Semi-Global Finite-Time Trajectory Tracking Realization for Disturbed Nonlinear Systems via Higher-Order Sliding Modes

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***Abstract*—This paper investigates an alternative non-recursive finite-time trajectory tracking control methodology for a class of nonlinear systems in the presence of general mismatched disturbances. By integrating a finite-time disturbance feedforward decoupling process via higher-order sliding modes (HOSMs), it is shown that, a novel non-recursive design framework resulting a simpler controller expression and easier gain tuning mechanism is presented. A new feature is that a quasi-linear inherent nonsmooth control law could be constructed straightforwardly from the system information, which is essentially detached from the determination of a series of virtual controllers. Moreover, by proposing a less ambitious semi-global tracking control objective, the synthesis procedure can be achieved without restrictive nonlinear growth constraints. Explicit stability analysis is given to ensure the theoretical justification. A numerical example and an application to the speed regulation of Permanent Magnet Synchronous Motor (PMSM) are provided to illustrate the simplicity and effectiveness of the proposed non-recursive control design approach.**

***Index Terms*—finite-time control, higher-order sliding mode (HOSM), active disturbance attenuation, homogeneous system theory, semi-global stability**

I. Introduction

In this paper, we specify the control objective as to realize the finite-time exact tracking task for the following nonlinear system

$$\begin{cases} \dot{x}_i(t) = x_{i+1}(t) + \phi_i(\bar{x}_i(t)) + d_i(t),\ i \in \mathbb{N}_{1:n-1}, \\ \dot{x}_n(t) = u(t) + \phi_n(x(t)) + d_n(t), \\ y(t) = x_1(t), \end{cases} \tag{1}$$

where $\bar{x}_i = (x_1, \cdots, x_i)^\top \in \mathbb{R}^i$ is the system partial state vector with $i \in \mathbb{N}_{1:n}$ ($\mathbb{N}_{j:i} := \{j, j+1, \cdots, i\}$ where j and i are integers satisfying $0 \leq j \leq i$), $x = \bar{x}_n$ is the full state vector, y is the system output, $\phi_i(\cdot),\ i \in \mathbb{N}_{1:n}$ is a known smooth nonlinear function (or, at least $\mathbb{C}^{n-i}$), $d_i(t),\ i \in \mathbb{N}_{1:n}$ represents a nonvanishing mismatched disturbance item. The output reference signal, denoted by y_r, and its $n-$th order derivative are assumed to be piecewise continuous, known and bounded. Without loss of generality, the initial time is set as zero.

The issue of exact trajectory tracking realization for nonlinear systems in the presence of mismatched uncertainties/disturbances has aroused great efforts in control community due to its significant application demands. Existing related results found in the literature could fall into two main categories: 1) *nonlinear output regulation* under a common assumption that the external disturbances are governed by certain deterministic exosystems, see. e.g., [1]–[3], etc. For instance, the disturbance is mostly supposed to be a harmonic one with unknown magnitude and phase but known frequency. In the case when the exosystems are completely unknown, the internal model is inaccessible in general owing to the missing information of the disturbance model; 2) *backstepping design* integrated with a HOSM observation/identification process, see, e.g., [4]–[8] and references therein. However, a well known side-effect of the backstepping based design approaches is the expanding complexity of the controller expression along with the increase of system order, which could possibly cause a costly implementation process.

Compared with the blossom of exact asymptotical tracking realization methods as partially mentioned above, it is also noted that there are fewer results in the literature to address the exact tracking problem for system (1) via an *inherent nonsmooth* (continuously non-differentiable) design, which could result in a finite-time convergence rate and stronger robustness [9]. Consider the case when system (1) is presented with $d_i(t) = 0$, finite-time control problem is actually well understood by referring to [10], [11], etc. However, it presents a nontrivial problem when the addressed nonlinear systems are perturbed by various non-vanishing disturbances, especially in a mismatched perturbation manner. Indeed, owing to the equilibrium drift caused by the adverse effects of disturbances, even asymptotical stabilization could not be achieved for system (1). In reference [12], the finite-time control problem can be solved under the assumption that $\phi_i + d_i$ is treated as a bounded lumped disturbance term. Nevertheless, the inherent nonlinearity characteristics are all sacrificed. A later result in [13] proposes a feedback domination method to solve the finite-time tracking problem for system (1), while a restrictive nonlinear growth constraint is necessarily required. It is also worth pointing out that, as one main character of all the

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existing related literatures mentioned above, a recursive design adopting series of virtual controllers is always essential to derive the final control scheme. Not surprisingly, for high order systems, recursive design will apparently cause heavy calculations of partial derivative terms and complex mathematical magnifying or reducing steps, see e.g., [5], [10], [12], [13], etc.

Following the discussions above, this paper is aiming to propose an alternative *non-recursive* synthesis strategy which could yield a simplified finite-time trajectory tracking controller design procedure. More distinguishably, unlike backstepping based approaches, it will be shown that the proposed control scheme could be straightforwardly derived from system (1) with largely reduced calculation burdens, owing to the fact that the proposed controller can be constructed separately from the Lyapunov function based stability analysis. To this end, firstly, we employ the higher-order sliding mode (HOSM) observer to provide a finite-time disturbance decoupling process. Secondly, we put forward a systematic feedfoward framework to transform the original system into a stabilizable system form, which facilitates the integration of homogeneous system theory. Thirdly, in order to present a simpler controller form and ease the practical implementations, a non-recursive composite control design strategy is therefore investigated. In this paper, it is also shown that, under a less demanding but more practical control objective, namely, *semi-global* instead of global control, the restrictive nonlinearity growth constraints in most of the existing related continuous finite-time control works including [13] can be fully removed. In other word, an essential smooth condition of ϕ_i in system (1) will be sufficient to derive an finite-time exact tracking control law. Theoretically, an explicit selection guideline for the bandwidth factor is formulated in a delicate semi-global attractivity analysis and the stability could be ensured after a rigourous contradiction argument. In addition, the proposed control law can also reduce to a simple linear composite controller only by tuning the homogeneous degree to zero, which could be regarded as a specific smooth control case.

Compared with existing related results, the main contribution is twofold:

- An *inherent nonsmooth* control law could be constructed straightforwardly from the system information in a non-recursive manner. Hence the controller could be essentially detached from the determination of a series of virtual controllers, which is a basic design principle of backstepping based approaches.
- By employing a less ambitious *semi-global* tracking control objective, the proposed nonsmooth synthesis procedure can be realized without any restrictive nonlinear growth constraints, which are currently essential in existing global control results.

To demonstrate the simplicity and effectiveness of the proposed control design scheme, both a numerical example and an application to the speed regulation of Permanent Magnet Synchronous Motor (PMSM) are provided.

Definitions and Notations:

i) The symbol $\mathbb{C}^i$ denotes the set of all differentiable functions whose first ith time derivatives are continuous. $\mathbb{R}_+$ represents the set of positive real numbers. A continuous function $\lfloor\cdot\rceil^a$ is defined by $\lfloor\cdot\rceil^a = \text{sign}(\cdot)|\cdot|^a$ where $a \in \mathbb{R}_+$ is a constant.

ii) (*Weighted Homogeneity* [14]): For fixed coordinates $x = (x_1, \cdots, x_n) \in \mathbb{R}^n$, and positive real numbers $(\gamma_1, \gamma_2, \cdots, \gamma_n) \triangleq \gamma$, a one-parameter family of dilation is a map $\Delta^\gamma : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$, defined by $\Delta_\epsilon^\gamma x = (\epsilon^{\gamma_1} x_1, \cdots, \epsilon^{\gamma_n} x_n)$ for any constant $\epsilon \in \mathbb{R}_+$. For a given dilation Δ^γ and a real number τ, a continuous function $V(x): \mathbb{R}^n \to \mathbb{R}$ is called $\Delta^\gamma-$homogeneous of degree τ, denoted by $V \in \mathbb{H}^\tau_{\Delta^\gamma}$ if $V(\Delta_\epsilon^\gamma x) = \epsilon^\tau V(x)$. A continuous vector field $f(x) = \sum f_j(x)(\frac{\partial}{\partial x_j})$ is $\Delta^\gamma-$ homogeneous of degree τ, if $f_j \in \mathbb{H}^{\tau+\gamma_j}_{\Delta^\gamma}$, $j \in \mathbb{N}_{1:n}$. In this paper, γ is given by $\gamma_1 = 1, \gamma_i = \gamma_{i-1} + \tau = 1 + (i-1)\tau,\ i \in \mathbb{N}_{2:n}$ with a degree $\tau \in (-\frac{1}{n}, 0)$. A homogeneous vector is defined as $\lfloor x\rceil^\tau_{\Delta^\gamma} = (\lfloor x_1\rceil^{\frac{\tau}{\gamma_1}}, \cdots, \lfloor x_n\rceil^{\frac{\tau}{\gamma_n}})^\top$. $\|x\|_{\Delta^\gamma} = (\sum_{i=1}^n |x_i|^{2/\gamma_i})^{1/2}$ denotes a homogeneous $2-$norm.

II. Problem Formulation

To begin with, the following assumption of the disturbances is essentially required.

Assumption 2.1: The mismatched disturbance $d_i(t) \in \mathbb{C}^{n-i+1}$ satisfies $\max_{i\in\mathbb{N}_{1:n},\ j\in\mathbb{N}_{0:n-i+1},\ t\in\mathbb{R}_+} \left\{\left|\frac{\partial d_i^j(t)}{\partial t^j}\right|\right\} \leq D$ where $D \in \mathbb{R}_+$ is a constant.

Remark 2.1: In most of the existing output regulation results, exosystems of the disturbances are always employed as a pre-condition in order to present exact tracking results, see e.g., [2], [3], etc. In this paper, the boundedness assumption made on the disturbances is much more general from a practical point of view.

Firstly, inspired by the higher-order sliding mode observer design in Section 5, [15], the following observer can be built to realize an accurate estimation of the mismatched disturbances

$$\begin{aligned}
\dot{z}_{i,0} &= \hbar_{i,0} + x_{i+1} + \phi_i(\bar{x}_i),\\
\hbar_{i,0} &= z_{i,1} - l_{i,0}\lambda_i^{\alpha_{i,0}}\lfloor z_{i,0} - x_i\rceil^{1-\alpha_{i,0}},\\
\dot{z}_{i,1} &= \hbar_{i,1},\ \cdots\ \dot{z}_{i,k} = \hbar_{i,k},\ k \in \mathbb{N}_{1:n-i+1},\\
\hbar_{i,j} &= z_{i,j+1} - l_{i,j}\lambda_i^{\alpha_{i,j}}\lfloor z_{i,j} - \hbar_{i,j-1}\rceil^{1-\alpha_{i,j}},\\
&\quad j \in \mathbb{N}_{1:n-i},\\
\hbar_{i,n-i+1} &= -l_{i,n-i+1}\lambda_i^{\alpha_{i,n-i+1}}\\
&\quad \times \lfloor z_{i,n-i+1} - \hbar_{i,n-i}\rceil^{1-\alpha_{i,n-i+1}},\ i \in \mathbb{N}_{1:n-1},\\
\dot{z}_{n,0} &= \hbar_{n,0} + u + \phi_n(x),\ \dot{z}_{n,1} = \hbar_{n,1}
\end{aligned} \tag{2}$$

where $\alpha_{i,j} = \frac{1}{n+2-i-j}$, $l_{i,j} \in \mathbb{R}_+$, $\lambda_i \in \mathbb{R}_+$ are design parameters, and $z_{i,0} = \hat{x}_i$, $z_{i,1} = \hat{d}_i$, $z_{i,j} = \widehat{d_i^{(j-1)}}$ which represent the estimates of x_i, d_i and $d_i^{(j-1)}$, respectively.

Denote $e_{i,0} = \hat{x}_i - x_i$ and $e_{i,j} = z_{i,j} - d_i^{(j-1)}$. Combining (1) and (2), the error dynamics gives

$$\begin{aligned}
\dot{e}_{i,0} &= e_{i,1} - l_{i,0}\lambda_i^{\alpha_{i,0}}\lfloor e_{i,0}\rceil^{1-\alpha_{i,0}},\\
\dot{e}_{i,j} &= e_{i,j+1} - l_{i,j}\lambda_i^{\alpha_{i,j}}\lfloor e_{i,j} - \dot{e}_{i,j-1}\rceil^{1-\alpha_{i,j}},\\
&\quad j \in \mathbb{N}_{1:n-i},\\
\dot{e}_{i,n-i+1} &\in [-D, D]\\
&\quad - l_{i,n-i+1}\lambda_i^{\alpha_{i,n-i+1}}\lfloor e_{i,n-i+1} - \dot{e}_{i,n-i}\rceil^{1-\alpha_{i,n-i+1}}.
\end{aligned} \tag{3}$$

Lemma 2.1: (Theorem 5, [15]) Assume the observer gain λ_i satisfies $\lambda_i > D$, $i \in \mathbb{N}_{1:n}$. For all possible well defined trajectories $x(t)$, all signals in (3) are uniformly bounded and there exists a finite-time $T_1 \in \mathbb{R}_+$ such that $e_{i,j}(t) = 0,\ t \in [T_1, \infty)$. ∎

Provided that all the disturbance terms $d_i's$ are exactly known, we are thereafter able to define an auxiliary variable $\bar{\chi}_i = (\chi_1, \cdots, \chi_i)^\top$, $i \in \mathbb{N}_{1:n+1}$, where each element χ_i is determined by the following output regulation equations

$$\begin{cases} \chi_1 = y_r, \\ \chi_i = \frac{d\chi_{i-1}}{dt} - \phi_{i-1}(\bar{\chi}_{i-1}) - d_{i-1},\ i \in \mathbb{N}_{2:n+1}. \end{cases} \tag{4}$$

Note that (4) is clearly unaccessible in practice. However, with the corresponding estimates from the disturbance observer (2), replacing $\frac{\partial d_i^j}{\partial t^j}$ by $z_{i,j+1}$ for $i \in \mathbb{N}_{1:n}$, $j \in \mathbb{N}_{0:n-i+1}$, one can therefore obtain the following implementable state trajectory reference function

$$x_i^* = \chi_i(z, \bar{y}_r),\ i \in \mathbb{N}_{1:n+1} \tag{5}$$

where $z = (z_{1,0}, z_{1,1}, \cdots, z_{1,n}, \cdots, z_{n,1})^\top$, $\bar{y}_r = (y_r, y_r^{(1)}, \cdots, y_r^{(n)})^\top$.

Further, let $\eta_i := x_i - x_i^*$, $i \in \mathbb{N}_{1:n}$ and $\eta = (\eta_1, \cdots, \eta_n)^\top$. By defining a change of coordinates

$$\xi_i = \eta_i / L^{i-1},\ i \in \mathbb{N}_{1:n},\ v = (u - x_{n+1}^*)/L^n \tag{6}$$

where $L \geq 1$ is a scaling gain to be made precise in the semi-global attractivity analysis later on, system (1) can be transformed to the following stabilizable form

$$\begin{cases} \dot{\xi}_i = L\xi_{i+1} + (\phi_i(\bar{x}_i) - \phi_i(\bar{x}_i^*) + \varepsilon_i)/L^{i-1},\ i \in \mathbb{N}_{1:n-1}, \\ \dot{\xi}_n = Lv + (\phi_n(x) - \phi_n(x^*) + \varepsilon_n)/L^{n-1} \end{cases} \tag{7}$$

where $\bar{x}_i^* = (x_1^*, \cdots, x_i^*)^\top$, $i \in \mathbb{N}_{1:n}$, $x^* = \bar{x}_n^*$, $\varepsilon_i = \phi_i(\bar{x}_i^*) - \phi_i(\bar{\chi}_i) + x_{i+1}^* - \chi_{i+1} + \dot{\chi}_i - \dot{x}_i^*$.

Up to now, we are able to show that, without going through a series of recursive design steps, a simple controller can be explicitly pre-built of the following form

$$v = -K\lfloor \xi \rceil_{\Delta^\gamma}^{\gamma_n + \tau},\ u = L^n v + x_{n+1}^* \tag{8}$$

with $K = [k_1, \cdots, k_n]$ is the coefficient vector of a Hurwitz polynomial $p(s) = s^n + k_n s^{n-1} + \cdots + k_2 s + k_1$.

Remark 2.2: In order to carry out a novel non-recursive controller design strategy, this paper proposes an alternative handling procedure with existing recursive design results, such as [1], [2], [13], etc. A direct benefit is that the controller can be directly derived as a simple form of (8) without going through the determination of a series of virtual controllers. It is also worth pointing out that by setting the homogeneous $\tau = 0$, the proposed controller (8) reduces to a conventional linear state feedback control law.

Remark 2.3: As illustrated in the design procedure presented above, the proposed control methodology provides the control engineers a more practical synthesis manner. More distinguishably, the requirement of a Hölder continuous condition (or, homogeneous growth condition) on the system nonlinearities which are always employed in finite-time control related literatures is essentially relaxed, see e.g., [10], [13], etc. In addition, the proposed non-recursive composite control design strategy could largely facilitate the practical implementations, by recalling that existing backstepping based approaches always employ exhaustive recursive calculations within nondetachable step-by-step Lyapunov stability analysis, see for details in references [5], [12], etc.

III. Main Result

The main result of this paper can be summarized by the following theorem.

Theorem 3.1: Consider the closed-loop system consisting of (1) under Assumption 2.1 and the dynamic compensator (2)-(8) with a sufficiently large scaling gain L. Then for any given constant $\rho \in \mathbb{R}_+$ which could be arbitrarily large, all trajectories of $x(t)$ starting from the compact set $\mho \triangleq [-\rho, \rho]^n$ will converge to the equilibrium point within a finite-time. ∎

Proof: Inspired by [14], [16], construct a candidate Lyapunov function $U(\xi) \in \mathbb{C}^1 \cap \mathbb{H}_{\Delta^\gamma}^{2-\tau}$ of the form

$$U(\xi) = \left(\lfloor \xi \rceil_{\Delta^\gamma}^{1-\frac{\tau}{2}}\right)^\top P \lfloor \xi \rceil_{\Delta^\gamma}^{1-\frac{\tau}{2}} \tag{9}$$

with P being a positive definite and symmetrical matrix satisfying $\Lambda^\top P + P\Lambda = -I$ and Λ being a companion matrix of K. The time derivative of $U(\xi)$ along the closed-loop system (7)-(8) is given by

$$\begin{aligned} \dot{U}(\xi) = &\frac{\partial U(\xi)}{\partial \xi^\top} L\left(\xi_2, \cdots, \xi_n, -K\lfloor \xi \rceil_{\Delta^\gamma}^{r_n+\tau}\right)^\top \\ &+ \sum_{i=1}^n \frac{\partial U(\xi)}{\partial \xi_i}\Big(\frac{\phi_i(\bar{x}_i) - \phi_i(\bar{x}_i^*)}{L^{i-1}} + \frac{\varepsilon_i}{L^{i-1}}\Big). \end{aligned} \tag{10}$$

By the definition of x_i^* in (5) and Lemma 2.1, we know that for any well defined $x(t)$, the signal x_i^* is uniformly bounded, that is, there exists a constant $\bar{\rho} > 0$ such that $\max_{i \in \mathbb{N}_{1:n}}\{\sup_{t \geq 0}\{x_i^*(t)\}\} \leq \bar{\rho}$.

Thereafter, for a given compact set $\mho \triangleq [-\rho, \rho]^n$, define a level set $\mathbf{\Omega} = \left\{\eta \in \mathbb{R}^n | \left(\lfloor \eta \rceil_{\Delta^\gamma}^{1-\frac{\tau}{2}}\right)^\top P \lfloor \eta \rceil_{\Delta^\gamma}^{1-\frac{\tau}{2}} \leq c_0\right\}$ where $c_0 \triangleq \sup_{\eta \in [-(\rho+\bar{\rho}), (\rho+\bar{\rho})]^n} \{\left(\lfloor \eta \rceil_{\Delta^\gamma}^{1-\frac{\tau}{2}}\right)^\top P \lfloor \eta \rceil_{\Delta^\gamma}^{1-\frac{\tau}{2}}\}$. On the other hand, with $L \geq 1$ and the relation (6) in mind, we know that $\forall \eta(t) \in \Omega \Rightarrow \xi(t) \in \Omega$.

In order to proceed, the following propositions, whose proofs are included in the Appendix, are required.

Proposition 3.1: There exist a constant $\alpha \in \mathbb{R}_+$ and a constant $\varsigma \in (0, \frac{1}{n})$, such that $\frac{\partial U(\xi)}{\partial \xi^\top}\left(\xi_2, \cdots, \xi_n, -K\lfloor \xi \rceil_{\Delta^\gamma}^{\gamma_n+\tau}\right)^\top \leq -\alpha \|\xi\|_{\Delta^r}^2$ holds for $\tau \in (-\varsigma, 0)$.

Proposition 3.2: There exists a constant $\tilde{\alpha} \in \mathbb{R}_+$ which is independent of L, such that $\sum_{i=1}^n \frac{\partial U(\xi)}{\partial \xi_i} \frac{\phi_i(\bar{x}_i) - \phi_i(\bar{x}_i^*)}{L^{i-1}} \leq \tilde{\alpha} \|\xi\|_{\Delta^\gamma}^2$ holds for $\eta \in \mathbf{\Omega}$.

With Lemma 2.1 and Assumption 2.1 in mind, we know that for any well defined $x(t)$, $\forall 0 \leq t < T_1$, there exists a bounded constant $\Gamma \in \mathbb{R}_+$, such that $\max_{i \in \mathbb{N}_{1:n}}\{|\varepsilon_i|\} \leq \Gamma$. Utilizing Lemmas A.1 and A.3, the following relation can be

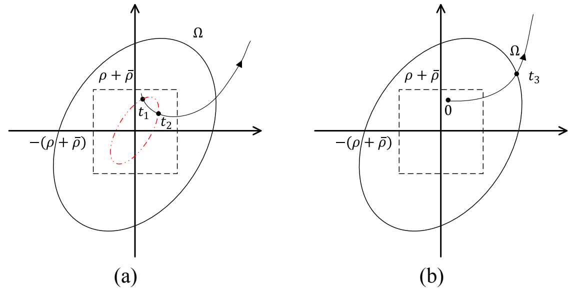

Fig. 1: Sketch figure of finite-time escaping phenomenon, (a): case 1, (b): case 2.

obtained for constants $\hat{\alpha} \in \mathbb{R}_+$ and $\bar{\alpha} \in \mathbb{R}_+$

$$\begin{aligned}\sum_{i=1}^{n} \frac{\partial U(\xi)}{\partial \xi_i} \frac{\varepsilon_i}{L^{i-1}} &\leq \hat{\alpha} \sum_{i=1}^{n} \|\xi\|_{\Delta^\gamma}^{1-i\tau} \left(\frac{\Gamma}{L^{i-1}}\right)^{\frac{1+i\tau}{1+i\tau}} \\ &\leq \bar{\alpha} \left(\|\xi\|_{\Delta^\gamma}^2 + \sum_{i=1}^{n} \frac{\Gamma^{\frac{2}{1+i\tau}}}{L^{\frac{2(i-1)}{1+i\tau}}} \right) \\ &\triangleq \bar{\alpha}\|\xi\|_{\Delta^\gamma}^2 + \Gamma^*, \ t \in [0, T_1). \end{aligned} \tag{11}$$

Using Lemma A.1 again, the following relation holds with a constant $\mu \in \mathbb{R}_+$

$$\|\xi\|_{\Delta^\gamma}^2 \geq \mu U^{\frac{2}{2-\tau}}(\xi). \tag{12}$$

Then for any arbitrarily small tolerance $\delta \in (0, c_0/2)$, now we are able to choose the scaling gain $L \geq 1$ under the following guideline

$$\alpha L - \tilde{\alpha} - \bar{\alpha} \geq 1/\mu, \ \Gamma^* \leq \frac{(\alpha L - \tilde{\alpha} - \bar{\alpha})\mu}{2} \delta^{\frac{2}{2-\tau}}. \tag{13}$$

In what follows, we will first show the uniform boundness of trajectory ξ, and then prove that a local finite-time convergence can be achieved.

1) *Uniform boundness:* In this regard, we shall prove that for any non-zero initial states satisfying $\eta(0) \in [-(\rho+\bar{\rho}), (\rho+\bar{\rho})]^n$, all the trajectories of $\eta(t)$ and $\xi(t)$ will stay in $\mathbf{\Omega}$ forever.

If the above statement is not true, that is, at least one trajectory of $\eta(t)$ will escape $\mathbf{\Omega}$ within a finite-time. Regarding the finite-time escaping phenomenon, two cases described in Fig. 1 will be discussed as follows.

Case 1: There exist two time instants $t_2 > t_1 > 0$, such that

$$\begin{aligned} &i) \ \dot{U}(\xi(t_1)) < 0, \\ &ii) \ U(\xi(t_2)) = U(\xi(t_1)) > \delta, \\ &iii) \ \dot{U}(\xi(t_2)) > 0. \end{aligned} \tag{14}$$

It is clear that for $t \in [t_1, t_2]$, $\eta(t) \in \mathbf{\Omega}$. Hence with Propositions 3.1-3.2, (11), (12) and (13), the following relation can be obtained from (10)

$$\begin{aligned}\dot{U}(\xi) &\leq -\alpha L \|\xi\|_{\Delta^r}^2 + \tilde{\alpha}\|\xi\|_{\Delta^\gamma}^2 + \bar{\alpha}\|\xi\|_{\Delta^\gamma}^2 + \Gamma^* \\ &\leq -(\alpha L - \tilde{\alpha} - \bar{\alpha}) \|\xi\|_{\Delta^\gamma}^2 + \Gamma^* \\ &\leq -(\alpha L - \tilde{\alpha} - \bar{\alpha}) \mu \left(U^{\frac{2}{2-\tau}}(\xi) - \frac{1}{2}\delta^{\frac{2}{2-\tau}} \right) \\ &< 0, \ t \in [t_1, t_2). \end{aligned} \tag{15}$$

It is noted that (15) implies

$$U(\xi(t_2)) - U(\xi(t_1)) = \int_{t_1}^{t_2} \dot{U}(\xi(s))ds < 0.$$

Recalling the relation ii) of (14), it will lead to an obvious contradiction, i.e., $0 = \int_{t_1}^{t_2} \dot{U}(\xi(s))ds < 0$.

Case 2: There exists one time instant $t_3 \geq 0$, such that

$$\begin{aligned} &i) \ \left(\lfloor \eta(t_3) \rceil_{\Delta^r}^{1-\frac{\tau}{2}}\right)^\top P \lfloor \eta(t_3) \rceil_{\Delta^r}^{1-\frac{\tau}{2}} = c_0, \\ &ii) \ \dot{U}(\xi(t_3)) > 0, \\ &iii) \ U(\xi(t_3)) > \delta. \end{aligned} \tag{16}$$

In this case, we know that $\forall t \in [0, t_3]$, $\eta(t) \in \mathbf{\Omega}$ and $\xi(t) \in \mathbf{\Omega}$. Similarly with (15), we have

$$\dot{U}(\xi(t_3)) \leq -(\alpha L - \tilde{\alpha} - \bar{\alpha}) \mu \left(U^{\frac{2}{2-\tau}}(\xi(t_3)) - \frac{1}{2}\delta^{\frac{2}{2-\tau}} \right) < 0$$

which clearly contradicts the claim ii) in (16).

In a summary of the above two cases, we can arrive at the conclusion that $\forall x(0) \in \mho \Rightarrow \eta(t) \in \mathbf{\Omega} \Rightarrow \xi(t) \in \mathbf{\Omega}, \ \forall t \geq 0$.

2) *Local finite-time convergence:* Now it is true that the relation (15) also holds for $t \in [0, T_1]$. With this relation in mind, we know that any trajectory of the closed-loop system (7)-(8) will be well defined. In the case when $t \geq T_1$, it concludes from Lemma 2.1 that $\varepsilon_i = 0, \ t \geq T_1, \ i \in \mathbb{N}_{1:n}$. Based on (13) and (15), one can also have

$$\dot{U}(\xi) + U^{\frac{2}{2-\tau}}(\xi)\Big|_{\mathbf{\Omega}} \leq 0, \ t \in [T_1, \infty). \tag{17}$$

By Lemma A.2 and with the fact that $0 < \frac{2}{2-\tau} < 1$ in mind, the relation (17) leads to a straightforward conclusion that there exists another time instant $T_2 > T_1 > 0$, such that $y(t) - y_r = 0, \ t \in [T_2, \infty)$. This completes the proof of Theorem 3.1. ■

Remark 3.1: In the proof of Theorem 3.1, a delicate contradiction argument is employed to guarantee the avoidance of finite-time escaping phenomenon. Under the framework of non-recursive homogeneous domination approach, we first show that under the guideline (13), the semi-global attractivity of the level set $\mathbf{\Omega}$ can be ensured via a contradiction argument, and then all signals in the closed-loop system will be uniformly bounded. Moreover, by Lyapunov function based analysis, the finite-time convergence property of the system states is eventually guaranteed.

IV. Numerical Simulations

A. A Numerical Example

Example 4.1: Consider the following disturbed nonlinear system:

$$\begin{cases} \dot{x}_1 = x_2 + \sin(x_1) + d_1(t), \\ \dot{x}_2 = x_3 + \ln(1 + x_1^2), \\ \dot{x}_3 = u + x_1 x_3^{1/3} + d_2(t), \\ \ y = x_1 \end{cases} \tag{18}$$

where $d_1(t)$ and $d_2(t)$ are mismatched and matched disturbances, respectively. The control objective is to realize finite-time exact tracking of a given reference signal $y_r =$

$1+\sqrt{2}\sin(t+\pi/4)$ while the disturbances are set as $d_1 = 0.1\sin(t),\ d_2 = 1$.

On one hand, the problem of finite-time trajectory tracking for system (18) presents a nontrivial task by referring to existing literature. Firstly, the existing nonsmooth control methods such as [10], [12], [17]–[19], etc. will only lead to a control result with practical stability. Secondly, the global design framework proposed in [13] cannot be applied due to the fact that the nonlinearity growth hypothesis cannot be pre-verified.

In this work, we show that by considering a semi-global control objective, the exact tracking control problem for Example 4.1 can now be solved by following a simple synthesis procedure depicted as follows: By skipping the pre-verifications of nonlinearity growth constraints, one can straightforwardly utilize the proposed control method with a series of pre-calculations as: $y_r^{(1)} = \cos(t)-\sin(t),\ y_r^{(2)} = -\sin(t)-\cos(t),\ y_r^{(3)} = -\cos(t)+\sin(t);\ x_1^* = y_r,\ x_2^* = y_r^{(1)} - \sin(y_r) - z_{1,1},\ x_3^* = y_r^{(2)} - \cos(y_r)y_r^{(1)} - z_{1,2} - \ln(1+y_r^2),\ u^* := x_4^* = y_r^{(3)} + \sin(y_r)(y_r^{(1)})^2 - \cos(y_r)y_r^{(2)} - z_{1,3} - 2y_r y_r^{(1)}/(1+y_r^2) - y_r(x_3^*)^{1/3} - z_{2,1}$. With the coordinates transformation $\xi_1 = x_1 - x_1^*,\ \xi_2 = (x_2 - x_2^*)/L,\ \xi_3 = (x_3 - x_3^*)/L^2,\ v = (u-u^*)/L^3$, the obtained exact tracking controller is depicted explicitly in Table I.

Table I: Finite-time controller design for system (18)

HOSM Disturbance Observer:

$$d_1:\begin{cases}\dot z_{1,0} = x_2 + \sin(x_1) + \hbar_{1,0}\\ \dot z_{1,i} = \hbar_{1,i},\ i\in\mathbb{N}_{1:3}\\ \hbar_{1,0} = -l_{1,0}\lambda_1^{1/4}\lfloor z_{1,0}-x_1\rceil^{3/4} + z_{1,1}\\ \hbar_{1,1} = -l_{1,1}\lambda_1^{1/3}\lfloor z_{1,1}-\hbar_{1,0}\rceil^{2/3} + z_{1,2}\\ \hbar_{1,2} = -l_{1,2}\lambda_1^{1/2}\lfloor z_{1,2}-\hbar_{1,1}\rceil^{1/2} + z_{1,3}\\ \hbar_{1,3} = -l_{1,3}\lambda_1\lfloor z_{1,3}-\hbar_{1,2}\rceil^{0};\end{cases}$$

$$d_2:\begin{cases}\dot z_{2,0} = u + x_1x_3^{1/3} + \hbar_{2,0}\\ \dot z_{2,1} = \hbar_{2,1}\\ \hbar_{2,0} = -l_{2,0}\lambda_2^{1/2}\lfloor z_{2,0}-x_3\rceil^{1/2} + z_{2,1}\\ \hbar_{2,1} = -l_{2,1}\lambda_2\lfloor z_{2,1}-\hbar_{2,0}\rceil^{0};\end{cases}$$

Exact Tracking Control Law:

$$\begin{cases} v = -K\left[\lfloor\xi_1\rceil^{1+3\tau}, \lfloor\xi_2\rceil^{\frac{1+3\tau}{1+\tau}}, \lfloor\xi_3\rceil^{\frac{1+3\tau}{1+2\tau}}\right]^\top,\\ u = L^3v + u^*.\end{cases}$$

On the other hand, in reference to existing backstepping based approaches, a clear improvement is the design simplicity under the proposed non-recursive design framework. For instance, following the backstepping based design in [6], a more complex control scheme with nested virtual controllers can be carried out via recursive design steps, as depicted sketchily in Table II.

In the simulation, by following the proposed design procedure, the gain vector K can be selected following the classical pole placement manner In the simulation, we choose $K = [27, 27, 9]$ to place the pole of the nominal system into $(-3, -3, -3)$. The scaling gain is selected as $L = 1.5$ according to the guideline (13). The designed homogeneous degree is set as $\tau = -0.1$. The observer gains are chosen as $\lambda_1 = 1,\ l_{1,0} = 5,\ l_{1,1} = 4,\ l_{1,2} = 2,\ l_{1,3} = 1$ and

Table II: Backstepping based controller design for system (18)

Disturbance Observer:

$$d_1:\begin{cases}\hat d_1 = \lambda_1(x_1 - p_1)\\ \dot p_1 = x_2 + \sin(x_1) + \hat d_1;\end{cases}\qquad d_2:\begin{cases}\hat d_2 = \lambda_2(x_3 - p_2)\\ \dot p_2 = u + x_1x_3^{1/3} + \hat d_2;\end{cases}$$

Backstepping Control Law:

$$\begin{cases} x_1^* = y_r,\ \zeta_1 = x_1 - x_1^*,\\ x_2^* = -k_1\zeta_1 - f_1 - \hat d_1, \zeta_2 = x_2 - x_2^*,\\ x_3^* = -\zeta_1 - \left(k_2 + \left(\frac{\partial x_2^*}{\partial x_1} + \frac{\partial x_2^*}{\partial \hat d_1}\lambda_1\right)^2\right)\zeta_2 - f_2\\ \qquad + \left(\frac{\partial x_2^*}{\partial x_1}\left(x_1 + f_1 + \hat d_1\right) + \frac{\partial x_2^*}{\partial x_1^*}\dot x_1^* + \frac{\partial x_2^*}{\partial \dot x_1^*}\ddot x_1^*\right),\\ \zeta_3 = x_3 - x_3^*,\\ u = -\zeta_2 - \left(k_3 + \left(\frac{\partial x_3^*}{\partial x_1} + \frac{\partial x_3^*}{\partial \hat d_1}\lambda_1\right)^2\right)\zeta_3 - f_3 - \hat d_2\\ \qquad + \left(\frac{\partial x_3^*}{\partial x_1}\left(x_1 + f_1 + \hat d_1\right) + \frac{\partial x_3^*}{\partial x_2}(x_3 + f_2)\right.\\ \qquad \left. + \frac{\partial x_3^*}{\partial x_1^*}\dot x_1^* + \frac{\partial x_3^*}{\partial \dot x_1^*}\ddot x_1^* + \frac{\partial x_3^*}{\partial \ddot x_1^*}\dddot x_1^*,\right.\\ k_1 > 1/2, k_2 > 0, k_3 > 1/2, \lambda_1 > 3/2, \lambda_2 > 1.\end{cases}$$

$\lambda_2 = 2,\ l_{2,0} = 2,\ l_{2,1} = 1$. The initial values are given as $[x(0); z(0)] = [3, 2, -4; 3, -1, 0, -1, -4, 0]$. For the backstepping based controller in Table II, the parameters are set as $k_1 = 0.51,\ k_2 = 0.02,\ k_3 = 0.51,\ \lambda_1 = 2.3,\ \lambda_2 = 5$ while the initial value for disturbance observer is $[p_1(0), p_2(0)] = [3, -4]$.

As shown in Fig. 2, the finite-time tracking objective is realized under the designed tracking scheme while the backstepping controller could only render a practical tracking result. Fig. 4 shows that under the proposed method, the states x_2 and x_3 also approach to their desired reference signal $x_2^*,\ x_3^*$ within a finite-time. The time histories of two control input signals are shown in Fig. 3. In Fig. 5, the performance of the finite-time disturbance observer is demonstrated.

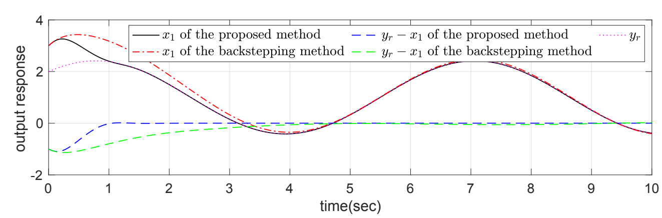

Fig. 2: Output tracking performance.

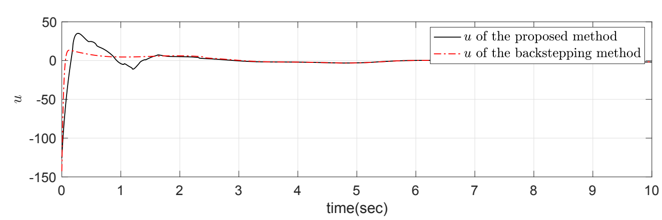

Fig. 3: Time histories of the control inputs.

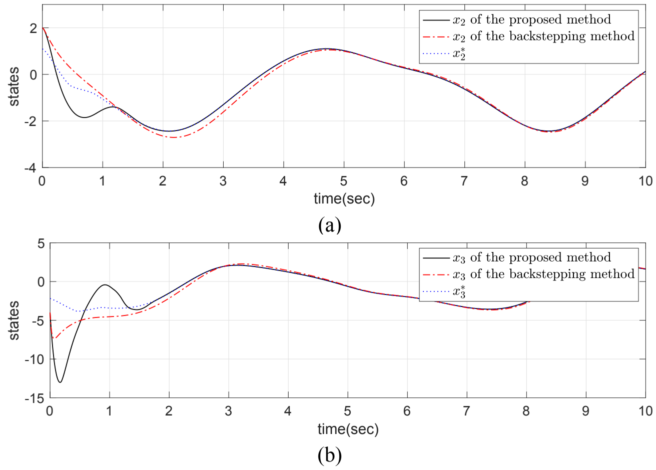

Fig. 4: State response curves, (a): x_2, (b): x_3.

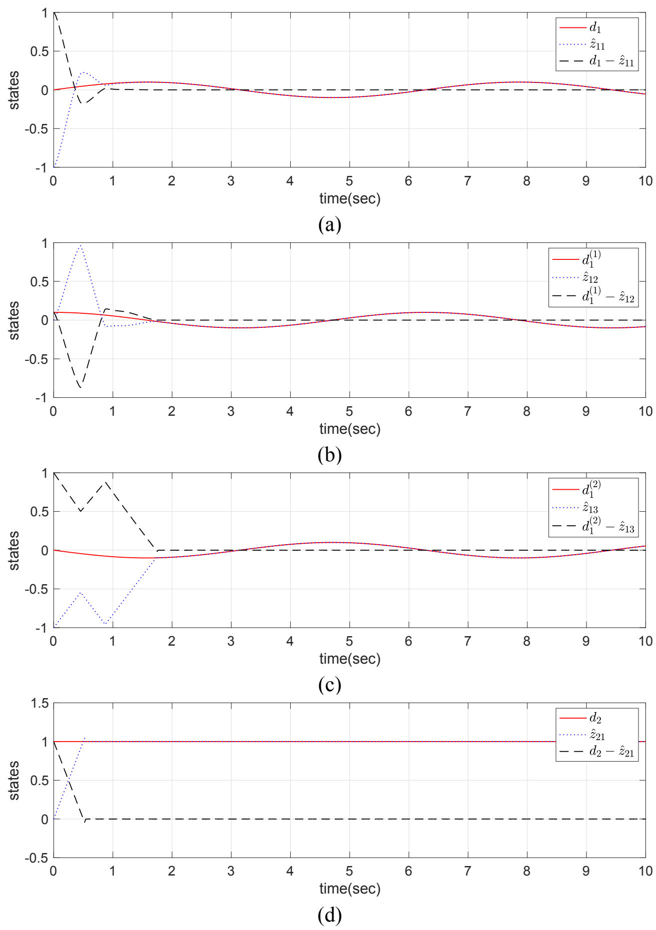

Fig. 5: Disturbance observation performance under the HOSM disturbance observer,
(a): d_1, (b): $d_1^{(1)}$, (c): $d_1^{(2)}$, (d): d_2.

B. Application to Speed Regulation of PMSM

The mathematical model of the permanent magnet synchronous motor (PMSM) in the rotor reference frame is presented as follows [20]

$$\begin{cases} \dot{i}_d = \dfrac{1}{L}(u_d - Ri_d + n_p L\omega i_q) \\ \dot{i}_q = \dfrac{1}{L}(u_q - Ri_q - n_p L\omega i_d - n_p\psi_f\omega) \\ \dot{\omega} = \dfrac{1}{J}\left(\dfrac{3}{2}n_p\psi_f i_q - B_v\omega - T_L\right) \end{cases} \quad (19)$$

where ω is the rotor angular velocity; i_d and i_q are d- and q-axis stator currents, respectively; u_d and u_q are d- and q-axis stator voltages, respectively; T_L is the load torque; n_p is the number of poles-pairs, equals 4; R is the stator resistance, equals 9.7Ω; L is the stator inductance, equals 26mH; ψ_f is the magnetic flux linkage, equals 0.084Wb; J is the moment of inertia, equals $1.35\times10^{-4}\text{kg}\cdot\text{m}^2$; B_v is the frictional coefficient, equals $7.4\times10^{-5}\text{N}\cdot\text{m}\cdot\text{s/rad}$. The control object is to realize finite-time exact tracking of a given speed reference signal $\omega_r = 100\text{rad/s}$ under an unknown load torque, assumed as

$$T_L = \begin{cases} 0.5\ \text{N}\cdot\text{m} & 1s < t \le 2s \\ 0.5 + 0.5\sin(20t)\ \text{N}\cdot\text{m} & 2s < t \le 4s \\ 0\ \text{N}\cdot\text{m} & \text{others.} \end{cases}$$

Under a semi-global stability criterion, one can utilize the proposed exact tracking control method while several auxiliary variables are calculated as: $i_d^* = 0, u_d^* = Li_d^{*(1)} + Ri_d^* - n_pL\omega^* i_q^*, \xi_d = i_d - i_d^*, v_d = (u_d - u_d^*)/l_d$; $\omega^* = \omega_r, i_q^* = \frac{2}{3n_p\psi_f}\left(J\omega_r^{(1)} + B_v\omega_r - Jz_1\right), u_q^* = \frac{2L}{3n_p\psi_f}\left(J\omega_r^{(2)} + B_v\omega_r^{(1)} - Jz_2\right) + Ri_q^* + n_pL\omega_r i_d^* + n_p\psi_f\omega_r; \xi_\omega = \omega - \omega^*,\ \xi_q = (i_q - i_q^*)/l_q, v_q = (u_q - u_q^*)/l_q^2$. The finite-time control scheme is then explicitly presented in Table III.

Table III: Finite-time controller design for PMSM system (19)

HOSM Disturbance Observer: $T_L : \begin{cases} \dot{z}_0 = \frac{1}{J}\left(\frac{3}{2}n_p\psi_f i_q - B_v\omega\right) + \hbar_0 \\ \dot{z}_1 = \hbar_1 \\ \dot{z}_2 = \hbar_2 \\ \hbar_0 = -l_0\lambda^{1/3}\lfloor z_0 - \omega\rceil^{2/3} + z_1 \\ \hbar_1 = -l_1\lambda^{1/2}\lfloor z_1 - \hbar_0\rceil^{1/2} + z_2 \\ \hbar_2 = -l_2\lambda\lfloor z_2 - \hbar_1\rceil^0 \end{cases}$
Exact Tracking Control Law: $\begin{cases} v_q = -K_q\left[\lfloor\xi_\omega\rceil^{1+2\tau_q}, \lfloor\xi_q\rceil^{\frac{1+2\tau_q}{1+\tau_q}}\right]^\top, \\ u_q = l_q^2 v_q + u_q^*. \end{cases}$ $\begin{cases} v_d = -K_d\lfloor\xi_d\rceil^{1+\tau_d}, \\ u_d = l_d v_d + u_d^*. \end{cases}$

In the simulation, the observer parameters are set as: $\lambda = 10^7$, $l_0 = 4$, $l_1 = 2$, $l_2 = 1$. The control parameters are set as: $K_q = [1, 2]$, $l_q = 1.1$, $\tau_q = -0.1$; $K_d = 10$, $l_d = 1.1$, $\tau_d = -0.1$. The initial values of the closed-loop system are chosen as 0.

As is clearly depicted by Fig. 6, in the presence of unknown load torque variation, the speed regulation objective can still be well achieved under the proposed finite-time controller. The response curves of i_q, i_d and the time histories of two control inputs u_q, u_d are presented in Figs. 7 and 8, respectively. The response curves of disturbances and disturbance estimates are given in Fig. 9, which clearly demonstrate the effectiveness of the designed HOSM disturbance observer depicted in Table III.

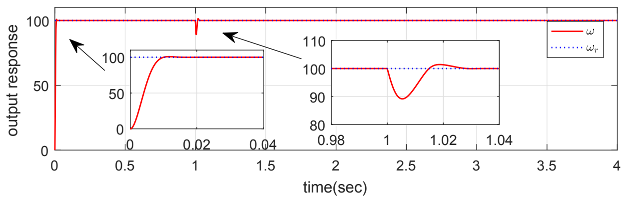

Fig. 6: Output tracking performance.

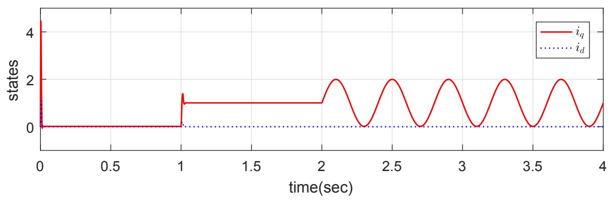

Fig. 7: State response curves.

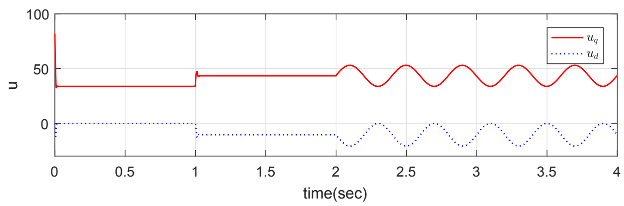

Fig. 8: Time history of the control input.

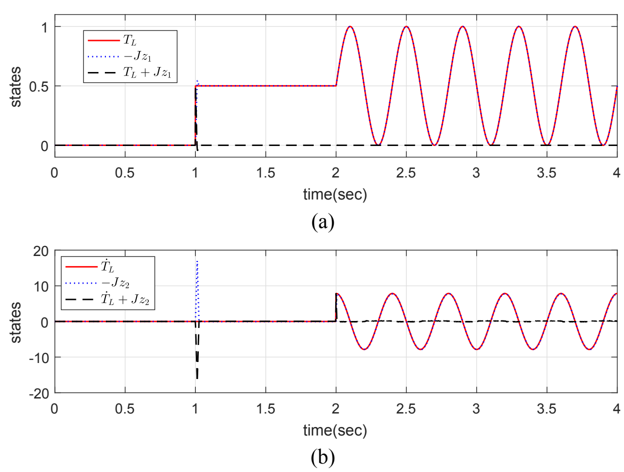

Fig. 9: Disturbance observation performance, (a) T_L, (b) $T_L^{(1)}$.

V. Conclusions

In this paper, we investigate a novel non-recursive tracking control design framework under a semi-global control objective. Compared with existing related results, several distinguishable improvements can be achieved. Firstly, the proposed control scheme is presented with simpler homogenous expression and gain tuning mechanisms. Secondly, it is shown that a finite-time trajectory tracking result can also be realized for disturbed smooth nonlinear systems without any additional nonlinearity growth condition hypothesis. Moreover, the proposed one-step control design and stability analysis under a new non-recursive synthesis manner will largely facilitate the practical implementations, as illustrated by both a numerical example and a PMSM system application.

Appendix

A. Useful Lemmas

Some useful lemmas are stated as follows.

Lemma A.1: (Lemmas 2.1 and 2.2, [21]) Let $V_1(x) \in \mathbb{H}^{\tau_1}_{\Delta^\gamma}$ and $V_2(x) \in \mathbb{H}^{\tau_2}_{\Delta^\gamma}$, respectively, then the following statements hold.

i) $V_1(x)V_2(x) \in \mathbb{H}^{\tau_1+\tau_2}_{\Delta^\gamma}$.

ii) $\frac{\partial V_1(x)}{\partial x_i} \in \mathbb{H}^{\tau_1 - r_i}_{\Delta^\gamma}, \ i \in \mathbb{N}_{1:n}$.

iii) There exists a constant $\bar{c} > 0$ such that $V_1(x) \le \bar{c}\|x\|^{\tau_1}_{\Delta^\gamma}$. Moreover, if $V_1(x)$ is positive definite, then there exists another constant $\underline{c} > 0$ such that $V_1(x) \ge \underline{c}\|x\|^{\tau_1}_{\Delta^\gamma}$.

Lemma A.2: (Theorem 2.1, [14]) Consider a dynamical system $\dot{\eta} = f(\eta), \ f(0) = 0$, where $f : \mathbb{D} \to \mathbb{R}^n$ is non-Lipschitz continuous on an open neighborhood $\mathbb{D}$ of the origin $\eta = 0$ in $\mathbb{R}^n$. If there exist an open neighborhood $\mathbb{U}$ of the origin, a $\mathbb{C}^1$ positive-definite and proper Lyapunov function $V : \mathbb{U}\backslash\{0\} \to \mathbb{R}_+$ and real numbers $c \in \mathbb{R}_+$ and $\iota \in (0,1)$, such that $\dot{V} + cV^\iota \le 0$ for $\eta \in \mathbb{U}$, then the origin $\eta = 0$ is a locally finite-time stable equilibrium with a settling time $T \le \frac{V^{1-\iota}(\eta_0, t_0)}{c(1-\iota)}$ for any given initial condition $\eta_0 = \eta(t_0)$.

Lemma A.3: (Lemma A.2, [10]) Let $c, \ d$ be positive constants. Given any positive constant π, the following inequality holds $|x|^c|y|^d \le \frac{c}{c+d}\pi|x|^{c+d} + \frac{d}{c+d}\pi^{-c/d}|y|^{c+d}$.

B. Proofs of Propositions

This subsection collects the proofs of propositions used in the paper.

Proof of Proposition 3.1: Firstly, by following a similar proof as Lemma 2.6 in [18], we know that $\frac{\partial U(\xi)}{\partial \xi^\top}(\xi_2, \cdots, \xi_n, -K\lfloor\xi\rceil^{\gamma_n+\tau}_{\Delta^\gamma})^\top$ is negative definite for $\tau \in (-\varsigma, 0)$. Secondly, with $U(\xi) \in \mathbb{H}^{2-\tau}_{\Delta^\gamma}$ and $\left(\xi_2, \cdots, \xi_n, -K\lfloor\xi\rceil^{\gamma_n+\tau}_{\Delta^\gamma}\right)^\top \in \mathbb{H}^{\tau}_{\Delta^\gamma}$ in mind, using Lemma A.1, Proposition 3.1 can be achieved for a constant $\alpha \in \mathbb{R}_+$.

Proof of Proposition 3.2: Noting that $\phi_i, \ i \in \mathbb{N}_{1:n}$ is a smooth function, by Mean-Value Theorem, one can obtain that $\phi_i(\bar{x}_i) - \phi_i(\bar{x}_i^*) \le \bar{c}_i(\bar{x}_i, \bar{x}_i^*)\Big(|x_1 - x_1^*| + \cdots + |x_i - x_i^*|\Big)$ where $\bar{c}_i(\bar{x}_i, \bar{x}_i^*)$ is a $\mathbb{C}^1$ nonnegative function.

If $\eta \in \mathbf{\Omega}$, we know there must exist a constant $N > 0$ such that $|x_j| \le N$ for $j \in \mathbb{N}_{1:i}$. Then subsequently, there exists a constant $\tilde{c}_i$ such that $\bar{c}_i(\bar{x}_i, \bar{x}_i^*) \le \tilde{c}_i$. In what follows, the following two cases will be studied.

In the case when $|x_j - x_j^*| \geq 1,\ \forall j \in \mathbb{N}_{1:i}$, by noting that $|x_j| \leq N$ and $|x_j^*| \leq \bar{\rho}$, the following relation holds $|x_j - x_j^*| \leq N + \bar{\rho} \leq (N+\bar{\rho})|x_j - x_j^*|^{\frac{1+i\tau}{1+(j-1)\tau}}$.

In the case when $|x_j - x_j^*| < 1,\ \forall j \in \mathbb{N}_{1:i}$, by noting $\frac{1+i\tau}{1+(j-1)\tau} \leq 1$, we know $|x_j - x_j^*| \leq |x_j - x_j^*|^{\frac{1+i\tau}{1+(j-1)\tau}}$.

By summarizing the above two cases and noting that $(j-1)\frac{1+i\tau}{1+(j-1)\tau} - (i-1) \leq 0, \forall j \in \mathbb{N}_{1:i}$ the following relation holds with constants $\check{c}_i = \tilde{c}_i \max\{N+\bar{\rho}, 1\}$ and $c_i \in \mathbb{R}_+$ which are independent of L

$$
\begin{aligned}
&(\phi_i(\bar{x}_i) - \phi_i(\bar{x}_i^*))/L^{i-1} \\
\leq &\check{c}_i \frac{\left(|L^0\xi_1|^{\frac{1+i\tau}{1}} + |L^1\xi_2|^{\frac{1+i\tau}{1+\tau}} + \cdots + |L^{i-1}\xi_i|^{\frac{1+i\tau}{1+(i-1)\tau}}\right)}{L^{i-1}} \\
\leq &c_i \|\xi\|_{\Delta^\gamma}^{1+i\tau},\ \eta \in \mathbf{\Omega}.
\end{aligned}
$$

It is clear that $U(\xi) \in \mathbb{H}_{\Delta^\gamma}^{2-\tau}$ and $\bar{f}_i \in \mathbb{H}_{\Delta^\gamma}^{1+i\tau}$. With Lemma A.1 in mind, it is straightforward to conclude that Proposition 3.2 holds with a constant $\tilde{\alpha} \in \mathbb{R}_+$.

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