of creativity.

The openness of PPI tasks determines these tasks' complexity, since an investigation can lead in unpredicted directions, conjectures can appear to be incorrect, or solving some posed problems can require knowledge and skills at a level that surpasses the level of problem-solving expertise of those who posed the problems. At the same time, the openness of the PPI tasks and their complexity determines the power of these tasks as tools for the investigation of creativity and problem-solving expertise. The requirement to solve the posed problems links PPI to the participants' problem-solving expertise. Thus, we evaluated both creativity-related skills and proof-related skills linked to participants' performance on PPI tasks (Leikin & Elgrably, 2020; 2022; Elgrably & Leikin, 2021).

The examined creativity components included fluency, flexibility and originality (Haylock, 1987; Silver, 1997; Leikin, 2009). Fluency was defined as the number of investigation strategies used / number of posed problems. Flexibility was defined as the number of different investigation strategies employed/ different posed problems. Originality was defined by the newness and rareness of the investigation strategies / posed problems. The model we employed for the evaluation of PPI is based on the model for evaluation of creativity using multiple solution tasks MSTs (Leikin, 2009).

The examined proof-related components included auxiliary constructions that led to the discovered property, appropriateness of proof and the complexity of the posed problems determined by the conceptual density of the problem (cf., Silver & Zawodjewsky, 1997) combined with the length of the required proof.

Leikin & Elgrably (2020) described an explorative study that examined PPI tasks as a tool for the development of PMTs' proof skills and their creativity components in geometry, and for exploring the relationships between problem-solving expertise and creativity skills. Elgrably & Leikin (2021) explored PPI performance by participants with different types and levels of problem-solving expertise and examined differences in their creativity and problem solving performance when engaged in PPI activity. In Leikin & Elgrably (2022) we made distinctions between outcome creativity (linked to posed problems) and strategy creativity (linked to processes of creation of the new problems through investigations).

The studies demonstrated that both proving skills and creativity components can be developed effectively through employing PPI activities, with significant changes seen in all the creativity components related to the posed problems (i.e., outcome-based creativity). There were no changes in strategy-related creativity linked to engagement with PPI tasks. With regard to types of creativity examined in the study, we found that higher strategy creativity did not necessarily lead to higher outcome creativity, while a higher level of strategy originality correlated with outcome flexibility. Thus creative product and creative process are two distinct characteristics of cognitive processing linked to creativity-directed problem solving.

Focusing on the links between types and levels of participants' problem-solving expertise and PPI we argue that problem-solving expertise at high level significantly influences the quality of PPI as reflected in proof skills and creativity components. Unfortunately, mathematical expertise related to studying mathematics in B.Sc.-level university courses does not affect mathematical creativity linked to PPI. In addition, creativity components of participants with a high level of problem-solving expertise significantly correlated with their proof skills and, moreover, problem posing and proving performed by them appeared to be interwoven. Finally, the role of DGE when conjecturing differed between participants with different levels and types of problem-solving expertise: High level experts tested hypotheses about additional properties and discovered new properties while searching for proofs of complex posed problems, whereas non-experts used a trial-and-error strategy. Correspondingly, high level experts performed auxiliary constructions consciously and with careful planning whereas non-experts used a trial-and-error approach to auxiliary constructions.

REFLECTING ON THE CONSEQUENCES OF USERS' ACTIONS

Reflecting on the consequences of users' actions focuses on a major role of technology as what can scaffold the posing of powerful problems. Often there is a substantial logical distance between the starting points offered by nature and our conjectures about nature and the detailed implications of our models. To help learners make chains of inferences, appropriately crafted software environments can aid dramatically in extending our ability to explore our formal models (Schwartz, 1992). *The Mathematics Imagery Trainer* provides a vivid example in which technology provides and environment in which students explore different perceptual orientations, and these perceptual orientations, in turn, ground prospective mathematical concepts (Abrahamson et al., 2014; Alberto et al., 2021).

In the design of Mathematics Imagery Trainer activities, the *problem* students work on is a motor-control problem—the student is tasked to figure out how to move their body or, by extension, how to operate selected objects in the activity space, such as cursors on a screen, voluminous solids in virtual space, or diagrammatic elements of a unit circle, so as to elicit some designated feedback, for example performing a coordinated bimanual movement that keeps a screen green. As such, Trainer tasks emulate embodied cultural practices, such as swimming, riding a bicycle, or operating a screwdriver—in all these, one must coordinate ongoing sensorimotor actions that maintain a consistent relation between the moving body and the environmental media, whether natural, artifactual, or some combination thereof, so as to perform goal-oriented actions. That is, the *solution* is not static or finite but inherently dynamic—students learn to move in a new way that instantiates the conservation of a mathematical invariant, such as a particular ratio, quadratic function, or sine function, even before they come to realize the mathematical modeling of this new movement form. This design approach implements a theoretical position by which the cognitive activity of engaging in mathematics is not epistemologically different from other cultural–historical practices (Abrahamson & Sánchez–García, 2016; Abrahamson & Trninic, 2015; Shvarts & Abrahamson, in press a, b). This epistemological position is grounded in an evolutionary argument for enactivist mathematics pedagogy, by which the biological cognitive capacity for engaging in mathematical reasoning is an *exaptation* (Gould & Vrba, 1982)—that is, a co-opting—of our species atavistic capacity of perception-for-action (Abrahamson, 2021). Hence, our natural capacity to develop new strategies for perception to guide action (Abrahamson & Mechsner, 2022) is a key idea in our pedagogical development. In turn, key to developing new strategies for perception to guide action is sensorimotor exploration, as we now elaborate.

In Trainer activities, the construct of problem-*solving* is theorized and, therefore, operationalized as pre-symbolic sensorimotor exploration. We view mathematics education essentially as the education of perception (Merleau–Ponty, 2005):

> Any mechanistic theory runs up against the fact that the learning process is systematic; the subject does not weld together individual movements and individual stimuli but acquires the power to respond with a certain type of solution to situations of a certain general form. The situations may differ widely from case to case, and the response movements may be entrusted sometimes to one operative organ, sometimes to another, both situations and responses in the various cases having in common not so much a partial identity of elements as a shared meaning (pp. 164–165)

In a similar guise, Piaget (1971) writes as follow:

> [W]e shall talk of "perception" in the case of a proximate structure of given sensorial evidence; as such, perception can already be seen to intervene in instinctive behavior and to be a no less essential part of kindred behavior. (p. 2)

That is, perception *per se* is an innate general cognitive faculty—the adaptive capacity to organize sensorimotor activity so as to engage the environment effectively. So, when we talk about educating perception, we mean, in fact, constructing new neural substrates by which perception can govern sensorimotor activity that is effective to accomplish some task of cultural significance. For example, when we learn to ride a bicycle, we are building the cerebral networks that let our perception respond rapidly to emergent environmental contingencies by activating sensorimotor behaviors that keep us moving stably—we develop what Bernstein (1996) called the *automatisms*. Importantly, educating perception is never about somehow improving the sensory organs or the muscles themselves but figuring out new Gestalts as mental structures that govern automatic sensorimotor activation (Mechsner 2003, 2004; Mechsner et al., 2001). In summary, we look to create learning experiences that emulate enactivist epistemology:

> In a nutshell, the enactive approach consists of two points: (1) perception consists in perceptually guided action and (2) cognitive structures emerge from the recurrent sensorimotor patterns that enable action to be perceptually guided. (Varela et al., 1991, p. 173)

Whereas Trainer activities appear similar to certain Dynamic Geometry Environments, such as GeoGebra (Leung et al., 2013), they also differ from these DGE modules. In Trainer activities, the objects that students operate are not computationally pre-constrained to maintain the targeted mathematical relations, for example, Trainer students can position the objects in a configuration that violates the targeted conceptual instantiation. In Trainer environments, only the system's feedback will indicate whether or not the student's proposed configuration abides with the hidden rule. Thus, Trainer students need to discover the rule and self-impose it as a new constraint on their sensorimotor actions: they assimilate the discovered rule and accommodate their sensorimotor behavior accordingly. That is, in Trainer environments the constraints are implicit to the feedback, so that students need to figure out what they may or may not do in order to receive the favorable feedback, whereas in classical DGE the constraints are inherent to the manipulability of the objects themselves, so that students can't help but operate within the constraints regardless of how they manipulate the objects. By way of gross analogy, Trainer activities allow you to fall off your bicycle as you learn to ride it, whereas classical DGE activities never remove the training wheels. Abrahamson and Abdu (2020) draw on the literature of the movement sciences to argue for the pedagogical advantages of Trainer "open" tasks (oDGE), where students need to discover and self-impose constraints on their perception–action loops, as compared to "closed" tasks (xDGE), where the constraints on action are preconditions of the interactive system. The relative advantages of these two approaches remain to be evaluated empirically.

Teachers or, more generally, human or AI pedagogical agents, can play active roles in guiding students' sensorimotor engagement and mathematical sense-making in enactive learning environments (Abrahamson et al., 2012). Using multimodal semiotic resources—words, gestures, diagrammatic structures, etc.—teachers introduce constraints suggest affordances that modify the student's engagement with the environment (cf. Churchill, 2022; Newell & Ranganathan, 2010). By way of analogy, a piano teacher might place coins on their student's hands and ask that they continue playing without dropping the coins. These interventions in the *micro-zone of proximal development* orient students toward elements of the environment whose perception could increase the students' grip on the world, ushering in normative cultural practice (Shvarts & Abrahamson, 2019). Next, teachers introduce supplementary mathematical instruments into the working space, such as a grid of lines, and create opportunities for students to discover how these new resources may serve as means of extending their existing sensorimotor activity.

Students discern in these new resources features affording potential utility for enhancing either the enactment, evaluation, or explanation of their strategy. Yet in the course of instrumentalizing these digital artifacts to improve on their existing task performance, students implicitly appropriate the cultural–historical practices potentiated by these epistemic forms, and consequently their strategy is modified, often from qualitative to quantitative procedures (Abrahamson et al., 2011). Teachers facilitate this semiotic process: they "re-voice" students' gestures and speech to highlight, objectify, elaborate, and stabilize students' mathematical interpretations of their own sensorimotor engagement (Flood et al., 2020). As such, the student–teacher dyad establishes mathematical notions through negotiating for the mutual intelligibility of their respective and joint actions (Flood, 2018).

THE IMPACT OF DIGITAL ENVIRONMENTS' ON LEARNERS' MATHEMATICAL KNOWLEDGE AND THE ROLE OF THE TEACHER

Students, and to some point their teachers, would use a pragmatic set of criteria for assessing intellectual worth, making sure knowledge of a particular piece of subject matter be turned to advantage in the outside world in which one lives. Yet Schwartz (1992) notes also that "people are engaged by more than just the pragmatic. They are often engaged by interesting complexity, particularly if it is complexity of their own making". We can think of environments as offering people the opportunity to fashion and explore complex situations in domains that our culture has come to regard as important, e. g. Seeing the Entire Picture - STEP (Olsher et al., 2016). However, it is well-documented that many teachers are challenged to support learners to engage productively in such environments, when most have never previously experienced authentic intellectual progress alongside technological tools (Clark-Wilson & Hoyles, 2017; Noss & Hoyles, 1996).

Since the early research on digital technologies in the 1980s and 1990s, which brought the role of the teacher into view, it is well-documented the introduction of epistemically rich technologies to school mathematics is influenced in a number of ways which include the teachers. One aspect is the prior experiences as learners of mathematics with, or without access to digital tools. The fact that many digital tools are available for many decades, does not imply that current teachers had a chance to use them as students. When teachers have hands-on experiences of the digital tool to work on the (successive) problems that they will subsequently offer to their students, it impacts the teacher's own knowledge and practice.

More is now known in some countries about the nature of the more impactful professional learning opportunities that pre- and in-service teachers engage with and value. Furthermore, as such professional learning opportunities also require careful design and implementation, which requires a group of expert others, who might be known as champions, or mathematics teacher educators. Research on the nature and role of professional learning communities within the specific context of student technology use for the purpose of problem solving is also emerging, with some promising findings that can inform new country, school and classroom contexts. In addition, it is well known that teachers' personal perspectives with respect to school mathematics curricula and teaching approaches have a great influence on their practices. Furthermore, their views of technology may align or conflict with these ideas. Writing this in 2023, we find ourselves in the midst of a global discourse on the impacts of artificially intelligent mathematics tools on that are rapidly negating the need for rote learning of traditional numeric and algebraic algorithms. This negates the type of problems that are present in most high-stakes assessments of school mathematics, and leaves wondering whether their new role will be to restrict students' access to digital technologies, or join the widening lobby to radically review school learners' educational experiences, and the place of mathematics within it. Furthermore, there are a number of institutional factors that also influence how teachers can come to know, develop, and sustain the student use of technology within more problem solving contexts. Factors that are known to increase the likelihood of teacher growth in this area include: Supportive national, regional and local mathematics curricula; mathematically- and pedagogically-aligned assessment methods and practices (especially high-stakes testing); high quality professional learning activities, resources and communities, which are designed and supported by suitable experts; and supportive national, regional and local policies and resourcing for digital tools.

The mathematics education research literature abounds with examples of small-scale, mostly exploratory studies that conclude promising findings with respect to teachers' motivational, confidence and epistemic growth within the context of student problem-solving with technology. However, few studies are followed up by larger-scale research initiatives that aim to understand effectiveness (and possibly efficacy) of such approaches in multiple schools and classrooms. This is a problem for the field, as in the real world of schools globally, there are multi-millions of free and paid-for digital educational resources available, some of which will be positioned towards problem solving in mathematics education. Although, these resources compete with the more research-informed resources, they are rarely accompanied by support resources (material and human) that can enable them to be implemented widely and with increasing effectiveness in diverse classrooms. However, the research field is mature enough to be able to propose more research-informed professional learning opportunities for teachers on a wide scale - enhanced and informed by technology itself.

One way to tackle the challenges in the pursuit of interesting mathematical knowledge in a mathematical classroom is demonstrated by design principles of the STEP platform. Beyond providing the students with rich tasks that enable them to express their own mathematical ideas using learner generated examples and explore different mathematical concepts through different patterns of example eliciting tasks (Olsher, 2022), STEP demonstrates means to increase the interaction with mathematical knowledge harnessing also digital tools for analysis of student work.

STEP analyzes student answers as mathematical objects. This analysis creates a set of automatically assessed characteristics for student's answers. For example, Abdu et al. (2021) presented a set of characteristics that were assessed for quadratic functions submitted by students as examples for quadratic functions that their graph passed through two given points. While the functions were assessed also for whether their graph indeed passed through the given points which indicate whether the answers was correct or not, other characteristics were also automatically assessed. Among these characteristics were the type of extremum point (minimum or maximum), the number of x-axis intercepts (zero, one or two), the form of the expression (polynomial, vertex or intercepts). While these characteristics are not critical for the correctness of the answer to the task, they provide the teachers and the students an accessible means of interaction over different mathematical ideas related to the task at hand, providing flexibility in terms of the mathematical knowledge that takes form in the interaction over the task. On a student level, one can learn to communicate about different mathematical objects in various ways either during the work on the task itself or after submitting it (Olsher & Thurm, 2022; Yerushalmy, Olsher, Harel & Chazan, 2022).

This analyzed data about student work in used to create different interactive tools that make different aspects in students work more accessible for the teacher. Using the different interactive visualizations, the teachers can choose whether they want to manually browse through their student's work, and drill down into specific students' work. Teachers can also further their analysis using statistical information, as well as interactive filtering tools resembling online shopping websites to filter student's work and inspect characteristics they find as relevant, as well as their relation to other characteristics (Abu-Raya & Olsher, 2021). In their use of STEP in their classrooms, we observed that teachers shift the focus of their teaching from mostly student mistakes to other characteristics of student answers (e. g. interesting non critical characteristics of student answers), thus providing an example of flexibility in what they find as relevant, worthwhile, mathematical knowledge (Olsher & Abu- Raya, 2019).

CONCLUDING REMARKS

In this paper we take a contemporary perspective to revisit the interplay between technology and conjecturing within the process of problem-solving. This perspective, rooted in sub-themes conceptualized over three decades ago, is demonstrated through the use of up-to-date technologies. The manifestation and evolution of the "classic" ideas and categorization provide a unique opportunity to value both the fundamental ideas in combining technology in conjecturing, and also the influence of the current developments on the evolution of the ways we perceive problem solving over the years.

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