Investigations into Semantics in Reductive Logic

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I, Alexander V. Gheorghiu, confirm that the work presented in this thesis is my own. Where information has been derived from other sources, I confirm that this has been indicated in the work.

Abstract

Logic is the study of reasoning. Typically, it proceeds in terms of inferring a conclusion from established premises. The systematic use of symbolic and mathematical techniques to determine the forms of valid reasoning on this plan determines *Deductive Logic*. *Reductive Logic* is the dual paradigm that proceeds by generating from a putative conclusion a set of sufficient premises. While logical consequence can be characterized through proof-theoretic and semantic approaches, work in Reductive Logic has traditionally focused on the former. This monograph is composed of three parts that illustrate the interplay between semantics and proof in Reductive Logic: Part I comprises a case-study, Part II develops tools and gives result for a more general account, and Part III considers a semantics entirely based on notions of proofs. These a briefly outlined below.

In Part I, the monograph examines proof-search in the logic of Bunched Implications (BI), presenting technical results such as cut-elimination, logic programming, and focusing. It also illustrates a novel approach to soundness and completeness (S&C) for BI that proceeds entirely through proof-search methods, eliminating the need for constructing term- and counter-models.

In Part II, the monograph introduces and develops the theory of a paradigm of proof system called 'algebraic constraint systems' (ACSs). Briefly, ACSs are sequent calculi enriched with an algebra over which constraints are generated during reduction that, when solved, determine a proof. They help bridge the gap between proof theory and semantics in Reductive Logic. In particular, the part uses ACSs for the following: to provide a general account of the approach to S&C studied in BI; to systematically generate relational calculi for logics that satisfy specific conditions;

and to derive a semantics of IPL from its proof theory.

In Part III, the monograph explores *proof-theoretic semantics* — the approach to meaning in logic based on proof rather than truth — and provides both general insights and a range of technical results.

While the monograph contains several contributions to logic across mathematics, informatics, and philosophy, its real contribution is to demonstrate the viability and merit of studying semantics from the perspective of Reductive Logic and to give methods, techniques, and tools for a systematic theory to be developed.

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I am deeply grateful to my supervisor, Prof. David J. Pym, who is a scientific inspiration, for the past few years. Perhaps the most important thing I have learned from him during this time is to ask, *What is the story?* This question is a reminder to ground the research and its presentation at a human level — What is the problem? What is the background? What are the tools? Why are they combined like this? etc.

When I arrived at UCL and began my research, I was most supported by Dr Simon Docherty and Dr Sonia Marin. I hope to support junior researchers in the future as they have supported me. Later, I was fortunate to work closely with Dr. Tao Gu, whose persistence and patience in working through technical detail is something I aim to emulate. This time is also marked by closer association with Dr Elaine Pimentel, who inspires me in her endurance and the astounding quality of her contributions, and her student Mr Yll Buzoku, whose career I watch with much interest.

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Impact Statement

This monograph is about logic. In particular, it develops the theory of logic (as opposed to its applications). Therefore, its impact is primarily academic. That the contributions it contains are substantial is witnessed by publications — see Gheorghiu et al. [79, 77, 82, 80, 81] — at a range of conferences and journals. The following are some of the fields to which this work contributes:

- MATHEMATICAL LOGIC. This work gives several technical results on the metatheory of *the logic of Bunched Implications* (BI) in Part I, and on the semantics of *intuitionistic propositional logic* (IPL) in part III.

- COMPUTATIONAL LOGIC. This work relates proof-search and semantics in Part I and Part II, where it also introduces a tool — *algebraic constraint systems* (ACSs) — for studying proof-search and semantics in computational logic.

- PHILOSOPHICAL LOGIC. This work develops Reductive Logic as a paradigm in which to study semantics, as well as the mathematical treatment of the inferentialist account of logic known as *proof-theoretic semantics*.

- PROOF THEORY. This work contains a range of results on the proof theory of BI in Part I, and introduces a new type of proof system (i.e., ACSs) in Part II. The work on proof-theoretic semantics in Part III may also be considered a part of proof theory.

- MODEL THEORY. This work provides techniques and tools for studying model-theoretic semantics in Part I and Part II. In particular, it provides a novel approach to proving soundness and completeness.

While the monograph contains a number of results supporting the above fields, its deeper impact is the development of techniques and tools that enable future work in those fields. In particular, it lays the foundation for a more systematic account of Reductive Logic and of the interplay between semantics and proof theory. This enables precise pathways for the work in the various fields above to mutually inform one another.

There are several *potential* impacts outside of the academic world. The technical work on proof-search for BI in Part I is valuable according to the applications of the logic — for example, BI is the assertion language of Separation Logic [108]. Similarly, ACS are a means to study control problems during proof-search — for example, Harland and Pym [99, 98] have used it to study the context-management problem in logic programming with linear logics. In general, the use of logic as a mathematical tool for problem-solving — including, for example, its application in program/systems verification — requires both proof- and model-theoretic techniques, whose relationship this monograph investigates. Finally, Reductive Logic lies at the heart of widely deployed proof assistants such as LCF [87], HOL [86], Isabelle [151], Coq [1, 19], Twelf [2, 155], and more; most of which can be seen as building on Milner's theory of *tactical proof* [143], which gave rise to the programming language ML [152]. The theory of tactical proof is shown in Part III to be implicit in apparently different systems of proof-search that have been developed, thereby providing a uniform framework for their use in practice.

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Chapter 1

General Introduction

Logic is the study of reasoning. It is characterized symbolically by two components: a language in which assertions are expressed and a judgement ($\vdash$) on assertions determining consequence. This view of logic did not mature in any mathematical way until the algebraic study in the nineteenth century by Boole [24] and de Morgan [42], and the symbolic approach of Frege [62, 75] — see Gabbay and Woods [69] for a summary. Almost immediately, however, a pantheon of logics became available, which includes, for example, logics that qualify assertions (i.e., modal logics) and those that capture intensional reasoning (i.e., substructural and relevant logics). Whenever a new logic is developed, the theoretical work of the logician lies in developing and relating two broad approaches of inquiry: *proof theory* (the study of reasoning systems) and *semantics* (the study of meaning). This monograph concerns investigating how these two approaches relate to each other from the perspective of *Reductive Logic*.

What is Reductive Logic? Traditionally, the definition of a system of logic is given in terms of asserting a conclusion from established propositions according to specific rules of inference,

$$\frac{\text{Established Premiss}_1 \quad \ldots \quad \text{Established Premiss}_k}{\text{Established Conclusion}} \Downarrow$$

This systematic use of symbolic and mathematical techniques to determine the forms of valid deductive argument defines *Deductive Logic*. This is all very well as a way of defining *proofs*, but it relatively rarely reflects how logic is used in practical reasoning problems. In contrast, one often begins with a putative conclusion and applies the inference rules 'backwards'. In this usage, the rules are sometimes called *reduction operators*, read from conclusion to premisses, and denoted

$$\frac{\text{Sufficient Premiss}_1 \quad \ldots \quad \text{Sufficient Premiss}_k}{\text{Putative Conclusion}}\Uparrow$$

The systematic use of symbolic and mathematical techniques to determine the forms of valid reductive argument defines *Reductive Logic*. Hence, Reductive Logic and Deductive Logic are dual to each other. Notably, the space of reductions of a proposition is larger than its space of proofs, including also failed searches. It appears that this idea of reduction was first explained *in these terms* by Kleene [112]. The first step in developing a mathematical theory of Reductive Logic — in the setting of intuitionistic and classical logic — has been given by Pym and Ritter [173].

The reductive paradigm more closely resembles the way in which mathematicians actually prove theorems and, more generally, the way in which people solve problems using formal representations. It also encapsulates diverse applications of logic in computer science, such as the programming paradigm known as *logic programming*, the proof-search problem at the heart of AI and automated theorem proving, precondition generation in program verification, and more — see, for example, Kowalski [121]. Though described in terms of *derivations*, they need not be understood as formal constructions in a formal system, so Reductive Logic is also reflected at the level of truth-functional semantics — the perspective on logic utilized for *model-checking* and thus verifying the correctness of industrial systems — wherein the truth value of a formula is calculated according to the truth values of its constituent parts. The idea that the *unfolding* of (an inductive definition of) a semantics is much like backward reasoning in a symbolic proof system is explored extensively in this monograph.

By introducing *control*, one may delegate work to a computer. The more control structure provided, the more work delegated: mechanical problem solving begets algorithmic theorem proving techniques begets a programming language paradigm known as logic programming. The witnessed interaction between logic, control, and computation has been put by Kowalski [120] as follows:

$$\text{Algorithm} = \text{Logic} + \text{Control}$$

Emphasising the reductive perspective, and identifying problem solving with *proof-search*, Pym and Ritter [173] instantiate the slogan as follows:

$$\text{Proof-search} = \text{Reductive Logic} + \text{Control}$$

This perspective demonstrates how Reductive Logic encapsulates diverse applications of logic described above.

Indeed, it is precisely *control* that determines the efficacy of a proof-search procedure: some procedures will be complete, some not, some will affect the shape of proofs being found, and some will affect the complexity of the procedure. Background to the issue of reduction, control, and proof-search in the context of capturing human reasoning is discussed extensively in, for example, the work of Kowalski [120] and Bundy [29]. Reductive reasoning lies at the heart of widely deployed proof assistants such as LCF [87], HOL [86], Isabelle [151], Coq [1, 19], Twelf [2, 155], and more. Most of this can be seen as building on the theory of *tactical proof* by Milner [143], which eventually gave rise to the programming language ML [152]. It is discussed in more detail in Part III.

This explains the perspective of Reductive Logic. It remains to explain what is meant by proof theory and by semantics.

By *proof theory*, we mean the study of formal systems of inference and their constructions as mathematical objects with static and dynamic properties. There are many paradigms of proof systems — for example, axiomatic systems, natural deduction systems, tableaux systems, display calculi, etc. — this monograph concentrates on *sequent calculi*, broadly conceived, but also uses *natural deduction* in the sense of Gentzen [200]. To clarify: the monograph concentrates on the sequent calculus *format* — that is, this restriction does not mean calculi with ‘left’ and 'right' rules — which is without loss of generality as the other paradigms are expressible in standard ways. A valuable property of sequent calculi, in the context of reduction, is that they have *local* correctness in the sense that whether or not a reduction

$$\frac{\Gamma_1 \triangleright \Delta_1 \quad \dots \quad \Gamma_n \triangleright \Delta_n}{\Gamma \triangleright \Delta} \Uparrow$$

is well-formed depends only on the content of the sequents involved. We defer further explanation of sequent calculi and their use to when it becomes relevant in the body of the monograph.

By *semantics*, we will, for the majority of the monograph, mean *model-theoretic semantics* (M-tS) in the sense of Tarski [201, 203] (especially in the form of *possible world semantics* — see Kripke [124, 125]). In M-tS, logical consequence is defined in terms of *models* — that is, abstract mathematical structures in which propositions are interpreted and their truth is judged. According to Tarski [201, 203], a propositional formula φ follows model-theoretically from a context Γ iff every model of Γ is a model of φ — that is,

$$\Gamma \vDash \varphi \quad \text{iff} \quad \text{for all models } \mathcal{M}\text{, if } \mathcal{M} \vDash \psi \text{ for all } \psi \in \Gamma\text{, then } \mathcal{M} \vDash \varphi$$

A logic is *sound* with respect to a semantics if consequence implies validity (i.e., $\Gamma \vdash \Delta$ implies $\Gamma \vDash \Delta$); a logic is *complete* with respect to a semantics if validity implies consequence (i.e., $\Gamma \vDash \varphi$ implies $\Gamma \vdash \varphi$). Typically, the validity judgement is defined inductively on the syntax of the formal language. For example, disjunction $(\vee)$ in *intuitionistic propositional logic* (IPL) is typically defined as follows:

$$\mathfrak{M}, x \vDash \varphi \vee \psi \qquad \text{iff} \qquad \mathfrak{M}, x \vDash \varphi \text{ or } \mathfrak{M}, x \vDash \psi$$

— here x denotes a certain *world* in the structure $\mathfrak{M}$. Checking whether a given sequent is valid according to this characterization requires one to *unfold* the semantics according to the inductive definition; for example, one asks if $\varphi \vee \psi$ is valid (at x in $\mathfrak{M}$) by asking if φ or ψ are valid (at x in $\mathfrak{M}$). The model-theoretic reading of logical consequence is central to the applications of logic, specifically in the industrial uses of logic, such as for systems and program verification.

Importantly, in M-tS, *meaning* and *validity* are characterized in terms of *truth*. We emphasize this because this monograph also considers an alternative approach to semantics, *proof-theoretic semantics* (P-tS). In P-tS meaning and validity are characterized *in terms of* proof.

The general idea underlying P-tS is discussed in more detail in Chapter 19 — see also Francez [61], Schroeder-Heister [193], and Wansing [213]. The philosophical paradigm underpinning P-tS is *inferentialism* — the view that the meaning of a proposition is determined by its inferential behaviour (see Brandom [26]). This may be viewed as a particular instantiation of the 'meaning as use' principle advanced by Wittgenstein [215].

To illustrate the paradigmatic shift from M-tS to P-tS, consider the proposition 'Tammy is a vixen'. What does it mean? Intuitively, it means, somehow, 'Tammy is female' *and* 'Tammy is a fox'. In inferentialism, its meaning is given by the rules,

$$\frac{\text{Tammy is a fox} \quad \text{Tammy is female}}{\text{Tammy is a vixen}} \qquad \frac{\text{Tammy is a vixen}}{\text{Tammy is female}} \qquad \frac{\text{Tammy is a vixen}}{\text{Tammy is a fox}}$$

These merit comparison with the laws governing $\wedge$ in IPL, which justify the sense in which the above proposition is a conjunction:

$$\frac{\varphi \quad \psi}{\varphi \wedge \psi} \qquad \frac{\varphi \wedge \psi}{\varphi} \qquad \frac{\varphi \wedge \psi}{\psi}$$

This is the sense of meaning that P-tS concerns.

In this monograph, we concentrate on two major branches of P-tS: *proof-theoretic validity* (P-tV) in the Dummett-Prawitz tradition — see, for example, Schroeder-Heister [190] — and *base-extension semantics* (B-eS) — see, for example, Sandqvist [184, 182, 183]. The terminology, which is taken for the two branches is taken from the referenced work, may be misleading as both define 'validity' and make use of 'base-extension' in doing so. For the purposes of this monograph, we regard the former as a particular stream of ideas for defining the semantics of arguments, and the latter as the same but for a semantics of a logic. This distinction is not part of P-tS broadly, but enables us to discuss with more immediate discernment recent developments in the field — see Part III.

This monograph investigates the interplay between semantics and proof theory from the perspective of Reductive Logic. It comprises three parts, each supporting the other, but which are separate domains of investigation and can be handled independently. We presently give a terse explanation of each part and a summary of the overall picture they deliver, as each has its own introduction and conclusion that gives a more detailed account of their contributions.

Part I of this thesis is a case study of *the logic of Bunched Implication* (BI) [150]. It investigates reduction and proof-search in BI using standard techniques in proof theory. We study BI because it has a relatively subtle meta-theory resulting from it using complex data structures of propositions — specifically, it uses *bunches* rather than lists or multisets for collections of formulae — and, therefore, applying the standard techniques in this setting exposes how they work. The technical contributions include cut-elimination, logic programming, and a focused system for the logic. The part culminates with a proof of soundness and completeness, with respect to a standard model-theoretic semantics for the logic, that proceeds entirely in terms of reduction and proof-search (as opposed to term- and counter-model constructions). The advantage is that constructing models can be quite subtle for BI because of its complex structure. A general account of this approach to soundness and completeness is given in Part II,

Part II introduces a type of proof system called an *algebraic constraint system* (ACS). Briefly, ACSs are a generalization of sequent calculi that carry labels on the data within sequents corresponding to elements of an algebra such that, during reduction, one generates constraints on those labels. The correctness of a construction in an ACSs is then *global* because one must solve the constraints to verify that the construction is a proof. What is the advantage? While ACSs described in this way seem more complex than sequent calculi, they can be much simpler, and several examples of ACSs and their uses across a range of logics are included in the text.

In particular, one can use the algebra as leverage to study proof-search in another sequent calculus as it can allow one to defer committing to certain choices to the end of construction when one solves the constraints. For example, using Boolean algebra, one can render proof-search for BI in its standard sequent calculus, with all the complex interactions between additive and multiplicative structures, as simple proof-search in IPL — this is illustrated in the body of the section. For this monograph, ACSs are useful as they help bridge the gap between semantics and proof theory: first, the novel approach to soundness and completeness in Part II can be expressed quite generally using the theory of ACSs; second, one can use ACSs to *derive* a model-theoretic semantics from a proof-theoretic specification of a logic — see Chapter 17.

Part III investigates P-tS from the perspective of Reductive Logic. In P-tS, validity is grounded in derivability in atomic systems, which (as we discuss) coheres with the background work on Reductive Logic by Pym and Ritter [173] in which validity is determined relative to atomic derivations called *indeterminates* — see Chapter 20. The part analyses the two major branches of P-tS — namely, P-tV in the Dummett-Prawitz tradition and B-eS — from the perspective of Reductive Logic. It demonstrates the B-eS for IPL by Sandqvist [181] can be understood as the declarative counterpart of the *basic* P-tV by Prawitz [164] and that its completeness can be understood in terms of logic programming, in which case the treatment of negation (a subtle issue in P-tS — see, for example, Kürbis [126]) can be understood in terms of the celebrated *negation-as-failure* protocol.

Overall, this monograph is about investigating semantics from the perspective of Reductive Logic. It contains several technical results on reduction and proof-search (especially for IPL and BI) and introduces a paradigm of proof systems that enables a more uniform and systematic study. Relative to these technical developments, it provides a new method for studying soundness and completeness in model-theoretic semantics. It also studies P-tS from the perspective of Reductive Logic, exposing deep connections between its two major branches and delivering an understanding of negation, which is subtle. These contributions come across mathematical, philosophical, and computational logic and open up the opportunity for new approaches to the theory and practice of logic.

Background

Chapter 2

Intuitionistic Propositional Logic

Central to this monograph is *intuitionistic propositional logic* (IPL), whose mathematical treatment serves both as motivation and provides a background of basic results on which the work is built. This chapter concerns the technical results of IPL to which we shall often refer. The history, development, and philosophy underpinning IPL, while interesting, are not important for this work and are, therefore, largely elided — see Dummett [52].

2.1 Syntax & Consequence

There are many presentations of IPL in the literature. Therefore, we begin by fixing the relevant concepts and terminology for this work.

Definition 2.1 (Formula)**.** *Fix a set of atomic propositions* $\mathbb{A}$*. The set of formulas* $\mathbb{F}$ *(over* $\mathbb{A}$*) is constructed by the following grammar:*

$$\varphi ::= \mathrm{p} \in \mathbb{A} \mid \varphi \vee \varphi \mid \varphi \wedge \varphi \mid \varphi \to \varphi \mid \bot$$

We use meta-variables Γ and Δ, possibly adorned with subscripts, to denote sets of formulae. We use the following abbreviations, where φ is a formula and Γ is a finite set:

$$\hat{\Gamma} := \bigwedge_{\varphi \in \Gamma} \varphi \qquad \neg\varphi := \varphi \to \bot$$

Definition 2.2 (Sequent)**.** *A sequent is a pair* $\Gamma \triangleright \Phi$ *in which* Γ *is a set of formulas, and* Φ *is either a formula* φ *or is the emptyset* $\varnothing$*.*

$$
\begin{array}{lll}
\mathfrak{M}, w \Vdash \mathrm{p} & \text{iff} & w \in [\![p]\!] \\
\mathfrak{M}, w \Vdash \varphi \wedge \psi & \text{iff} & \mathfrak{M}, w \Vdash \varphi \text{ and } \mathfrak{M}, w \Vdash \psi \\
\mathfrak{M}, w \Vdash \varphi \vee \psi & \text{iff} & \mathfrak{M}, w \Vdash \varphi \text{ or } \mathfrak{M}, w \Vdash \psi \\
\mathfrak{M}, w \Vdash \varphi \to \psi & \text{iff} & \text{for any } v \in \mathbb{V},\ \text{if } w \preceq v \text{ and } \mathfrak{M}, v \Vdash \varphi,\ \text{then } \mathfrak{M}, v \Vdash \psi \\
\mathfrak{M}, w \vDash \bot & & \text{never}
\end{array}
$$

Figure 2.1: Kripke's Semantics for IPL

Intuitionistic propositional logic (IPL) is a certain judgement $\vdash$ on sequents, called *consequence* — we write $\Gamma \vdash \varphi$ to denote that $\Gamma \triangleright \varphi$ is a consequence of IPL. This judgement exists *a priori*, but may be characterized in various ways — in particular, through semantics and proof theory. Such characterizations are presented in the following sections.

2.2 Model-theoretic Semantics

A well-known characterization of IPL is the *possible world semantics* given by Kripke [125]. We assume general familiarity, and give a brief summary only to keep the monograph self-contained.

Definition 2.3 (Frame)**.** *A frame is a pair $\mathcal{F} := \langle \mathbb{V}, \preceq \rangle$ in which $\preceq$ is a partial order on $\mathbb{V}$.*

Definition 2.4 (Interpretation)**.** *An interpretation of the atoms $\mathbb{A}$ in the frame $\langle \mathbb{V}, \preceq \rangle$ is a mapping $[\![-]\!] : \mathbb{A} \to \mathscr{P}(\mathbb{V})$.*

Definition 2.5 (Satisfaction)**.** *Satisfaction is the least relation satisfying the clauses of Figure 2.1 in which $\mathfrak{M} := \langle \langle \mathbb{V}, \preceq \rangle, [\![-]\!] \rangle$ such that $\langle \mathbb{V}, \preceq \rangle$ is a frame, $[\![-]\!]$ is an interpretation of $\mathbb{A}$ in $\mathcal{F}$, and $w \in \mathbb{V}$.*

Definition 2.6 (Model)**.** *A model is a pair $\mathfrak{M} := \langle \mathcal{F}, [\![-]\!] \rangle$ that is persistent: for any $u, w \in \mathbb{V}$ and $\varphi \in \mathbb{F}$, if $w \preceq u$ and $\mathfrak{M}, w \Vdash \varphi$, then $\mathfrak{M}, u \Vdash \varphi$. The set of all models is denoted $\mathbb{K}$.*

This concept of model induces a *validity* judgement as follows:

$$\Gamma \Vdash \varphi \qquad \text{iff} \qquad \text{for any } \mathfrak{M} \in \mathbb{K} \text{ and any } w \in \mathfrak{M}, \text{ if } \mathfrak{M}, w \Vdash \Gamma, \text{ then } \mathfrak{M}, w \Vdash \varphi$$

This judgement is IPL-consequence:

Theorem 2.7 (Kripke [125])**.** $\Gamma \vdash \varphi$ *iff* $\Gamma \Vdash \varphi$.

This summarizes the typical possible world semantics for IPL. Another similar semantics of IPL that actually predates it is the one given by Beth [20]. The problem with this earlier semantics is that it has a complicated clause for disjunction ($\vee$). We give a terse account of it presently, as it will be referred to in later chapters (particularly in Part I).

A frame $\langle \mathbb{V}, \preceq \rangle$ is a *Beth frame* if it is a rooted tree — that is, there is a distinguished node, the root r, such that for any other node x there is a sequence of worlds $x_0 \leq x_1 \leq \ldots \leq x_{n-1} \leq x_n$ such that $x_0 = r$ and $x_n = x$. In a Beth frame, a set $\mathbb{B} \subseteq \mathbb{V}$ *bars* $x \in \mathbb{V}$ if $\mathbb{B}$ is a maximal linearly ordered subset of $\mathbb{V}$ containing x. The notion of satisfaction and model is as above, except for the treatment of disjunction:

$$\begin{array}{lll} \mathfrak{M}, w \Vdash \varphi \vee \psi & \text{iff} & \text{any } \mathbb{B} \subseteq \mathbb{V} \text{ barring } w \text{ contains } v \text{ such that} \\ & & v \succeq w \text{ and either } \mathfrak{M}, v \Vdash \varphi \text{ or } \mathfrak{M}, v \Vdash \psi \end{array}$$

2.3 Natural Deduction

This section contains a terse but complete characterization of IPL in terms of *natural deduction*. We assume general familiarity with the subject — see, for example, Troelstra and Schwichtenberg [207], Negri and von Plato [149] — and, therefore, elide justification and elaboration accordingly.

Definition 2.8 (Argument)**.** *An argument is a rooted tree of formulae with some leaves marked as discharged with an associated edge understood as the place where the leaf is discharged.*

An argument is *open* if it has undischarged assumptions; otherwise, it is *closed*. The leaves of an argument are its *assumptions*, and the root is its *conclusion*. An

$$\frac{\varphi \quad \psi}{\varphi\wedge\psi}\wedge_{\mathsf{I}} \qquad \frac{\varphi\wedge\psi}{\varphi}\wedge_{\mathsf{E}}^{1} \quad \frac{\varphi\wedge\psi}{\psi}\wedge_{\mathsf{E}}^{2} \qquad \frac{\begin{matrix}[\varphi]\\ \psi\end{matrix}}{\varphi\to\psi}\to_{\mathsf{I}} \qquad \frac{\bot}{\varphi}\,\mathsf{EFQ}$$

$$\frac{\varphi}{\varphi\vee\psi}\vee_{\mathsf{I}}^{1} \quad \frac{\psi}{\varphi\vee\psi}\vee_{\mathsf{I}}^{2} \qquad \frac{\varphi\vee\psi \quad \begin{matrix}[\varphi]\\ \chi\end{matrix} \quad \begin{matrix}[\psi]\\ \chi\end{matrix}}{\chi}\vee_{\mathsf{E}} \qquad \frac{\varphi \quad \varphi\to\psi}{\varphi}\to_{\mathsf{E}}$$

Figure 2.2: Natural Deduction System NJ

argument $\mathcal{A}$ is an argument for a sequent $\Gamma \rhd \varphi$ iff the open assumptions of $\mathcal{A}$ are a subset of Γ and the conclusion of $\mathcal{A}$ is φ. We use the following notations to express that $\mathcal{A}$ is an argument for $\Gamma \rhd \varphi$:

$$\begin{matrix}\mathcal{A}\\ \varphi\end{matrix} \qquad \begin{matrix}\Gamma\\ \mathcal{A}\end{matrix} \qquad \begin{matrix}\Gamma\\ \mathcal{A}\\ \varphi\end{matrix}$$

An argument regulated by rules for a *natural deduction system* is a *derivation* in that system.

Definition 2.9 (Natural Deduction System NJ)**.** *The natural deduction system* NJ *is composed of the rules in Figure 2.2.*

Definition 2.10 (Derivation in NJ)**.** *The set of* NJ*-derivations is defined inductively as follows:*

- Base Case. *If φ is a formula, then the one element tree consisting of just φ is an* NJ*-derivation.*

- Inductive Step. *Let* $\mathsf{r} \in \mathsf{NJ}$ *be the following rule:*

$$\frac{\begin{matrix}[\Gamma_1]\\ \varphi_1\end{matrix} \quad \dots \quad \begin{matrix}[\Gamma_n]\\ \varphi_n\end{matrix}}{\varphi}$$

— the sets $\Gamma_1,...,\Gamma_n$ may be empty. For $i = 1,...,n$, let $\mathcal{D}_i$ be an NJ*-derivation of φ_1 containing open assumptions Γ_i (and possible others). The tree with root φ and immediate sub-trees $\mathcal{D}_1,...,\mathcal{D}_n$ with the the elements of Γ_1 possibly discharged is an*

NJ*-derivation.*

Observe that we do not insist in the base case of Definition 2.10 that the formula is an atom.

An argument is closed iff it contains no open assumptions. A closed derivation in NJ is a *proof* in NJ. We write $\vdash_{\mathsf{NJ}} \varphi$ to denote that there is an NJ-proof of φ.

Theorem 2.11 (Gentzen [200]). *There is an* NJ*-proof of* φ *iff* φ *is a consequence of IPL — that is,*

$$\vdash_{\mathsf{NJ}} \varphi \qquad \textit{iff} \qquad \vdash \varphi$$

More generally, we write $\Gamma \vdash_{\mathsf{NJ}} \varphi$ iff there is an NJ-derivation all of which open assumptions are Γ and whose root is φ. This enables (using the Deduction Theorem — see Herbrand [101]) the following restatement of the above theorem:

Proposition 2.12. *There is an* NJ*-derivation of* $\Gamma \triangleright \varphi$ *iff* $\Gamma \vdash \varphi$.

2.4 Sequent Calculus

While natural deduction gives a proof-theoretic characterization of IPL, it is difficult to analyze; for example, it is difficult to use NJ to check whether or not an arbitrary sequent $\Gamma \triangleright \varphi$ is a consequence of IPL. The problem is the non-determinism afforded by the elimination rules; for example, an NJ-derivation of $\Gamma \triangleright \chi$ may conclude by the use of $\vee_{\mathsf{E}}$ on some formula $\varphi \vee \psi$, but it is not clear from analyzing χ what formulae φ and ψ to choose. An alternative proof-theoretic treatment more convenient for computation (especially proof-search) is given by *sequent calculi.* As in Section 2.3, we assume familiarity with typical presentations — see, for example, Gentzen [200], Troelstra and Schwichtenberg [207], and Negri and von Plato [149].

Definition 2.13 (Sequent Calculus LJ). *Sequent calculus* LJ *is comprised of the rules in Figure 2.3, in which* φ *and* ψ *are formulae,* Γ *is a set of formulae,* $\varnothing$ *is the emptyset, and* Φ *either a formula or the emptyset.*

Definition 2.14 (Proof in LJ). *The set of* LJ*-proofs is the smallest set of rooted trees of sequents satisfying the following:*

$$\frac{}{\varphi \rhd \varphi}\,\mathsf{ax} \qquad \frac{\Gamma \rhd \Phi}{\varphi, \Gamma \rhd \Phi}\,\mathsf{w_L} \qquad \frac{\Gamma \rhd}{\Gamma \rhd \varphi}\,\mathsf{w_R}$$

$$\frac{\Gamma \rhd \varphi \quad \Gamma \rhd \psi}{\Gamma \rhd \varphi \wedge \psi}\,\wedge_{\mathsf{R}} \qquad \frac{\varphi, \Gamma \rhd \Phi}{\varphi \wedge \psi, \Gamma \rhd \Phi}\,\wedge^1_{\mathsf{L}} \qquad \frac{\psi, \Gamma \rhd \Phi}{\varphi \wedge \psi, \Gamma \rhd \Phi}\,\wedge^2_{\mathsf{L}} \qquad \frac{\varphi, \Gamma \rhd \varnothing}{\Gamma \rhd \neg \varphi}\,\neg_{\mathsf{R}}$$

$$\frac{\varphi, \Gamma \rhd \Phi \quad \psi, \Gamma \rhd \Phi}{\varphi \vee \psi, \Gamma \rhd \Phi}\,\vee_{\mathsf{L}} \qquad \frac{\Gamma \rhd \varphi}{\Gamma \rhd \varphi \vee \psi}\,\vee^1_{\mathsf{R}} \qquad \frac{\Gamma \rhd \psi}{\Gamma \rhd \varphi \vee \psi}\,\vee^2_{\mathsf{R}} \qquad \frac{\Gamma \rhd \varphi}{\neg \varphi, \Gamma \rhd \varnothing}\,\neg_{\mathsf{L}}$$

$$\frac{\varphi, \Gamma \rhd \psi}{\Gamma \rhd \varphi \to \psi}\,\to_{\mathsf{R}} \qquad \frac{\Gamma_1 \rhd \varphi \quad \psi, \Gamma_2 \rhd \Phi}{\varphi \to \psi, \Gamma_1, \Gamma_2 \rhd \Phi}\,\to_{\mathsf{L}}$$

Figure 2.3: Sequent Calculus LJ

- Base Case. *The one element tree consisting of a sequent $\varphi \rhd \varphi$, for any formula φ, is an* LJ*-proof.*

- Inductive Step. *Suppose* LJ *contains a rule instantiating to the following:*

$$\frac{\Gamma_1 \rhd \Phi_1 \quad \ldots \quad \Gamma_n \rhd \Phi_n}{\Gamma \rhd \Phi}$$

For $i = 1, \ldots, n$, let $\mathcal{D}_i$ be an LJ*-proof with root $\Gamma_i \rhd \Phi_i$. The tree with root $\Gamma \rhd \varphi$ and immediate sub-trees $\mathcal{D}_1, \ldots, \mathcal{D}_n$ is an* LJ*-proof.*

Theorem 2.15 (Gentzen [200])**.** *There is an* LJ*-proof with root $\Gamma \rhd \varphi$ iff $\Gamma \vdash \varphi$.*

The proof of this makes essential use of the *admissibility* of the following rule:

$$\frac{\Gamma \rhd \chi \quad \chi, \Gamma \rhd \Phi}{\Gamma \rhd \Phi}\,\mathsf{cut}$$

That is, let $\mathsf{LJ} + \mathsf{cut}$ denote the sequent calculus resulting from LJ extended with cut. The following result follows from Gentzen [200]:

Proposition 2.16 (cut-Admissibility)**.** *There is an* LJ*-proof of $\Gamma \rhd \Phi$ iff there is an* $\mathsf{LJ} + \mathsf{cut}$*-proof of $\Gamma \rhd \varphi$.*

Troelstra and Schwichtenberg [207]. One has an *effective procedure* way of transforming an LJ-proof of a sequent into an $\mathsf{LJ} + \mathsf{cut}$-proof of the same sequent, following the ideas and methods introduced by Gentzen [200]. The procedure is closely related to the reduction operators given by Prawitz [168] (Section 2.3) on natural deduction arguments.

Apart from its use in semantics, proofs in $\mathsf{LJ}+\mathsf{cut}$ can be much shorter than in LJ — see, for example, Boolos [25]. So, why not simply include cut? The sequent calculus without the cut is easier to analyze. In particular, its proofs are *analytic*, meaning that all the formulae appearing sequents are sub-formulae of the root — this is known as the *sub-formula property*. The sub-formula property makes the sequent calculus a powerful computational device in terms of reductive proof-search since the space of arguments is limited by the number of ways sub-formulae can be combined to form sequents. Indeed, for IPL, the space becomes restricted in that one has a decision procedure for consequence via proof-search — see Gentzen [200].

2.5 Logic Programming

Proof-search may be used to give an *operational semantics* to logics; that is, expressing the logical constants in terms of how they are arrived at (as opposed to how they affect truth relative to algebraic structures as in M-tS). This is explored in this section. We closely follows work by Miller [139] (see also Harland [96]) as the results and techniques are used in other parts of the monograph — see Chapter 8 and Chapter 23. There are, of course, other notable proof-theoretic approaches to logic programming — see, for example, Gabbay and Reyle [64, 65] and Schroeder-Heister and Hallnäs [94, 95].

The logical constant need to be proof-theoretically well-behaved for there to be a tractable operational semantics. This is mostly the case for IPL and LJ. For example, the $\wedge_{\mathsf{I}}$ rule is invertible — that is, $\Gamma \vdash \varphi \wedge \psi$ iff $\Gamma \vdash \varphi$ and $\Gamma \vdash \psi$ — so one may read the assertion of $\varphi \wedge \psi$ relative to a context Γ as asserting both φ and ψ relative to Γ. However, disjunction is more subtle. One may not, in general, read an assertion $\varphi \vee \psi$ relative to a context Γ as saying that one either has φ or ψ relative to Γ; for example, an LJ-proof witnessing $\varphi \vee \psi \vdash \psi \vee \varphi$ must begin with

$\vee_{\mathsf{E}}$ (not $\vee_{\mathsf{I}}$). Under certain conditions, however, disjunction is more tractable from the perspective of proof-search. For example, IPL has the *Disjunctive Property* according to which $\vee_{\mathsf{I}}$ is invertible when the context is empty,

$$\vdash \varphi \vee \psi \qquad \text{iff} \qquad \vdash \varphi \text{ or } \vdash \psi$$

Intuitively, the problem with $\vee_{\mathsf{I}}$ is when the context is not *definite* in the sense that it contains the *potential* of a disjunction — for example, as discussed above, it fails above when the context literally contains a disjunction. More generally, disjunction is proof-theoretically well-behaved when the context is composed of *definite* formulae.

The *propositional hereditary Harrop fragment of IPL* is determined by the following grammar in which $A \in \mathbb{A}$ is an atomic proposition, D is a *definite formula*, and G is a *goal* formula:

$$\begin{array}{rcl} D & ::= & A \mid G \to A \mid D \wedge D \\ G & ::= & A \mid D \to G \mid G \wedge G \mid G \vee G \end{array}$$

A set of definite formulae $\mathscr{P}$ is a *program* — typically, it is a finite set, but we shall have cause to consider infinite sets. The set of all programs is $\mathbb{P}$. This terminology arises as one thinks of programs $\mathscr{P}$ as a set of instructions relative to which one *computes* goal formula G.

The judgement $\mathscr{P} \vdash_o G$ obtains iff there is an execution in the operational semantics for the hereditary Harrop fragment of IPL that concludes the sequent $\mathscr{P} \triangleright G$. The semantics in question is given by *uniform* proof-search in LJ — see Miller et al. [140]. For purely technical reasons, we require a decomposition function $\mathsf{cl}(-) : \mathbb{P} \to \mathbb{P}$ that closes programs according to their conjunctions. Let $\mathsf{cl}(\mathscr{P})$ be the least set satisfying the following:

- $\mathscr{P} \subseteq \mathsf{cl}(\mathscr{P})$

- If $D_1 \wedge D_2 \in \mathsf{cl}(\mathscr{P})$, then $D_1 \in \mathsf{cl}(\mathscr{P})$ and $D_2 \in \mathsf{cl}(\mathscr{P})$.

$\mathscr{P} \vdash_o A$	if	$A \in \mathsf{cl}(\mathscr{P})$	(IN)
$\mathscr{P} \vdash_o A$	if	$G \to A \in \mathsf{cl}(\mathscr{P})$ and $\mathscr{P} \vdash_o G$	(CLAUSE)
$\mathscr{P} \vdash_o G_1 \vee G_2$	if	$\mathscr{P} \vdash_o G_1$ or $\mathscr{P} \vdash_o G_2$	(OR)
$\mathscr{P} \vdash_o G_1 \wedge G_2$	if	$\mathscr{P} \vdash_o G_1$ and $\mathscr{P} \vdash_o G_2$	(AND)
$\mathscr{P} \vdash_o D \to G$	if	$\mathscr{P} \cup \{D\} \vdash_o G$	(LOAD)

Figure 2.4: Operational Semantics for Hereditary Harrop IPL

Definition 2.17 (Operational Semantics)**.** *The operational semantics witnessing a judgement $\mathscr{P} \vdash_o G$ is given in Figure 2.4.*

Theorem 2.18 (Miller et al.[139, 140])**.** *$\mathscr{P} \vdash G$ iff $\mathscr{P} \vdash_o G$.*

This completes the operational semantics of definite formulae. How does it relate to the model-theoretic semantics of IPL presented in Section 2.2? Rather than give a completely abstract characterization of models, one can take advantage of the proof-search behaviour to construct a model from the definite formulae composing the program relative to which goals may be evaluated. This is the subject of the remainder of this section. It illustrates how one may use Reductive Logic to connect proof theory and semantics.

Definition 2.19 (Interpretation)**.** *An interpretation is a mapping $I : \mathbb{P} \to \mathscr{P}(\mathbb{A})$ such that $\mathscr{P} \subseteq \mathscr{Q}$ implies $I(\mathscr{P}) \subseteq I(\mathscr{Q})$.*

Definition 2.20 (Satisfaction)**.** *The satisfaction judgement is given by the clauses of Figure 2.5.*

We desire a particular interpretation J such that the following holds:

$$J, \mathscr{P} \vDash G \qquad \text{iff} \qquad \mathscr{P} \vdash G$$

To this end, we consider an operator T (mapping interpretations to interpreta-

$$\begin{array}{lll}
I, \mathscr{P} \vDash A & \text{iff} & A \in I(\mathscr{P}) \\
I, \mathscr{P} \vDash G_1 \vee G_2 & \text{iff} & I, \mathscr{P} \vDash G_1 \text{ or } I, \mathscr{P} \vDash G_2 \\
I, \mathscr{P} \vDash G_1 \wedge G_2 & \text{iff} & I, \mathscr{P} \vDash G_1 \text{ and } I, \mathscr{P} \vDash G_2 \\
I, \mathscr{P} \vDash D \to G & \text{iff} & I, \mathscr{P} \cup \{D\} \vDash G
\end{array}$$

Figure 2.5: Satisfaction for Hereditary Harrop IPL

tions) that unfolds programs:

$$\begin{array}{rcl}
T(I)(\mathscr{P}) & := & \{A \mid A \in \mathsf{cl}(\mathscr{P})\} \cup \\
& & \{A \mid (G \to A) \in \mathsf{cl}(\mathscr{P}) \text{ and } I, \mathscr{P} \vDash G\}
\end{array}$$

Interpretations form a lattice under point-wise union ($\sqcup$), point-wise intersection ($\sqcap$), and point-wise subset ($\sqsubseteq$); the bottom of the lattice is given by $I_\perp : \mathscr{P} \mapsto \varnothing$. It is easy to see that T is monotonic and continuous on this lattice, and, by the Knaster-Tarski Theorem [7], its least fixed-point is given as follows:

$$T^\omega I_\perp := I_\perp \sqcup T(I_\perp) \sqcup T^2(I_\perp) \sqcup \ldots$$

Intuitively, each application of T corresponds to the application of a clause from the program so that T^ω corresponds to arbitrarily many applications.

Theorem 2.21 (Miller [139], Harland [96])**.** $\mathscr{P} \vdash_o G$ *iff* $T^\omega I_\perp, \mathscr{P} \vDash G$

The proximal relationship between the denotational semantics and proof-search for the hereditary Harrop fragment for IPL supports and motivates the monograph. Is the relationship always there? Yes — this is one of the theses of this monograph and is developed explicitly in Part II. Since both proof-search and semantics concern the unfolding of the logic one expects a close relationship. The challenge is that the proof-search is usually so complex that the relationship is obscured.

2.6 The BHK Interpretation

To conclude this introduction on IPL, we give a typical simplified account of the Brouwer-Heyting-Kolmogorov interpretation, which involves numerous complex ideas, following Troelstra and van Dalen [208].

Intuitionism, as defined by Brouwer [28], is the view that an argument is valid when it provides sufficient evidence for its conclusion. This *defines* IL. A distinguishing feature is that IL differs from classical logic by rejecting *tertium non datur* — that is, the ability to assert a proposition for the rejection of its negation — as such an inference does not constitute sufficient evidence for its conclusion. An important question is, what is meant by *sufficient evidence*?

Heyting [102] and Kolmogorov [114] provided a semantics for intuitionistic proof, which captures the evidential character of intuitionism. This is the BHK interpretation of IL. It is now the standard explanation of the logic — see, for example, Dummett [52]. Supposing a notion of proof for atomic formulae,

- a proof $\mathcal{A}$ of $\varphi \land \psi$ is a pair $\langle \mathcal{B}_1, \mathcal{B}_2 \rangle$ such that $\mathcal{B}_1$ is a proof of φ and $\mathcal{B}_2$ is a proof of ψ
- a proof $\mathcal{A}$ of $\varphi \lor \psi$ is either a pair $\langle 0, \mathcal{B} \rangle$ such that $\mathcal{B}$ is a proof of φ or a pair $\langle 0, \mathcal{B} \rangle$ such that $\mathcal{B}$ is a proof of ψ
- a proof of $\varphi \to \psi$ is a method of f for constructing a proof of ψ from a proof of φ
- nothing is a proof of $\bot$

We observe in this characterization apparent similarities with the characterization of P-tV through satisfaction in Figure 20.1. Indeed, the motivating question is the same as the one asked by Tennant [204] that serves as general motivation for P-tS — see Chapter 19.

The *propositions-as-types* correspondence — see Howard [106] — gives an standard way of instantiating the denotation of proofs in the BHK interpretation of intuitionistic propositional logic (IPL) (see Section 2.6) as terms in the simply-typed

λ-calculus. Technically, the setup can be sketched as follows: a judgement that Φ is an NJ-proof of the sequent $\varphi_1, \ldots, \varphi_k \triangleright \varphi$ corresponds to a typing judgement

$$x_1 : A_1, \ldots, x_k : A_k \vdash M(x_1, \ldots, x_k) : A$$

where the A_is are types corresponding to the φ_is, the x_is correspond to placeholders for proofs of the φ_is, the λ-term $M(x_1, \ldots, x_k)$ corresponds to Φ, and the type A corresponds to φ.

Lambek [129] gave a more abstract account by showing that simply-typed λ-calculus is the internal language of *Cartesian Closed Categories* (CCCs), thereby giving a categorical semantics of proofs for IPL. In this setup, a morphism

$$[\![\varphi_1]\!] \times \ldots \times [\![\varphi_k]\!] \xrightarrow{[\![\Phi]\!]} [\![\varphi]\!]$$

in a CCC (where × denotes Cartesian product) that interprets the NJ-proof Φ of $\varphi_1, \ldots, \varphi_k \triangleright \varphi$ also interprets the term M, where the $[\![\varphi_i]\!]$s interpret also the A_is and $[\![\varphi]\!]$ also interprets A.

To generalize to full IL (and beyond), Seely [195] modified this categorical setup and introduced *hyperdoctrines* — indexed categories of CCCs with coproducts over a base with finite products. Martin-Löf [136] gave a formulae-as-types correspondence for predicate logic using dependent type theory. Barendregt [13] gave a systematic treatment of type systems and the propositions-as-types correspondence. A categorical treatment of dependent types came with Cartmell [31] — see also, for examples among many, work by Streicher [199], Pavlović [153], Jacobs [109], and Hofmann [105]. In total, this gives a semantic account of *proof* for first- and higher-order predicate intuitionistic logic based on the BHK interpretation.

That is all very well as explaining what a proof is for IL, but the space of objects considered when finding an argument also contains things that are not proofs and cannot be continued to form proofs. Constructed by backward inference, we call these objects *reductions*. In IPL, an example of a reduction that fails to be a proof

$$\begin{tikzcd}
{[\Gamma] \vdash [\Phi] : [\varphi]} \arrow[rr, leftrightarrow] \arrow[dr, leftrightarrow] & & {\llbracket \Gamma \rrbracket \overset{\llbracket \Phi \rrbracket}{\rightsquigarrow} \llbracket \varphi \rrbracket} \arrow[dl, leftrightarrow] \\
& \Phi \Rightarrow \Gamma \rhd \varphi &
\end{tikzcd}$$

Figure 2.6: The Constructions-as-Realizers-as-Arrows Correspondence

is an NJ-derivation whose open assumptions are not theorems of IPL. While such an argument is well-constructed according to intuitionism, it is not valid since it is not closed and cannot be closed (i.e., the open assumptions cannot be substituted for proofs).

To address this problem, Pym and Ritter [173] have provided a general semantics of Reductive Logic that is close to P-tV. This is summarized in a *constructions-as-realizers-as-arrows correspondence* — see Figure 2.6. The judgement $\Phi \Rightarrow \Gamma \rhd \varphi$ denotes that Φ is a proof-search — that is, an attempt at constructing a valid argument — for the sequent $\Gamma \rhd \varphi$; the judgement $[\Gamma] \vdash [\Phi] : [\varphi]$ denotes that $[\Phi]$ is a *realizer* of $[\varphi]$ with respect to the assumptions $[\Gamma]$; and $\llbracket \Gamma \rrbracket \overset{\llbracket \Phi \rrbracket}{\rightsquigarrow} \llbracket \varphi \rrbracket$ denotes that $\llbracket \Gamma \rrbracket$ is a morphism from $\llbracket \Gamma \rrbracket$ to $\llbracket \varphi \rrbracket$. In the setting of Pym and Ritter [173], the categorical interpretation of Reductive Logic employs polynomial constructions over the underlying category in order to capture the use of *indeterminates* — that is, variables that stand for uncompleted fragments of proofs.

This concludes the summary of IPL.

Chapter 3

First-order Classical Logic

Throughout this monograph, first-order classical logic (FOL) is used as a tool to study metatheory. This is akin to the use of logic in, for example, universal algebra, where it provides a platform on which one can uniformly express and investigate different kinds of structures. We assume general familiarity with FOL — see, for example, van Dalen [211] and Troelstra and Schwichtenberg [207] — and give a terse but complete summary so we may refer to it without ambiguity.

As we wish to reserve traditional symbols such as $\vdash$ and $\rightarrow$ for the other logics, we will use $\blacktriangleright$ and $\Rightarrow$ for FOL. In both cases, we use the symbol $\triangleright$ as the sequents symbol, regarding $\vdash$ and $\blacktriangleright$ as *consequence* relations.

3.1 Syntax and Consequence

There are many presentations of FOL in the literature. Therefore, we begin by fixing the relevant concepts and terminology for this work.

Definition 3.1 (First-order Language)**.** *An alphabet is a tuple $A := \langle \mathbb{R}, \mathbb{F}, \mathbb{K}, \mathbb{V} \rangle$ in which $\mathbb{R}$, $\mathbb{F}$, $\mathbb{K}$, and $\mathbb{V}$ are pairwise disjoint countable sets of symbols, and each element of $\mathbb{R}$, $\mathbb{F}$ and $\mathbb{K}$ has a fixed arity.*

The set $\mathbb{TERM}(A)$ of terms *T from A is the least set containing $\mathbb{K}$ and $\mathbb{V}$ such that, for any $F \in \mathbb{F}$, if F has arity n and $T_1, ..., T_n \in \mathbb{TERM}(A)$, then $F(T_1, ..., T_n) \in \mathbb{TERM}(A)$.*

The set $\mathbb{ATOMS}(\mathbb{A})$ is set of strings $R(T_1, ..., T_n)$ such that $R \in \mathbb{R}$ has arity n and $T_1, ..., T_n \in \mathbb{TERM}(A)$.

The set $\mathbb{WFF}(A)$ *of* formulae *from A is defined by the following grammar, in which* $X \in \mathbb{V}$*:*

$$\Phi := A \in \mathbb{ATOMS}(\mathbb{A}) \mid \Phi \Rightarrow \Phi \mid \Phi \& \Psi \mid \Phi \invamp \Psi \mid \forall X \Phi \mid \exists X \Phi \mid \# \mid \Box$$

The symbols $\Rightarrow$, $\&$, $\invamp$, #, and $\Box$ are *implication*, *conjunction*, *disjunction*, and *absurdity*, and *top*, respectively, in FOL. They are inter-definable in the usual way; for example, $\Box = A \Rightarrow A$ for any $A \in \mathbb{ATOMS}(\mathbb{A})$ — see, for example, van Dalen [211]. Typically, $\invamp$ is reserved for multiplicative disjunction in *Linear Logic* [83] (LL), but since we do not study LL in this monograph, the symbol is liberated. It is suitable for denoting disjunction because it is aesthetically *dual* to $\&$, reflecting their algebraic relationship, and other standard symbols (e.g., $\oplus$) are used for other purposes in later chapters.

We may use the usual convention for suppressing brackets; that is, conjunction ($\&$) and disjunction ($\invamp$) bind more strongly than implication ($\Rightarrow$). Moreover, we may use the usual auxiliary terminology for first-order languages (e.g., *sub-formula*, *closed-formula*, *sentence*, etc.) without further explanation. Let be X a variable, T be a term, and Φ a wff; we write $\Phi[X \mapsto T]$ to denote the result of replacing every free occurrence of X by the term T so that no variable in T becomes bound in Φ after the substitution.

Definition 3.2 (First-order Sequent)**.** *A first-order sequent (FO-sequent) is a pair* $\Pi \triangleright \Sigma$ *in which* Π *and* Σ *are multi-sets of first-order formulae.*

We write $\Pi \blacktriangleright \Sigma$ to denote that $\Pi \triangleright \Sigma$ is a consequence of FOL. We think of sequents as unjudged structures; in particular, we have $\varnothing \triangleright \varnothing$ as a well-formed sequent in FOL, but it is not a consequence of FOL.

3.2 Proof Theory

One way to characterize FOL is by provability in a sequent calculus.

Definition 3.3 (Sequent Calculus $\mathsf{G3c}$)**.** *The sequent calculus* $\mathsf{G3c}$ *is composed of the rules in Figure 3.1 in which* T *is an arbitrary term, and* Y *is an eigenvariable.*

$$\frac{}{\Phi,\Pi \rhd \Sigma,\Phi}\ \mathsf{ax} \qquad \frac{}{\#,\Pi \rhd \Sigma}\ \#$$

$$\frac{\Phi,\Psi,\Pi \rhd \Sigma}{\Phi \& \Psi,\Pi \rhd \Sigma}\ \&_{\mathsf{L}} \qquad \frac{\Pi \rhd \Sigma,\Phi \quad \Pi \rhd \Sigma,\Psi}{\Pi \rhd \Sigma,\Phi \& \Psi}\ \&_{\mathsf{R}}$$

$$\frac{\Phi,\Pi \rhd \Sigma \quad \Psi,\Pi \rhd \Sigma}{\Phi \parr \Psi,\Pi \rhd \Sigma}\ \parr_{\mathsf{L}} \qquad \frac{\Pi \rhd \Sigma,\Phi,\Psi}{\Pi \rhd \Sigma,\Phi \parr \Psi}\ \parr_{\mathsf{R}}$$

$$\frac{\Pi \rhd \Sigma,\Phi \quad \Psi,\Pi \rhd \Sigma}{\Phi \Rightarrow \Psi,\Pi \rhd \Sigma}\ \Rightarrow_{\mathsf{L}} \qquad \frac{\Phi,\Pi \rhd \Sigma,\Psi}{\Pi \rhd \Sigma,\Phi \Rightarrow \Psi}\ \Rightarrow_{\mathsf{R}}$$

$$\frac{\forall X\Phi,\Phi[x \mapsto T],\Pi \rhd \Sigma}{\forall X\Phi,\Pi \rhd \Sigma}\ \forall_{\mathsf{L}} \qquad \frac{\Pi \rhd \Sigma,\Phi[X \mapsto Y]}{\Pi \rhd \Sigma,\forall X\Phi}\ \forall_{\mathsf{R}}$$

$$\frac{\Phi[x \mapsto Y],\Pi \rhd \Sigma}{\exists X\Phi,\Pi \rhd \Sigma}\ \exists_{\mathsf{L}} \qquad \frac{\Pi \rhd \Sigma,\exists X\Phi,\Phi[X \mapsto T]}{\Pi \rhd \Sigma,\exists X\Phi}\ \exists_{\mathsf{R}}$$

Figure 3.1: Sequent Calculus G3c

We write $\Pi \vdash_{\mathsf{G3c}} \Sigma$ to denote that there is a G3c-proof of $\Pi \rhd \Sigma$. Troelstra and Schwichtenberg [207] proved that G3c-provability characterizes classical consequence:

Theorem 3.4. $\Pi \blacktriangleright \Sigma$ *iff* $\Pi \vdash_{\mathsf{G3c}} \Sigma$

We use G3c to characterize FOL, as opposed to other proof systems, because of its desirable proof-theoretic properties — for example, Troelstra and Schwichtenberg [207] have shown that the rules of the calculus are (height-preserving) invertible and that the following rules are admissible:

$$\frac{\Pi \rhd \Sigma}{\Phi,\Pi \rhd \Sigma}\ \mathsf{w_L} \quad \frac{\Pi \rhd \Sigma}{\Pi \rhd \Sigma,\Phi}\ \mathsf{w_R} \quad \frac{\Phi,\Phi,\Pi \rhd \Sigma}{\Phi,\Pi \rhd \Sigma}\ \mathsf{c_L} \quad \frac{\Pi \rhd \Sigma,\Phi,\Phi}{\Pi \rhd \Sigma,\Phi}\ \mathsf{c_R}$$

3.3 Model-theoretic Semantics

Another way to characterize FOL is by validity in a semantics; in particular, by validity in a model-theoretic semantics. As mentioned above, we assume familiarity with the subject and therefore give a terse but complete account of definitions to keep the paper self-contained.

Definition 3.5 (First-order Structure)**.** *A first-order structure is a tuple $\mathcal{S} = \langle \mathbb{U}, \mathbb{R}, \mathbb{F}, \mathbb{K} \rangle$ in which $\mathbb{U}$ is a countable set, $\mathbb{K}$ is a subset of $\mathbb{U}$ (i.e., $\mathbb{K} \subseteq \mathbb{U}$), $\mathbb{F}$ is a countable set of operators on $\mathbb{U}$ (i.e., maps $f : \mathbb{U}^n \to \mathbb{U}$, for finite n), and $\mathbb{R}$ is a countable set of relations on $\mathbb{U}$ (i.e., relations $r \in \mathbb{U}^n$ for $n > 1$).*

We may write $\alpha(f)$ to denote the arity of a function or a relation. The same notation may be used for the symbols in a first-order alphabet.

Definition 3.6 (Interpretation)**.** *Let $\mathcal{S} := \langle \mathbb{U}, \mathbb{R}, \mathbb{F}, \mathbb{K} \rangle$ be a structure, and let $A := \langle \mathbb{R}', \mathbb{F}', \mathbb{K}', \mathbb{V} \rangle$ be an alphabet. An interpretation of A in $\mathcal{S}$ is a function $[\![-]\!]$ satisfying the following:*

- *if $x \in \mathbb{V}$, then $[\![x]\!] \in \mathbb{U}$;*
- *if $c \in \mathbb{K}'$, then $[\![c]\!] \in \mathbb{K}$;*
- *if $f \in \mathbb{F}'$, then $[\![f]\!] \in \mathbb{F}$, and $\alpha([\![f]\!]) = \alpha(f)$;*
- *if $R \in \mathbb{R}'$, then $[\![R]\!] \in \mathbb{R}$, and $\alpha([\![R]\!]) = \alpha(R)$.*

We may write $[\![-]\!] : A \to \mathcal{S}$ to denote that $[\![-]\!]$ is an interpretation of the alphabet A in the first-order structure $\mathcal{S}$. They extend to terms as follows:

$$[\![F(T_1,...,T_n)]\!] := [\![F]\!]([\![T_1]\!],...,[\![T_n]\!])$$

Let $[\![-]\!] : A \to \mathcal{S}$ be an abstraction, $X \in \mathbb{V}$ a variable, and $A \in \mathbb{U}$. The A/X-variant $[\![-]\!]'$ of $[\![-]\!]$ is defined as follows:

$$[\![-]\!]' := \begin{cases} A & \text{if } Y = X \\ [\![Y]\!] & \text{otherwise} \end{cases}$$

We may write $[\![-]\!][X \mapsto A]$ to denote the A/X-variant of $[\![-]\!]$.

We use the term *abstraction* for what is traditionally referred to as a *model* — see, for example, van Dalen [211]. This is to avoid confusion as we reserve ‘model’ for the semantics of the object-logics.

$\mathfrak{A} \Vvdash R(T_1,...,T_n)$	iff	$\langle [\![t_1]\!],...,[\![t_n]\!] \rangle \in [\![P]\!]$
$\mathfrak{A} \Vvdash \Phi \Rightarrow \Psi$	iff	not $\mathfrak{A} \Vvdash \Phi$ or $\mathfrak{A} \Vvdash \Psi$
$\mathfrak{A} \Vvdash \Phi \& \Psi$	iff	$\mathfrak{A} \Vvdash \Phi$ and $\mathfrak{A} \Vvdash \Psi$
$\mathfrak{A} \Vvdash \Phi \parr \Psi$	iff	$\mathfrak{A} \Vvdash \Phi$ or $\mathfrak{A} \Vvdash \Psi$
$\mathfrak{A} \Vvdash \forall X \Phi$	iff	$\mathfrak{A}[X \mapsto A] \Vvdash \Phi$ for any $A \in \mathbb{U}$
$\mathfrak{A} \Vvdash \exists X \Phi$	iff	$\mathfrak{A}[X \mapsto A] \Vvdash \Phi$ for some $A \in \mathbb{U}$
$\mathfrak{A} \Vvdash \#$		never
$\mathfrak{A} \Vvdash \Box$		always

Figure 3.2: Truth in an Abstraction

Definition 3.7 (Abstraction)**.** *An abstraction of an alphabet A is a pair $\mathfrak{A} := \langle \mathcal{S}, [\![-]\!] \rangle$ in which $\mathcal{S}$ is a structure and $[\![-]\!] : A \to \mathcal{S}$ be an interpretation.*

Let $\mathfrak{A} := \langle \mathcal{S}, [\![-]\!] \rangle$ be an abstraction. We may write $\mathfrak{A}[X \mapsto A]$ to denote $\langle \mathcal{S}, [\![-]\!][X \mapsto A] \rangle$.

Definition 3.8 (Truth in an Abstraction)**.** *Let A be an alphabet, let φ be a formula over A, and let $\mathfrak{A} = \langle \mathcal{S}, [\![-]\!] \rangle$ be an abstraction of A. The formula φ is true in $\mathfrak{A}$ iff $\mathfrak{A} \Vvdash \varphi$, which is defined inductively by the clauses in Figure 3.2.*

We can define truth of sequents as follows:

$\Pi \Vvdash \Sigma$	iff	for any abstraction $\mathfrak{A}$,
		if $\mathfrak{A} \Vvdash \varphi$ for every $\varphi \in \Pi$,
		then $\mathfrak{A} \Vvdash \psi$ for some $\psi \in \Sigma$

Theorem 3.9 (Gödel [84])**.** $\Pi \blacktriangleright \Sigma$ *iff* $\Pi \Vvdash \Sigma$.

This concludes the summary of FOL.

Chapter 4

The Theory of Tactical Proof

A general framework supporting the mechanization of Reductive Logic is the theory of *tactical proof* introduced by Milner [143]. While little developed mathematically, the theory is sufficiently general to encompass as diverse reasoning activities as proving a formula in a formal system and seeking to meet a friend before noon on Saturday. It does not concern finding the *best* way to reason about a goal (e.g., minimizing the prospect of failure), though these things are important, instead it makes precise how concepts used during reasoning — such as 'goal,' 'strategy,' 'achievement,' 'failure,' etc. — relate to one another. The point is that all such reasoning activities, different in domain and formality, can be articulated in terms of a uniform language that a user may express insight into reasoning methods and delegate routine, but error-prone, work to a machine. In the words of Milner [143]:

> Here it is a matter of taste whether the human prover wishes to see this performance done by the machine, in all its frequently repulsive detail, or wishes only to see the highlights, or is merely content to let the machine announce the result (a theorem!).

Following Milner [143], we introduce the theory at the full level of generality in Section 4.1. Following this, we illustrate it in the context of proof-search for IPL in Section 4.2, and then for logics in general in Section 4.3. It is tactical proof that delivers systematically the various proof-assistants mentioned in Chapter 1

4.1 Tactics and Tacticals

One has two classes of prime entities: *goals* and *events*. The two classes are carried by the sets $\mathbb{GOALS}$ and $\mathbb{EVENTS}$, respectively. The goals and events are related by a notion of *achievement* $\propto \subseteq \mathbb{GOALS} \times \mathbb{EVENTS}$ that determines what events witness what goals. The idea is that an event $\mathscr{E}$ achieves goal G as it satisfies the description that the goal has designated. Heuristically, an event achieves a goal when it satisfies the description that the goal has designated. For example, the goal G that 'Alice and Bob meet before noon on Saturday' is achieved by the event $\mathscr{E}$ is that 'Alice and Bob meet under the clock at Waterloo station at 11:53 on Saturday.'

We take reasoning about a goal as the process of replacing it with new goals that suffice to produce the original. In the nomenclature of Reductive Logic, such replacements are captured by *reduction operators*, which may be taken as a partial function from goals to lists of goals:

$$\rho : \mathbb{GOALS} \rightharpoonup \mathcal{L}\mathbb{GOALS}$$

The goals produced by applying a reduction operator to a given goal are said to be *subgoals*.

What renders a reduction from a goal to a list of subgoals valid is that any events possibly witnessing the subgoals yield an event possibly witnessing the original goal. This justification is witnessed by a *procedure*,

$$\pi : \mathcal{L}\mathbb{EVENTS} \rightharpoonup \mathbb{EVENTS}$$

Returning to the example above concerning Alice and Bob, the goal G may be reduced to the following sub-goals:

G_1 : Alice arrives under the clock at Waterloo Station before noon on Saturday
G_2 : Bob arrives under the clock at Waterloo Station before noon on Saturday.

This reduction is justified by the fact that G_1 and G_2 are achieved by the following events, respectively, which yield $\mathscr{E}$ through the procedure of *waiting*:

$$\begin{aligned} \mathscr{E}_1 &: \text{Alice arrives at Waterloo Station at 11:57 on Saturday} \\ \mathscr{E}_2 &: \text{Bob arrives at Waterloo Station at 11:53 on Saturday.} \end{aligned}$$

Thus, one step of reasoning amounts to applying a (partial) mapping takings goals to subgoals together with a procedure,

$$\tau : G \mapsto \langle [G_1, \ldots, G_n], \pi \rangle$$

These mappings are called *tactics*. According to the above discussion, we have the following notion of validity:

Definition 4.1 (Valid Tactic)**.** *Let $\propto$ be a notion of achievement. A tactic τ is $\propto$-valid iff, for any $G, G_1, \ldots, G_n \in \mathbb{GOALS}$ and $\mathscr{E}, \mathscr{E}_1, \ldots, \mathscr{E}_n \in \mathbb{EVENTS}$, if $\tau : G \mapsto \langle [G_1, \ldots, G_n], \pi \rangle$ and $\mathscr{E} := \pi(\mathscr{E}_1, \ldots, \mathscr{E}_n)$, and $\mathscr{E}_i \propto G_i$ obtains for $1 \leq i \leq n$, then $\mathscr{E} \propto G$ obtains.*

Of course, a goal typically requires several iterations of reasoning of the above form such that subgoals are resolved into further subgoals, and so on. For example, suppose Alice starts from Andover and Bob starts from Birmingham; then, to reason about G, one requires many component tactics that collectively bridge the distance both physical and temporal — for instance, one may have the subgoal G_1' for G_1 that 'Bob takes the tube to Waterloo Station from Euston Station', which is witnessed by the event 'Bob takes the 11:46 southbound Northern Line service from Euston to Waterloo on Saturday.' Hence, we require a notion of composition of tactics.

A composition of tactics is called a *tactical*. A tactical is valid when it preserves the validity of the tactics it combines:

Definition 4.2 (Valid Tactical)**.** *A tactical is valid iff it preserves the validity of tactics; that is, if $\circ$ is a tactical and $\tau_1, \ldots, \tau_n$ are $\propto$-valid, then $\circ(\tau_1, \ldots, \tau_n)$ are $\propto$-valid.*

The foregoing is a complete account of tactical reasoning as introduced by Milner [143]. To be precise in the semantics presented in this chapter, we supplement the above with some additional definitions.

Definition 4.3 (Tactical System)**.** *A tactical system* T *is a collection of tactics and tacticals that are valid relative to some notion of achievement.*

We have opted to present the theory in its full generality. In the sequel, we apply it to the context of the use of logic as a reasoning technology. We follow the account in Milner [143], which is the basis of many automated reasoning technologies using logic, such as the proof assistants mentioned in Section 19.

Let $\vdash$ be the consequence relation for IPL — see Chapter 2. We have the following setup:

- a goal is a sequent $\Gamma \triangleright \varphi$ in which Γ is a list of formulas and φ is a formula
- an event is a sequent $\Delta \triangleright \psi$ such that $\Delta \vdash \psi$ obtains
- the achievement relation $\propto$ is as follows:

$$(\Delta \triangleright \psi) \propto (\Gamma \triangleright \varphi) \qquad \text{iff} \qquad \varphi = \psi \text{ and } \Delta \sqsubseteq \Gamma \text{ and } \Delta \vdash \psi$$

(We write $\Delta \sqsubseteq \Gamma$ to denote that the set of elements in Δ is a subset of the set of elements of Γ)

In this context, a tactic is valid iff it corresponds to an admissible rule for IPL. For example, in NJ the $\wedge_{\mathsf{I}}$-rule determines the tactic $\tau_{\wedge_{\mathsf{I}}}$ which has the following components:

$$\underbrace{\frac{\Gamma \triangleright \varphi \quad \Gamma \triangleright \psi}{\Gamma \triangleright \varphi \wedge \psi}\Uparrow}_{\text{reduction operator}} \qquad \underbrace{\frac{\Delta_1 \vdash \varphi \quad \Delta_2 \vdash \psi}{\Delta_1, \Delta_2 \vdash \varphi \wedge \psi}\Downarrow}_{\text{procedure}}$$

This concludes the overview of the theory of tactical proof as used in this monograph (Part III).

4.2 Tactical Proof and Intuitionistic Propositional Logic

Having presented the theory of tactics as a metatheoretical framework in which one studies reasoning — the construction of arguments — in its full generality, it is informative to consider how it applies in the concrete setting of natural deduction for IPL.

In Section 4.1, we witnessed the following tactic corresponding to the $\wedge_{\mathsf{I}}$ rule:

$$\tau_{\wedge} : (\Gamma \triangleright \varphi \wedge \psi) \mapsto \langle [(\Gamma \triangleright \varphi), (\Gamma \triangleright \psi)], \wedge_{\mathsf{R}} \rangle$$

The analogous treatment of $\to_{\mathsf{I}}$ yields the following:

$$\tau_{\to} : (\Gamma \triangleright \varphi \to \psi) \mapsto \langle [(\varphi, \Gamma \triangleright \psi)], \to_{\mathsf{R}} \rangle$$

These individual reasoning steps are combined with a *tactical* $\fatsemi$ that corresponds to the sequential application of rules in natural deduction:

$$\tau_{\wedge} \fatsemi \tau_{\to} : \big(\Gamma \triangleright \chi \wedge (\varphi \to \psi)\big) \mapsto \langle [(\Gamma \triangleright \chi), (\varphi, \Gamma \triangleright \psi)], \wedge_{\mathsf{R}} \otimes \to_{\mathsf{R}} \rangle$$

The procedure $\wedge_{\mathsf{R}} \otimes \to_{\mathsf{R}}$ is the product of the procedures for $\wedge_{\mathsf{R}}$ and $\to_{\mathsf{R}}$.

We have thus related natural deduction and consequence using tactics. And yet, something is missing in this setup. What argument does $\tau_{\wedge} \fatsemi \tau_{\to}$ witness? This question demands an interpretation of tactics as arguments, understood as abstract entities such as natural deduction arguments. We return to this problem in Part III (Chapter 20)

4.3 Tactical Proof and Logic

The theory of tactical proof is an engine relating the search for arguments. It is, essentially, a system of reduction operators that are step-wise justified by rules in a sequent calculus. That is how we regard them in this chapter. The application of tactics drives the computation of arguments, which is to say the *search* for arguments, and a sequent calculus (in the sense of Chapter 14) defines the procedures of the tactics. As such, supplying a sequent calculus amounts to supplying a notion of *inference* against which the computation of an argument is justified. This is captured in the semantic framework of this chapter in Section 20.4.1

The dichotomy between proof and search actually predates the tactical proof. The components were historically called *analysis* and *synthesis*, respectively — see Pólya [162] for a general discussion of this study for mathematical practice. In *analysis*, one repeatedly asks from what conditions could the desired result, which is to say goal, be obtained; during *synthesis*, one derives from the analysis a solution to the problem.

The shift from analysis to synthesis (i.e., the shift from computing subgoals to using procedures, from reduction to deduction) is captured by a *synthesizer*.

Definition 4.4 (Synthesizer)**.** *Let* L *be a sequent calculus and* T *be tactical system with achievement* $\propto$ *whose events are* L*-sequents. The achievement* $\propto$ *is an* L*-synthesizer for* T *iff the procedures of* T *are the rules of* L*.*

An example of a synthesizer is offered at the end of Section 4.1. Here the reduction operators correspond to NJ rules, and the procedures correspond to LJ rules.

In the same way that tactics are implicit in much of the literature on logic, synthesizers also appear implicitly anywhere one considers the inferential content within arguments in a certain space. The running case of natural deduction for IPL discussed above is a key example; we give some others in Part III (Chapter 20).

Part I

Reduction, Control, and Semantics in the Logic of Bunched Implications

Chapter 5

Introduction to Part I

This part of the monograph investigates reduction and proof-search in the *logic of Bunched Implications* (BI) [150]. While the background on IPL in Chapter 2 suggests that Reductive Logic is a perspective in which one may see the interplay between semantics and proof theory, it is a simple setting. In contrast, BI has a relatively subtle meta-theory, with respect to both semantics and proof theory, meaning that studying similar phenomena (e.g., reduction, control, and the relationship to semantics) exposes the more subtle aspects of the interplay between semantics and proof theory. Significantly, this part culminates with a novel approach to soundness and completeness for BI, entirely based on proof-search, with respect to model-theoretic semantics. This supports the more general investigation into semantics and proof theory in Reductive Logic in Part II.

One way to understand BI is by contrast with *Linear Logic* (LL) [83], which is well-known for its computational interpretation developed by Abramsky [3], and others. In LL, the structural rules of weakening and contraction are regulated by modalities, ! (related to □ in modal logic) and ? (related to ◊ in modal logic). To see how this works, consider the left and right (single-conclusioned) sequent calculus rules for the ! modality,

$$\frac{\Gamma, \varphi \rhd \psi}{\Gamma, !\,\varphi \rhd \psi} \quad \text{and} \quad \frac{!\,\Gamma \rhd \varphi}{!\,\Gamma \rhd !\,\varphi}$$

and note that weakening and contraction arise as

$$\frac{\Gamma \rhd \psi}{\Gamma, !\varphi \rhd \psi}\, W \quad \text{and} \quad \frac{\Gamma, !\varphi, !\varphi \rhd \psi}{\Gamma, !\varphi \rhd \psi}\, C$$

respectively.

There are two important consequences of this setup. First, proof-theoretically, the relationship between intuitionistic (additive) implication, $\rightarrow$, and linear (multiplicative) implication, $\multimap$, is given by Girard's translation [83],

$$\varphi \rightarrow \psi \equiv (!\varphi) \multimap \psi$$

Second, more semantically, LL has a rudimentary interpretation as a logic of resource via the so-called *number-of-uses* reading, in which, in a sequent $\Gamma \triangleright \varphi$, the number of occurrences of a formula ψ in Γ determines the number of times ψ may be 'used' in φ. The significance of the modality ! can now be seen: if $!\psi$ is in Γ, then ψ may be used any number of times in φ (including zero). This reading is wholly consistent with the forms of weakening and contraction above.

The relationship between logic and structure offered in the above reading has been called by Abramsky [4] the *intrinsic* view of logic. By comparison, BI is, perhaps, the prime example of the contrasting *descriptive* view of resource — that is, in the resource interpretation of BI, a proposition is not a resource itself, but a declaration about the state of some resources.

Informally, in BI, a judgement $\mathfrak{M}, m \vDash \varphi \wedge \psi$ is a declaration that the resource m (in the model $\mathfrak{M}$) satisfies both φ and ψ (i.e., $\mathfrak{M}, m \vDash \varphi$ and $\mathfrak{M}, m \vDash \psi$); meanwhile, the judgement $\mathfrak{M}, m \vDash \varphi * \psi$ says that m can be *split* into two parts n and n' that satisfy φ and ψ, respectively (i.e., $\mathfrak{M}, n \vDash \varphi$ and $\mathfrak{M}, n' \vDash \psi$). We may illustrate this idea by an example. Suppose chocolate bar A costs 2 gold coins, and chocolate bar B costs 3 gold coins. We may write $3 \vDash A \wedge B$ to say that three gold coins suffice for both chocolates in the sense that one could freely choose to have either A or B; meanwhile, we may write $7 \vDash A * B$ to say that seven gold coins may be split into two piles that suffice to purchase A and B — in particular, $3 \vDash A$ and $4 \vDash B$. Notice the persistence of the judgement; that is, since $2 < 3$ and $2 \vDash A$, we also have $3 \vDash A$. It is this kind of resource reasoning for which BI is, intuitively, suitable. Yet, this does not define BI.

The original motivation for BI is, in contrast to LL, to have two primitive (intuitionistic) implications, one additive and one multiplicative. As a consequence, contexts in BI are not the typical *flat* data structures typically used in logic (e.g., lists or multisets), but instead are layered structures called *bunches*. As a result, BI has a well-motivated proof-theoretic formulation, but a more complex model-theoretic formulation.

Of course, BI is one of many *bunched* logics — see, for example, Docherty [44]. It is closely related to (and may even be understood as part of) the family of relevance logics, where from the term *bunch* comes, which has received extensive study — see, for example, Belnap [15]. and Read [176]. However, the literature on BI departs slightly from relevance logics because its motivations from substructural logic; for example, one often uses the terms *additive* and *multiplicative* in BI for what ought properly to be called *intensional* and *extensional* (cf. Belnap [16]). These difference are largely superficial. Indeed, bunches are generally useful algebraic structures in proof theory; for example, Schroeder-Heister [186, 187] has used bunches as a structural entity in connection with structural extensional and intensional implications to define introduction and elimination rules for logical connectives corresponding to the extensional 'higher-level' rules — see Part III.

This part begins in Chapter 6 with a definition of BI and a brief survey of some of its meta-theory to motivate later discussion. Chapter 7 provides a cut-elimination argument for a sequent calculus for BI — while cut-admissibility is known for the sequent calculus, it is through indirect means such as through the logic's display calculus (see Brotherston [27]) and semantically (see Frumin et al. [63]). Chapter 8 studies logic programming in BI, closely following the treatment of IPL in Chapter 2, but also considers a denotational perspective via coalgebra. Chapter 9 provides a *focused calculus* for BI — briefly, a calculus in which proof-search is strictly controlled — and illustrates its soundness and completeness through a cut-elimination argument. Chapter 10 uses proof-search to demonstrates the soundness and completeness of BI with respect to a model-theoretic semantics. Significantly, this approach entirely avoids term- and counter-model constructions, which are subtle for BI because of the more complex structure of contexts (i.e., bunches) and the interaction between the additive and multiplicative connectives — and is explored further in Part II. The part concludes in Chapter 11 with a summary of contributions.

Chapter 6

The Logic of Bunched Implications

This chapter contains a background on BI that supports the subsequent chapters. It provides the syntax of the logic in Section 6.1, and a proof-theoretic characterization of it in Section 6.2. There is also a survey of some results on its semantics in Section 6.3 which indicate the subtlety of the logic's meta-theory.

6.1 Syntax & Consequence

Having already motivated BI in Chapter 5, we proceed with a formal, syntactic, definition. Essentially, BI is free combination (i.e., the fibration — see Gabbay [68]) of (additive) intuitionistic logic, with connectives $\land, \lor, \to, \top, \bot$, and multiplicative intuitionistic logic, with connectives $*, \mathrel{-\!\!*}, \top^*$.

Definition 6.1 (Formulas)**.** *Let $\mathbb{A}$ be a denumerable set of propositional letters. The set of formulas $\mathbb{F}$ (over $\mathbb{A}$) is defined by the following grammar:*

$$\varphi ::= \mathrm{p} \in \mathbb{A} \mid \top \mid \bot \mid \top^* \mid \varphi \land \varphi \mid \varphi \lor \varphi \mid \varphi \to \varphi \mid \varphi * \varphi \mid \varphi \mathrel{-\!\!*} \varphi$$

A distinguishing feature of BI is that contexts are not one of the familiar structures of lists, multisets, or sets, since the two context-formers $\fatsemi$ and $,$ representing the two conjunctions $\land$ and $*$, respectively, do not commute with each other, though individually they behave as usual; contexts are instead *bunches*.

Definition 6.2 (Bunch)**.** *The set of bunches* $\mathbb{B}$ *is defined by the following:*

$$\Gamma ::= \varphi \in \mathbb{F} \mid \varnothing_{+} \mid \varnothing_{\times} \mid \Gamma \fatsemi \Gamma \mid \Gamma , \Gamma$$

The $\fatsemi$ *is the additive context-former, and the* $\varnothing_{+}$ *is the additive unit; the* $,$ *is the multiplicative context-former, and the* $\varnothing_{\times}$ *is the multiplicative unit.*

The use of $\fatsemi$ and $,$ as the 'additive' and 'multiplicative' context-formers are standard practice for BI (see, for example, O'Hearn and Pym [150] and Docherty [44]). This stands in contrast to the practice in relevance logics where the roles are usually reversed — see, for example, Read [176].

A bunch Δ is a *sub-bunch* of a bunch Γ iff Δ is a sub-tree of Γ. We may write $\Gamma(\Delta)$ to express that Δ is a sub-bunch of Γ. The operation $\Gamma(\Delta)[\Delta \mapsto \Delta']$ — abbreviated to $\Gamma(\Delta')$, where no confusion arises — is the result of replacing the occurrence of Δ by Δ'.

Definition 6.3 (Sequent)**.** *A sequent is a pair* $\Gamma \triangleright \varphi$ *in which* Γ *is a bunch and* φ *is a formula.*

Since contexts — the left-hand side of sequents — are more complex than in many of the more familiar logics (e.g., FOL, IPL, etc.), the following is an explicit characterization of equivalence of bunches.

Definition 6.4 (Coherent Equivalence)**.** *Two bunches* $\Gamma, \Gamma' \in \mathbb{B}$ *are coherently equivalent when* $\Gamma \equiv \Gamma'$*, where* $\equiv$ *is the least relation satisfying:*

- *commutative monoid equations for* $\fatsemi$ *with unit* $\varnothing_{+}$
- *commutative monoid equations for* $,$ *with unit* $\varnothing_{\times}$
- *coherence — that is, if* $\Delta \equiv \Delta'$ *then* $\Gamma(\Delta) \equiv \Gamma(\Delta')$.

By commutative monoid equations for $\circ$ with unit 1, we mean the following:

$$\underbrace{x \circ y \equiv y \circ x}_{\text{commutativity}} \qquad \underbrace{x \circ (y \circ z) \equiv (x \circ y) \circ z}_{\text{associativity}} \qquad \underbrace{x \circ 1 \equiv x}_{\text{unitality}}$$

We have not included any contraction or weakening in this definition. When defining BI's consequence judgement, $\fatsemi$ will indeed have both structural properties and $,$ will have neither; this is what determines the former as 'additive' and the latter as 'multiplicative'.

Bunches are typically understood as the syntax trees provided by Definition 6.2 modulo coherent equivalence, in the same way that sets or multisets for the contexts of FOL or IPL may be understood as lists modulo permutation. The idea that the context-formers represent the conjunctions provides the following transformation:

Definition 6.5 (Compacting)**.** *The compacting function* $\lfloor \cdot \rfloor : \mathbb{B} \to \mathbb{F}$ *is defined as follows:*

$$\lfloor \Gamma \rfloor := \begin{cases} \varphi & \textit{if } \Gamma = \varphi \in \mathbb{F} \\ \top & \textit{if } \Gamma = \varnothing_+ \\ \top^* & \textit{if } \Gamma = \varnothing_\times \\ \lfloor \Delta_1 \rfloor * \lfloor \Delta_2 \rfloor & \textit{if } \Gamma = \Delta_1 , \Delta_2 \\ \lfloor \Delta_1 \rfloor \land \lfloor \Delta_2 \rfloor & \textit{if } \Gamma = \Delta_1 \fatsemi \Delta_2 \end{cases}$$

That a sequent $\Gamma \triangleright \varphi$ is a consequence of BI is denoted $\Gamma \vdash \varphi$.

6.2 Proof Theory

The motivations for the proof-theoretic presentation of BI are treated extensively in O'Hearn and Pym [150, 171]; therefore, for the sake of brevity, we elide it and proceed directly with the relevant definitions. We shall concentrate on the sequent calculus presentation since we are primarily concerned with proof-search, but BI admits all the usual proof-theoretic treatments such as axiomatic systems and analytic tableaux — see, for example, Galmiche et al. [73, 71, 74, 72, 70].

Definition 6.6 (System LBI)**.** *Sequent calculus* LBI *comprises the rules of Figure 6.1, in which* $\varphi, \psi, \chi \in \mathbb{F}$*,* $\Gamma, \Delta, \Delta' \in \mathbb{B}$*, and* e *has the side-condition* $\Delta \equiv \Delta'$*.*

Proof in a sequent calculus is defined in the usual way — see, for example, Troelstra and Schwichtenberg [207].

$$\frac{}{\varphi \triangleright \varphi}\,\mathsf{ax} \qquad \frac{}{\Gamma(\bot) \triangleright \varphi}\,\bot_{\mathsf{L}} \qquad \frac{}{\varnothing_{\times} \triangleright \top^{*}}\,\top^{*}_{\mathsf{R}} \qquad \frac{}{\varnothing_{+} \triangleright \top}\,\top_{\mathsf{R}}$$

$$\frac{\Delta' \triangleright \varphi \quad \Gamma(\Delta , \psi) \triangleright \chi}{\Gamma(\Delta , \Delta' , \varphi \mathbin{-\!\!*} \psi) \triangleright \chi}\,\mathbin{-\!\!*}_{\mathsf{L}} \qquad \frac{\Delta , \varphi \triangleright \psi}{\Delta \triangleright \varphi \mathbin{-\!\!*} \psi}\,\mathbin{-\!\!*}_{\mathsf{R}}$$

$$\frac{\Delta(\varphi , \psi) \triangleright \chi}{\Delta(\varphi * \psi) \triangleright \chi}\,*_{\mathsf{L}} \qquad \frac{\Delta \triangleright \varphi \quad \Delta' \triangleright \psi}{\Delta , \Delta' \triangleright \varphi * \psi}\,*_{\mathsf{R}} \qquad \frac{\Delta(\varnothing_{\times}) \triangleright \chi}{\Delta(\top^{*}) \triangleright \chi}\,\top^{*}_{\mathsf{L}}$$

$$\frac{\Delta(\varphi \fatsemi \psi) \triangleright \chi}{\Delta(\varphi \wedge \psi) \triangleright \chi}\,\wedge_{\mathsf{L}} \qquad \frac{\Delta \triangleright \varphi \quad \Delta \triangleright \psi}{\Delta \triangleright \varphi \wedge \psi}\,\wedge_{\mathsf{R}} \qquad \frac{\Delta(\varnothing_{+}) \triangleright \chi}{\Delta(\top) \triangleright \chi}\,\top_{\mathsf{L}}$$

$$\frac{\Delta(\varphi) \triangleright \chi \quad \Delta(\psi) \triangleright \chi}{\Delta(\varphi \vee \psi) \triangleright \chi}\,\vee_{\mathsf{L}} \qquad \frac{\Delta \triangleright \varphi}{\Delta \triangleright \varphi \vee \psi}\,\vee_{\mathsf{R1}} \qquad \frac{\Delta \triangleright \psi}{\Delta \triangleright \varphi \vee \psi}\,\vee_{\mathsf{R2}}$$

$$\frac{\Delta \triangleright \varphi \quad \Gamma(\Delta \fatsemi \psi) \triangleright \chi}{\Gamma(\Delta \fatsemi \varphi \to \psi) \triangleright \chi}\,\to_{\mathsf{L}} \qquad \frac{\Delta \fatsemi \varphi \triangleright \psi}{\Delta \triangleright \varphi \to \psi}\,\to_{\mathsf{R}}$$

$$\frac{\Delta(\Delta') \triangleright \chi}{\Delta(\Delta' \fatsemi \Delta'') \triangleright \chi}\,\mathsf{w} \qquad \frac{\Delta \triangleright \chi}{\Delta' \triangleright \chi}\,\mathsf{e}_{(\Delta \equiv \Delta')} \qquad \frac{\Delta(\Delta' \fatsemi \Delta') \triangleright \chi}{\Delta(\Delta') \triangleright \chi}\,\mathsf{c}$$

Figure 6.1: Sequent Calculus LBI

We write $\Gamma \vdash_{\mathsf{LBI}} \varphi$ to denote that there is a LBI-proof of $\Gamma \triangleright \varphi$. We may express that $\mathcal{D}$ is a LBI-proof of a sequent $\Gamma \triangleright \varphi$, by writing $\mathcal{D} : \Gamma \vdash_{\mathsf{LBI}} \varphi$ or $\mathcal{D} : \Gamma \triangleright \varphi$. To say that LBI characterizes BI is to say the following:

$$\Gamma \vdash \varphi \qquad \text{iff} \qquad \Gamma \vdash_{\mathsf{LBI}} \varphi$$

The proof of this by Pym [171] makes essential use of the following rule:

$$\frac{\Delta \triangleright \chi \quad \Gamma(\chi) \triangleright \varphi}{\Gamma(\Delta) \triangleright \varphi}\,\mathsf{cut}$$

Brotherston [27] established the admissibility of cut in BI by an argument that passes through the logic's display calculus; and Frumin [63] has given a semantic proof of cut-admissibility that has been verified in Coq. Therefore, we may leave it out. However, as neither of these proofs go through the traditional permutation style argument for LBI, we give one in Chapter 7.

The remainder of this section concerns some simple results about BI that follow immediately from analysis LBI and will be useful below.

Proposition 6.7. *The following rules are derivable in* LBI*, and replacing* w *with them does not affect the completeness of the system.*

$$\frac{}{\Delta \fatsemi \varphi \triangleright \varphi}\ \mathsf{Taut} \quad \frac{}{\Delta \fatsemi \varnothing_{\times} \triangleright \top^{*}}\ {\top^{*}}'_{\mathsf{R}} \quad \frac{}{\Delta \fatsemi \varnothing_{+} \triangleright \top}\ \top'_{\mathsf{R}}$$

$$\frac{\Delta \triangleright \varphi \quad \Delta' \triangleright \psi}{(\Delta \fatcomma \Delta') \fatsemi \Delta'' \triangleright \varphi * \psi}\ *'_{\mathsf{R}} \quad \frac{\Delta' \triangleright \varphi \quad \Delta(\Delta'' \fatcomma \psi) \triangleright \chi}{\Delta(\Delta' \fatcomma \Delta'' \fatcomma (\Delta''' \fatsemi \varphi \mathrel{-\!\!*} \psi)) \triangleright \chi}\ {\mathrel{-\!\!*}}'_{\mathsf{L}}$$

Proof Sketch. We can construct in LBI-proofs with the same premisses and conclusion as these rules by use of the structural rules. Let LBI' be LBI without w but with these new rules (retaining also $*_{\mathsf{R}}$, $\mathrel{-\!\!*}_{\mathsf{L}}$, $\top^{*}_{\mathsf{R}}$, $\top_{\mathsf{R}}$, and Taut), then w is admissible in LBI' using a standard permutation argument. □

One may regard the above modification to LBI as forming a new calculus, but since all the new rules are derivable it is really a restriction of the calculus, in the sense that all proofs in the new system have equivalent proofs in LBI differing only by explicitly including instances of weakening (w).

Despite LBI being analytic (i.e., only sub-formulae of a sequent will appear in an LBI-proof of it), proof-search in LBI remains a difficult problem. Before any of the logical problems, the syntax of BI already provides a challenge as it uses a more complex data structure than many other logics (i.e., bunches rather than lists) thereby rendering the notions of control more complex. Indeed, the complex structure of bunches means that the space of reductions generated by LBI for any given putative conclusion is infinite. Nonetheless, one may still study proof-search in BI, as shown in Chapter 9 and Chapter 8.

6.3 Monoidal Semantics

To complete this introduction to BI, we provide a brief survey of its model-theoretic semantics. This is background and none of the results are directly used in later parts of the monograph, but it motivates the work in Chapter 10.

Throughout we use the syntax of FOL (Chapter 3) as notation for the ambient logic. That is, we employ $\Rightarrow$ for 'implies', $\&$ for 'and', and $\invamp$ for 'or', without further reference.

Preordered Commutative Monoids

In contrast to classical logic, in the model theory for non-classical logics, one thinks of statements (i.e., formulas of the logic) not as being universally true, but instead true with respect to a certain state of affairs, such as at a certain time or with respect to some information. For intuitionistic propositional logic (IPL), a formula is true when one can provide a method for witnessing (or constructing) it; the states in the model-theoretic semantics for IPL are sometimes understood as witnesses of these methods, and the clauses of the satisfaction relation specify how the witnesses relate to each other — see, for example, Dummett [52]. The model-theoretic semantics of BI is, intuitively, an extension of the model-theoretic semantics for IPL (see Chapter 2) that allows these witnesses to be *decomposed*.

Decomposition is witnessed by a monoidal product — that is, w is a composition of u and v iff $w = u \circ v$. A witness w satisfies an additive conjunction $\varphi \wedge \psi$ when it satisfies both φ and ψ, and a witness w satisfies a multiplicative conjunction $\varphi * \psi$ when there are two states u and v such that $w = u \circ v$, u satisfies φ, and v satisfies ψ.

Definition 6.8 (Preordered Commutative Monoid)**.** *A preordered commutative monoid (PCM) is a tuple $\mathcal{M} = \langle \mathbb{M}, \preceq, \circ, e \rangle$, in which $\preceq$ is a preorder on $\mathbb{M}$, and $\circ$ is a commutative monoidal product on $\mathbb{M}$ with unit $e \in \mathbb{M}$.*

Definition 6.9 (Interpretation)**.** *Let $\mathcal{M} := \langle \mathbb{M}, \preceq, \circ, e \rangle$ be a PCM. An interpretation $[\![-]\!] : \mathbb{A} \to \mathcal{M}$ is a mapping $[\![-]\!] : \mathbb{A} \to \mathscr{P}(\mathbb{M})$.*

Let $\mathcal{M} = \langle \mathbb{M}, \preceq, \circ, e \rangle$ be a PCM. We may write $w \in \mathcal{M}$ to denote that $w \in \mathbb{M}$. Using PCMs as the semantics for BI is coherent with the idea of it as a *fibration* (see Gabbay [67]), which determines the *bifunctoriality* condition:

$$m \preceq m' \& n \preceq n' \Rightarrow m \circ n \preceq m' \circ n'$$

Definition 6.10 (PCM Pre-model)**.** *A PCM pre-model is a pair $\mathfrak{M} := \langle \mathcal{M}, [\![-]\!] \rangle$ in which $\mathcal{M}$ is a PCM that is bifunctorial and $[\![-]\!] : \mathbb{A} \to \mathcal{M}$ is an interpretation.*

$$\begin{array}{lll}
w \Vdash \mathrm{p} & \text{iff} & w \in [\![\mathrm{p}]\!] \\
w \Vdash \top & & \text{always} \\
w \Vdash \bot & & \text{never} \\
w \Vdash \top^* & \text{iff} & e \preceq w \\
w \Vdash \varphi \land \psi & \text{iff} & w \Vdash \varphi \text{ and } w \Vdash \psi \\
w \Vdash \varphi \lor \psi & \text{iff} & w \Vdash \varphi \text{ or } w \Vdash \psi \\
w \Vdash \varphi \to \psi & \text{iff} & \text{for any } v, \text{ if } w \preceq v \text{ and } v \Vdash \varphi, \text{ then } v \Vdash \psi \\
w \Vdash \varphi * \psi & \text{iff} & \text{there are } u, v \text{ st.} u \circ v \preceq w \text{ and } u \Vdash \varphi \text{ and } v \Vdash \psi \\
w \Vdash \varphi \mathrel{-\!\!*} \psi & \text{iff} & \text{for any } u, v, \text{ if } u \Vdash \varphi, \text{ then } u \circ v \Vdash \psi
\end{array}$$

Figure 6.2: Satisfaction for BI in Preordered Commutative Monoids

Let $\mathcal{M} = \langle \mathbb{M}, \preceq, \circ, e \rangle$ be a PCM. We may write $w \in \mathcal{M}$ to denote that $w \in \mathbb{M}$.

Definition 6.11 (Satisfaction)**.** *Satisfaction is the least relation satisfying the clauses of Figure 6.2 in which $\mathfrak{M} := \langle \mathcal{M}, [\![-]\!] \rangle$ is a PCM pre-model.*

Definition 6.12 (PCM Model)**.** *A PCM pre-model $\mathfrak{M} := \langle \mathcal{M}, [\![-]\!] \rangle$ is a PCM model iff it is atomically persistent — that is, for any $p \in \mathbb{A}$ and any $u, w \in \mathbb{M}$,*

$$\textit{if } \mathfrak{M}, w \Vdash p \textit{ and } w \preceq u, \textit{ then } \mathfrak{M}, u \Vdash p$$

The class of PCM-models is $\mathbb{K}$.

Let $\mathfrak{M} = \langle \mathcal{M}, [\![-]\!] \rangle$ be a PCM. We may write $w \in \mathcal{M}$ to denote that $w \in \mathbb{M}$.

PCM models provide a notion of consequence as follows:

$$\Gamma \vDash \varphi \qquad \text{iff} \qquad \forall \mathfrak{M} \in \mathbb{K} \forall w \in \mathfrak{M} (\mathfrak{M}, w \Vdash \Gamma \Rightarrow \mathfrak{M}, w \Vdash \varphi)$$

The soundness of BI with respect to this semantics (i.e., $\Gamma \vdash \varphi \Rightarrow \Gamma \vDash \varphi$) has been known for a while (see, for example, Pym [171]) and is easy to prove using familiar methods, but completeness (i.e., $\Gamma \vDash \varphi \Rightarrow \Gamma \vdash \varphi$) is more subtle. The following shows that BI is not complete:

Proposition 6.13 (Pym et al. [171, 172]). *Let* $\Gamma := (\varphi \mathrel{-\!\!*} \bot) \to \bot \,; (\psi \mathrel{-\!\!*} \bot) \to \bot$ *and* $\chi := ((\varphi * \psi) \mathrel{-\!\!*} \bot) \to \bot$*, for some* φ *and* ψ*,*

$$\Gamma \vDash \chi \qquad \textit{but} \qquad \Gamma \nvdash \chi$$

The form of Proposition 6.13 is pathological in that it expresses the incompatibility of the consistency condition with the totality of the monoids: if there are u and v such that $u \Vdash \mathrm{p}$ and $v \Vdash \mathrm{p} \mathrel{-\!\!*} \bot$, then $u \circ v \Vdash \mathrm{p} * \mathrm{p} \mathrel{-\!\!*} \bot$, but then $u \circ v \Vdash \bot$, which is absurd. However, BI is complete with respect to this semantics apart from $\bot$:

Proposition 6.14 (Pym et al. [171, 172]). *Let* Γ *and* φ *not contain* $\bot$*. If* $\Gamma \vDash \varphi$*, then* $\Gamma \vdash \varphi$*.*

The Inconsistency Semantics

How do we modify the semantics so that BI is complete? One approach is to make a slight concession to the absurd: include a distinguished element π dominating the PCMs — $\forall w (w \in \mathfrak{M} \Rightarrow w \preceq \pi)$ — which satisfies absurdity ($\bot$). One varies the satisfaction relation in Figure 6.2, to have the following clause for absurdity:

$$\mathfrak{M}, w \Vdash^{\bot} \bot \qquad \text{iff} \qquad w = \pi$$

This induces a validity judgement $\vDash^{\bot}$ relative to $\Vdash^{\bot}$ and the dominated PCMs. However, BI is incomplete with respect to this semantics too:

Proposition 6.15 (Pym [171]). *Let* $\varphi = ((\psi \mathrel{-\!\!*} \bot) \mathrel{-\!\!*} \bot) \vee (\psi \mathrel{-\!\!*} \bot)$*. The judgement* $\varnothing_{\times} \vDash^{\bot} \varphi$ *but* $\varnothing_{\times} \vdash \varphi$ *does not.*

Consequently, one must modify the clause for disjunction too, effectively using Beth's clause instead of Kripke's — see Chapter 2. A term-model construction exists with respect to the Grothendieck sheaf-theoretic models studied by Pym et al. [171, 172].

Partial and Non-deterministic Monoids

Galmiche et al. [74] have considered variants of the monoidal semantics in which the monoidal product is *partial*, which is both sound and complete. A more general perspective is offered by Docherty and Pym [47, 44], who considered the option of having non-deterministic monoids. This consideration arises naturally from the setting up of a uniform metatheory for bunched logics by extending the metatheory for intuitionistic layered graph logic — see Docherty and Pym [45, 46].

The structures involved in the semantics of Docherty and Pym [47, 44] are similar to the ordered monoids above except rather than have a unit e, they have a set of elements E at least one of which is a unit, which further satisfies the following:

$$\underbrace{e \in E \,\&\, e' \succeq e \Rightarrow e' \in E}_{\text{Closure}} \qquad \underbrace{e \in E \,\&\, x \in y \circ e \Rightarrow y \preceq x}_{\text{Coherence}}$$

$$\underbrace{t' \succeq t \in x \circ y \,\&\, w \in t' \circ z \Rightarrow \exists s, s', w' (s' \succeq s \in y \circ z \,\&\, w \succeq w' \in x \circ s')}_{\text{Strong Associativity}}$$

Accordingly, one takes the following variations of the clauses for satisfaction:

$$\begin{array}{lll}
x \Vdash \top^* & \text{iff} & x \in E \\
x \Vdash \varphi * \psi & \text{iff} & \text{there exists } x', y, z \text{ st. } x \succeq x' \in y \circ z, y \Vdash \varphi \text{ and } z \Vdash \psi \\
x \Vdash \varphi -\!\!* \, \psi & \text{iff} & \text{for any } x', y, z, \text{ if } x \preceq x', z \in x' \circ y \text{ and } y \Vdash \varphi, \text{ then } z \Vdash \psi
\end{array}$$

As above, given an interpretation, such structures have been shown to be sound and complete for BI. Moreover, one has soundness and completeness for related logics upon suitable augmentation — for example, replacing the preorder with equality one produces models for Boolean BI (see Docherty [44]).

The clauses used here may be simplified as follows:

$$\begin{array}{lll}
x \Vdash \varphi * \psi & \text{iff} & \text{there exists } y, z \text{ st. } x \in y \circ z, y \Vdash \varphi \text{ and } z \Vdash \psi \\
x \Vdash \varphi -\!\!* \, \psi & \text{iff} & \text{for any } y, z, \text{ if } z \in x \circ y \text{ and } y \Vdash \varphi, \text{ then } z \Vdash \psi
\end{array}$$

Soundness and completeness requires persistent models, but checking that a model satisfies this criterion or constructing one that does can be challenging. Fortunately, there are results in the literature that address this issue.

In the deterministic case the problem can be resolved by assuming bifunctoriality, but generalizing the property to non-deterministic case is a delicate matter. Cao et al. [30] have considered the following conditions:

$$z \in x \circ y \& z \preceq z' \Rightarrow \exists x', y'(z' \in x' \circ y' \& x \preceq x' \& y \preceq y')$$
$$z \in x \circ y \& x' \preceq x \& y' \preceq y \Rightarrow \exists z'(z' \preceq z \& z' \in x' \circ y')$$

Assuming these properties, called *Upward Closed* and *Downward Closed*, respectively, one recovers soundness with both the direct and indirect clauses for $*$ and $-\!\!*$, respectively. Moreover, Cao et al. [30] showed that any structure satisfying either condition together with *Simple Associativity* — $t \in x \circ y \& w \in t \circ z \Rightarrow \exists s(s \in y \circ z \& w \in x \circ s)$ — can be conservatively transformed into sound models of BI satisfying all three. Docherty and Pym [48, 44] has further shown that strong associativity for the non-deterministic models suffices for the same result without assuming the model to be either upward or downward closed.

The Relational Semantics

Galmiche et al. [74] attempted to put the partial semantics within a more general framework. The structures are similar to those above. The monoidal product is generalized to a relation R, such that $R(w,u,v)$ means that w decomposes into u and v, and one has a distinguished element π satisfying absurdity, satisfying the following:

$$\underbrace{R(\pi,x,y)}_{\pi\text{-max}} \qquad \underbrace{R(y,x,\pi) \Rightarrow \pi \preceq y}_{\pi\text{-abs}}$$

The preorder is now defined in terms of the relation (i.e., $x \preceq y \Longrightarrow R(y,x,e)$) , and there are some additional conditions beyond commutativity and associativity:

$$\underbrace{R(z,x,y) \& x \preceq x' \Rightarrow R(z,x',y)}_{\text{Compatibility}} \qquad \underbrace{R(z,x,y) \& z \preceq z' \Rightarrow R(z',x,y)}_{\text{Transitivity}}$$

The relational structures form models under an interpretation $[\![-]\!]$ of the atoms when they are atomically persistent and, for any world w and atom a, if $\pi \preceq w$, then

$w \in [\![\mathrm{a}]\!]$. The resulting semantics was shown sound and complete via a term-model construction — see Galmiche et al. [74].

Chapter 7

The Admissibility of Cut

In Chapter 6, BI was characterized by the sequent calculus LBI without cut —

$$\frac{\Delta \triangleright \chi \quad \Gamma(\chi) \triangleright \varphi}{\Gamma(\Delta) \triangleright \varphi}\ \mathsf{cut}$$

Versions of cut are studied in many logic and are typically used when studying meta-theory (e.g., it is used in translating natural deduction proofs into sequent calculi proofs and thereby showing the soundness of the latter), also in more computational questions such as complexity of proofs — see, for example, Boolos [25]. Nonetheless, the presence of cut has the undesirable effect of rendering proof-search non-analytic — that is, given a putative conclusion, one may have to guess the formula χ not evident from analysisng the conclusion. Hence, one wishes to show that one may appeal to cut when desired, but not have to rely on it to prove a formula. This is what it means to be admissible; more precisely, while LBI characterizes BI, so does the the extended system LBI cut-rule to it without changing the logic.

Brotherston [27] has given a proof of admissibility for the rule using display calculi, and Frumin [63] has given a semantic proof of it. However, it is not only the admissibility of cut in itself that is useful, the proof of it can be too. The Curry-Howard Correspondence Theorem [106] means that 'rewrite' proofs of cut-admissibility correspond to normalization of some typed λ-calculus. Type systems corresponding to BI have been studied by O'Hearn and Pym [150], and they have been deployed by Petricek [154] to model context-aware programming languages. This chapter provides a procedure for gradually removing cuts from sequent calculi proofs.

The earliest and still most traditional method of demonstrating the admissibility of cut in a sequent calculus is due to Gentzen [200]. The *Hauptsatz*, as it is often called, is proved for first-order classical and intuitionistic logic. The proof proceeds by recursively rewriting sequent calculus proofs in such a way the cuts gradually disappear while leaving the conclusion unchanged. There are at least two attempts in the literature on BI on rewriting proofs for cut-admissibility: one by Pym [171] and one by Arisaka and Qin [8]. Both contain flaws, which are discussed below.

7.1 Previous Attempts

The rewriting method for cut-admissibility in a sequent calculus is characterized by providing a transformation that permutes cuts upward (i.e., towards the leaves of the proof) in an arbitrary proof from the sequent calculus by stating how they may be permuted with other rules. This delivers admissibility when one can show that some sequence of rewrites terminates in a cut-free proof, which is typically done by establishing some well-founded measure that decreases with each transformation.

A typical measure used for this purposes is the multiset ordering induced by lexicographic ordering (see Dershowitz and Manna [43]) on *cut rank* — see, for example, Troelstra and Schwichtenberg [207].

The traditional *cut-rank* is a pair of the size of the cut-formula and the maximal distance from the leaves of the proof.

Definition 7.1 (Size)**.** *The size of a formula φ, denoted $\sigma(\varphi)$, is the number of binary connectives that it contains; that is,*

$$\sigma(\varphi) := \begin{cases} 0 & \textit{if } \varphi \in \mathbb{A} \cup \{\top, \bot, \top^*\} \\ \sigma(\psi_1) + \sigma(\psi_2) + 1 & \textit{if } \varphi = \psi_1 \circ \psi_2 \textit{ for } \circ \in \{\land, \lor, \to, *, -\!\!*\} \end{cases}$$

Definition 7.2 (Height)**.** *The height of a proof $\mathcal{D}$, denoted $h(\mathcal{D})$, is the maximal* *number of nodes from root to leaf; that is,*

$$h(\mathcal{D}) := \begin{cases} 0 & \text{if } \mathcal{D} : \Gamma \rhd \Delta \text{ is an instance of an axiom} \\ \max\{h(\mathcal{D}_1), ..., h(\mathcal{D}_n)\} + 1 & \text{if } \mathcal{D} \text{ has immediate sub-trees } \mathcal{D}_1, ..., \mathcal{D}_n \end{cases}$$

Definition 7.3 (Cut)**.** *A cut in a* $\mathsf{LBI}+\mathsf{cut}$*-proof* $\mathcal{D}$ *is a triple* $k = \langle \mathcal{L}, \mathcal{R}, \chi \rangle$ *such that the following is a sub-proof of* $\mathcal{D}$*:*

$$\dfrac{\mathcal{L} : \Delta \rhd \chi \quad \mathcal{R} : \Gamma(\chi) \rhd \varphi}{\Gamma(\Delta) \rhd \varphi}\ \mathsf{cut}$$

Definition 7.4 (Traditional cut-rank)**.** *Let* $k = \langle \mathcal{L}, \mathcal{R}, \chi \rangle$ *be a cut in a* $\mathsf{LBI}+\mathsf{cut}$*-proof* $\mathcal{D}$*. Its cut-rank is a pair* $\langle \sigma(\chi), \max\{h(\mathcal{L}), h(\mathcal{R}))\} \rangle$*.*

Definition 7.5 (Multiset Ordering)**.** *Let* $\mathbb{V}$ *be ordered by a relation* $\prec$*. Let* M *and* N *be multi-sets over* $\mathbb{V}$*. The multi-set ordering* $<$ *is the least ordering on multisets such that* $M < N$ *iff there are* X *and* Y *such that* $M = (N - X) \sqcup Y$ *with* $X \sqsubseteq N$ *and* $X \neq \varnothing$ *and for any* $y \in Y$ *there is* $x \in X$ *such that* $y \prec x$*.*

Dershowitz and Manna [43] have shown that if $\prec$ is a well-order, then the induced multiset ordering is also a well-order.

To show cut-admissibility, it suffices to provide a collection of operations on proofs that replace cuts with cuts of smaller cut-rank.

Example 7.6. *The following transformation shows how a cut may be permuted with* $\wedge_\mathsf{R}$ *and* $\wedge_\mathsf{L}$ *when their principal formulae is the cut-formula:*

$$\dfrac{\dfrac{\mathcal{D}_1 : \Delta \rhd \varphi \quad \mathcal{D}_2 : \Delta \rhd \psi}{\Delta \rhd \varphi \wedge \psi}\ \wedge_\mathsf{R} \quad \dfrac{\mathcal{D}_3 : \Gamma(\varphi \,;\, \psi) \rhd \chi}{\Gamma(\varphi \wedge \psi) \rhd \chi}\ \wedge_\mathsf{L}}{\Gamma(\Delta) \rhd \chi}\ \mathsf{cut} \quad \mapsto$$

$$\dfrac{\dfrac{\mathcal{D}_2 : \Delta \rhd \psi \quad \dfrac{\mathcal{D}_1 : \Delta \rhd \varphi \quad \mathcal{D}_3 : \Gamma(\varphi \,;\, \psi) \rhd \chi}{\Gamma(\Delta \,;\, \psi) \rhd \chi}\ \mathsf{cut}}{\Gamma(\Delta \,;\, \Delta) \rhd \chi}\ \mathsf{cut}}{\Gamma(\Delta) \rhd \chi}\ \mathsf{c}$$

Though there are more cuts after the transformation than before it, both of *them are on formulas of smaller cut-size.* ∎

When attempting permutations of cut with the other rules one eventually has to handle the interaction with contraction (c):

$$\dfrac{\mathcal{D}_1 : \Delta \rhd \varphi \qquad \dfrac{\mathcal{D}_2 : \Gamma(\varphi \fatsemi \varphi) \rhd \chi}{\Gamma(\varphi) \rhd \chi}\,\mathsf{c}}{\Gamma(\Delta) \rhd \chi}\,\mathsf{cut}$$

The most obvious transformation is to produce the following:

$$\dfrac{\dfrac{\mathcal{D}_1 \rhd \Delta \rhd \varphi \qquad \dfrac{\mathcal{D}_2 : \Delta \rhd \varphi \quad \mathcal{D}_3 : \Gamma(\varphi \fatsemi \varphi) \rhd \chi}{\Gamma(\Delta \fatsemi \varphi) \rhd \chi}\,\mathsf{cut}}{\Gamma(\Delta \fatsemi \Delta) \rhd \chi}\,\mathsf{cut}}{\Gamma(\Delta) \rhd \chi}\,\mathsf{c}$$

Unfortunately, this transformation does not result in a decrease in the usual termination measure outlined above. This is not unusual. The same problem arises for FOL, which led Gentzen [200] to introduce the *multi*-cut rule that absorb contractions in the cut-rule. The approach is suggested by Pym [171] for BI, but the complex structures of bunches renders it difficult to implement in BI correctly.

The naive way to implement a multi-cut rule in BI is by cut′: let $\varphi^0 := \varphi$ and $\varphi^n := \varphi^{n-1} \fatsemi \varphi$,

$$\dfrac{\Delta \rhd \varphi \quad \Gamma(\varphi^n) \rhd \chi}{\Gamma(\Delta) \rhd \chi}\,\mathsf{cut}'$$

The interaction with c shown above can indeed be handled by this rule; that is, one transforms the given proof into the following strictly shorter proof:

$$\dfrac{\mathcal{D}_1 : \Delta \rhd \varphi \quad \mathcal{D}_3 : \Gamma(\varphi \fatsemi \varphi) \rhd \chi}{\Gamma(\Delta) \rhd \chi}\,\mathsf{cut}'$$

What makes it naive is that it fails to consider other more complex interactions with c, as observed by Arisaka and Qin [8]. For example, suppose that a proof contains a cut of the following form:

$$\dfrac{\mathcal{D}_1 : \Sigma \triangleright \varphi \qquad \dfrac{\mathcal{D}_2 : \Gamma(\Delta(\varphi) \fatsemi \Delta(\varphi)) \triangleright \chi}{\Gamma(\Delta(\varphi)) \triangleright \chi}\ \mathsf{c}}{\Gamma(\Delta(\Sigma)) \triangleright \chi}\ \mathsf{cut}$$

Here cut′ is no more useful than cut was before since the cut formula is contained in two different sub-bunches.

Arisaka and Qin [8] proposed a less naive multi-cut rule to resolve this situation. Define $\Delta^0 := \Delta$ and $\Delta^n := \Delta^{n-1} \fatsemi \Delta$, then bunched multi-cut is as follows:

$$\dfrac{\Sigma \triangleright \varphi \quad \Gamma(\Delta(\varphi)^n) \triangleright \chi}{\Gamma(\Delta(\Sigma)) \triangleright \chi}\ \mathsf{cut}''$$

Unfortunately, this rule also fails to give the desired result. Consider what happens when one of the bunches on which the multi-cut is applied is modified in the next step; for example, one has a proof containing a cut of the following form:

$$\dfrac{\mathcal{D}_1 : \Pi \triangleright \varphi \qquad \dfrac{\dfrac{\mathcal{D}_2 : \Gamma(\Delta'(\varphi) \fatsemi \Delta(\varphi)) \triangleright \chi}{\Gamma(\Delta(\varphi) \fatsemi \Delta(\varphi)) \triangleright \chi}\ \mathsf{w}}{\Gamma(\Delta(\varphi)) \triangleright \chi}\ \mathsf{c}}{\Gamma(\Delta(\Pi)) \triangleright \chi}\ \mathsf{cut}''$$

The cut″-rule can indeed handle the contraction, but not the weakening,

$$\dfrac{\mathcal{D}_1 : \Pi \triangleright \varphi \qquad \dfrac{\mathcal{D}_2 : \Gamma(\Delta'(\varphi) \fatsemi \Delta(\varphi)) \triangleright \chi}{\Gamma(\Delta(\varphi) \fatsemi \Delta(\varphi)) \triangleright \chi}\ \mathsf{w}}{\Gamma(\Delta(\Pi)) \triangleright \chi}\ \mathsf{cut}''$$

Thus, the idea of multi-cut in BI merely postpones the problem of replication in the interaction with c, rather than handling it.

We take another approach to cut-admissibility entirely: we modify the termination measure to track the number of replications of cuts due to contractions rather than to absorb the contractions within the cut-rule.

7.2 The Rewrite Transformation

The rewrite relation on proofs is defined by substituting sub-proofs that conclude by uses of the cut-rule. To simplify later discussion, it is useful to make formal what is meant by *a cut* in a proof.

Definition 7.7 (Cut)**.** *A cut in a* $\mathsf{LBI}+\mathsf{cut}$*-proof* $\mathcal{D}$ *is a triple* $k = \langle \mathcal{L}, \mathcal{R}, \chi \rangle$ *such that the following is a sub-proof of* $\mathcal{D}$*:*

$$\frac{\mathcal{L} : \Delta \triangleright \chi \quad \mathcal{R} : \Gamma(\chi) \triangleright \varphi}{\Gamma(\Delta) \triangleright \varphi}\ \mathsf{cut}$$

It is useful to classify cuts into three groups: base, commutative, and principal. This taxonomy is typical in cut-admissibility proofs proceeding by a rewrite procedure — see, for example, Troelstra and Schwichtenberg [207]. The classes may be distinguished structurally:

- A cut $\langle \mathcal{L}, \mathcal{R}, \varphi \rangle$ is a *base* cut iff either $\mathcal{L}$ or $\mathcal{R}$ consists of a single node (i.e., is an instance of an axiom).

- A cut $\langle \mathcal{L}, \mathcal{R}, \varphi \rangle$ is a *commutative* cut iff it is not a base cut and the cut-formula is not principal in the last inference of either $\mathcal{L}$ or $\mathcal{R}$.

- A cut $\langle \mathcal{L}, \mathcal{R}, \varphi \rangle$ is a *principal* cut iff the cut-formula φ is principal in the last inference of both $\mathcal{L}$ and $\mathcal{R}$.

To be able to specify the cut-transformations, we require a small technical result.

Proposition 7.8. *The following equivalences hold:*

$$\Gamma(\varnothing_\times) \vdash \varphi \qquad \textit{iff} \qquad \Gamma(\top^*) \vdash \varphi$$

and

$$\Gamma(\varnothing_+) \vdash \varphi \qquad \textit{iff} \qquad \Gamma(\top) \vdash \varphi$$

Proof. This is observed by substitution of $\varnothing_\times$ with $\top^*$ and $\varnothing_+$ and $\top$ on proofs. That is, consider an arbitrary LBI-proof $\mathcal{D}$ witnessing $\Gamma(\varnothing_\times)\vdash\varphi$. By case analysis on the possible rules concluding $\mathcal{D}$, one sees that there is a $\mathcal{D}'$ witnessing $\Gamma(\top^*)\vdash\varphi$. The same holds in the reverse direction, and *mutatis mutandis* for $\varnothing_+$ and $\top$. □

The transformation of cuts is defined by the following relation:

Definition 7.9 (Cut-reduction)**.** *The relation $\mapsto$ on proofs, cut-reduction, is defined as follows:*

- *The reduction of base cuts is given in Figure 7.1a.*

- *The reduction of commutative cuts is given in Figure 7.1b with the trivial commutative cuts elided — that is, we suppress commutative cuts in which one simply permutes* cut *with the rule for a connective, such as in the following:*

$$\dfrac{\dfrac{\mathcal{D}_1 : \Delta \fatsemi \chi_1 \rhd \chi_2}{\Delta \rhd \chi_1 \to \chi_2}\,{\to_{\mathsf{R}}} \qquad \dfrac{\mathcal{D}_2 : \Gamma(\chi_1 \to \chi_2) \rhd \varphi}{\Gamma(\chi_1 \to \chi_2) \rhd \varphi \lor \psi}\,{\lor_{\mathsf{R}}}}{\Gamma(\Delta) \rhd \varphi \lor \psi}\,\mathsf{cut} \qquad \mapsto$$

$$\dfrac{\dfrac{\dfrac{\mathcal{D}_1 : \Delta \fatsemi \chi_1 \rhd \chi_2}{\Delta \rhd \chi_1 \to \chi_2}\,{\to_{\mathsf{R}}} \qquad \mathcal{D}_2 : \Gamma(\chi_1 \to \chi_2) \rhd \varphi}{\Gamma(\Delta) \rhd \varphi}\,\mathsf{cut}}{\Gamma(\Delta) \rhd \varphi \lor \psi}\,{\lor_{\mathsf{R}}}$$

- *The reduction of principal cuts is given in Figure 7.1c.*

Definition 7.10 (Rewrite)**.** *Let $\mathcal{D}$ be a* $\mathsf{L}+\mathsf{cut}$*-proof. An* $\mathsf{L}+\mathsf{cut}$*-proof $\mathcal{D}'$ is a rewrite of $\mathcal{D}$ — denoted $\mathcal{D} \rightsquigarrow \mathcal{D}'$ — iff there are* $\mathsf{L}+\mathsf{cut}$*-proofs δ and δ' such that $\delta \mapsto \delta'$ and $\mathcal{D}'$ is result of replacing a sub-tree δ with δ' in $\mathcal{D}$.*

It remains to show that the rewrite relation is terminating, under some sequence of reductions, and that it terminates in cut-free proofs.

7.3 Termination of Rewrite

In this section, we show that the the rewrite transformation eventually (i.e., in finitely many steps) yields cut-free proofs. To this end, we show that each step

$$\dfrac{\overline{\varphi \rhd \varphi}\;\mathsf{ax} \qquad \mathcal{D} : \Gamma(\varphi) \rhd \chi}{\Gamma(\varphi) \rhd \chi}\;\mathsf{cut} \qquad \mapsto \qquad \mathcal{D} : \Gamma(\varphi) \rhd \chi$$

$$\dfrac{\mathcal{D} : \Gamma \rhd \varphi \qquad \overline{\varphi \rhd \varphi}\;\mathsf{ax}}{\Gamma \rhd \varphi}\;\mathsf{cut} \qquad \mapsto \qquad \mathcal{D} : \Gamma \rhd \varphi$$

$$\dfrac{\overline{\varnothing_{+} \rhd \top}\;\top_{\mathsf{R}} \qquad \mathcal{D}_1 : \Gamma(\top) \rhd \chi}{\Gamma(\varnothing_{+}) \rhd \chi}\;\mathsf{cut} \qquad \mapsto \qquad \underbrace{\mathcal{D}_1' : \Gamma(\varnothing_{+}) \rhd \top}_{\text{(Proposition 7.8 on } \mathcal{D}_1 \text{)}}$$

$$\dfrac{\overline{\varnothing_{\times} \rhd \top^{*}}\;\top^{*}_{\mathsf{R}} \qquad \mathcal{D}_1 : \Gamma(\top^{*}) \rhd \chi}{\Gamma(\varnothing_{\times}) \rhd \chi}\;\mathsf{cut} \qquad \mapsto \qquad \mathcal{D}_1' : \underbrace{\Gamma(\varnothing_{\times}) \rhd \chi}_{\text{(Proposition 7.8 on } \mathcal{D}_1 \text{)}}$$

(a) Reduction of Base Cuts

$$\dfrac{\mathcal{D}_1 : \Pi \rhd \varphi \qquad \dfrac{\mathcal{D}_2 : \Gamma(\Delta) \rhd \chi}{\Gamma(\Delta \fatsemi \Sigma(\varphi)) \rhd \chi}\;\mathsf{w}}{\Gamma(\Delta \fatsemi \Sigma(\Pi)) \rhd \chi}\;\mathsf{cut} \qquad \mapsto \qquad \dfrac{\mathcal{D}_2 : \Gamma(\Delta) \rhd \chi}{\Gamma(\Delta \fatsemi \Sigma(\Pi)) \rhd \chi}\;\mathsf{w}$$

$$\dfrac{\mathcal{D}_1 : \Delta \rhd \varphi \qquad \dfrac{\mathcal{D}_2 : \Gamma(\varphi \fatsemi \varphi) \rhd \chi}{\Gamma(\varphi) \rhd \chi}\;\mathsf{c}}{\Gamma(\Delta) \rhd \chi}\;\mathsf{cut} \qquad \mapsto$$

$$\dfrac{\dfrac{\mathcal{D}_1 : \Delta \rhd \varphi \qquad \dfrac{\mathcal{D}_1 : \Delta \rhd \varphi \quad \mathcal{D}_2 : \Gamma(\varphi \fatsemi \varphi) \rhd \chi}{\Gamma(\Delta \fatsemi \varphi) \rhd \chi}\;\mathsf{cut}}{\Gamma(\Delta \fatsemi \Delta) \rhd \chi}\;\mathsf{cut}}{\Gamma(\Delta) \rhd \chi}\;\mathsf{c}$$

$$\dfrac{\mathcal{D}_1 : \Delta \rhd \varphi \qquad \dfrac{\mathcal{D}_2 : \Gamma'(\varphi) \rhd \chi}{\Gamma(\varphi) \rhd \chi}\;\mathsf{e}}{\Gamma(\Delta) : \chi}\;\mathsf{cut} \qquad \mapsto \qquad \dfrac{\dfrac{\mathcal{D}_1 : \Delta \rhd \varphi \quad \mathcal{D}_2 : \Gamma'(\varphi) \rhd \chi}{\Gamma'(\varphi) \rhd \chi}\;\mathsf{cut}}{\Gamma(\varphi) \rhd \chi}\;\mathsf{e}$$

(b) Reduction of Commutative Cuts

Figure 7.1: Transformations of Cuts in LBI (Part 1)

$$\dfrac{\dfrac{\mathcal{D}_1 : \Delta \rhd \varphi \quad \mathcal{D}_2 : \Delta' \rhd \psi}{\Delta \fatsemi \Delta' \rhd \varphi \wedge \psi}\wedge_{\mathsf{R}} \qquad \dfrac{\mathcal{D}_3 : \Gamma(\varphi \fatsemi \psi) \rhd \chi}{\Gamma(\varphi \wedge \psi) \rhd \chi}\wedge_{\mathsf{L}}}{\Gamma(\Delta \fatsemi \Delta') : \chi}\,\mathsf{cut} \qquad \mapsto$$

$$\dfrac{\mathcal{D}_2 : \Delta' \rhd \psi \qquad \dfrac{\mathcal{D}_1 : \Delta \rhd \varphi \quad \mathcal{D}_2 : \Gamma(\varphi \fatsemi \psi) \rhd \chi}{\Gamma(\Delta \fatsemi \psi) \rhd \chi}\,\mathsf{cut}}{\Gamma(\Delta \fatsemi \Delta') \rhd \chi}\,\mathsf{cut}$$

$$\dfrac{\dfrac{\mathcal{D}_1 : \Sigma \fatsemi \varphi \rhd \psi}{\Sigma \rhd \varphi \to \psi}\to_{\mathsf{R}} \qquad \dfrac{\mathcal{D}_2 : \Delta \rhd \varphi \quad \mathcal{D}_3 : \Gamma(\psi) \rhd \chi}{\Gamma(\Delta \fatsemi \varphi \to \psi) \rhd \chi}\to_{\mathsf{L}}}{\Gamma(\Delta \fatsemi \Sigma) \rhd \chi}\,\mathsf{cut} \qquad \mapsto$$

$$\dfrac{\dfrac{\dfrac{\mathcal{D}_2 : \Delta \rhd \varphi \quad \mathcal{D}_1 : \Sigma \fatsemi \varphi \rhd \chi}{\Sigma \fatsemi \Delta \rhd \psi}\,\mathsf{cut} \qquad \mathcal{D}_3 : \Gamma(\psi) \rhd \chi}{\Gamma(\Sigma \fatsemi \Delta) \rhd \chi}\,\mathsf{cut}}{\Gamma(\Delta \fatsemi \Sigma) \rhd \chi}\,\mathsf{e}$$

$$\dfrac{\dfrac{\mathcal{D}_3 : \Delta \rhd \varphi_i}{\Delta \rhd \varphi_1 \vee \varphi_2}\vee_{\mathsf{R}} \qquad \dfrac{\mathcal{D}_1 : \Gamma(\varphi_1) \rhd \chi \quad \mathcal{D}_2 : \Gamma(\varphi_2) \rhd \chi}{\Gamma(\varphi_1 \vee \varphi_2) \rhd \chi}\vee_{\mathsf{L}}}{\Gamma(\Delta) \rhd \chi}\,\mathsf{cut} \qquad \mapsto$$

$$\dfrac{\mathcal{D}_3 : \Delta \rhd \varphi_i \quad \mathcal{D}_i : \Gamma(\varphi_i) \rhd \chi}{\Gamma(\Delta) \rhd \chi}\,\mathsf{cut}$$

$$\dfrac{\dfrac{\mathcal{D}_1 : \Delta \rhd \varphi \quad \mathcal{D}_2 : \Delta' \rhd \psi}{\Delta , \Delta' \rhd \varphi * \psi}*_{\mathsf{R}} \qquad \dfrac{\mathcal{D}_3 : \Gamma(\varphi , \psi) \rhd \chi}{\Gamma(\varphi * \psi) \rhd \chi}*_{\mathsf{L}}}{\Gamma(\Delta , \Delta') \rhd \chi}\,\mathsf{cut} \qquad \mapsto$$

$$\dfrac{\mathcal{D}_2 : \Delta' \rhd \psi \qquad \dfrac{\mathcal{D}_1 : \Delta \rhd \varphi \quad \mathcal{D}_3 : \Gamma(\varphi , \psi) \rhd \chi}{\Gamma(\Delta , \psi) \rhd \chi}\,\mathsf{cut}}{\Gamma(\Delta , \Delta') \rhd \chi}\,\mathsf{cut}$$

$$\dfrac{\dfrac{\mathcal{D}_1 : \Sigma , \varphi \rhd \psi}{\Sigma \rhd \varphi \mathrel{-\!\!*} \psi}\mathrel{-\!\!*}_{\mathsf{R}} \qquad \dfrac{\mathcal{D}_2 : \Delta \rhd \varphi \quad \mathcal{D}_3 : \Gamma(\Delta' , \psi) \rhd \chi}{\Gamma(\Delta , \Delta' , \varphi \mathrel{-\!\!*} \psi) \rhd \chi}\mathrel{-\!\!*}_{\mathsf{L}}}{\Gamma(\Delta , \Delta' , \Sigma) \rhd \chi}\,\mathsf{cut} \qquad \mapsto$$

$$\dfrac{\dfrac{\dfrac{\mathcal{D}_2 : \Delta \rhd \varphi \quad \mathcal{D}_1 : \Sigma , \varphi \rhd \psi}{\Sigma , \Delta \rhd \psi}\,\mathsf{cut} \qquad \mathcal{D}_2 : \Gamma(\Delta' , \psi) \rhd \chi}{\Gamma(\Delta' , \Sigma , \Delta) \rhd \chi}\,\mathsf{cut}}{\Gamma(\Delta , \Delta' , \Sigma) \rhd \chi}\,\mathsf{e}$$

(c) Reduction of Principal Cuts

Figure 7.1: Transformations of Cuts in LBI (Part 2)

(i.e., each transformation) corresponds to a descent in some well-order on proofs whose bottom elements do not contain cuts. The well-order in question is over a refined notion of cut-rank that also tracks interactions with contractions.

Definition 7.11 (Contraction Potential). *Let $k = \langle \mathcal{L}, \mathcal{R}, \varphi \rangle$ be a cut in a proof $\mathcal{D}$. The contraction potential of φ — denoted $\kappa(k)$ — is the number of times a contraction is used in $\mathcal{R}$ on a sub-bunch containing φ.*

Essentially, contraction potential tracks the occurrence of a formula. Extending the rewriting method used by Gentzen [200] (see Section 7.1) by such measures has been considered before — see, for example, Schroeder-Heister [189].

Definition 7.12 (Cut-rank). *The rank of a cut $k = \langle \mathcal{L}, \mathcal{R}, \chi \rangle$ in a proof $\mathcal{D}$ is the following triple:*

$$\langle \sigma(\chi), \kappa(k), \max\{h(\mathcal{L}), h(\mathcal{R})\} + 1 \rangle$$

Let $\prec$ be the lexicographic ordering on ranks. Since it is a well-order, the induced multi-set ordering, also denoted $\prec$, is a well-order. If $\mathcal{D}$ is $\mathsf{LBI} + \mathsf{cut}$-proofs then denote $\rho(\mathcal{D})$ for the multiset of ranks of cuts in $\mathcal{D}$. It remains to give a procedure for rewriting such that each step corresponds to a reduction in the multi-set ordering. We do this in two steps: first, we show that the reduction of a proof containing at most one cut reduces in the multi-set ordering; second, transform proofs with cuts into proofs without cuts by gradually removing the top-most cuts.

Proposition 7.13. *Let $\mathcal{D}$ and $\mathcal{D}'$ be $\mathsf{LBI} + \mathsf{cut}$-proofs containing at most one cut,*

$$\text{if } \mathcal{D} \rightsquigarrow \mathcal{D}', \text{ then } \rho(\mathcal{D}') \prec \rho(\mathcal{D})$$

Proof. The result follows by case analysis on the possible transformations. We will give four key cases: the interaction with contraction, a base cut, a commutative cut, and a principal cut. The rest of the transformations are similar to at least one of these cases.

- CONTRACTION. Suppose $\mathcal{D} \rightsquigarrow \mathcal{D}'$ is witnessed by a transformation for the interaction between c and cut — that is, the second transformation in Figure 7.1b,

$$\dfrac{\mathcal{D}_1 : \Delta \rhd \varphi \qquad \dfrac{\mathcal{D}_2 : \Gamma(\varphi \fatsemi \varphi) \rhd \chi}{\Gamma(\varphi) \rhd \chi}\,\mathsf{c}}{\Gamma(\Delta) \rhd \chi}\,\mathsf{cut} \qquad \mapsto$$

$$\dfrac{\dfrac{\mathcal{D}_1 : \Delta \rhd \varphi \qquad \dfrac{\mathcal{D}_1 : \Delta \rhd \varphi \quad \mathcal{D}_2 : \Gamma(\varphi \fatsemi \varphi) \rhd \chi}{\Gamma(\Delta \fatsemi \varphi) \rhd \chi}\,\mathsf{cut}}{\Gamma(\Delta \fatsemi \Delta) \rhd \chi}\,\mathsf{cut}}{\Gamma(\Delta) \rhd \chi}\,\mathsf{c}$$

The one cut has been replaced by two cuts each of the same size of cut formula but with a smaller contraction potential, and therefore each of which has a smaller rank. Hence, $\rho(\mathcal{D}) \preceq \rho(\mathcal{D})$, as required.

- BASE CUT. Let that $\mathcal{D} \rightsquigarrow \mathcal{D}'$ be justified by the transformation $\delta \mapsto \delta'$ in which ax is used on the left branch — that is, the first transformation in Figure 7.1a,

$$\dfrac{\dfrac{}{\varphi \rhd \varphi}\,\mathsf{ax} \qquad \mathcal{D}_1 : \Gamma(\varphi) \rhd \chi}{\Gamma(\varphi) \rhd \chi}\,\mathsf{cut} \qquad \mapsto \qquad \mathcal{D}_1 : \Gamma(\varphi) \rhd \chi$$

The one cut has been removed. Indeed, $\rho(\mathcal{D}') \prec \rho(\mathcal{D})$, as required.

- COMMUTATIVE CUT. Let that $\mathcal{D} \rightsquigarrow \mathcal{D}'$ be witnessed by the transformation for the interaction between w and cut —- that is, the first transformation in Figure 7.1b,

$$\dfrac{\mathcal{D}_1 : \Pi \rhd \varphi \qquad \dfrac{\mathcal{D}_2 : \Gamma(\Delta) \rhd \chi}{\Gamma(\Delta \fatsemi \Sigma(\varphi)) \rhd \chi}\,\mathsf{w}}{\Gamma(\Delta \fatsemi \Sigma(\Pi)) \rhd \chi}\,\mathsf{cut} \qquad \mapsto \qquad \dfrac{\mathcal{D}_2 : \Gamma(\Delta) \rhd \chi}{\Gamma(\Delta \fatsemi \Sigma(\Pi)) \rhd \chi}\,\mathsf{w}$$

By the same argument as in the above case, $\rho(\mathcal{D}') \prec \rho(\mathcal{D})$. Typically, it is the cut-height that decreases for commutative cuts. The only exceptions are in the interaction with weakening, which is treated here and in the interaction with contraction, which is treated above.

- Principal Cut. Let that $\mathcal{D} \rightsquigarrow \mathcal{D}'$ be justified by the transformation in which the principal connective of the cut-formula is $\land$ — that is, the first transformation in Figure 7.1c,

$$\dfrac{\dfrac{\mathcal{D}_1 : \Delta \triangleright \varphi \quad \mathcal{D}_2 : \Delta' \triangleright \psi}{\Delta \fatsemi \Delta' \triangleright \varphi \land \psi}\,\land_{\mathsf{R}} \qquad \dfrac{\mathcal{D}_3 : \Gamma(\varphi \fatsemi \psi) \triangleright \chi}{\Gamma(\varphi \land \psi) \triangleright \chi}\,\land_{\mathsf{L}}}{\Gamma(\Delta \fatsemi \Delta') : \chi}\,\mathsf{cut} \qquad \mapsto$$

$$\dfrac{\mathcal{D}_2 : \Delta' \triangleright \psi \qquad \dfrac{\mathcal{D}_1 : \Delta \triangleright \varphi \quad \mathcal{D}_2 : \Gamma(\varphi \fatsemi \psi) \triangleright \chi}{\Gamma(\Delta \fatsemi \psi) \triangleright \chi}\,\mathsf{cut}}{\Gamma(\Delta \fatsemi \Delta') \triangleright \chi}\,\mathsf{cut}$$

  The one cut has been replaced by two cuts each of which has a smaller cut-formula, and therefore each of which has a smaller rank. Hence, $\rho(\mathcal{D}') \preceq \rho(\mathcal{D})$, as required.

  Typically, it is size of the cut-formula that decreases for principal cuts.

This completes the case analysis. □

Proposition 7.14. *If $\mathcal{D}$ is an* $\mathsf{LBI} + \mathsf{cut}$*-proof containing only one cut, then there is an* LBI*-proof $\mathcal{D}'$ such that $\mathcal{D} \rightsquigarrow^* \mathcal{D}'$.*

Proof. We proceed by induction on $\rho(\mathcal{D})$.

- Base Case. If the rank of $\rho(\mathcal{D})$ is minimal then, by Proposition 7.13, a reduction of it must yield a proof without cuts. Such a proof is an LBI-proof.

- Inductive Step. The induction hypothesis (IH) is as follows: for any $\mathcal{D}'$ such that $\rho(\mathcal{D}') \prec \rho(\mathcal{D})$, if $\mathcal{D}'$ is an $\mathsf{LBI} + \mathsf{cut}$-proof containing at most one cut, then there is an LBI-proof $\mathcal{D}''$ such that $\mathcal{D}' \rightsquigarrow^* \mathcal{D}''$. We proceed by sub-induction on the number of cuts n in $\mathcal{D}'$. The sub-induction hypothesis (sub-IH) is as follows: for any $\mathcal{D}''$ such that $\rho(\mathcal{D}'') \preceq \rho(\mathcal{D}')$ containing $k < n$ cuts, there is an LBI-proof $\mathcal{D}'''$ such that $\mathcal{D}'' \rightsquigarrow^* \mathcal{D}'''$.

  - Base Case. $n = 1$. Let $\mathcal{D}''$ be such that $\mathcal{D}' \rightsquigarrow \mathcal{D}''$. By Proposition 7.13, $\rho(\mathcal{D}'') \prec \rho(\mathcal{D}')$. Hence, by the induction hypothesis (IH), there is a cut-free $\mathcal{D}'''$ such that $\mathcal{D}'' \rightsquigarrow^* \mathcal{D}'''$. By transitivity, we have $\mathcal{D} \rightsquigarrow^* \mathcal{D}'''$.

- INDUCTIVE STEP. $n > 1$. Let $\mathcal{D}''$ be arbitrary such that $\mathcal{D}' \rightsquigarrow^* \mathcal{D}''$. Consider a sub-proof δ of $\mathcal{D}''$ containing fewer than n cuts. Observe that $\rho(\delta) \prec \rho(\mathcal{D}')$, hence (by the sub-IH) there is an LBI-proof δ' such that $\delta \rightsquigarrow^* \delta'$. Let $\mathcal{D}'''$ be the result of replacing δ by δ' in $\mathcal{D}''$. Observe that $\mathcal{D}'' \rightsquigarrow \mathcal{D}'''$ and $\mathcal{D}'''$ has at most $n-1$ cuts. By the sub-IH, there is an LBI-proof $\mathcal{D}''''$ such that $\mathcal{D}''' \rightsquigarrow^* \mathcal{D}''''$. By transitivity, we have $\mathcal{D}'' \rightsquigarrow^* \mathcal{D}''''$. By transitivity again, we have $\mathcal{D} \rightsquigarrow^* \mathcal{D}''''$

This completes the sub-induction.

This completes the induction. □

It remains to extend the result to $\mathsf{LBI} + \mathsf{cut}$-proofs with an arbitrary number of cuts.

Theorem 7.15. *The rewrite relation is terminating.*

Proof. Let $\mathcal{D}$ be a $\mathsf{LBI} + \mathsf{cut}$-proof. We require to show that there is a LBI-proof $\mathcal{D}'$ such that $\mathcal{D} \rightsquigarrow^* \mathcal{D}$. We proceed by induction on the number of cuts n in $\mathcal{D}$. The induction hypothesis (IH) is as follows: if a $\mathsf{LBI} + \mathsf{cut}$-proof $\mathcal{D}$ contains $k < n$ cuts, then there is an LBI-proof $\mathcal{D}'$ such that $\mathcal{D} \rightsquigarrow^* \mathcal{D}'$.

- BASE CASE. $n = 0$. This follows from the reflexivity of $\rightsquigarrow^*$.

- INDUCTIVE STEP. $n > 1$. Let $\mathcal{D}$ be a LBI-proof with k cuts. Let δ be a sub-proof of $\mathcal{D}$ containing precisely one cut. By Proposition 7.14, there is an LBI-proof δ' such that $\delta \rightsquigarrow^* \delta'$. Let $\mathcal{D}'$ be the result of replacing δ by δ' in $\mathcal{D}$. Observe that $\mathcal{D}'$ contains at most $n-1$ cuts. Hence, by the IH, there is an LBI-proof $\mathcal{D}''$ such that $\mathcal{D}' \rightsquigarrow^* \mathcal{D}''$. Since $\mathcal{D} \rightsquigarrow^* \mathcal{D}'$, it follows by transitivity that $\mathcal{D} \rightsquigarrow^* \mathcal{D}''$.

This completes the induction. □

Corollary 7.16 (cut-Admissibility)**.** $\Gamma \vdash_{\mathsf{LBI}} \varphi$ *iff* $\Gamma \vdash_{\mathsf{LBI+cut}} \varphi$

When Gentzen [200] proves the *Hauptsatz* for FOL, it made proof-search in logic tractable because it yielded the sub-formula property: the space of reductions of a putative conclusion need only contain sequents containing formulas that appear in the putative conclusion. Importantly, however, having an analytic sequent calculus — that is, a sequent calculus with the sub-formula property — does not automatically mean that one has a proof-search algorithm which is guaranteed to find a proof whenever there is one. For example, the complexity of proof-search in FOL is forbidding: FOL is Turing complete!

Chapter 8

Logic Programming

This chapter considers *logic programming* (LP) with BI. It is based on the following article:

> Gheorghiu, A. V., Docherty, S., and Pym, D. J. Reductive Logic, Coalgebra, and Proof-search: A Perspective from Resource Semantics. In *Samson Abramsky on Logic and Structure in Computer Science and Beyond*, A. Palmigiano and M. Sadrzadeh, Eds., Springer Outstanding Contributions to Logic Series. Springer, 2021

Logic programming arose through work by Colmerauer, Keuhner, Kowalski, and Philippe [41, 119]. In particular, it was the procedural interpretation of implications in clausal form by Kowalski and Keuhner [122] that centred it on proof-search in FOL. Since FOL is undecidable, they achieved this by restricting to a fragment of the logic that is sufficiently expressive to be interesting while being complete with respect to a single rule under a fixed backtracking strategy — the Horn clause fragment of FOL. This rendered logic central to symbolic artificial intelligence. Eventually, this led Kowalski [120] to propose the following slogan:

$$\text{Algorithm} = \text{Logic} + \text{Control}$$

In this chapter, we concentrate on the proof-theoretic approach by Miller et al. [139, 140] (cf. Gabbay and Reyle [64, 65], and Schroeder-Heister and Hallnäs [94, 95]). The idea is that one can give an operational semantics for a logic programming language by restricting to a special kind of *goal-directed* proof-search which finds *uniform* proofs — see Chapter 2. This strictly generalizes the earlier work by Kowalski and Keuhner by showing that one can have multiple rules as long as one has a control regime dictating when to apply which rule.

Recall from Chapter 2 that having an operational reading of formulae enables one to give a particular canonical model-theoretic semantics that is informed by that reading. This model-theoretic semantics is significant as it explains the sense in which clauses in the context of a sequent are *definitional*, which underpins the use of LP in symbolic artificial intelligence. Altogether, this indicates how Reductive Logic can inform semantics for a logic in a serious way.

In this section we study the hereditary Harrop fragment for BI and its use as a logic programming language through uniform proof-search. We are not really studying computation, but rather restricting attention to a fragment of BI in which proof-search is well-behaved; the point is that we get a particularly simple semantics, closely related to proof-search, as a result. This illustrates the central idea of the monograph: one can usefully study the relationship between semantics and proof from the perspective of Reductive Logic.

The work in this chapter is based on earlier work by Armelín [9, 10]. In Section 8.1, we define definite clauses and goal formulae and give their operational reading. In Section 8.2, we give a model-theoretic semantics for the hereditary Harrop fragment of BI based on their operational reading, using a fixed point construction. In Section 8.3, we give a denotational semantics of the operational reading in terms of coalgebra. Finally, in Section 8.4, we discuss what it is that is actually *computed* by the logic programming language,

8.1 The Hereditary Harrop Fragment

The hereditary Harrop fragment of BI (hHBI) is composed to two forms of formulae defined by mutual induction: *definite clauses* and *goal formulae*. The intuition is that a set of definite formulae define atoms proof-theoretically, while goal formulae are complex statements over atoms requiring verification relative to a set of definite formulae.

Definition 8.1 (Definite Clauses, Goal Formulae, and Programs)**.** *The definite clauses D and goal formulae G are defined by mutual induction as follows:*

$$\begin{array}{rcl} D & ::= & \top^* \mid \top \mid \mathrm{A} \in \mathbb{A} \mid D \land D \mid D * D \mid G \to \mathrm{A} \mid G \mathrel{-\!\!*} \mathrm{A} \\ G & ::= & \top^* \mid \top \mid \mathrm{A} \in \mathbb{A} \mid G \land G \mid G \lor G \mid G * G \mid D \to G \mid D \mathrel{-\!\!*} G \end{array}$$

A program *P is any bunch comprised of definite clauses. The set of all programs is denoted by $\mathbb{P}$.*

In Chapter 2, the operational reading of definite clause for IPL may be understood proof-theoretically as a strict control pattern on reductions in LJ in which right rules are prioritised over left rules. The class of proofs thus defined are called *uniform proofs* — see Miller et al. [140]. Unfortunately, uniform proof-search understood as the application of right rules before left rules is not sufficient for goal-directedness in LBI, even for hHBI. For example, the following sequence of reductions is not goal-directed as the atom A is not principal in the first $\mathrel{-\!\!*}_{\mathsf{L}}$, but it is uniform in the informal reading given so far:

$$\dfrac{B \triangleright B \qquad \dfrac{\dfrac{A \triangleright A}{\varnothing_\times \fatsemi A \triangleright A}\,\mathsf{e}}{\top^* \fatsemi A \triangleright A}\,\top^*_{\mathsf{L}}}{B \fatsemi B \mathrel{-\!\!*} \top^* \fatsemi A \triangleright A}\,\mathrel{-\!\!*}_{\mathsf{L}}$$

This problem can be remedied by the introduction of a cut as follows:

$$\dfrac{\dfrac{B \triangleright B \qquad \dfrac{\varnothing_\times \triangleright \top^*}{\top^* \triangleright \top^*}\,\top^* L}{B \fatsemi B \mathrel{-\!\!*} \top^* \triangleright \top^*}\,\mathrel{-\!\!*} L \qquad \dfrac{\dfrac{A \triangleright A}{\varnothing_\times \fatsemi A \triangleright A}\,E}{\top^* \fatsemi A \triangleright A}\,\top^* L}{B \fatsemi B \mathrel{-\!\!*} \top^* \fatsemi A \triangleright A}\,\mathrm{Cut}$$

More complicated cases include the possibility that A is the atom defined by the implication, in which case one can also make a judicious use of a cut to keep the proof goal-directed.

The modification thus provided gives a goal-directed proof-search procedure for hHBI. That is, we restrict to *resolution proofs* that are like uniform proofs, but the right sub-proof of an implication rule consists of a series of weakening followed by a cut on the atom defined by the implication.

The control regime of resolution proofs can be enforced by augmenting $\to_{\mathsf{L}}$, $-\!\!*_{\mathsf{L}}$ and ax to rules that encode the uses of cut. To this end, we use a particular *canonical* presentation of bunches.

Definition 8.2 (Canonical Bunch)**.** *A bunch* Γ *is in canonical form iff the left-hand branch of* Γ *is either a proposition, a unit or a canonical bunch of the opposite (additive or multiplicative) type, and the right-hand branch of* Γ *is a canonical bunch.*

Proposition 8.3 (Armelín [9, 10])**.** *For every bunch* Γ*, there is a canonical bunch* Γ' *such that* $\Gamma \equiv \Gamma'$.

The purpose of canonical forms is to let us represent bunches in a convenient way. Instead of writing a bunch like $\Gamma_1 \fatcomma (\Gamma_2, (\ldots \fatcomma (\Gamma_{n-1} \fatcomma \Gamma_n)))$ we can write it as $\Gamma_1, \Gamma_2, \ldots., \Gamma_{n-1}, \Gamma_n$ with no loss of information, where Γ_i is an additive bunch for $i = 1 \ldots n$. To emphasize the shift in notation, we reserve the context-formers (i.e., $\fatcomma$ and $\fatsemi$) for the traditional presentation and use the comma and the semicolon for the special presentation.

Proposition 8.4 (Based on Armelín [9, 10])**.** *The following rules, where* $\alpha \in \{\top, \top^*\} \cup \mathbb{A}$ *and all the bunches are in canonical form, are admissible, and replacing* $\to_{\mathsf{L}}$*,* $-\!\!*_{\mathsf{L}}$ *and* ax *with them in* LBI *does not affect the completeness of the system, with respect to hHBI:*

$$\frac{P; G \to \alpha \triangleright G \qquad Q \triangleright \top^*}{Q, (P; G \to \alpha) \triangleright \alpha}\,\mathsf{res}_1 \qquad \frac{P \triangleright G}{P, G -\!\!* \alpha \triangleright \alpha}\,\mathsf{res}_2 \qquad \frac{\Gamma \triangleright \top^*}{\Gamma, \alpha \triangleright \alpha}\,\mathsf{res}_3$$

Moreover, uniform proofs in the resulting system are complete for the hereditary Harrop fragment of BI.

To suppress the use of left rules that are not resolutions, we use *clausal decomposition* — that is, the left inverse of compacting (Definition 6.5) for definite formulae

Definition 8.5 (Decomposition of Definite Clauses)**.** *The decomposition function on definite clauses is defined as follows:*

$$[D] := \begin{cases} A & \textit{if } D = A \in \mathbb{A} \\ G \to A & \textit{if } D = G \to A \\ G \mathrel{-\!\!*} A & \textit{if } D = G \mathrel{-\!\!*} A \\ \varnothing_{+} & \textit{if } D = \top \\ \varnothing_{\times} & \textit{if } D = \top^{*} \\ [D_1] \mathbin{;} [D_2] & \textit{if } D = D_1 \land D_2 \\ [D_2] \mathbin{,} [D_2] & \textit{if } D = D_1 * D_2 \end{cases}$$

We are left with the structural rules of w and e, beyond the goal-directed rules. They cannot be eliminated from LBI (or, rather, LBI modified with the resolution rules). Therefore, we introduce an ordering on programs able to capture these structural relationships.

Definition 8.6 (Bunch-extension)**.** *The* bunch-extension *relation* $\geqq$ *is the smallest relation satisfying:*

- *if* $\Gamma \equiv \Gamma'[\Delta \mapsto \Delta \mathbin{;} \Delta']$*, then* $\Gamma \geqq \Gamma'$

- *if* $\Gamma \geqq \Gamma'$ *and* $\Gamma' \geqq \Gamma''$*, then* $\Gamma \geqq \Gamma''$

Though the use of this ordering seems to reintroduce a lot of non-determinism, the choice it offers is still captured by the use of a control structure. Thus, once a clause has been *selected* in the program, the weakening may be performed in a *goal-directed* way to bring the clause to the top of the bunch. Anything additively combined with the clause is removed (using weakening), and when something is multiplicatively combined, the bunch is re-ordered so that the context-former of the

$$
\begin{array}{lll}
P \vdash_{\mathrm{o}} \top & & \text{Always} \\
P \vdash_{\mathrm{o}} \top^{*} & \text{if} & [P] \equiv \varnothing_{\times} ; R \\
P \vdash_{\mathrm{o}} A & \text{if} & [P] \geqq S, A \text{ and } S \vdash_{\mathrm{o}} \top^{*} \\
P \vdash_{\mathrm{o}} A & \text{if} & [P] \geqq Q, G \mathrel{-\!\!*} A \text{ and } Q \vdash_{\mathrm{o}} G \\
P \vdash_{\mathrm{o}} A & \text{if} & [P] \geqq S, (Q; G \to A) \text{ and } S \vdash_{\mathrm{o}} \top^{*} \text{ and } Q; G \to A \vdash_{\mathrm{o}} G \\
P \vdash_{\mathrm{o}} G_1 \vee G_2 & \text{if} & P \vdash_{\mathrm{o}} G_1 \text{ or } P \vdash_{\mathrm{o}} G_2 \\
P \vdash_{\mathrm{o}} G_1 \wedge G_2 & \text{if} & P \vdash_{\mathrm{o}} G_1 \text{ and } P \vdash_{\mathrm{o}} G_2 \\
P \vdash_{\mathrm{o}} G_1 * G_2 & \text{if} & P \equiv R; (Q, R) \text{ and } Q \vdash_{\mathrm{o}} G_1 \text{ and } R \vdash_{\mathrm{o}} G_2 \\
P \vdash_{\mathrm{o}} D \to G & \text{if} & [D]; P \vdash_{\mathrm{o}} G \\
P \vdash_{\mathrm{o}} D \mathrel{-\!\!*} G & \text{if} & [D], P \vdash_{\mathrm{o}} G
\end{array}
$$

Figure 8.1: Resolution System RBI

clause is principal. For example, we have the following sequence:

$$
\begin{array}{rcl}
(Q_0 \,,\, (Q_1 \fatsemi Q_2)) \,,\, G \mathrel{-\!\!*} \alpha & \leqq & Q_0 \,,\, ((Q_1 \fatsemi Q_2) \,,\, G \mathrel{-\!\!*} \alpha) \\
& \leqq & Q_0 \,,\, ((Q_1 \fatsemi Q_2) \,,\, (Q_3 \fatsemi (G \mathrel{-\!\!*} \alpha)) \\
& \leqq & Q_0 \,,\, ((Q_1 \fatsemi Q_2) \,,\, (Q_3 \fatsemi (Q_4 \fatsemi G \mathrel{-\!\!*} \alpha))
\end{array}
$$

Altogether, this allows us to give an operational semantics for configurations via goal-directed proof-search.

Definition 8.7 (Resolution System RBI)**.** *The resolution system is comprised of rules of Figure 8.1 in which $\mathscr{P}$ is a program, D is a definite formula, G is a goal formula, and A is an atom.*

The first two clauses of Figure 8.1 are called the initial rules, the following three the resolution rules (cut-resolution, $\mathrel{-\!\!*}$-resolution, and $\to$-resolution respectively), and the final five are the decomposition rules.

Theorem 8.8. *For any configuration $P \rhd G$,*

$$P \vdash G \qquad \textit{iff} \qquad P \vdash_{\mathrm{o}} Q$$

Proof. This follows immediately from Proposition 8.4. □

This concludes the operational reading of the hereditary Harrop fragment of BI. Properly speaking, to make it a logic *programming language*, one needs to fix a selection function and backtracking strategy. For intuitionistic logic, in which programs are lists of definite clauses, one may simply choose to always attempt the leftmost clause first when using a resolution rule; and, having made a choice, one then progresses until success or failure of the search, returning to an earlier stage in the computation in case of the latter. This forms one possibility which has been called *depth-first search with leftmost selection* — see Lloyd [132] and Kowalski [121]. Another example of a backtracking schedule and selection function is *breadth-first search with leftmost selection*, where after one step of reduction one immediately backtracks so that every possibility is tried as soon as possible. These two choices are the extremes of a range of possibilities which have different advantages and disadvantages including complexity — see Plaistead and Zhu [161].

8.2 Model-theoretic Semantics

As in Chapter 2, the operational reading of definite formulae yields a model-theoretic semantics that precisely encodes it. This is the subject of the present section, closely following the method presented by Miller [139] (originating with Apt [7]).

Interpretation and Satisfaction

We interpret a given program as the set of atomic formulae which it satisfies. That is, relative to mappings $I : \mathbb{P} \to \mathscr{P}(\mathbb{A})$, we define the judgement $I, P \Vdash G$. This is closely related to the monoidal reading of BI — see Chapter 6.

Definition 8.9 (Monoid of Programs, Interpretation)**.** *The monoid of programs is the structure $\langle \mathbb{P}, \geqq, \circ, \varnothing_\times \rangle$, where $\circ$ is multiplicative composition.*

Definition 8.10 (Interpretation)**.** *An interpretation of the monoid of programs is an order reversing mapping $I : \mathbb{P} \to \mathscr{P}(\mathbb{A})$.*

$I,P \Vdash \top$	iff	Always
$I,P \Vdash \top^*$	iff	$P \equiv \varnothing_\times ; R$
$I,P \Vdash A$	iff	$A \in I(P)$
$I,P \Vdash G_1 \vee G_2$	iff	$I,P \Vdash G_1$ or $I,P \Vdash G_2$
$I,P \Vdash G_1 \wedge G_2$	iff	$I,P \Vdash G_1$ and $I,P \Vdash G_2$
$I,P \Vdash G_1 * G_2$	iff	there exists $Q,R \in \mathbb{P}$ such that $P \geqq Q \circ R$ and $I,Q \Vdash G_1$ and $I,R \Vdash G_2$
$I,P \Vdash D \rightarrow G$	iff	$I,(P;[D]) \Vdash G$
$I,P \Vdash D \mathrel{-\!\!*} G$	iff	$I,(P,[D]) \Vdash G$

Figure 8.2: Satisfaction for Hereditary Harrop BI

Definition 8.11 (Satisfaction for hHBI)**.** *Satisfaction $\Vdash$ is the least relation satisfying the clauses of Figure 8.2, in which $\mathscr{P}$ is a program, D is a definite formula, G is a goal formula, and A is an atom.*

The aim is to define a special mapping J such that satisfaction coincides with consequence — that is,

$$\mathscr{P} \vdash G \qquad \text{iff} \qquad J, \mathscr{P} \Vdash G.$$

The clauses of the semantics are sufficiently close to the resolution system that the semantics is adequate for every interpretation that is adequate for atoms:

$$A \in I(P) \implies P \vdash_{\mathrm{o}} A$$

A primitive example of such an interpretation is $I_\perp : P \mapsto \varnothing$.

Proposition 8.12 (Adequacy)**.** *Suppose for any $A \in \mathbb{A}$, if $A \in I(P)$, then $P \vdash_{\mathrm{o}} A$. If $I,P \Vdash G$ implies $P \vdash_{\mathrm{o}} G$.*

Proof. We proceed by induction on the satisfaction relation for the hereditary Harrop fragment of BI.

- BASE CASE. There are three the sub-cases to consider for $G \in \{\top, \top^*\} \cup \mathbb{A}$. In each one, the result follows by equivalence of the respective clauses between program satisfaction and the resolution system.

- INDUCTIVE CASE. We consider each case given according to the structure of the goal formula separately.

  - $G = G_1 \vee G_2$. By Definition 8.11, we have $I, P \Vdash G_1$ or $I, P \Vdash G_2$. Therefore, by the induction hypothesis (IH), $P \vdash_{\mathrm{O}} G_1$ or $P \vdash_{\mathrm{BI}} G_2$. Hence, $P \vdash_{\mathrm{O}} G_1 \vee G_2$.

  - $G = G_1 \wedge G_2$. By Definition 8.11, we have $I, P \Vdash G_1$ and $I, P \Vdash G_2$. Therefore, by the IH, $P \vdash_{\mathrm{O}} G_1$ and $P \vdash_{\mathrm{BI}} G_2$. Hence $P \vdash_{\mathrm{O}} G_1 \wedge G_2$.

  - $G = G_1 * G_2$. By Definition 8.11, we have $P \geqq Q \circ R$ with $I, Q \Vdash G_1$ and $I, R \Vdash G_1$. Therefore, by the IH, $Q \vdash_{\mathrm{BI}} G_1$ and $R \vdash_{\mathrm{O}} G_2$. Hence, $Q, R \vdash_{\mathrm{BI}} G_1 * G_2$. Whence $P \vdash_{\mathrm{O}} G_1 * G_2$.

  - $G = D \to G'$. By Definition 8.11, we have $P; [D] \Vdash G'$. Therefore, by the IH, $P; [D] \vdash_{\mathrm{BI}} G'$. Hence, $P \vdash_{\mathrm{O}} D \to G'$.

  - $G = D \mathrel{-\!\!*} G'$. By Definition 8.11, we have $P, [D] \Vdash G'$. Therefore, by the IH, $P, [D] \vdash_{\mathrm{O}} G'$. Hence, $P \vdash_{\mathrm{O}} D \to G$.

This completes the induction. □

It remains to construct J such that the semantics is also faithful to the resolution system. We do this by essentially teaching $I_\perp$ the resolution system through the indefinite application of a transformation T representing *one step* of the operational semantics.

The Least Fixed Point Interpretation

Since $\mathcal{P}(\mathbb{A})$ forms a complete lattice, so do interpretations — that is,

$$\begin{array}{lcl} I_1 \sqsubseteq I_2 & \text{iff} & \forall w (I_1(P) \subseteq I_2(P)) \\ (I_1 \sqcup I_2)(P) & := & I_1(P_1) \cup I_2(P_2) \\ (I_1 \sqcap I_2)(P) & := & I_1(P_1) \cap I_2(P_2) \end{array}$$

The interpretation $I_\bot$ is the bottom element of this lattice. We use the operational semantics to move through step-wise incorporating more of the resolution system. To this end, consider the following T-operator on interpretations:

Definition 8.13 (T-Operator)**.** *The T operator is defined as follows:*

$$\begin{aligned}
T(I)(P) :=& \{A \mid [P] \geqq Q, A \text{ and } I, u \Vdash \top^*\} \cup & (1)\\
& \{A \mid [P] \geqq Q, G \twoheadrightarrow A \text{ and } I, Q \Vdash G\} \cup & (2)\\
& \{A \mid [P] \geqq Q, (R; G \to A) \text{ and} \\
& \quad I, Q \Vdash \top^* \text{ and } I, (R; G \to A) \Vdash G\} & (3)
\end{aligned}$$

Observe that the three parts of the definition correspond exactly to the resolution clauses of execution, so one application of T precisely incorporates *one* resolution step. Applying it indefinitely, therefore, corresponds to performing an arbitrary number of resolutions. This is handled mathematically by the use of a least fixed-point.

It follows from the Knaster-Tarski Theorem [113, 202, 7] that if T is monotone and continuous (see Proposition 8.15), then the following limit operator is well-defined:

$$T^\omega(I_\bot) := I_\bot \sqcup T(I_\bot) \sqcup T^2(I_\bot) \sqcup T^3(I_\bot) \sqcup \ldots$$

We may write $P \Vdash G$ to abbreviate $T^\omega(I_\bot), P \Vdash G$.

The completeness of the semantics follows immediately from the definition of T, since it simply observes the equivalence between resolution and application of the T-operator; that is, that T extends correctly. Soundness, on the other hand, requires showing that every path during execution is eventually considered during the unfolding.

Theorem 8.14 (Soundness and Completeness)**.** *$P \vdash_{\mathrm{o}} G$ iff $P \Vdash G$*

Proof of Completeness ($\Longleftarrow$). From Proposition 8.12, it suffices to show that the adequacy condition holds:

$$A \in T^{\omega} I_{\bot}(P) \implies P \vdash_{\mathsf{o}} A$$

It follows from Proposition 8.15 that the antecedent holds only if there exists $k \in \mathbb{N}$ such that $T^k I_{\bot}, P \Vdash A$. We proceed by induction on k.

- BASE CASE. $k = 0$. This case is vacuous as $I_{\bot}(P) = \varnothing$.
- INDUCTION STEP. Only by at least one of (1), (2), or (3) in Definition 8.13 does $A \in T^k I_{\bot}(P)$ obtain. In either case, the result follows from the induction hypothesis and the correspondence between (1), (2), and (3), with the resolution clauses of Figure 8.1.

This completes the induction. □

Proof of Soundness ($\implies$). We proceed by induction on the length N of executions in the resolution system.

- BASE CASE. $N = 1$. It must be that the proof of $P \vdash_{\mathsf{o}} G$ follows from the application of an initial rule. Hence, either $G = \top$, or $G = \top^*$ and $P \equiv \varnothing_{\times}; R$. In either case, $P \Vdash G$ follows immediately.
- INDUCTIVE STEP. We consider each of the the clauses of Figure 8.1 separately.
  - Cut-resolution. We have $[P] \geqq Q, A$, where $Q \vdash_{\mathsf{o}} \top^*$ with height $N' < N$. Therefore, by the induction hypothesis (IH), $Q \Vdash \top^*$. Hence, $A \in T(T^{\omega}(I_{\bot}))(P) = T^{\omega}(I_{\bot})(P)$. Whence, $P \Vdash A$.
  - $-\!\!*$-resolution. We have $[P] \geqq Q, G' -\!\!* A$, where $Q \vdash_{\mathsf{o}} G'$ with height $N' < N$. Therefore, by the IH, $Q \Vdash G'$. Hence, $A \in T(T^{\omega}(I_{\bot}))(P) = T^{\omega}(I_{\bot})(P)$. Whence $P \Vdash A$.
  - $\rightarrow$-resolution. We have $[P] \equiv Q, (R; G' \rightarrow A)$, where $Q \vdash_{\mathsf{o}} \top^*$ and $R \vdash_{\mathsf{o}} G'$, with heights $N', N'' < N$. Therefore, by the IH, $Q \Vdash \top^*$ and $R \Vdash G'$. Hence, $A \in T(T^{\omega} I_{\bot})(P) = T^{\omega} I_{\bot}(P)$. Whence, $P \Vdash A$.

- $G = G_1 \vee G_2$. We have $P \vdash_{\mathrm{o}} G_i$ for some $i \in \{1,2\}$ with height $N' < N$. Therefore, by the IH, $P \Vdash G_i$ for some $i \in \{1,2\}$. Hence, $P \Vdash G_1 \vee G_2$.
- $G = G_1 \wedge G_2$. We have $P \vdash_{\mathrm{o}} G_i$ for all $i \in \{1,2\}$ with heights $N', N'' < N$. Therefore, by the IH, $P \Vdash G_i$ for all $i \in \{1,2\}$. Hence, $P \Vdash G_1 \wedge G_2$.
- $G = G_1 * G_2$. We have $P \geqq Q \circ R$ where $Q \vdash_{\mathrm{o}} G_1$ and $R \vdash_{\mathrm{o}} G_2$ with heights $N', N'' < N$. Therefore, by the IH, $Q \Vdash G_1$ and $R \Vdash G_2$. Hence, $P \Vdash G_1 * G_2$.
- $G = D \to G'$. We have $P; [D] \vdash_{\mathrm{BI}} G'$ with height $N' < N$. Therefore, by the IH, $[D]; P \Vdash G'$. Hence, $P \Vdash D \to G'$.
- $G = D \mathrel{-\!\!*} G'$. We have $P, [D] \vdash_{\mathrm{BI}} G'$ with height $N' < N$. Therefore, by the IH, $[D], P \Vdash G'$. Hence, $P \Vdash D \mathrel{-\!\!*} G'$.

This completes the induction. □

It remains to show that $T^{\omega}(I_{\perp})$ is well-defined; that is, that T is monotone and continuous.

Proposition 8.15. *Let $I_0 \sqsubseteq I_1 \sqsubseteq \ldots$. be a collection of interpretations, let $P \in \mathbb{P}$, and let G be a goal. The following all hold:*

- *Persistence.* $I_1, P \Vdash G \implies I_2, P \Vdash G$.
- *Compactness.* $\bigsqcup_{i=1}^{\infty} I_i, P \Vdash G \implies \exists k \in \mathbb{N} : I_k, w \Vdash G$.
- *Monotonicity.* $T(I_0) \sqsubseteq T(I_1)$.
- *Continuity.* $T(\bigsqcup_{i=0}^{\infty} I_i) = \bigsqcup_{i=0}^{\infty} T(I_i)$.

Proof of Persistence. We proceed by induction on satisfaction.

- BASE CASE. The cases $G \in \{\top, \top^*\}$ are immediate since satisfaction is independent of interpretation. The case $G \in \mathbb{A}$ follows from the definition of the ordering on interpretations.
- INDUCTIVE STEP. We proceed by case analysis on the clauses of Figure 8.2.

- $G_1 \vee G_2$. By Definition 8.11, $I_1, P \Vdash G_1$ or $I_1, P \Vdash G_2$. Therefore, by the IH, $I_2, P \Vdash G_1$ or $I_2, P \Vdash G_2$. Hence, $I_2, P \Vdash G_1 \vee G_2$.

- $G_1 \wedge G_2$. By Definition 8.11, $I_1, P \Vdash G_1$ and $I_1, P \Vdash G_2$. Therefore, by the IH, $I_2, w \Vdash G_1$ and $I_2, w \Vdash G_2$. Hence, $I_2, w \Vdash G_1 \wedge G_2$.

- $G_1 * G_2$. By Definition 8.11, $P \geqq Q, R$ such that $I_1, Q \Vdash G_1$ and $I_1, R \Vdash G_2$. Therefore, by the IH, $I_2, Q \Vdash G_1$ and $I_2, R \Vdash G_2$. Hence, $I_2, P \Vdash G_1 * G_2$.

- $D \to G$. By Definition 8.11, $I_1, ([D]; P) \Vdash G$. Therefore, by the induction hypothesis (IH), $I_2, (P; [D]) \Vdash G$. Hence, $I_2, P \Vdash D \to G$.

- $D \mathrel{-\!\!*} G$. By Definition 8.11, $I_1, ([D], P) \Vdash G$. Therefore, by the IH, $I_2, ([D], P) \Vdash G$. Hence, $I_2, P \Vdash D \mathrel{-\!\!*} G$.

This completes the induction. □

Proof of Compactness. We proceed by induction on satisfaction.

- BASE CASE. The cases $G \in \{\top, \top^*\}$ are immediate since satisfaction is independent of interpretation. For the case in which $G \in \mathbb{A}$, note that $\left(\bigsqcup_{i=0}^{\infty}\right)(w) = \bigcup_{i=0}^{\infty} I_i(w)$, so by the definition of satisfaction $G \in I_k(w)$, for some k, and so $I_k, w \Vdash G$.

- INDUCTIVE STEP. We proceed by case analysis on the clauses of Figure 8.2.

  - $G_1 \vee G_2$. By Definition 8.11, $\bigsqcup_{i=1}^{\infty} I_i, P \Vdash G_1$ or $\bigsqcup_{i=1}^{\infty} I_i, P \Vdash G_2$. Therefore, by the induction hypothesis (IH), $\exists k \in \mathbb{N}$ such that $I_k, P \Vdash G_1$ or $I_k, P \Vdash G_2$. Hence, $I_k, w \Vdash G_1 \vee G_2$.

  - $G_1 \wedge G_2$. By Definition 8.11, $\bigsqcup_{i=1}^{\infty} I_i, P \Vdash G_1$ and $\bigsqcup_{i=1}^{\infty} I_i, P \Vdash G_2$. Therefore, by the IH, $\exists m, n \in \mathbb{N}$ such that $I_m, P \Vdash G_1$ and $I_n, P \Vdash G_2$. Let $k = \max(m, n)$. By persistence, $I_k, P \Vdash G_1$ and $I_k \Vdash G_2$. Hence, $I_k, P \Vdash G_1 \wedge G_2$.

  - $G_1 * G_2$. By Definition 8.11, $P \geqq Q \circ R$ such that $\bigsqcup_{i=1}^{\infty} I_i, P \Vdash G_1$ and $\bigsqcup_{i=1}^{\infty} I_i, Q \Vdash G_2$. Therefore, by the IH, $\exists m, n \in \mathbb{N}$ such that $I_m, Q \Vdash G_1$

and $I_n, E \Vdash G_1$. Let $k = \max(m,n)$. By persistence, $I_k, Q \Vdash G_1$ and $I_k, R \Vdash G_2$. Hence, $I_k, (Q,R) \Vdash G_1 * G_2$. Whence, $I_k, P \Vdash G_1 * G_2$.

- $D \to G'$. By Definition 8.11, $\sqcup_{i=1}^{\infty} I_i, ([D];P) \Vdash G_1$. Therefore, by the IH, $\exists k \in \mathbb{N}$ such that $I_k, ([D];P) \Vdash G'$. Hence, $I_k, w \Vdash D \to G'$.

- $D \mathrel{-\!\!*} G'$. By Definition 8.11, $\sqcup_{i=1}^{\infty} I_i, ([D],P) \Vdash G'$. Therefore, by the IH, $\exists k \in \mathbb{N}$ with $I_k, ([D],P) \Vdash G'$. Hence, $I_k, P \Vdash D \mathrel{-\!\!*} G'$.

This completes the induction. □

Proof of Monotonicity. Let $P \in \mathbb{P}$ be arbitrary, and suppose $A \in T(I_0)(P)$. We require to show $A \in T(I_1)(P)$. We proceed by case analysis on Definition 8.13:

(1) Suppose $[P] \geqq Q, A$ such that $I_0, Q \Vdash \top^*$. By persistence, $I_1, Q \Vdash \top^*$. Hence, by Definition 8.13, $A \in T(I_1)(P)$.

(2) Suppose $[P] \geqq Q, G \mathrel{-\!\!*} A$ such that $I_0, Q \Vdash G$. By persistence, $I_1, Q \Vdash G$. Hence, by Definition 8.13, $A \in T(I_1)(P)$.

(3) Suppose $[P] \geqq Q, (R; G \to A)$ such that $I_0, Q \Vdash \top^*$ and $I_0, (R; G \to A) \Vdash G$. By persistence, $I_1, Q \Vdash \top^*$ and $I_1, (R; G \to A) \Vdash G$. Hence, by Definition 8.13, $A \in T(I_1)(P)$.

This completes the case analysis. □

Proof of Continuity. We consider each direction of the inclusion separately.

- $\sqcup_{i=0}^{\infty} T(I_i) \sqsubseteq T(\sqcup_{i=0}^{\infty} I_i)$. Let $j \geq 0$, then $I_j \sqsubseteq \sqcup_{i=1}^{\infty} I_i$. By monotonicity, $T(I_j) \sqsubseteq T(\sqcup_{i=1}^{\infty} I_i)$. Since j was arbitrary, $\sqcup_{i=1}^{\infty} T(I_i) \sqsubseteq T(\sqcup_{i=1}^{\infty} I_i)$.

- $T(\sqcup_{i=0}^{\infty} I_i) \sqsubseteq \sqcup_{i=0}^{\infty} T(I_i)$. Let $P \in \mathbb{P}$ be arbitrary, and suppose $A \in T(\sqcup_{i=1}^{\infty} I_i)(P)$. It suffices to show $\exists k \in \mathbb{N}$ such that $A \in T(I_k)$. We proceed by case analysis on Definition 8.13.

  (1) Suppose $[P] \geqq Q, A$ and $\sqcup_{i=1}^{\infty} I_i, Q \Vdash \top^*$. By compactness, $\exists k \in \mathbb{N}$ such that $I_k, u \Vdash \top^*$. Hence, by Definition 8.13, $A \in TI_k(w)$.

(2) Suppose $[P] \geqq Q, G \twoheadrightarrow A$ and $\sqcup_{i=1}^{\infty} I_i, u \Vdash G$. By compactness, $\exists k \in \mathbb{N}$ such that $I_k, Q \Vdash \top^*$. Hence, by Definition 8.13, $A \in TI_k(w)$.

(3) Suppose $[P] \geqq Q, (R; G \to A)$ and $\sqcup_{i=1}^{\infty} I_i, Q \Vdash \top^*$ and $\sqcup_{i=1}^{\infty} I_i, (R; G \to A) \Vdash G$. By compactness, $\exists m, n \in \mathbb{N}$ such that $I_m, Q \Vdash \top^*$ and $I_m, (R; G \to A) \Vdash G$. Let $k = \max(m, n)$, then (by persistence) $I_k, Q \Vdash \top^*$ and $I_k, (R; G \to A) \Vdash G$. Hence, by Definition 8.13, $A \in TI_k(w)$.

This completes the case analysis.

This demonstrates both inclusions, hence equality. □

This concludes the model-theoretic semantics for the hereditary Harrop fragment of BI, which is precisely analogous to the one for the hereditary Harrop fragment of IPL in Chapter 2. In the next section, we consider an alternate *coalgebraic* semantics that exposes more of the *control* structures involved.

8.3 Coalgebraic Semantics

Deductive Logic is inherently algebraic. That is, rules in a proof system in the deductive paradigm can be understood as functions — deduction operators — that canonically determine algebras for a functor. The space of valid sequents is defined by the recursive application of these operators, which instantiates the inductive definition of proofs. In contrast, Reductive Logic is inherently coalgebraic. That is, reduction operators can be mathematically understood as coalgebras. The much larger space of sequents (or configurations) explored during reductive proof-search is corecursively generated, and reductions are coinductively defined. A terse, but complete, summary of coalgebra for this section is given in Appendix A.

The operational reading of hHBI proceeds through proof-search. There is some literature on denotational semantics of Reductive Logic. For example, Pym and Ritter [174, 173] have studied categorical semantics for classical and intuitionistic logic, and game-theoretic semantics in the spirit of Abramsky's full abstraction for PCF [5] — see also work by Miller [142]. In logic programming, Komendantskaya et al. [115, 116, 117] have successfully implemented coalgebraic models for Horn clause Logic Pogramming; Bonchi and Zanasi [23] extended the work to a precise bi-algebraic semantics. This is close to the the uniform treatment of denotational readings of operational semantics given by Turi and Plotkin [209]. In this section we give a coalgebraic semantics of the operational reading of hHBI following the above tradition.

Abstracting the Setup

We begin by modelling the syntax of the hereditary Harrop fragment of BI. Following this, we can define reduction operators as coalgebras from rules.

We use the standard free-construction to model language. We being with a standard example to illustrate the idea.

Example 8.16. *Consider the definition of a list of elements x from a set $\mathbb{A}$,*

$$\ell ::= \mathsf{nil} \mid x :: \ell$$

There are two kinds of constructors here: nil, which denotes the empty list, and the $::$ constructor represents the pairing of the two components. The $\mid$ represents a choice of constructors. Intuitively, therefore, the grammar is modelled by the least fixed point of the functor $\mathcal{F}_{\mathbb{A}} : X \mapsto \mathsf{nil} + \mathbb{A} \times X$, where nil is the emptyset. That is, the functor $\hat{\mathcal{F}}$ defined by the following ω-chain:

$$\mathsf{nil} \longrightarrow \mathsf{nil} + \mathbb{A} \times \mathsf{nil} \longrightarrow \mathsf{nil} + \mathbb{A} \times (\mathsf{nil} + \mathbb{A} \times \mathsf{nil}) \longrightarrow \ldots$$

The arrows are inductively defined by extending with the unique function out of the emptyset. Thus lists have the following coalgebraic denotation: $\mathcal{L} : X \mapsto \hat{\mathcal{F}}_{\mathbb{A}}(X)$. In particular $\mathcal{L}(\mathbb{A})$ are lists of elements from $\mathbb{A}$. Observe that $\mathcal{L}$ is a monad whose unit is the single-element list constructor $a \mapsto a :: (\mathsf{nil} :: \mathsf{nil} \ldots)$ and whose multiplication is concatenation of lists. ■

Sequents $P \triangleright G$ can be modelled analogously to Example 8.16. To begin, one models the grammar in Definition 8.1. There are several constructors in these grammar(s) that need to be distinguished (i.e., they cannot all simply be the product of the category). Therefore, we *sign* the products with symbols distinguishing the kind of pair. For economy in the notation, we shall use the logical constants both as syntactic object and as the name for their denotations — for example,

$$\wedge : (X,Y) \mapsto X \times \{\wedge\} \times Y \qquad \text{and} \qquad * : (X,Y) \mapsto X \times \{*\} \times Y$$

To simplify presentation, we use infix notation for these functors. The sets X and Y are variables upon which the construction takes place, and at the end of the construction they will be instantiated by $\mathbb{A}$.

Modelling the syntax of the hereditary Harrop fragment of BI is slightly more elaborate than in Example 8.16 because of the mutual induction taking place over definite clauses and goal formulate. Consequently, we first consider mappings of two variables delineating the structural difference of the two types of formulae:

$$\begin{array}{rcl}
\mathcal{F}_D(X,Y) & := & \{\top^*\} + \{\top\} + \mathcal{K}_{\mathbb{A}}(X) + Y \wedge Y + Y * Y + \\
 & & X \to Y + X \mathrel{-\!\!*} Y \\
\mathcal{F}_G(X,Y) & := & \{\top^*\} + \{\top\} + \mathcal{K}_{\mathbb{A}}(Y) + X \wedge X + X \vee X + X * X + \\
 & & Y \to X + Y \mathrel{-\!\!*} X
\end{array}$$

These definitions merit comparison with Definition 8.1.

We now model the unfolding of the inductive definition of definite clauses and goal formulae simultaneously. To this end, we use the first and second projection function π_1 and π_2 to put the right formulae in the right place. That is, we apply the free construction to the following:

$$\mathcal{F}(Z) := \mathcal{F}_G(\pi_1 Z, \pi_2 Z) \times \mathcal{F}_D(\pi_1 Z, \pi_2 Z)$$

Here Z is a product of some sets X and Y. The functors defining $\mathcal{F}$ are sufficiently simple and the category sufficiently well-behaved that the free construction yields a limit,

$$\begin{aligned}&Z \longrightarrow Z+\mathcal{F}Z = \\ &Z+\underbrace{\mathcal{F}_G\big(\pi_1(Z+\mathcal{F}Z),\, \pi_2(Z+\mathcal{F}Z)\big)}_{=\mathcal{F}_G(\pi_1 Z+\mathcal{F}_G Z, \pi_2 Z+\mathcal{F}_D Z)} \times \underbrace{\mathcal{F}_D\big(\pi_1(Z+\mathcal{F}Z),\, \pi_2(Z+\mathcal{F}Z)\big)}_{=\mathcal{F}_D(\pi_1 Z+\mathcal{F}_G(Z), \pi_2 Z+\mathcal{F}_D(Z))} \longrightarrow \ldots\end{aligned}$$

Each transition in the construction is the embedding of the previously constructed set within the next which contains it as a component of its disjoint union; for example, the first arrow embeds Z in $Z+\mathcal{F}(Z)$. This embedding is in fact a natural transformation $\mathcal{I} \to \mathcal{I}+\mathcal{F}$. Hence, as the construction continues more and more stages of the inductive definition of definite clauses and goal formulae are captured. The limit, therefore, contains all the possible goals and definite clauses at once.

Let $\hat{\mathcal{F}}$ denote the free functor for $\mathcal{F}$, then the goal formulae and definite clauses are recovered via the first and second projections,

$$\hat{\mathcal{F}}_G(X) := \pi_1 \hat{\mathcal{F}}(X,X) \qquad \hat{\mathcal{F}}_D(X) := \pi_2 \hat{\mathcal{F}}(X,X)$$

By fixing the set of atomic proposition $\mathbb{A}$ (i.e., the base of the inductive construction), the disjoint union present in the free construction means that all goal formulae and definite clauses are present in $\hat{\mathcal{F}}_G(\mathbb{A})$ and $\hat{\mathcal{F}}_D(\mathbb{A})$, respectively.

Modelling bunches is comparatively simple since there is no mutual induction, so it simply requires the free monad $\hat{\mathcal{B}}$ for the following functor:

$$\mathcal{B} := \hat{\mathcal{F}} + \fatsemi + , + \{\varnothing_\times\} + \{\varnothing_+\}$$

Configurations are pairs of programs and goals, so their abstract data-structure is given by the functor $\mathcal{G}(X) := \hat{\mathcal{B}}(X) \times \hat{\mathcal{F}}(X)$. In particular, configurations are modelled by elements of $\mathcal{G}(\mathbb{A})$.

This concludes the modelling of the syntax for the hereditary Harrop fragment of BI. Henceforth, we may use syntax and their denotations interchangeably. We turn now to the modelling of rules as reduction operators.

Example 8.17. *The configuration* $(A \wedge \top) \mathrel{-\!\!*} A \triangleright A$*, where* $A \in \mathbb{A}$*, is modelled by the* *tuple* $\big(((A,\wedge,\top),\mathrel{-\!\!*},A),A\big)$*, which is nothing more than the typical encoding of the syntax-tree as an ordered tree.* ■

A rule r in a sequent calculus can be understood mathematically as relation $\mathbf{r}$ on sequents that holds only when the first sequent (the conclusion) is inferred from the remaining (the premisses) by the rule.

Example 8.18. *The rule* e *in* LBI*, it is expressed by the following rule figure with the condition* $\Delta \equiv \Delta'$*:*

$$\frac{\Delta \rhd \varphi}{\Delta' \rhd \varphi}\,\mathsf{e}$$

Observe that $\varnothing_\times \equiv \varnothing_\times ; \varnothing_\times$*, but* $\varnothing_\times \not\equiv \varnothing_\times \fatsemi \varnothing_\times$*. Therefore, this rule uderstood as a relation* $\boldsymbol{e}$*, we obtain* $(\varnothing_\times \rhd A)\,\boldsymbol{e}\,((\varnothing_\times ; \varnothing_\times)\rhd A)$*, but do not obtain* $(\varnothing_\times \rhd A)\,\boldsymbol{e}\,((\varnothing_\times \fatsemi \varnothing_\times)\ \rhd A)$*.* ■

These relations, in turn, can be canonically understood as a non-deterministic partial functions, *deduction operators*. A relation $\mathbf{r}$ generated by a rule r yields the function $\delta_{\mathsf{r}} : \mathcal{L}(\mathcal{G}(\mathbb{A})) \rightharpoonup \mathscr{P}(\mathcal{G}(\mathbb{A}))$,

$$\delta_{\mathsf{r}} : \ell \mapsto \{(P,G) \in \mathcal{G}(\mathbb{A}) \mid (P,G)\,\mathbf{r}\,\ell\}$$

However, in Reductive Logic, we are interested in the inverse action. Maps $\delta_{\mathsf{r}}^{-1} : \mathcal{G}(\mathbb{A}) \to \mathscr{P}(\mathscr{P}(\mathcal{G}(\mathbb{A})))$, defined as follows:

$$\delta^{-1} : (P,G) \mapsto \{\ell \in \mathscr{P}(\mathcal{G}(\mathbb{A})) \mid \delta(\ell) = (P,G)\}$$

These map a putative conclusion to sufficent premisses.

Example 8.19. *Let* r *be the* $\to$*-resolution rule. Its deduction operator and its inverse are as follows:*

$$\begin{array}{rcll} \delta & : & \{(P;G\to A \rhd G),(Q\rhd A)\} & \mapsto \quad (Q,(P;G\to A))\rhd A \\ \delta^{-1} & : & (Q,(P;G\to A))\rhd A & \mapsto \quad \{(P;G\to A\rhd G),(Q\rhd A)\} \end{array}$$

Recall that bunches are represented in canonical form. ■

In this analysis we recognise that deduction operators are intuitively understood as algebras for the $\mathscr{P}$-functor, and reduction operators as coalgebras for the $\mathscr{P}\mathscr{P}$-functor. We make now an important assumption for proof-search, as presented herein, to be effective: reduction operators are *finitely branching*. That is, proceeding onward, we exclude rules whose upward reading renders an infinite set of potential collections of sufficient premisses. At first this assumption seems to be hugely restrictive, since almost all common proof-systems seemingly fail this criterion; for example, e does not satisfy this condition! What is required is a means to control the rule; for example, the e rule is confined in the resolution system (Figure 8.1) to the weak coherence ordering in Section 8.1 which is used in a controlled way. Hence, we drop the $\mathscr{P}$ structure in deduction operators and their inverses, the reduction operators, and replace it with the finite powerset monad $\mathscr{P}_f$.

Definition 8.20 (Reduction Operator)**.** *A reduction operator over sequents $\mathcal{G}(\mathbb{A})$ is a coalgebra $\rho : \mathcal{G}(\mathbb{A}) \to \mathscr{P}_f\mathscr{P}_f\mathcal{G}(\mathbb{A})$.*

In this setup, the use of a reduction operators in Reductive Logic is the application of the coalgebra together with a choice function that instantiates a particular set of sufficient premisses from the available,

$$\begin{tikzcd} \mathcal{G}(\mathbb{A}) \arrow[r, "\rho"] & \mathscr{P}_f\mathscr{P}_f\mathcal{G}(\mathbb{A}) \arrow[r, "\sigma"] & \mathscr{P}_f\mathcal{G}(\mathbb{A}) \end{tikzcd}$$

The choices presented by these steps represent the control problems of proof-search, as discussed in Chapter 1; that is, the choice of rule, and the choice of instance.

When the construction in Example 8.19 is performed for the resolution system in Figure 8.1, with the simplification on the structure of states and collections of premisses, one forms the coalgebras in Figure 8.3. These are the *reduction operators* for the resolution system understood as coalgebras.

This presentation already offers insight into the proof-search behaviour. For example, the disjointedness of the defined portions of the reduction operators for the operational and initial rules means the choice of application is deterministic for a non-atomic goal. Therefore, one may coalesce the operators into a single reduction

$$
\begin{array}{rlcl}
\rho_{\top} : & P \triangleright \top & \mapsto & \{\varnothing\} \\
\rho_{\top^*} : & P \triangleright \top^* & \mapsto & \{\varnothing \mid [P] \equiv \varnothing_{\times};R\} \\
\rho_{\mathsf{res}_1} : & P \triangleright A & \mapsto & \{\{Q \triangleright A\} \mid [P] \geqq Q,A\} \\
\rho_{\mathsf{res}_2} : & P \triangleright A & \mapsto & \{\{Q \triangleright G\} \mid [P] \geqq Q, G \mathrel{-\!\!*} A\} \\
\rho_{\mathsf{res}_3} : & P \triangleright A & \mapsto & \{\{Q \triangleright \top^*, R;G \to A \triangleright G\} \mid [P] \geqq Q, (R;G \to A)\} \\
\rho_{\vee} : & P \triangleright G_1 \vee G_2 & \mapsto & \{\{P \triangleright G_1\}, \{P \triangleright G_2\}\} \\
\rho_{\wedge} : & P \triangleright G_1 \vee G_2 & \mapsto & \{\{P \triangleright G_1, P \triangleright G_2\}\} \\
\rho_{*} : & P \triangleright G_1 * G_2 & \mapsto & \{\{Q \triangleright G_1, R \triangleright G_2\} \mid P \equiv (Q,R);S\} \\
\rho_{\to} : & P \triangleright D \to G_2 & \mapsto & \{\{P;[D] \triangleright G\}\} \\
\rho_{\mathrel{-\!\!*}} : & P \triangleright D \mathrel{-\!\!*} G_2 & \mapsto & \{\{P,[D] \triangleright G\}\}
\end{array}
$$

Figure 8.3: Coalgebraic Representation of RBI

operator as a goal destructor:

$$\rho_{op} := \rho_{\vee} + \rho_{\wedge} + \rho_{*} + \rho_{\to} + \rho_{\mathrel{-\!\!*}} + \rho_{\top} + \rho_{\top^*}$$

Moreover, since there is no *a priori* way to know which resolution rule to use, one in principle tests all of them, so uses a reduction operator of the shape:

$$\rho_{\mathsf{res}} : P \triangleright A \mapsto \rho_{\mathsf{res}_1}(P \triangleright A) \cup \rho_{\mathsf{res}_2}(P \triangleright A) \cup \rho_{\mathsf{res}_3}(P \triangleright A)$$

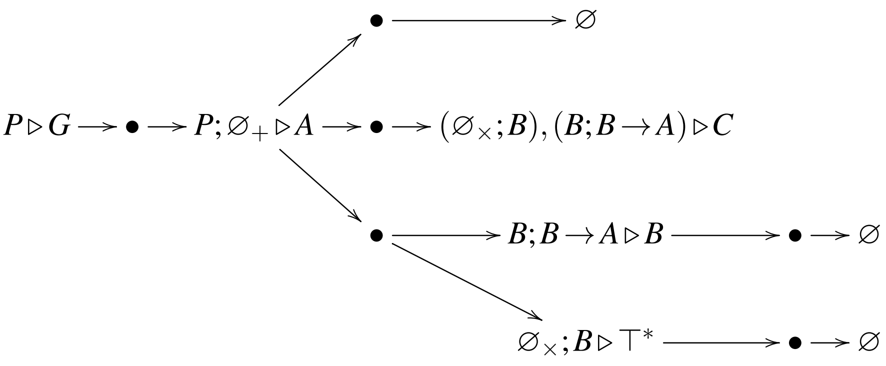
The presentation of ρ_{op} and ρ_{res} as operators is simple when working with coalgebras, but it is not at all clear how to present them as a single rule figures. Attempts toward such presentations are the *synthetic* rules derived from focused proof systems — see, for example, Chaudhuri et al. [35, 34].

The Proof-search Space

The construction of reduction operators as above is an interpretation of a reduction step. We now turn to modelling reduction proper; that is, we construct a coalgebraic interpretation of the *proof-search space* — the structure explored during proof-search. In practice, we simply formalize the heuristic exploratory approach of reductive inference as the corecursive application of reduction operators.

Computations, such as proof-search, can be understood as sequences of actions, and the possible traces can be collected into tree structures where different paths represent particular threads of execution. In logic programming, such trees appear in the literature as *coinductive derivation trees* (CD-trees) — see, for example, Komendantskaya et al. [115, 116, 117], and Bonchi and Zanasi [23] — and an *action* is one step of reductive inference. Typically one distinguishes the two components, the reduction operator and the choice function, by using an intermediary node labelled with a •, sometimes called an *or*-node, as it represents the disjunction of sets of sufficient premisses.

Example 8.21. *Let $P = ((\varnothing_\times;B), C \mathrel{-\!\!*} A, (B;B \to A));A$ and $G = \top \to A$. The CD-tree for $P \triangleright G$ in the resolution system is the following:*


$\bullet \to \varnothing$
$P \triangleright G \to \bullet \to P;\varnothing_+ \triangleright A \to \bullet \to (\varnothing_\times;B),(B;B \to A) \triangleright C$
$\bullet \to B;B \to A \triangleright B \to \bullet \to \varnothing$
$\varnothing_\times;B \triangleright \top^* \to \bullet \to \varnothing$

At the first bifurcation point, the three • nodes represent from top to bottom the choice of the unique member of $\rho_{\mathsf{res}_i}(P;\varnothing_\times \triangleright A)$ for $i = 1,2,3$. In the case of ρ_{res_1} and ρ_{res_3}, the procedure continues and terminates successfully; meanwhile, for ρ_{res_2}, it fails. ■

The bullets • serve only as *punctuation* separating the possible choice functions, so the actual coinductive derivation tree is the tree without them.

Definition 8.22 ((Punctuated) Coinductive Derivation Tree)**.** *A punctuated coinductive derivation tree (PCD-tree) for a sequent S is a tree satisfying the following:*

- *The root of the tree is S*

- *The root has $|\rho(S)|$ children labelled $\bullet$*

- *For each $\bullet$ there is a unique set $\{S_0,...,S_n\} \in \rho(S)$ so that the the children are PCD-trees for the S_i.*

A coinductive derivation (CD-tree) is the tree constructed from the PCD-tree by connecting the parents of $\bullet$-nodes directly to the children, removing the node itself.

The CD-trees model reduction only (as opposed to proof-search) since the representation of a control regime is lacking. It remains to give a coalgebraic account of CD-trees, which we do following Komendantskaya et al. [115, 116, 117] — see also work by Bonchi and Zanasi [23].

The CD-structure on a set X of sequents is formally the cofree comonad $\mathcal{C}(X)$ on the $\mathscr{P}_f\mathscr{P}_f$ functor, the behaviour-type of reduction. It is constructed inductively as follows:

$$\begin{cases} Y_0 & := X \\ Y_{\alpha+1} & := X \times \mathscr{P}_f\mathscr{P}_f Y_\alpha \end{cases}$$

Each stage of the construction yields a coalgebra $\rho_\alpha : X \to Y_\alpha$ defined inductively as follows, where I is the identify function:

$$\begin{cases} \rho_0 & := I \\ \rho_{\alpha+1} & := I \times (\mathscr{P}_f\mathscr{P}_f \rho_\alpha \circ \rho) \end{cases}$$

For some limit ordinal λ — see Worrell [216] — the coalgebra $\rho_\lambda : \mathcal{G}(\mathbb{A}) \to \mathcal{C}(\mathcal{G}(\mathbb{A}))$ precisely maps a configuration to its CD-tree.

To show that this model of the proof-search space is faithful we must show that every step, represents a valid reduction; meanwhile, to show that it is adequate we must prove that every proof is present. A proof is witnessed in a CD-tree by choosing a particular path.

Definition 8.23 (Controlled Subtree of CD-tree)**.** *A subtree $\mathcal{R}$ of $\rho_\lambda(P,G)$ is controlled iff it is a tree extracted from the PCD-tree for $P \triangleright G$ by taking the root node* *and connecting it to the heads of the reduction trees of all the children of one •-node. It is successful if and only if the leaves are all $\varnothing$.*

Observe that the application of a choice function, determined by a control regime, is precisely the choosing of a particular •-node at each stage of the extraction.

Example 8.24. *The following is an example of a controlled sub-tree from the example (P)CD-tree above:*

$$\begin{tikzcd}
P \triangleright G \arrow[r] & P;\varnothing_{\times} \triangleright A \arrow[r] & \varnothing
\end{tikzcd}$$

The first choice of • is trivial (as there is only one) and the second choice is the upper path. ▪

We do not claim that every controlled sub-tree in a CD-tree is finite; in fact, this is demonstrably not the case.

Example 8.25. *Consider the PCD-tree for $A;A \to A \triangleright A$:*

$$\begin{tikzcd}
A \to A;A \triangleright A \arrow[r] & \bullet \arrow[r] \arrow[dr] & A \to A;A \triangleright A \arrow[r] & \bullet \arrow[r] \arrow[dr] & A \to A;A \triangleright A \arrow[r] & \dots \\
& & \varnothing & & \varnothing &
\end{tikzcd}$$

Every finite execution of the configuration is successful; however, there is an infinite path which represents an attempt at proof-search that never terminates, but also never reaches a invalid configuration. This further demonstrates the care that is required when implementing controls because the depth-first search with leftmost selection *regime here fails.* ▪

Theorem 8.26 (Soundness and Completeness)**.** *A tree labelled with sequents is a proof of a configuration $P \triangleright G$ in the resolution system if and only if it is a successful controlled sub-tree of $\rho_\lambda(P,G)$.*

Proof. Immediate by induction on the height of proofs and the definition of reduction tree. □

Choice and Control

The difference between modelling the proof-search space and modelling proof-search is subtle. It comes down to whether or not control admits a denotation. We have thus far restricted attention to a traditional sequential paradigm for computation, where one chooses one particular collection of sufficient premisses; now, we now show that an alternative parallel model is immediately available from the constructions above.

Recall that we have a two-step understanding of reductive inference: first, apply a reduction operator (understood as a coalgebra); then choose a collection of sufficient premisses (by applying a choice function). The usual way to interpret it is to perform these action in sequence as read, yielding the backtracking-schedule approach to computation studied so far; however, it may be interpreted simply *extracting* a correct reduction from the proof-search space. In this latter reading the traces of the proof-search space can be understood as being in a superposition, forming a parallel semantics of computation. This reading has been thoroughly studied for Horn clause logic programming by Gupta et al. [90, 91, 89] and this idea of parallelism can be captured proof-theoretically by hypersequent calculi —see, for example, work by Harland and Kurokawa [98, 127].

The coalgebraic model of the proof-search space immediately offers a coalgebraic model of parallel proof-search; that is, the controlled sub-tree extraction from $\rho_\lambda : \mathcal{G}(\mathbb{A}) \to \mathcal{G}(\mathcal{G}(\mathbb{A}))$. Indeed, this is precisely analogous to the parallel model of Horn clause logic programming studied by Komendantskaya et al. [115, 116, 117]. The coalgebraic approach has the advantage over more traditional algebraic models in that it allows for infinite searches, thereby extending the power of logic programming to include features such as corecursion.

Indeed, the parallel semantics is amenable to a more accurate model by unpacking the algebraic structure of the state-space, yielding a bialgebraic semantics. Observe then that in the structure $\mathscr{P}_f\mathscr{P}_f$ for reduction operators, the external functor structures the set of choices, and the internal one structures the states themselves (i.e., the collections of sufficient premisses). The outer one is disjunctive, captured diagrammatically by the *or*-nodes represented by • in the PCD-trees; meanwhile, the inner one is conjunctive since every premiss needs to be verified. There is no distributive law of $\mathscr{P}_f$ over $\mathscr{P}_f$, but there is a distributive law of $\mathcal{L}$ over $\mathscr{P}_f$ which coheres with this analysis:

$$[X_1, ..., X_N] \mapsto \{[x_1, ..., x_n] \mid x_i \in X_i\}$$

This suggests a bialgebraic model for the parallel reading of the operational semantics obtained by performing the same cofree comonad construction for the behaviour, but with $\mathscr{P}_f\mathcal{L}$ instead. This has already been studied in the case of Horn clause logic programming by Bonchi and Zanasi [23].

These models are studied partly to let one reason about computation, and perhaps use knowledge to improve behaviour. For example, in the sequential semantics a programmer may purposefully tailor the program to the selection function to have better behaviour during execution, meanwhile in the parallel approach the burden is shifted to the machine (or, rather, theorem prover) which may give more time to branches that are more promising. For example, while generating the (P)CD-tree in Example 8.21 a theorem prover can ignore the branch choosing the $C \multimap A$ in the program since it is clear that it will never be able to justify C as the atom appears nowhere else in the context.

The problem being handled in either case is how best to explore the space of possible reductions. The two approaches, parallel and sequential, both suffer from the *amount* of non-determinism in the system. In fact, this problem is exponentially increasing with each inference made as each collection of premisses represents another branch in the CD-tree. Moreover, in practice, with any additional features in the logic the problem compounds so that such reasoning becomes increasingly intractable.

8.4 Computation

So far we have given an operational reading of hHBI. Taking it as a programming language, however, there ought to be something that it *computes*. Traditionally —

for example, in the setting of Horn clause logic programming — what is comuted is a unifier which states what object satisfy the condition of the goal according to a program. This demands extending present setup to a predicate logic. ‘

Predicate hereditary Harrop BI

The standard approach for turning a propositional logic into predicate logic begins with the introduction of a set of terms $\mathbb{T}$, typically given by a context-free grammar which has three disjoint types of symbols: variables, constants, and functions. The propositional letters are then partitioned $\mathbb{A} := \bigcup_{i<\omega} \mathbb{A}_i$ into classes of predicates/relations of different arities, such that the set of atomic formulae is given by elements $A(t_0,...,t_i)$, where $t_0,...,t_i$ are terms and $A \in \mathbb{A}_i$. In the model theory, the Herbrand universe is the set of all ground terms $\overline{\mathbb{T}}$ (terms not containing free variables), and the Hebrand base is the set of all atomic formulae (instead of atomic propositions).

The extra expressivity of predicate logic comes from the presence of two quantifiers: the universal quantifier $\forall$ and the existential quantifier $\exists$, which for the hereditary Harrop fragment of BI gives the following grammar for formulae:

$$\begin{array}{rcl} D & ::= & ... \mid \forall x(G \to A) \mid \forall x(G \mathrel{-\!\!*} A) \\ G & ::= & ... \mid \exists x G \end{array}$$

Formally, programs and goals are not constructed out of arbitrary formulae, but only out of *sentences*: formulae containing no free-variables. However, since in this fragment the quantifiers are restricted to different types of formulae (and the sets of variables and constants are disjoint) they may be suppressed without ambiguity. For example, the formulae $A(x)$ regarded as a goal is unambiguously existentially quantified, whereas when regarded as a definite clause it is unambiguously universally quantified.

Rules for the quantifiers require the use of a mappings from $\theta : \mathbb{T} \to \overline{\mathbb{T}}$ that are fixed on $\overline{\mathbb{T}} \subseteq \mathbb{T}$, which are uniquely determined by their assignment of variables.

$$\begin{array}{lll}
P \vdash_{\mathrm{o}} \exists x G & \text{if} & P \vdash_{\mathrm{o}} G\theta \text{ for some susbtitution } \theta \\
P \vdash_{\mathrm{o}} A & \text{if} & P \geqq R; (Q, \forall x (G \mathrel{-\!\!*} B)) \text{ and} \\
 & & \text{there is } \theta \text{ such that } Q \vdash_{\mathrm{o}} G\theta \text{ and } A\theta = B\theta \\
P \vdash_{\mathrm{o}} A & \text{if} & P \geqq (Q, (R; \forall x (G \to B))) \text{ and } Q \vdash_{\mathrm{o}} \top^* \text{ and} \\
 & & \text{there is } \theta \text{ such that } R; \forall x (G \to B) \vdash_{\mathrm{o}} G\theta \text{ and } A\theta = B\theta
\end{array}$$

Figure 8.4: Unification in Predicate Hereditary Harrop BI

Such a function becomes a *substitution* under the following action:

$$\varphi\theta := \begin{cases} A(\theta(t_0), ..., \theta(t_n)) & \text{if } \varphi = A(t_0, ..., t_n) \\ \psi_0\theta \circ \psi_1\theta & \text{if } \varphi := \psi_0 \circ \psi_1 \text{ for any } \circ \in \{\land, \lor, \to, *, \mathrel{-\!\!*}\} \\ \varphi & \text{if } \varphi \in \{\top, \top^*\} \end{cases}$$

The resolution system (Figure 8.1) is thus extended with the operators in Figure 8.4 which incorporate the quantifier rules. Observe that substitution is used to match a definite clause with the goal, and for this reason is traditionally called a *unifier*. Since execution is about finding some term (some element of the Herbrand universe) which satisfies the goal, one may regard the thing being computed as the combined effect of the substitutions witnessed along the way, often called the *most general unifier*.

The introduction of quantifiers into the logic programming language offered here is minimal, and much more development is possible — see, for example, work by Armelín [9, 10]. An attempt at a full predicate BI has been given by Pym [170, 171], but its metatheory is not currently adequate.

The intuition follows from the intimate relationship between implication and quantification in intuitionistic logic — see, for example, Dummett [52]. The intended reading of an implication $A \to B$ in BI is a constructive claim of the existence of a procedure which turns a proof A into a proof of B. Therefore, a proof of an existential claim $\exists x A(x)$ involves generating (or showing how to generate) an object t for which one can prove $A(t)$; similarly, a proof of a universal claim $\forall x A(x)$ is an procedure which takes any object t and yields a proof of $A(t)$.

The presence of both additive ($\rightarrow$) and multiplicative ($-\!\!*$) implications in BI results in the possibility of both additive (resp. $\{\exists, \forall\}$) and multiplicative (resp. $\{\exists_{\text{new}}, \forall_{\text{new}}\}$) quantifiers. Intuitively, the difference between $\forall$ and $\forall_{\text{new}}$ is that for $\forall_{\text{new}} x \varphi$ makes a claim about objects *x separate* from any other term appearing in φ, thus it may be read*for all new*; similarly for the relationship between the $\exists$ and $\exists_{\text{new}}$ quantifiers. This behaviour is similar to the freshness quantifier from nominal logic, which is the familiar universal quantifier together with an exclusivity condition, and has a well understood metatheory — see Pitts [160].

Example: Databases

We have thus developed a concept of logic programming using BI. Our motivations are mathematical and technical, but they have practical consequences, which we illustrate presently. In this section, we show that the concept of LP for BI in this chapter is interesting and useful for working with databases.

Logic programming has historically had a profound effect on databases both theoretically, providing a logical foundation, and practically, by extending the power to incorporate reasoning capabilities — see, for example, Grant [88]. Standard relational database systems are the fundamental information storage solution in data management, but have no reasoning abilities meaning information is either stored explicitly or is not stored at all. One may combine a database with a logic prohramming language resulting in a *deductive* database, which extends the capabilities of such relational databases to included features such as multiple file handling, concurrency, security, and inference.

A deductive database combines two components. The *extensional* part contains the atomic facts (ground atoms), and is the type of data that can exist in a relational database; meanwhile the *intensional* part contains inference rules and represents primitive reasoning abilities relative to a knowledgebase. In the case of the predicate version of the hereditary Harrop fragment of BI, if there are no recursive rules in the intensional database then it corresponds to views in a relational database. However,

Column 1	Column 2	Column 3
Al(gebra)	Lo(gic)	Da(tabases)
Pr(obability)	Ca(tegories)	Co(mpilers)
Gr(aphs)	Au(tomata)	AI

Figure 8.5: Informatics Electives at Unseen University

even without recursion the two connectives offers extra abilities as demonstrated by the following example.

Suppose Unseen University offers a computer science course. To have a well-rounded education, students must select one module from each of three columns in Figure 8.5, with the additional constraint that to complete the course students must belong to a particular stream, A or B. Stream A contains $Al, Gr, Lo, Ca, Au, Co, AI$, and students must pick one from each column, stream B contains the complement. This compatibility information for modules may be stored as an extensional database provided by the bunch ED $=$ (Col1,Col2,Col3) with each Col$i$ bunch defined as follows:

$$\begin{aligned}
\mathrm{Col1} &:= \mathrm{A(Al)}\ ;\ \mathrm{A(Gr)}\ ;\ \mathrm{B(Pr)}\ ;\ \mathrm{B(Gr)} \\
\mathrm{Col2} &:= \mathrm{A(Lo)}\ ;\ \mathrm{A(Ca)}\ ;\ \mathrm{A(Au)}\ ;\ \mathrm{B(Ca)}\ ;\ \mathrm{B(Au)} \\
\mathrm{Col3} &:= \mathrm{A(Co)}\ ;\ \mathrm{A(AI)}\ ;\ \mathrm{B(Da)}\ ;\ \mathrm{B(Co)}\ ;\ \mathrm{B(AI)}
\end{aligned}$$

Let x be a list of subjects, then the logic determining CS courses for the $Astr$(eams) and $Bstr$(eams) respectively is captured by an intensional database ID given by the following bunch, where π_i is the ith projection function:

$$Astr\big(\pi_0(x),\pi_1(x),\pi_2(x)\big) \to str(x) \quad ; \quad Bstr\big(\pi_0(x),\pi_1(x),\pi_2(x)\big) \to str(x) \quad ;$$
$$A(x) * A(y) * A(z) \mathrel{-\!\!*} Astr(x,y,z) \quad ; \quad B(x) * B(y) * B(z) \mathrel{-\!\!*} Bstr(x,y,z)$$

The equivalent implementation in the predicate version of hHBI would require a tagging system to show compatibility of the columns; meanwhile the computation can be handled easily and (more importantly) *logically* in the present system.

Chapter 9

Focused Proof-search

The *focusing principle* was introduced for *Linear Logic* (LL) by Andreoli [6]. It is a generalization of uniform proof-search, the procedure underlying standard approaches to logic programming — see Chapter 2 and Chapter 8. In this chapter, we prove that LBI satisfies the focusing principle by showing the soundness and completeness of a *focused calculus* FBI in which the focusing control regime is enforced by control symbols. This establishes the property in LBI because FBI-proofs are canonically turned into *focused* LBI-proofs. This chapter is based on the following paper:

> Gheorghiu, A. V., and Marin, S. Focused Proof-search in the Logic of Bunched Implications. In *Foundations of Software Science and Computation Structures - FOSSACS 24* (2021), S. Kiefer and C. Tasson, Eds., vol. 12650 of *Lecture Notes in Computer Science*, Springer, pp. 247–267

Focused proof-search is characterised by alternating *focused* and *unfocused* phases of goal-directed proof-search. This alternation can be enforced by a partition of the set of formulas into two classes, *positive* and *negative*. For negative formulas, provability is invariant with respect to the application of a right rule; and for positive formulas, provability is invariant with respect to the application of a left rule. The unfocused phases of a focused proof comprise the exhaustive (reductive) application of rules which are *safe* to apply (e.g., $\wedge_{\mathsf{R}}$ in LJ — see Chapter 2); conversely, the focused phase comprises the hereditary reduction (i.e., focused reduction) of a formula and its sub-formulas where potentially invalid sequents may arise, and backtracking may be required (e.g., $\vee_{\mathsf{R}}$ in LJ — see Chapter 2). During focused proof-search the unfocused phases are performed eagerly, followed by controlled goal-directed focused phases, until safe reductions are available again. A sequent calculus satisfies the focusing principle when every provable sequent admits a focused proof in that calculus.

The original proof of the focusing principle in LL by Andreoli [6] was via long and tedious permutations of rules. In this chapter, we use a different methodology that was originally presented by Laurent [130] and has since been implemented for a variety of logics — see, for example, work by Liang and Miller [131], and Chaudhuri et al. [34, 35]. The method is as follows: given a sequent calculus, first one polarises the syntax according to positive and negative behaviours; second, one gives a focused variation of the sequent calculus in which the control flow of proof-search is managed by polarization; third, one shows that this system admits cut; and, finally, one shows that in the presence of cut the original sequent calculus may be simulated in the focused one. When the focused system is complete with respect to the logic, the focusing principle holds.

In LBI certain rules (the structural rules) have no natural placement in either the focused or the unfocused phases of proof-search. Thus, a design choice must be made: to eliminate/constrain these rules, or to permit them without restriction. In this paper, we choose the former as our motivation is to study computational behaviour of proof-search in BI, the latter being recovered by familiar admissibility results. The only case where confinement is not possible is the *exchange* rule. In standard sequent calculi the exchange rule is made implicit by working with a more convenient data-structure such as multisets as opposed to lists; however, the specific structure of bunches in BI means that a more complex alternative is required. The solution presented is to use *nested multisets of two kinds* (additive and multiplicative) corresponding to the two different context-formers and conjunctions.

9.1 Re-presenting the Logic of Bunched Implications

In this section, we define the space $\mathbb{B}/_{\equiv}$ (i.e., bunches modulo coherent equivalence) as containing *nested multisets of two kinds* — henceforth, *nests*. This reading was suggested by Donnelly [50], though never formally realised. It allows us to suppresses uses of exchange in proof-search. We expect no obvious difficulty in studying focused proof-search with bunches instead of nested multisets; the design choice is simply to reduce the complexity of the argument by supressing all uses of exchange.

For readability, henceforth we use Γ to denote nests and to denote Δ for bunches.

Definition 9.1 (Two-sorted Nest)**.** *Nests* (Γ) *are formulas or multisets, ascribed either additive* (Σ)*, or multiplicative* (Π) *kind, containing nests of the opposite kind:*

$$\Gamma := \Sigma \mid \Pi \qquad \Sigma := \varphi \mid \{\Pi_1, ..., \Pi_n\}_+ \qquad \Pi := \varphi \mid \{\Sigma_1, ..., \Sigma_n\}_\times$$

The constructors are signed multiset constructors which may be empty in which case the nests are denoted $\varnothing_+$ and $\varnothing_\times$, respectively. No multiset is a singleton. The set of all nests is denoted $\mathbb{B}/_{\equiv}$.

Let Λ and Γ be nests. We write $\Lambda \in \Gamma$ to denote one of the following: either $\Lambda = \Gamma$, when Γ is a formula; or Λ is an element of Γ regarded as a multiset. We write $\Lambda \subseteq \Gamma$ to denote that Λ is a sub-multiset of Γ — that is, if $\gamma \in \Lambda$ and $\gamma \neq \Lambda$, then $\gamma \in \Gamma$. We write $\Gamma\{\Lambda\}_+$ (resp. $\Gamma\{\Lambda\}_\times$), to denote that Λ is a sub-nest of Γ of additive (resp. multiplicative) kind, and may write $\Gamma\{\Lambda\}$ when the kind is not specified. In either case $\Gamma\{\Lambda'\}$ denotes the substitution of Λ for Λ'. A promotion in the syntax tree may be required after a substitution either to handle a singleton or an improper alternation of constructor types, as witnessed in the following example:

Example 9.2. *The following inclusions are valid,*

$$\{\varphi, \chi\}_\times \in \Big\{\{\varphi, \chi\}_\times, \psi\Big\}_+ \subseteq \Big\{\{\varphi, \chi\}_\times, \psi, \psi, \varnothing_\times\Big\}_+ = \Gamma\{\{\varphi, \chi\}_\times\}_+$$

We write $\Gamma\{\{\varphi,\varphi\}_+\}_+$ *to denote the substitution yielding* $\{\varphi,\varphi,\psi,\psi,\varnothing_\times\}_+$. *Note the absence of the* $\{\cdot\}_+$ *constructor after substitution, this is due to a promotion in the syntax tree to avoid having two nested additive constructors. Similarly, since* $\varnothing_\times$ *denotes the empty multiset of multiplicative kind, substituting* χ *with it gives* $\{\varphi,\psi,\psi,\varnothing_\times\}_+$ *— that is, first the improper* $\{\varphi,\varnothing_\times\}_\times$ *becomes* $\{\varphi\}_\times$*; then, the resulting singleton* $\{\varphi\}_\times$ *is promoted to* φ. ∎

We have the following abuse of notation, where $\circ \in \{+,\times\}$, to denote promotions:

$$\Gamma\{\{\Pi_1,...,\Pi_i\}_\circ,\Pi_{i+1},..,\Pi_n\}_\circ := \Gamma\{\Pi_1,...,\Pi_n\}_\circ$$

The following are useful for analysing the structure of nests:

Definition 9.3 (Depth)**.** *The depth* δ *of a nest* Γ *is as follows:*

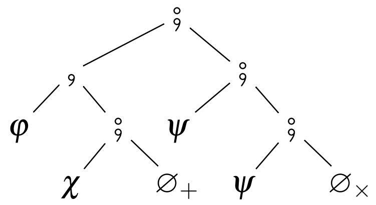
$$\delta(\Gamma) := \begin{cases} 0 & \text{\emph{if}}\Gamma \in \mathbb{F} \cup \{\varnothing_\times,\varnothing_+\} \\ \max\{\delta(\Gamma_1),...,\delta(\Gamma_n)\}+1 & \text{\emph{if}}\ \Gamma = \{\Gamma_1,...,\Gamma_n\}_+ \\ \max\{\delta(\Gamma_1),...,\delta(\Gamma_n)\}+1 & \text{\emph{if}}\ \Gamma = \{\Gamma_1,...,\Gamma_n\}_\times \end{cases}$$

It remains to relate bunches and nests. To this end, it will be useful to have a measure on sub-bunches which can identify their distance from the root node:

Definition 9.4 (Rank)**.** *If* Δ' *is a sub-bunch of* Δ*, then* $\rho(\Delta')$ *is the number of alternations of additive and multiplicative context-formers between the principal context-former of* Δ'*, and the root context-former of* Δ.

Let Δ be a complex bunch, we write $\Delta' \in \Delta$ to denote that Δ' is a (proper) top-most sub-bunch; that is, Δ is a sub-bunch satisfying $\Delta \neq \Delta'$ but $\rho(\Delta') = 0$.

Example 9.5. *Let* $\varphi,\psi,\chi \in \mathbb{F}$ *and let* $\Delta = (\varphi\,,(\chi\,;\varnothing_+))\,;(\psi\,;(\psi\,;\varnothing_\times))$. *Consider the parse-tree of* Δ,



Reading upward from $\varnothing_+$ one encounters first $\fatsemi$ which changes into $;$ and then back to $\fatsemi$, thus $\rho(\varnothing_+) = 2$; whereas, counting up from $\varnothing_\times$ one only encounters $\fatsemi$, thus $\rho(\varnothing_\times) = 0$. Hence, $\psi, \varnothing_\times, (\varphi \,;(\chi \,; \varnothing_\times)) \in \Delta$. ∎

The equivalence of bunches and nests is captures in terms of a canonical translation between them, witnessed by a *nestifying* function η and a *bunching* function β, which are inverses of each other. The transformation β is simply going from a tree with arbitrary branching to a binary one, and η is the reverse.

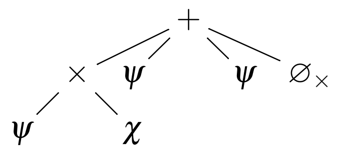
Definition 9.6 (Canonical Translation)**.** *The canonical translation $\eta : \mathbb{B} \to \mathbb{B}/_\equiv$ is defined recursively as follows,*

$$\eta(\Delta) := \begin{cases} \Delta & \textit{if } \Delta \in \mathbb{F} \cup \{\varnothing_+, \varnothing_\times\} \\ \{\eta(\Delta') \in \mathbb{B}/_\equiv \mid \rho(\Delta') = 1 \textit{ and } \Delta' \in \mathbb{B}^\times\}_+ & \textit{if } \Delta \in \mathbb{B}^+ \\ \{\eta(\Delta') \in \mathbb{B}/_\equiv \mid \rho(\Delta') = 1 \textit{ and } \Delta' \in \mathbb{B}^+\}_\times & \textit{if } \Delta \in \mathbb{B}^\times \end{cases}$$

The canonical translation $\beta : \mathbb{B}/_\equiv \to \mathbb{B}$ is defined recursively as follows,

$$\beta(\Gamma) := \begin{cases} \Gamma & \textit{if } \Gamma \in \mathbb{F} \cup \{\varnothing_+, \varnothing_\times\} \\ \beta(\Pi_1); (\beta(\Pi_2); ...) & \textit{if } \Gamma = \{\Pi_1, \Pi_2, ...\}_+ \\ \beta(\Sigma_1), (\beta(\Sigma_2), ...) & \textit{if } \Gamma = \{\Sigma_1, \Sigma_2, ...\}_\times \end{cases}$$

Example 9.7. *Applying η to the bunch in Example 9.5 gives the nest in Example 9.2:*



∎

That η and β are correct in terms of relating bunches to bunches modulo coherent equivalence is witnessed by the following result:

Proposition 9.8. *The following hold:*

1. *if* $\Delta \in \mathbb{B}$ *then* $(\beta \circ \eta)(\Delta) \equiv \Delta$*;*

2. *if* $\Gamma \in \mathbb{B}/_{\equiv}$ *then* $(\eta \circ \beta)(\Gamma) \equiv \Gamma$*;*

3. *let* $\Delta, \Delta' \in \mathbb{B}$*, then* $\Delta \equiv \Delta'$ *if and only if* $\eta(\Delta) = \eta(\Delta')$.

Proof. Statements 1 and 2 follow by induction on the depth and rank, where one must take care to consider the case of a context consisting entirely of units. Statement 3 employs the first in the forward direction, and proceeds by induction on depth in the reverse direction. □

Being able to systematically recast bunches as nests means that we can re-present LBI in terms of the latter, without loss of expressive power. The result is ηLBI.

Definition 9.9 (System ηLBI)**.** *The nested sequent calculus* ηLBI *is composed of the rules in Figure 9.1, where the metavariables denote possibly empty nests.*

Observe the use of metavariable Γ' instead of Π (resp. Σ) as sub-nests in Figure 9.1. This allows classes of inferences such as

$$\dfrac{\{\Sigma_0,\ldots,\Sigma_i\}_\times \rhd \varphi \quad \{\Sigma_{i+1},\ldots,\Sigma_n\}_\times \rhd \varphi}{\{\Sigma_0,\ldots,\Sigma_n\}_\times \rhd \varphi * \psi}\ {*}\mathsf{R}$$

to be captured by a single figure because we identify $\{\Sigma_0,\ldots,\Sigma_i\}_\times$ and $\{\Sigma_{i+1},\ldots,\Sigma_n\}_\times$ with some multiplicative nests Π_1 and Π_2, respectively, and regard the context as the nest $\{\Pi_1,\Pi_2\}_+$.

Example 9.10. *The following is a proof in* ηLBI*:*

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{A \rhd A}\,\mathsf{ax} \quad \dfrac{}{\{B,C\}_+ \rhd B}\,\mathsf{ax}}{\{A,\{B,C\}_+\}_\times \rhd A * B}\,{*}\mathsf{R}}{\{A,(B \wedge C)\}_\times \rhd A * B}\,{\wedge}\mathsf{L} \quad \dfrac{\dfrac{}{A \rhd A}\,\mathsf{ax} \quad \dfrac{\dfrac{}{\{B,C\}_+ \rhd C}\,\mathsf{ax}}{B \wedge C \rhd C}\,{\wedge}\mathsf{L}}{\{A,(B \wedge C)\}_\times \rhd A * C}\,{*}\mathsf{R}}{\{A,(B \wedge C)\}_\times \rhd (A * B) \wedge (A * C)}\,{\wedge}\mathsf{R}}{A * (B \wedge C) \rhd (A * B) \wedge (A * C)}\,{*}\mathsf{L}}{\varnothing_\times \rhd (A * (B \wedge C)) \mathrel{-\!\!*} ((A * B) \wedge (A * C))}\,{-\!\!*}\mathsf{R}$$

■

$$\frac{}{\{\Gamma,\varphi\}_{+}\triangleright\varphi}\,\mathsf{ax} \qquad \frac{}{\Gamma\{\bot\}\triangleright\chi}\,\bot_{\mathsf{L}} \qquad \frac{}{\varnothing_{\times}\triangleright\top^{*}}\,\top^{*}_{\mathsf{R}} \qquad \frac{}{\Gamma\triangleright\top}\,\top_{\mathsf{R}}$$

$$\frac{\Gamma'\triangleright\varphi \quad \Gamma\{\Gamma'',\psi\}_{\times}\triangleright\chi}{\Gamma\{\Gamma',\Gamma'',\{\Gamma''',\varphi\mathrel{-\!\!*}\psi\}_{+}\}_{\times}\triangleright\chi}\,\mathrel{-\!\!*}_{\mathsf{L}} \qquad \frac{\{\Gamma,\varphi\}_{\times}\triangleright\psi}{\Gamma\triangleright\varphi\mathrel{-\!\!*}\psi}\,\mathrel{-\!\!*}_{\mathsf{R}}$$

$$\frac{\Gamma\{\{\varphi,\psi\}_{\times}\}\triangleright\chi}{\Gamma\{\varphi*\psi\}\triangleright\chi}\,*_{\mathsf{L}} \qquad \frac{\Gamma\triangleright\varphi \quad \Gamma'\triangleright\psi}{\{\{\Gamma,\Gamma'\}_{\times},\Gamma''\}_{+}\triangleright\varphi*\psi}\,*_{\mathsf{R}} \qquad \frac{\Gamma\{\varnothing_{\times}\}\triangleright\chi}{\Gamma\{\top^{*}\}\triangleright\chi}\,\top^{*}_{\mathsf{L}}$$

$$\frac{\Gamma\{\{\varphi,\psi\}_{+}\}\triangleright\chi}{\Gamma\{\varphi\wedge\psi\}\triangleright\chi}\,\wedge_{\mathsf{L}} \qquad \frac{\Gamma\triangleright\varphi \quad \Gamma\triangleright\psi}{\Gamma\triangleright\varphi\wedge\psi}\,\wedge_{\mathsf{R}} \qquad \frac{\Gamma\{\varnothing_{+}\}\triangleright\chi}{\Gamma\{\top\}\triangleright\chi}\,\top_{\mathsf{L}}$$

$$\frac{\Gamma\{\varphi\}\triangleright\chi \quad \Gamma\{\psi\}\triangleright\chi}{\Gamma\{\varphi\vee\psi\}\triangleright\chi}\,\vee_{\mathsf{L}} \qquad \frac{\Gamma\triangleright\varphi}{\Gamma\triangleright\varphi\vee\psi}\,\vee_{\mathsf{R1}} \qquad \frac{\Gamma\triangleright\psi}{\Gamma\triangleright\varphi\vee\psi}\,\vee_{\mathsf{R2}}$$

$$\frac{\Gamma'\triangleright\varphi \quad \Gamma\{\Gamma',\psi\}_{+}\triangleright\chi}{\Gamma\{\Gamma',\varphi\to\psi\}_{+}\triangleright\chi}\,\to_{\mathsf{L}} \qquad \frac{\{\Gamma,\varphi\}_{+}\triangleright\psi}{\Gamma\triangleright\varphi\to\psi}\,\to_{\mathsf{R}} \qquad \frac{\Gamma\{\Gamma',\Gamma'\}_{+}\triangleright\chi}{\Gamma\{\Gamma'\}_{+}\triangleright\chi}\,\mathsf{c}$$

Figure 9.1: Sequent Calculus $\eta\mathsf{LBI}$

Note, $\eta\mathsf{LBI}$ is a re-presentation of LBI using the nested syntax. That is, it is not really a development in the proof theory for BI.

Proposition 9.11 (Soundness and Completeness of $\eta\mathsf{LBI}$)**.** *Systems* LBI *and* $\eta\mathsf{LBI}$ *are equivalent:*

- *Soundness: If* $\Gamma\vdash_{\eta\mathsf{LBI}}\Gamma\varphi$*, then* $\beta(\Gamma)\vdash_{\mathsf{LBI}}\triangleright\varphi$*;*

- *Completeness: If* $\Delta\vdash_{\mathsf{LBI}}\varphi$*, then* $\eta(\Delta)\vdash_{\eta\mathsf{LBI}}\varphi$*.*

Proof. Each claim follows by induction on the context, appealing to Proposition 9.8 to organise the data structure for the induction hypothesis, without loss of generality. □

9.2 The Polarised Syntax

Andreoli [6] witnessed in LL a partition of logical constants according to their behaviour, there are those that are *synchronous* and those that are *asynchronous*. The former class corresponds to the behaviours for which the structure of the context affects the applicability of the rule, and the later to those for which it does not. The logical connectives admitting synchronous behaviour are *positive*, and the logical connectives admitting asynchronous behaviour are *negative*.

The two classes of formulae are not exclusionary — for example, $\varphi \land \psi$ is both synchronous and asynchronous in LBI. That is, the following both hold for BI:

- If $\Delta \vdash \varphi$ and $\Delta \vdash \psi$, then $\Delta \vdash \varphi \land \psi$
- If $\Delta \vdash \varphi$ and $\Delta' \vdash \psi$, then $\Delta \fatsemi \Delta' \vdash \varphi \land \psi$

It cannot be said that either one of these is *the* behaviour of $\land$ in BI since they are both valid. The former behaviour is what is captured by $\land_\mathsf{R}$ in LBI; but, in the presence of the structural rules e, w, and c, the rule may be replaced by the following rule corresponding to the latter behaviour:

$$\frac{\Delta \triangleright \varphi \quad \Delta' \triangleright \psi}{\Delta \fatsemi \Delta' \triangleright \varphi \land \psi}$$

Nonetheless, in terms of proof-search, the former behaviour is preferable as the converse implication also holds; that is,

$$\Delta \vdash \varphi \text{ and } \Delta \vdash \psi \qquad \text{iff} \qquad \Delta \vdash \varphi \land \psi$$

We have so far concentrated to the behaviour on the *right* of sequents. But the same phenomenon can be studied in terms of the behaviour on the *left* of sequents. For conjunction, we may study the following two behaviours:

- If $\Delta(\varphi) \vdash \chi$ and $\Delta(\psi) \vdash \chi$, then $\Delta(\varphi \land \psi) \vdash \chi$
- If $\Delta(\varphi \fatsemi \psi) \vdash \chi$, then $\Delta(\varphi \land \psi) \vdash \chi$

This time, it is the latter that that is preferred because the converse implication holds; that is,

$$\Delta(\varphi \fatsemi \psi) \vdash \chi \qquad \text{iff} \qquad \Delta(\varphi \land \psi) \vdash \chi$$

In summary, we have two set of behaviours:

$$
\begin{array}{c|c}
\dfrac{\Gamma\{\varphi\} \triangleright \chi}{\Gamma\{\varphi \wedge \psi\} \triangleright \chi}\ \wedge^-_{\mathsf{L1}} \qquad \dfrac{\Gamma\{\psi\} \triangleright \chi}{\Gamma\{\varphi \wedge \psi\} \triangleright \chi}\ \wedge^-_{\mathsf{L2}} & \dfrac{\Gamma\{\{\varphi, \psi\}_+\} \triangleright \chi}{\Gamma\{\varphi \wedge \psi\} \triangleright \chi}\ \wedge^+_{\mathsf{L}} \\[2ex]
\dfrac{\Gamma \triangleright \varphi \quad \Gamma \triangleright \psi}{\Gamma \triangleright \varphi \wedge \psi}\ \wedge^-_{\mathsf{R}} & \dfrac{\Gamma \triangleright \varphi \quad \Gamma' \triangleright \psi}{\{\Gamma, \Gamma'\}_+ \triangleright \varphi \wedge \psi}\ \wedge^+_{\mathsf{R}}
\end{array}
$$

All of these rules are sound (i.e., if the premisses are consequence of BI, then so is the conclusion), and replacing the conjunction rules in LBI with any pair of a left and right rule will result in a sound and complete system. In LBI, the two rules with the best behaviours in terms of proof-search (i.e., $\wedge^-_{\mathsf{R}}$ and $\wedge^+_{\mathsf{L}}$) are taken. Indeed, the rules are inter-derivable when the structural rules are present.

To maximize the power of the results of this chapter, we shall give the ultimate choice to the user as to whether they want a certain occurrence of $\wedge$ in their putative conclusion to be understood as positive ($\wedge^+$) or negative ($\wedge^-$).

Definition 9.12 (Polarised Syntax)**.** *Let $\mathbb{A}^+ \sqcup \mathbb{A}^-$ be a partition of $\mathbb{A}$. The polarised formulas are defined by the following grammar,*

$$
\begin{array}{ll}
P, Q ::= L \mid P \vee Q \mid P * Q \mid P \wedge^+ Q \mid \top^+ \mid \top^* \mid \bot & L ::= \downarrow N \mid A \in \mathbb{A}^+ \\
N, M ::= R \mid P \to N \mid P \mathrel{-\!\!*} N \mid N \wedge^- M \mid \top^- & R ::= \uparrow P \mid A \in \mathbb{A}^-
\end{array}
$$

The set of positive formulas P is denoted $\mathbb{F}^+$; the set of negative formulas N is denoted $\mathbb{F}^-$; and the set of all polarised formulas is denoted $\mathbb{F}^\pm$. The sub-classifications L and R are left-neutral and right-neutral formulas respectfully.

Definition 9.13 (Unfocused, Neutral, and Focused Nests)**.** *A nest composed of only positive formulae is an unfocused nest; the nests $\varnothing_\times$ and $\varnothing_+$ are also unfocused. An unfocused nest in which all the formulae are left-neutral is a neutral nest. Let $\Gamma\{L\}$ be neutral nest, and let N be a negative formula; the nest $\Gamma\{N\}$ is a focused nest.*

We shall use the meta-variable Γ to denote unfocused nests, $\vec{\Gamma}$ to denote neutral nests, and $\vec{\Gamma}\{\langle N\rangle\}$ to denote a focused nest with negative formula N — that is, $\vec{\Gamma}(\downarrow N)$ is a neutral nest. The highlighting of the negative formula aids in this notation aids readability and is not formally part of the syntax.

Definition 9.14 (Unfocused, Neutral, and Focused Sequents)**.** *An unfocused sequent is a pair $\Gamma \triangleright N$ in which Γ is an unfocused nest and N is a negative formula. A neutral sequent is a pair $\vec{\Gamma} \triangleright R$ in which $\vec{\Gamma}$ is a neutral nest and R is a neutral formula. A focused sequent is either a pair $\vec{\Gamma}\{\langle N\rangle\} \triangleright R$ in which $\vec{\Gamma}\{\langle N\rangle\}$ is a focused nest and R is a right-neutral formula, or a pair $\vec{\Gamma} \triangleright \langle P\rangle$ in which Γ is a neutral nest and P is a positive formula.*

The shift operators $\uparrow$ and $\downarrow$ have no logical meaning; they simply mediate the exchange of polarity, and thus the *shifting* into a new phase of proof-search. Consequently, to reduces cases in subsequent proofs, we will consider formulas of the form $\uparrow\downarrow N$ and $\downarrow\uparrow P$, but not $\downarrow\uparrow\downarrow N$, $\downarrow\uparrow\downarrow\uparrow P$, etc. A polarised formula canonically determines a formula by eliminating all the control structure.

Definition 9.15 (Depolarization)**.** *The depolarization function $\lfloor\cdot\rfloor : \mathbb{F}^{\pm} \to \mathbb{F}$ is defined as follows:*

$$\lfloor\varphi\rfloor := \begin{cases} \top & \textit{if } \varphi \in \{\top^{+}, \top^{-}\} \\ \bot & \textit{if } \varphi = \bot \\ \top^{*} & \textit{if } \varphi = \top^{*} \\ A & \textit{if } \varphi = A \in \mathbb{A} \\ \lfloor\psi\rfloor & \textit{if } \varphi = \uparrow\psi \textit{ or } \varphi = \downarrow\psi \\ \lfloor\psi_1\rfloor \circ \lfloor\psi_2\rfloor & \textit{if } \varphi = \psi_1 \circ \psi_2 \textit{ for } \circ \in \{\vee, \wedge, *, \to, -\!\!*\} \end{cases}$$

The depolarization map extends to polarised nests $\lfloor\cdot\rfloor : \mathbb{B}/^{\pm}_{\equiv} \to \mathbb{B}/_{\equiv}$ as follows:

$$\lfloor\{\Pi_1,...,\Pi_n\}_{+}\rfloor = \{\lfloor\Pi_1\rfloor,...,\lfloor\Pi_n\rfloor\}_{+} \qquad \lfloor\{\Sigma_1,...,\Sigma_n\}_{\times}\rfloor = \{\lfloor\Sigma_1\rfloor,...,\lfloor\Sigma_n\rfloor\}_{\times}$$

The polarised syntax thus given allows us to express focusing syntactically using the types of sequents and the shift operators as control structures. Importantly, for any nested sequent $\Gamma \triangleright \varphi$, there is an unfocused sequent $\Gamma' \triangleright N$ such that $\lfloor \Gamma' \rfloor = \Gamma$ and $\lfloor N \rfloor = \varphi$.

This means that given a sequent for which one wants a focused proof in ηLBI (or LBI) one begins by *choosing* a polarity, and different choice may return different focused proofs. Taking arbitrary formula φ, the process by which one polarises it is as follows: first, fix partion the propositional letters into positive and negative sets, then assign a polarity to φ with the following steps:

- if φ is a propositional atom, it must be polarised by default;
- If $\varphi = \top$, then *choose* polarization $\top^+$ or $\top^-$
- if $\varphi = \psi_1 \land \psi_2$, first polarise ψ_1 and ψ_2, then *choose* an additive conjunction (i.e., either $\land^+$ or $\land^-$) and combine accordingly, using shifts to ensure the formula is well-formed
- if $\varphi = \psi_1 \circ \psi_2$ where $\circ \in \{*, -\!\!*, \to, \lor\}$, then polarise ψ_1 and ψ_2 and combine with $\circ$ accordingly, using shifts where necessary.

Example 9.16. *Suppose A is negative and B is positive, then $(A * B) \land A$ may be polarised by choosing the additive conjunction to be positive resulting in $(\downarrow A * B) \land^+ \downarrow A$ (when $\downarrow(A * \downarrow B) \land^+ A)$ would not be well-formed). Choosing to shift one can ascribe a negative polarization $\uparrow((\downarrow A * B) \land^+ \downarrow A)$.* ■

The above generates the set of all polarised formulas when all possible choices are explored. The free assignment of polarity to formulas means several distinct focusing procedures are captured by the completeness theorem; uniform proof-search, the operational semantics delivering proof-search, corresponds to the choice in which all the formulas are negative — see Andreoli [6].

9.3 The Focused Calculus

The polarised syntax enables us to give a focused system; that is, a system in which only focused proofs can be constructed. Moreover, according to the polarization chosen, every focused proof of a given sequent can be constructed in the focused system.

Definition 9.17 (System FBI)**.** *The focused system* FBI *is composed of the rules on Figure 9.2.*

It is, perhaps, proof-theoretically displeasing to incorporate weakening into the operational rules as in $-\!\!*_{\mathsf{L}}$ and $*_{\mathsf{R}}$ in FBI, but it has good computational behaviour during focused proof-search since the reduction of $\varphi -\!\!* \psi$ can only arise out of an explicit choice made earlier in the computation. The following example illustrates how FBI enforces focusing during reduction:

Example 9.18. *Let A and C be negative, and B be positive. The following is an* FBI*-proof:*

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\langle A\rangle \triangleright A}\,\mathsf{Ax}^-}{{\downarrow}A \triangleright A}\,{\downarrow}_{\mathsf{L}}}{{\downarrow}A \triangleright \langle {\downarrow}A\rangle}\,{\downarrow}_{\mathsf{R}} \quad \dfrac{}{B \triangleright \langle B\rangle}\,\mathsf{Ax}^+}{\{{\downarrow}A, B\}_\times \triangleright \langle {\downarrow}A * B\rangle}\,{*}_{\mathsf{R}}}{\{{\downarrow}A,\ B\}_\times \triangleright {\uparrow}({\downarrow}A * B)}\,{\uparrow}_{\mathsf{R}}}{\{{\downarrow}A, \langle {\uparrow}B\rangle\}_\times \triangleright {\uparrow}({\downarrow}A * B)}\,{\uparrow}_{\mathsf{L}}}{\{{\downarrow}A, \langle {\uparrow}B \wedge^- C\rangle\}_\times \triangleright {\uparrow}({\downarrow}A * B)}\,\wedge^-_{\mathsf{L1}}}{\{{\downarrow}A, {\downarrow}({\uparrow}B \wedge^- C)\}_\times \triangleright {\uparrow}({\downarrow}A * B)}\,{\uparrow}_{\mathsf{L}}\ (1) \quad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\langle A\rangle \triangleright A}\,\mathsf{Ax}^-}{{\downarrow}A \triangleright A}\,{\downarrow}_{\mathsf{L}}}{{\downarrow}A \triangleright \langle {\downarrow}A\rangle}\,{\downarrow}_{\mathsf{R}} \quad \dfrac{\dfrac{\dfrac{\dfrac{}{\langle C\rangle \triangleright C}\,\mathsf{Ax}^-}{\langle {\uparrow}B \wedge^- C\rangle \triangleright C}\,\wedge^-_{\mathsf{L2}}}{{\downarrow}({\uparrow}B \wedge^- C) \triangleright C}\,{\downarrow}_{\mathsf{L}}}{{\downarrow}({\uparrow}B \wedge^- C)\}_\times \triangleright \langle {\downarrow}C\rangle}\,{\downarrow}_{\mathsf{R}}}{\{{\downarrow}A, {\downarrow}({\uparrow}B \wedge^- C)\}_\times \triangleright \langle {\downarrow}A * {\downarrow}C\rangle}\,{*}_{\mathsf{R}}}{\{{\downarrow}A, {\downarrow}({\uparrow}B \wedge^- C)\}_\times \triangleright {\uparrow}({\downarrow}A * {\downarrow}C)}\,{\uparrow}_{\mathsf{R}}\ (2)}{\dfrac{\{{\downarrow}A, {\downarrow}({\uparrow}B \wedge^- C)\}_\times \triangleright {\uparrow}({\downarrow}A * B) \wedge^- {\uparrow}({\downarrow}A * {\downarrow}C)}{\dfrac{{\downarrow}A * {\downarrow}({\uparrow}B \wedge^- C) \triangleright {\uparrow}({\downarrow}A * B) \wedge^- {\uparrow}({\downarrow}A * {\downarrow}C)}{\varnothing_\times \triangleright ({\downarrow}A * {\downarrow}({\uparrow}B \wedge^- C)) -\!\!* ({\uparrow}({\downarrow}A * B) \wedge^- {\uparrow}({\downarrow}A * {\downarrow}C))}\,-\!\!*_{\mathsf{R}}}\,{*}_{\mathsf{L}}}\,\wedge^-_{\mathsf{R}}$$

This is, intuitively, an FBI*-proof equivalent to the* ηLBI*-proof in Example 9.10. Observe that the only non-deterministic choices are which formula to focus on, such as in steps (1) and (2), where different choices have been made for the sake of demonstration. The point of focusing is that* only *at such points do choices that affect termination occur. The assignment of polarity to the propositional letters is what forced the shape of the proof; for example, if B had been negative the above would not have been well-formed. This phenomenon is typical in focused systems — see, for example, Chaudhuri [35, 34].* ■

Soundness of FBI follows immediately from depolarization:

Theorem 9.19 (Soundness of FBI)**.** *Let Γ be a polarised nest and N a negative formula. If $\Gamma \vdash_{\mathsf{FBI}} N$ then $\lfloor \Gamma \rfloor \vdash_{\eta\mathsf{LBI}} \lfloor N \rfloor$*

Focused Phase

$$\frac{}{\{\vec{\Gamma}, P\}_+ \rhd \langle P \rangle}\,\mathsf{Ax}^+ \qquad \frac{}{\{\vec{\Gamma}, \langle N \rangle\}_+ \rhd N}\,\mathsf{Ax}^-$$

$$\frac{}{\{\vec{\Gamma}, \downarrow\uparrow P\}_+ \rhd \langle P \rangle}\,\mathsf{P} \qquad \frac{}{\{\vec{\Gamma}, \langle N \rangle\}_+ \rhd \uparrow\downarrow N}\,\mathsf{N}$$

$$\frac{}{\vec{\Gamma} \rhd \langle \top^+ \rangle}\,\top^+_{\mathsf{R}} \qquad \frac{}{\{\vec{\Gamma}, \varnothing_\times\}_+ \rhd \langle \top^* \rangle}\,\top^*_{\mathsf{R}}$$

$$\frac{\vec{\Gamma}\{\langle N_i \rangle\}_+ \rhd R}{\vec{\Gamma}\{\langle N_1 \wedge^- N_2 \rangle\}_+ \rhd R}\,\wedge^-_{\mathsf{Li}} \qquad \frac{\vec{\Gamma} \rhd \langle P_i \rangle}{\vec{\Gamma} \rhd \langle P_1 \vee P_2 \rangle}\,\vee_{\mathsf{Ri}} \qquad \frac{\vec{\Gamma}\{\varnothing_+\} \rhd R}{\vec{\Gamma}\{\langle \top^- \rangle\} \rhd R}\,\top^-_{\mathsf{L}}$$

$$\frac{\vec{\Gamma} \rhd \langle P \rangle \quad \vec{\Gamma}' \rhd \langle Q \rangle}{\{\vec{\Gamma}, \vec{\Gamma}'\}_+ \rhd \langle P \wedge^+ Q \rangle}\,\wedge^+_{\mathsf{R}} \qquad \frac{\vec{\Delta} \rhd \langle P \rangle \quad \vec{\Gamma}\{\vec{\Delta}, \langle N \rangle\}_+ \rhd R}{\vec{\Gamma}\{\vec{\Delta}, \langle P \to N \rangle\}_+ \rhd R}\,\to_{\mathsf{L}}$$

$$\frac{\vec{\Gamma} \rhd \langle P \rangle \quad \vec{\Gamma}' \rhd \langle Q \rangle}{\{\{\vec{\Gamma}, \vec{\Gamma}'\}_\times, \vec{\Gamma}''\}_+ \rhd \langle P * Q \rangle}\,*_{\mathsf{R}} \qquad \frac{\vec{\Delta} \rhd \langle P \rangle \quad \vec{\Gamma}\{\vec{\Delta}', \langle N \rangle\}_\times \rhd R}{\vec{\Gamma}\{\vec{\Delta}, \vec{\Delta}', \{\vec{\Delta}'', \langle P \mathrel{-\!\!*} N \rangle\}_+\}_\times \rhd R}\,\mathrel{-\!\!*}_{\mathsf{L}}$$

Neutral Phase

$$\frac{\vec{\Gamma} \rhd \langle P \rangle}{\vec{\Gamma} \rhd \uparrow P}\,\uparrow_{\mathsf{R}} \qquad \frac{\vec{\Gamma}\{P\} \rhd R}{\vec{\Gamma}\{\langle \uparrow P \rangle\} \rhd R}\,\uparrow_{\mathsf{L}} \qquad \frac{\vec{\Gamma} \rhd N}{\vec{\Gamma} \rhd \langle \downarrow N \rangle}\,\downarrow_{\mathsf{R}} \qquad \frac{\vec{\Gamma}\{\langle N \rangle\} \rhd R}{\vec{\Gamma}\{\downarrow N\} \rhd R}\,\downarrow_{\mathsf{L}}$$

$$\frac{\vec{\Gamma}\{\{\vec{\Delta}, \vec{\Delta}\}_+\} \rhd R}{\vec{\Gamma}\{\vec{\Delta}\} \rhd R}\,\mathsf{C}$$

Unfocused Phase

$$\frac{}{\Gamma \rhd \top^-}\,\top^-_{\mathsf{R}} \qquad \frac{}{\Gamma\{\bot\} \rhd N}\,\bot_{\mathsf{L}}$$

$$\frac{\Gamma \rhd N \quad \Gamma \rhd M}{\Gamma \rhd N \wedge^- M}\,\wedge^-_{\mathsf{R}} \qquad \frac{\Gamma\{P\} \rhd N \quad \Gamma\{Q\} \rhd N}{\Gamma\{P \vee Q\} \rhd N}\,\vee_{\mathsf{L}}$$

$$\frac{\Gamma\{\{P, Q\}_+\} \rhd N}{\Gamma\{P \wedge^+ Q\} \rhd N}\,\wedge^+_{\mathsf{L}} \qquad \frac{\{\Gamma, P\}_+ \rhd N}{\Gamma \rhd P \to N}\,\to_{\mathsf{R}} \qquad \frac{\Gamma\{\varnothing_+\} \rhd N}{\Gamma\{\top^+\} \rhd N}\,\top^+_{\mathsf{L}}$$

$$\frac{\Gamma\{\{P, Q\}_\times\} \rhd N}{\Gamma\{P * Q\} \rhd N}\,*_{\mathsf{L}} \qquad \frac{\{\Gamma, P\}_\times \rhd N}{\Gamma \rhd P \mathrel{-\!\!*} N}\,\mathrel{-\!\!*}_{\mathsf{R}} \qquad \frac{\Gamma\{\varnothing_\times\} \rhd N}{\Gamma\{\top^*\} \rhd N}\,\top^*_{\mathsf{L}}$$

Figure 9.2: Focused System FBI

Proof. Every rule in FBI except the shift rules become an admissible rule in $\eta\mathsf{LBI}$ when the antecedent(s) and consequent are depolarised, using Proposition 6.7. Instance of the shift rule can be ignored since the depolarised versions of the consequent and antecedents are the same. □

It remains to show that FBI is complete — that is, let $\Gamma \triangleright N$ be an arbitrary polarised sequent,

$$\text{if } \lfloor \Gamma \rfloor \vdash_{\eta\mathsf{LBI}} \lfloor N \rfloor, \text{ then } \Gamma \vdash_{\mathsf{FBI}} N$$

To show the completeness of FBI, we introduce a cut-rule so that any $\eta\mathsf{LBI}$-proof can be *simulated* in FBI. Let $\vec{\varphi}$ denote a formula that is either φ or φ prenexed with an additional shift. The cut-rule in question is the following:

$$\frac{\Delta \triangleright \varphi \quad \Gamma\{\vec{\varphi}\} \triangleright \chi}{\Gamma\{\Delta\} \triangleright \chi}\ \mathsf{cut}$$

Admissibility follows by a permutation argument similar to the on in Chapter 7. Therefore, we shall give a comparatively terse account.

We separate the treatment of commutative cuts and principal cuts into two different propositions. To witness the requisite transformation we require weakening in the focused system.

Proposition 9.20. *If* $\vec{\Gamma}\{\vec{\Delta},\vec{\Delta}'\}_\times \vdash_{\mathsf{FBI}} \chi$, *then* $\Gamma\{\vec{\Delta},\{\vec{\Delta}',\vec{\Delta}''\}_+\}_\times \vdash_{\mathsf{FBI}} \chi$.

Proof. This follows from the usual permutation arguments — see, for example, Troelstra and Schwichtenberg [207]. □

The setup for cut-admissibility is similar to that of Chapter 7. As before, a cut κ in a $\mathsf{FBI} + \mathsf{cut}$-proof $\mathcal{D}$ is a triple $\langle \mathcal{L}, \mathcal{R}, \varphi \rangle$ such that $\mathcal{D}$ contains the following inference:

$$\frac{\mathcal{L} : \Delta \triangleright \varphi \quad \mathcal{R} : \Gamma\{\vec{\varphi}\} \triangleright \chi}{\Gamma\{\Delta\} \triangleright \chi}\ \mathsf{cut}$$

Recall that the rank of a cut $k = \langle \mathcal{L}, \mathcal{R}, \varphi \rangle$ in a LBI-proof $\mathcal{D}$ is the triple $\langle \sigma(\varphi), \kappa(k), \max\{h(\mathcal{L}), h(\mathcal{R})\} + 1 \rangle$ — that is, the size of the cut formula, contraction potential (i.e., the number of times a constractions used in $\mathcal{R}$ on a sub-bunch containing the cut-formula), and the height of the cut.

We use the taxonomy of cuts in Chapter 7, but with the terms *good* cuts for principal cuts and *bad* cuts for commutative cuts. We further sub-divide bad cuts according to the branch on which the cut commutes.

A cut $\langle \mathcal{L}, \mathcal{R}, \varphi \rangle$ is classified as follows:

Good - If φ is principal in both $\mathcal{L}$ and $\mathcal{R}$.

Bad - If φ is not principal in one of $\mathcal{L}$ and $\mathcal{R}$.

Type 1: If φ is not principal in $\mathcal{L}$.

Type 2: If φ is not principal in $\mathcal{R}$.

As in Chapter 7, let $\preceq$ denote the multiset ordering derived from the lexicographic ordering on cut rank. Denote $\mathcal{D} \preceq \mathcal{D}'$ iff $M \preceq N$, where M is the multiset of cuts in $\mathcal{D}$ and N is the multiset of cuts in $\mathcal{D}'$.

Proposition 9.21. *Let $\mathcal{D}$ be a* $\mathsf{FBI} + \mathsf{cut}$ *proof of S. There is an* $\mathsf{FBI} + \mathsf{cut}$ *proof $\mathcal{D}'$ of S containing no good cuts such that $\mathcal{D}' \preceq \mathcal{D}$.*

Proof. Let $\mathcal{D}$ be as in hypothesis. If $\mathcal{D}$ contains no good cuts then $\mathcal{D} = \mathcal{D}'$ gives the desired proof. Otherwise, $\mathcal{D}$ contains at least one good cut $\langle \mathcal{L}, \mathcal{R}, \varphi \rangle$. Let δ be the sub-proof in $\mathcal{D}$ concluding with this cut. Assume there is a transformation of δ yielding $\mathsf{FBI} + \mathsf{cut}$-proof δ' with the same conclusion such that $\delta' \preceq \delta$. Since $\preceq$ is a well-order, indefinitely replacing δ with δ' in $\mathcal{D}$ for various cuts yields the desired $\mathcal{D}'$.

It remains to justify the assumption. These are provided by permutation of cuts analogous to those in Chapter 7, which we elide for economy. We give two examples below to illustrate the idea; we use a double-line adorned with w to denote an appeal to Proposition 9.20:

$$\dfrac{\dfrac{}{\{\vec{\Gamma}',A^+\}_+ \triangleright \langle A^+\rangle}\,\mathsf{Ax}^+ \qquad \mathcal{R} : \vec{\Gamma}\{A^+\} \triangleright \langle A^+\rangle}{\vec{\Gamma}\{\{\vec{\Gamma}',A^+\}_+\} \triangleright \langle A^+\rangle}\,\mathsf{cut} \qquad \text{becomes}$$

$$\begin{array}{c} \mathcal{R} : \vec{\Gamma}\{A^+\} \triangleright \langle A^+\rangle \\ \hline\hline \vec{\Gamma}\{\{\vec{\Gamma}', A^+\}_+\} \triangleright \langle A^+\rangle \end{array}\ \mathsf{w}$$

$$\dfrac{\dfrac{\mathcal{L}_1 : \{\vec{\Delta}'', P\}_\times \triangleright N}{\vec{\Delta}'' \triangleright P \mathrel{-\!\!*} N}\ {-\!\!*}\mathsf{R} \qquad \dfrac{\mathcal{R}_1 : \vec{\Delta} \triangleright \langle P\rangle \quad \mathcal{R}_2 : \Gamma\{\vec{\Delta}', \langle N\rangle\}_\times \triangleright R}{\vec{\Gamma}\{\vec{\Delta}, \vec{\Delta}', \{\vec{\Delta}''', \langle P \mathrel{-\!\!*} N\rangle\}_+\}_\times \triangleright R}\ {-\!\!*}\mathsf{L}}{\vec{\Gamma}\{\vec{\Delta}, \vec{\Delta}', \{\vec{\Delta}'', \vec{\Delta}'''\}_+\}_\times \triangleright R}\ \mathsf{cut} \qquad \text{becomes}$$

$$\begin{array}{c} \dfrac{\mathcal{R}_1 : \vec{\Delta} \triangleright \langle P\rangle \qquad \dfrac{\mathcal{L}_1 : \{\vec{\Delta}'', P\}_\times \triangleright N \quad \mathcal{R}_2 : \vec{\Gamma}\{\vec{\Delta}', \langle N\rangle\}_\times \triangleright R}{\vec{\Gamma}\{\vec{\Delta}, \vec{\Delta}'', P\}_\times \triangleright R}\ \mathsf{cut}}{\vec{\Gamma}\{\vec{\Delta}, \vec{\Delta}', \vec{\Delta}''\}_\times \triangleright R}\ \mathsf{cut} \\ \hline\hline \vec{\Gamma}\{\vec{\Delta}, \vec{\Delta}', \{\vec{\Delta}'', \vec{\Delta}'''\}_+\}_\times \triangleright R \end{array}\ \mathsf{w}$$

□

Proposition 9.22. *Let $\mathcal{D}$ be an* $\mathsf{FBI} + \mathsf{cut}$ *proof containing only one cut, and let that cut be bad. There is an* $\mathsf{FBI} + \mathsf{cut}$*-proof $\mathcal{D}'$ such that $\mathcal{D}' \prec \mathcal{D}$.*

Proof. Let the cut in question be $\langle \mathcal{L}, \mathcal{R}, \varphi\rangle$. Without loss of generality, it is the last inference in $\mathcal{D}$. We show that it may be replaced by other cuts that are smaller in cut rank. As in the proof of Proposition 9.21, we use a double-line adorned with a w to denote an appeal to Proposition 9.20.

First, consider the case in which both $\mathcal{L}$ and $\mathcal{R}$ are both axioms. There are no Type 1 bad cuts on axioms as the formula is always principal; meanwhile, the Type 2 bad cuts can trivially be permuted upwards or ignored — for example,

$$\dfrac{\dfrac{}{\{\vec{\Delta}''', A_+\}_+ \triangleright \langle A_+\rangle}\ \mathsf{Ax}^+ \qquad \dfrac{\mathcal{R}_1 : \vec{\Delta} \triangleright \langle P\rangle \quad \mathcal{R}_2 : \vec{\Gamma}\{\vec{\Delta}', \langle N\rangle\}_\times \triangleright R}{\vec{\Gamma}\{\vec{\Delta}, \vec{\Delta}', \{\vec{\Delta}'', A_+, \langle P \mathrel{-\!\!*} N\rangle\}_+\}_\times \triangleright R}\ {-\!\!*}\mathsf{L}}{\vec{\Gamma}\{\vec{\Delta}, \vec{\Delta}', \{\vec{\Delta}'', \vec{\Delta}''', A_+, \langle P \mathrel{-\!\!*} N\rangle\}_+\}_\times \triangleright R}\ \mathsf{cut} \qquad \text{becomes}$$

$$\begin{array}{c} \dfrac{\mathcal{R}_1 : \vec{\Delta} \triangleright \langle P\rangle \quad \mathcal{R}_2 : \vec{\Gamma}\{\vec{\Delta}', \langle N\rangle\}_\times \triangleright R}{\vec{\Gamma}\{\vec{\Delta}, \vec{\Delta}', \{\vec{\Delta}'', A_+, \langle P \mathrel{-\!\!*} N\rangle\}_+\}_\times \triangleright R}\ {-\!\!*}\mathsf{L} \\ \hline\hline \vec{\Gamma}\{\vec{\Delta}, \vec{\Delta}', \{\vec{\Delta}'', \vec{\Delta}''', A_+, \langle P \mathrel{-\!\!*} N\rangle\}_+\}_\times \triangleright R \end{array}\ \mathsf{w}$$

Second, in the remaining cases the cuts are commutative in the sense that they may be permuted upward thereby reducing the second or third component of rank. A case in which the third component decreases is given as follows:

$$\dfrac{\dfrac{\mathcal{L}_1 : \vec{\Delta}\{\langle N_1\rangle\} \rhd M}{\vec{\Delta}\{\langle N_1 \wedge^- N_2\rangle\} \rhd M}\,\wedge^-_{\mathsf{L1}} \qquad \mathcal{R} : \vec{\Gamma}\{M\} \rhd R}{\vec{\Gamma}\{\vec{\Delta}\{\langle N_1 \wedge^- N_2\rangle\}\} \rhd R}\,\mathsf{cut} \qquad \text{becomes}$$

$$\dfrac{\dfrac{\mathcal{L}_1 : \Delta\{\langle N_1\rangle\} \rhd M \qquad \mathcal{R} : \Gamma\{M\} \rhd R}{\Gamma\{\Delta\{\langle N_1\rangle\}\} \rhd R}\,\mathsf{cut}}{\Gamma\{\Delta\{\langle N_1 \wedge^- N_2\rangle\}\} \rhd R}\,\wedge^-_{\mathsf{L1}}$$

The exceptional case is the interaction with contraction where the cut is replaced by cuts whose rank are smaller in the second component — for example,

$$\dfrac{\vec{\Delta}' \rhd \langle L\rangle \qquad \dfrac{\mathcal{R}_1 : \vec{\Gamma}\{\{\vec{\Delta}\{L\}, \vec{\Delta}\{L\}\}_+\} \rhd R}{\vec{\Gamma}\{\vec{\Delta}\{L\}\} \rhd R}\,\mathsf{c}}{\vec{\Gamma}\{\vec{\Delta}\{\vec{\Delta}'\}\} \rhd R}\,\mathsf{cut} \qquad \text{becomes}$$

$$\dfrac{\dfrac{\mathcal{L} : \vec{\Delta}' \rhd \langle L\rangle \qquad \dfrac{\mathcal{L} : \vec{\Delta}' \rhd \langle L\rangle \qquad \mathcal{R}_1 : \vec{\Gamma}\{\{\vec{\Delta}\{L\}, \vec{\Delta}\{L\}\}_+\} \rhd R}{\vec{\Gamma}\{\{\vec{\Delta}\{\vec{\Delta}'\}, \vec{\Delta}\{L\}\}_+\} \rhd R}\,\mathsf{cut}}{\vec{\Gamma}\{\{\vec{\Delta}\{\vec{\Delta}'\}, \vec{\Delta}\{\vec{\Delta}'\}\}_+\} \rhd R}\,\mathsf{cut}}{\vec{\Gamma}\{\vec{\Delta}\{\vec{\Delta}'\}\} \rhd R}\,\mathsf{c}$$

□

Theorem 9.23 (Admissibility of Polarised Cut in FBI)**.** *Let* Γ *be a positive nest and* N *a negative formula. Then,* $\Gamma \vdash_{\mathsf{FBI}} N$ *if and only if* $\Gamma \vdash_{\mathsf{FBI+cut}} N$*.*

Proof. If $\Gamma \vdash_{\mathsf{FBI}} N$, then $\Gamma \vdash_{\mathsf{FBI+cut}} N$ trivially since $\mathsf{FBI} \subseteq \mathsf{FBI} + \mathsf{cut}$. It remains to show that $\Gamma \vdash_{\mathsf{FBI+cut}} N$ implies $\Gamma \vdash_{\mathsf{FBI}} N$.

Let $\mathcal{D}$ be a $\mathsf{FBI} + \mathsf{cut}$-proof of $\Gamma \rhd N$. If $\mathcal{D}$ has no cuts, then it is a FBI-proof, so we are done. Otherwise, $\mathcal{D}$ has at least one cut. We proceed by induction on $\preceq$ to show that there is a proof $\Gamma \rhd N$ containing no cuts.

- BASE CASE. Assume $\mathcal{D}$ is minimal with respect to $\prec$ with at least one cut. By Proposition 9.21, we can assume that the cut is bad. It follows from Proposition 9.22 that there is a proof strictly smaller in $\prec$-ordering with the sam conclusion, but this proof must be cut-free as $\mathcal{D}$ is minimal.

- Inductive Step. By Proposition 9.21, there is a proof $\mathcal{D}'$ of $\Gamma \triangleright N$ containing no good cuts such that $\mathcal{D}' \preceq \mathcal{D}$. Either $\mathcal{D}'$ is cut-free and we are done; otherwise, $\mathcal{D}'$ contains bad cuts. Let δ be a sub-proof of $\mathcal{D}'$ containing precisely one cut. By Proposition 9.22, there is a proof δ' of the same sequent such that $\delta' \prec \delta$. Hence, by inductive hypothesis, there is a cut-free proof δ'' of the sequent. Replacing δ by δ'' proof gives a proof of $\Gamma \triangleright \varphi$ strictly smaller in $\prec$-ordering; this follows as there are fewer cuts and their rank has not increased. Thus, by inductive hypothesis there is a cut-free proof of $\Gamma \triangleright \varphi$ as required.

This completes the induction. □

It remains to show that $\mathsf{FBI} + \mathsf{cut}$ can simulate $\eta\mathsf{LBI}$.

Proposition 9.24 (Completeness of $\mathsf{FBI} + \mathsf{cut}$)**.** *Let Γ be a polarised nest and N a negative formula. If $\lfloor \Gamma \rfloor \vdash_{\eta\Gamma\mathsf{LBI}} \lfloor N \rfloor$, then $\vdash_{\mathsf{FBI+cut}} \Gamma \triangleright N$.*

Proof. Suppose $\eta\mathsf{LBI}$ contains a rule

$$\frac{\Gamma_1 \triangleright N_1 \quad \dots \quad \Gamma_k \triangleright N_k}{\Gamma_0 \triangleright N_0}$$

Let Γ'_i be an unfocused nest such that $\lfloor \Gamma'_i \rfloor = \Gamma_i$ and N_i be a negative formula such that $\lfloor N \rfloor$, for $0 \leq i \leq k$. We show that following is derivable in $\mathsf{FBI} + \mathsf{cut}$:

$$\frac{\Gamma'_1 \triangleright N'_1 \quad \dots \quad \Gamma'_k \triangleright N'_k}{\Gamma'_0 \triangleright N'_0}$$

For $\rightarrow_{\mathsf{R}}, \mathrel{-\!\!*}_{\mathsf{R}}, \wedge_{\mathsf{R}}, \wedge_{\mathsf{L}}, \vee_{\mathsf{L}}, *_{\mathsf{L}}, \bot_{\mathsf{L}}, \top_{\mathsf{R}}, \top_{\mathsf{L}}, \top^*_{\mathsf{L}}, \mathsf{ax}, \mathsf{c} \in \eta\mathsf{LBI}$, this this is immediate. as well as for ax and c. In remains to consider $\rightarrow_{\mathsf{L}}$, $\mathrel{-\!\!*}_{\mathsf{L}}$, $\vee_{\mathsf{R1}}$, $\vee_{\mathsf{R2}}$, $*_{\mathsf{R}}$, $\top^*_{\mathsf{R}}$. In each case, these rules have corresponding version in FBI, but the correspond rules are focused; the unfocused version are simulated by using a cut to break the focusing phase. We illustrate this be an example.

Consider the $*_{\mathsf{R}}$-rule in $\eta\mathsf{NBI}$,

$$\dfrac{\Gamma \rhd \varphi \quad \Delta \rhd \psi}{\{\{\Gamma,\Delta\}_{\times},\Delta'\}_{+} \rhd \varphi * \psi}\ *_{\mathsf{R}}$$

It is simulated in $\mathsf{FBI}+\mathsf{cut}$ by the following derivation:

$$\dfrac{\dfrac{\dfrac{\Gamma \rhd \uparrow\varphi^{+}}{\Gamma \rhd \langle\downarrow\uparrow\varphi^{+}\rangle}\ \downarrow_{\mathsf{R}} \quad \dfrac{\Delta \rhd \uparrow\psi^{+}}{\Delta \rhd \langle\downarrow\uparrow\psi^{+}\rangle}\ \downarrow_{\mathsf{R}}}{\{\Gamma,\Delta\}_{\times} \rhd \langle\downarrow\uparrow\varphi^{+} * \downarrow\uparrow\psi^{+}\rangle}\ *_{\mathsf{R}} \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{\downarrow\uparrow\varphi^{+} \rhd \langle\varphi^{+}\rangle}\ \mathsf{P} \quad \dfrac{}{\downarrow\uparrow\psi^{+} \rhd \langle\psi^{+}\rangle}\ \mathsf{P}}{\{\{\downarrow\uparrow\varphi^{+},\downarrow\uparrow\psi^{+}\}_{\times},\Delta'\}_{+} \rhd \langle\varphi^{+} * \psi^{+}\rangle}\ *_{\mathsf{R}}}{\{\{\downarrow\uparrow\varphi^{+},\downarrow\uparrow\psi^{+}\}_{\times},\Delta'\}_{+} \rhd \uparrow(\varphi^{+} * \psi^{+})}\ \uparrow_{\mathsf{R}}}{\{\downarrow\uparrow\varphi^{+} * \downarrow\uparrow\psi^{+},\Delta'\}_{+} \rhd \uparrow(\varphi^{+} * \psi^{+})}\ *_{\mathsf{L}}}{\{\{\Gamma,\Delta\}_{\times},\Delta'\}_{+} \rhd \uparrow(\varphi^{+} * \psi^{+})}\ \mathsf{cut}$$

Thus as each rule in $\eta\mathsf{NBI}$ may be simualted in $\mathsf{FBI}+\mathsf{cut}$, each -$\eta\mathsf{NBI}$-proof can be simulated by FBI-proof. □

The admissibility of polarised cut together with the simulation result together yield the completeness of FBI.

Theorem 9.25 (Completeness of FBI)**.** *Let Γ be a polarised nest and N a negative formula. If $\lfloor\Gamma\rfloor \vdash_{\eta\mathsf{LBI}} \lfloor N\rfloor$, then $\Gamma \vdash_{\mathsf{FBI}} N$.*

Proof. From the hypohtesis, by Proposition 9.24, there is a $\mathsf{FBI}+\mathsf{cut}$-proof of $\Gamma \rhd N$ in $\mathsf{FBI}+\mathsf{cut}$. From this, by Proposition 9.23, there is a proof of $\Gamma \rhd N$ in FBI. □

Given an arbitrary sequent the above theorem guarantees the existence of a focused proof, thus the focusing principle holds for $\eta\mathsf{LBI}$ and therefore for LBI.

Chapter 10

Semantical Analysis of the Logic of Bunched Implications

This chapter illustrates a novel approach to proving completeness of proof system with respect to a semantics that is symmetric to the traditional approach to proving soundness. Typically, when proving soundness, one proceeds by showing that validity ($\vDash$) is invariant with respect to the rules of the proof system; for example, given the $\wedge_{\mathsf{I}}$-rule in LBI, one would show if $\Gamma \vDash \varphi$ and $\Delta \vDash \psi$, then $\Gamma \,;\, \Delta \vDash \varphi * \psi$. Meanwhile, when proving completeness, one typically provides a method for constructing counter-models out of terms; that is, if $\Gamma \nvdash \varphi$, then one constructs a model witnessing $\Gamma \nvDash \varphi$. Such constructions can be subtle and challenging for BI because of the complex structure of bunches and the interaction between the additive and multiplicative parts. In this chapter, we offer an alternative approach that avoids counter- and term-model constructions entirely. Instead, we treat the *unfolding* of the semantics according to its inductive definition analogously to the unfolding of the proof systems in the proof of soundness. The view of these systems as unfolding sets this work in Reductive Logic. This work is based on the following paper:

> Gheorghiu, A. V., and Pym, D. J. Semantical Analysis of the Logic of Bunched Implications. *Studia Logica* (2023)

The main ideas are later generalized in Part II — specifically, Chapter 16 and Chapter 17.

The intuition behind the approach to soundness and completeness in this chapter is that the ways in which proof theory and model theory define the connectives coincide; for example, in both paradigms, additive conjunction is characterized by the behaviour that relative to some available information Γ one has the conjunction $\varphi \land \psi$ if and only if, relative to the same information, one has each of φ and ψ, independently. Essentially, the approach in this chapter proceeds by showing that one may restrict to a sufficiently systematic unfolding of the semantics that it can be *simulated* in the proof system.

To characterize validity in terms of a space of reductions, one needs a proof system for it, which is handled by encoding it in *meta-logic* (FOL — see Chapter 3) such that worlds and formulas become terms and satisfaction becomes a relation symbol. This is similar to other uses of logic to reason about mathematical objects and structures. Therefore, the central part of the paper concerns giving a sequent calculus for a restriction of the meta-logic expressive enough to reason about validity, while tractable enough to characterize its space of reduction. Since we work with eigenvariables representing worlds, dubbed *eigenworlds*, we bypass truth-in-a-model.

10.1 Model-theoretic Semantics

While various model-theoretic semantics for BI have been given in Chapter 6, those are left as motivational background. We consider a variation of relational semantics discussed at the end of Chapter 6. This handling of the semantics is in the style of the style of Routley and Meyer [178] and Urquhart [210] for relevant logics.

Definition 10.1 (Frame)**.** *A frame is a quintuple $\mathcal{F} := \langle \mathbb{U}, e, \pi, \preceq, R \rangle$ in which $\mathbb{U}$ is a set, e and π are distinguished element of the set, $\preceq$ is a preorder on the set dominated by π — that is, for any w in the set, $w \preceq \pi$ — and R is a ternary relation on the set satisfying the following conditions:*

- *(*Unitality*)* $R(w,w,e)$

- *(*Commutativity*)* $R(x,y,z)$ *iff* $R(x,z,y)$

$$
\begin{array}{lll}
w \Vdash \mathrm{p} & \text{iff} & \mathrm{p} \in [\![w]\!] \\
w \Vdash \top & \text{iff} & w \in \mathbb{U} \\
w \Vdash \bot & \text{iff} & w = \pi \\
w \Vdash \top^{*} & \text{iff} & e \preceq w \\
w \Vdash \varphi \wedge \psi & \text{iff} & w \Vdash \varphi \text{ and } w \Vdash \psi \\
w \Vdash \varphi \vee \psi & \text{iff} & w \Vdash \varphi \text{ or } w \Vdash \psi \\
w \Vdash \varphi \to \psi & \text{iff} & \text{for any } v, \text{ if } w \preceq v \text{ and } v \Vdash \varphi, \text{ then } v \Vdash \psi \\
w \Vdash \varphi * \psi & \text{iff} & \text{there are } u, v \text{ st.} R(w,u,v) \text{ and } u \Vdash \varphi \text{ and } v \Vdash \psi \\
w \Vdash \varphi \mathrel{-\!\!*} \psi & \text{iff} & \text{for any } u, v, \text{ if } R(v,w,u) \text{ and } u \Vdash \varphi, \text{ then } v \Vdash \psi
\end{array}
$$

Figure 10.1: Satisfaction for BI

- *(*Associativity*) if $R(x,w,y)$ and $R(y,u,v)$, then there exists a z such that $R(z,w,u)$ and $R(x,z,v)$.*

The elements of $\mathbb{U}$ in a frame $\mathcal{F} := \langle \mathbb{U}, e, \pi, \preceq, R \rangle$ are called *worlds*. We may write $w \in \mathcal{F}$ to denote $w \in \mathbb{U}$.

Definition 10.2 (Interpretation)**.** *Let $\mathcal{F} := \langle \mathbb{U}, e, \pi, \preceq, R \rangle$ be a frame. An interpretation $[\![-]\!] : \mathbb{A} \to \mathcal{F}$ is a mapping $[\![-]\!] : \mathbb{A} \to \mathscr{P}(\mathbb{U})$.*

Definition 10.3 (Pre-model)**.** *A pre-model is a pair $\mathfrak{M} := \langle \mathcal{F}, [\![-]\!] \rangle$ in which $\mathcal{F}$ is a frame and $[\![-]\!] : \mathbb{A} \to \mathcal{F}$ is an interpretation.*

Definition 10.4 (Satisfaction)**.** *Satisfaction is the least relation satisfying the clauses in Figure 10.1, where $\mathfrak{M}$ is a pre-model and $w \in \mathfrak{M}$.*

We require the following *(general) persistence* condition on satisfaction, to model BI:

$$\text{for any } \varphi \in \mathbb{F} \text{ and any } w, u \in \mathbb{U}, \text{ if } w \preceq u \text{ and } w \Vdash \varphi, \text{ then } u \Vdash \varphi$$

Moreover, we require the special world π to be *absurd* — that is, $\pi \Vdash \varphi$ for any φ.

This concept of a model here actually arises from the approach to completeness that this chapter demonstrates. That is, the clauses are designed to reflect the proof-theoretic behaviour of the connectives. This is explored further in Part II.

Definition 10.5 (Model)**.** *A pre-model is a pair* $\mathfrak{M} := \langle \mathcal{F}, [\![-]\!] \rangle$ *is a model when it is persistent and* π *is absurd. The set of all models is* $\mathbb{M}$.

Definition 10.6 (Validity)**.** *A sequent* $\Gamma \rhd \varphi$ *is valid — denoted* $\Gamma \vDash \varphi$ *— if, for any model* $\mathfrak{M} \in \mathbb{M}$*, at any world* w *in* $\mathfrak{M}$*, if* $w \Vdash \Gamma$*, then* $w \Vdash \varphi$.

The intuition for why BI is complete for this class of frames (i.e., $\Gamma \vDash \varphi$ implies $\Gamma \vdash \varphi$) is that it has precisely the structure required to simulate the proof-theoretic characterization of BI. By definition, $\Gamma \vDash \varphi$ means that for an arbitrary world in an arbitrary model such that $w \Vdash \Gamma$, it is also the case that $w \Vdash \varphi$. Since w is arbitrary, this must hold by the contents of Γ, as expressed in terms of the structure of models, not by certain properties of the world at a certain model.

10.2 Analysis of the Semantics

We give a *formal* analysis of the model-theoretic semantics above in a *classical first-order logic* (FOL — see Chapter 3) regarded as meta-logic. This requires the notations established in Chapter 3. Briefly, $\binampersand$ is meta-conjunction, $\bindnasrepma$ is meta-disjunction, $\Rightarrow$ is meta-implication, $\square$ is meta-top, and $\blacktriangleright$ is meta-consequence.

Encoding in a Meta-logic

In short, we express the definitions in Section 10.1 as a theory of FOL. To economize on notation, we will overload a lot of symbols. In particular, we shall use as a symbol in the meta-logic the mathematical object it intends to denote; for example, the symbol $\Vdash$ is overloaded as both a relation symbol in the meta-logic and the satisfaction relation of BI. That this is without confusion follows from the correctness of the encoding.

Definition 10.7 (Meta-alphabet for BI)**.** *The meta-alphabet for BI is the first-order alphabet* $\langle \mathbb{R}, \mathbb{F}, \mathbb{K}, \mathbb{U} \rangle$ *in which the sets are as follows:*

- $\mathbb{R} := \{R, \preceq, \Vdash\}$, *with each symbol of arity* 2

- $\mathbb{F} := \{\top, \bot, \top^*, \wedge, \vee, \to, -\!\!*, *\}$, *and each of* $\top$, $\bot$, *and* $\top^*$ *of artity* 1, *and each of* $\wedge, \vee, \to, -\!\!*, *$ *of arity* 2

- $\mathbb{K} := \mathbb{A} \cup e, \pi$, *with* $\mathbb{A}$ *the atoms of BI.*

- $\mathbb{V} := \mathbb{V}_w \cup \mathbb{V}_f$, *with* $\mathbb{V}_w$ *a denumerable set of* world-variables *and* $\mathbb{V}_f$ *a denumerable set of* formula-variables.

Terms constructed without $\mathbb{V}_w$ are called *formula-terms*. Hence, we use φ to denote a formula-term, which is intuitively a BI-formula in which some subformulas may be formula-variables. Let Γ be a bunch, we may write $(w \Vdash \Gamma)$ to abbreviate the meta-atom $(w \Vdash \gamma)$, where γ is the formula-term corresponding to the formula reading of Γ (Definition 6.5). We call meta-atoms of the form $(w \Vdash \varphi)$ *assertions*.

We desire a collection of meta-formulas Ω such that the following holds:

$$\Omega, (w \Vdash \Gamma) \blacktriangleright (w \Vdash \Delta) \qquad \text{iff} \qquad \Gamma \vDash \Delta$$

Intuitively, the theory Ω is a *definition* of the proposed semantics of BI in the meta-logic in the following sense: A FOL-model of Ω determines a BI-model and *vice versa*. Here w is an eigenvariable; that is, it does not occur anywhere in Ω and, therefore, represents an arbitrary world in the a BI-model corresponding to a given FOL-model of Ω. Constructing Ω is the task of the remainder of this section: it yields completeness if we can also show equivalence to consequence (as defined by provability in LBI) in BI,

$$\Omega, (w \Vdash \Gamma) \blacktriangleright (w \Vdash \Delta) \qquad \text{iff} \qquad \Gamma \vdash \varphi$$

Ultimately, therefore, the approach to completeness in this chapter does not entirely avoid model-existence — of course, connecting models and provability requires for there to be a model *at some point* — but instead outsources it to model-existence for FOL, which is simpler and well-established, and uses the expressiveness of the logic to capture the proposed semantics of BI.

While the task of constructing Ω may appear daunting, it is not. The definitions of the previous section can be encoded in the meta-logic; that is, one may regard the model theory of BI *qua* a theory in the meta-logic. There are two parts to capture: the sentences governing frames $\Omega_{\mathfrak{M}}$ (Definition 10.5) and sentences governing satisfaction $\Omega_{\Vdash}$ (Definition 10.4).

The sentences in $\Omega_{\mathfrak{M}}$ are the universal closure of the following, in which u,v,w,x,y,z are world-variables and φ is a formula variable:

$$\underbrace{R(x,x,e)}_{\text{unitality}} \quad \underbrace{\big(R(x,y,z) \Leftrightarrow R(x,z,y)\big)}_{\text{commutativity}} \quad \underbrace{\big(w \preceq u \Rightarrow (w \Vdash \varphi \Rightarrow u \Vdash \varphi)\big)}_{\text{persistence}}$$

$$\underbrace{\big(R(x,w,y) \& R(y,u,v) \Rightarrow \exists z(R(x,z,v) \& R(z,w,u))\big)}_{\text{associativity}} \quad \underbrace{w = \pi \Rightarrow w \Vdash \varphi}_{\text{absurdity}}$$

The sentences in $\Omega_{\Vdash}$ are given by the universal closure of the meta-formulas in Figure 10.2, which merits comparison with Figure 10.1, in which quantifiers are taken to be over each implicit conjunct separately in the bi-implications — that is, the universal closure of $\Phi \Leftrightarrow \Psi$ means the universal closure of $\Phi \Rightarrow \Psi$ conjoined with the universal closure of the $\Phi \Leftarrow \Psi$. There are two significant differences between Figure 10.1 and Figure 10.2: first, there is no clause for $(w \Vdash \mathrm{p})$, where $\mathrm{p} \in \mathbb{A}$; second, there is no clause for $(w \Vdash \top^*)$. This is an effort to simplify computations about satisfaction in subsequent parts of the paper.

The elimination of a clause for atomic satisfaction $(w \Vdash \mathrm{p})$ follows from working with validity directly (i.e., without passing though truth-in-a-model) as interpretations are no longer required; that is, atomic satisfaction is captured by an atomic tautology, $\Omega, (w \Vdash \mathrm{p}) \triangleright (w \Vdash \mathrm{p})$. We justify the omission of an encoding for the $\top^*$-clause at end of this section.

The concatenation of $\Omega_{\Vdash}$ and $\Omega_{\mathfrak{M}}$ is the desired theory Ω. It is easy to see by its model-theoretic semantics of FOL (see Chapter 3) that the desired condition holds,

$$\Omega, (w \Vdash \Gamma) \blacktriangleright (w \Vdash \varphi) \qquad \text{iff} \qquad \Gamma \vDash \varphi$$

$w \Vdash \top$	iff	
$w \Vdash \bot$	iff	$w = \pi$
$w \Vdash \varphi \land \psi$	iff	$(w \Vdash \varphi)\&(w \Vdash \psi)$
$w \Vdash \varphi \lor \psi$	iff	$(w \Vdash \varphi)\parr(w \Vdash \psi)$
$w \Vdash \varphi \to \psi$	iff	$\forall u(w \preceq u \Rightarrow (u \Vdash \varphi \Rightarrow u \Vdash \psi))$
$w \Vdash \varphi * \psi$	iff	$\exists u, v(R(w,u,v)\&u \Vdash \varphi\&v \Vdash \psi)$
$w \Vdash \varphi \mathrel{-\!\!*} \psi$	iff	$\forall u, w'(R(w',w,u) \Rightarrow (u \Vdash \varphi \Rightarrow w' \Vdash \psi))$

Figure 10.2: Satisfaction for BI (Symbolic)

The significance of this is that all the familiar tools of classical logic become available, including sequent calculi for reasoning about when the above implication holds. The details of the proof-theoretic tools used for the meta-logic in this paper is reserved for Section 10.2.

Definition 10.8 (Basic Validity Sequent)**.** *A basic validity sequent (BVS) is a meta-sequent* $\Omega, (w \Vdash \Gamma) \rhd (w \Vdash \varphi)$.

Definition 10.9 (Complex Validity Sequent)**.** *A complex validity sequent (CVS) is a meta-sequent* $\Omega, \bar{\Sigma} \rhd \bar{\Pi}$ *in which* $\bar{\Sigma}$ *and* $\bar{\Pi}$ *are sets of assertions.*

To conclude this section, we explain why the $\top^*$-clause of satisfaction may be omitted in Ω. Let $\Phi_{\top^*} := \forall x\big((x \Vdash I) \Leftrightarrow (e \preceq x)\big)$, we claim $\Omega, (w \Vdash \Gamma) \blacktriangleright (w \Vdash \top^*)$ iff $\Omega, \Phi^{*}_{\top}, (w \Vdash \Gamma) \blacktriangleright (w \Vdash \top^*)$. This follows from the fact that $\Omega, \Phi_{\top^*}, (w \Vdash \Gamma) \vdash (w \Vdash \top^*)$ iff $\Gamma \vdash \top^*$, which is what we would expect for a model of BI, but then we already have $\Omega, (w \Vdash \Gamma) \blacktriangleright (w \Vdash \top^*)$. In short, the $\top^*$-clause can be removed from frames without loss of generality when encoding in the meta-logic because the sequent calculus rule governing $\top^*$ requires that $\top^*$ is already part of the context — indeed, this is the same reason satisfaction of atoms could be eliminated. The other atomic rules, such as $\top$ and $\bot$ do not satisfy this condition, therefore their clauses are required.

Meta-Logic Proof Theory

Having encoded the (putative) semantics of BI as a theory of the meta-logic, all of the tools of FOL become available. In particular, that a meta-sequent is a consequence of the meta-logic may be established by witnessing a proof for it in a proof system for FOL. In this section, we develop a sequent calculus for the meta-logic that is tractable for analyzing the semantics.

The logic of BI is constructive. Consequently, one expects satisfaction to be constructive in the sense that, if $\Omega, (w \Vdash \Gamma) \blacktriangleright (w \Vdash \varphi)$ obtains, then there should be a constructive proof of it. Therefore, we may restrict to an *intuitionistic* sequent calculus for the meta-logic. This sequent calculus need only be sound and complete for BVSs that are valid in classical logic, we do not require it to be sound and complete for the whole logic.

The following is based on Dummett's [52] multiple-conclusioned system for first-order intuitionistic logic:

Definition 10.10 (Meta-sequent Calculus DLJ). *Meta-sequent Calculus* DLJ *is composed of the rules in Figure 10.3 in which θ_X denotes a substitution for X and $\hat{\theta}_X$ denotes a substitution for X by an eigenvariable.*

We elide rules for negation from Figure 10.3 as Ω is negation-free, so they will not be required at any point. We may use double-lines to suppress the use of multiple inference; for example, we may write

$$\frac{\Phi, \Phi', \Phi'' \triangleright \Pi}{\overline{\overline{(\Phi \,\&\, \Phi') \,\&\, \Phi'' \triangleright \Pi}}}\ \&_{\mathsf{R}}$$

to denote compactly the repeated use of $\&_{\mathsf{L}}$ — that is, to express the following:

$$\dfrac{\dfrac{\Phi, \Phi', \Phi'' \triangleright \Pi}{\Phi \,\&\, \Phi', \Phi'' \triangleright \Pi}\ \&_{\mathsf{R}}}{(\Phi \,\&\, \Phi') \,\&\, \Phi'' \triangleright \Pi}\ \&_{\mathsf{R}}$$

The rule of DLJ are sound for FOL, but not complete — for example, it cannot prove $\varphi \lor \neg\varphi$. We do not require full completeness, but only for it to be complete for BVSs. To this end, it suffices to show that the following rules are admissible

$$\frac{\Sigma \rhd \Pi}{\Phi, \Sigma \rhd \Pi}\,\mathsf{w} \quad \frac{\Sigma \rhd \Pi}{\Sigma \rhd \Pi, \Phi}\,\mathsf{w} \quad \frac{\Sigma_1, \Psi, \Phi, \Sigma_2 \rhd \Pi}{\Sigma_1, \Phi, \Psi, \Sigma_2 \rhd \Pi}\,\mathsf{e} \quad \frac{\Sigma \rhd \Pi_1, \Psi, \Phi, \Pi_2}{\Sigma \rhd \Pi_1, \Phi, \Psi, \Pi_2}\,\mathsf{e}$$

$$\frac{\Sigma \rhd \Phi, \Pi \quad \Sigma \rhd \Psi, \Pi}{\Sigma \rhd \Pi, \Phi \,\&\, \Psi}\,\&_{\mathsf{R}} \qquad \frac{\Phi, \Psi, \Sigma \rhd \Pi}{\Phi \,\&\, \Psi, \Sigma \rhd \Pi}\,\&_{\mathsf{L}}$$

$$\frac{\Phi, \Sigma \rhd \Pi \quad \Psi, \Sigma \rhd \Pi}{\Phi \parr \Psi, \Sigma \rhd \Pi}\,\parr_{\mathsf{L}} \qquad \frac{\Sigma \rhd \Pi, \Phi, \Psi}{\Sigma \rhd \Pi, \Phi \parr \Psi}\,\parr_{\mathsf{R}}$$

$$\frac{\Phi, \Sigma \rhd \Psi}{\Sigma \rhd \Phi \Rightarrow \Psi}\,\Rightarrow_{\mathsf{R}} \qquad \frac{\Phi \Rightarrow \Psi, \Sigma \rhd \Pi, \Phi \quad \Psi, \Phi \Rightarrow \Psi, \Sigma \rhd \Pi}{\Phi \Rightarrow \Psi, \Sigma \rhd \Pi}\,\Rightarrow_{\mathsf{L}}$$

$$\frac{\Phi\theta_X, \forall X\Phi, \Sigma \rhd \Pi}{\forall X\Phi, \Sigma \rhd \Pi}\,\forall_{\mathsf{L}} \qquad \frac{\Sigma \rhd \forall X\Phi, \Phi\hat{\theta}_X}{\Sigma \rhd \forall X\Phi}\,\forall_{\mathsf{R}}$$

$$\frac{\Phi\hat{\theta}_X, \exists X\Phi, \Sigma \rhd \Pi}{\exists X\Phi, \Sigma \rhd \Pi}\,\exists_{\mathsf{L}} \qquad \frac{\Sigma \rhd \Pi, \exists X\Phi, \Phi\theta_X}{\Sigma \rhd \Pi, \exists X\Phi}\,\exists_{\mathsf{R}}$$

$$\frac{}{\Phi \rhd \Phi}\,\mathsf{ax} \qquad \frac{}{\Phi \rhd \Box}\,\Box_{\mathsf{R}}$$

Figure 10.3: Meta-sequent Calculus DLJ

for DLJ-proofs of BVSs as including them recovers a meta-sequent calculus for classical logic:

$$\frac{\Sigma \rhd \Pi, \Phi}{\Sigma \rhd \Pi, \forall x\Phi}\,\forall^{\mathsf{K}}_{\mathsf{R}} \qquad \frac{\Phi, \Sigma \rhd \Pi, \Psi}{\Sigma \rhd \Pi, \Phi \Rightarrow \Psi}\,\Rightarrow^{\mathsf{K}}_{\mathsf{R}} \qquad \frac{\Sigma \rhd \Pi, \Phi, \Phi}{\Sigma \rhd \Pi, \Phi}\,\mathsf{c}_{\mathsf{R}} \qquad \frac{\Phi, \Phi, \Sigma \rhd \Pi}{\Phi, \Sigma \rhd \Pi, \Phi}\,\mathsf{c}_{\mathsf{L}}$$

Two rules are immediate:

Proposition 10.11. *The* c_{R}*-rule and* c_{L}*-rule are admissible in* DLJ.

Proof. Follows from the idempotency of intuitionistic disjunction and intuitionistic conjunction. That is, since $\Phi \Leftrightarrow \Phi \parr \Phi$ is valid in intuitionistic logic, $\Sigma \vdash_{\mathsf{DLJ}} \Pi, \Phi$ obtains iff $\Sigma \vdash_{\mathsf{DLJ}} \Pi, \Phi \parr \Phi$ obtains. Similarly, since $\Phi \Leftrightarrow \Phi \,\&\, \Phi$ is valid in intuitionistic logic, $\Phi, \Sigma \vdash_{\mathsf{DLJ}} \Pi$ obtains iff $\Phi \,\&\, \Phi, \Sigma \vdash_{\mathsf{DLJ}} \Pi$ obtains. The result follows from application of the application of $\parr_{\mathsf{R}}$ and $\&_{\mathsf{L}}$. □

The remaining two rules (i.e.,$\forall^{\mathsf{K}}_{\mathsf{R}}$ and $\Rightarrow^{\mathsf{K}}_{\mathsf{R}}$) are generalized versions of $\forall_{\mathsf{R}}$ and $\Rightarrow_{\mathsf{R}}$, respectively. Define $\mathsf{DLJ}^{\mathsf{K}} := \mathsf{DLJ} \cup \{\forall^{\mathsf{K}}_{\mathsf{R}}, \Rightarrow^{\mathsf{K}}_{\mathsf{R}}, \mathsf{c}_{\mathsf{R}}, \mathsf{c}_{\mathsf{L}}\}$. The relationship between DLJ and $\mathsf{DLJ}^{\mathsf{K}}$ is the same as the relationship between Dummett's [52] (multiple-conclusioned) sequent calculus for intuitionistic logic and Gentzen's [200] sequent calculus for classical logic — that is, that certain rules in the former system are guarded by a single-conlusioned condition that, if relaxed to be multiple-conclusioned, recovers the latter system. Why can this guard be removed for proofs of BVSs without effecting completeness of the calculus? A sufficient guard is already captured in the change of world that takes place, as illustrated in Example 10.17 below.

Rather than consider proofs (or reductions) for the meta-logic in general, we restrict attention to proofs of CVSs and, eventually, BVSs. To this end, reductions that *use* a meta-formula in Ω are controlled.

Definition 10.12 (Resolution)**.** *A resolution is a derivation that instantiates a clause from Ω by applying the $\forall_{\mathsf{L}}$-rule, then applies the $\Rightarrow_{\mathsf{L}}$-rule on the resulting sub-formula, and then removes the sub-formula — that is, a derivation of the following form:*

$$
\begin{prooftree}
\AxiomC{$\Omega, \Sigma \triangleright \Pi, \Phi$}
\RightLabel{w}
\UnaryInfC{$\Omega, \Phi \Rightarrow \Psi, \Sigma \triangleright \Pi, \Phi$}
\AxiomC{$\Omega, \Psi, \Sigma \triangleright \Pi$}
\RightLabel{w}
\UnaryInfC{$\Omega, \Phi \Rightarrow \Psi, \Psi, \Sigma \triangleright \Pi$}
\RightLabel{$\Rightarrow_{\mathsf{L}}$}
\BinaryInfC{$\Omega, \Phi \Rightarrow \Psi, \Sigma \triangleright \Pi$}
\doubleLine
\RightLabel{$\forall_{\mathsf{L}}$}
\UnaryInfC{$\Omega, \Sigma \triangleright \Pi$}
\end{prooftree}
$$

A resolution is closed if the head of the clause matches with an assertion already present in the meta-sequent, and one removes (by using w_{L} or w_{R}) the head in the non-axiom premiss,

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\Omega, \Phi, \Sigma \triangleright \Pi, \Phi$}
\AxiomC{$\Omega, \Psi, \Sigma \triangleright \Pi$}
\RightLabel{w_{L}}
\UnaryInfC{$\Omega, \Psi, \Phi, \Sigma \triangleright \Pi$}
\doubleLine
\RightLabel{$\Uparrow$ *(resolution)*}
\BinaryInfC{$\Omega, \Phi, \Sigma \triangleright \Pi$}
\end{prooftree}
$$

$$
\begin{prooftree}
\AxiomC{$\Omega, \Sigma, \triangleright \Pi, \Phi$}
\RightLabel{w_{R}}
\UnaryInfC{$\Omega, \Sigma, \triangleright \Pi, \Psi, \Phi$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\Omega, \Psi, \Sigma \triangleright \Pi, \Psi$}
\doubleLine
\RightLabel{$\Uparrow$ *(resolution)*}
\BinaryInfC{$\Omega, \Sigma \triangleright \Pi, \Psi$}
\end{prooftree}
$$

It is without loss of generality that we reduce with $\Rightarrow_{\mathsf{L}}$ immediately after reducing with $\forall_{\mathsf{L}}$ as the quantifier rule is invertible. Intuitively, a *closed* resolution is a resolution in which the consequent of the implication replaces the formula that matches the antecedent. A resolution is *open* iff it is not closed.

When a resolution is closed, we may denote the reduction by the premiss that is not a tautology, labeling it by the name of the justifying meta-formula. That is, let f be name of some formula in Ω that instantiates to $\Phi \Rightarrow \Psi$, then closed resolutions using f are as follows:

$$\frac{\Omega,\Psi,\Sigma \rhd \Pi}{\Omega,\Phi,\Sigma \rhd \Pi}\,\mathsf{f}_\mathsf{L} \qquad\qquad \frac{\Omega,\Sigma \rhd \Pi,\Phi}{\Omega,\Sigma \rhd \Pi,\Psi}\,\mathsf{f}_\mathsf{R}$$

When no confusion arises, we may suppress the left or right subscript on these inference. Denoted in this way, resolutions may be thought of as rules (or, more precisely, as reduction operators). This allows us to emphazise the steps that make use of the theory Ω while de-emphasizing the meta-logical ones.

Example 10.13. *Reasoning that $\Gamma \vDash \varphi \wedge \psi$ arrives from $\Gamma \vDash \varphi$ and $\Gamma \vDash \psi$ is represented in the meta-logic by a closed resolution using the $\wedge$-clause:*

$$\dfrac{\dfrac{\dfrac{\Omega,(w \Vdash \Gamma) \rhd (w \Vdash \varphi) \qquad \Omega,(w \Vdash \Gamma) \rhd (w \Vdash \psi)}{\Omega,(w \Vdash \Gamma) \rhd (w \Vdash \varphi) \mathbin{\&} (w \Vdash \psi)}}{\Omega,(w \Vdash \Gamma) \rhd (w \Vdash \varphi) \mathbin{\&} (w \Vdash \psi),(w \Vdash \varphi \wedge \psi)}\,\mathsf{w}_\mathsf{R} \qquad \overline{}\;\mathsf{ax}}{\Omega,(w \Vdash \Gamma) \rhd (w \Vdash \varphi \wedge \psi)}$$

We suppress the conclusion of the ax*-rule for readability. Using the compact notation for closed-resolutions, the same derivation may be denoted as follows:*

$$\dfrac{\dfrac{\Omega,(w \Vdash \Gamma) \rhd (w \Vdash \varphi) \qquad \Omega,(w \Vdash \Gamma) \rhd (w \Vdash \psi)}{\Omega,(w \Vdash \Gamma) \rhd (w \Vdash \varphi) \mathbin{\&} (w \Vdash \psi)}\,\mathbin{\&}_\mathsf{R}}{\Omega,(w \Vdash \Gamma) \rhd (w \Vdash \varphi \wedge \psi)}\,\wedge\text{-}clause$$

■

Since the theory Ω is conserved in DLJ-reductions, henceforth we may suppress it without further comment.

It is reasoning by resolution that captures what it means *to use* a clause of satisfaction, hence the sequent calculus for the meta-logic ought to have resolutions as the primary operational step during reduction. The fact that resolution is how semantic reasoning is conducted is not surprising; after all, that a theory composed of clauses may be used to *define* a predicate is the idea underpinning Horn clause logic programming (LP) — see Kowalski [121].

Resolutions can be used not only to perform computation about the satisfaction relation, but to break up the structure of bunches such that they may be read in the form of a classical context. We may think of this as *unpacking* the bunch. Of course, it is essential that no information is lost in this process.

Definition 10.14 (Unpacking, Packing)**.** *An unpacking of a meta-atom* $(w \Vdash \Gamma)$ *in a meta-sequent* $\Omega, \Sigma, (w \Vdash \Gamma) \rhd \Pi$ *is a sequence of closed resolutions using* $\land$*- and* $*$*-clauses in the context with* $\exists_{\mathsf{L}}$ *and* $\&_{\mathsf{L}}$ *applied eagerly. A packing is the reverse of an unpacking.*

Example 10.15. *The following computation constitutes an unpacking of the meta-formula* $(w \Vdash \Gamma \mathbin{,} (\Delta \fatsemi \Delta'))$ *in the meta-sequent* $(w \Vdash \Gamma \mathbin{,} (\Delta \fatsemi \Delta')), (u \Vdash \Gamma') \rhd (w \Vdash \varphi), (u \Vdash \psi)$*:*

$$\begin{array}{cl}
R(w,x,y), (x \Vdash \Gamma), (y \Vdash \Delta) \& (y \Vdash \Delta' \fatsemi \Delta'), (u \Vdash \Gamma') \rhd (w \Vdash \varphi), (u \Vdash \psi) & \\ \hline
R(w,x,y), (x \Vdash \Gamma), (y \Vdash \Delta \fatsemi \Delta'), (u \Vdash \Gamma') \rhd (w \Vdash \varphi), (u \Vdash \psi) & \land\textit{-clause} \\ \hline\hline
R(w,x,y) \& (x \Vdash \Gamma) \& (y \Vdash \Delta \fatsemi \Delta'), (u \Vdash \Gamma') \rhd (w \Vdash \varphi), (u \Vdash \psi) & \&_{\mathsf{L}} \\ \hline
\exists x, y (R(w,x,y) \& x \Vdash \Gamma \& y \Vdash \Delta \fatsemi \Delta'), (u \Vdash \Gamma') \rhd (w \Vdash \varphi), (u \Vdash \psi) & \exists_{\mathsf{L}} \\ \hline
(w \Vdash \Gamma \mathbin{,} (\Delta \fatsemi \Delta')), (u \Vdash \Gamma') \rhd (w \Vdash \varphi), (u \Vdash \psi) & *\textit{-clause}
\end{array}$$

■

The notation $\Sigma_{w,\Gamma}$ denotes a theory that arises from an unpacking of $(w \Vdash \Gamma)$. Unpackings do not have to be total — that is, one can have $w \Vdash \Gamma(\varphi)$ unpack to a theory $\Sigma_{w,\Gamma(\varphi)}$ containing a meta-formula $x \Vdash \varphi$. In this case, the theory may be denoted $\Sigma_{w,\Gamma(\varphi),x \Vdash \varphi}$. It is partial in the sense that the unpacking does not continue on the assertion $w \Vdash \varphi$,

Proposition 10.16 (Packing)**.** *Both packing and unpackings are invertible.*

Proof. The result follows from the invertibility of $\&_{\mathsf{L}}$ and $\exists_{\mathsf{L}}$, as witnessed by the following computations in which we use dashed lines to represent the inverse of a rule:

$$\begin{array}{cl}
\Sigma, (w \Vdash \varphi \wedge \psi) \triangleright \Pi & \\ \hline
\Sigma, (w \Vdash \varphi) \& (w \Vdash \psi) \triangleright \Pi & \wedge\text{-clause} \\ \hdashline
\Sigma, (w \Vdash \varphi), (w \Vdash \psi) \triangleright \Pi & \&_{\mathsf{R}}{}^{-1} \\ \hline
\Sigma, (w \Vdash \varphi) \& (w \Vdash \psi) \triangleright \Pi & \&_{\mathsf{R}} \\ \hline
\Sigma, (w \Vdash \varphi \wedge \psi) \triangleright \Pi & \wedge\text{-clause}
\end{array}$$

$$\begin{array}{cl}
\Sigma, (w \Vdash \varphi * \psi) \triangleright \Pi & \\ \hline
\Sigma, \exists u, v \big(R(w,u,v) \& (u \Vdash \varphi) \& (v \Vdash \psi)\big) \triangleright \Pi & *\text{-clause} \\ \hdashline
\Sigma, R(w,u,v) \& (u \Vdash \varphi) \& (v \Vdash \psi) \triangleright \Pi & \exists_{\mathsf{L}}{}^{-1} \\ \hdashline\hdashline
\Sigma, R(w,u,v), (u \Vdash \varphi), (v \Vdash \psi) \triangleright \Pi & \&_{\mathsf{R}}{}^{-1} \\ \hline\hline
\Sigma, R(w,u,v) \& (u \Vdash \varphi) \& (v \Vdash \psi) \triangleright \Pi & \&_{\mathsf{R}} \\ \hline
\Sigma, \exists u, v \big(R(w,u,v) \& (u \Vdash \varphi) \& (v \Vdash \psi)\big) \triangleright \Pi & \exists_{\mathsf{L}} \\ \hline
\Sigma, (w \Vdash \varphi * \psi) \triangleright \Pi & *\text{-clause}
\end{array}$$

□

We may now return to the question of the adequacy of DLJ for BVSs despite being intuitionistic. Heuristically, one expects DLJ to be adequate because the guard distinguishing $\Rightarrow_{\mathsf{R}}$ and $\forall_{\mathsf{R}}$ from $\Rightarrow_{\mathsf{R}}^{\mathsf{K}}$ and $\forall_{\mathsf{R}}^{\mathsf{K}}$ is captured by the *change of world* when encountering an implication formula in the extract of a CVS. This idea is witnessed in the following example:

Example 10.17. *To see how the change of world acts as a sufficient guard for BI-validity to be constructive, we may see how* $\mathsf{DLJ}^{\mathsf{K}}$ *avoids the law of the excluded middle (i.e., why* $(w \Vdash \varnothing_\times) \triangleright (w \Vdash \varphi \vee (\varphi \to \bot))$ *from holding in BI:*

$$\begin{array}{cl}
(w \Vdash \varnothing_\times), (u \Vdash \varnothing_\times \fatsemi \varphi) \triangleright (w \Vdash \varphi), (u \Vdash \bot) & \\ \hline
(w \Vdash \varnothing_\times), (u \Vdash \varnothing_\times), (u \Vdash \varphi) \triangleright (w \Vdash \varphi), (u \Vdash \bot) & \wedge\textit{-clause} \\ \hline
(w \Vdash \varnothing_\times), u \preceq w, (u \Vdash \varphi) \triangleright (w \Vdash \varphi), (u \Vdash \bot) & \text{pers.} \\ \hline\hline
(w \Vdash \varnothing_\times) \triangleright (w \Vdash \varphi), (w \preceq u \Rightarrow (u \Vdash \varphi \Rightarrow u \Vdash \bot)) & \Rightarrow_{\mathsf{R}}^{\mathsf{K}} \\ \hline
(w \Vdash \varnothing_\times) \triangleright (w \Vdash \varphi), \forall u (w \preceq u \Rightarrow (u \Vdash \varphi \Rightarrow u \Vdash \bot)) & \forall_{\mathsf{R}}^{\mathsf{K}} \\ \hline
(w \Vdash \varnothing_\times) \triangleright (w \Vdash \varphi), (w \Vdash (\varphi \to \bot)) & \to\textit{-clause} \\ \hline
(w \Vdash \varnothing_\times) \triangleright (w \Vdash \varphi) \parr (w \Vdash (\varphi \to \bot)) & \parr_{\mathsf{R}} \\ \hline
(w \Vdash \varnothing_\times) \triangleright (w \Vdash \varphi \vee (\varphi \to \bot)) & \vee\textit{-clause}
\end{array}$$

Moving to u and using persistence means that one has all the contextual information about w available (i.e., that $w \Vdash \Gamma$ *is in the context enables* $u \Vdash \Gamma$ *to be assumed); but since* $u \Vdash \varphi$ *in the context and* $w \Vdash \varphi$ *in the extract are different atoms since u and w are distinct, one has not reached an axiom. In short despite working in* *a classical system (i.e.,* $\mathsf{DLK}^{\mathsf{K}}$*), suppressing an additional computational step, the above calculation witnesses that* $\varnothing_\times \vDash \varphi \vee \varphi \to \bot$ *if* $\varnothing_\times \vDash \varphi$ *or* $\varphi \vDash \bot$*, which is what one would expect of entailment for a constructive logic such as BI.* ∎

The way the change-of-world guard works is that the CVS to which one reduces when resolving an implications assertion contains two *independent* claims about validity, as witnessed in Example 10.17, which may be separated out.

Definition 10.18 (World-independent)**.** *Sets of meta-formulas* Σ *and* Σ' *are world-independent if no free world-variable appearing in one appears in the other.*

Proposition 10.19. *Let* $\Sigma, \Sigma', \Pi, \Pi'$ *be sets of propositional meta-formulas such that* Σ, Π *and* Σ', Π' *are world-independent:*

$$\Omega, \Sigma, \Sigma' \blacktriangleright \Pi, \Pi' \quad \textit{iff} \quad \Omega, \Sigma \blacktriangleright \Pi \textit{ or } \Omega, \Sigma' \blacktriangleright \Pi'$$

Recall that $\Omega, \Sigma, \Sigma' \blacktriangleright \Pi, \Pi'$ obtains iff there there is a $\mathsf{DLJ}^{\mathsf{K}}$-proof of $\Omega, \Sigma, \Sigma' \rhd \Pi, \Pi'$. We proceed by by induction on $\mathsf{DLJ}^{\mathsf{K}}$-proofs. Note, the proof is sensitive to she shape of formulas in Ω; for example, the induction step would fail if we had the linearity axiom $\forall x, y(x \preceq y \& y \preceq x)$.

Proof. The *if* direction follows immediately by $\mathsf{w_L}$ and w. For the *only if* direction, assume $\Omega, \Sigma, \Sigma' \blacktriangleright \Pi, \Pi'$ and let $\mathcal{D}$ be a $\mathsf{DLJ}^{\mathsf{K}}$-proof of it. We proceed by induction the number of resolutions in $\mathcal{D}$.

Base Case. If $\mathcal{D}$ contains no resolutions, then $\Omega, \Sigma, \Sigma' \rhd \Pi, \Pi'$ is proved by ax together with the rules for the meta-connectives. But then there are proofs for $\Omega, \Sigma \rhd \Pi$ or $\Omega, \Sigma' \rhd \Pi'$ since the rules for the connectives cannot affect what world- or formula-variables.

Induction Step. Recall, without loss of generality, in $\mathsf{DLJ}^{\mathsf{K}}$, the $\forall_{\mathsf{L}}$-rule is always followed by $\Longrightarrow_{\mathsf{L}}^{\mathsf{K}}$. If a resolution of $\Omega, \Sigma, \Sigma' \rhd \Pi, \Pi'$ yields a meta-sequent of the form $\Omega, \Sigma \rhd \Pi$ or $\Omega, \Sigma' \rhd \Pi'$, then the result follows from the induction hypothesis. We show that this is the case.

The only non-obvious case is in the case of a closed resolution using the $\to$-clause or $-\!\!*$-clause in the extract because they have universal quantifiers that would allow one to produce a meta-atom in the extract that contains both a world from Σ,Π and Σ',Π' simultaneously, thereby breaking world-independence. We show the $\to$-case, the other being similar.

Let $\Sigma = \Sigma'', w \Vdash \varphi \to \psi$ and suppose u is a world variable appearing in Σ',Π', then we have the following computation:

$$\dfrac{\dfrac{\dfrac{\Omega,\Sigma'',\Sigma' \rhd \Pi,\Pi', w \preceq u \qquad \Omega,\Sigma'',\Sigma' \rhd \Pi,\Pi',(u \Vdash \varphi) \qquad \Omega,\Sigma'',\Sigma',(u \Vdash \psi) \rhd \Pi,\Pi'}{\Omega,\Sigma'',(w \preceq u \Rightarrow (u \Vdash \varphi \Rightarrow u \Vdash \psi)),\Sigma' \rhd \Pi,\Pi'}\,{\Rightarrow_{\mathsf{L}}}^{\mathsf{K}}}{\Omega,\Sigma'',\forall x(w \preceq x \Rightarrow (x \Vdash \varphi \Rightarrow x \Vdash \psi)),\Sigma' \rhd \Pi,\Pi'}\,\forall_{\mathsf{L}}}{\Omega,\Sigma'',(w \Vdash \varphi \to \psi),\Sigma' \rhd \Pi,\Pi'}\,\to\text{-clause}$$

The meta-atom $(w \preceq u)$ may be removed from the leftmost premiss because the only way for the meta-atom to be used in the remainder of the proof is if $w \preceq u$ appears in the context, but this is impossible. Hence, without loss of generality, $\mathcal{D}$ applies w_{R} to the branch, yielding $\Sigma'',\Sigma' \rhd \Pi,\Pi'$, as required.

□

To prove that DLJ is adequate for proofs of CVSs it only remains to argue that the change-of-world guard is implemented whenever it is required, and that it indeed results in a world-independent situation.

Proposition 10.20. *The $\forall^{\mathsf{K}}_{\mathsf{R}}$ and $\Rightarrow^{\mathsf{K}}_{\mathsf{R}}$ rules are admissible for* DLJ*-proofs of CVSs:*

Proof. By case analysis on Ω, the only place on may require ${\forall_{\mathsf{R}}}^K$ or ${\Rightarrow_{\mathsf{R}}}^K$ over $\forall_{\mathsf{R}}$ or $\Rightarrow_{\mathsf{R}}$ is when resolving an implicational assertion (i.e., using the $\to$-clause or the $-\!\!*$-clause). This is because they are the only clauses whose bodies contain implications; notably, the clause for $*$ does *not* contain a meta-implication, and this is so that it behaves like a conjunction, which delivers the packing and unpacking above, as well as completeness.

In the case of $\to$-clause, without loss of generality, the resolution may be taken to be required for the proof such that persistence is applied eventually to the meta-atom $(w \preceq u)$ producing by the resolution. By permuting resolutions, we may assume that it is used immediately. By Proposition 10.16, these reductions are followed by a packing:

$$\begin{array}{c}
\bar{\Sigma},(w \Vdash \Gamma),(u \Vdash \Gamma \fatsemi \varphi) \triangleright \bar{\Pi},(u \Vdash \psi) \\ \hline
\bar{\Sigma},(w \Vdash \Gamma),(u \Vdash \Gamma),(u \Vdash \varphi) \triangleright \bar{\Pi},(u \Vdash \psi) \\ \hline
\bar{\Sigma},(w \Vdash \Gamma), w \preceq u,(u \Vdash \varphi) \triangleright \bar{\Pi},(u \Vdash \psi) \\ \hline\hline
\bar{\Sigma},(w \Vdash \Gamma) \triangleright \bar{\Pi},(w \preceq u \Rightarrow (u \Vdash \varphi \Rightarrow u \Vdash \psi) \\ \hline
\bar{\Sigma},(w \Vdash \Gamma) \triangleright \bar{\Pi}, \forall u(w \preceq u \Rightarrow (u \Vdash \varphi \Rightarrow u \Vdash \psi))
\end{array}
\begin{array}{l}
\wedge\text{-clause} \\ \text{pers.} \\ \Rightarrow^{\mathsf{K}}_{\mathsf{R}} \\ \forall^{\mathsf{K}}_{\mathsf{R}}
\end{array}$$

Arguing similarly, in the case of the $-\!\!*$-clause, one has the following derivation:

$$\begin{array}{c}
\bar{\Sigma},(w \Vdash \Gamma),(w' \Vdash \Gamma \mathbin{,} \psi) \triangleright \bar{\Pi},(w' \Vdash \psi) \\ \hline
\bar{\Sigma},(w \Vdash \Gamma), R(w',w,u), u \Vdash \varphi \triangleright \bar{\Pi},(w' \Vdash \psi) \\ \hline\hline
\bar{\Sigma},(w \Vdash \Gamma) \triangleright \bar{\Pi},(R(w',w,u) \Rightarrow (u \Vdash \varphi \Rightarrow w' \Vdash \psi)) \\ \hline
\bar{\Sigma},(w \Vdash \Gamma) \triangleright \bar{\Pi}, \forall w',u(R(w',w,u) \Rightarrow (u \Vdash \varphi \Rightarrow w' \Vdash \psi))
\end{array}
\begin{array}{l}
*\text{-clause} \\ \Rightarrow^{\mathsf{K}}_{\mathsf{R}} \\ \forall^{\mathsf{K}}_{\mathsf{R}}
\end{array}$$

In either case, by the eigenvariable condition on universal instantiations, the premiss is a meta-sequent of the form $\Omega, \Sigma, \Sigma' \triangleright \Pi, \Pi'$ in which Σ, Π and Σ', Π' are world-independent. Hence, by Proposition 10.19, one yields premisses that one may have reached using the single-conclusioned variants of the rules; whence, the multiple-conclusioned variants are admissible. □

The adequacy of DLJ follows as a corollary from the preceeding work.

Proposition 10.21. *A CVS holds iff it admits a* DLJ*-proof.*

Proof. Immediate by Proposition 10.20 and Proposition 10.11. □

In the remainder of this section, we eliminate a particular behaviours from DLJ in order to simplify the analysis of the space of reductions for a given CVS: the possibility of introducing world-variables that are irrelevant.

When reducing a CVS, it is possible to instantiate a meta-formula in Ω with a world not present in the meta-sequent, but such a world-variable represents an arbitrary world alien to information about models available in the sequent and therefore, intuitively, it cannot be a required part of the reasoning used to establish or refute the CVS.

Example 10.22. *The following derivation is a reduction of a BVS that begins with a resolution introducing a world alien to the original meta-sequent:*

$$\begin{array}{c}
\Omega,(w \Vdash \mathrm{p} \wedge \mathrm{q}) \triangleright (u \Vdash \top) \qquad \Omega,(w \Vdash \mathrm{p} \wedge \mathrm{q}) \triangleright (w \Vdash \mathrm{p} \vee \mathrm{q}) \\ \hline
\Omega,(u \Vdash \top) \Rightarrow,(w \Vdash \mathrm{p} \wedge \mathrm{q}) \triangleright (w \Vdash \mathrm{p} \vee \mathrm{q}) \\ \hline
\Omega,(w \Vdash \mathrm{p} \wedge \mathrm{q}) \triangleright (w \Vdash \mathrm{p} \vee \mathrm{q})
\end{array}
\begin{array}{l}
\Rightarrow_{\mathsf{L}} \\ \forall_{\mathsf{L}}
\end{array}$$

■

We eliminate computation such as in Example 10.22 so that after resolutions way may always interpret meta-sequents as BI-sequents (see Section 10.2, below).

Definition 10.23 (World-conservative). *A* DLJ*-proof of a CVS is said to be world-conservative if in any instance of* $\forall_{\mathsf{L}}$ *or* $\exists_{\mathsf{R}}$*, every world-variable occurring in the premiss occurs in the conclusion.*

Proposition 10.24. *A CVS holds iff it admits a world-conservative* DLJ*-proof.*

Proof. Since $\forall_{\mathsf{L}}$ has no pre-conditions, the result follows from Proposition 10.21 by renaming variables. That is, suppose a DLJ-proof contains the following inference that is not world-conservative (i.e., $\theta_u : u \mapsto x$ and x does not appear in Σ or Π):

$$\frac{\Omega, \Sigma, \Psi\theta_u \triangleright \Pi}{\Omega, \Sigma, \forall u \Psi \triangleright \Pi}$$

The proof can be made world-conservative by replacing all hereditary occurrences of x in the proof by a world-variable y that does appear in either Σ or Π — for example, the above inference becomes the following, where $\theta'_u : u \mapsto y$:

$$\frac{\Omega, \Sigma, \Psi\theta'_u \triangleright \Pi}{\Omega, \Sigma, \forall u \Psi \triangleright \Pi}$$

This substitution is then propagated up through the reduction.

□

Kreisel [123] has shown that there is no constructive proof of completeness for IPL with respect to its frame semantics. In this paper, the actual proof of completeness (i.e., Corollary 10.33) is certainly not constructive just because DLJ is constructive.

Since the theory Ω is conserved in DLJ-reductions of BVSs, henceforth we may suppress it without further comment.

Reduction & Control

In this section, we give a meta-sequent calculus VBI, which is a restriction of DLJ that we use to characterize reasoning about validity. In particular, it is one in which closed resolutions are enforced precisely where we desire them.

In the meta-logic, we address validity, bypassing truth-at-a-world, because the world-variables in meta-sequents do not stand for particular worlds, but rather are generic representatives of worlds. As such, we may call them *eigenworlds*.

Example 10.25. *Consider a meta-sequent* $(w \Vdash \mathrm{r}) \triangleright (w \Vdash \mathrm{p} * \mathrm{q})$*, in which* p, q*, and* $\mathrm{r} \in \mathbb{P}$*, to which we wish to apply the* $*$*-clause,*

$$\forall \varphi \forall \psi \forall x \big(\exists y \exists z (R(x,y,z) \& x \Vdash \varphi \& y \Vdash \psi \& z \Vdash \psi) \Rightarrow x \Vdash \varphi * \psi \big)$$

Resolving with this clause produces the following meta-sequent:

$$(w \Vdash \mathrm{r}) \triangleright \exists y, z \big(R(w,y,z) \& (y \Vdash \mathrm{p}) \& (z \Vdash \mathrm{q}) \big)$$

In the absence of any specific worlds, one introduces eigenworlds u and v to eliminate the existential quantifiers for y and z, respectively, yielding the following:

$$(w \Vdash \mathrm{r}) \triangleright \big(R(w,u,v) \& (u \Vdash \mathrm{p}) \& (v \Vdash \mathrm{q}) \big)$$

This reasoning can take place at any world in any model; that is, suppose one were given an actual model $\mathfrak{M}$*, then the above shows that if it holds for actual worlds* a,b,c *in* $\mathfrak{M}$ *that* $R(c,a,b)$*,* $a \Vdash \mathrm{p}$*, and* $b \Vdash \mathrm{q}$ *obtain, then* necessarily $c \Vdash \mathrm{p} * \mathrm{q}$ *also obtains.* ■

Our aim is to restrict to a meta-seqeunt calculus in which the possible ways in which one may reason about a BVS can be analyzed. To this end, we introduce a calculus for validity:

Definition 10.26 (System VBI)**.** *System* VBI *is composed of the rules in Figure 10.4 in which the theory* Ω *has been suppressed in the context and* $\mathsf{cl}(\mathsf{asso})$ *is invertible.*

$$\dfrac{w \Vdash \Gamma(\varphi \fatsemi \psi) \triangleright w \Vdash \chi}{w \Vdash \Gamma(\varphi \wedge \psi) \triangleright w \Vdash \chi}\ \mathsf{cl}(\wedge)_{\mathsf{L}} \qquad \dfrac{w \Vdash \Gamma \triangleright w \Vdash \varphi \quad w \Vdash \Gamma \triangleright w \Vdash \psi}{w \Vdash \Gamma \triangleright w \Vdash \varphi \wedge \psi}\ \mathsf{cl}(\wedge)_{\mathsf{R}}$$

$$\dfrac{w \Vdash \Gamma(\varphi , \psi) \triangleright w \Vdash \chi}{w \Vdash \Gamma(\varphi * \psi) \triangleright w \Vdash \chi}\ \mathsf{cl}(*)_{\mathsf{L}} \qquad \dfrac{w \Vdash \Delta_1 \triangleright w \Vdash \varphi_1 \quad w \Vdash \Delta_2 \triangleright w \Vdash \varphi_2}{w \Vdash \Gamma \fatsemi (\Delta_1 , \Delta_2) \triangleright w \Vdash \varphi_1 * \varphi_2}\ \mathsf{cl}(*)_{\mathsf{R}}$$

$$\dfrac{w \Vdash \Gamma(\varphi) \triangleright w \Vdash \chi \quad w \Vdash \Gamma(\psi) \triangleright w \Vdash \chi}{w \Vdash \Gamma(\varphi \vee \psi) \triangleright w \Vdash \chi}\ \mathsf{cl}(\vee)_{\mathsf{L}} \qquad \dfrac{w \Vdash \Gamma \triangleright w \Vdash \varphi_i}{w \Vdash \Gamma \triangleright w \Vdash \varphi_1 \vee \varphi_2}\ \mathsf{cl}(\vee)_{\mathsf{R}}$$

$$\dfrac{w \Vdash \Delta \triangleright w \Vdash \varphi \quad w \Vdash \Gamma(\Delta, \psi) \triangleright w \Vdash \chi}{w \Vdash \Delta \fatsemi \varphi \to \psi \triangleright w \Vdash \chi}\ \mathsf{cl}(\to)_{\mathsf{L}} \qquad \dfrac{w \Vdash \Gamma \fatsemi \varphi \triangleright w \Vdash \psi}{w \Vdash \Gamma \triangleright w \Vdash \varphi \to \psi}\ \mathsf{cl}(\to)_{\mathsf{R}}$$

$$\dfrac{w \Vdash \Delta_2 \triangleright w \Vdash \varphi \quad w \Vdash \Gamma(\Delta_1 , \psi) \triangleright w \Vdash \chi}{w \Vdash \Gamma(\Delta_1 , \Delta_2, \varphi \mathrel{-\!\!*} \psi) \triangleright w \Vdash \chi}\ \mathsf{cl}(\mathrel{-\!\!*})_{\mathsf{L}} \qquad \dfrac{w \Vdash \Gamma , \varphi \triangleright w \Vdash \psi}{w \Vdash \Gamma \triangleright w \Vdash \varphi \mathrel{-\!\!*} \psi}\ \mathsf{cl}(\mathrel{-\!\!*})_{\mathsf{R}}$$

$$\dfrac{w \Vdash \Gamma(\Delta) \triangleright w \Vdash \chi}{w \Vdash \Gamma(\Delta , \varnothing_\times) \triangleright w \Vdash \chi}\ \mathsf{cl}(\top^*)_{\mathsf{L}} \qquad \dfrac{w \Vdash \Gamma(\Delta , \varnothing_\times) \triangleright w \Vdash \chi}{w \Vdash \Gamma(\Delta) \triangleright w \Vdash \chi}\ \mathsf{cl}(\top^*)_{\mathsf{L}}$$

$$\dfrac{w \Vdash \Gamma(\Delta \fatsemi \varnothing_+) \triangleright w \Vdash \chi}{w \Vdash \Gamma(\Delta) \triangleright w \Vdash \chi}\ \mathsf{cl}(\top)_{\mathsf{L}} \qquad \dfrac{}{w \Vdash \Gamma \triangleright w \Vdash \top}\ \mathsf{cl}(\top)_{\mathsf{R}}$$

$$\dfrac{w \Vdash \Gamma(\varphi) \triangleright w \Vdash \chi}{w \Vdash \Gamma(\bot) \triangleright w \Vdash \chi}\ \mathsf{cl}(\bot)_{\mathsf{L}} \qquad \dfrac{}{w \Vdash \Gamma \fatsemi \varphi \triangleright w \Vdash \varphi}\ \mathsf{ax}$$

$$\dfrac{w \Vdash \Gamma(\Delta_2 , \Delta_1) \triangleright w \Vdash \chi}{w \Vdash \Gamma(\Delta_1 , \Delta_2) \triangleright w \Vdash \chi}\ \mathsf{cl(comm)} \qquad \dfrac{w \Vdash \Gamma((\Delta_1 , (\Delta_2) , \Delta_3) \triangleright w \Vdash \chi}{w \Vdash \Gamma(\Delta_1 , (\Delta_2 , \Delta_3)) \triangleright w \Vdash \chi}\ \mathsf{cl(asso)}$$

$$\dfrac{w \Vdash \Gamma(\Delta_2 \fatsemi \Delta_1) \triangleright w \Vdash \chi}{w \Vdash \Gamma(\Delta_1 \fatsemi \Delta_2) \triangleright w \Vdash \chi}\ \mathsf{e}_1 \qquad \dfrac{w \Vdash \Gamma((\Delta_1 \fatsemi \Delta_2) \fatsemi \Delta_3) \triangleright w \Vdash \chi}{w \Vdash \Gamma(\Delta_1 \fatsemi (\Delta_2 \fatsemi \Delta_3)) \triangleright w \Vdash \chi}\ \mathsf{e}_2$$

$$\dfrac{w \Vdash \Gamma(\Delta \fatsemi \Delta) \triangleright w \Vdash \chi}{w \Vdash \Gamma(\Delta) \triangleright w \Vdash \chi}\ \mathsf{c} \qquad \dfrac{w \Vdash \Gamma(\Delta) \triangleright w \Vdash \chi}{w \Vdash \Gamma(\Delta \fatsemi \Sigma) \triangleright w \Vdash \chi}\ \mathsf{w}$$

$$\dfrac{w \Vdash \Delta \triangleright w \Vdash \varphi \quad w \Vdash \Gamma(\varphi) \triangleright w \Vdash \chi}{w \Vdash \Gamma(\Delta) \triangleright w \Vdash \chi}\ \mathsf{cut}$$

Figure 10.4: Meta-sequent Calculus VBI

Theorem 10.27. *A BVS is valid iff it admits a* VBI*-proof.*

Proof. The soundness of VBI is immediate by observing that each rule follows as the application of a meta-formula in Ω; for example, the admissibility of $\mathsf{cl}(\wedge)_\mathsf{R}$ is witnessed in this way in Example 10.13. It remains to argue for the completeness of VBI.

By Proposition 10.24, a BVS is a consequence of the meta-logic iff it admits a world-conservative DLJ-proof. But since DLJ is an intuitionistic calculus, we have the same result for the single-conlusioned variant GLJ (i.e., the rules of DLJ with only one meta-formula in the extract and $\invamp_\mathsf{L}$ forcing one to choose one disjunct). We proceed by case analysis on the possible reductions for the BVS in GLJ and show that they correspond to reductions in VBI.

We may express that there is a reduction taking C to $P_1,...,P_n$ as follows:

$$\frac{P_1 \quad \dots \quad P_n}{C} \Uparrow$$

In particular, if the reduction continues by taking P_i $Q_1,...,Q_m$ the effect may be expressed as follows@

$$\frac{P_1 \quad \dots \quad P_{i-1} \quad Q_1 \quad \dots \quad Q_n \quad P_i \quad \dots \quad P_n}{C} \Uparrow$$

Without loss of generality, each reduction begins with an unpacking of the BVS. We may write $\Pi_{w,\Gamma(\Delta),x}, x \Vdash \Delta$ to denote a theory $\Sigma_{w,\Gamma(\Delta),x \Vdash \Delta}$. Moreover, without loss of generality, in the case of closed resolution, we assume that the resulting sub-formulas are immediately decomposed (i.e., are principal in the next reduction), as otherwise the resolution could have been postponed until this is the case.

By Proposition 10.16, we apply packing eagerly; that is, we apply it whenever it results in a sequent different from the original. Of course, there are more than one possible ways to pack a meta-sequent, but this is no concern as the possible choices simply correspond to $\mathsf{e}_2 \in \mathsf{VBI}$. An example is offered by the following:

$$\begin{array}{c}
(w \Vdash \Gamma((\Delta_1 \fatsemi \Delta_2) \fatsemi \Delta_3) \triangleright (w \Vdash \chi) \\
\hline
\Pi_{w,\Gamma(\Delta_1 \fatsemi (\Delta_2 \fatsemi \Delta_2)),x}, (x \Vdash \Delta_1), (x \Vdash \Delta_2), (x \Vdash \Delta_3) \triangleright (w \Vdash \chi) \\
\hline
(w \Vdash \Gamma(\Delta_1 \fatsemi (\Delta_2 \fatsemi \Delta_3))) \triangleright (w \Vdash \chi)
\end{array}
\begin{array}{c} \Uparrow \\ \Uparrow \end{array}$$

The reductions of BVSs GLJ begin with one of the following: an axiom, an open resolutions, a clause on an assertion in the extract, a clause on an assertion in the context, a frame law, or a structural rule. We structure the case-analysis into these groups for readability.

1. AXIOM. System GLJ contains two axioms: ax and $\square$. Only one of them is applicable to the unpacking of a BVS — namely, ax. If the reduction used ax, then the unpacking of the BVS was of the form $\Sigma_{w,\Gamma}, (w \Vdash \varphi) \triangleright (w \Vdash \varphi)$. This is only possible if the original BVS was of the form $w \Vdash (\Gamma \fatsemi \varphi) \triangleright (w \Vdash \varphi)$. These reductions are captured by VBI as an instance of its version of id.

2. OPEN RESOLUTIONS. Recall, an open resolution is a resolution that is not closed — that is, one in which neither the antecedent nor the consequent of an instantiation of a meta-formula in Ω matches with any meta-formula in the meta-sequent. We consider the generic meta-sequent $w \Vdash \Gamma(\Delta) \triangleright w \Vdash \varphi$, which is unpacked to $\Sigma_{w,\Gamma(\Delta),x \Vdash \Delta} \triangleright (w \Vdash \varphi)$. A generic open resolution is as follows:

$$\dfrac{\Sigma_{w,\Gamma(\Delta),x \Vdash \Delta} \triangleright \Phi \quad \Sigma_{w,\Gamma(\Delta),x \Vdash \Delta}, \Psi \triangleright (w \Vdash \varphi)}{\Sigma_{w,\Gamma(\Delta),x \Vdash \Delta} \triangleright (w \Vdash \varphi)} \Uparrow$$

By world-conservativity and by case-analysis on Ω, it must be that, for some χ, either $\Phi = (x \Vdash \chi)$ or $\Psi = (x \Vdash \chi)$. By the invertibility of the resolutions, we may continue with a closed-resolution to yield the following:

$$\dfrac{\Sigma_{w,\Gamma(\Delta),(x \Vdash \Delta)} \triangleright (x \Vdash \chi) \quad \Sigma_{w,\Gamma(\Delta),(x \Vdash \Delta)}, (x \Vdash \chi) \triangleright (w \Vdash \varphi)}{\Sigma_{w,\Gamma(\Delta),(x \Vdash \Delta)} \triangleright (w \Vdash \varphi)} \Uparrow$$

By Proposition 10.19 and by Proposition 10.16, each branch is then weakened and packed. In total, the reduction from the original BVS is as follows:

$$\frac{(x \Vdash \Delta) \rhd (x \Vdash \chi) \quad (w \Vdash \Gamma(\Delta \fatsemi \chi)) \rhd (w \Vdash \varphi)}{(w \Vdash \Gamma(\Delta)) \rhd (w \Vdash \varphi)} \Uparrow$$

Such reductions are captured by VBI as an instance of c followed by cut.

3. EXTRACT-CLOSED RESOLUTIONS. An extract-closed resolution is a closed resolution on a meta-formula in the extract. Without loss of generality, the unpacking at the beginning of each reduction is trivial when it is not required for the reduction to take place.

$\land$ - Reductions beginning with the $\land$-clause are as follows:

$$\cfrac{\cfrac{(w \Vdash \Gamma) \rhd (w \Vdash \varphi) \quad (w \Vdash \Gamma) \rhd (w \Vdash \psi)}{(w \Vdash \Gamma) \rhd (w \Vdash \varphi) \,\&\, (w \Vdash \psi)} \&_{\mathsf{R}}}{(w \Vdash \Gamma) \rhd (w \Vdash \varphi \land \psi)} \Uparrow$$

These reductions are captured by VBI as $\mathsf{cl}(\land)_{\mathsf{R}}$.

$\lor$ - Reduction beginning with the $\lor$-clause are as follows:

$$\cfrac{\cfrac{(w \Vdash \Gamma) \rhd (w \Vdash \varphi_i)}{(w \Vdash \Gamma) \rhd (w \Vdash \varphi_1) \parr (w \Vdash \varphi_2)} \parr_{\mathsf{R}}}{(w \Vdash \Gamma) \rhd (w \Vdash \varphi \lor \psi)} \Uparrow$$

These reductions are captured by VBI as $\mathsf{cl}(\lor)_{\mathsf{R}}$.

$\to$ - Reductions beginning with the $\to$-clause are as follows:

$$\frac{(w \Vdash \Gamma) \rhd (w \preceq u) \Rightarrow \big((u \Vdash \varphi) \Rightarrow (u \Vdash \psi)\big)}{(w \Vdash \Gamma) \rhd (w \Vdash \varphi \to \psi)} \Uparrow$$

By the invertibility of $\Rightarrow_{\mathsf{R}}$, this is continued to yield the following:

$$\frac{(w \Vdash \Gamma), (w \preceq u), (u \Vdash \varphi) \rhd (u \Vdash \psi)}{(w \Vdash \Gamma) \rhd (w \Vdash \varphi \to \psi)} \Uparrow$$

Without loss of generality, this reduction is continued by persistence. This follows by Proposition 10.19 as, if not, then $(w \Vdash \Gamma)$ and $(w \preceq u)$ may be removed without loss of completeness, but this removal can still happen after persistence. Moreover, by Proposition 10.16, the reduction is thence continued by a packing. In total, the reduction is as follows:

$$\frac{(w \Vdash \Gamma \fatsemi \varphi) \triangleright (u \Vdash \psi)}{(w \Vdash \Gamma) \triangleright (w \Vdash \varphi \to \psi)} \Uparrow$$

These reductions are captured by VBI as $\mathsf{cl}(\to)_{\mathsf{R}}$.

$\top$ - Reductions beginning with the $\top$-clause are as follows:

$$\frac{(w \Vdash \Gamma) \triangleright}{(w \Vdash \Gamma) \triangleright (w \Vdash \top)} \Uparrow$$

Without loss of generality, the reduction ends by application of the $\square_{\mathsf{R}}$-axiom. These reductions are captured in VBI as instances of $\mathsf{cl}(\top)_{\mathsf{R}}$.

$\bot$ - Reductions beginning with the $\bot$-clause are as follows:

$$\frac{(w \Vdash \Gamma) \triangleright (w = \bot)}{(w \Vdash \Gamma) \triangleright (w \Vdash \bot)} \Uparrow$$

Without loss of generality, this is continued by the same reduction in reverse, but this is equivalent to doing no reduction at all. Hence, we do not require a rule in VBI corresponding to this case.

$*$ - Reductions beginning with the $*$-clause are as follows:

$$\frac{\dfrac{\Sigma_{w,\Gamma} \triangleright R(w,u,v) \quad \Sigma_{w,\Gamma} \triangleright (u \Vdash \varphi) \quad \Sigma_{w,\Gamma} \triangleright (v \Vdash \psi)}{\Sigma_{w,\Gamma} \triangleright R(w,u,v) \& (u \Vdash \varphi) \& (v \Vdash \psi)} \Uparrow}{\Sigma_{w,\Gamma} \triangleright (w \Vdash \varphi * \psi)} \Uparrow$$

This can only lead to a proof if there were $R(w,u,v), (u \Vdash \varphi), (v \Vdash \psi) \in \Sigma_{w,\Gamma}$, in which case $\Gamma = \Gamma' \fatsemi (\Delta \,,\, \Delta')$. But then, without loss of generality, ax is applied to one branch and Proposition 10.19 to the others, so that the reduction yields the following:

$$\frac{\Sigma_{u \Vdash \Delta_1} \rhd (u \Vdash \varphi_1) \quad \Sigma_{v \Vdash \Delta_2} \rhd (v \Vdash \varphi_2)}{\Sigma_{w,\Gamma' ; (\Delta_1 ; \Delta_2)} \rhd (w \Vdash \varphi_1 * \varphi_2)} \Uparrow$$

Without loss of generality, by Proposition 10.16, the reduction is continued by packing. These reductions are captured in VBI as instances of $\mathsf{cl}(*)_{\mathsf{R}}$.

$-\!\!*$ - Reductions beginning with the $-\!\!*$-clause are as follows:

$$\frac{(w \Vdash \Gamma) \rhd R(w', w, u) \& (u \Vdash \varphi) \Rightarrow (w' \Vdash \psi)}{(w \Vdash \Gamma) \rhd (w \Vdash \varphi -\!\!* \psi)} \Uparrow$$

By invertibility of $\Rightarrow_{\mathsf{R}}$ and $\&_{\mathsf{L}}$, this is continued to yield the following:

$$\frac{(w \Vdash \Gamma), R(w', w, u), (u \Vdash \varphi) \rhd (w' \Vdash \psi)}{(w \Vdash \Gamma) \rhd (w \Vdash \varphi -\!\!* \psi)} \Uparrow$$

Without loss of generality, by Proposition 10.16, this is continued with a packing. These reductions are captured in VBI as instances of $\mathsf{cl}(-\!\!*)_{\mathsf{R}}$.

4. Context-closed Resolutions. A context-closed resolution is a closed resolution on a meta-formula in the context. Each case begins with an unpacking that produces some assertion $x \Vdash \chi$ on which the clause defining the case is applied.

$\wedge$ - Reductions beginning with the $\wedge$-clause are as follows:

$$\frac{\Pi_{w,\Gamma(\varphi \wedge \psi)}, (x \Vdash \varphi), (x \Vdash \psi) \rhd (w \Vdash \chi)}{\Pi_{w,\Gamma(\varphi \wedge \psi)}, (x \Vdash \varphi \wedge \psi) \rhd (w \Vdash \chi)} \Uparrow$$

Without loss of generality, by Proposition 10.16, it is continued by a packing. These reductions are captured in VBI as instances of $\mathsf{cl}(\wedge)_{\mathsf{L}}$.

$\vee$ - Reductions beginning with the $\vee$-clause are as follows:

$$\dfrac{\dfrac{\Pi_{\Gamma(\varphi\vee\psi),x}, (x \Vdash \varphi) \rhd (w \Vdash \chi) \quad \Pi_{\Gamma(\varphi\vee\psi),x}, (x \Vdash \varphi) \rhd (w \Vdash \chi)}{\Pi_{\Gamma(\varphi\vee\psi),x}, (x \Vdash \varphi) \parr (x \Vdash \psi) \rhd (w \Vdash \chi)}\;\parr_{\mathsf{L}}}{\Pi_{\Gamma(\varphi\vee\psi),x}, (x \Vdash \varphi \vee \psi) \rhd w \Vdash \chi}\;\Uparrow$$

Without loss of generality, by Proposition 10.16, each branch is continued by a packing. These reductions are captured in VBI as instances of $\mathsf{cl}(\vee)_{\mathsf{L}}$.

$\to$ - Reductions beginning with the $\to$-clause are as follows:

$$\dfrac{\Pi_{w,\Gamma(\Delta\fatsemi\varphi\to\psi),x}, (x \Vdash \Delta), \forall y\big((x \preceq y) \Rightarrow \big((y \Vdash \varphi) \Rightarrow (y \Vdash \psi)\big)\big) \rhd w \Vdash \chi}{\Pi_{w,\Gamma(\Delta\fatsemi\varphi\to\psi),x}, (x \Vdash \Delta), (x \Vdash \varphi \to \psi) \rhd w \Vdash \chi}\;\Uparrow$$

The only choice of instantiation that can terminate in a proof is to instantiate the quantified world-variable as x. At this point the resulting sub-formula can be decomposed or else the resolution could be permuted with the next resolution. Hence, the reduction is continued as follows:

$$\dfrac{\dfrac{\Pi_{w,\Gamma(\Delta\fatsemi\varphi\to\psi),x}, (x \Vdash \Delta) \rhd (x \Vdash \varphi) \quad \Pi_{w,\Gamma(\Delta\fatsemi\varphi\to\psi),x}, (x \Vdash \Delta), (x \Vdash \psi) \rhd (w \Vdash \chi)}{\Pi_{w,\Gamma(\Delta\fatsemi\varphi\to\psi),x}, (x \Vdash \Delta), \big((x \preceq x) \Rightarrow \big((x \Vdash \varphi) \Rightarrow (x \Vdash \psi)\big)\big) \rhd (w \Vdash \chi)}\;\Rightarrow_{\mathsf{L}}}{\Pi_{w,\Gamma(\Delta\fatsemi\varphi\to\psi),x}, (x \Vdash \Delta), (x \Vdash \varphi \to \psi) \rhd (w \Vdash \chi)}\;\Uparrow$$

Without loss of generality, by Proposition 10.16, each branch is continued by a packing. These reductions are captured in VBI as instances of $\mathsf{cl}(\to)_{\mathsf{L}}$.

$\top$ - There are two possible reduction patterns beginning with the $\top$-clause. First, one may have the following:

$$\dfrac{\Pi_{w,\Gamma(\Delta\fatsemi\varnothing_{+}),x}, (x \Vdash \Delta) \rhd (w \Vdash \chi)}{\Pi_{w,\Gamma(\Delta\fatsemi\varnothing_{+}),x}, (x \Vdash \Delta), (x \Vdash \varnothing_{+}) \rhd (w \Vdash \chi)}\;\Uparrow$$

Without loss of generality, by Proposition 10.16, each branch is continued by a packing. These reductions are captured in VBI as instances of w.

Second, one may have the following:

$$\frac{\Pi_{w,\Gamma(\Delta),x}, (x \Vdash \Delta), (x \Vdash \varnothing_{+}) \rhd (w \Vdash \chi)}{\Pi_{w,\Gamma(\Delta),x}, (x \Vdash \Delta) \rhd (w \Vdash \chi)} \Uparrow$$

Without loss of generality, by Proposition 10.16, each branch is continued by a packing. These reductions are captured in VBI as instances of $\mathsf{cl}(\top)_{\mathsf{L}}$.

$\bot$ - Reductions beginning with the $\bot$-clause are as follows:

$$\frac{\Pi_{w,\Gamma(\bot),x}, (x = \pi) \rhd (w \Vdash \chi)}{\Pi_{w,\Gamma(\bot),x}, (x \Vdash \bot) \rhd (w \Vdash \chi)} \Uparrow$$

If another resolution is made then the the two resolution could have been permuted, unless the resolution was with the absurdity law, in which case the reduction continued to yield the following:

$$\frac{\Pi_{w,\Gamma(\bot),x}, (x \Vdash \varphi) \rhd (w \Vdash \chi)}{\Pi_{w,\Gamma(\bot),x}, (x \Vdash \bot) \rhd (w \Vdash \chi)} \Uparrow$$

Without loss of generality, by Proposition 10.16, each branch is continued by a packing. These reductions are captured in VBI as instances of $\mathsf{cl}(\bot)_{\mathsf{L}}$.

$*$ - There are two possible reduction patterns beginning with the $*$-clause. First, one may have the following:

$$\frac{\Pi_{w,\Gamma(\varphi*\psi),x}, R(x,u,v), (u \Vdash \varphi), (v \Vdash \psi) \rhd (w \Vdash \chi)}{\Pi_{w,\Gamma(\varphi*\psi),x}, (x \Vdash \varphi * \psi) \rhd (w \Vdash \chi)} \Uparrow$$

Without loss of generality, by Proposition 10.16, each branch is continued by a packing. These reductions are captured in VBI as instances of $\mathsf{cl}(*)^{1}_{\mathsf{L}}$.

Second, one may have the following:

$$\frac{\Pi_{w,\Gamma(\Delta,\top^{*}),x}, R(x,x,e), (x \Vdash \Delta), (e \Vdash \top^{*}) \rhd (w \Vdash \chi)}{\Pi_{w,\Gamma(\Delta,\top^{*}),x}, (x \Vdash \Delta * \top^{*}) \rhd (w \Vdash \chi)} \Uparrow$$

Without loss of generality, by Proposition 10.19 and Proposition 10.16, this is continued to yield the following:

$$\frac{(w \Vdash \Gamma(\Delta)) \rhd (w \Vdash \chi)}{\Pi_{w,\Gamma(\Delta ; \top^*),x}, (x \Vdash \Delta * \top^*)), (x \Vdash \Delta ; \top^*) \rhd (w \Vdash \chi)} \Uparrow$$

These reductions are captured in VBI as instances of $\mathsf{cl}(*)^2_{\mathsf{L}}$.

$-\!\!*$ - Reductions beginning with the $-\!\!*$-clause are as follows, in which $\Sigma := \{R(x,y,z), R(z,u,v)\}$ and $\Psi := \forall a, b\big(R(b,v,a) \Rightarrow (a \Vdash \varphi \Rightarrow b \Vdash \psi)\big)$:

$$\frac{\Pi_{w,\Gamma(\Delta ; \Delta' ; \varphi -\!\!* \psi),x}, (y \Vdash \Delta), (u \Vdash \Delta'), \Sigma \rhd (w \Vdash \chi)}{\Pi_{w,\Gamma(\Delta ; \Delta' ; \varphi -\!\!* \psi),x}, \Sigma, \Psi, (y \Vdash \Delta), (u \Vdash \Delta'), (v \Vdash \varphi -\!\!* \psi) \rhd (w \Vdash \chi)} \Uparrow$$

There is only one choice of instantiation for a and b that can terminate in a proof, which yields the the following reduction pattern, in which $\Psi' := R(x,y,z), R(z,u,v), R(z,v,u) \Rightarrow (u \Vdash \varphi \Rightarrow z \Vdash \psi)$:

$$\frac{\Pi_{w,\Gamma(\Delta ; \Delta' ; \varphi -\!\!* \psi),x}, \Sigma, (y \Vdash \Delta), (u \Vdash \Delta'), \Psi' \rhd (w \Vdash \chi)}{\Pi_{w,\Gamma(\Delta ; \Delta' ; \varphi -\!\!* \psi),x}, \Sigma, (y \Vdash \Delta), (u \Vdash \Delta'), (v \Vdash \varphi -\!\!* \psi) \rhd w \Vdash \chi} \Uparrow$$

The sub-formula is immediately decomposed or else this resolution and the next could have been permuted. Hence, the reduction continues to yield subgoals

$$\Pi_{w,\Gamma(\Delta ; \Delta' ; \varphi -\!\!* \psi),x}, R(x,y,z), R(z,u,v), (y \Vdash \Delta), (u \Vdash \Delta') \rhd (u \Vdash \varphi)$$

and

$$\Pi_{w,\Gamma(\Delta ; \Delta' ; \varphi -\!\!* \psi),x}, R(x,y,z), R(z,u,v), (y \Vdash \Delta), (u \Vdash \Delta') \rhd (w \Vdash \chi)$$

Without loss of generality, by Proposition 10.16, each branch is continued by a packing. These reductions are captured in VBI as instances of $\mathsf{cl}(-\!\!*)_{\mathsf{L}}$.

5. Case Analysis on the Frame Laws. The frame laws are unitality of

e, commutative of R, associativity of R, persistence of $\preceq$, dominance of $\preceq$ and the absurdity of π. Except for the first three frame laws, the clauses can only be used after a particular resolution has occurred that introduces the appropriate atom, and these cases have been considered above; for example, persistence requires $(w \preceq u)$ to appear in the context, which can only happen if $(w \Vdash \varphi \to \psi)$ was resolved in the extract. We consider here the remaining cases.

Unit. - Reductions beginning with unitality are as follows:

$$\frac{\Sigma_{\Gamma(\Delta),x}, (x \Vdash \Delta), R(x,x,e) \rhd (w \Vdash \chi)}{\Sigma_{\Gamma(\Delta),x}, (x \Vdash \Delta) \rhd (w \Vdash \chi)} \Uparrow$$

Without loss of generality, by Proposition 10.16, the reduction is continued with a packing. But, this simply yields the original sequent. Otherwise, it may be that a weakening on $x \Vdash \Delta$ and $R(x,x,e)$ is performed and then the packing occurs. These reductions are captured in VBI as instances of $\mathsf{cl}(\top^*)_{\mathsf{L}}$.

Comm. - Reductions beginning with commutativity of R are as follows:

$$\frac{\Pi_{\Gamma(\Delta,\Delta'),x}, R(x,v,u), (u \Vdash \Delta), (v \Vdash \Delta') \rhd (w \Vdash \chi)}{\Pi_{\Gamma(\Delta,\Delta'),x}, R(x,u,v), (u \Vdash \Delta), (v \Vdash \Delta') \rhd (w \Vdash \chi)} \Uparrow$$

Without loss of generality, by Proposition 10.16, this is continued by a packing. These reductions are captured in VBI as instances of $\mathsf{cl}(\mathsf{comm})_{\mathsf{L}}$.

Asso. - Reductions beginning with associativity of R are as follows:

$$\frac{\Pi_{\Gamma(\Delta,(\Delta',\Delta''))}, R(x,a,v), (y \Vdash \Delta), R(a,z,u), (u \Vdash \Delta'), (v \Vdash \Delta'') \rhd (w \Vdash \chi)}{\Pi_{\Gamma(\Delta,(\Delta',\Delta''))}, R(x,y,z), (y \Vdash \Delta), R(z,u,v), (u \Vdash \Delta'), (v \Vdash \Delta'') \rhd (w \Vdash \chi)} \Uparrow$$

Without loss of generality, by Proposition 10.16, this is continued by a packing. These reductions are captured in VBI as instances of $\mathsf{cl}(\mathsf{asso})_{\mathsf{L}}$.

6. CASE ANALYSIS OF THE STRUCTURAL RULES. There are instances of the structural rules that do not result in a change of sequent after packing; for example, permuting meta-atoms that are not assertions is without effect. In the following we

restrict attention to the cases where the use of the structural rule affects the packing of the sequent.

e - Reductions beginning with an exchange are as follows:

$$\frac{\Pi_{\Gamma(\Delta ; \Delta'),x}, (x \Vdash \Delta'), (x \Vdash \Delta') \triangleright (w \Vdash \chi)}{\Pi_{\Gamma(\Delta ; \Delta'),x}, (x \Vdash \Delta), (x \Vdash \Delta') \triangleright (w \Vdash \chi)} \Uparrow$$

Without loss of generality, by Proposition 10.16, this is continued by a packing. These reductions are captured in VBI as instances of e_1.

c - Reductions beginning with contractions are as follows:

$$\frac{\Pi_{\Gamma(\Delta),x}, (x \Vdash \Delta), (x \Vdash \Delta) \triangleright (w \Vdash \chi)}{\Pi_{\Gamma(\Delta),x}, (x \Vdash \Delta) \triangleright (w \Vdash \chi)} \Uparrow$$

Without loss of generality, by Proposition 10.16, this is continued by a packing. These reductions are captured in VBI as instances of c.

w - Reductions beginning with weakening are as follows:

$$\frac{\Pi_{\Gamma(\Delta),x}, (x \Vdash \Delta) \triangleright (w \Vdash \chi)}{\Pi_{\Gamma(\Delta ; \Delta'),x}, (x \Vdash \Delta), (x \Vdash \Delta') \triangleright (w \Vdash \chi)} \Uparrow$$

Without loss of generality, by Proposition 10.16, this is continued by a packing. These reductions are captured in VBI as instances of c.

This completes the proof.

□

It is useful to make precise how to read BI content from a BVS.

Definition 10.28 (State)**.** *The state of a BVS* $\Omega, (w \Vdash \Gamma) \triangleright (w \Vdash \varphi)$ *is the BI-sequent* $\Gamma \triangleright \varphi$.

Each rule in VBI can be directly read in terms of its effect on states. In this way, it is then a calculus of validity; for example, the $\mathsf{cl}(\wedge)_\mathsf{R}$-rule captures the following action on states:

$$\frac{\Gamma \triangleright \varphi \quad \Gamma \triangleright \psi}{\Gamma \triangleright \varphi \wedge \psi}$$

This we recognize as the $\wedge_{\mathsf{R}}$-rule in sLBI. In this way, we may compare the behaviour of validity and the behaviour of provability, thereby establishing behavioural equivalence, for which extensional equivalence (i.e., soundness and completeness) is a corollary.

10.3 Completeness via Meta-logic Proof-search

We show that the semantics in this chapter, which is fully described by VBI, characterizes BI. We do this by showing that proof-search in VBI amounts to proof-search in a sequent calculus for BI. Specifically the following calculus:

Definition 10.29 (System sLBI)**.** *System* sLBI *is composed of the rules in Figure 10.5 in which* asso *is invertible.*

Proposition 10.30. $\Gamma \vdash_{\mathsf{LBI}} \varphi$ *iff* $\Gamma \vdash_{\mathsf{sLBI}} \varphi$.

Proof. This is shown by routine results showing that all the rules of LBI are derivable in VBI, and *vice versa* — see Gheorghiu [82]. □

In reductive logic, we may regard sequent calculi as *transition systems* that dictate how sets of propositions may be transformed. This is how the proof-search delivers an operational semantics of LP. In general, suppose desire to prove that a sequent S is a consequence of a logic. We apply a rule of the sequent calculus (reductively) to get a set of putative consequence $S_1, ..., S_n$, each of which we desire to show is a consequence of the logic — that is, if S_i reduces to $\{S_i^1, ..., S_i^k\}$, then we reduce to $\{S_1, ..., S_i^1, ..., S_i^k, ..., S_n\}$. The process terminates when we reach the empty set, which happens when all of the putative premisses are eventually reduced to axioms. In this view, our approach to soundness and completeness is to show that VBI and sLBI are *equivalent* as transition systems.

There are many notions of equivalence between transition system. Here we are concerned with the subset that pertain to *behavioural* equivalence; that is, how

$$\frac{}{\Gamma \fatsemi \varphi \triangleright \varphi}\ \mathsf{taut} \qquad \frac{}{\bot \triangleright \varphi}\ \bot_{\mathsf{L}} \qquad \frac{}{\varnothing_{\times} \triangleright \top^{*}}\ \top^{*}_{\mathsf{R}} \qquad \frac{}{\Gamma \fatsemi \varnothing_{+} \triangleright \top}\ \top_{\mathsf{R}}$$

$$\frac{\Delta' \triangleright \varphi \quad \Gamma(\Delta \,,\, \psi) \triangleright \chi}{\Gamma(\Delta \,,\, \Delta' \,,\, \varphi \mathrel{-\!\!*} \psi) \triangleright \chi}\ \mathrel{-\!\!*}_{\mathsf{L}} \qquad \frac{\Delta \,,\, \varphi \triangleright \psi}{\Delta \triangleright \varphi \mathrel{-\!\!*} \psi}\ \mathrel{-\!\!*}_{\mathsf{R}}$$

$$\frac{\Delta(\varphi \,,\, \psi) \triangleright \chi}{\Delta(\varphi * \psi) \triangleright \chi}\ *_{\mathsf{L}} \qquad \frac{\Delta \triangleright \varphi \quad \Delta' \triangleright \psi}{\Gamma \fatsemi (\Delta \,,\, \Delta') \triangleright \varphi * \psi}\ *_{\mathsf{R}} \qquad \frac{\Delta(\varnothing_{\times}) \triangleright \chi}{\Delta(\top^{*}) \triangleright \chi}\ \top^{*}_{\mathsf{L}}$$

$$\frac{\Delta(\varphi \fatsemi \psi) \triangleright \chi}{\Delta(\varphi \wedge \psi) \triangleright \chi}\ \wedge_{\mathsf{L}} \qquad \frac{\Delta \triangleright \varphi \quad \Delta \triangleright \psi}{\Delta \triangleright \varphi \wedge \psi}\ \wedge_{\mathsf{R}} \qquad \frac{\Delta(\varnothing_{+}) \triangleright \chi}{\Delta(\top) \triangleright \chi}\ \top_{\mathsf{L}}$$

$$\frac{\Delta(\varphi) \triangleright \chi \quad \Delta(\psi) \triangleright \chi}{\Delta(\varphi \vee \psi) \triangleright \chi}\ \vee_{\mathsf{L}} \qquad \frac{\Delta \triangleright \varphi}{\Delta \triangleright \varphi \vee \psi}\ {\vee_{\mathsf{R}}}^{1} \qquad \frac{\Delta \triangleright \psi}{\Delta \triangleright \varphi \vee \psi}\ {\vee_{\mathsf{R}}}^{2}$$

$$\frac{\Delta \triangleright \varphi \quad \Gamma(\Delta \fatsemi \psi) \triangleright \chi}{\Gamma(\Delta \fatsemi \varphi \to \psi) \triangleright \chi}\ \to_{\mathsf{L}} \qquad \frac{\Delta \fatsemi \varphi \triangleright \psi}{\Delta \triangleright \varphi \to \psi}\ \to_{\mathsf{R}}$$

$$\frac{\Gamma(\Delta \fatsemi \varnothing_{+}) \triangleright \chi}{\Gamma(\Delta) \triangleright \chi}\ \mathsf{c}_{\varnothing_{+}} \qquad \frac{\Gamma(\Delta \,,\, \varnothing_{\times}) \triangleright \chi}{\Gamma(\Delta) \triangleright \chi}\ \mathsf{c}_{\varnothing_{\times}}$$

$$\frac{\Gamma(\Delta) \triangleright \chi}{\Gamma(\Delta \fatsemi \varnothing_{+}) \triangleright \chi}\ \mathsf{w}_{\varnothing_{+}} \qquad \frac{\Gamma(\Delta) \triangleright \chi}{\Gamma(\Delta \,,\, \varnothing_{\times}) \triangleright \chi}\ \mathsf{w}_{\varnothing_{\times}}$$

$$\frac{\Gamma(\Delta' \fatsemi \Delta) \triangleright \chi}{\Gamma(\Delta \fatsemi \Delta') \triangleright \chi}\ \mathsf{comm}_{+} \qquad \frac{\Gamma(\Delta' \,,\, \Delta) \triangleright \chi}{\Gamma(\Delta \,,\, \Delta') \triangleright \chi}\ \mathsf{comm}_{\times}$$

$$\frac{\Gamma((\Delta' \fatsemi \Delta) \fatsemi \Delta'') \triangleright \chi}{\Gamma(\Delta \fatsemi (\Delta' \fatsemi \Delta'') \triangleright \chi}\ \mathsf{asso}_{+} \qquad \frac{\Gamma((\Delta' \,,\, \Delta) \,,\, \Delta'') \triangleright \chi}{\Gamma(\Delta \fatsemi (\Delta' \,,\, \Delta'')) \triangleright \chi}\ \mathsf{asso}_{\times}$$

$$\frac{\Delta(\Delta') \triangleright \chi}{\Delta(\Delta' \fatsemi \Delta'') \triangleright \chi}\ \mathsf{w} \qquad \frac{\Delta \triangleright \varphi \quad \Gamma(\varphi) \triangleright \chi}{\Gamma(\Delta) \triangleright \chi}\ \mathsf{cut} \qquad \frac{\Delta(\Delta' \fatsemi \Delta') \triangleright \chi}{\Delta(\Delta') \triangleright \chi}\ \mathsf{c}$$

Figure 10.5: Sequent Calculus sLBI

transitions in one system may be understood as transitions in the other. The finest notion of behavioural equivalence is *bisimulation*.

Definition 10.31 (Bisimulation of Transition Systems). *Let $\mathfrak{T}_1 := \langle \mathbb{S}_1, \rightsquigarrow_1 \rangle$ and $\mathfrak{T}_n := \langle \mathbb{S}_2, \rightsquigarrow_2 \rangle$ be transition systems. A relation $\sim \subseteq \mathbb{S}_1 \times \mathbb{S}_2$ is a bisimulation between $\mathfrak{T}_1$ and $\mathfrak{T}_2$ iff, for any $\sigma_1 \in \mathbb{S}_1$ and $\sigma_2 \in \mathbb{S}_2$ such that $\sigma_1 \sim \sigma_2$:*

- *if there is $\sigma_1' \in \mathbb{S}_1$ such that $\sigma_1 \rightsquigarrow_1 \sigma_1'$, then there is $\sigma_2 \in \mathbb{S}_2$ such that $\sigma_2 \rightsquigarrow_2 \sigma_2'$ and $\sigma_1' \sim \sigma_2'$;*
- *if there is $\sigma_2' \in \mathbb{S}_2$ such that $\sigma_2 \rightsquigarrow_2 \sigma_2'$, then there is $\sigma_1 \in \mathbb{S}_1$ such that $\sigma_1 \rightsquigarrow_1 \sigma_1'$ and $\sigma_1' \sim \sigma_2'$.*

The transition systems are bisimilar iff there is a bisimulation between them.

Theorem 10.32. *Reduction in* sLBI *is bisimilar to reduction in* VBI.

Proof. Let $\sim$ be the least relation between BVSs and states,

$$\{\Omega, (w \Vdash \Gamma) \triangleright (w \Vdash \varphi)\} \sim \{\Gamma \triangleright \varphi\}$$

By observing the symmetry of the rules in Figure 10.5 and Figure 10.4, we see that $\sim$ is a bisimulation. □

By unpacking the soundness proof of Theorem 10.27 within the proof of Theorem 10.32, one recovers the usual inductive proof of soundness. The contribution of this paper is to demonstrate an analogous technique for proving completeness. In this case, unpacking the completeness proof of Theorem 10.27 within the proof of Theorem 10.32 one recovers a *co-inductive* proof of completeness. This highlights the duality between soundness and completeness.

Corollary 10.33. *Provability is extensionally equivalent to validity,*

$$\Gamma \vdash \varphi \qquad \textit{iff} \qquad \Gamma \vDash \varphi$$

We say *extensionally* equivalent in Corollary 10.33 to emphasize the work in this paper in delivering *behavioural* equivalence.

Of course, other completeness results follows too. Particularly interesting are those pertaining to the additive and multiplicative fragments of BI — that is, sound and completeness for IPL and MILL, respectively.

To end this chapter, we consider the difference the treatment of disjunction by Kripke [125] and Beth [20] — see Chapter 2. The clause used by Kripke means is such that validity of $\Gamma \triangleright \varphi \vee \psi$ behaves as the $\vee_{\mathsf{R}}$-rule of LBI. However, this proof-theoretic definition for disjunction is not necessarily the most natural one. Intuitively, $\leq$ represents $\vdash_{\mathsf{IPL}}$ (by persistence). We may understand $\Gamma \vdash_{\mathsf{IPL}} \varphi_1 \vee \varphi_2$ to mean that, if at some day one know Γ, then at a latter day one knows $\varphi_1 \vee \varphi_2$. Therefore, one may understand that there is a sequence of days in which one constructs from the information in Γ either φ_1 or φ_2. In this reading, the following rules are a natural way to understand disjunction:

$$\frac{\Gamma \triangleright \Delta \quad \Delta \triangleright \varphi_i}{\Gamma \triangleright \varphi_1 \vee \varphi_2}$$

The semantic clause corresponding to these rules is Beth's clause for disjunction — see Chapter 2.

Chapter 11

Conclusion to Part I

This part of the monograph concerns reduction and proof-search in the *logic of Bunched Implications* (BI) [150], and their relationship to the model-theoretic semantics of the logic. While the technical results largely develop the proof theory of BI, the part illustrates that semantics may be profitably studied from the point of view of Reductive Logic. In this way, it may be regarded as a motivating case-study to the thesis of the monograph — that is, that the interplay between semantics and proof theory may be witnessed from the perspective of Reductive Logic. The choice of BI for the case-study arises from the fact that the logic has a relatively subtle meta-theory due to the interaction between the additive and multiplicative connectives through the bunched structure of contexts. This makes it a useful setting to review the traditional techniques and intuitions on the meta-theory of IPL (see Chapter 2) as the more complex setting exposes subtleties not otherwise apparent.

Chapter 6 serves as an introduction to BI. It contains a brief summary of some of the work on the interpretation of BI as the logic over pre-ordered monoids, which coheres with its resource reading. This background illustrates the subtlety of the logic's metatheory.

Chapter 7 provides a proof of cut-admissibility in the standard sequent calculus for BI. More precisely, it provies a rewrite method that transforms a proof with cut into a proof without cuts. While cut-admissibility is know for BI — originally proved by Brotherston [27] — it was not through a rewrite method, but rather in-directly through display calculi. Such a method is valuable for several reasons; for example, it provides a method for turning a non-analytic proof into an analytic one.

Chapter 8 begins the proper study of proof-search in BI. It provides an operational reading of the hereditary Harrop fragment of the logic, which becomes a logic programming language when the usual control structures are applied (e.g., a backtracking scheme). This operational reading yields a model-theoretic semantics for the fragment. This exposes that the semantics of the logical constants is quite close to their reductive behaviour, but that meta-theoretic constructions (e.g., the T-operator) are required to precisely encapsulate the depth and breadth of the proof-search space. The chapter also gives a coalgebraic interpretation of the operational reading of the hereditary Harrop fragment. This shifts the discussion into a setting in which tools from coalgebra (and category theory, more generally) can be used to ask and address questions of control.

Chapter 9 shows that BI has the *focusing property*. Essentially, this generalizes the proof-theoretic foundations of the operational reading of the hereditary Harrop fragment in Chapter 9 to the entire logic. Moreover, it illustrate the use of the rewrite-method for cut-admissibility in Chapter 6 as it is this method (applied to a focused system) that yields the focusing principle, by gradually transforming unfocused proofs into focused proofs. This illustrates that, while subtle, the proof theory for BI is relatively well-behaved.

Chapter 10 studies the semantics of BI from the perspective of Reductive Logic. It provides a proof of soundness and completeness of BI with respect to a model-theoretic semantics using proof-search in a meta-logic. Importantly, this avoid term- and counter-model constructions, which are complex in BI because of the bunched structure of contexts. The observation is that the unfolding of validity judgements according to the inductive definition of the semantics is behaviourally equivalent to the unfolding of a proof system for the logic. In this way, this generalizes the observations in Chapter 8 regarding the proximity of the meaning of the logical constants and their reductive behaviour. The relationship between meta-logic and object-logic exposed in this chapter is what motivates Part II.

Overall, this part provides several technical results for BI that together illustrate the relationship between proof(-search) and semantics in Reductive Logic. While this relationship is studied for BI, it does not appear to use anything in particular from that logic. Instead, that the investigation works despite the subtleties and complexities in the meta-theory of BI suggests that it may be conducted in general. While we have entirely concentrated on this question — that is, the relationship between semantics and proof theory via Reductive Logic — there are other questions that should be addressed relative to the technical results herein; in particular, having concentrated on 'computational' results (e.g., a rewriting proof of cut-admissibility, logic programming, etc.), one expects associated computational analysis of the methods/results (e.g., decidability, complexity, etc).

Part II

Algebraic Constraint Systems

Chapter 12

Introduction to Part II

This part is about a method for a general, uniform approach to studying the proof-theoretic presentations of logics which provides insight into how they work and how they are related to one another. For example, studying substructural systems helps expose subtleties in fundamental results in proof theory such as cut-admissibility (see, for example, Miller and Pimentel [141]) and proof-search (see, for example, Andreoli [6]). Specifically, this part introduce a framework in which one can represent the reasoning in a logic, as captured by a concept of proof for that logic, in terms of the reasoning within another logic through an algebra of constraints — as a slogan,

$$\text{Proof in } \mathsf{L} = \text{Proof in } \mathsf{L}' + \text{Algebra of Constraints } \mathcal{A}$$

Such decompositions of L' into L and $\mathcal{A}$ allow us to study the metatheory of the former by analyzing the latter. The advantage is that the latter is typically simpler in some desirable way — for example, it may relax the side conditions on the use of certain rules — which facilitates, in particular, the study of proof-search with the original logic of interest. We shall refer back to this slogan often and will use the following abbreviated form:

$$\mathsf{L} = \mathsf{L}' \oplus \mathcal{A}$$

The $\oplus$ is not formal — that is, $\mathsf{L}' \oplus \mathcal{A}$ may be used for several ways of applying constraints $\mathcal{A}$ to L'. This work is based on the following paper:

Gheorghiu, A. V., and Pym, D. J. Defining Logical Systems via Algebraic Constraints on Proofs. *Journal of Logic and Computation* (2023). (to appear)

The original example of the kind of decomposition in this part is the *resource-distribution via Boolean constraints* (RDvBC) mechanism introduced by Harland and Pym [99, 98] for the study of proof-search in substructural logics. It is explained in detail in Chapter 13, but we give a brief account here to illustrate what the decomposition above means. Let $V = [x_1,...,x_n]$ be a list of (Boolean) variables. Let $\Gamma \cdot V$ denote the pointwise distributions of the variables over the formulae of Γ — that is, $[\varphi_1,...,\varphi_n] \cdot V := [\varphi_1 \cdot x_1,....,\varphi_n \cdot x_n]$. Let $\bar{V}$ denote the dual of V — that is, $\bar{V} := [\bar{x}_1,...,\bar{x}_n]$. The rule governing multiplicative conjunction can then be expressed as follows:

$$\frac{\Gamma \cdot V \triangleright \varphi \quad \Gamma \cdot V \triangleright \psi}{\Gamma \triangleright \varphi \& \psi} \qquad \frac{\Gamma \cdot V \triangleright \varphi \quad \Gamma \cdot \bar{V} \triangleright \psi}{\Gamma \triangleright \varphi \otimes \psi}$$

One *applies* the constraints by giving an assignment of the Boolean variables and evaluating formulae $\varphi \cdot x$ by keeping φ when $x = 1$ and deleting φ when $x = 0$. Thus, $\Gamma \cdot V$ represents, at the meta-level, a decomposition of Γ. Through this system, one may analyze the various context-management strategies as deciding how to split the context is postponed until the end of reduction. In terms of this slogan above, RDvBC witnesses the decomposition of a substructural logic into classical/intuitionistic logic together with Boolean constraints.

The decompositions expressed by the slogan above may be iterated in valuable ways. Each time we do such a decomposition, the combinatorics of the proof system becomes simpler as more and more is delegated to the algebraic constraint. Eventually, the combinatorics becomes as simple as possible, and one recovers something with all the flexibility of the proof theory for classical logic. Thus, we advance the view that, in general, classical logic forms a combinatorial core of syntactic reasoning since its proof theory is comparatively relaxed — that is, possibly after iterating decompositions, one eventually witnesses the following:

$$\text{Proof in } \mathsf{L} = \text{Classical Proof} + \text{Algebra of Constraints } \mathcal{A}$$

The view of classical logic as the core of logic has, of course, been advanced before — see, for example, Gabbay [66].

Using techniques from universal algebra, we define the algebraic constraints by a theory of first-order classical logic; for example, we may define Boolean algebra by its axiomatization — see Chapter 13. We then enrich the rules of a system L with expressions from $\mathcal{A}$ to express the rules of another system L'. There are several examples presented within, but we give a brief account for clarity presently. A system comprising rules enriched in this way is called a *(algebraic) constraint system* (ACS).

We consider two kinds of relationships an ACS may have with the logic of interest. A constraint system is *sound and complete* when the evaluation of construction from the constraints system concludes a sequent iff that sequent is valid in the logic. A stronger relationship is *faithfulness and adequacy*:

- *Faithfulness*. The evaluation of a construction from ACS is a proof in the logical system of interest.

- *Adequacy*. Every proof in the logical system of interest is the evaluation of some construction from the ACS.

Both relationships are important, as illustrated by several examples.

The point of ACSs is that they allow us to study the metatheory of the logic of interest. There are two principal such activities: first, one may use ACSs to study proof-search in the logic of interest; second, they may be used to bridge the gap between the proof theory and model theory of a logic. On the latter use, ACSs allow a general account of the novel approach to soundness and completeness proofs for BI in Part I, which by-passes term- and counter-model constructions; furthermore, they give a principled way of generating a correct-by-design model-theoretic semantics for the logic of interest by analyzing a proof system for it, making essential use of algebraic constraints and the aforementioned decomposition to classical logic.

Of course, the idea that one may use labels to internalize the semantics of logics within proof systems has taken several forms and goes back as far as Kanger [111]. It underpins a systematic development of analytic tableaux (see, for example, Fitting [59, 60], Catach [32], Massacci [138], Baldoni [12], Docherty and Pym [47, 49, 44], and Galmiche and Méry [73, 71, 74, 72]), natural deduction systems (see, for example, Simpson [196], and Basin et al. [14]), sequent calculi (see, for example, Mints [144, 145], Viganò [212], Kushida and Okada [128]). Particularly significant within this stream are the relational calculi studied by Negri [148, 146, 147].

The notion of ACS is closely related to Gabbay's *Labelled Deductive Systems* (LDSs) [67] — see also Russo [180]. However, the paper deviates from the established theory of LDSs in two fundamental ways: First, one may choose any syntactic structure in the grammar of the object-logic (e.g., data composed of formulae, such as sets, multisets, bunches), not just formulae, to annotate; second, the labels do not only express additional information but have an action on the structure. Note, there are other proof systems in the literature in which one labels data composed of formulae — see, for example, Marx et al. [137]. Consequently, more subtle examples are also available, not otherwise captured by LDSs.

We begin in Chapter 13 by explaining RDvBC to give a clear idea of what ACSs are and how they work. In Chapter 14, we give a general account of *propositional logic* relative to which the theory of ACSs is developed. The formal account of ACS follows in Chapter 15. A systematic account for generating relational calculi — in the sense of Negri [147] — is given in terms of ACSs and illustrates their use for metatheory. Chapter 17 uses ACSs to *derive* the semantics for IPL given by Kripke [125] (see Chapter 2) from its proof-theoretic characterizations; this illustrates how ACS (more generally, Reductive Logic) bridges the gap between proof theory and semantics. Importantly, it exposes that the meaning of the logical constants is implicit in their rules, which is the subject of Part III. The part concludes in Chapter 18 with a summary of the ideas and results.

Chapter 13

Example: Resource-distribution via Boolean Constraints

This chapter provides a terse but complete account of *resource-distribution via Boolean constraints* (RDvBC) mechanism used to study proof-search in substructural logics. The purpose of including it is to provide intuition and motivate the idea and use of *(algebraic) constraint systems* (ACSs). It summarizes parts of the following paper:

> Harland, J., and Pym, D. J. Resource-distribution via Boolean Constraints. *ACM Transactions on Computational Logic 4*, 1 (2003), 56–90

In that paper, RDvBC is developed for LL, BI, and (uniformly) for a family of relevant/relevance logics. This chapter only concerns BI as its background was established in Part I.

Ultimately, what makes proof-search in LBI complex are the multiplicative connectives (i.e., $*$ and $-\!\!*$) — or, more generally, the synchronous rules for any of the connectives — as they demand deciding how to break up the context during reduction.

Example 13.1. *The following proof-search attempts differ only in the choice of* *distribution, one successfully produces a proof and the other fails:*

$$\underbrace{\dfrac{\dfrac{}{\mathsf{p} \rhd \mathsf{p}}\,\mathsf{ax} \quad \dfrac{\dfrac{}{\mathsf{q} \rhd \mathsf{q}}\,\mathsf{ax} \quad \dfrac{}{\mathsf{r} \rhd \mathsf{r}}\,\mathsf{ax}}{\mathsf{q} , \mathsf{r} \rhd \mathsf{q} * \mathsf{r}}\,{*}\mathsf{R}}{\mathsf{p} , \mathsf{q} , \mathsf{r} \rhd \mathsf{p} * (\mathsf{q} * \mathsf{r})}\,{*}\mathsf{R}}_{\textit{succeeds}} \qquad \underbrace{\dfrac{\dfrac{?}{\mathsf{p} , \mathsf{q} \rhd \mathsf{p}} \quad \dfrac{?}{\mathsf{r} \rhd \mathsf{q} * \mathsf{r}}}{\mathsf{p} , \mathsf{q} , \mathsf{r} \rhd \mathsf{p} * (\mathsf{q} * \mathsf{r})}\,{*}\mathsf{R}}_{\textit{fails}}$$

How can we analyze the various distribution strategies? ■

There is a literature of intricate rules of inference in multiplicative logics that are used to keep track of the relevant information to enable proof-search, but they are generally tailored for one particular distribution method — see, for example, Hodas and Miller [104, 103], Winikoff and Harland [214], Cervasto [33], and Lopez [133]. In this context, Harland and Pym [99, 100] introduced RDvBC. The idea is that rather than commit to a particular strategy for managing the distribution, one uses Boolean expressions to express that a resource distribution needs to be made and the conditions it needs to satisfy.

Briefly, in the RDvBC mechanism, one assigns a Boolean expression to each formula requiring distribution. Constraints on the possible values of this expression are then generated during the proof-search and propagated up the search tree, resulting in a set of Boolean equations. A successful proof-search in the enriched system will generate a set of equations such that each solution corresponds to a distribution of formulae across the branches of the structure. Instantiating that distribution results in an actual LBI-proof.

We begin by defining the constraint algebra that delivers RDvBC:

Definition 13.2 (Boolean Algebra). *A* Boolean algebra *is a structure* $\mathcal{B} := \langle \mathbb{B}, \{+, \times, \bar{\cdot}\}, 0, 1\rangle$ *in which* $\mathbb{B}$ *is a set,* $+ : \mathbb{B}^2 \to \mathbb{B}$*,* $\times : \mathbb{B}^2 \to \mathbb{B}$*,* $\bar{\cdot} : \mathbb{B} \to \mathbb{B}$ *are operators on* $\mathbb{B}$*, and* $0, 1 \in \mathbb{B}$*, satisfying the following conditions in Figure 13.1 for any* $a, b, c \in \mathbb{B}$*:*

A *presentation* of the Boolean algebra is a first-order alphabet (see Chapter 3) with equality for which the Boolean algebra is a model. We use the following, in which X is a set of *variables*, e are *expressions* (i.e., meta-terms), and φ are *Boolean*

$$\begin{aligned}
a+(b+c) &= (a+b)+c & a\times(b\times c) &= (a\times b)\times c \\
a+b &= b+c & a\times b &= b\times a \\
a+(a\times b) &= a & a\times(a+b) &= a \\
a+0 &= a & a\times 1 &= a \\
a+(b\times c) &= (a+b)\times(a+c) & a\times(b+c) &= (a\times b)+(a\times c) \\
a+\bar{a} &= 1 & a\times\bar{a} &= 0
\end{aligned}$$

Figure 13.1: Boolean Algebra

formulae (i.e., meta-formulae):

$$e ::= x \in X \mid e+e \mid e\times e \mid \bar{e} \mid 0 \mid 1 \qquad \varphi ::= (e=e) \mid \varphi \& \varphi \mid \varphi \invamp \varphi \mid \neg\varphi \mid \forall x\varphi \mid \exists x\varphi$$

We are overloading $+$ and $\times$ to be both function-symbols in the term language and their corresponding operators in the Boolean algebra; similarly, we are overloading 0 and 1 to be both constants in the term language and the bottom and top element of the Boolean algebra. This is to economize on notation. We may suppress the $\times$ when no confusion arises — that is, $t_1 \times t_2$ may be expressed $t_1 t_2$. For a list of Boolean expressions $V = [e_1, \dots e_n]$, let $\bar{V} := [\bar{e}_1, \dots \bar{e}_n]$; we may write $V = e$ to denote that V is a list containing only e. We may write $V = V_1 \sqcup V_2$ to denote that V is a concatenation of V_1 and V_2.

Let $\mathbb{A}$ be the set of atoms, and let $\mathbb{F}$ be the set of BI-formulae (over $\mathbb{A}$) — see Chapter 6. An *annotated BI-formula* is a BI-formula φ together with a Boolean expression e, denoted $\varphi \cdot e$ — for example, $p \cdot x$ is an annotation of the p by the Boolean variable x. The annotation of a bunch Γ by a list of Boolean expressions V is defined inductively as follows:

- if $\Gamma = \gamma$, where $\gamma \in \mathbb{F} \cup \{\varnothing_+, \varnothing_\times\}$ and $V = [e]$, then $\Gamma \cdot V := \gamma \cdot e$;

- if $\Gamma = (\Delta_1 \fatsemi \Delta_2)$, and $V = [e]$, then $\Gamma \cdot V := (\Delta_1 \fatsemi \Delta_2) \cdot e$;

- if $\Gamma = (\Delta_1 \mathbin{,} \Delta_2)$, and $V = V_1 \sqcup V_2$, then $\Gamma \cdot V := (\Delta_1 \cdot V_1 \fatsemi \Delta_2 \cdot V_2)$.

For example, $p \fatcomma (q \fatsemi r) \cdot [x,y] := p \cdot x \fatcomma (q \fatsemi r) \cdot y$. Intuitively, the annotation of bunches only acts on the top-level of multiplicative connectives and treats everything below (e.g., additive sub-bunches) as formulae. This makes sense as all of the distributions in LBI take place at this level of the bunch.

The idea in RDvBC is that Boolean constraints are used to *mark* the distribution of formulae during reduction. This mechanism is captured by working a sequent calculus $\mathsf{LBI}_{\mathcal{B}}$ over sequents enriched with Boolean expressions. The same names are used for rules in $\mathsf{LBI}_{\mathcal{B}}$ and LBI to economize on notation.

Definition 13.3 (Constraint System $\mathsf{LBI}_{\mathcal{B}}$)**.** *Constraint System* $\mathsf{LBI}_{\mathcal{B}}$ *comprised of the rules in Figure 13.2, in which V is a list of Boolean variables that do not appear in any sequents present in the tree.*

An $\mathsf{LBI}_{\mathcal{B}}$-reduction is a tree constructed by applying the rules of $\mathsf{LBI}_{\mathcal{B}}$ reductively, beginning with a sequent in which each formula is annotated by 1.

Example 13.4. *The following is is an* $\mathsf{LBI}_{\mathcal{B}}$*-reduction* $\mathcal{D}$*:*

$$\dfrac{\dfrac{(x_1 = 1) \& (x_2 = 0) \& (x_3 = 0)}{(\mathrm{p} \cdot x_1) \fatcomma (\mathrm{q} \cdot x_2) \fatcomma (\mathrm{r} \cdot x_3) \triangleright \mathrm{p} \cdot 1}\,\mathsf{ax} \qquad \mathcal{D}'}{(\mathrm{p} \cdot 1) \fatcomma (\mathrm{q} \cdot 1) \fatcomma (\mathrm{r} \cdot 1) \triangleright \mathrm{p} * (\mathrm{q} * \mathrm{r}) \cdot 1}\,*_{\mathsf{R}}$$

— here $\mathcal{D}'$ is the following:

$$\dfrac{\dfrac{(\bar{x}_1 y_1 = 0) \& (\bar{x}_2 y_2 = 1) \& (\bar{x}_3 y_3 = 0)}{(\mathrm{p} \cdot \bar{x}_1 y_1) \fatcomma (\mathrm{q} \cdot \bar{x}_2 y_2) \fatcomma (\mathrm{r} \cdot \bar{x}_3 y_3) \triangleright \mathrm{q}}\,\mathsf{ax} \quad \dfrac{(\bar{x}_1 \bar{y}_1 = 0) \& (\bar{x}_2 \bar{y}_2 = 0) \& (\bar{x}_3 \bar{y}_3 = 1)}{\mathrm{p} \cdot (\bar{x}_1 \bar{y}_1) \fatcomma \mathrm{q} \cdot (\bar{x}_2 \bar{y}_2) \fatcomma \mathrm{r} \cdot (\bar{x}_3 \bar{y}_3) \triangleright \mathrm{r}}\,\mathsf{ax}}{(\mathrm{p} \cdot \bar{x}_1) \fatcomma (\mathrm{q} \cdot \bar{x}_2) \fatcomma (\mathrm{r} \cdot \bar{x}_3) \triangleright \mathrm{q} * \mathrm{r} \cdot 1}\,*_{\mathsf{R}}$$

It is mostly easily understood when read reductively as one thinks of the constraints as being generated during proof-search instead of guessed at the beginning of a deductive construction. ∎

Having produced an $\mathsf{LBI}_{\mathcal{B}}$-reduction, if the constraints are consistent, their solutions correspond to interpretations of the variables such that the constraints are satisfied. Such interpretations I induce a valuation v_I that acts on formulae by keeping formulae whose label evaluate to 1 and deleting (i.e., producing the empty-string

ε) for formulae whose label evaluate to 0; that is, let φ be a BI-formula and e a Boolean expression,

$$v_I(\varphi \cdot e) := \begin{cases} \varphi & \text{if } I(e) = 1 \\ \varepsilon & \text{if } I(e) = 0 \end{cases}$$

A valuation extends to sequents by acting on each formulae occurring in it; it extends to $\mathsf{LBI}_{\mathcal{B}}$-reductions by acting on each sequent occurring in it and removing the constraints.

Example 13.5 (Example 13.4 cont'd)**.** *The constraints on $\mathcal{D}$ are satisfied by any interpretation $I(z) = 1$ for $z \in \{x_1, y_2\}$ and $I(z) = 0$ for $z \in \{x_2, x_3, y_1, y_3\}$. For any such I, the tree $v_I(D)$ is as follows:*

$$\dfrac{\dfrac{}{\mathrm{p} \rhd \mathrm{p}}\ \mathsf{ax} \qquad \dfrac{\dfrac{}{\mathrm{q} \rhd \mathrm{q}}\ \mathsf{ax} \quad \dfrac{}{\mathrm{r} \rhd \mathrm{r}}\ \mathsf{ax}}{\mathrm{q}, \mathrm{r} \rhd \mathrm{q} * \mathrm{r}}\ *_{\mathsf{R}}}{\mathrm{p}, \mathrm{q}, \mathrm{r} \rhd \mathrm{p} * (\mathrm{q} * \mathrm{r})}\ *_{\mathsf{R}}$$

This is the successful derivation in LBI *in Example 13.1. Significantly, by observing the constraints, we see that a distribution strategy results in a successful proof-search only if it sends only the first formula to the left branch.* ■

Harland and Pym [99, 100] proved that $\mathsf{LBI}_{\mathcal{B}}$ is *faithful* and *adequate* for LBI in the following sense:

- *Faithfulness*. If $\mathcal{R}$ is an $\mathsf{LBI}_{\mathcal{B}}$-reduction and I is an interpretation satisfying those constraints, then $v_I(\mathcal{R})$ is a LBI-proof.
- *Adequacy*. If $\mathcal{D}$ is an LBI-proof, then there is a $\mathsf{LBI}_{\mathcal{B}}$-reduction $\mathcal{R}$ and an interpretation I satisfying the constraints on $\mathcal{R}$ such that $v_I(\mathcal{R}) = \mathcal{D}$.

Recall that we may think of BI as the free combination of *intuitionistic propositional logic* (IPL) and *intuitionistic multiplicative linear logic* (IMLL) — see Chapter 6. Accordingly, we regard LBI as the combination of sequent calculi for these two logics (i.e., LJ and IMLL, respectively) — that is,

$$\mathsf{LBI} = \mathsf{LJIMLL}$$

Heuristically, the RDvBC mechanism outsources the substructurallity in IMLL to Boolean constraints. Hence, in the form of the slogan of this paper,

$$\mathsf{IMLL} = \mathsf{LJ} \oplus \mathcal{B}$$

In this section, we have chosen to study BI (as opposed to just IMLL) to illustrate the modularity of constraint systems. That is, we only have constraints participating actively in part of the sequent calculus for BI, but with the same overall effect since the other part conserves them. Abusing the slogan somewhat, we may express the work of this section as follows:

$$\mathsf{LBI} = \mathsf{LJ} \cup (\mathsf{LJ} \oplus \mathcal{B})$$

The subsequent chapters of this part aim to define the use of constraints in proof systems in general.

$$\frac{y = 1 \& V := 0}{\varphi \cdot y \fatcomma \Delta \cdot V \triangleright \varphi \cdot 1}\ \mathsf{ax} \qquad \frac{y = 1 \& V = 0}{\bot \cdot y \fatcomma \Delta \cdot V \triangleright \varphi}\ \bot_{\mathsf{L}}$$

$$\frac{y = 1 \& V = 0}{\varnothing_{\times} \cdot y \fatcomma \Delta \cdot V \triangleright \top^{*}}\ \top^{*}_{\mathsf{R}} \qquad \frac{y = 1 \& V = 0}{\varnothing_{+} \cdot y \fatcomma \Delta \cdot V \triangleright \top}\ \top_{\mathsf{R}}$$

$$\frac{\Delta \cdot V \triangleright \varphi \cdot e \quad \Gamma(\Delta \cdot \bar{V} \fatcomma \psi \cdot e) \triangleright \chi \quad e = 1}{\Gamma(\Delta \fatcomma \varphi \mathrel{-\!\!*} \psi \cdot e) \triangleright \chi}\ \mathrel{-\!\!*}_{\mathsf{L}} \qquad \frac{\Delta \fatcomma \varphi \cdot e \triangleright \psi \cdot e \quad e = 1}{\Delta \triangleright (\varphi \mathrel{-\!\!*} \psi) \cdot e}\ \mathrel{-\!\!*}_{\mathsf{R}}$$

$$\frac{\Delta(\varphi \cdot e \fatcomma \psi \cdot e) \triangleright \chi \quad e = 1}{\Delta((\varphi * \psi) \cdot e) \triangleright \chi}\ *_{\mathsf{L}} \qquad \frac{\Delta \cdot V \triangleright \varphi \quad \Delta' \cdot \bar{V} \triangleright \psi}{\Delta \fatcomma \Delta' \triangleright \varphi * \psi}\ *_{\mathsf{R}}$$

$$\frac{\Delta(\varphi \cdot e \fatsemi \psi \cdot e) \triangleright \chi \quad e = 1}{\Delta((\varphi \wedge \psi) \cdot e) \triangleright \chi}\ \wedge_{\mathsf{L}} \qquad \frac{\Delta \triangleright \varphi \quad \Delta \triangleright \psi}{\Delta \triangleright \varphi \wedge \psi}\ \wedge_{\mathsf{R}}$$

$$\frac{\Delta(\varnothing_{\times} \cdot e) \triangleright \chi \quad e = 1}{\Delta(\top^{*} \cdot e) \triangleright \chi}\ \top^{*}_{\mathsf{L}} \qquad \frac{\Delta(\varnothing_{+} \cdot e) \triangleright \chi \quad e = 1}{\Delta(\top \cdot e) \triangleright \chi}\ \top_{\mathsf{L}}$$

$$\frac{\Delta(\varphi \cdot e) \triangleright \chi \quad \Delta(\psi \cdot e) \triangleright \chi \quad e = 1}{\Delta((\varphi \vee \psi) \cdot e) \triangleright \chi}\ \vee_{\mathsf{L}} \qquad \frac{\Delta \triangleright \varphi}{\Delta \triangleright \varphi \vee \psi}\ \vee_{\mathsf{R1}} \qquad \frac{\Delta \triangleright \psi}{\Delta \triangleright \varphi \vee \psi}\ \vee_{\mathsf{R2}}$$

$$\frac{\Delta \triangleright \varphi \quad \Gamma(\Delta \fatsemi \psi \cdot e) \triangleright \chi \quad e = 1}{\Gamma(\Delta \fatsemi (\varphi \to \psi) \cdot e) \triangleright \chi}\ \to_{\mathsf{L}} \qquad \frac{\Delta \fatsemi \varphi \cdot e \triangleright \psi \quad e = 1}{\Delta \triangleright \varphi \to \psi}\ \to_{\mathsf{R}}$$

$$\frac{\Gamma(\Delta \cdot e) \triangleright \chi \quad e = 1}{\Gamma(\Delta \cdot e \fatsemi \Delta \cdot e) \triangleright \chi}\ \mathsf{w} \qquad \frac{\Delta \triangleright \chi}{\Delta' \triangleright \chi}\ \mathsf{e} \qquad \frac{\Gamma(\Delta \cdot e \fatsemi \Delta \cdot e) \triangleright \chi \quad e = 1}{\Gamma(\Delta \cdot e) \triangleright \chi}\ \mathsf{c}$$

Figure 13.2: Constraint System $\mathsf{LBI}_{\mathcal{B}}$

Chapter 14

Propositional Logic

We desire to study constraint systems in *general*; that is, to define and study algebraic constraint systems (ACSs) for arbitrary propositional logics rather than specific ones. To this end, we require a general notion of *propositional logic* over which constraint systems may be defined. This is the subject of the present chapter. In the sense that Chapter 3 concerns a formal account of *meta-logic*, this chapter concerns a formal account of *object-logics*. There are various doxastic philosophical desiderata that one may expect a propositional logic to satisfy, but they do not matter for the technical developments herein, so a quite encompassing notion of propositional logic is provided.

We begin by defining what a syntax for a propositional logic may be in Section 14.1. The consequence judgement is a relation on sequents for the propositional logic that exists *a priori* — that is, we do not take the stance that semantics is prior to proof, or *vice versa*. In Section 14.3, we give a generic notion of *model-theoretic semantics* that will allow us to generalize the ideas of Chapter 10; that is, analyze consequence in a logic through reduction in a meta-logic (i.e., FOL).

14.1 Syntax and Consequence

We include context-formers explicitly as a part of the language of propositional logic. This enables us to move between the propositional logics and the meta-logic without ambiguity; moreover, it enables us to handle propositional logics that are expressed in terms of more complex data structures of formulae than lists, multisets, or sets, such as the family of relevance logics — see, for example, Read [176] — and the family of bunched logics — see, for example, work by Docherty, Pym and O'Hearn [150, 171, 44] and Part I. More precisely, we include *data*-formers as they may appear on either the left or right of sequents for the propositional logics, and we use 'context' to refer to the left of sequents.

Definition 14.1 (Propositional Alphabet)**.** *A propositional alphabet is a triple $P := \langle \mathbb{A}, \mathbb{O}, \mathbb{C} \rangle$ in which $\mathbb{A}$, $\mathbb{O}$, and $\mathbb{C}$ are pairwise disjoint sets of symbols such that $\mathbb{A}$ is countable and $\mathbb{O}$ and $\mathbb{C}$ are finite. The symbols in $\mathbb{O}$ and $\mathbb{C}$ have a fixed arity.*

The elements of $\mathbb{A}$ are *atomic propositions*, the elements of $\mathbb{O}$ are *operators*, and the elements of $\mathbb{C}$ are *data-constructors*. We use the term *operators* to subsume 'connectives' and 'modalities' in the traditional terminology. Moreover, we use the term 'data-constructor' as a neutral term for what is sometimes called a 'context-former' as we shall have data both on the left and right of sequents with possibly different constructors and reserve the term 'context' for the left-hand side.

Definition 14.2 (Formula, Data, Sequent)**.** *Let $P := \langle \mathbb{A}, \mathbb{O}, \mathbb{C} \rangle$ be a propositional alphabet. The set of propositional formulae $\mathbb{FORM}(P)$ is the least set containing $\mathbb{A}$ such that, for any $\varphi_1, ..., \varphi_k \in \mathbb{FORM}(P)$ and $\circ \in \mathbb{O}$, if $\circ$ has arity n, then $\circ(\varphi_1, ..., \varphi_n) \in \mathbb{FORM}(P)$. The set $\mathbb{DATA}(P)$ is the least set extending $\mathbb{FORM}(P)$ such that, for any $\delta_1, ..., \delta_n \in \mathbb{DATA}(P)$ and $\bullet \in \mathbb{C}$, if $\bullet$ has arity n, then $\bullet(\delta_1, ..., \delta_n) \in \mathbb{DATA}(P)$. A P-sequent is a pair $\Gamma \triangleright \Delta$ in which $\Gamma, \Delta \in \mathbb{DATA}(P)$.*

Example 14.3. *The basic modal alphabet is $B = \langle \mathbb{A}, \{\wedge, \vee, \Box\}, \{\varnothing, ,, \fatsemi\} \rangle$. The arities of $\wedge$, $\vee$, $\neg$, $,$, and $\fatsemi$ is 2; the arity of $\neg$ and $\Box$ is 1; and, the arity of $\varnothing$ is 0. We may write $\varphi \supset \psi$ to denote $\neg\varphi \vee \psi$, for any formula φ and ψ. Let $\mathrm{p}_1, \mathrm{p}_2, \mathrm{p}_3 \in \mathbb{A}$. Using infix notation, the following are examples of elements from $\mathbb{FORM}(B)$:*

$$\mathrm{p}_3 \qquad (\mathrm{p}_1 \wedge \mathrm{p}_2) \qquad (\mathrm{p}_3 \supset (\mathrm{p}_1 \wedge \mathrm{p}_1))$$

These are also elements in $\mathbb{DATA}(B)$. Another example of an element from $\mathbb{DATA}(B)$ is the following:

$$\mathrm{p}_3 \, , (\mathrm{p}_3 \supset (\mathrm{p}_1 \wedge \mathrm{p}_2))$$

The following is an example of a B-sequent:

$$\mathsf{p}_3 \, , \, \mathsf{p}_3 \supset (\mathsf{p}_1 \land \mathsf{p}_2) \rhd \mathsf{p}_1 \land \mathsf{p}_2$$

■

This completes the definition of the language of a propositional logic generated by an alphabet. What makes language into a logic is a notion of consequence.

Definition 14.4 (Consequence)**.** *A consequence judgement over a propositional language P is a relation $\vdash$ on P-sequents.*

Typically, we write consequence with *infix* notation; that is, we write $\Gamma \vdash \Delta$ to denote that the judgement $\vdash$ obtains for the sequent $\Gamma \rhd \Delta$.

14.2 Sequent Calculus

One way to characterize consequence is by proof in a formal system. In this section, we give a generic account of sequent calculus format relative to which we may perform the investigation. Of course, all the other proof formats such as natural deduction, axiomatic systems, and analytic tableaux may be presented in terms of sequent calculi in standard ways, so this restriction is without loss of generality. The advantage over using the sequent calculus format over any other paradigm is that all the meta-logical structure required to expresses it is already included in the definition of a logic above (e.g., by making data-formers explicit).

An inference is the process of beginning with some sequents — thought of as a putative consequence of a logic — and ending with another sequent — thought of as being entailed, according to our logic, by the original sequents. These inferences are understood as instances of rules.

Definition 14.5 (Rule)**.** *A rule is a relation* r *on sequents.*

That $\mathsf{r}(s, s_1, \ldots, s_n)$ obtains may be denoted by inference schemas:

$$\frac{s_1 \ldots s_n}{s}\, r$$

The sequent s is said to be *conclusion* and the sequents $s_1,\ldots,s_n$ the *premisses*. A rule that has no premisses (i.e., a predicate on sequents) is called an *axiom*-rule.

With this notion of rule, we give the standard treatment of sequent calculi proofs — see, for example, Troestra and Schwichtenberg [207].

Definition 14.6 (Calculus)**.** *A calculus is a set of rules containing at least one axiom-rule.*

Definition 14.7 (Proof)**.** *Let* L *be a calculus. The set of* L*-proofs is the set of rooted trees of sequents inductively constructed as follows:*

– BASE CASE. *If there is an axiom-rule* $\mathsf{a} \in \mathsf{L}$ *such that* $\mathsf{a}(s)$*, then the tree of just the node* s *is an* L*-proof.*

– INDUCTIVE STEP. *If* $\mathcal{P}_1,\ldots,\mathcal{P}_n$ *are* L*-proofs with roots* $s_1,\ldots,s_n$*, respectively, and there is a sequent* s *and a rule* $\mathsf{r} \in \mathsf{L}$ *such that* $\mathsf{r}(s,s_1,\ldots,s_n)$ *obtains, then the argument* $\mathcal{P}$ *with root* s *and immediate sub-trees* $\mathcal{P}_1,\ldots,\mathcal{P}_n$ *is an* L*-proof.*

The notion of proof from a calculus induces a notion of provability from the calculus, which is a consequence relation:

Definition 14.8 (Provability)**.** *A sequent is* L*-provable iff there is a* L*-proof that concludes it.*

We say that a calculus L *characterizes* a logic when L-provability coincides with the consequence relation of the logic. To be precise, this coincidence has two directions: *soundness* and *completeness*. The calculus is *sound* for the logic when it only proves the consequences of the logic, and it is *complete* when it can prove all of the consequences of the logic.

Definition 14.9 (Soundness and Completeness)**.** *Let* $\vdash$ *be a consequence relation and* L *be a calculus.*

- *Calculus* L *is* sound *for* $\vdash$ *iff, for any sequent* $\Gamma \triangleright \Delta$*, if* $\Gamma \vdash_{\mathsf{L}} \Delta$*, then* $\Gamma \vdash \Delta$*.*

- *Calculus* L *is* complete *for* $\vdash$ *iff, for any sequent* $\Gamma \triangleright \Delta$*, if* $\Gamma \vdash \Delta$*, then* $\Gamma \vdash_{\mathsf{L}} \Delta$.

Of course, several different calculi may characterize a logic, some of which may differ substantially. One way to generate new calculi from old is by including rules that are conservative over provability. Such rules are said to be *admissible* — that is, r is admissible in L iff $\vdash_{\mathsf{L} \cup \{\mathsf{r}\}}$ is sound with respect to $\vdash_{\mathsf{L}}$. More generally, a rule can be admissible for a logic when it preserves validity in the logic — that is, that is, r is admissible for $\vdash$ iff, for any sequents $s, s_1, ..., s_n$, if $\mathsf{r}(s, s_1, ..., s_n)$ obtains and $\vdash s_1, \ldots, \vdash s_n$, then $\vdash s$. Observe that if L is sound for $\vdash$, then it is necessarily the case that all the rules in L are admissible for $\vdash$.

14.3 Model-theoretic Semantics

We now give a generic account *model-theoretic semantics* (M-tS) that can be used to characterize consequence. By M-tS, we mean a possible world semantics *á la* Kripke [124, 125] — see also Beth [20]. We follow Blackburn et al. [22] in the approach for a general account of M-tS.

Definition 14.10 (Type). *A type τ is a list of non-negative integers.*

Definition 14.11 (Frame). *Let $\tau := \langle t_1, ..., t_n \rangle$ be a type. A τ-frame is a tuple $\langle \mathbb{U}, k_1, ..., k_n \rangle$ in which $\mathbb{U}$ is a set and k_i is a relation on $\mathbb{U}$ of arity t_i.*

Definition 14.12 (Assignment). *Let $P = \langle \mathbb{A}, \mathbb{O}, \mathbb{C} \rangle$ be a propositional alphabet. An assignment of P to $\mathcal{F}$ is a mapping from propositional atoms to sets of worlds, $I : \mathbb{A} \to \mathscr{P}(\mathbb{U})$.*

We use the term *assignment* to distinguish it from the term *interpretation* used for the meta-logic — see Chapter 3.

Definition 14.13 (Pre-model). *A τ-pre-model over P is a pair $\mathfrak{M} := \langle \mathcal{F}, I \rangle$, in which $\mathcal{F}$ is a τ-frame and I is an assignment of P to $\mathcal{F}$.*

The elements of $\mathbb{U}$ are called possible worlds. One possible objection to the definition of frames is the absence of operators (i.e., endomorphism $f : \mathbb{U}^n \to \mathbb{U}$). This is to simplify the setup and is without loss of generality as operators may be regarded as special types of relations; that is, the operator $f : \mathbb{U}^n \to \mathbb{U}$ corresponds to the $n+1$-ary relation R satisfying $R(w, u_1, ..., u_n)$ iff $w = f(u_1, ..., u_n)$.

Intuitively, a formula φ is true in a model $\mathfrak{M}$ at a world w if the world w satisfies the formulas.

Definition 14.14 (Satisfaction for a Type)**.** *Let τ be a type and P a propositional alphabet. A τ-satisfaction relation for P is a relation $\Vdash$ parameterized by τ-pre-models between worlds w in the pre-models $\mathfrak{M} = \langle \mathcal{F}, I \rangle$ and P-data such that the following holds:*

$$\mathfrak{M}, w \Vdash \mathrm{p} \qquad \textit{iff} \qquad w \in I(\mathrm{p})$$

Definition 14.15 (Semantics, Validity)**.** *Let τ be a type and P a propositional alphabet. A semantics is a pair $\mathfrak{S} := \langle \mathbb{M}, \Vdash \rangle$ in which $\mathbb{M}$ is a set of τ-pre-models and $\Vdash$ is a τ-satisfaction relation for P.*

Definition 14.16 (Validity)**.** *A sequent $\Gamma \triangleright \Delta$ is valid in $\mathfrak{S}$ — denoted $\Gamma \vDash_{\mathfrak{S}} \Delta$ — iff, for any $\mathfrak{M} \in \mathbb{M}$ and any $w \in \mathfrak{M}$, if $\mathfrak{M}, w \Vdash \Gamma$, then $\mathfrak{M}, w \Vdash \Delta$.*

Example 14.17. *Fix the type $\tau := \langle 2 \rangle$. An example of a τ-frame is a pair $\langle \{x, y\}, R \rangle$ in which R is a binary relation on $\{x, y\}$. Partition the atoms $\mathbb{A}$ into two classes $\mathbb{A}_1$ and $\mathbb{A}_2$; an example of an assignment $I : \mathbb{A} \to \mathscr{P}(\{x, y\})$ is given as follows:*

$$I(\mathrm{p}) := \begin{cases} x & \textit{if}\, \mathrm{p} \in \mathbb{A}_1 \\ y & \textit{if}\, \mathrm{p} \in \mathbb{A}_2 \end{cases}$$

The pair $\mathfrak{M} := \langle \mathcal{F}, I \rangle$ is an example of a model over B. The basic semantics $\mathfrak{K}$ is the pair $\langle \mathbb{M}, \Vdash \rangle$ in which $\mathbb{M}$ is the set of all τ-pre-models and $\Vdash$ is the least relation satisfying the clauses in Figure 14.1 together with the following:

$$\begin{array}{lcl} \mathfrak{M}, w \Vdash \Delta \fatcomma \Delta' & \textit{iff} & \mathfrak{M}, w \Vdash \Delta \textit{ and } \mathfrak{M}, w \Vdash \Delta' \\ \mathfrak{M}, w \Vdash \Delta \fatsemi \Delta' & \textit{iff} & \mathfrak{M}, w \Vdash \Delta \textit{ or } \mathfrak{M}, w \Vdash \Delta' \end{array}$$

The validity judgement $\vDash_{\mathfrak{K}}$ defines the modal logic K — see, for example, Kripke [124], Blackburn et al. [22], and Fitting and Mendhelohn [60].

$\mathfrak{M}, w \Vdash \mathrm{p}$	iff	$w \in I(\mathrm{p})$
$\mathfrak{M}, w \Vdash \varphi \land \psi$	iff	$\mathfrak{M}, w \Vdash \varphi$ and $\mathfrak{M}, w \Vdash \psi$
$\mathfrak{M}, w \Vdash \varphi \lor \psi$	iff	$\mathfrak{M}, w \Vdash \varphi$ or $\mathfrak{M}, w \Vdash \psi$
$\mathfrak{M}, w \Vdash \neg\varphi$	iff	not $\mathfrak{M}, w \Vdash \varphi$
$\mathfrak{M}, w \Vdash \Box\varphi$	iff	for any u, if wRu, then $\mathfrak{M}, u \Vdash \varphi$

Figure 14.1: Satisfaction for Modal Logic K

■

The notion of semantics in this paper is generous, including many relations that, perhaps, one would not typically accept as semantics. This is to keep the presentation simple and intuitive. In subsequent chapters, we impose additional restrictions that result in certain behaviours desirable for analysis — see, in particular, Chapter 16. While the running example of this chapter has been modal logic K, this work coheres with the presentation of IPL in Chapter 2 and BI in Part I (Chapter 6 and Chapter 10).

Chapter 15

Algebraic Constraint Systems

Having defined both object-logics (i.e., propositional logics — see Chapter 14) and meta-logics (i.e., FOL — see Chapter 3), it remains to define *(algebraic) constraint systems* (ACSs). The heuristics have already been given in Chapter 12 followed by an instructive example in Chapter 13. This chapter provides a general and formal definition of them. Importantly, ACSs (e.g., $\mathsf{LBI}_{\mathcal{B}}$) sit more naturally within the perspective of Reductive Logic than Deductive Logic, with the intuition that one generates constraints as one applies rules backwards. Therefore, when we speak of *using* a rule, we mean it in the reductive sense. This chapter is short and technical, but it provides the framework for the remaining chapters of this part, in which semantics are studied from the perspective of Reductive Logic using ACSs.

The chapter begins in Section 15.1 with a definition of an ACS for a propositional logic. It also defines *soundness* and *completeness* in terms of the ACS witnessing consequence of the logic. A more refined relationship to the propositional logic is defined —- *faithfulness* and *adequacy* — are defined in Section 15.2. Here the ACS may be used to reason about *proof* in a sequent calculus for the object logic — see, for example, the relationship between LBI and $\mathsf{LBI}_{\mathcal{B}}$ in Chapter 13. In this way, soundness and completeness concern the global correctness of ACSs with respect to the logic of interest; meanwhile, faithfulness and adequacy concern the local correctness of ACSs.

15.1 Constraints and Reductions

When we may say *algebra* to mean a first-order structure (see Chapter 3), we call the terms and formulae from an alphabet in which that algebra is interpreted *expressions* and *constraints*, respectively.

Definition 15.1 (Expression)**.** *An A-expression is a term over A.*

Definition 15.2 (Constraint)**.** *An A-constraint is a formula over A.*

We use the terms 'expression' and 'constraint' to draw attention to the fact that we have a certain algebra in mind and a certain way that the constants and functions of the alphabet are meant to be interpreted. For example, in Chapter 13, we always take the symbol $+$ to be interpreted as Boolean addition. What may change is the interpretation of variables. In short, we have some intended interpretations that are *coherent*.

Definition 15.3 (Coherent Interpretations)**.** *Let $\mathbb{I}$ be a set of interpretations of an algebra $\mathcal{A}$ in A. The set $\mathbb{I}$ is coherent iff any $I_1, I_2 \in \mathbb{I}$ are equivalent except possibly for their action on variables.*

Typically, the set of intended interpretations is maximal in that any interpretation of the algebra in the alphabet is either in the set or is not a variant of an interpretation in the set. This becomes clear in the examples below; presently, it is instructive to recall the handling of Boolean algebra in Chapter 13.

We use expressions to enrich the language of the propositional logic and thereby express meta-theoretic conditions on formulae and sequents. Let P be a propositional alphabet.

Definition 15.4 (Labelled Data)**.** *The set of labelled P-data is defined inductively as follows:*

- BASE CASE. *If φ is a formula and e is an A-expression, then $\varphi \cdot e$ is a A-labelled P-datum.*

- INDUCTIVE STEP. *If δ_1, ... , δ_n are labelled P-data, $\bullet$ is a data-constructor in P with arity n, and e is an A-expression, then $\bullet(\delta_1,...,\delta_n)\cdot e$ is a A-labelled P-datum.*

Definition 15.5 (Enriched Sequent)**.** *An A-enriched P-sequent is a pair $\Pi \rhd \Sigma$, in which Π and Σ are multisets of A-labelled P-data and constraints.*

We may suppress A and P when the alphabet for what algebra is labelling what propositional language is clear. Observe that we have shifted from the object-logic to the meta-logic; enriched sequents are a restricted form of meta-logic sequents that encapsulate object-logic sequents with conditions expressed by expressions from the algebra. This setup differs slightly from the presentation of RDvBC in Chapter 13 to simplify the presentation of the general case. Recall that 'data' is the general name for 'contexts' in the propositional logic, which may be bunches. Consequently, the presentation of RDvBC in terms of enriched sequents would consist of pairs of multisets, each containing only *one* element, the labelled bunch.

Example 15.6. *The are various enriched sequents in RDvBC (Chapter 13). An additional example is as follows:*

$$\mathrm{p}\cdot x \mathbin{,} (q \fatsemi r)\cdot y \rhd (\mathrm{p}\land\mathrm{q})\cdot x$$

■

An *algebraic constraint system* (ACS) is a generalization of a sequent calculus that uses enriched sequents and constraints.

Definition 15.7 (Algebraic Constraint System)**.** *A constraint rule is a relation between an enriched sequent and a list of enriched sequents and constraints. An algebraic constraint system (ACS) is a set of constraint rules.*

The constructions of an ACS are generated reductively on enriched sequents, producing constraints along the way. That $\mathsf{r}(C,P_1,...,P_n)$ obtains for the enriched sequents C,P_1,..., and P_n may be expressed as follows:

$$\frac{P_1 \quad ... \quad P_n}{C}\,\mathsf{r}$$

In this case, C is an enriched sequent and $P_1,...,P_n$ are either enriched sequents or constraints. We use the terms *premiss* and *conclusion* analogously with sequenuset-for calculus rules. We assume the convention of putting constraints after enriched sequents in the list of premisses.

Example 15.8 (Example 15.6 cont'd)**.** *System* $\mathsf{LBI}_{\mathcal{B}}$ *in Chapter 13 is an ACS. An example of a constraint rule is given by the following:*

$$\frac{\Delta \cdot V \triangleright \varphi \quad \Delta' \cdot \bar{V} \triangleright \psi}{\Delta \mathbin{,} \Delta' \triangleright \varphi * \psi}\, {*}\mathsf{R}$$

Here Δ denotes a labelled datum. If e is the expression in Δ, then $\Delta \cdot V$ denotes the result of replacing e by a product of e and V — in particular, presently, by product in the Boolean algebra.

The following inference is an instance of the rule:

$$\frac{\mathrm{p} \cdot (1x) \mathbin{,} (q \fatsemi r) \cdot (1y) \triangleright \mathrm{p} \cdot 1 \quad \mathrm{p} \cdot (1\bar{x}) \mathbin{,} (q \fatsemi r) \cdot (1\bar{y}) \triangleright \mathrm{q} \cdot 1}{\mathrm{p} \cdot 1 \mathbin{,} (q \fatsemi r) \cdot 1 \triangleright (\mathrm{p} * \mathrm{q}) \cdot 1}$$

■

Unlike sequent calculi, an ACS does not necessarily contain axioms (i.e., predicates on enriched sequents). This is possible because the set of things an ACS generates is defined co-inductively, so the restriction is unnecessary. We define reductions co-inductively because ACSs sit within the paradigm of Reductive Logic.

Definition 15.9 (Reduction in an Algebraic Constraint System)**.** *Let* C *be an ACS and S be an enriched sequent. A tree of enriched sequents $\mathcal{R}$ is a* C*-reduction of S iff there is a rule* $\mathsf{r} \in \mathsf{C}$ *such that* $\mathsf{r}(S, P_1, .., P_n)$ *obtains and the immediate sub-trees* R_i*, with root P_i, are as follows: if P_i is an enriched sequent, it is a* C*-reduction of P_i; the single node P_i, otherwise (i.e., if P_i is a constraint).*

Example 15.10 (Example 15.8 cont'd)**.** *A reduction in an ACS is given in Example 13.4 in Chapter 13.* ■

The distinguishing feature of reductions in an ACS is the constraints. In particular, we are interested in the constraints in the premisses of the (reductive) application of a constraint rule because we think of them as saying something about the reduction as a whole as opposed to the constraints within an enriched sequents whose scope is only that sequents.

Definition 15.11 (Side-condition). *Let C be a ACS and let $\mathcal{R}$ be a C-reduction. A side-condition of $\mathcal{R}$ is a constraint that is a leaf of $\mathcal{R}$.*

The side-conditions are global constraints on the reduction, determining the conditions for which the structure is meaningful.

Definition 15.12 (Coherent Reduction). *Let C be an ACS, let $\mathcal{R}$ be a C-reduction, and let $\mathbb{S}$ be the set of side-conditions of $\mathcal{R}$. The set $\mathbb{S}$ is coherent iff there is an interpretation in which all of the side-conditions in $\mathbb{S}$ are valid; the reduction $\mathcal{R}$ is coherent iff $\mathbb{S}$ is coherent.*

We may regard coherent reductions as proofs of certain sequents, but this requires a method of reading what sequent of the propositional logic the reduction asserts.

Definition 15.13 (Ergo). *An ergo is a map ν_I, parameterized by intended interpretations I, from enriched sequents to sequents.*

Let C be an ACS and ν an ergo. We write $\Gamma \vdash_{\mathsf{C}}{}^{\nu} \Delta$ to denote that there is a coherent C-reduction $\mathcal{R}$ of an enriched sequent S such that $\nu_I(S) = \Gamma \triangleright \Delta$, where I is an interpretation satisfying all the side-conditions of $\mathcal{R}$.

These definitions all manifest in the presentation of the RDvBC mechanism in Chapter:13, and are worth recalling for clarity. In particular, Example 13.4 illustrates a coherent reduction with constraints, and Example 13.5 illustrates the result of applying an ergo to that reduction.

Definition 15.14 (Soundness and Completeness of Algebraic Constraint Systems). *An ACS may have the following relationships to a propositional logic:*

- *Soundness: If* $\Gamma \vdash_{\mathsf{C}}^{v} \Delta$*, then* $\Gamma \vdash \Delta$.

- *Completeness: If* $\Gamma \vdash \Delta$*, then* $\Gamma \vdash_{\mathsf{C}}^{v} \Delta$.

This defines ACSs and their relationship to logics. In Chapter 16, we give an algorithmic method for producing sound and complete ACS for a large class of propositional logic, including modal and substructural logics. In the next section, we provide a stronger relationship between an ACS and a logic, which means that the ACS can be used to reason about proofs in sequent calculus, as was the case for RDvBC in Chapter 13. This is used in Chapter 17 to derive a semantics for IPL from a proof-theoretic characterization.

15.2 Faithfulness & Adequacy

The soundness and completeness conditions (Definition 15.14) mean that ACSs are simply an elaborate proof-theoretic characterization of a consequence relation in a labelled sequent calculus. We desire to use ACSs to study other proof-theoretic specifications of a logic. To this end, we use the side-conditions generated during reduction to determine a set of interpretations that allow one to evaluate the reduction as a proof in a sequent calculus for the logic. This is the subject of the present section.

Fix a propositional alphabet P, an algebra $\mathcal{A}$, an alphabet A for that algebra, and a set ▯ of intended interpretations of $\mathcal{A}$ in A. Fix an ACS C and an ergo v. The ergo extends to C reductions by pointwise application to the tree's enriched sequents and deleting all the constraints. Using this extension, ACSs may be regarded as computational devices capturing sequent calculi. For this reason, we do not use the terms *soundness* and *completeness* but rather use the more computational terms of *faithfulness* and *adequacy*.

Definition 15.15 (Faithful & Adequate). *Let* C *be an ACS, let* L *be a sequent calculus, let* v *be a valuation.*

- *System* C *is* faithful *to* L *if, for any* C*-reduction* $\mathcal{R}$ *and interpretation* I *satisfying the constraints of* $\mathcal{R}$*, the application* $v_I(\mathcal{R})$ *is an* L*-proof.*

- *System* C *is* adequate *for* L *if, for any* L*-proof* $\mathcal{D}$*, there is a* C*-reduction* $\mathcal{R}$ *and an interpretation I satisfying the constraints of* R *such that* $\mathsf{v}_I(\mathcal{R}) = \mathcal{D}$.

Intuitively, ACS for a logic (more precisely, those that are faithful and adequate with respect to a sequent calculus for a logic) distinguish the combinatorial and idiosyncratic aspects of that logic. The former refers to how rules manipulate the data in sequents, while the latter refers to the constraints generated by the rules. Typically, the manipulation of data is precisely analogous to that of sequent calculi for classical propositional logic (e.g., to $\mathsf{G3c}$ without the quantifier rules); for example, the $*_{\mathsf{R}} \in \mathsf{LBI}_{\mathcal{B}}$ (see Figure 13.2) has the same form as the $\&_{\mathsf{R}}$-rule in $\mathsf{G3c}$ (Figure 3.1), but with only a formula on the right-hand side). This is what we mean by saying that classical logic is the *combinatorial core* of a logic.

Chapter 16

Systematic Generation of Relational Calculi

Relational calculi were introduced by Negri [147] as a systematic way to give sequent-like calculi for modal logics. They can be viewed as constraint systems; that is, the *constraint algebra* is provided by a first-order theory capturing an M-tS for a logic, and the labelling action captures satisfaction in that semantics. Traditionally, $x : \varphi$ is used in place of $\varphi \cdot x$ for relational calculi, and we shall adopted this notation for this section to be consistent with the existing work. The change in notation is a *aide-mémoire* that we are working with a particular form of constraint systems, in contrast to the fully general perspective in Chapter 15. This chapter gives sufficient conditions for a sequent calculus to admit a relational calculus. We further give conditions under which these relational calculi (regarded as constraint systems) are faithful and adequate for a sequent calculus for the logic. We continue the study of the modal logic K in Chapter 14 as a running example.

First, we define what it means for a semantics of a propositional logic to be first-order definable; this is a pre-condition for producing relational calculi that express the semantics. We call the propositional logic we are studying the *object-logic*; and, we call FOL the *meta-logic*. For clarity, we use the convention prefixing *meta-* for structures at the level of the meta-logic where the terminology might otherwise overlap; for example, *formulae* are syntactic construction at the object level, and *meta-formulae* are syntactic construction in the meta-logic.

Second, we give a sufficient condition, called *tractability*, for us to take a first-order definition Ω of a semantics and produce a relational calculus from it. Essentially, the condition amounts to unfolding Ω within $\mathsf{G3c}$ so that we can suppress all the logical structures from the meta-logic, leaving only a labelled calculus for the propositional logic — namely, the relational calculus.

Third, we give a method for transforming tractable definitions into sequent calculi and prove that the result is sound and complete for the semantics. This is a general account of the approach to semantics in Chapter 10.

16.1 Tractable Propositional Logics

The frames in the model-theoretic semantics of normal modal logics are typically comprised of a universe $\mathbb{U}$ structured by an *accessibility* relation R — see Kripke [124] and Blackburn et al. [22]. In the work by Negri [147] on relational calculi for normal modal logics, the formulae over which the relational calculi operates come in two forms: they are either of the form $(x : \varphi)$, in which x is a variable denoting an arbitrary world, φ is a formula, and : is a pairing symbol intuitively saying that φ is satisfied at x; or, they are of the form xRy, in which x and y are variables denoting worlds and R is a relation denoting the accessibility relation of the semantics. To generalize this approach, we begin by *fusing* the language P of the propositional logic of interest (i.e., the object-logic) with a first-order language F able to express frames for the semantics.

Definition 16.1 (Fusion)**.** *Let* $F := \langle \mathbb{R}, \varnothing, \mathbb{K}, \mathbb{V} \rangle$ *be a first-order alphabet and let* $P := \langle \mathbb{A}, \mathbb{O}, \mathbb{C} \rangle$*. The fusion* $F \otimes P$ *is the first-order alphabet* $\langle \mathbb{R} \cup \{:\}, \mathbb{O} \cup \mathbb{C}, \mathbb{K} \cup \mathbb{A}, \mathbb{V} \rangle$

To aid readability, we shall use the convention of writing $\hat{\varphi}$ for meta-variables that we intend to be interpreted as object-formulae and $\hat{\Gamma}$ or $\hat{\Delta}$ for meta-variables that we intend to be interpreted as object-data.

Observe that P-formulae and F-terms both becomes terms in $F \otimes P$, and : is a relation. In particular, the object-logic operators (i.e., connectives, modalities) are function-symbols in the fusion. Further note that both $(x : \varphi)$ and $(\varphi : x)$ are well-formed formulae in the fusion, the former is desirable and the latter is not.

In principle, one could work over a typed language in which these definitions are enforced in the grammar (e.g., by using a typed language), but doing so does not actually simplify matters.

As in Chapter 10, we desire a *theory* Ω over the fused language such that : is interpreted as satisfaction in the semantics. Note, relative to such a theory, while well-formed, a meta-formula $(\varphi : x)$ is indeed nonsense.

Definition 16.2 (Definition of a Semantics)**.** *Let Ω be a set of sentences from a fusion $F \otimes P$ and let $\mathfrak{S}$ be a semantics over P. The set Ω defines the semantics $\mathfrak{S}$ iff the following holds: for any Γ, Δ,*

$$\Omega, (x : \Gamma) \blacktriangleright (x : \Delta) \qquad \textit{iff} \qquad \Gamma \vDash \Delta$$

Though a seemingly strong condition, such theories Ω are fairly systematically constructed for the semantics present in the literature. Intuitively, the abstraction of Ω — see Chapter 3 — are composed of models from the semantics together with an interpretation of the satisfaction relation. That is, Ω is typically composed of two theories Ω_1 and Ω_2, where Ω_1 captures frames and Ω_2 captures the conditions of the satisfaction relation. For example, in modal logic, if the accessibility relation is transitive, then Ω_1 contains $\forall x, y, z(xRy \& yRz \Rightarrow xRz)$, and if the object-logic contains an additive conjunction $\land$, then Ω_2 may contain $\forall x, \hat{\varphi}, \hat{\psi}((x : \hat{\varphi} \land \hat{\psi}) \Rightarrow (x : \hat{\varphi}) \& (x : \hat{\psi}))$ and $\forall x, \hat{\varphi}, \hat{\psi}((x : \hat{\varphi}) \& (x : \hat{\psi}) \Rightarrow (x : \hat{\varphi} \land \hat{\psi}))$. This is the situation for BI in Chapter 10.

Example 16.3. *By the universal closure of $(\Phi \Leftrightarrow \Psi)$ we mean the meta-formulae Θ and Θ' in which Θ is the universal closure of $\Phi \Rightarrow \Psi$ and Θ' is the universal closure of $\Psi \Rightarrow \Phi$. Consider the semantics $\mathfrak{K} = \langle \mathbb{K}, R \rangle$ in Example 14.17. It is defined by the universal closures of the formulae in Figure 16.1, which merits comparison with Figure 14.1, together with the universal closure of the following:*

$$\begin{array}{ccc} (x : \hat{\Gamma} \, , \, \hat{\Delta}) & \textit{iff} & (x : \hat{\Gamma}) \& (x : \hat{\Delta}) \\ (x : \hat{\Gamma} \, ⨾ \, \hat{\Delta}) & \textit{iff} & (x : \hat{\Gamma}) \parr (x : \hat{\Delta}) \end{array}$$

$$
\begin{array}{lcl}
(w : \hat{\varphi} \wedge \hat{\psi}) & \text{iff} & (w : \hat{\varphi}) \& (w : \hat{\psi}) \\
(w : \hat{\varphi} \vee \hat{\psi}) & \text{iff} & (w : \hat{\varphi}) \parr (w : \hat{\psi}) \\
(w : \neg \hat{\varphi}) & \text{iff} & \big((w : \hat{\varphi}) \Rightarrow \bot\big) \\
(w : \Box \hat{\varphi}) & \text{iff} & \forall u (wRu \Rightarrow u : \hat{\varphi})
\end{array}
$$

Figure 16.1: Satisfaction for Modal Logic K (Symbolic)

Every model of K (see Chapter 14) arises as an abstraction of these formulas, and vice versa*. This theory is denoted* $\Omega_{\mathfrak{K}}$.

As in the treatment of BI in Chapter 10, there is no meta-formula corresponding to atomic satisfaction — that is, $(w : \mathrm{p})$*, where* p *is atomic — because it is handled by the structure of meta-sequents. Again, it follows from working with validity directly (i.e., without passing though truth-in-a-model): atomic satisfaction is captured by tautology,*

$$\Omega, (w \Vdash \mathrm{p}) \triangleright (w \Vdash \mathrm{p})$$

■

We may use the meta-logic to characterize those propositional logics whose semantics is particularly amenable to analysis; first-order definability is, perhaps, the most general condition we may demand. What are some other properties of Ω that may be useful? Since we are interested in a *computational* analysis of the semantics, we require that Ω is finite, among other things. In particular, we restrict the structure of the theory to something amenable to proof-theoretic analysis according to G3c.

There is literature on generating proof systems for propositional logics defined axiomatically; see, for example, work by Ciabattoni et al. [36, 39, 38]. Within this tradition, Marin et al. [135] have used *focusing* in intuitionistic and classical logic, thought of as a meta-logic, as a general tool to express uniformly an algorithm for turning axioms into rules applicable across different domains. We use a similar method and, therefore, polarize the syntax for the meta-logic.

Recall that $\mathbb{ATOMS}(F \otimes P)$ is the set of meta-atoms — that is, atoms of the first-order language generated from the alphabet $F \otimes P$. The *positive* meta-formulae P and *negative* meta-formulae N are defined as follows:

$$\begin{array}{lcl} P & ::= & A \in \mathbb{ATOMS}(F \otimes P) \mid \bot \mid P \& P \mid N \Rightarrow P \mid P \parr P \mid \exists X P \\ N & ::= & A \in \mathbb{ATOMS}(F \otimes P) \mid N \& N \mid P \Rightarrow N \mid \forall X N \end{array}$$

This taxonomy arises from behaviour; specifically,using this taxonomy we can define a class of formulae that we can systematically transform into *synthetic rules* using focusing in G3c. While closely related to the taxonomy used by Marin et al. [135], whence the approach comes, it is not the same as they work over a syntax that has positive and negative connectives.

Definition 16.4 (Polarity Alternation)**.** *The number of* polarity alternations *in a polarized formula* Φ *is* $\pi(\Phi)$ *defined as follows:*

$$\pi(\Phi) := \begin{cases} 0 & \textit{if } \Phi \in \mathbb{ATOMS}(F \otimes P) \\ \max\{\pi(\Phi_1), \pi(\Phi_2)\} & \textit{if } \Phi = \Phi_1 \circ \Phi_2 \textit{ and } \circ \in \{\&, \parr\} \\ \pi(\Psi) & \textit{if } \Phi = \forall X \Psi \textit{or} \Phi = \exists X \Psi \\ 1 + \max\{\pi(\Phi_1), \pi(\Phi_2)\} & \textit{if } \Phi = \Phi_1 \Rightarrow \Phi_2 \end{cases}$$

Definition 16.5 (Tractable Meta-formula)**.** *A meta-formula* Φ *is tractable iff* Φ *is negative and* $\pi(\Phi) \leq 1$*, or* Φ *is positive and* $\pi(\Phi) \leq 2$*.*

This definition is allows us to fully leverage focusing; that is, positive and negative formulae correspond to unfocused and focused phases of reduction. A combination of focused and unfocused phase may yield a formula in which the choice of reduction affect termination; this is used in the proof of Proposition 16.9. Thus, the definition of tractability is not necessarily a theoretical limit for converting theories into systems of rules, but only the limit of using focusing to do so.

The class of *geometric implications* studied by Negri [146] for the generation of sequent calculus rules from axioms defining propositional logics is a subset of the tractable formulae — a meta-formula Θ is a geometric implication iff Θ is the universal closure of a meta-formula of the form $(\Phi_1 \& ... \& \Phi_m) \Rightarrow (\exists Y_1 \Psi_1 \parr ... \parr \exists Y_n \Psi_n)$ such that $\Psi_i := \Psi^i_1 \& ... \& \Psi^i_{m_i}$, with the Ψ^i_j meta-atoms for $1 \leq j \leq m_i$ and $1 \leq i \leq n$, and Φ_i meta-atoms for $1 \leq i \leq m$. Docherty and Pym [47, 44] have similarly used this class of meta-formulae to give a uniform account of proof systems for the family of bunched logics, with application to separation logics.

The motivation for tractability is to make a certain step in the generation of relational calculi possible, as seen in the proof of Proposition 16.9 (below).

Definition 16.6 (Tractable Theory, Semantics, Logic)**.** *A set of meta-formulae Ω is a tractable theory iff Ω is finite and any $\Phi \in \Omega$ is a negative tractable meta-sentence. A semantics $\mathfrak{S}$ is tractable iff it is defined by a tractable theory Ω. A propositional logic is tractable iff it admits a tractable semantics $\mathfrak{S}$.*

Example 16.7. *The semantics for modal logic in Example 14.17 is tractable, as witnessed by the tractable definition in Example 16.3.* ■

It remains to give an algorithm that generates a relational calculus given a tractable definition and to prove correctness of that algorithm. Fix a semantics $\mathfrak{S} := \langle \mathbb{M}, \Vdash \rangle$ with a tractable definition Ω. Recall that $\Gamma \vDash \Delta$ obtains iff $\Omega, (x : \Gamma) \blacktriangleright (x : \Delta)$ obtains. The relational calculus we generate is a meta-sequent calculus R for the meta-logic expressive enough to capture all instances $\Omega, (x : \Gamma) \blacktriangleright (x : \Delta)$, but sufficiently restricted such that all the meta-connectives and quantifiers may be suppressed.

16.2 Generating Relational Calculi

Intuitively, relational calculi work by *unfolding* validity according to clauses of semantics. Therefore, proof-theoretically, they are more natural understood in terms of *Reductive Logic*. Given a tractable theory Ω, we systematically turn the elements into rules by reduction in the meta-logic (i.e., FOL). More accurately, we collapse sequences of invertible reductions in FOL into rules that have the same overall effect. Doing this for all the clauses in Ω yields a proof system that encapsulates its content.

By *generic hereditary reduction* on a meta-formula Φ we mean the indefinite use of reduction operators from G3c on Φ and the generated sub-formulae, until they are meta-atoms, beginning with a meta-sequent $\Phi, \Pi \rhd \Sigma$, with generic Π and Σ. For example, the following is a generic hereditary reduction for $(A \& B) \parr (C \& D)$ with A, B, C, and D as meta-atoms:

$$\dfrac{\dfrac{A, B, \Pi \rhd \Sigma}{(A \& B), \Pi \rhd \Sigma} \Uparrow \&_{\mathsf{R}} \quad \dfrac{C, D, \Pi \rhd \Sigma}{(C \& D), \Pi \rhd \Sigma} \Uparrow \&_{\mathsf{R}}}{(A \& B) \parr (C \& D), \Pi \rhd \Sigma} \Uparrow \parr_{\mathsf{R}}$$

Such reductions are collapsed into *synthetic* rules, which is the rule-relation taking the putative conclusion to the premisses — see Chaudhuri et al. [35, 34] and Marin [135]. The above instance collapses to the following:

$$\frac{A, B, \Pi \rhd \Sigma \quad C, D, \Pi \rhd \Sigma}{(A \& B) \parr (C \& D), \Pi \rhd \Sigma}$$

The quantifier rules have side-conditions in order to be applicable, and we assert these conditions in the synthetic rule. For example, when using $\forall_{\mathsf{L}}$ when doing generic hereditary reduction on $\forall X \Phi$, we require that the term T for which the variable X is substituted in Φ is already present in the meta-sequent; for example let $\Phi := (A(X) \& B(X)) \parr (C(X) \& D(X))$, we have the following synthetic rule for $\forall X \Phi$ with the side condition that T occurs in either Π or Σ:

$$\frac{A(T), B(T), \Pi \rhd \Sigma \quad C(T), D(T), \Pi \rhd \Sigma}{\forall X (A(X) \& B(X) \parr (C(X) \& D(X)), \Pi \rhd \Sigma}$$

Definition 16.8 (Sequent Calculus for a Tractable Theory). *Let Ω be a tractable theory. The sequent calculus* $\mathsf{G3c}(\Omega)$ *is composed* $\mathsf{ax}, \bot, \mathsf{c_L}$, $\mathsf{c_R}$, *and the the synthetic rules for the meta-formulae in Ω.*

The tractability condition is designed such that the following holds:

Proposition 16.9. *Let Ω be a tractable definition and let Π and Σ be multisets of meta-atoms,*

$$\Omega, \Pi \vdash_{\mathsf{G3c}} \Sigma \qquad \textit{iff} \qquad \Omega, \Pi \vdash_{\mathsf{G3c}(\Omega)} \Sigma$$

Proof. Assume $\Omega, \Pi \vdash_{\mathsf{G3c}} \Sigma$. Without loss of generality — see, for example, Liang and Miller [131] and Marin et al. [135] — there is a focused $\mathsf{G3c} + \mathsf{c_R} + \mathsf{c_L}$-proof $\mathcal{D}$ of $\Omega, \Pi \triangleright \Sigma$, meaning that invertible rules are used eagerly and non-invertible rules are used hereditarily. We can assume that $\mathcal{D}$ is *focused* upto possibly using instance of $\mathsf{c_L}$ or $\mathsf{c_R}$. That is, $\mathcal{D}$ is structured by sections of alternating phases of the following kind:

- an instance of $\mathsf{c_L}$ or $\mathsf{c_R}$
- hereditary reduction on positive meta-formulae on the right and negative meta-formulae on the left
- eager reduction on negative meta-formulae on the right and positive meta-formulae on the left.

Since Π and Σ are composed of meta-atoms and Ω is composed of negative meta-formulae, $\mathcal{D}$ begins by a contraction and then hereditary reducing on some $\Phi \in \Omega$. Since Φ is tractable, this section in $\mathcal{D}$ may be replaced by the synthetic rule for Φ. Doing this to all the phases in $\mathcal{D}$ yields a tree of sequents $\mathcal{D}'$ that witnesses $\Omega, \Pi \vdash_{\mathsf{G3c}(\Omega)} \Sigma$.

Assume $\Omega, \Pi \vdash_{\mathsf{G3c}(\Omega)} \Sigma$. Since all the rules in $\mathsf{G3c}(\Omega)$ are admissible in $\mathsf{G3c}$, we immediately have $\Omega, \Pi \vdash_{\mathsf{G3c}} \Sigma$. □

Example 16.10. *Consider the tractable theory $\Omega_{\mathfrak{K}}$ in Example 16.3. The sequent calculus $\mathsf{G3c}(\Omega_{\mathfrak{K}})$ contains, among other things, the following rules corresponding to the clause for $\wedge$ in Figure 16.1 in which w, φ, and ψ already occur in Ω, Π, or Σ:*

$$\frac{\Omega, \Pi \triangleright \Sigma, (w : \varphi \wedge \psi) \quad \Omega, (w : \varphi), (w : \psi), \Pi \triangleright \Sigma}{\Omega, \Pi \triangleright \Sigma}$$

$$\frac{\Omega, \Pi, (w : \varphi \wedge \psi) \triangleright \Sigma \quad \Omega, \Pi \triangleright \Sigma, (w : \varphi) \quad \Omega, \Pi \triangleright \Sigma, (w : \psi)}{\Omega, \Pi \triangleright \Sigma}$$

Of course, in practice, one does not use the rules in this format. Rather, one would only apply the rules if one already knew that the left-branch would terminate; that *is, one uses the following:*

$$\frac{\Omega,(w:\varphi),(w:\psi),(w:\varphi\wedge\psi),\Pi\rhd\Sigma}{\Omega,(w:\varphi\wedge\psi),\Pi\rhd\Sigma}$$

$$\frac{\Omega,\Pi\rhd\Sigma,(w:\varphi\wedge\psi),(w:\varphi)\quad\Omega,\Pi\rhd\Sigma,(w:\varphi\wedge\psi),(w:\psi)}{\Omega,\Pi\rhd\Sigma,(w:\varphi\wedge\psi)}$$

This simplification can be made systematically according to the shape of the meta-formula generating the rules; it corresponds to forward-chaining *and* back-chaining *in the proof-theoretic analysis of the meta-formula — see, for example, Marin et al. [135].*

One desires a systematic account of the transformation of rules of arbitrary shape into rules of other (more desirable) shape. This remains to be considered in the context of relational calculi and demands further analysis on the structure of Ω*. Some results of such transformations for arbitrary sequent calculi have been provided by Indrejczak [107].* ■

The calculus $\mathsf{G3c}(\Omega)$ is a restriction of G3c precisely encapsulating the proof-theoretic behaviours of the meta-formulae in Ω. It remains to suppress the logical constants of the meta-logic entirely, and thereby yield a relational calculus expressed as a labelled sequent calculus for the propositional logic.

Definition 16.11 (Relational Calculus for a Tractable Theory)**.** *Let* Ω *be a tractable theory. The relational calculus for* Ω *is the sequent calculus* $\mathsf{R}(\Omega)$ *that results from* $\mathsf{G3c}(\Omega)$ *by suppressing* Ω*.*

Theorem 16.12 (Soundness & Completeness)**.** *Let* $\mathfrak{S}$ *be a tractable semantics and let* Ω *be a tractable definition for* $\mathfrak{S}$*.*

$$\Gamma\vDash_{\mathfrak{S}}\Delta\qquad \textit{iff}\qquad (x:\Gamma)\vdash_{\mathsf{R}(\Omega)}(x:\Delta)$$

Proof. We have the following:

$$
\begin{array}{llll}
\Gamma \vDash_{\mathfrak{G}} \Delta & \text{iff} & \Omega, (x:\Gamma) \blacktriangleright (x:\Delta) & \text{(Definition 16.6)} \\
& \text{iff} & \Omega, (x:\Gamma) \vdash_{\mathsf{G3c}} (x:\Delta) & \text{(Proposition 3.4)} \\
& \text{iff} & \Omega, (x:\Gamma) \vdash_{\mathsf{G3c}(\Omega)} (x:\Delta) & \text{(Proposition 16.9)}
\end{array}
$$

It remains to show that $\Omega, (x:\Gamma) \vdash_{\mathsf{G3c}(\Omega)} (x:\Delta)$ iff $(x:\Gamma) \vdash_{\mathsf{R}(\Omega)} (x:\Delta)$.

Let $\mathcal{D}$ be a $\mathsf{G3c}(\Omega)$-proof of $\Omega, (x:\Gamma) \triangleright (x:\Delta)$, and let $\mathcal{D}'$ be the result of removing Ω from every meta-sequent in $\mathcal{D}$. By Definition 16.11, we have that $\mathcal{D}'$ is a $\mathsf{R}(\Omega)$-proof of $(x:\Gamma) \triangleright (x:\Delta)$. Thus, $\Omega, (x:\Gamma) \vdash_{\mathsf{G3c}(\Omega)} (x:\Delta)$ implies $(x:\Gamma) \vdash_{\mathsf{R}(\Omega)} (x:\Delta)$.

Let $\mathcal{D}$ be a $\mathsf{R}(\Omega)$-proof of $(x:\Gamma) \triangleright (x:\Delta)$, and let $\mathcal{D}'$ be the result of putting Ω in every meta-sequent in $\mathcal{D}$. By Definition 16.11, we have that $\mathcal{D}'$ is a $\mathsf{G3c}(\Omega)$-proof of $\Omega, (x:\Gamma) \triangleright (x:\Delta)$. Thus, $(x:\Gamma) \vdash_{\mathsf{R}(\Omega)} (x:\Delta)$ implies $\Omega, (x:\Gamma) \vdash_{\mathsf{G3c}(\Omega)} (x:\Delta)$. □

This theorem is deceptively simple. The setup is designed to yield it, in particular the notion of tractability. It is this theorem that delivers the approach to completeness in Chapter 10, bypassing term- and counter-model constructions. In Chapter 17, we use it to study the semantics of IPL.

Example 16.13. *The sequent calculus in Example 16.10 becomes a relational calculus* $\mathsf{R}(\Omega_{\mathfrak{K}})$ *by suppressing* Ω *in the rules; for example,*

$$
\frac{\Omega, \Pi \triangleright \Sigma, (w:\varphi \wedge \psi) \quad \Omega, (w:\varphi), (w:\psi), \Pi \triangleright \Sigma}{\Omega, \Pi \triangleright \Sigma}
$$

becomes

$$
\frac{\Pi \triangleright \Sigma, (w:\varphi \wedge \psi) \quad (w:\varphi), (w:\psi), \Pi \triangleright \Sigma}{\Pi \triangleright \Sigma}
$$

Abbreviating $\neg\Box\neg\Phi$ *by* $\Diamond\Phi$ *and doing some proof-theoretic analysis on* $\mathsf{R}(\Omega_{\mathfrak{K}})$*, we have the simplified system* RK *in Figure 16.2. This is, essentially, the relational calculus for K introduced by Negri [147].* ■

While we have effectively transformed (tractable) semantics into relational calculi, giving a general, uniform, and systematic proof theory to an ample space of

$$\frac{}{\Phi,\Pi \rhd \Sigma,\Phi}\ \mathsf{ax} \qquad\qquad \frac{}{\bot,\Pi \rhd \Sigma}\ \bot_{\mathsf{R}}$$

$$\frac{(x:\varphi),(x:\psi),\Pi \rhd \Sigma}{(x:\varphi \wedge \psi),\Pi \rhd \Sigma}\ \wedge_{\mathsf{L}} \qquad\qquad \frac{\Pi \rhd \Sigma,(x:\varphi) \quad \Pi \rhd \Sigma,(x:\psi)}{\Pi \rhd \Sigma,(x:\varphi \wedge \psi)}\ \wedge_{\mathsf{R}}$$

$$\frac{(x:\varphi),\Pi \rhd \Sigma \quad (x:\psi),\Pi \rhd \Sigma}{(x:\varphi \vee \psi),\Pi \rhd \Sigma}\ \vee_{\mathsf{L}} \qquad\qquad \frac{\Pi \rhd \Sigma,(x:\varphi),(x:\psi)}{\Pi \rhd \Sigma,(x:\varphi \vee \psi)}\ \vee_{\mathsf{R}}$$

$$\frac{(y:\varphi),(x:\Box\varphi),xRy,\Pi \rhd \Sigma}{(x:\Box\varphi),xRy,\Pi \rhd \Sigma}\ \Box_{\mathsf{L}} \qquad\qquad \frac{xRy,\Pi \rhd \Sigma,(y:\varphi)}{\Pi \rhd \Sigma,(x:\Box\varphi)}\ \Box_{\mathsf{R}}$$

$$\frac{xRy,(y:\varphi),\Pi \rhd \Sigma}{(x:\Diamond\varphi),\Pi \rhd \Sigma}\ \Diamond_{\mathsf{L}} \qquad\qquad \frac{xRy,\Pi \rhd \Sigma,(x:\Diamond\varphi),(y:\varphi)}{xRy,\Pi \rhd \Sigma,(x:\Diamond\varphi)}\ \Diamond_{\mathsf{R}}$$

$$\frac{\bot,\Pi \rhd \Sigma}{(x:\bot),\Pi \rhd \Sigma}\ \bot_{\mathsf{L}} \qquad\qquad \frac{\bot,\Pi \rhd \Sigma,\bot}{\Pi \rhd \Sigma,(x:\bot)}\ \bot_{\mathsf{R}}$$

$$\frac{(x:\Gamma),(x:\Gamma'),\Pi \rhd \Sigma}{(x:\Gamma \mathbin{,} \Gamma'),\Pi \rhd \Sigma}\ {,}_{\mathsf{L}} \qquad\qquad \frac{\Pi \rhd \Sigma,(x:\Delta),(x:\Delta')}{\Pi \rhd \Sigma,(x:\Delta \fatsemi \Delta')}\ \fatsemi_{\mathsf{R}}$$

Figure 16.2: Relational Calculus RK

logics, significant analysis remains to be done. In Example 16.13, we showed that under relatively mild conditions, one expects the relational calculus to have a particularly good shape. This begs for further characterization of the definitions of semantics and what properties one may expect the resulting relational calculus to have; the beginnings of such an analysis are given below Definition 16.2 in which we require Ω to contain a first-order definition of frames together with an inductive definition of the semantics. Thus, the algorithm herein presents a wide platform in which logics and their proof theory may be analysed. In particular, following the treatment of BI in Chapter 10, this platform offers a new approach to proving soundness and completeness that bypasses term- or counter-model constructions entirely.

16.3 Faithfulness & Adequacy

In this section, we give sufficient conditions for faithfulness and adequacy of a relational calculus with respect to a sequent calculus. More precisely, we give conditions under which one may transform a relational calculus into a sequent calculus for the object-logic. The result is immediate proof of soundness and completeness for the sequent calculus concerning the semantics; significantly, it bypasses term- or counter-model construction. This is precisely the method used in Chapter 10.

While the work of the preceding section generates a relational calculus, one may require some proof theory to yield a relational calculus that meets the conditions in this section, *faithfulness* and *adequacy*. Likewise, one may require proof theory on the generated sequent calculus to yield a sequent calculus one recognizes as sound and complete with respect to a logic of interest. We do not consider these problems here.

Our objective is to systematically transform (co-)inferences in the relational calculus into (co-)inferences of the propositional logic. Regarded as constraint systems, relational calculi do not have any side-conditions on inferences; instead, all of the constraints are carried within sequents. Thus we do not need to worry about assignments and aim only to develop a valuation v.

Fix a propositional logic $\vdash$ and relational calculus R. We assume the propositional logic has data-constructors $\circ$ and $\bullet$ such that

$$\Gamma \circ \Gamma' \vdash \Delta \quad \text{iff} \quad (w : \Gamma) \,\&\, (w : \Gamma') \vdash_{\mathsf{R}} (w : \Delta)$$

and

$$\Gamma \vdash \Delta \bullet \Delta' \quad \text{iff} \quad (w : \Gamma) \vdash_{\mathsf{R}} (w : \Delta) \parr (w : \Delta')$$

This means that the weakening, contraction, and exchange structural rules are admissible for $\circ$ and $\bullet$ on the left and right, respectively. In particular, these data-constructors behave like classical conjunction and disjunction, respectively.

Example 16.14. *The logic with relational calculus* RK *satisfies the data-constructor condition — specifically,* $\fatcomma$ *is conjunctive and* $\fatsemi$ *is disjunctive.* ■

A list of meta-formulae is *monomundic* iff it only contains one world-variable (but possibly many occurrences of that world-variable; we write Π^w or Σ^w to denote monomundic lists containing the world-variable w. A monomundic list is *basic* iff it only contains meta-atoms of the form $(w : \Gamma)$, which is denoted $\bar{\Pi}^w$ or $\bar{\Sigma}^w$.

Definition 16.15 (Basic Validity Sequent)**.** *A basic validity sequent (BVS) is a pair of basic monomundic lists,* $\bar{\Pi}^w \triangleright \bar{\Sigma}^w$*.*

Definition 16.16 (Basic Rule)**.** *A rule in a relational calculus is basic iff it is a rule over BVSs — that is, it has the following form:*

$$\frac{\bar{\Pi}_1^{w_1} \triangleright \Sigma_1^{w_1} \quad ... \quad \bar{\Pi}_n^{w_n} \triangleright \Sigma_n^{w_n}}{\bar{\Pi}^w \triangleright \Sigma^w}$$

Definition 16.17 (Basic Relational Calculus)**.** *A relational calculus* r *is basic iff it is composed of basic rules.*

Using the data-structures $\bullet$ and $\circ$, a BVS intuitively corresponds to a sequent in the propositional logic. Define $\lfloor - \rfloor_\circ$ and $\lfloor - \rfloor_\bullet$ on basic monomundic lists as follows:

$$\lfloor (w : \Gamma_1), ..., (w : \Gamma_m) \rfloor_\circ := \Gamma_1 \circ ... \circ \Gamma_m \qquad \lfloor (w : \Delta_1), ..., (w : \Delta_n) \rfloor_\bullet := \Delta_1 \bullet ... \bullet \Delta_n$$

We can define ν on BVSs by this encoding,

$$\nu(\bar{\Pi}^w \triangleright \bar{\Sigma}^w) := \lfloor \bar{\Pi}^w \rfloor \triangleright \lfloor \bar{\Sigma}^w \rfloor$$

The significance is that whatever inference is made in the semantics using BVSs immediately yields an inference it terms of propositional sequents.

Let r be a basic rule, its propositional encoding $\nu(\mathsf{r})$ is the following:

$$\frac{\nu(\bar{\Pi}_1^{w_1} \triangleright \Sigma_1^{w_1}) \quad ... \quad \nu(\bar{\Pi}_n^{w_n} \triangleright \Sigma_n^{w_n})}{\nu(\bar{\Pi}^w \triangleright \Sigma^w)}$$

This extends to basic relational calculi pointwise,

$$\nu(\mathsf{R}) := \{\nu(\mathsf{r}) \mid \mathsf{r} \in \mathsf{R}\}$$

Despite their restrictive shape, the possible world semantics of logics in the literature typically yield basic rules. For example, if the body of a clause is composed of only conjunctions and disjunctions of assertions, the rules generated by the algorithm presented above will be basic. Sets of basic rules can sometimes replace more complex rules in relational calculi to yield a basic relational calculus from a non-basic relational calculus — see Chapter 10 and Chapter 17 for examples.

We are thus in a situation where the rules of a reduction system intuitively correspond to the rules of a sequent calculus. The formal statement of this is below.

Theorem 16.18. *A basic relational calculus* R *is faithful and adequate with respect to its propositional encoding* $\nu(\mathsf{R})$.

Proof. The result follows by Definition 15.15 because a valuation of an instance of a rule in R corresponds to an instance of a rule of R on the states of the sequents involved. In paritcular, faithfulness follows by application of ν on R-proofs; that is, for any R-reduction $\mathcal{D}$, one produces a corresponding $\nu(\mathcal{R})$-proof by apply ν to each sequent in $\mathcal{D}$. Meanwhile, adequacy follows by introducing arbitrary world-variables into a $\nu(\mathsf{R})$-proof. Let $\mathcal{D}$ be a $\nu(\mathsf{R})$-proof, it concludes by an inference of the following form:

$$\frac{\begin{array}{c}\mathcal{D}_1\\ \nu(\bar{\Pi}_1^{w_1} \triangleright \Sigma_1^{w_1})\end{array} \quad \ldots \quad \begin{array}{c}\mathcal{D}_n\\ \nu(\bar{\Pi}_n^{w_n} \triangleright \Sigma_n^{w_n})\end{array}}{\nu(\bar{\Pi}^{w} \triangleright \Sigma^{w})}$$

We can co-inductively define a corresponding R-with the following co-recursive step in which $\mathcal{R}_i$ is the reduction corresponding to $\mathcal{D}_i$:

$$\frac{\begin{array}{c}\mathcal{R}_1\\ \bar{\Pi}_1^{w_1} \triangleright \Sigma_1^{w_1}\end{array} \quad \ldots \quad \begin{array}{c}\mathcal{R}_n\\ \bar{\Pi}_n^{w_n} \triangleright \Sigma_n^{w_n}\end{array}}{\bar{\Pi}^{w} \triangleright \Sigma^{w}}$$

Hence, for any $\nu(\mathsf{R})$-proof, there is a R-reduction $\mathcal{R}$ such that $\nu(\mathcal{R}) = \mathcal{D}$, as required. □

Of course, despite basic rules being relatively typical, many relational calculi are not comprised of *only* basic rules. Nonetheless, the phenomenon does occur for even quite complex logic. It can be used for the semantical analysis of that logic in those instances — see, for example, Chapter 10.

Chapter 17

Synthesizing Semantics for Intuitionistic Propositional Logic

In Chapter 16, we gave a general, uniform, and systematic procedure for generating proof systems for logics with model-theoretic semantics satisfying certain conditions. What about the reverse problem? Given a proof-theoretic characterization of a propositional logic, can we to *derive* a model-theoretic semantics for it? This chapter provides an example of such a derivation for intuitionistic propositional logic (IPL).

We begin from a naive position on IPL. Our *definition* of IPL is by LJ — see Gentzen [200]. We choose this over other systems (e.g., G3i — see Troelstra and Schwichtenberg [207]) because we assume that we do not even know much about its proof theory so that we may explain through the analysis what we require. In the end, we recover the model-theoretic semantics by Kripke [124] using constraint systems as the enabling technology. Of course, we shall imagine that we do not know the semantics.

Reflecting on Chapter 16, we expect the semantics we synthesize for IPL will be tractable. Therefore, we intend to build a relational calculus to bridge the proof theory and semantics of IPL as in Chapter 16, but this time we build it from the proof theory side. Recall that relational calculi are fragments of proof systems for FOL (i.e., the meta-logic). Therefore, in Section 17.1, we begin by building a constraint system for IPL that is *classical* in shape using ACSs. The system is derived in a principled way from this desire, but it is only sound and complete for IPL. We require it to be *faithful* and *adequate* for LJ because we hope to generate a clause for each connective from its rules. Hence, in Section 17.2, we analyze the constraint system to recover a faithful and adequate constraint system for IPL. In Section 17.3, we study the reductive behaviour of connectives of IPL in this constraint systems and write tractable FOL-formulae that capture the same behaviour in $\mathsf{G3c}$. The resulting theory Ω determines a model-theoretic semantics for IPL, as shown in Section 17.4.

17.1 Multiple-conclusions via Boolean Constraints

While we have given the background to IPL in Chapter 2 according to the setup of ACSs. This allows us to expose essential details in its behaviour. Its syntax is provided by an alphabet J.

Definition 17.1 (Alphabet J)**.** *The alphabet* $J := \langle \mathbb{P}, \{\wedge,\vee,\to,\neg\}, \{\mathbin{,},\fatsemi,\varnothing\}\rangle$*, in which symbols* $\wedge,\vee,\to,\mathbin{,},\fatsemi$ *have arity* 2*, the symbol* $\neg$ *has arity* 1*, and* $\varnothing$ *has arity* 0*.*

Let $\equiv$ be the smallest relation satisfying commutative monoid equations for $\mathbin{,}$ and $\fatsemi$ with unit $\varnothing$.

Definition 17.2 (Sequent Calculus LJ)**.** *Sequent calculus* LJ *is comprised of the rules in Figure 17.1, in which* Δ *is either a J-formula* φ *or* $\varnothing$*, and* $\Gamma \equiv \Gamma'$ *and* $\Delta \equiv \Delta'$ *in* e.

In this chapter, provability in LJ *defines* the consequence relation $\vdash_{\mathsf{LJ}}$ for IPL. We desire to use ACSs to present IPL in a sequent calculus with *classical* shape. The constraints inform us of where the semantics of IPL diverge from those of FOL, which enables us to write down a theory that restricts abstractions to some class of structures that are the *models* of IPL.

The sequent calculus for classical logic closest to LJ is Gentzen's LK [200]. The points of departure are in c_R, $\to_\mathsf{L}$, and $\neg_\mathsf{L}$. These rules enable multiple-conclusion sequents to appear in the latter but not the former. We enrich LK with Boolean constraints (similarily to Chapter 13) to express this difference.

$$\frac{\Gamma \triangleright \Delta}{\varphi \,,\, \Gamma \triangleright \Delta}\ \mathsf{w_L} \qquad \frac{\Gamma \triangleright \varnothing}{\Gamma \triangleright \varphi}\ \mathsf{w_R} \qquad \frac{\varphi \,,\, \varphi \,,\, \Gamma \triangleright \Delta}{\varphi \,,\, \Gamma \triangleright \Delta}\ \mathsf{c_L} \qquad \frac{\Gamma' \triangleright \Delta'}{\Gamma \triangleright \Delta}\ \mathsf{e}$$

$$\frac{\Gamma \triangleright \varphi \quad \Gamma \triangleright \psi}{\Gamma \triangleright \varphi \land \psi}\ \land_{\mathsf{R}} \qquad \frac{\varphi \,,\, \Gamma \triangleright \Delta}{\varphi \land \psi \,,\, \Gamma \triangleright \Delta}\ \land^1_{\mathsf{L}} \qquad \frac{\psi \,,\, \Gamma \triangleright \Delta}{\varphi \land \psi \,,\, \Gamma \triangleright \Delta}\ \land^2_{\mathsf{L}} \qquad \frac{\varphi \,,\, \Gamma \triangleright \varnothing}{\Gamma \triangleright \neg \varphi}\ \neg_{\mathsf{R}}$$

$$\frac{\varphi \,,\, \Gamma \triangleright \Delta \quad \psi \,,\, \Gamma \triangleright \Delta}{\varphi \lor \psi \,,\, \Gamma \triangleright \Delta}\ \lor_{\mathsf{L}} \qquad \frac{\Gamma \triangleright \varphi}{\Gamma \triangleright \varphi \lor \psi}\ \lor^1_{\mathsf{R}} \qquad \frac{\Gamma \triangleright \psi}{\Gamma \triangleright \varphi \lor \psi}\ \lor^2_{\mathsf{R}} \qquad \frac{\Gamma \triangleright \varphi}{\neg \varphi \,,\, \Gamma \triangleright \varnothing}\ \neg_{\mathsf{L}}$$

$$\frac{\varphi \,,\, \Gamma \triangleright \psi}{\Gamma \triangleright \varphi \to \psi}\ \to_{\mathsf{R}} \qquad \frac{\Gamma_1 \triangleright \varphi \quad \psi \,,\, \Gamma_2 \triangleright \Delta}{\varphi \to \psi \,,\, \Gamma_1 \,,\, \Gamma_2 \triangleright \Delta}\ \to_{\mathsf{L}} \qquad \frac{}{\varphi \triangleright \varphi}\ \mathsf{ax}$$

Figure 17.1: Sequent Calculus LJ

Example 17.3. *The following is an enriched J-sequent.*

$$(\Gamma \cdot 1) \,,\, (\varphi \cdot x) \triangleright (\Delta \cdot \bar{x}) \fatsemi (\psi \cdot x)$$

■

For the rules that are the same across both systems (e.g., $\land_{\mathsf{I}}$), the expressions are inherited from the justifying sub-formulae of the conclusion; for example, one has the following:

$$\frac{\Gamma \triangleright \varphi \quad \Gamma \triangleright \psi}{\Gamma \triangleright \varphi \land \psi}\ \land_{\mathsf{R}} \quad \text{becomes} \quad \frac{\Gamma \triangleright \Delta \fatsemi (\varphi \cdot x) \quad \Gamma \triangleright \Delta \fatsemi (\psi \cdot x)}{\Gamma \triangleright \Delta \fatsemi (\varphi \land \psi \cdot x)}\ \land^{\mathcal{B}}_{\mathsf{R}}$$

For readability, we may suppress the Boolean expressions on these rules.

Definition 17.4 (Constraint System $\mathsf{LK} \oplus \mathcal{B}$)**.** *Constraint System* $\mathsf{LK} \oplus \mathcal{B}$ *is comprised of the rules in Figure 17.2, in which* Γ *and* Δ *are enriched J-data, and* $\Gamma \equiv \Gamma'$ *and* $\Delta \equiv \Delta'$ *in* $\mathsf{e}^{\mathcal{B}}$.

The ergo rendering $\mathsf{LK} \oplus \mathcal{B}$ sound and complete for IPL is the same as for RDvBC — see Chapter 13. In this setting, it may be regarded as the demand to choose which formulae in the succedent of a sequent to assert as a consequence of the context. This reading is closely related to the semantics of intuitionistic proof-

$$\dfrac{\Gamma \rhd \Delta}{\varphi \,, \Gamma \rhd \Delta}\ \mathsf{w}_{\mathsf{L}}^{\mathcal{B}} \qquad \dfrac{\Gamma \rhd \Delta}{\Gamma \rhd \Delta \,, \varphi}\ \mathsf{w}_{\mathsf{R}}^{\mathcal{B}} \qquad \dfrac{\varphi \,, \varphi \,, \Gamma \rhd \Delta}{\varphi \,, \Gamma \rhd \Delta}\ \mathsf{c}_{\mathsf{L}}^{\mathcal{B}} \qquad \dfrac{\Gamma \rhd \Delta \fatsemi \varphi \cdot x \fatsemi \varphi \cdot \bar{x}}{\Gamma \rhd \Delta \fatsemi \varphi}\ \mathsf{c}_{\mathsf{R}}^{\mathcal{B}}$$

$$\dfrac{\Gamma' \rhd \Delta'}{\Gamma \rhd \Delta}\ \mathsf{e}^{\mathcal{B}} \qquad \dfrac{\Gamma \rhd \Delta \cdot \bar{x} \,, \varphi \cdot x}{\neg \varphi \,, \Gamma \rhd \Delta}\ \neg_{\mathsf{L}}^{\mathcal{B}} \qquad \dfrac{\varphi \,, \Gamma \rhd \Delta}{\Gamma \rhd \Delta \fatsemi \neg \varphi}\ \neg_{\mathsf{R}}^{\mathcal{B}}$$

$$\dfrac{\Gamma \rhd \varphi \fatsemi \Delta \quad \Gamma \rhd \psi \fatsemi \Delta}{\Gamma \rhd \Delta \fatsemi \varphi \wedge \psi}\ \wedge_{\mathsf{R}}{}^{\mathcal{B}} \qquad \dfrac{\varphi \,, \Gamma \rhd \Delta}{\varphi \wedge \psi \,, \Gamma \rhd \Delta}\ \wedge_{\mathsf{L}}^{\mathcal{B}_1} \qquad \dfrac{\psi \,, \Gamma \rhd \Delta}{\varphi \wedge \psi \,, \Gamma \rhd \Delta}\ \wedge_{\mathsf{L}}^{\mathcal{B}_2}$$

$$\dfrac{\varphi \,, \Gamma \rhd \Delta \quad \psi \,, \Gamma \rhd \Delta}{\varphi \vee \psi \,, \Gamma \rhd \Delta}\ \vee_{\mathsf{L}}^{\mathcal{B}} \qquad \dfrac{\Gamma \rhd \Delta \fatsemi \varphi}{\Gamma \rhd \Delta \fatsemi \varphi \vee \psi}\ \vee_{\mathsf{R}}^{\mathcal{B}_1} \qquad \dfrac{\Gamma \rhd \Delta \fatsemi \psi}{\Gamma \rhd \Delta \fatsemi \varphi \vee \psi}\ \vee_{\mathsf{R}}^{\mathcal{B}_2}$$

$$\dfrac{\varphi \,, \Gamma \rhd \Delta \fatsemi \psi}{\Gamma \rhd \Delta \fatsemi \varphi \to \psi}\ \to_{\mathsf{R}}^{\mathcal{B}} \qquad \dfrac{\Gamma_1 \rhd \Delta_1 \cdot \bar{x} \,, \varphi \cdot x \quad \psi \,, \Gamma_2 \rhd \Delta_2}{\varphi \to \psi \,, \Gamma_1 \,, \Gamma_2 \rhd \Delta_1 \fatsemi \Delta_2}\ \to_{\mathsf{L}}^{\mathcal{B}} \qquad \dfrac{x = y = 1}{\varphi \cdot x \rhd \varphi \cdot y}\ \mathsf{ax}^{\mathcal{B}}$$

Figure 17.2: (Algebraic) Constraint System $\mathsf{LK} \oplus \mathcal{B}$

search provided by Pym and Ritter [173, 174].

Definition 17.5 (Choice Ergo). *Let $I : X \to \mathcal{B}$ be an interpretation of the language of the Boolean algebra. The choice ergo is the function σ_I which acts on J-formulae as follows:*

$$\sigma_I(\varphi) \mapsto \begin{cases} \varphi & \text{if } \varphi \text{ unlabelled} \\ \sigma_I(\psi) & \text{if } I(x) = 1 \text{ and } \varphi = \psi \cdot x \\ \varnothing & \text{if } I(x) = 0 \text{ and } \varphi = \psi \cdot x \end{cases}$$

The choice ergo acts on enriched J-data by acting pointwise on the formulae, and it acts on enriched sequents by acting on each component independently — that is, $\sigma_I(\Gamma \rhd \Delta) = \sigma_I(\Gamma) \rhd \sigma_I(\Delta)$.

Example 17.6. *Let S be the sequent in Example 17.3. If $I(x) = 1$, then $\sigma_I(S) = \Gamma \,, \varphi \rhd \psi$* ■

Proposition 17.7. *System $\mathsf{LK} \oplus \mathcal{B}$, with the choice ergo σ, is sound and complete for IPL,*

$$\Gamma \vdash^{\sigma}_{\mathsf{LK} \oplus \mathcal{B}} \Delta \qquad \textit{iff} \qquad \Gamma \vdash_{\mathsf{LJ}} \Delta$$

Proof of Soundness. Suppose $\Gamma \vdash^{\sigma}_{\mathsf{LK} \oplus \mathcal{B}} \Delta$, then there is a coherent $\mathsf{LK} \oplus \mathcal{B}$-reduction

$\mathcal{R}$ of an enriched sequent S such that $\sigma_I(S) := \Gamma \triangleright \Delta$, where I is any assignment satisfying $\mathcal{R}$. It follows that S is equivalent (up to exchange) to the sequent $\Gamma' \fatsemi \Pi \triangleright \Sigma \fatsemi \Delta'$ in which Γ' and Δ' are like Γ and Δ but with labelled data and I applied to the expressions in Π and Σ evaluates to 0 but applied to expressions in Γ' and Δ' evaluates to 1. We proceed by induction n on the height of $\mathcal{R}$ — the maximal number of reductive inferences in a branch of the tree.

- Base Case. If $n = 1$, then $\Gamma', \Pi \triangleright \Sigma, \Delta'$ is an instances of $\mathsf{ax}^{\mathcal{B}}$. But then $S = \varphi \cdot x \triangleright \varphi \cdot y$, for some formula φ. We have $\varphi \vdash_{\mathsf{LJ}} \varphi$ by ax.

- Inductive Step. The induction hypothesis (IH) is as follows: if $\Gamma' \vdash^{\sigma}_{\mathsf{LK} \oplus \mathcal{B}} \Delta'$ is witnessed by $\mathsf{LK} \oplus \mathcal{B}$-reductions of $k \leq n$, then $\Gamma \vdash_{\mathsf{LJ}} \Delta$.

  Suppose that the shortest reduction witnessing $\Gamma' \vdash^{\sigma}_{\mathsf{LK} \oplus \mathcal{B}} \Delta'$ is of height $n+1$. Let $\mathcal{R}$ be such a reduction. Without loss of generality, we assume the root of $\mathcal{R}$ is of the form $\Gamma' \fatsemi \Pi \triangleright \Sigma \fatsemi \Delta'$, as above. It follows by case analysis on the final inferences of $\mathcal{R}$ (i.e., reductive inferences applied to the root) that $\Gamma \vdash_{\mathsf{LJ}} \Delta$. We show two cases, the rest being similar.

  – Suppose the last inference of $\mathcal{R}$ was by $\mathsf{c_R}^{\mathcal{B}}$. In this case, $\mathcal{R}$ has an immediate sub-tree $\mathcal{R}'$ that is a coherent $\mathsf{LK} \oplus \mathcal{B}$-reduction of either $\Gamma' \fatsemi \Sigma \triangleright \Sigma' \fatsemi \Delta'$ or $\Gamma' \fatsemi \Sigma \triangleright \Sigma' \fatsemi \Delta''$, in which Σ' and Δ'' are like Σ and Δ', respectively, but with some formula repeated such that one occurrence carries an additional expression x and the other occurrence with an $\bar{x}$. Since the two reductions have the same constraints, the coherent assignment of $\mathcal{R}$ are the same as those $\mathcal{R}'$. We observe that under these coherent assignment $\mathcal{R}'$ witnesses $\Gamma' \vdash^{\sigma}_{\mathsf{LK} \oplus \mathcal{B}} \Delta'$. By the IH, since $\mathcal{R}'$ is of height n, it follows that $\Gamma \vdash_{\mathsf{LJ}} \Delta$.

  – Suppose the last inference of $\mathcal{R}$ was by $\rightarrow_{\mathsf{R}}{}^{\mathcal{B}}$. In this case, $\mathcal{R}$ has an immediate sub-tree $\mathcal{R}'$. If the principal formula of the inference is not in Δ', then $\mathcal{R}'$ witnesses $\Gamma' \vdash^{\sigma}_{\mathsf{LK} \oplus \mathcal{B}} \Delta'$. Hence, by the IH, we conclude that $\Gamma \vdash_{\mathsf{LJ}} \Delta$. If the principal formula of the inference is in Δ', then $\mathcal{R}'$ is a proof of $\varphi \fatsemi \Gamma' \fatsemi \Pi \triangleright \Sigma \fatsemi \Delta'' \fatsemi \psi$, where $\Delta' := \Delta'' \fatsemi \varphi \rightarrow \psi$. It follows, by the IH, that $\varphi, \Gamma' \vdash_{\mathsf{LJ}} \Delta'' \fatsemi \psi$. By the $\rightarrow_{\mathsf{R}}$-rule in LJ, we have $\Gamma \vdash_{\mathsf{LJ}} \varphi \rightarrow \psi$ — that is, $\Gamma \vdash_{\mathsf{LJ}} \Delta$, as required.

This completes the case analysis.

This completes the induction. □

Proof of Completeness. This follows immediately from the fact that all the rules of LJ may be simulated in $\mathsf{LK} \oplus \mathcal{B}$. □

The point of this work is that $\mathsf{LK} \oplus \mathcal{B}$ characterizes IPL in a way that is combinatorially comparable to FOL. This is significant as the semantics of IPL is given classically; hence $\mathsf{LK} \oplus \mathcal{B}$ bridges the proof-theoretic and model-theoretic characterizations of IPL.

17.2 Faithfulness & Adequacy

Though we may use $\mathsf{LK} \oplus \mathcal{B}$ to reason about IPL with classical combinatorics, the system does not immediately reveal the meaning of the connectives of IPL in terms of their counterparts in FOL. The problem is that $\mathsf{LK} \oplus \mathcal{B}$-proofs are only *globally* valid for IPL, for the choice ergo σ. Therefore, to conduct a semantical analysis of IPL in terms of FOL, we require a constraint system based on FOL whose proofs are *locally* valid — that is, a system which is not only sound and complete for IPL, but *faithful* and *adequate*. In this section, we analyze the relationship between $\mathsf{LK} \oplus \mathcal{B}$ and LJ to produce such a system.

A significant difference between LK and LJ is the use of richer data-structures for the suceedent in the former than in the latter (i.e., list or multi-sets vs formulae). Intuitively, the data-constructor in the succeedent acts as a meta-level disjunction, thus we may investigate how $\mathsf{LK} \oplus \mathcal{B}$ captures IPL by considering how $\mathsf{c}^{\mathcal{B}}_{\mathsf{R}}$ interacts with $\vee^{\mathcal{B}}_{\mathsf{R}}$. We may restrict attention to interactions of the following form:

$$\dfrac{\dfrac{\dfrac{\Gamma \triangleright \Delta \fatsemi (\varphi \cdot x) \fatsemi (\psi \cdot \bar{x})}{\Gamma \triangleright \Delta \fatsemi (\varphi \cdot x) \fatsemi (\varphi \vee \psi \cdot \bar{x})}\,\vee^{\mathcal{B}_2}_{\mathsf{R}}}{\Gamma \triangleright \Delta \fatsemi (\varphi \vee \psi \cdot x) \fatsemi (\varphi \vee \psi \cdot \bar{x})}\,\vee^{\mathcal{B}_1}_{\mathsf{R}}}{\Gamma \triangleright \Delta \fatsemi \varphi \vee \psi}\,\mathsf{c}^{\mathcal{B}}_{\mathsf{R}}$$

These may be collapsed into single inference rules,

$$\frac{\Gamma \rhd \varphi \cdot xy \fatsemi \psi \cdot x\bar{y} \fatsemi \Delta}{\Gamma \rhd \varphi \vee \psi \cdot x \fatsemi \Delta}$$

The other connectives either make use of constraints and, therefore, have no significant interaction with disjunction or can be permuted without loss of generality — for example, a typical interaction between $\mathsf{c}_{\mathsf{R}}^{\mathcal{B}}$ and $\wedge_{\mathsf{R}}^{\mathcal{B}}$,

$$\dfrac{\dfrac{\dfrac{\dfrac{\Gamma \rhd \Delta_1 \fatsemi \varphi \fatsemi \varphi \quad \Gamma \rhd \Delta_2 \fatsemi \varphi \fatsemi \psi}{\Gamma \rhd \Delta_1 \fatsemi \Delta_2 \fatsemi \varphi \fatsemi \varphi \wedge \psi}\,\wedge_{\mathsf{R}}^{\mathcal{B}}}{\Gamma \rhd \Delta_1 \fatsemi \Delta_2 \fatsemi \varphi \wedge \psi \fatsemi \varphi}\,\mathsf{e}_{\mathsf{R}}^{\mathcal{B}} \qquad \Gamma \rhd \Delta_3 \fatsemi \psi}{\Gamma \rhd \Delta_1 \fatsemi \Delta_2 \fatsemi \Delta_3 \fatsemi \varphi \wedge \psi \fatsemi \varphi \wedge \psi}\,\wedge_{\mathsf{R}}^{\mathcal{B}}}{\Gamma \rhd \Delta_1 \fatsemi \Delta_2 \fatsemi \Delta_3 \fatsemi \varphi \wedge \psi}\,\mathsf{c}_{\mathsf{R}}^{\mathcal{B}}$$

may be replaced by the derivation, which permutes the inferences,

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\Gamma \rhd \Delta_1 \fatsemi \varphi \fatsemi \varphi}{\Gamma \rhd \Delta_1 \fatsemi \varphi \fatsemi \varphi \fatsemi \Delta_2}\,\mathsf{w}_{\mathsf{R}}^{\mathcal{B}}}{\Gamma \rhd \Delta_1 \fatsemi \varphi \fatsemi \Delta_2 \fatsemi \varphi}\,\mathsf{e}_{\mathsf{R}}^{\mathcal{B}}}{\Gamma \rhd \Delta_1 \fatsemi \Delta_2 \fatsemi \varphi \fatsemi \varphi}\,\mathsf{e}_{\mathsf{R}}^{\mathcal{B}}}{\Gamma \rhd \Delta_1 \fatsemi \Delta_2 \fatsemi \varphi}\,\mathsf{c}_{\mathsf{R}}^{\mathcal{B}} \qquad \Gamma \rhd \Delta_3 \fatsemi \psi}{\Gamma \rhd \Delta_1 \fatsemi \Delta_2 \fatsemi \Delta_3 \fatsemi \varphi \wedge \psi}\,\wedge_{\mathsf{R}}^{\mathcal{B}}$$

This analysis allows us to eliminate $\mathsf{c}_{\mathsf{R}}^{\mathcal{B}}$ as it is captured wherever the augmented rule for disjunction needs it; similarly, we may eliminate $\mathsf{c}_{\mathsf{L}}^{\mathcal{B}}$ by incorporating it in the other rules. This yields a new constraint system, $\mathsf{LK}^{+} \oplus \mathcal{B}$.

Definition 17.8 (System $\mathsf{LK}^{+} \oplus \mathcal{B}$)**.** *System* $\mathsf{LK}^{+} \oplus \mathcal{B}$ *is given in Figure 17.3, in which* Γ *and* Δ *are enriched J-datum, and* $\Gamma \equiv \Gamma$ *and* $\Delta \equiv \Delta'$ *in* e.

System $\mathsf{LK}^{+} \oplus \mathcal{B}$ characterizes IPL locally — that is, it is faithful and adequate for *some* sequent calculus for IPL. That sequent calculus, however, is not LJ, but rather a multiple-conclusion system LJ^{+}. Essentially, LJ^{+} is the multiple-conclusion sequent calculus introduced by Dummett [52] with particular instances of the structural rules incorporated into the operational rules.

$$\frac{}{\varphi \mathbin{,} \Gamma \rhd \Delta \fatsemi \varphi}\,\mathsf{ax} \qquad \frac{\Gamma \rhd \Delta}{\varphi \mathbin{,} \Gamma \rhd \Delta}\,\mathsf{w}_{\mathsf{L}}^{\mathcal{B}} \qquad \frac{\Gamma \rhd \Delta}{\Gamma \rhd \Delta \fatsemi \varphi}\,\mathsf{w}_{\mathsf{R}}^{\mathcal{B}} \qquad \frac{\varphi \mathbin{,} \varphi \mathbin{,} \Gamma \rhd \Delta}{\varphi \mathbin{,} \Gamma \rhd \Delta}\,\mathsf{c}_{\mathsf{L}}^{\mathcal{B}} \qquad \frac{\Gamma' \rhd \Delta'}{\Gamma \rhd \Delta}\,\mathsf{e}^{\mathcal{B}}$$

$$\frac{\Gamma \rhd \Delta \fatsemi \varphi}{\neg \varphi \fatsemi \Gamma \rhd \Delta}\,\neg_{\mathsf{L}}^{\mathcal{B}} \qquad \frac{\varphi \cdot xy \mathbin{,} \Gamma \rhd \Delta \quad xy = 1}{\Gamma \rhd \Delta \fatsemi \neg \varphi \cdot x}\,\neg_{\mathsf{R}}^{\mathcal{B}}$$

$$\frac{\varphi \mathbin{,} \psi \mathbin{,} \Gamma \vdash \Delta}{\varphi \land \psi \mathbin{,} \Gamma \rhd \Delta}\,\land_{\mathsf{L}}^{\mathcal{B}} \qquad \frac{\Gamma \rhd \Delta \fatsemi \varphi \quad \Gamma \rhd \Delta \fatsemi \psi}{\Gamma \rhd \Delta \fatsemi \varphi \land \psi}\,\land_{\mathsf{R}}^{\mathcal{B}}$$

$$\frac{\varphi \mathbin{,} \Gamma \rhd \Delta \quad \psi \mathbin{,} \Gamma \rhd \Delta}{\varphi \lor \psi \mathbin{,} \Gamma \rhd \Delta}\,\lor_{\mathsf{L}}^{\mathcal{B}} \qquad \frac{\Gamma \rhd \varphi \cdot xy \fatsemi \psi \cdot x\bar{y} \fatsemi \Delta}{\Gamma \rhd \varphi \lor \psi \cdot x \fatsemi \Delta}\,\lor_{\mathsf{R}}^{\mathcal{B}}$$

$$\frac{\Gamma \rhd \Delta \fatsemi \varphi \quad \psi \mathbin{,} \Gamma \rhd \Delta}{\varphi \to \psi \mathbin{,} \Gamma \rhd \Delta}\,\to_{\mathsf{L}}^{\mathcal{B}} \qquad \frac{\Gamma \mathbin{,} \varphi \cdot xy \rhd \psi \cdot xy \fatsemi \Delta \quad xy = 1}{\Gamma \rhd \varphi \to \psi \cdot x \fatsemi \Delta}\,\to_{\mathsf{R}}^{\mathcal{B}}$$

Figure 17.3: Constraint System $\mathsf{LK}^{+} \oplus \mathcal{B}$

$$\frac{}{\varphi \mathbin{,} \Gamma \rhd \Delta \fatsemi \varphi}\,\mathsf{ax} \qquad \frac{\Gamma \rhd \Delta}{\varphi \mathbin{,} \Gamma \rhd \Delta}\,\mathsf{w}_{\mathsf{L}} \qquad \frac{\Gamma \rhd \Delta}{\Gamma \rhd \Delta \fatsemi \varphi}\,\mathsf{w}_{\mathsf{R}} \qquad \frac{\varphi \mathbin{,} \varphi \mathbin{,} \Gamma \rhd \Delta}{\varphi \mathbin{,} \Gamma \rhd \Delta}\,\mathsf{c}_{\mathsf{L}} \qquad \frac{\Gamma' \rhd \Delta'}{\Gamma \rhd \Delta}\,\mathsf{e}$$

$$\frac{\Gamma \rhd \Delta \fatsemi \varphi}{\neg \varphi \mathbin{,} \Gamma \rhd \Delta}\,\neg_{\mathsf{L}} \qquad \frac{\varphi \mathbin{,} \Gamma \rhd \varnothing}{\Gamma \rhd \Delta \fatsemi \neg \varphi}\,\neg_{\mathsf{R}}$$

$$\frac{\varphi \mathbin{,} \psi \mathbin{,} \Gamma \rhd \Delta}{\varphi \land \psi \mathbin{,} \Gamma \rhd \Delta}\,\land_{\mathsf{L}} \qquad \frac{\Gamma \rhd \Delta \fatsemi \varphi \quad \Gamma \rhd \Delta \fatsemi \psi}{\Gamma \rhd \Delta \fatsemi \varphi \land \psi}\,\land_{\mathsf{R}}$$

$$\frac{\varphi \mathbin{,} \Gamma \rhd \Delta \quad \psi \mathbin{,} \Gamma \rhd \Delta}{\varphi \lor \psi \mathbin{,} \Gamma \rhd \Delta}\,\lor_{\mathsf{L}} \qquad \frac{\Gamma \rhd \varphi \fatsemi \psi \fatsemi \Delta}{\Gamma \rhd \varphi \lor \psi \fatsemi \Delta}\,\lor_{\mathsf{R}}$$

$$\frac{\Gamma \rhd \Delta \fatsemi \varphi \quad \psi \mathbin{,} \Gamma \rhd \Delta}{\varphi \to \psi \mathbin{,} \Gamma \rhd \Delta}\,\to_{\mathsf{L}} \qquad \frac{\Gamma \fatsemi \varphi \rhd \psi}{\Gamma \rhd \varphi \to \psi \fatsemi \Delta}\,\to_{\mathsf{R}}$$

Figure 17.4: Sequent Calculus LJ^{+}

Definition 17.9 (Sequent Calculus LJ^{+}). *Sequent calculus* LJ^{+} *is given by the rules in Figure 17.4, in which* Γ *and* Δ *are J-datum, and* $\Gamma \equiv \Gamma'$ *and* $\Delta \equiv \Delta'$ *in* e.

Proposition 17.10. *Sequent calculus* LJ^{+} *is sound and complete for IPL,*

$$\Gamma \vdash_{\mathsf{LJ}} \varphi \qquad \textit{iff} \qquad \Gamma \vdash_{\mathsf{LJ}^{+}} \varphi$$

Proof. Follows from Dummett [52]. □

The choice ergo σ extends to a valuation from $\mathsf{LK}^+ \oplus \mathcal{B}$ to LJ^+ by pointwise application to every sequent within the reduction.

Proposition 17.11. *System $\mathsf{LK}^+ \oplus \mathcal{B}$, with valuation σ, is faithful and adequate with respect to LJ^+.*

Proof. Faithfulness follows from the observation that each rule in $\mathsf{LK}^+ \oplus \mathcal{B}$ produces the corresponding rule in LJ^+ when its constraints are observed. Adequacy follows from the observation that every instance of every rule in LJ^+ is an evaluation of an instance of a rule of $\mathsf{LK}^+ \oplus \mathcal{B}$ respecting its constraints. □

The force of this result is that we may use $\mathsf{LK}^+ \oplus \mathcal{B}$ to study intuitionistic connectives in terms of their classical counterparts. This analysis allows us to synthesize a model-theoretic semantics for IPL from the semantics of FOL.

17.3 Synthezising the Semantics

We desire to construct a set of meta-formulae Ω that forms a tractable definition of a semantics for IPL. The idea is that we use $\mathsf{LK}^+ \oplus \mathcal{B}$ to determine meta-formulae for each connective such that *simulates* the rules for IPL in LJ^+ within LK^+; for example, the meta-formula for $\to$ in Ω should impose the single-conclusion condition for $\to_{\mathsf{R}}$.

We observe in $\mathsf{LK}^+ \oplus \mathcal{B}$ that intuitionistic conjunction has the same inferential behaviour as classical conjunction,

$$\frac{\Gamma \triangleright \Delta \fatsemi \varphi \quad \Gamma \triangleright \Delta \fatsemi \psi}{\Gamma \triangleright \Delta \fatsemi \varphi \wedge \psi}\wedge^{\mathcal{B}}_{\mathsf{R}} \qquad \text{vs.} \qquad \frac{\Pi \triangleright \Sigma, \Phi \quad \Pi \triangleright \Sigma, \Psi}{\Pi \triangleright \Sigma, \Phi \& \Psi}\&_{\mathsf{R}}$$

Therefore, $\wedge$ in IPL should be defined as $\&$ in FOL. A candidate meta-formula governing the connective is the universal closure of the following:

$$(w : \hat{\varphi} \wedge \hat{\psi}) \qquad \text{iff} \qquad (w : \hat{\varphi}) \& (w : \hat{\psi})$$

— we use the convention in Section 16 in which $\hat{\varphi}$ and $\hat{\psi}$ are used as meta-variables for formulae of the object-logic. This is the appropriate clause for the connective as it enables the following behaviour in the meta-logic:

$$\dfrac{\dfrac{\Omega,\Pi,(w:\Gamma)\rhd(w:\varphi),\Sigma \quad \Omega,\Pi,(w:\Gamma)\rhd(w:\psi),\Sigma}{\Omega,\Pi,(w:\Gamma)\rhd(w:\varphi)\&(w:\psi),\Sigma}\;\&_{\mathsf{R}}}{\Omega,\Pi,(w:\Gamma)\rhd(w:\varphi\wedge\psi),\Sigma}\;\mathsf{res}_{\mathsf{R}}$$

Recall, such derivations correspond to the *use* of the clause — see Section 16.2 — which may be collapsed into rules themselves. In this case, it becomes the following:

$$\dfrac{\Omega,\Pi,(w:\Gamma)\rhd(w:\varphi),\Sigma \quad \Omega,\Pi,(w:\Gamma)\rhd(w:\psi),\Sigma}{\Omega,\Pi,(w:\Gamma)\rhd(w:\varphi\wedge\psi),\Sigma}\;\wedge\text{-clause}$$

Intuitively, this rule precisely recovers $\wedge_{\mathsf{R}} \in \mathsf{LJ}^{+}$,

$$\dfrac{\Gamma\rhd\Delta\fatsemi\varphi \quad \Gamma\rhd\Delta\fatsemi\psi}{\Gamma\rhd\Delta\fatsemi\varphi\wedge\psi}\;\wedge_{\mathsf{R}}$$

Of course, it is essential to check that the clause also has the correct behaviour on the left-hand side of sequents; that is, this clause also simulates $\wedge_{\mathsf{L}}$.

We obtain the universal closure of the following for the clauses governing disjunction ($\vee$) and ($\bot$) analogously:

$$(x:\hat{\varphi}\vee\hat{\psi}) \Leftrightarrow \big((x:\hat{\varphi})\parr(x:\hat{\psi})\big) \qquad (x:\bot)\Leftrightarrow\bot$$

It remains to analyze implication ($\to$). The above reasoning does not follow *mutatis mutandis* because the constraints in $\mathsf{LK}^{+}\oplus\mathcal{B}$ becomes germane, so we require something additional to get the appropriate simulation. Consider $\to_{\mathsf{R}}^{\mathcal{B}}$ in $\mathsf{LK}^{+}\oplus\mathcal{B}$,

$$\dfrac{\Gamma\,,\,(\varphi\cdot xy)\rhd(\psi\cdot xy)\fatsemi\Delta \quad xy=1}{\Gamma\rhd(\varphi\to\psi\cdot x)\fatsemi\Delta}\;\to_{\mathsf{R}}^{\mathcal{B}}$$

Since $\mathsf{LK}^{+}\oplus\mathcal{B}$ is not only sound and complete for IPL but faithful and adequate, we know that this rule characterizes the connective. The rule admits two assignment classes: $x \mapsto 0$ or $x \mapsto 1$. Super-imposing these valuations can capture the behaviour we desire in the meta-logic on each other by using *possible worlds* to distinguish the possible cases,

$$\frac{\Omega, \Pi[w \mapsto u], \Pi[w \mapsto v], (u : \varphi) \triangleright (u : \psi), \Sigma[w \mapsto v], \Sigma[w \mapsto u]}{\Omega, \Pi \triangleright (w : \varphi \to \psi), \Sigma}$$

We assume that since u and v are distinct, they do not interact so that the rule captures the following possibilities:

$$\frac{\Omega, \Pi[w \mapsto u], (u : \varphi) \triangleright (u : \psi), \Sigma[w \mapsto u]}{\Omega, \Pi \triangleright (w : \varphi \to \psi), \Sigma} \qquad \frac{\Omega, \Pi[w \mapsto v] \triangleright \Sigma[w \mapsto v]}{\Omega, \Pi \triangleright (w : \varphi \to \psi), \Sigma}$$

The assumption is proved valid below — see Proposition 17.18. Intuitively, these capture the possible cases of $\to_{\mathsf{R}}^{\mathcal{B}}$. This justifies that this super-imposing behaviour is what we desire of the clause governing implication. It remains only to find that clause.

One of these possible behaviours amounts to a weakening — that is, reading reductively, we remove $(w : \varphi \to \psi)$ from the succeedent — a behaviour already present through interpreting the data-structures as classical conjunction and disjunction.

The other possible behaviour we recognize as having the combinatorial behaviour of classical implication concerns creating a meta-formula in the antecedent of the premiss by taking part in a meta-formula in the succeedent of the conclusion. Naively, we may consider the following as the clause:

$$(w : \hat{\varphi} \to \hat{\psi}) \qquad \text{iff} \qquad \big((w : \hat{\varphi}) \Rightarrow (w : \hat{\psi})\big)$$

However, this fails to account for the change in the world. Thus, we require the clause to have a universal quantifier over worlds and a precondition that enables the $\Pi[w \mapsto u]$ substitution. Analyzing the possible use cases, we observe that R must satisfy reflexivity so that the substitution for u may be trivial (e.g., when validating $(w : \varphi \land (\varphi \to \psi)) \triangleright (w : \psi)$). In total, we have the universal closure of the following meta-formulae:

$$\begin{array}{rcl}(x : \hat{\varphi} \to \hat{\psi}) & \text{iff} & \forall y\big((xRy) \& (y : \hat{\varphi}) \Rightarrow (y : \hat{\psi})\big) \\ xRx & & xRy \Rightarrow \forall \hat{\Gamma}\big((x : \hat{\Gamma}) \Rightarrow (y : \hat{\Gamma})\big)\end{array}$$

We have introduced an ancillary relation R precisely to recover the behaviour determined by the algebraic constraints. Since the data-constructors behave exactly as conjunction ($\wedge$) and disjunction ($\vee$), we may replace $\hat{\Gamma}$ with $\hat{\varphi}$ without loss of generality.

This concludes the analysis. Altogether, the meta-formulae thus generated comprise a tractable definition for a model-theoretic semantics for IPL, called Ω_{IPL}. Any abstraction of this theory gives the semantics.

Definition 17.12 (Intuitionistic Frame, Satisfaction, and Model)**.** *An intuitionistic frame is a pair $\mathcal{F} := \langle \mathbb{U}, R \rangle$ in which R is a reflexive relation on $\mathbb{U}$.*

Let $[\![-]\!]$ be an interpretation mapping J-atoms to $\mathbb{U}$. Intuitionistic satisfaction is the relation between elements $w \in \mathbb{U}$ and $\varphi \in \mathbb{F}$ defined by the clauses of Figure 17.5.

A pair $\langle \mathcal{F}, [\![-]\!] \rangle$ is an intuitionistic model iff it is persistent — that is, for any J-formula φ and worlds w and v,

$$\textit{if } wRv \textit{ and } w \Vdash \varphi\textit{, then } v \Vdash \varphi$$

The class of all intuitionistic models is $\mathbb{K}$.

This semantics generates the following validity judgment:

$$\Gamma \vDash \Delta \qquad \text{iff} \qquad \text{for any } \mathfrak{M} \in \mathbb{K} \text{ and any } w \in \mathfrak{M}\text{, if } w \Vdash \Gamma\text{, then } w \Vdash \Delta$$

It remains to prove soundness and completeness for the semantics — see Section 17.4 below. Of course, we have *designed* the semantics so that it corresponds to LJ^{+}.

As a remark, *tertium non datur* is known not to apply in IPL. How does encod-

$w \Vdash \mathrm{p}$	iff	$w \in [\![\mathrm{p}]\!]$
$w \Vdash \varphi \wedge \psi$	iff	$w \Vdash \varphi$ and $w \Vdash \psi$
$w \Vdash \varphi \vee \psi$	iff	$w \Vdash \varphi$ or $w \Vdash \psi$
$w \Vdash \varphi \to \psi$	iff	for any u, if wRu and $u \Vdash \varphi$, then $u \Vdash \psi$
$w \Vdash \bot$		never

Figure 17.5: Satisfaction for IPL

ing of IPL within classical logic avoid it? It is instructive to study this question as it explicates the clause for implication, which defines the intuitionistic connective in terms of a (meta-level) classical one.

Example 17.13. *The following reduction is a canonical instance of using the clause:*

$$\begin{array}{cl}
\Omega_{IPL}, (wRu), (u:\varphi) \triangleright (w:\varphi), \bot & \\ \cline{1-1}
\Omega_{IPL}, (wRu \& u:\varphi) \triangleright (w:\varphi), \bot & \&_{\mathsf{L}} \\ \cline{1-1}
\Omega_{IPL} \triangleright (w:\varphi), (wRu \& u:\varphi \Rightarrow \bot) & \Rightarrow_{\mathsf{R}} \\ \cline{1-1}
\Omega_{IPL} \triangleright (w:\varphi), \forall x (wRx \& x:\varphi \Rightarrow \bot) & \forall_{\mathsf{R}} \\ \cline{1-1}
\Omega_{IPL} \triangleright (w:\varphi), (w:\varphi \to \bot) & \to\text{-}\mathit{clause} \\ \cline{1-1}
\Omega_{IPL} \triangleright (w:\varphi) \parr (w:\varphi \to \bot) & \parr_{\mathsf{R}} \\ \cline{1-1}
\Omega_{IPL} \triangleright (w:\varphi \vee \varphi \to \bot) & \vee\text{-}\mathit{clause}
\end{array}$$

Since $(u:\varphi)$ in the antecedent and $(w:\varphi)$ in the succedent are different atoms since u and w are different world-variables, one has not reached an axiom. In short, despite working in a classical system, the above calculation witnesses that $\varphi \vee \neg\varphi$ is valid in IPL if and only if one already knows that φ is valid in IPL or one already knows that $\neg\varphi$ is valid in IPL. ▪

The underlying conceit of this construction of the semantics is that the rules of LJ^{+} *define* the logical constants of IPL, in some sense. The idea that the meaning of a logical constant is determined by its rules is the subject of Part III.

17.4 Soundness & Completeness

Since the semantics generated in this chapter is the one given by Kripke [124], its soundness and completeness is known. Nonetheless, the method by which the semantics was determined gives a different method for establishing this relationship — namely, as an instance of the work in Chapter 16 (see also Chapter 10). We shall concentrate on completeness.

Theorem 17.14 (Soundness). *If* $\Gamma \vdash \varphi$*, then* $\Gamma \vDash \varphi$*.*

Proof. This was shown by Kripke [125]. One may also apply the traditional inductive proof — see, for example, Van Dalen [211]. Otherwise, use the bisimulation techniques of Chapter 6. □

In this paper, we shall prove completeness symmetrically. We show that the relational calculus generated by the semantics is *adequate* for a sequent calculus characterizing IPL. This suffices because the relational calculus is sound and complete for the semantics, as per Theorem 16.12. To simplify the presentation, we shall have contraction explicit in the relational calculus for Ω_{IPL} rather than implicit.

Definition 17.15 (Relational Calculus RJ). *The relational calculus* RJ *is comprised of the rules in Figure 17.6 —* Φ *denotes a meta-formula,* Π *and* Σ *denote multiset of meta-formulae, x and y denote world-variables,* Δ *denotes object-logic data,* φ *and* ψ *denote object-logic formulae. The rules* $,_{\mathsf{L}}$ *and* $\fatsemi_{\mathsf{R}}$ *are invertible, and the world-variable y does not appear elsewhere in the sequents in* $\rightarrow_{\mathsf{R}}$.

Corollary 17.16. $\Gamma \vDash \Delta$ *iff* $\Gamma \vdash^{\sigma}_{\mathsf{IPL}} \Delta$

Proof. Instance of Theorem 16.12. □

We desire to transform RJ into a sequent calculus for which it is adequate, which we may then show is a characterization of IPL. Such transformations are discussed in Section 17.2, but the rules of IPL are slightly too complex for the procedure of that section to apply immediately. Therefore, we require some additional meta-theory.

$$\frac{\Phi,\Phi,\Pi\rhd\Sigma}{\Phi,\Pi\rhd\Sigma}\,\mathsf{c} \qquad \frac{}{\Phi,\Pi\rhd\Sigma,\Phi}\,\mathsf{ax} \qquad \frac{}{\bot,\Pi\rhd\Sigma}\,\bot \qquad \frac{}{\Pi\rhd\Sigma,xRx}\,\mathsf{ref}$$

$$\frac{(x:\varphi),(x:\psi),\Pi\rhd\Sigma}{(x:\varphi\land\psi),\Pi\rhd\Sigma}\,\land_{\mathsf{L}} \qquad \frac{\Pi\rhd\Sigma,(x:\varphi) \quad \Pi\rhd\Sigma,(x:\psi)}{\Pi\rhd\Sigma,(x:\varphi\land\psi)}\,\land_{\mathsf{R}}$$

$$\frac{(x:\varphi),\Pi\rhd\Sigma \quad (x:\psi),\Pi\rhd\Sigma}{(x:\varphi\lor\psi),\Pi\rhd\Sigma}\,\lor_{\mathsf{L}} \qquad \frac{\Pi\rhd\Sigma,(x:\varphi),(x:\psi)}{\Pi\rhd\Sigma,(x:\varphi\lor\psi)}\,\lor_{\mathsf{R}}$$

$$\frac{\Pi\rhd\Sigma,(xRy) \quad \Pi\rhd\Sigma,(y:\varphi) \quad (y:\psi),\Pi\rhd\Sigma}{(x:\varphi\to\psi),\Pi\rhd\Sigma}\,\to_{\mathsf{L}}$$

$$\frac{(xRy),(y:\varphi),\Pi\rhd\Sigma,(y:\psi)}{\Pi\rhd\Sigma,(x:\varphi\to\psi)}\,\to_{\mathsf{R}} \qquad \frac{(xRy),(x:\Gamma),(y:\Gamma),\Pi\rhd\Sigma}{(xRy),(x:\Gamma),\Pi\rhd\Sigma}\,\mathsf{pers}$$

$$\frac{\bot,\Pi\rhd\Sigma}{(x:\bot),\Pi\rhd\Sigma}\,\bot_{\mathsf{L}} \qquad \frac{\Pi\rhd\Sigma,\bot}{\Pi\rhd\Sigma,(x:\bot)}\,\bot_{\mathsf{L}}$$

$$\frac{(x:\Gamma),(x:\Gamma'),\Pi\rhd\Sigma}{(x:\Gamma\fatsemi\Gamma'),\Pi\rhd\Sigma}\,\fatsemi_{\mathsf{L}} \qquad \frac{\Pi\rhd\Sigma,(x:\Delta),(x:\Delta')}{\Pi\rhd\Sigma,(x:\Delta\fatsemi\Delta')}\,\fatsemi_{\mathsf{R}}$$

Figure 17.6: Relational Calculus RJ

The complexity comes from the →-clause as it may result in non-BVSs. However, we immediately use persistence to create a composite behaviour that a basic rule can capture. This is because the combined effect yields BVSs whose contents may be partitioned; by design, persistence uses world-variables that do not, and cannot, interact throughout the rest of the proof.

Definition 17.17 (World-independence)**.** *Let Π and Σ be lists of meta-formulae. The lists Π and Σ are world-independent iff the set of world-variable in Π is disjoint from the set of world-variables in Σ.*

Let $\mathfrak{S}$ be a tractable semantics and let Ω be a tractable definition. Let Π_1,Σ_1 and Π_2,Σ_2 be world-independent lists meta-formulae. The semantics $\mathfrak{S}$ has world-independence iff, if $\Omega,\Pi_1,\Pi_2 \blacktriangleright \Sigma_1,\Sigma_2$, then either $\Omega,\Pi_1 \blacktriangleright \Sigma_1$ or $\Omega,\Pi_2 \blacktriangleright \Sigma_2$.

Intuitively, world-independence says that whatever is true at a world in the semantics does not depend on truth at a world not related to it.

Let $\Pi_i^1,\Pi_i^2,\Sigma_i^1$, and Σ_i^2 be lists of meta-formulae, for $1\leq i\leq n$, and suppose that

Π^1_i, Σ^1_i is world-independent from Π^2_i, Σ^2_i. Consider a rule of the following form:

$$\frac{\Pi^1_1, \Pi^2_1 \triangleright \Sigma^1_1, \Sigma^2_1 \quad \dots \quad \Pi^1_n, \Pi^2_n \triangleright \Sigma^1_n, \Sigma^2_n}{\Pi \triangleright \Sigma}$$

Assuming world-independence of the semantics, this rule can be replaced by the following two rules:

$$\frac{\Pi^1_1 \triangleright \Sigma^1_1 \quad \dots \quad \Pi^1_n \triangleright \Sigma^1_n}{\Pi \triangleright \Sigma} \qquad \frac{\Pi^1_2 \triangleright \Sigma^1_2 \quad \dots \quad \Pi^2_n \triangleright \Sigma^2_n}{\Pi \triangleright \Sigma}$$

If all the lists were basic, iterating these replacements may yield a set of basic rules with the same expressive power as the original rule.

Proposition 17.18. *The semantics of IPL — that is, the semantics $\langle \mathbb{K}, \Vdash \rangle$ defined by Ω_{IPL} — has world-independence.*

An analogous proposition appears for the treatment of BI in Chapter 10 as Proposition 10.19.

Proof. If $\Omega_{\text{IPL}}, \Pi_1, \Pi_2 \vdash \Sigma_1, \Sigma_2$, then there is a G3-proof $\mathcal{D}$ of it. We proceed by induction the number of resolutions in such a proof.

Base Case. Recall, without loss of generality, an instantiation of any clause from Ω_{IPL} is a resolution. Therefore, if $\mathcal{D}$ contains no resolutions, then $\Omega_{\text{IPL}}, \Pi_1, \Pi_2 \vdash \Sigma_1, \Sigma_2$ is an instance of ax. In this case, either $\Omega_{\text{IPL}}, \Pi_1 \vdash \Sigma_1$ or $\Omega_{\text{IPL}}, \Pi_2 \vdash \Sigma_2$ is also an instance of ax, by world-independence.

Induction Step. After a resolution of a sequent of the form $\Omega_{\text{IPL}}, \Pi_1, \Pi_2 \vdash \Sigma_1, \Sigma_2$, one returns a meta-sequent of the same form — that is, a meta-sequent in which we may partition the meta-formulae in the antecedent and succeedent into world-independent multi-sets. This being the case, the result follows immediately from the induction hypothesis.

The only non-obvious case is in the case of a closed resolution using the $\to$-clause in the antecedent because they have universal quantifiers that would allow one to produce a meta-atom that contains both a world from Σ_1, Π_1 and Σ_2, Π_2 simultaneously, thereby breaking world-independence.

Let $\Pi_1 = \Pi_1', (w \Vdash \varphi \to \psi)$ and suppose u is a world-variable appearing in Σ_2, Π_2. Consider the following computation — for readability, we suppress Ω_{IPL}:

$$\dfrac{\dfrac{\dfrac{\dfrac{\Pi_1', \Pi_2 \rhd \Sigma_1, \Sigma_2, wRu \qquad \Pi_1', \Pi_2 \rhd \Sigma_1, \Sigma_2, (u : \varphi)}{\Pi_1', \Pi_2 \rhd \Sigma_1, \Sigma_2, (wRu) \& (u : \varphi)}\,\&_{\mathsf{L}} \qquad \Pi_1', \Pi_2, (u : \psi) \rhd \Sigma_1, \Sigma_2}{\Pi_1', (wRu \& (u : \varphi) \Rightarrow u : \psi), \Pi_2 \rhd \Sigma_1, \Sigma_2}\,\Rightarrow_{\mathsf{L}}}{\Pi_1', \forall x (wRx \& x : \varphi \Rightarrow x : \psi), \Pi_2 \rhd \Sigma_1, \Sigma_2}\,\forall_{\mathsf{L}}}{\Pi_1', (w : \varphi \to \psi), \Pi_2 \rhd \Sigma_1, \Sigma_2}\,\to\text{-clause}$$

The wRu may be deleted (by w_{L}) from the leftmost premiss because the only way for the meta-atom to be used in the remainder of the proof is if wRu appears in the context, but this is impossible (by world-independence). Hence, without loss of generality, this branch reduces to $\Sigma_1', \Sigma_2 \vdash \Pi_1, \Pi_2$. Each premiss now has the desired form. □

Using world-independence, we may give a relational calculus RJ^+ characterizing the semantics comprised of basic rules. It arises from analyzing the rôle of the atom xRy in RJ to get rid of it. Essentially, we incorporate it in $\to_{\mathsf{R}}$, which was always its purpose — see Section 17.3.

Definition 17.19 (Relational Calculus RJ^+)**.** *Relational calculus* RJ^+ *is comprised of the rules in Figure 17.7, in which* $,_{\mathsf{L}}$ *and* $\fatsemi_{\mathsf{R}}$ *are invertible.*

Proposition 17.20. $\Gamma \vdash_{\mathsf{RJ}} \Delta$ *iff* $\Gamma \vdash_{\mathsf{RJ}^+} \Delta$

Proof. Every RJ-proof can be simulated in RJ^+ by using (i.e., reduce with) pers eagerly after using $\to_{\mathsf{L}}$. Thus, $\Gamma \vdash_{\mathsf{RJ}} \Delta$ implies $\Gamma \vdash_{\mathsf{RJ}^+} \Delta$. It remains to show that $\Gamma \vdash_{\mathsf{RJ}^+} \Delta$ implies $\Gamma \vdash_{\mathsf{RJ}} \Delta$.

Without loss of generality, in RJ^+ one may always use pers immediately after $\to_{\mathsf{R}}$, as otherwise the use of $\to_{\mathsf{L}}$ could be postponed. Similarly, without loss of generality, $\to_{\mathsf{L}}$ always instantiates the wRu with $u \mapsto w$ — this follows as we require the leftmost branch of the following to close, which it does by reflexivity:

$$\dfrac{\Pi \rhd \Sigma, (wRu) \qquad \Pi \rhd \Sigma, (w : \varphi) \qquad \Pi, (w : \psi) \rhd \Sigma}{\Pi, (w : \varphi \to \psi) \rhd \Sigma}$$

A RJ^+-proof following these principles maps to a RJ-proof simply by collapsing the instances of $\to_{\mathsf{L}}$ and $\to_{\mathsf{R}}$ in the former to capture $\to_{\mathsf{L}}$ and $\to_{\mathsf{R}}$ in the latter. □

$$\frac{\Phi,\Phi,\Pi \rhd \Sigma}{\Phi,\Pi \rhd \Sigma}\,\mathsf{c} \qquad \frac{}{\Phi,\Pi \rhd \Sigma,\Phi}\,\mathsf{ax} \qquad \frac{}{\bot,\Pi \rhd \Sigma}\,\bot \qquad \frac{}{\Pi \rhd \Sigma, xRx}\,\mathsf{ref}$$

$$\frac{(x:\varphi),(x:\psi),\Pi \rhd \Sigma}{(x:\varphi \land \psi),\Pi \rhd \Sigma}\,\land_{\mathsf{L}} \qquad \frac{\Pi \rhd \Sigma,(x:\varphi) \quad \Pi \rhd \Sigma,(x:\psi)}{\Pi \rhd \Sigma,(x:\varphi \land \psi)}\,\land_{\mathsf{R}}$$

$$\frac{(x:\varphi),\Pi \rhd \Sigma \quad (x:\psi),\Pi \rhd \Sigma}{(x:\varphi \lor \psi),\Pi \rhd \Sigma}\,\lor_{\mathsf{L}} \qquad \frac{\Pi \rhd \Sigma,(x:\varphi),(x:\psi)}{\Pi \rhd \Sigma,(x:\varphi \lor \psi)}\,\lor_{\mathsf{R}}$$

$$\frac{\Pi \rhd \Sigma,(x:\varphi) \quad (x:\psi),\Pi \rhd \Sigma}{(x:\varphi \to \psi),\Pi \rhd \Sigma}\,\to_{\mathsf{L}} \qquad \frac{(y:\varphi),\Pi[x \mapsto y] \rhd (x:\psi)}{\Pi \rhd \Sigma,(x:\varphi \to \psi)}\,\to_{\mathsf{R}}$$

$$\frac{\bot,\Pi \rhd \Sigma}{(x:\bot),\Pi \rhd \Sigma}\,\bot_{\mathsf{L}} \qquad \frac{\Pi \rhd \Sigma,\bot}{\Pi \rhd \Sigma,(x:\bot)}\,\bot_{\mathsf{R}}$$

$$\frac{(x:\Gamma),(x:\Gamma'),\Pi \rhd \Sigma}{(x:\Gamma \fatsemi \Gamma'),\Pi \rhd \Sigma}\,\fatsemi_{\mathsf{L}} \qquad \frac{\Pi \rhd \Sigma,(x:\Delta),(x:\Delta')}{\Pi \rhd \Sigma,(x:\Delta \fatsemi \Delta')}\,\fatsemi_{\mathsf{R}}$$

Figure 17.7: Relational Calculus RJ^+

Observe that the propositional encoding of RJ^+ is precisely LJ^+. The connexion to IPL follows immediately:

Corollary 17.21. *System* RJ^+ *is faithful and adequate with respect to* LJ^+.

Proof. Instance of Theorem 16.18. □

Theorem 17.22 (Completeness)**.** *If* $\Gamma \vDash \Delta$*, then* $\Gamma \vdash \Delta$.

Proof. We have the following:

$$\begin{array}{rlll}
\Gamma \vDash \Delta & \text{implies} & (w:\Gamma) \vdash_{\mathsf{RJ}} (w:\Delta) & \text{(Corollary 17.16)} \\
& \text{implies} & (w:\Gamma) \vdash_{\mathsf{RJ}^+} (w:\Delta) & \text{(Proposition 17.20)} \\
& \text{implies} & \Gamma \vdash_{\mathsf{LJ}^+} \Delta & \text{(Corollary 17.21)} \\
& \text{implies} & \Gamma \vdash_{\mathsf{LJ}} \Delta & \text{(Proposition 17.10)}
\end{array}$$

Since LJ characterizes IPL, this completeness the proof. □

Thus, we have derived a semantics of IPL for LJ and proved its soundness and completeness using the constraints systems. This semantics is not quite Kripke's one [124], which insists that R be transitive, thus rendering it a pre-order. This requirement is naturally seen from the connexion to Heyting algebra and the modal logic S4. In the analysis of Section 17.3, from which the semantics in this chapter comes, there was no need for transitivity; the proofs of soundness and completeness go through without it.

One may add transitivity — that is, the meta-formula $\forall x, y, z(xRy \& yRz \implies xRz)$ — to Ω_{IPL} and proceed as above, adding the following rule to RJ:

$$\frac{xRy, yRz, xRz, \Pi \triangleright \Sigma}{xRy, yRz, \Pi \triangleright \Sigma}$$

The proof of completeness passes again through RJ^+ by observing that eagerly using persistence does all the work required of transitivity; that is, according to the eager use of pers, in a sequent $xRy, yRz, \Pi \triangleright \Sigma$, the set Π is of the form $\Pi'[x \mapsto y] \cup \Pi'[y \mapsto z]$, so that whatever information was essential about (the world denoted by) x is already known about (the world denoted by) z by passing through (the world denoted by) y.

Chapter 18

Conclusion to Part II

This part introduced a paradigm of proof system called an *(algebraic) constraint system* (ACSs). They serves as a uniform tool for studying the metatheory of one logic in terms of the metatheory of another. The advantage is that the latter may be simpler or more well-understood in some practical sense. In short, a constraint system is a labelled sequent calculus in which the labels carry an algebraic structure to determine correctness conditions on proof structures. Among other things, they give a uniform setting for the approach to soundness and completeness for BI in Part I.

Chapter 13 provides a motivating example of a class of constraint systems already present in the literature. The *resource-distribution via Boolean constraints* (RDvBC) mechanism by Harland and Pym [99, 98] proceeds by enriching a sequent calculus for a substructural logic — in particular, LL, BI, and a class of relevance logics — with boolean variables that allow one to defer context-management to end of the (reductive) construction of a proof, where it appears a set of Boolean equations that may be passed to a solver. These systems are algebraic constraint systems. While the systems may be used for proof-search, the point is really that they be used to *analyze* the possible context-management strategies during proof-search. The chapter summarizes work by Harland and Pym [99, 98] where such an analysis is provided.

Chapter 14 defines *propositional logic*. This cannot be done without controversy. The chapter provides a simple and intuitive notion that clearly encapsulates all of the typical ‘propositional logics’ in the literature, but is perhaps over-encompassing. The point is that constraint systems may be defined uniformly for this class. The chapter also provides a notion of a model-theoretic semantics for a propositional logic relative to which we may illustrate the use of constraint systems for bridging the gap between proof theory and semantics.

Chapter 15 defines (algebraic) constraint systems for the class of propositional logics in Chapter 14. Importantly, ACSs more naturally fit in Reductive Logic (as opposed to Deductive Logic) since one generates constraints when moving from putative conclusion to sufficient premisses. The chapter uses RDvBC for BI (Chapter 13) as a running example to illustrate the definitions, which shows that the method indeed amount to constructing an ACS.

There are two possible relationships an ACS may have with a logic of interest: soundness and completeness, and faithfulness and adequacy. The former is a *global* correctness condition that says that completed reductions in the constraint system (i.e., constructions to which one cannot apply further reduction operators) whose constraints are coherent (i.e., admit a solution) characterize the consequence of a logic. Meanwhile, the latter is a *local* correctness condition in that each reduction step in the constraint system corresponds to a valid inference for the logic, when its constraints are satisfied; consequently, a completed reduction corresponds to a proof in a sequent calculus for the logic. Both correctness criteria are valuable in applications of constraint systems for studying meta-theory.

Chapter 16 uses ACSs to systematically produce relational calculi for propositional logics satisfying the *tractability* condition. In short, a logic is tractable when it admits a model-theoretic semantics that is first-order definable with a theory that is well-behaved proof-theoretically. The tractability condition is defined explicitly on the structure of the first-order definition of the model-theoretic semantics. The chapter also includes a condition one the relational calculus that automatically yields soundness and completeness of a sequent calculus for the propositional logic with respect to the model-theoretic semantics. While the chapter delivers the analysis uniformly, it is limited in the sense that further analysis is required in order to understand the structural properties of the relational calculi generated — in particular, one desires conditions on top of tractability so that the relational calculi are analytic.

This work generalizes the work on the semantics of BI in Chapter 10. It is not clear how the approach to soundness and completeness in this paper relates to the more traditional approaches, but such investigation may aid in understanding the principles underlying them and is left for future work. These questions are not addressed in this chapter as the point is to use ACSs, which are most naturally regarded reductively, to bridge the gap between proof theory and semantics.

Chapter 17 similarly uses ACS to bridge the gap between proof theory and semantics, but in a stronger sense. While Chapter 10 and Chapter 16 move from semantics to proof theory, this chapter considers the reverse problem: to derive a model-theoretic semantics of a logic from its proof-theoretic specification. It does not provide a general analysis, but rather it considers the problem in detail for IPL. It illustrates how one can analyse the proof theory of IPL using ACSs (i.e., in Reductive Logic) to recover Kripke's [125] semantics. In this way, it illustrate very strongly the relationship between semantics and proof theory in Reductive Logic. A general, uniform, and systematic account is desirable, but is left as future work.

Overall, this part introduces ACS and uses them to study metatheory. Notably, ACSs sit most naturally within Reductive Logic (as opposed to Deductive Logic). They are a powerful tool for investigating the interplay between semantics and proof theory, as witnessed by the systematic generation of relational calculi. What is more, they allow one to derive a semantics for a logic from a proof-theoretic characterization by capturing choices during proof-search in terms of an algebraic structure. In this way, this part provides a tool that may, perhaps, yield a precise characterization of (model-theoretic) semantics in Reductive Logic, in contrast to the exploratory investigations of this monograph. Finally, it illustrates that the semantics of a logical constant is contained in the rule of inference governing it, which is the basis of Part III.

Part III

Proof-theoretic Semantics

Chapter 19

Introduction to Part III

In Part I and Part II, logical consequence is understood according to the plan by Tarski [201, 203]: a propositional formula φ follows from a context Γ iff every model of Γ is a model of φ — that is,

$$\Gamma \vDash \varphi \quad \text{iff} \quad \text{for all models } \mathcal{M}\text{, if } \mathcal{M} \vDash \psi \text{ for all } \psi \in \Gamma\text{, then } \mathcal{M} \vDash \varphi$$

This approach to logical consequence determines *model-theoretic semantics* (M-tS). It is the dominant approach to semantics in logic, especially in terms of applications; for example, in the application of logic in program/systems verification, programs/systems are regarded mathematically as algebraic structures and formulae of the logic express properties about the structures — see, for example, the use of BI to reason about pointer-based programming languages by Ishtiaq and O'Hearn [108]. Of course, the practicalities of this deployment typically proceed through Reductive Logic; that is, model-checking — the principle activity involved — proceeds by calculating the truth value of a formula according to the truth values of its constituent parts determined by unfolding the M-tS.

While M-tS is an intuitive, useful, and powerful approach to semantics in logic, it is not the only one. Instead, one may ask more directly what logic says about *reasoning* (instead of *structure*). Accordingly, Tennant [204] provides an alternative, proof-theoretic (as opposed to model-theoretic), reading of logical consequence: a propositional formula φ follows proof-theoretically from a context Γ iff φ follows by some reasoning from Γ. This demands a notion of *valid argument* that encapsulates what the forms of valid reasoning are. That is, we require explicating the semantic conditions required for an argument that witnesses

$$\psi_1, \ldots, \psi_n; \text{ therefore, } \varphi$$

to be valid. In short, rather than leave *reasoning* as something that occurs at the meta-level and expressed in terms of logic, we may ask what logic says *about* reasoning, and *vice versa*.

This determines the field of *proof-theoretic semantics* (P-tS) [61, 193, 213]. There are two foundational problems:

- to define the validity of an argument in terms of notions of inference determined by a given logic; and
- to explicate the meaning of logical constants in terms of valid arguments.

Importantly, P-tS is not about giving proof systems or interpretations of proofs in a particular proof system, though these are related issues — for example, in Chapter 20, we discuss the BHK interpretation of intuitionistic logic and its relevance to P-tS and Reductive Logic, but BHK is not a P-tS in itself — instead it is about meaning relative to the *notion* of proof.

Observe that in M-tS, a rule in a formal system is justified by showing that it derives only consequences of the logic; that is, it is truth-preserving in all models. Hence, inference follows from validity. Meanwhile, P-tS sits in the semantic paradigm of *inferentialism* in which meaning (or validity) arises *from* rules of inference (see Brandom [26]). Nonetheless, as Schroeder-Heister [191] observes, since no formal system is fixed (only notions of inference), the relationship between semantics and provability remains the same as it has always been with soundness and completeness are desirable features of formal systems. What differs is that *proofs* in P-tS — understood as objects denoting collections of acceptable inferences from accepted premisses — serve the role of truth in M-tS. This is clear in Chapter 20 and Chapter 21.

To illustrate the paradigmatic shift from M-tS to P-tS, consider the proposition 'Tammy is a vixen'. What does it mean? Intuitively, it means, somehow, 'Tammy is female' *and* 'Tammy is a fox'. In inferentialism, its meaning is given by the rules,

$$\frac{\text{Tammy is a fox} \quad \text{Tammy is female}}{\text{Tammy is a vixen}} \qquad \frac{\text{Tammy is a vixen}}{\text{Tammy is female}} \qquad \frac{\text{Tammy is a vixen}}{\text{Tammy is a fox}}$$

These merit comparison with the laws governing $\wedge$ in NJ, which justify the sense in which the above proposition is a conjunction:

$$\frac{\varphi \quad \psi}{\varphi \wedge \psi} \qquad \frac{\varphi \wedge \psi}{\varphi} \qquad \frac{\varphi \wedge \psi}{\psi}$$

Hence, in P-tS, validity is grounded in terms of proof in systems of rules governing atomic propositions — that is, formulae without logical structure — called *atomic systems* or *bases*.

In this monograph, we consider two streams of work within P-tS that align with the problems outlined above:

- Proof-theoretic Validity (P-tV) in the Dummett-Prawitz tradition (see, for example, Schroeder-Heister [190]) is a semantics *of* arguments;

- Base-extension semantics (Be-S) — in the sense of, for example, Sandqvist [184, 182, 183] — is a semantics of logical constants *in terms of* arguments.

The terminology used for these two streams within the literature is somewhat misleading: both concern *validity* and both make use of *base-extension* in doing so. The terminology used to distinguish them here is taken from their associated literature, but is not intended to denote sub-fields of P-tS. We distinguish the branches only to be able to formally relate the two parts of the literature, which are intended to speak of the same subject: meaning in terms of proof.

The mathematical treatment of both P-tV and B-eS has largely concentrated on classical and intuitionistic propositional logic, and a terse account of this background is provided in Chapter 20 and Chapter 21. Presently, we give a brief overview of the major ideas involved in each branch.

Proof-theoretic Valdity (P-tV). In this monograph, we follow the presentation by Schroeder-Heister [190], who distinguished the computational and semantic aspects of the earlier work.

In general, P-tV can be viewed as an attempt to execute the following programmatic remarks by Gentzen [200]:

> The introductions represent, as it were, the 'definitions' of the symbols concerned, and the eliminations are no more, in the final analysis, than the consequences of these definitions. This fact may be expressed as follows: In eliminating a symbol, we may use the formula with whose terminal symbol we are dealing only 'in the sense afforded it by the introduction of that symbol'.

The idea that rules are definitions sets P-tV in inferentialism; the priority of the introduction rules involves Prawitz's [168] normalization theory — see Chapter 20.

A consequence $\Gamma \vdash \varphi$ is read as saying there is a *direct* argument beginning with Γ and concluding φ. An argument is a natural deduction object in the sense of Gentzen [200], a tree of formulas with some marked as discharged. Some arguments can be seen as representing other arguments; for example an NJ-derivation is *indirect* (i.e., not direct) if it contains a *detour* — that is, an elimination rule is used to remove a logical constant that was added by an introduced rule earlier in the derivation — which may be removed through normalisation — for example, the detour

$$\dfrac{\begin{array}{c}\mathcal{D}_1\\ \varphi\end{array} \qquad \dfrac{\begin{array}{c}[\varphi]\\ \mathcal{D}_2\\ \psi\end{array}}{\varphi \to \psi}\,{\to}_{\mathsf{I}}}{\psi}\,{\to}_{\mathsf{E}}$$

can be *reduced* as follows:

$$\begin{array}{c}\mathcal{D}_1\\ \varphi\\ \mathcal{D}_2\\ \psi\end{array}$$

The study of such reductions is the technical background to P-tV, provided by

Prawitz [168]. The idea is that the normal form represents the *suasive content* of the derivation with detours. Therefore: Arguments without assumptions and detours are said to be *canonical proofs* and are inherently valid; the validity of an arbitrary argument is determined by whether or not it represents, according to some fixed operations (e.g., through a reduction in the sense of Prawitz [168]), one of these canonical proofs. An argument containing open assumptions is valid when arbitrary (valid) closures yield valid arguments. We call this condition *cut*:

> if the open assumptions of a valid argument admit valid arguments, then composing these arguments with the original yields an overall valid argument.

The case in which the open assumptions are atomic requires us to consider the validity of arguments relative to systems of rules governing such atoms. Thus, validity is grounded in terms of *atomic systems* (also known as *bases*).

Base-extension Semantics (B-eS). The alternative branch of P-tS is B-eS. In short, one defines a *support* judgement inductively according to the structure of formulas with the base case (i.e., the support of atoms) given by proof in a base (also known as *atomic systems*). This gives rise to consequence as the transmission of support: a propositional formula φ follows proof-theoretically from a context Γ iff a base supporting Γ also supports φ — that is,

$$\Gamma \Vdash \varphi \quad \text{iff} \quad \text{for all bases } \mathscr{B}\text{, if } \Vdash_{\mathscr{B}} \psi \text{ for all } \psi \in \Gamma\text{, then } \Vdash_{\mathscr{B}} \varphi$$

Of course, restricted to just this, B-eS appears to be no more than a redressing of M-tS — see, for example, Goldfarb [85] and Makinson [134]. However, it can differ remarkably according to specific setups motivated by inferentialism. This is where the subject becomes subtle. There are several incompleteness results for (super-)intuitionistic logics — see, for example, Piecha et al. [157, 156, 159], Goldfarb [85], Sandqvist [181, 182, 184, 183], Stafford [197].

Specifically, the problems arise around the semantics of *disjunction* and completeness. Briefly, the treatment of disjunction in the standard Kripke semantics for

IPL — that is, $w \vDash \varphi \vee \psi$ iff $w \vDash \varphi$ or $w \vDash \psi$ — corresponds only weakly to NJ (characterizing IPL) and,if such a clause is taken in the definition of validity in a B-eS for IPL, it leads to incompleteness — see Piecha and Schroeder-Heister [157, 156]. Sandqvist [183] (following ideas by Dummett [51]) showed completeness when the following clause is taken instead:

$$\begin{aligned} \Vdash_{\mathscr{B}} \varphi \vee \psi \quad \text{iff} \quad & \text{for all } \mathscr{C} \supseteq \mathscr{B} \text{ and all atomic } p, \\ & \varphi \Vdash_{\mathscr{C}} p \text{ and } \psi \Vdash_{\mathscr{C}} p \text{ implies } \Vdash_{\mathscr{C}} p \end{aligned}$$

This clause corresponds more closely to the proof theory of IPL — specifically, to the elimination rule in NJ.

It is perhaps not immediately apparent what it has to do with Reductive Logic. Essentially, the connections arise from how construction in Reductive Logic and arguments in P-tV are validated. Recall the constructions-as-realizers-as-arrows correspondence from Reductive Logic by Pym et al. [175] described in Chapter 2. The judgements there have analogues in the context of P-tS: the judgement $\Phi \Rightarrow \Gamma \triangleright \varphi$ corresponds to P-tV, the judgement $[\Gamma] \vdash [\Phi] : \varphi$ corresponds to *satisfaction* (see Chapter 20), and the judgement $[\![\Gamma]\!] \overset{[\![\Phi]\!]}{\rightsquigarrow} [\![\varphi]\!]$ corresponds to B-eS — see Pym et al. [175]. In particular, we move from the realizers perspective, in which the witnessing arguments must be constructed explicitly, to the 'types' perspective, in which the witnessing arguments are observed implicitly as arrows. More precisely, the arrows characterize an inductively defined judgement $W \vDash_{\Theta} (\Phi : \varphi)\Gamma$ in which W is a state of knowledge (i.e., the analogue of a base) and Θ is a set of indeterminates. There are also substantial connexions between P-tS and logic programming — see, for example, Hallnäs and Schroeder-Heister [94, 95] and Chapter 23.

This part begins in Chapter 20 with a presentation P-tV. Following this, Chapter 21 provides a terse but complete account of the B-eS of IPL. Chapter 22 demonstrates that the B-eS for IPL captures the declarative content of a basic version of P-tV presented in Chapter 20. Chapter 23 investigates the B-eS for IPL from the perspective of logic programming (see Chapter 2). The part ends in Chapter 24 with a conclusion and summary of results. In Appendix B, we show how P-tS arises when logic is regarded as the science of inference (as opposed to truth).

Chapter 20

Proof-theoretic Validity

This chapter presents proof-theoretic validity (P-tV) in the Dummett-Prawtiz tradition. As explained in Chapter 19, the foundations of P-tV are both computational (through normalization) and philosophical (through inferentialism). We follow the presentation by Schroeder-Heister [190] in distinguishing the relevant parts.

We are working relative to the background on IPL in Chapter 2. In particular, an *argument* is a rooted tree of formulas in which some (possibly no) leaves are marked as discharged. The leaves of an argument are its *assumptions*, and the root is its *conclusion*. An argument is *open* if it has undischarged assumptions; otherwise, it is *closed*. An argument $\mathcal{A}$ is an argument for a sequent $\Gamma \triangleright \varphi$ iff the open assumptions of $\mathcal{A}$ are a subset of Γ and the conclusion of $\mathcal{A}$ is φ. We use the following notations to express that $\mathcal{A}$ is an argument for $\Gamma \triangleright \varphi$:

$$\begin{matrix} \mathcal{A} \\ \varphi \end{matrix} \qquad \begin{matrix} \Gamma \\ \mathcal{A} \end{matrix} \qquad \begin{matrix} \Gamma \\ \mathcal{A} \\ \varphi \end{matrix}$$

Let $\mathcal{A}$ be an argument with open assumptions $\Gamma = \{\varphi_1, ..., \varphi_n\}$; and let $\mathcal{B}_1$,...,$\mathcal{B}_n$ be arguments with open assumptions $\Gamma_1, ..., \Gamma_n$, respectively, and conclusions φ_1,...,φ_n, respectively. We write $\mathsf{cut}(\mathcal{B}_1, ..., \mathcal{B}_n, \mathcal{A})$ to denote the result of composing $\mathcal{A}$ by $\mathcal{B}_1, ..., \mathcal{B}_n$ at the assumptions; that is,

$$\mathsf{cut}(\mathcal{B}_1, ..., \mathcal{B}_n, \mathcal{A}) := \begin{matrix} \mathcal{B}_1 \quad \mathcal{B}_n \\ \varphi_1 \cdots \varphi_n \\ \mathcal{A} \end{matrix}$$

This is a helpful abbreviation for the composition of arguments.

We briefly outline the chapter. First, in Section 20.1, we define *atomic systems*, which ground P-tV. Second, in Section 20.2, we define a *basic* notion of P-tV that heavily relies on the normalization theory of NJ. Third, in Section 20.3, we define a general notion of P-tV with references to the basic version. Finally, in Section 20.4, we give yet another account of P-tV as a uniform account of the validity of construction relative to the theory of tactical proof (see Chapter 4).

20.1 Atomic Systems

The base case of P-tV is given by *atomic systems*. These supply the meaning, on inferentialism, of atomic propositions analogous to the use of interpretation in M-tS. Piecha and Schroeder-Heister [194, 158] have given an inductive hierarchy of atomic rules and systems, which we include here. In some works on P-tS, $\bot$ is regarded as an atom, but in this monograph, 'atom' means propositional letter (i.e., an element of $\mathbb{A}$).

Definition 20.1 (Atomic Rule)**.** *An nth-level atomic rule is defined as follows:*

- *A zeroth-level atomic rule is a rule of the following form in which* $\mathrm{c} \in \mathbb{A}$*:*

$$\frac{}{\mathrm{c}}$$

- *A first-level atomic rule is a rule of the following form in which* $\mathrm{p}_1, ..., \mathrm{p}_n, \mathrm{c} \in \mathbb{A}$,

$$\frac{\mathrm{p}_1 \quad \cdots \quad \mathrm{p}_n}{\mathrm{c}}$$

- *An* $(n+1)$*th-level atomic rule is a rule of the following form in which* $\mathrm{p}_1, ..., \mathrm{p}_n, \mathrm{c} \in \mathbb{A}$ *and* $\Sigma_1, ..., \Sigma_n$ *are (possibly empty) sets of nth-level atomic rules:*

$$\frac{\begin{matrix}[\Sigma_1] \\ \mathrm{p}_1\end{matrix} \quad \cdots \quad \begin{matrix}[\Sigma_n] \\ \mathrm{p}_n\end{matrix}}{\mathrm{c}}$$

Since the premisses may be empty, an *m*th-level atomic rule is an *n*th-level atomic rule for any $n > m$. We say that a rule is *properly* nth-level iff it is nth-level and at least one of the premisses is a set of $(n-1)$th-level rules which are not $(n-2)$th-level rules. For example, the atomic rule

$$\frac{p \quad \begin{array}{c}[p_1]\\ c\end{array} \quad \begin{array}{c}[p_2]\\ c\end{array}}{c}$$

is both second- and third-level, but only *properly* second-level (i.e., not properly third-level).

Having sets of atomic rules as hypotheses is more general than having sets of atomic propositions as hypotheses; the former captures the latter by taking zeroth-order atomic rules. Significantly, atomic rules are *not* closed under substitution.

Definition 20.2 (Atomic System). *An atomic system is a set of atomic rules.*

Atomic systems may have infinitely many rules (at most, countably infinite). An atomic system $\mathscr{A}$ is *properly* nth-level iff, for any $\mathsf{r} \in \mathscr{A}$, there is $k \leq n$ such that r is properly kth-level.

Piecha and Schroeder-Heister [194, 158] have defined a notion of derivation in an arbitrary atomic system that generalizes Definition 2.10.

Definition 20.3 (Derivation in an Atomic System). *Let $\mathscr{A}$ be an atomic system. The set of $\mathscr{A}$-derivations is defined inductive as follows:*

- BASE CASE. *If $\mathscr{A}$ contains a zeroth-level rule concluding* c*, then the natural deduction argument consisting of just the node* c *is a $\mathscr{A}$-derivation.*

- INDUCTIVE STEP. *Suppose $\mathscr{A}$ contains an $(n+1)$th-level rule* r *of the following form:*

$$\frac{\begin{array}{c}[\Sigma_1]\\ \mathsf{p}_1\end{array} \quad \dots \quad \begin{array}{c}[\Sigma_n]\\ \mathsf{p}_n\end{array}}{\mathsf{c}}$$

And suppose that for each $1 \leq i \leq n$ there is an $\mathscr{A}$-derivation $\mathcal{D}_i$ of the following form:

$$\begin{array}{c}\Gamma_i, \Sigma_i\\ \mathcal{D}_i\\ \mathsf{p}_i\end{array}$$

Then the natural deduction argument with root c *and immediate sub-trees* $\mathcal{D}_1,...,\mathcal{D}_n$ *is an* $\mathscr{A}$*-argument of* c *from* $\Gamma_1 \cup \ldots \cup \Gamma_n \cup \mathscr{A}$*.*

An atom c *is derivable from* Γ *in* $\mathscr{A}$ *— denoted* $\Gamma \vdash_{\mathscr{A}} \mathrm{c}$ *— iff there is a* $\mathscr{A}$*-derivation of* c *from* $\Sigma \cup \mathscr{A}$*.*

Observe that atomic rules are taken *per se* and not closed under substitution. An argument $\mathcal{A}$ is an $\mathsf{NJ} \cup \mathscr{A}$-derivation for an atomic system $\mathscr{A}$, iff it is regulated by the rules of NJ and $\mathcal{A}$ both. In particular, when $\mathscr{A}$ is properly second-level, this amounts to natural deduction in the sense of Gentzen [200] (see also Troelstra and Schwichtenberg [207]).

There is a significant question on what kind of atomic systems should be considered — see, for example, Piecha and Schroeder-Heister [158]. We take that *some* class of atomic systems has been fixed and proceed relative to this class. To distinguish the elements of the classes from the general notion of atomic systems, we call them *bases*.

20.2 Basic Proof-theoretic Validity

There are many versions of P-tV in the literature. One of the original ones is provided by Prawitz [164] but is limited according to the general program of P-tS as it fixed NJ as the calculus of arguments. We call this original work *basic* P-tV. It is defined in the section as it informs the more general version of P-tV in Section 20.3 and is the background to the work in Chapter 22.

The idea behind basic P-tV is the question, what *proof* does a given NJ-derivations represent? Here, ‘proof’ is understood as the underlying suasive content of the NJ-derivation, not merely a closed NJ-derivation. That is, heuristically, basic P-tV may be thought to concern the *equivalence* of arguments (as opposed to the *validity* of arguments).

We provide a brief outline of this section. First, Section 20.2.1 gives the requisite background normalization theory for NJ. Second, Section 20.2.2 defines basic P-tV. Finally, Section 20.2.3 represents basic P-tV in terms of a *satisfaction* relation between arguments, atomic systems, and sequents.

20.2.1 Normalization in NJ

Observe that in NJ (Figure 2.2), there are rules with subscripts I and E. The former are the *introduction rules* (I-rules), and the latter are the *elimination rules* (E-rules). They sometimes come in pairs that, when composed, yield no additional results; for example, the derivation

$$\dfrac{\dfrac{\begin{matrix}\mathcal{D}_1 & \mathcal{D}_2\\ \varphi & \psi\end{matrix}}{\varphi \wedge \psi}\,\wedge_{\mathsf{I}}}{\varphi}\,\wedge_{\mathsf{E}}$$

contains superfluous argumentation for φ as it is already concluded by $\mathcal{D}_1$. This is the foundation of the normalization theory and, therefore, of P-tV.

Definition 20.4 (Detour)**.** *A* detour *in a derivation is a sub-derivation in which a formula is obtained by an I-rule and is then the major premise of the corresponding E-rule.*

Definition 20.5 (Canonical)**.** *A derivation is canonical iff it contains no detours.*

Prawitz [168] proved that canonical NJ-proofs are complete for IPL; that is, we may refine Theorem 2.11— that is, $\vdash_{\mathsf{NJ}}\varphi$ iff $\vdash\varphi$, where $\vdash$ is the consequence judgement for IPL — as follows:

Proposition 20.6 (Prawitz [168])**.** *There is a canonical* NJ*-derivation of* $\varnothing \rhd \varphi$ *iff* $\varnothing \vdash \varphi$.

The proof of this statement uses a reduction relation $\rightsquigarrow$ that precisely eliminates detours; for example, detours with implication ($\to$) are reduced as follows:

$$\dfrac{\begin{matrix}\mathcal{D}_1\\ \varphi\end{matrix} \qquad \dfrac{\begin{matrix}[\varphi]\\ \mathcal{D}_2\\ \psi\end{matrix}}{\varphi \to \psi}\,\to_{\mathsf{I}}}{\psi}\,\to_{\mathsf{E}} \qquad \rightsquigarrow \qquad \begin{matrix}\mathcal{D}_1\\ \varphi\\ \mathcal{D}_2\\ \psi\end{matrix}$$

The reflexive and transitive closure of $\rightsquigarrow$ is denoted $\rightsquigarrow^*$.

Prawitz [168] proved that this reduction relation is *normalizing* and that the *normal forms* contain no detours. If the normal form is closed, it is called *canonical proof*.

Proposition 20.7 (Prawitz [168]). *If $\mathcal{A}$ is an* NJ*-proof for $\varnothing \triangleright \varphi$, then there is canonical* NJ*-proof $\mathcal{A}'$ for $\varnothing \triangleright \varphi$ such that $\mathcal{A} \rightsquigarrow^* \mathcal{A}'$.*

Corollary 20.8 (Prawitz [168]). *There is a canonical* NJ*-proof $\mathcal{A}$ for $\varnothing \triangleright \varphi$ that concludes by the use of an introduction rule iff $\varnothing \vdash \varphi$.*

This establishes the relevant normalization theory for P-tV in this monograph. It is this last corollary that renders the normalization theory relevant for P-tS relative to Gentzen's [200] programmatic remarks quoted in Chapter 19 as it gives particular priority to the introduction rules of NJ.

20.2.2 Basic Proof-theoretic Validity

It remains to define basic P-tV. A discussion of how these ideas arise relative to the computational consideration of normalization in NJ has been provided by Schroeder-Heister [190].

Heuristically, basic P-tV explains that an NJ-derivation is not valid because it is regulated by the rules of NJ, but rather because it represents some *proof* whose validity arises from the *a priori* validity of the introduction rules. These rules *a priori* are valid simply as we regard them, following the remarks by Gentzen [200] in Chapter 19, as *definitions* of the logical constants.

Definition 20.9 (Basic Validity in a Base). *Let $\mathscr{B}$ be a base. An $\mathcal{A}$ is $\mathscr{B}$-valid iff one of the following holds:*

(1) $\mathcal{A}$ is a closed $\mathscr{B}$-derivation

(2) $\mathcal{A}$ is a closed canonical $\mathsf{NJ} \cup \mathscr{B}$-derivation whose immediate sub-derivations $\mathcal{A}_1, \ldots \mathcal{A}_n$ are $\mathscr{B}$-valid

(3) $\mathcal{A}$ is a closed non-canonical $\mathsf{NJ} \cup \mathscr{B}$-derivation that reduces to a $\mathscr{B}$-valid canonical $\mathsf{NJ} \cup \mathscr{B}$-derivation $\mathcal{A}'$

(4) $\mathcal{A}$ is an open derivation and, for every $\mathscr{C} \supseteq \mathscr{B}$, any extension of $\mathcal{A}$ by $\mathscr{C}$-valid arguments of the assumptions $\mathcal{C}_1 \ldots, \mathcal{C}_n$ is a $\mathscr{C}$-valid argument.

Note that reduction here is in the sense of Prawtiz [168] — see Section 20.2.1.

Definition 20.10 (Valid Argument)**.** *An argument $\mathcal{A}$ is valid iff, for every base $\mathscr{B}$, the argument $\mathcal{A}$ is $\mathscr{B}$-valid*

This defines P-tV. How does it relate to IPL ($\vdash$)? It is easy to see that validity is monotonic with respect to bases; that is, an argument $\mathcal{A}$ is $\mathscr{B}$-valid iff, for every $\mathscr{C} \supseteq \mathscr{B}$, $\mathcal{A}$ is $\mathscr{C}$-valid. Consequently, all NJ-derivations are valid. This yields soundness:

$$\text{if } \Gamma \vdash \varphi, \text{ then there is a valid argument for } \Gamma \rhd \varphi.$$

The converse (i.e., completeness) is not currently known. Schroeder-Heister [193] suggests that, under a particular reading, basic P-tV is incomplete for IPL — see also Piecha et al. [157, 156, 159]. In Chapter 22, we argue that such classical readings do, perhaps, not accurately capture basic P-tV. The subtlety is how one moves from a validity condition on derivations to a consequence judgement. An initial step to this end is made in Section 20.2.3.

While basic P-tV does offer a notion of validity of arguments, it is limited in that the derivations it considers are already regulated by NJ-derivations. Therefore, it may instead be seen as explicating the suasive content of a given NJ-derivation. It may then be thought of as providing a notion of equality on NJ-derivations; that is, $\mathcal{A}$ and $\mathcal{B}$ are equivalent iff they represent (i.e., after reduction and substitution of proofs for the assumptions) the same proof. The insistence on NJ comes from the use of reductions, which are all parameterized to give the general notion of P-tV in Section 20.3.

20.2.3 From Derivations to Sequents

We desire to express the declarative content of P-tV in terms of the logical constants of IPL. Equivalently, we desire to explicate the meaning of the logical constants in terms of P-tV. This is the subject of Chapter 22. To bridge the gap between P-tV and B-eS, we introduce the *satisfaction* relation:

Definition 20.11 (Satisfaction in a Base)**.** *The satisfaction judgment $\mathcal{A} : \Gamma \Vdash_{\mathscr{B}} \varphi$ obtains iff $\mathcal{A}$ is a $\mathscr{B}$-valid argument for $\Gamma \rhd \varphi$.*

This section concerns characterizing satisfaction in terms of the logical constants of IPL. In doing so, we shift the emphasis from arguments to the logic. To this end, we perform a case analysis on the defining conditions. Observe that Definition 20.9 has distinct clauses for closed derivations and open derivation, meaning that we distinguish two classes of sequents $\Gamma \triangleright \varphi$, those with $\Gamma = \varnothing$ and with $\Gamma \neq \varnothing$.

If $\Gamma \neq \varnothing$, the condition for validity passes through clause (4) of Definition 20.9. The clause for arguments containing open assumptions may be expressed as follows:

$$\begin{aligned} \mathcal{A} : \Gamma \Vdash_{\mathscr{B}} \varphi \quad \text{iff} \quad & \text{for any } \mathscr{C} \supseteq \mathscr{B}, \text{ if for each } \psi_i \in \Gamma \text{ there is } \mathcal{B}_i \text{ such that} \\ & \mathcal{B}_i \Vdash_{\mathscr{C}} \psi_i, \text{ then } \mathsf{cut}(\mathcal{B}_1, ..., \mathcal{B}_n, \mathcal{A}) : \Gamma_1, ..., \Gamma_n \Vdash_{\mathscr{C}} \varphi \end{aligned}$$

To simplify things, without loss of generality, the $\wedge_\mathsf{I}$, we may replace the $\mathcal{B}_1$,...,$\mathcal{B}_n$ by a single argument:

$$\begin{aligned} \mathcal{A} : \Gamma \Vdash_{\mathscr{B}} \varphi \quad \text{iff} \quad & \text{for any } \mathscr{C} \supseteq \mathscr{B}, \text{ if there is } \mathcal{B} \\ & \text{such that } \mathcal{B} : \varnothing \Vdash_{\mathscr{C}} \Gamma, \text{ then } \mathsf{cut}(\mathcal{B}, \mathcal{A}) \Vdash_{\mathscr{C}} \varphi \end{aligned}$$

This expresses satisfaction for sequents $\Gamma \triangleright \varphi$ with $\Gamma \neq \varnothing$ in terms of satisfaction, as required.

If $\Gamma = \varnothing$, the condition for validity passes through clauses (1), (2), (3) of Definition 20.9. For clause (3), we only need to make sure to normalize at each stage, so it remains only to consider clauses (1) and (2). We proceed by case analysis on the structure of φ.

First, $\varphi = \mathrm{p} \in \mathbb{A}$, we appeal to (1) of Definition 20.9,

$$\mathcal{A} : \varnothing \Vdash_{\mathscr{B}} \mathrm{p} \qquad \text{iff} \qquad \mathcal{A} \text{ is a } \mathscr{B}\text{-proof of } \varnothing \triangleright \mathrm{p}$$

This is sufficiently simple as it moves from the definition of validity in a base to provability in an atomic system (Definition 20.3).

If $\varphi \notin \mathbb{A}$, then it is complex — that is, it contains at least one logical constant. We proceed by case analysis on the structure of φ, using clause (2) of Defini-

$$\begin{array}{lll}
\mathcal{A} : \Gamma \Vdash_{\mathscr{B}} \varphi & \text{iff} & \text{for any } \mathscr{C} \supseteq \mathscr{B}\text{, for any argument } \mathcal{B}\text{,} \\
 & & \text{if } \mathcal{B} : \varnothing \Vdash_{\mathscr{C}} \Gamma\text{, then } \mathtt{cut}(\mathcal{B},\mathcal{A}) : \varnothing \Vdash_{\mathscr{C}} \varphi \\
\mathcal{A} : \varnothing \Vdash_{\mathscr{B}} \mathrm{p} & \text{iff} & \mathcal{A} \text{ is a } \mathscr{B}\text{-proof of } \varnothing : \mathrm{p} \\
\mathcal{A} : \varnothing \Vdash_{\mathscr{B}} \varphi \to \psi & \text{iff} & \mathcal{A} \rightsquigarrow^{*} \bar{\mathcal{A}} \in \to_{\mathsf{I}}(\mathcal{B}) \text{ and } \mathcal{B} : \varphi \Vdash_{\mathscr{B}} \psi \\
\mathcal{A} : \varnothing \Vdash_{\mathscr{B}} \varphi \wedge \psi & \text{iff} & \mathcal{A} \rightsquigarrow^{*} \bar{\mathcal{A}} \in \wedge_{\mathsf{I}}(\mathcal{A}_1,\mathcal{A}_2) \text{ and} \\
 & & \mathcal{A}_1 : \varnothing \Vdash_{\mathscr{B}} \varphi \text{ and } \mathcal{A}_2 : \varnothing \Vdash_{\mathscr{B}} \psi \\
\mathcal{A} : \varnothing \Vdash_{\mathscr{B}} \varphi \vee \psi & \text{iff} & \mathcal{A} \rightsquigarrow^{*} \bar{\mathcal{A}} \in \vee_{\mathsf{I}}(\mathcal{B}) \text{ and either} \\
 & & \mathcal{B} : \varnothing \Vdash_{\mathscr{B}} \varphi \text{ or } \mathcal{B} : \varnothing :\Vdash_{\mathscr{B}} \psi \\
\mathcal{A} : \varnothing \Vdash_{\mathscr{B}} \bot & & \text{never}
\end{array}$$

Figure 20.1: Proof-theoretic Validity as a Satisfaction Relation

tion 20.9.

First, let $\varphi = \bot$. Since bases do not contain falsum ($\bot$) and no rules that produce it, one never witnesses satisfaction of $\bot$. Therefore,

$$\mathcal{A} : \varnothing \Vdash_{\mathscr{B}} \bot \qquad \text{never}$$

Second, let $\varphi = \psi \circ \chi$ for $\circ \in \{\to, \wedge, \vee\}$. By Corollary 20.8, since we have reduced to a canonical argument, if $\mathcal{A} : \varnothing \Vdash_{\mathscr{B}} \varphi$, then $\mathcal{A}$ ends by the use of an introduction rule. By clause (2) of Definition 20.9, the immediate sub-trees of $\mathcal{A}$ are also $\mathscr{B}$-valid arguments. Thus, we have the following clauses:

$$\begin{array}{lll}
\mathcal{A} : \varnothing \Vdash_{\mathscr{B}} \varphi \to \psi & \text{iff} & \mathcal{A} \rightsquigarrow^{*} \bar{\mathcal{A}} \in \to_{\mathsf{I}}(\mathcal{B}) \text{ and } \mathcal{B} : \varphi \Vdash_{\mathscr{B}} \psi \\
\mathcal{A} : \varnothing \Vdash_{\mathscr{B}} \varphi \wedge \psi & \text{iff} & \mathcal{A} \rightsquigarrow^{*} \bar{\mathcal{A}} \in \wedge_{\mathsf{I}}(\mathcal{B}_1,\mathcal{B}_2) \text{ and } \mathcal{B}_1 : \varnothing \Vdash_{\mathscr{B}} \varphi \text{ and } \mathcal{B}_2 : \varnothing \Vdash_{\mathscr{B}} \psi \\
\mathcal{A} : \varnothing \Vdash_{\mathscr{B}} \varphi \vee \psi & \text{iff} & \mathcal{A} \rightsquigarrow^{*} \bar{\mathcal{A}} \in \vee_{\mathsf{I}}(\mathcal{B}) \text{ and either } \mathcal{B} : \varnothing \Vdash_{\mathscr{B}} \varphi \text{ or } \mathcal{B} : \varnothing \Vdash_{\mathscr{B}} \psi
\end{array}$$

This completes the investigation of Definition 20.9.

Proposition 20.12. *Satisfaction in $\mathscr{B}$ satisfies the clauses in Figure 20.1*

A significant feature of satisfaction is its *constructiveness*: when $\mathcal{A} : \Gamma \Vdash_{\mathscr{B}} \varphi$

with $\Gamma \neq \varnothing$, the argument $\mathcal{A}$ acts a function that takes a $\mathscr{C}$-valid argument of the assumption and yields a $\mathscr{C}$-valid argument for the conclusion, for arbitrary $\mathscr{C} \supseteq \mathscr{B}$. This recalls the BHK interpretation of IPL, discussed in Section 2.6, and is an essential observation in Chapter 22.

20.3 The Validity of Arguments

In the rest of this monograph, we are concerned with basic P-tV. However, for completeness, we briefly define a general notion of P-tV to show how the basic notion evolves into something suitable as a semantics of arguments in general (i.e., not just *NJ*-derivations).

In a series of papers, Prawitz [164, 166, 167] developed the notion of *general proof theory* as the study of the *notion* of proof. This stands in contrast to other traditions of proof theory that concern reducing mathematics to systems of rules — see Prawitz [165] for a discussion. It was Dummett [51] who first realized the philosophical significance of this program as giving a *semantics of proofs*.

Following the work in Chapter 20.2, we have the following heuristic is as follows:

- canonical proofs are *a priori* valid
- closed arguments are valid as a consequence of them reducing to canonical proofs
- open arguments are regarded as placeholders for closed arguments according to their possible closures; they are valid according to the ways that the open assumptions may be proved.

Clearly, basic P-tV is a limited implementation of this plan in which arguments are restricted to NJ-derivations and *reduction* is limited to normalization *à la* Prawitz [168] (see Section 20.2.1). However, while we take the introduction rules as *a priori* valid — following the programmatic remarks by Gentzen [200] quoted in Chapter 19 — we need not acquiesce to the elimination rules or their corresponding reductions. Parameterizing over possible choices of reduction recovers a general notion of P-tV that is conceptionally prior to proof systems.

There are many versions of P-tV in the literature. In this monograph, we follow Schroeder-Heister [190], who has provided a succinct definition.

Definition 20.13 (Reduction)**.** *A reduction is a pair $\langle \mathcal{A}_1, \mathcal{A}_2 \rangle$ in which $\mathcal{A}_1$ and $\mathcal{A}_2$ are arguments such that $\mathcal{A}_2$ has the same end formula as $\mathcal{A}_1$ and its open assumptions are a subset of the open assumptions of $\mathcal{A}_1$.*

Schroeder-Heister [190] provides a detailed account of how reduction applies. These details are unimportant here as we will not develop these ideas mathematically. Intuitively, from a set of reductions $\mathcal{J}$, one induces a reducibility relation $\rightsquigarrow_{\mathcal{J}}$ (with reflexive and transitive closure $\rightsquigarrow^*_{\mathcal{J}}$) between arguments by applying reduction from $\mathcal{J}$ to sub-argument. The only constraint for such sets of justifications is that they are closed under substitution in the following sense: if there is $R \in \mathcal{J}$ such that

$$\begin{matrix} \varphi_1 & \dots & \varphi_n \\ & \mathcal{A}_1 & \end{matrix} \qquad R \qquad \begin{matrix} \varphi_1 & \dots & \varphi_n \\ & \mathcal{A}_2 & \end{matrix}$$

then there is $R' \in \mathcal{J}$ such that

$$\begin{matrix} \mathcal{B}_1 & & \mathcal{B}_n \\ \varphi_1 & \dots & \varphi_n \\ & \mathcal{A}_1 & \end{matrix} \qquad R' \qquad \begin{matrix} \mathcal{B}_1 & & \mathcal{B}_n \\ \varphi_1 & \dots & \varphi_n \\ & \mathcal{A}_2 & \end{matrix}$$

This restriction makes sense as it means that justifications behave as a higher-order term rewriting system.

Definition 20.14 (Validity in a Base)**.** *Let $\mathscr{B}$ be a base and $\mathcal{J}$ be a set of reductions. An $\mathcal{A}$ is $\mathscr{B}$-valid with respect to $\mathcal{J}$ iff one of the following holds:*

(1) $\mathcal{A}$ is a closed $\mathscr{B}$-derivation

(2) $\mathcal{A}$ is a closed argument that is $\mathcal{J}$-irreducible and concludes with an introduction rule from NJ *such that its immediate sub-arguments $\mathcal{A}_1, \dots \mathcal{A}_n$ are $\mathscr{B}$-valid with respect to $\mathcal{J}$*

(3) *$\mathcal{A}$ is a closed argument and there is $\mathcal{A}'$ that is $\mathscr{B}$-valid with respect to $\mathcal{J}$ such that $\mathcal{A} \rightsquigarrow^{*}_{\mathcal{J}} \mathcal{A}'$*

(4) *$\mathcal{A}$ is an open derivation and, for every $\mathscr{B}' \supseteq \mathscr{B}$ and $\mathcal{J}' \supseteq \mathcal{J}$, any extension of $\mathcal{A}$ by arguments $\mathcal{B}_1 ...., \mathcal{B}_n$, which are $\mathscr{B}'$-valid with respect to $\mathcal{J}'$, is a $\mathscr{B}'$-valid argument with respect to $\mathcal{J}'$.*

Schroeder-Heister [190] remarks that extensions $\mathcal{B}' \supseteq \mathcal{B}$ and $\mathcal{J}' \supseteq \mathcal{J}$ are intended as monotonicity constraints.

Definition 20.15 (Valid)**.** *An argument $\mathcal{A}$ is valid with respect to a set of reduction $\mathcal{J}$ (closed under substitution) iff, for every base $\mathscr{B}$, the argument $\mathcal{A}$ is $\mathscr{B}$-valid with respect to $\mathcal{J}$.*

The *soundness* of IPL (i.e., the validity of NJ-derivations) is sensitive to the choice of reductions; of course, when $\mathcal{J}$ is the set of standard reductions, all NJ-derivations are valid (with respect to $\mathcal{J}$) — this recovers *basic* P-tV. We may also ask about the *completeness* of IPL. This is known as Prawitz's Conjecture:

Conjecture 20.16 (Prawitz [166, 169])**.** *Let $\mathcal{J}$ be a set of reductions closed under substitution. If there is an argument for $\Gamma \triangleright \varphi$ that is valid with respect to $\mathcal{J}$, then $\Gamma \vdash \varphi$.*

There may be natural restrictions on the notion of $\mathcal{J}$ and atomic systems that render this conjecture true or false; for example, one may sensibly restrict sets of reduction to be those for which IPL is sound. Piecha et al. [157, 156, 159] have shown that the conjecture is false with respect to certain interpretations of P-tV. We discuss these results further in Chapter 22.

20.4 Proof-theoretic Validity and Tactical Proof

In this section, we use P-tS and the theory of tactical proof to give a general account of the relationship between arguments and inference in Reductive Logic. We do not mean to say that this framework is how one *should* go about using logic as a mathematics of reasoning, but instead we aim to *describe* how logic is typically used in the literature.

The essential question in P-tV is, *what is a valid argument?* By framing the relationship between arguments and consequence in the theory of tactical proof, we can define various notions of validity on arguments according to the priority, by fiat, of some sequent calculus characterizing consequence and some transformations of arguments. There is nothing about tactics that pertains, in particular, to natural deduction, so the notion of argument herein can be generalized to other paradigms, too. These generalizations deliver the semantic framework for logic as a reasoning technology that this chapter is about.

In Section 20.4.1, we propose a semantic framework that considers the entire space of reductions, which proceeds through a general account of P-tV. We justify the framework by a correctness property in Section 20.4.2 and through a series of examples from the literature in which it is implicit in Section 20.4.3.

20.4.1 Proof-theoretic Validity, generalized

We begin with a space of *arguments* $\mathbb{A}$. Within this space, there is a subset $\mathbb{P} \subseteq \mathbb{A}$ arguments that are *a priori* valid; these are called *canonical proofs*. These canonical proofs are the basis on which the validity of all the other arguments is derived.

A given argument $\mathcal{A} \in \mathbb{A}$ may represent another argument $\mathcal{A}' \in \mathbb{A}$ in some way. For example, in the setting of natural deduction in the sense of Gentzen [200] (Chapter 2), an argument containing a detour can be thought of as representing a natural deduction argument arising reduction *à la* Prawtiz [168]. Thus, we take the space of arguments to be equipped with some *justification operators* of the form $j : \mathbb{A} \rightharpoonup \mathbb{A}$ that transform one argument to another.

It may be that arguments are left *open* in some sense. The idea is that the argument contains all the suasive content they require but has left something unjustified, which can be filled in arbitrarily by valid arguments. Returning to the case of natural deduction (Section 2), this was the state of open derivations, which have left the justification of their open assumptions unstated (by the very fact of them being open). Therefore, we further equip the space of arguments with *closure operators* of the form $c : \mathbb{A} \rightharpoonup \mathbb{A}$, mapping arguments to arguments.

To summarize:

Definition 20.17 (Argument Space)**.** *An argument space is a tuple* $\mathsf{A} := \langle \mathbb{A}, \mathbb{P}, \mathcal{J}, \mathcal{C} \rangle$ *in which* $\mathbb{A}$ *is the set of arguments,* $\mathbb{P}$ *is the set of proofs,* $\mathcal{J}$ *is the set of justification operators* $j : \mathbb{A} \rightharpoonup \mathbb{A}$*, and* $\mathcal{C}$ *is the set of closure operators* $c : \mathbb{A} \rightharpoonup \mathbb{A}$*.*

A notion of proof-theoretic validity precisely analogous to the treatment of IPL in Section 20 follows immediately:

Definition 20.18 (Proof-theoretic Validity)**.** *Let* $\mathsf{A} := \langle \mathbb{A}, \mathbb{P}, \mathcal{J}, \mathcal{C} \rangle$ *be an argument space. An argument* $\mathcal{A}$ *is* A*-valid iff one of the following holds:*

- *it is a canonical proof —* $\mathcal{A} \in \mathbb{P}$*;*
- *there is* $j \in \mathcal{J}$ *such that* $j(\mathcal{A})$ *is* A*-valid;*
- *for any* $c \in \mathcal{C}$*, the closure* $c(\mathcal{A})$ *is* A*-valid.*

Example 20.19 (Proof-theoretic Validity for IPL)**.** *Consider the arguments space* $\mathsf{N} := \langle \mathbb{A}, \mathbb{D}, \mathbb{P}, \mathcal{J}, \mathcal{C} \rangle$ *in which the components are as follows:*

$\mathbb{A}$ - *Comprises natural deduction arguments*

$\mathbb{P}$ - *Comprises canonical* NJ*-proofs*

$\mathcal{J}$ - *Comprises the reduction transformations by Prawitz [168]*

$\mathcal{C}$ - *Comprises maps that substitute open assumptions for derivation in a base.*

The validity condition from Definition 20.18 instantiated to N *is precisely Definition 14.16.* ■

Of course, the point of the generalization of proof-theoretic validity in this section is that other examples may be captured, too.

Example 20.20 (Proof-search Games)**.** *Consider the game-semantics of proof-search for IPL by Pym and Ritter [174]; see also work by Miller and Saurin [142]. Succinctly, a (partial) strategy is a (partial) function that extends plays — sequences of moves — that end on an opponent move, which satisfy certain conditions. Each strategy represents an attempt at proof-search in* LJ*. A* winning *strategy (i.e., a* *strategy satisfying certain conditions) represents a successful proof-search — that is, proof-search that finds an* LJ*-proof — but it may include backtracking. We have the argument space* $\mathsf{S} = \langle \mathbb{A}, \mathbb{D}, \mathbb{P}, \mathcal{J}, \mathcal{C} \rangle$ *in which the components are as follows:*

$\mathbb{A}$ - *Arguments are partial strategies*

$\mathbb{P}$ - *Canonical proofs are winning strategies without backtracking*

$\mathcal{J}$ - *The justification operators collapse backtracking sections of strategies*

$\mathcal{C}$ - *The closure operators extends partial strategies to total strategies.*

The validity condition from Definition 20.18 instantiated to $\mathcal{L}$ *renders a strategy valid when it represents an* LJ*-proof.* ■

Suppose one has a logic and a notion of argument for that logic; for example, IPL and natural deduction. The setup of argument spaces still needs to allow us to relate the two. As in P-tV for IPL (Chapter 20), we require a function determining what sequents are witnessed by a particular argument in the space.

Definition 20.21 (Ergo)**.** *Let* A *be an argument space with arguments* $\mathbb{A}$*. An ergo is a map from arguments to sequents,* $\mathsf{e} : \mathbb{A} \to \mathbb{S}$.

Definition 20.22 (Logical Argument Space)**.** *A logical argument space (LAS) is a pair* $\mathcal{L} := \langle \mathsf{A}, \mathsf{e} \rangle$ *in which* A *is an argument space and* e *is an ergo.*

Observe that the use of an ergo turns proof-theoretic validity from a *semantics of proofs* into a *semantics in terms of proofs*. A natural deduction argument with open assumptions Γ and conclusion φ has the consequence $\Gamma \triangleright \varphi$; this describes an ergo. The notion of validity of arguments renders a LAS a characterize of *some* logic; namely, the logic whose consequence relation consists of all those sequents admitting valid arguments. Given a LAS $\mathcal{L} = \langle \mathsf{A}, \mathsf{e} \rangle$, we write $\vdash_{\mathcal{L}}$ to denote the consequence relation of the logic it induces — that is,

$$\Gamma \vdash_{\mathcal{L}} \Delta \qquad \text{iff} \qquad \text{there is an } \mathsf{A}\text{-valid argument } \mathcal{A} \text{ such that } \mathsf{e}(\mathcal{A}) = \Gamma \triangleright \Delta$$

Recall that we take logics to be characterized by sequent calculi, generally conceived — see Chapter 14. The relationship between the proof-theoretic semantics and the logic is then captured by standard soundness and completeness conditions:

Definition 20.23 (Soundness and Completeness of Sequent Calculi)**.** *Let $\mathcal{L}$ be a LAS and let L be a sequent calculus over sequents $\mathbb{S}$.*

- *The calculus* L *is* sound *for $\mathcal{L}$ iff, for any sequent $\Gamma \triangleright \Delta \in \mathbb{S}$, if $\Gamma \vdash_{\mathsf{L}} \Delta$, then $\Gamma \vdash_{\mathcal{L}} \Delta$.*

- *The calculus* L *is* complete *for $\mathcal{L}$ iff for any sequent $\Gamma \triangleright \in \mathbb{S}$, if $\vdash_{\mathcal{L}} \Delta$, then $\Gamma \vdash_{\mathsf{L}} \Delta$.*

This relates logic to argument spaces in general. It remains to relate tactics to argument spaces. This addresses the question at the end of Section 4.2, *what argument does a tactic represent?* We have an interpretation from a system of tactics to a space of arguments.

Definition 20.24 (Interpretation)**.** *Call a functions from $\mathbb{A}$ to* $\mathrm{LIST}(\mathbb{A})$ *an* abstract reduction operator *(ARO).*

An interpretation of is a function $[\![-]\!]$ that maps goals to arguments, tactics to AROs, and tacticals to functions from AROs to AROs, such that $\tau : G \mapsto \langle [G_1, ..., G_n], \pi \rangle$, then $[\![\tau]\!]([\![G]\!]) \mapsto [[\![G_1]\!], ..., [\![G_n]\!]]$.

Example 20.25. *Let $\tau_{\wedge}$ and $\tau_{\rightarrow}$ and $⨾$ be as in Section 4.2. To this setup, we add the interpretation $[\![-]\!]$, which answers the questions of what arguments are represented by what tactics.*

The interpretation $[\![-]\!]$ acts on goals (i.e., IPL sequents) $\Gamma \triangleright \varphi$ by mapping them to the natural deduction argument consisting of nodes of formulas from Γ going directly to a node for φ. It maps tactics to their actions on arguments. For example,

$$[\![\tau_{\rightarrow}]\!](\varphi \rightarrow \psi) \mapsto \begin{matrix} \varphi \\ \psi \end{matrix}$$

The tactical $⨾$ interprets the composition of rules, respecting also discharge. Thus, *we have the overall composite action:*

$$[\![\tau_\wedge \fatsemi \tau_\to]\!](\chi \wedge (\varphi \to \psi)) = \dfrac{\chi \qquad \dfrac{\begin{array}{c}[\varphi]\\ \psi\end{array}}{\varphi \to \psi}}{\chi \wedge (\varphi \to \psi)}$$

■

This completes the framework of this chapter. We have arguments, logic, and tactics that are pairwise connected in simple, intuitive ways, faithful to mathematical practices, but presented generally. The following section demonstrates that validity, tactical proof, and consequence are coherent throughout the framework.

20.4.2 Correctness

Thus, We have presented a tripartite logic framework as a mathematics of reasoning: arguments, sequent calculi, and tactics. Above, we defined their relationships. It is summarized in the following diagram:

$$\begin{tikzcd}
& \mathcal{L} \arrow[dl, dotted, leftrightarrow, "{[\![-]\!]}"'] \arrow[dr, dotted, leftrightarrow, "\mathsf{e}"] & \\
\mathsf{T} \arrow[rr, leftrightarrow, "\sigma"'] & & \mathsf{L}
\end{tikzcd}$$

VALIDITY

SOUNDNESS & COMPLETENESS

PROVABILITY

Heuristically, tactics T represent (through an interpretation $[\![-]\!]$) the constructions of arguments $\mathcal{L}$ that assert (through an ergo e) sequents of a logic and that the reasoning steps involved are justified (through a synthesizer σ) by the rules of a sequent calculus L.

By fixing a sequent calculus, we declare a notion of inference for a logic. This notion of inference justifies a system of tactics if there is a synthesizer. The following coherence result captures this:

Proposition 20.26. *Let* L *be a sequent calculus; let* $\mathcal{L} = \langle \mathsf{A}, \mathsf{e} \rangle$ *be a LAS; and let* T *be a tactical system with achievement* α. *Let* $[\![-]\!]$ *be an interpretation of* T *in* A *and let* $\propto$ *satisfy the following coherence condition:*

$$\mathsf{e}[\![G]\!] \propto G$$

Let α be an L*-synthesizer for* T*, then the application of a tactic corresponds precisely to an inference in* L *— that is, if* $[\![\tau]\!] : [\![G]\!] \mapsto [[\![G_1]\!], ..., [\![G_n]\!]]$*, then there is an* L*-rule witnessing the following:*

$$\frac{\mathsf{e}[\![G_1]\!] \quad \ldots \quad \mathsf{e}[\![G_n]\!]}{\mathsf{e}[\![G]\!]}\ \pi$$

Proof. By Definition 20.24, if $\tau : G \mapsto \langle [G_1, ..., G_n], \pi \rangle$, then $[\![\tau]\!] : [\![G]\!] \mapsto [[\![G_1]\!], ..., [\![G_n]\!]]$. By the coherence condition: $\pi : [\mathsf{e}[\![G_1]\!], ..., \mathsf{e}[\![G_n]\!]] \mapsto \mathsf{e}[\![G]\!]$. Since α is a synthesizer, the result follows from Definition 4.4. □

Corollary 20.27. *Calculus* L *is sound for* $\mathcal{L}$*.*

Proof. Proposition 20.26 states that every rule in L is admissible for the logic induced by $\mathcal{A}$, which is the soundness condition in Definition 20.23. □

In this way, a sequent calculus characterizes inference, and a tactical system characterizes the construction of arguments. This means the notion of inference for a logic can be as rough or refined as one desires. For example, one may take the trivial sequent calculus for a consequence relation, which has the consequence of the logic as axioms, but then one admits no tactics. Notably, one has no way of constructing arguments. Though permissible, this situation is quite degenerate. Instead, one may use some notion of argument to inform what inferences are to be permitted.

We have thus shown that the tripartite framework captured by tactical proof and proof-theoretic semantics is coherent in the sense that arguments, sequent calculi, and tactics have the expected relationship. This framework does not arise from doxastic considerations of what these things should be but rather from how they are used in practice. So far, we have been led by a heuristic account and now justify it with a series of examples drawn from the literature on logic.

20.4.3 Examples of the Framework

We provide a brief survey of how various proof-search activities in the literature are instances of the framework in this chapter. This survey is far from complete and left at a quite high level as it only illustrates the descriptive power of the framework we have presented — namely, the relationship between proof-theoretic validity of arguments and inference as witnessed through the Reductive Logic carried by tactical proof. Of course, in addition, there are also the examples of natural deduction (Example 20.19) dialogue games (Example 20.20) in Section 20.4.1.

Example 20.28 (Focused Systems)**.** *The problem of proof-search is handling the various choices involved, such as* inter alia*, the choice of a rule to use and the choice of an instance of that rule. This problem motivates the concept of* focused *proof-search, introduced by Andreoli [6], where these things are largely determined. We review a typical approach for studying focusing — see, for example, Chaudhuri [35, 34] and Chapter:9.*

One begins a sequent calculus L *for which one wishes to establish the focusing property (i.e., that the class of focused proofs is complete for the logic). One introduces an augmented version* FL*, called the focused system, which arises from enriching the original calculus with control structures and introducing* cut*. In the framework of this chapter, we can describe the situation as follows: one has a system of tactics* T *that is validated by* L *(i.e., one has a synthesizer from* T *to* L*) such that a tactic is interpreted as an* FL*-proof. The space of arguments contains all* FL*-proofs, and the justification operators are given by cut-reduction. This is set up so that the canonical proofs represent* focused L*-proofs.* ■

Example 20.29 (Hyper-sequent Calculi)**.** *Reasoning in substructural and modal logics is often difficult because they seemingly do not admit analytic sequent calculi that do not have extra-logical structures (e.g., labels). Many such logics do admit* hyper-*sequent calculi; that is, calculi over finite multisets of sequents — see, for example, Baaz et al. [11] and Ciabattoni et al. [37]. We can use the framework of this chapter to describe the relationship of hyper-sequent calculi to the logic.*

One has a system of tactics T *of goals that are hyper-sequents such that the tactics are interpreted as reductions in the hyper-sequent calculus. These tactics are valid relative to a notion of achievement defined as follows: a consequence of the logic achieves a hyper-sequent iff it is among the sequents in the multiset.* ■

Example 20.30 (Analytic Tableaux)**.** *Analytic tableaux give a computationally useful paradigm of proof in logic. It has been extensively treated for modal logic (see, for example, Fitting and Mendelsohn [60]) and has been used to provide a uniform and modular proof theory for the family of bunched logics (see Docherty and Pym [47, 44]). Typically, these systems make use of prefixed signed formulas. The framework of this chapter can be used to describe the relationship between a tableaux system and a logic.*

One has a system of tactics T *whose goals are prefix signed formulas and whose tactics represent expansion rules for the system, and tacticals are sequential composition. These tactics are interpreted in the space of arguments containing the tableaux, in which the canonical proofs are closed tableaux, possibly satisfying a particular expansion scheme. The tactical system is valid relative to a notion of inference supplied by a* relational *sequent calculus in the form of Negri [147] and Chapter 16.* ■

Chapter 21

Base-extension semantics for Intuitionistic Propositional Logic

This chapter presents the base-extension semantics (B-eS) for IPL given by Sandqvist [183]. It summarizes, and generalizes in a minor way, the work in the following paper:

> Sandqvist, T. Base-extension Semantics for Intuitionistic Sentential Logic. *Logic Journal of the IGPL 23*, 5 (2015), 719–731

In Section 21.1, we give a terse but complete definition of the B-eS for IPL. In Section 21.2, we summarize the completeness proof. Finally, in Section 21.3, we discuss a modification of the treatment of conjunction that helps explain how and why the semantics works. This is subtle, as witnessed by a range of incompleteness results — see, for example, Piecha et al. [157, 156, 159], Goldfarb [85], Sandqvist [181, 182, 184, 183], Stafford [197].

Throughout this section, we fix a denumerable set of atomic propositions $\mathbb{A}$, and the following conventions: $\mathrm{p}, \mathrm{q}, \ldots$ denote atoms; $\mathrm{P}, \mathrm{Q}, \ldots$ denote finite sets of atoms; $\varphi, \psi, \theta, \ldots$ denote formulas; $\Gamma, \Delta, \ldots$ denote finite sets of formulas.

21.1 Support in a Base

The B-eS for IPL given by Sandqvist [183] only admits *properly* second-level atomic systems — that is, bases are composed of atomic rules of the form

$$\frac{}{\mathrm{c}} \qquad \frac{\begin{matrix}[\Sigma_1] & & [\Sigma_n] \\ \mathrm{p}_1 & \dots & \mathrm{p}_n\end{matrix}}{\mathrm{c}}$$

in which $\Sigma_1,...,\Sigma_n$ are possibly empty sets of atoms. They may be expressed inline as $(\mathrm{Q}_1 \triangleright \mathrm{q}_1, \dots, \mathrm{Q}_n \triangleright \mathrm{q}_n) \Rightarrow \mathrm{q}$ — note, the axiom case is the special case when the left-hand side is empty, $\Rightarrow \mathrm{q}$. Intuitively, $\Rightarrow q$ means that the atom q may be concluded whenever, while $(\mathrm{Q}_1 \triangleright \mathrm{q}_1, \dots, \mathrm{Q}_n \triangleright \mathrm{q}_n) \Rightarrow \mathrm{q}$ means that one may derive q from a set of atoms S if one has derived q_i from S assuming Q_i for $i = 1,...,n$. While this has already been treated formally in Chapter 20, we shall give an explicit treatment of the same things here in the restricted setting of second-level systems.

Definition 21.1 (Base)**.** *A* base *is a set of (properly second-level) atomic rules.*

We write $\mathscr{B},\mathscr{C},\dots$ to denote bases. We say $\mathscr{C}$ is an *extension* of $\mathscr{B}$ if $\mathscr{C}$ is a superset of $\mathscr{B}$, denoted $\mathscr{C} \supseteq \mathscr{B}$.

Definition 21.2 (Derivability in a Base)**.** Derivability in a base *$\mathscr{B}$ is the least relation $\vdash_{\mathscr{B}}$ satisfying the following:*

Ref $\mathrm{S},\mathrm{q} \vdash_{\mathscr{B}} \mathrm{q}$.

App *If atomic rule* $(\mathrm{Q}_1 \triangleright \mathrm{q}_1, \dots, \mathrm{Q}_n \triangleright \mathrm{q}_n) \Rightarrow \mathrm{q}$ *is in $\mathscr{B}$, and* $\mathrm{S},\mathrm{Q}_i \vdash_{\mathscr{B}} \mathrm{q}_i$ *for all $i = 1,\dots,n$, then* $\mathrm{S} \vdash_{\mathscr{B}} \mathrm{q}$.

This forms the base case of the B-eS for IPL:

Definition 21.3 (Sandqvist's Support in a Base)**.** Sandqvist's support in a base *$\mathscr{B}$ is the least relation $\Vdash_{\mathscr{B}}$ defined by the clauses of Figure 21.1. A sequent $\Gamma \triangleright \varphi$ is* valid *— denoted $\Gamma \Vdash \varphi$ — iff it is supported in every base,*

$$\Gamma \Vdash \varphi \qquad \textit{iff} \qquad \Gamma \Vdash_{\mathscr{B}} \varphi \textit{ for any } \mathscr{B}$$

$$
\begin{array}{llll}
\text{(At)} & \Vdash_{\mathscr{B}} \mathrm{p} & \text{iff} & \vdash_{\mathscr{B}} \mathrm{p} \\
(\to) & \Vdash_{\mathscr{B}} \varphi \to \psi & \text{iff} & \varphi \Vdash_{\mathscr{B}} \psi \\
(\wedge) & \Vdash_{\mathscr{B}} \varphi \wedge \psi & \text{iff} & \Vdash_{\mathscr{B}} \varphi \text{ and } \Vdash_{\mathscr{B}} \psi \\
(\vee) & \Vdash_{\mathscr{B}} \varphi \vee \psi & \text{iff} & \text{for any } \mathscr{C} \text{ such that } \mathscr{B} \subseteq \mathscr{C} \text{ and any } \mathrm{p} \in \mathbb{A}, \\
 & & & \text{ if } \varphi \Vdash_{\mathscr{C}} \mathrm{p} \text{ and } \psi \Vdash_{\mathscr{C}} \mathrm{p}, \text{ then } \Vdash_{\mathscr{C}} \mathrm{p} \\
(\bot) & \Vdash_{\mathscr{B}} \bot & \text{iff} & \Vdash_{\mathscr{B}} \mathrm{p} \text{ for any } \mathrm{p} \in \mathbb{A} \\
\text{(Inf)} & \Gamma \Vdash_{\mathscr{B}} \varphi & \text{iff} & \text{for any } \mathscr{C} \text{ such that } \mathscr{B} \subseteq \mathscr{C}, \\
 & & & \text{if } \Vdash_{\mathscr{C}} \gamma \text{ for any } \gamma \in \Gamma, \text{ then } \Vdash_{\mathscr{C}} \varphi
\end{array}
$$

Figure 21.1: Sandqvist's Support in a Base

Every base is an extension of the empty base ($\varnothing$), therefore $\Gamma \Vdash \varphi$ iff $\Gamma \Vdash_{\varnothing} \varphi$. Sandqvist [183] showed that this semantics characterizes IPL:

Theorem 21.4 (Sandqvist [183])**.** $\Gamma \vdash \varphi$ *iff* $\Gamma \Vdash \varphi$

We require a small generalization that follows exactly the same proof but with an appropriate parameter tracked throughout:

Theorem 21.5. $\Gamma \vdash_{\mathsf{NJ} \cup \mathscr{B}} \varphi$ *iff* $\Gamma \Vdash_{\mathscr{B}} \varphi$

Soundness — that is, $\Gamma \vdash \mathsf{NJ} \cup \mathscr{B} \varphi$ implies $\Gamma \Vdash_{\mathscr{B}} \varphi$ — follows from showing that $\Vdash_{\mathsf{NJ} \cup \mathscr{B}}$ respects the rules of $\mathsf{NJ} \cup \mathscr{B}$ in the traditional way; for example, $\Gamma \Vdash_{\mathscr{B}} \varphi$ and $\Delta \Vdash_{\mathscr{B}} \psi$ implies $\Gamma, \Delta \Vdash_{\mathscr{B}} \varphi \wedge \psi$. Completeness — that is, $\Gamma \Vdash_{\mathscr{B}} \varphi$ implies $\Gamma \vdash_{\mathsf{NJ} \cup \mathscr{B}} \varphi$ — is more subtle. We present the argument in Section 21.2.

21.2 Completeness of IPL

We require to show that $\Gamma \Vdash_{\mathscr{B}} \varphi$ implies that there is an $\mathsf{NJ} \cup \mathscr{B}$-proof witnessing $\Gamma \vdash_{\mathsf{NJ} \cup \mathscr{B}} \varphi$. To this end, we associate to each sub-formula ρ of Γ or φ a unique atom r, and construct a base $\mathscr{N}$ such that r behaves in $\mathscr{N}$ as ρ behaves in NJ. Moreover, formulas and their atomizations are semantically equivalent in any extension of $\mathscr{N}$ so that support in $\mathscr{N}$ characterizes both validity and provability. When $\rho \in \mathbb{A}$, we take $\mathrm{r} := \rho$, but for complex ρ we choose r to be alien to Γ and φ.

$$\frac{\rho^\flat \quad \sigma^\flat}{(\rho\wedge\sigma)^\flat}\wedge_{\mathsf{I}}{}^{\flat} \qquad \frac{(\rho\wedge\sigma)^\flat}{\rho^\flat}\wedge_{\mathsf{E}}{}^{\flat} \quad \frac{(\rho\wedge\sigma)^\flat}{\sigma^\flat}\wedge_{\mathsf{E}}{}^{\flat} \quad \frac{\rho^\flat \quad (\rho\to\sigma)^\flat}{\sigma^\flat}\to_{\mathsf{E}}{}^{\flat}$$

$$\frac{\rho^\flat}{(\rho\vee\sigma)^\flat}\vee_{\mathsf{I}}{}^{\flat} \quad \frac{\sigma^\flat}{(\rho\vee\sigma)^\flat}\vee_{\mathsf{I}}{}^{\flat} \quad \frac{(\rho\vee\sigma)^\flat \quad \begin{array}{c}[\rho^\flat]\\ \mathrm{p}\end{array} \quad \begin{array}{c}[\sigma^\flat]\\ \mathrm{p}\end{array}}{\mathrm{p}}\vee_{\mathsf{E}}{}^{\flat} \quad \frac{\begin{array}{c}[\rho^\flat]\\ \sigma^\flat\end{array}}{(\rho\to\sigma)^\flat}\to_{\mathsf{I}}{}^{\flat} \quad \frac{\bot^\flat}{\mathrm{p}}\ \mathsf{EFQ}^\flat$$

Figure 21.2: Atomic System $\mathscr{N}$

Example 21.6. *Suppose* $\rho := \mathrm{p}\wedge\mathrm{q}$ *is a sub-formula of* Γ *or* φ*. Associate to it a fresh atom* r*. Since the principal connective of* ρ *is* $\wedge$*, we require* $\mathscr{N}$ *to contain the following rules:*

$$\frac{\mathrm{p} \quad \mathrm{q}}{\mathrm{r}} \qquad \frac{\mathrm{r}}{\mathrm{p}} \quad \frac{\mathrm{r}}{\mathrm{q}}$$

We may write $(\mathrm{p}\wedge\mathrm{q})^\flat$ *for* r *so that these rules may be expressed as follows:*

$$\frac{\mathrm{p} \quad \mathrm{q}}{(\mathrm{p}\wedge\mathrm{q})^\flat} \qquad \frac{(\mathrm{p}\wedge\mathrm{q})^\flat}{\mathrm{p}} \quad \frac{(\mathrm{p}\wedge\mathrm{q})^\flat}{\mathrm{q}}$$

■

Formally, given a judgement $\Gamma \Vdash \varphi$, to every sub-formula ρ associate a unique atomic proposition $\rho^\flat$ as follows:

- if $\rho \notin \mathbb{A}$, then $\rho^\flat$ is an atom that does not occur in Γ or φ or $\mathscr{B}$;

- if $\rho \in \mathbb{A}$, then $\rho^\flat = \rho$.

By *unique* we mean that $(\cdot)^\flat$ is injective — that is, if $\rho \neq \sigma$, then $\rho^\flat \neq \sigma^\flat$. The left-inverse of $(\cdot)^\flat$ is $(\cdot)^\natural$, and the domain may be extended to the entirety of $\mathbb{A}$ by identity on atoms not in the codomain of $(\cdot)^\flat$. Both functions act on sets point-wise — that is, $\Sigma^\flat := \{\varphi^\flat \mid \varphi \in \Sigma\}$ and $\mathrm{P}^\natural := \{\mathrm{p}^\natural \mid \mathrm{p} \in \mathrm{P}\}$. Relative to $(\cdot)^\flat$, let $\mathscr{N}$ be the base containing the rules of Figure 21.2 for any sub-formulas ρ and σ of Γ and φ, and any $\mathrm{p} \in \mathbb{A}$.

Sandqvist [183] establishes three claims that deliver completeness and may be suitably generalized to the version of the theorem we desire to prove:

- **AtComp.** Let $\mathrm{S} \subseteq \mathbb{A}$ and $\mathrm{p} \in \mathbb{A}$ and let $\mathscr{B}$ be a base: $\mathrm{S} \Vdash_{\mathscr{B}} \mathrm{p}$ iff $\mathrm{S} \vdash_{\mathscr{B}} \mathrm{p}$.

- **Flat.** For any sub-formula ξ of Γ or φ and $\mathscr{N}' \supseteq \mathscr{N}$: $\Vdash_{\mathscr{N}'} \xi^\flat$ iff $\Vdash_{\mathscr{N}'} \xi$.

- **Nat.** Let $\mathrm{S} \subseteq \mathbb{A}$ and $\mathrm{p} \in \mathbb{A}$: if $\mathrm{S} \vdash_{\mathscr{N} \cup \mathscr{B}} p$, then $\mathrm{S}^\natural \vdash \mathscr{N} \cup \mathscr{B} p^\natural$.

The first claim is completeness in the atomic case. The second claim is that $\xi^\flat$ and ξ are equivalent in $\mathscr{N}$ — that is, $\xi^\flat \Vdash_{\mathscr{N}} \xi$ and $\xi \Vdash_{\mathscr{N}} \xi^\flat$. Consequently,

$$\Gamma^\flat \Vdash_{\mathscr{N}'} \varphi^\flat \qquad \text{iff} \qquad \Gamma \Vdash_{\mathscr{N}'} \varphi$$

The third claim is the simulation statement which allows us to make the final move from derivability in $\mathscr{N}$ to derivability in NJ.

Theorem 21.5 — Completeness. Assume $\Gamma \Vdash_{\mathscr{B}} \varphi$ and let $\mathscr{N}$ be its bespoke base. By **Flat**, $\Gamma^\flat \Vdash_{\mathscr{N} \cup \mathscr{B}} \varphi^\flat$. Hence, by **AtComp**, $\Gamma^\flat \vdash_{\mathscr{N} \cup \mathscr{B}} \varphi^\flat$. Whence, by **Nat**, $(\Gamma^\flat)^\natural \vdash_{\mathsf{NJ} \cup \mathscr{B}} (\varphi^\flat)^\natural$ — that is, $\Gamma \vdash_{\mathsf{NJ} \cup \mathscr{B}} \varphi$, as required. □

21.3 Support in a Base, revisited

One can, of course, simply mimic model-theoretic semantics for IPL using P-tS structures — see, for example, Goldfarb [85], and Stafford and Nascimento [198]. However, what is interesting about the B-eS in Sandqvist [183] is the way in which it is *not* a representation of the possible world semantics. This is most clearly seen in $(\vee)$, which takes the form of the 'second-order' definition of disjunction — that is, $U + V = \forall X\,((U \to X) \to (V \to X) \to X)$ (see Prawitz [168]).

Proof-theoretically, the clause recalls the elimination rule for the connective restricted to atomic conclusions,

$$\frac{\varphi \vee \psi \qquad \begin{array}{c}[\varphi] \\ \mathrm{p}\end{array} \qquad \begin{array}{c}[\psi] \\ \mathrm{p}\end{array}}{\mathrm{p}}$$

This is adumbrated by Dummett [51], but is, perhaps, surprising since the restriction to atoms is not possible within NJ (e.g., one cannot prove the commutativity of disjunction). Indeed, *all* of the clauses in Figure 21.1 may be regarded as taking the form of the corresponding elimination rules restricted in this way. There is a strong link between the structures here and those investigated extensively by Ferreira et al. [55, 56, 57, 58]

One justification for the clauses is the principle of *definitional reflection* (DR) (see Hallnäs [92, 93] and Schroeder-Heister [188]):

> Whatever follows from all the defining conditions of an assertion also follows from the assertion itself.

Taking the perspective that the introduction rules are definitions, DR provides an answer for the way in which the elimination rules follow. Similarly, it justifies that the clauses for the logical constants take the form of their elimination rules. Note, we here take the definitional view of the introduction rules for the logical constants of IPL, and not of bases themselves, thus do not contradict the distinctions made by Piecha and Schroeder-Heister [194, 158].

Following this observation, an alternative candidate clause for conjunction is as follows:

$$(\wedge^*) \quad \Vdash_{\mathscr{B}} \varphi \wedge \psi \quad \text{iff} \quad \text{for any } \mathscr{C} \supseteq B \text{ and any } \mathrm{p} \in \mathbb{A}, \text{ if } \varphi, \psi \Vdash_{\mathscr{C}} \mathrm{p}, \text{ then } \Vdash_{\mathscr{C}} \mathrm{p}$$

Definition 21.7. *The relation* $\Vdash_{\mathscr{B}}$ *is defined by the clauses of Figure 21.1 with* $(\wedge^*)$ *in place of* $(\wedge)$*. The judgement* $\Gamma \Vdash \varphi$ *obtains iff* $\Gamma \Vdash_{\mathscr{B}} \varphi$ *for every* $\mathscr{B}$*.*

The resulting semantics is sound and complete for IPL:

Theorem 21.8. $\Gamma \Vdash \varphi$ *iff* $\Gamma \vdash \varphi$.

Proof. We require to results which are proved at the end of this chapter. For arbitrary base $\mathscr{B}$, and formulas φ, ψ, χ,

- **Monotone*****.** If $\vDash_{\mathscr{B}} \varphi$, then $\vDash_{\mathscr{C}} \varphi$ for any $\mathscr{C} \supseteq \mathscr{B}$.

- **AndCut.** If $\vDash_{\mathscr{B}} \varphi \wedge \psi$ and $\varphi, \psi \vDash_{\mathscr{B}} \chi$, then $\vDash_{\mathscr{B}} \chi$.

The first claim follows easily from (Inf). The second is a generalization of $(\wedge^*)$; it follows by induction on the structure of χ — an analogous treatment of disjunction was given by Sandqvist [183].

By Theorem 21.5, it suffices to show that $\Gamma \vDash \varphi$ iff $\Gamma \Vdash \varphi$. For this it suffices to show $\vDash_{\mathscr{B}} \theta$ iff $\Vdash_{\mathscr{B}} \theta$ for arbitrary $\mathscr{B}$ and θ. We proceed by induction on the structure of θ. Since the two relations are defined identically except in the case when the θ is a conjunction, we restrict attention to this case.

First, we show $\Vdash_{\mathscr{B}} \theta_1 \wedge \theta_2$ implies $\vDash_{\mathscr{B}} \theta_1 \wedge \theta_2$. By $(\wedge^*)$, the conclusion is equivalent to the following: for any $\mathscr{C} \supseteq \mathscr{B}$ and $\mathrm{p} \in \mathbb{A}$, if $\theta_1, \theta_2 \vDash_{\mathscr{C}} \mathrm{p}$, then $\vDash_{\mathscr{C}} \mathrm{p}$. Therefore, fix $\mathscr{C} \supseteq \mathscr{B}$ and $\mathrm{p} \in \mathbb{A}$ such that $\theta_1, \theta_2 \vDash_{\mathscr{C}} \mathrm{p}$. By (Inf), this entails the following: if $\vDash_{\mathscr{C}} \theta_1$ and $\vDash_{\mathscr{C}} \theta_2$, then $\vDash_{\mathscr{C}} \mathrm{p}$. By $(\wedge)$ on the assumption (i.e., $\Vdash_{\mathscr{B}} \theta_1 \wedge \theta_2$), we obtain $\Vdash_{\mathscr{B}} \theta_1$ and $\Vdash_{\mathscr{B}} \theta_2$. Hence, by the induction hypothesis (IH), $\vDash_{\mathscr{B}} \theta_1$ and $\vDash_{\mathscr{B}} \theta_2$. Whence, by **Monotone***, $\vDash_{\mathscr{C}} \theta_1$ and $\vDash_{\mathscr{C}} \theta_2$. Therefore, $\vDash_{\mathscr{C}} \mathrm{p}$. We have thus shown $\vDash_{\mathscr{B}} \theta_1 \wedge \theta_2$, as required.

Second, we show $\vDash_{\mathscr{B}} \theta_1 \wedge \theta_2$ implies $\Vdash_{\mathscr{B}} \theta_1 \wedge \theta_2$. It is easy to see that $\theta_1, \theta_2 \vDash_{\mathscr{B}} \theta_i$ obtains for $i = 1, 2$. Applying **AndCut** (setting $\varphi = \theta_1$, $\psi = \theta_2$) once with $\chi = \theta_1$ and once with $\chi = \theta_2$ yields $\vDash_{\mathscr{B}} \theta_1$ and $\vDash_{\mathscr{B}} \theta_2$. By the IH, $\Vdash_{\mathscr{B}} \theta_1$ and $\Vdash_{\mathscr{B}} \theta_2$. Hence, $\Vdash_{\mathscr{B}} \theta_1 \wedge \theta_2$, as required. □

A curious feature of the new semantics is that the meaning of the context-former (i.e., the comma) is no longer interpreted as $\wedge$; that is, we define the context-former as follows:

$$\vDash_{\mathscr{B}} \Gamma, \Delta \qquad \text{iff} \qquad \vDash_{\mathscr{B}} \Gamma \text{ and } \vDash_{\mathscr{B}} \Delta$$

This differs from the definition of $\wedge$ in the new semantics. Nonetheless, as shown in the proof of Theorem 21.8, they are equivalent at every base — that is, $\vDash_{\mathscr{B}} \varphi, \psi$ iff $\vDash_{\mathscr{B}} \varphi \wedge \psi$ for any $\mathscr{B}$.

Having defined the context-former, we may express (Inf) as follows:

$$\Gamma \vDash_{\mathscr{B}} \varphi \qquad \text{iff} \qquad \text{for any } \mathscr{C} \supseteq \mathscr{B}\text{, if } \vDash_{\mathscr{C}} \Gamma\text{, then } \vDash_{\mathscr{C}} \varphi$$

This illustrates that support in a base of a sequent is the transmission of the support of the context in a base to support of the formula in a bigger base.

This equivalence of the two semantics yields the following:

Corollary 21.9. *For arbitrary base $\mathscr{B}$ and formula φ, $\Vdash_{\mathscr{B}} \varphi$ iff, for every $\mathscr{X} \supseteq B$ and every atom* p, *if $\varphi \Vdash_{\mathscr{X}}$* p, *then $\Vdash_{\mathscr{X}}$* p.

Proof. Let $\top$ be any formula such that $\Vdash \top$ — for example, $\top := \mathrm{p} \wedge (\mathrm{p} \to \mathrm{q}) \to \mathrm{q}$.

We apply the two equivalent definitions of $\wedge$ to the neutrality of $\top$.

$$\begin{aligned}
\Vdash_{\mathscr{B}} \varphi \text{ iff } & \Vdash_{\mathscr{B}} \varphi \text{ and } \Vdash_{\mathscr{B}} \top && \text{(def. of } \top)\\
\text{iff } & \Vdash_{\mathscr{B}} \varphi \wedge \top && (\wedge)\\
\text{iff } & \text{for any } \mathscr{X} \supseteq B,\ \text{for any } \mathrm{p} \in \mathbb{A}, \varphi, \top \Vdash_{\mathscr{X}} \mathrm{p} \text{ implies } \Vdash_{\mathscr{B}} \mathrm{p} && (\wedge^{*})\\
\text{iff } & \text{for any } \mathscr{X} \supseteq B,\ \text{for any } \mathrm{p} \in \mathbb{A}, \varphi \Vdash_{\mathscr{X}} \mathrm{p} \text{ implies } \Vdash_{\mathscr{B}} \mathrm{p} && \text{(def. of } \top)
\end{aligned}$$

This establishes the desired equivalence. □

The significance of this result is that we see that formulas in the B-eS are precisely characterized by their support of atoms.

It remains to prove the claims **Monotone*** and **AndCut** in the proof of Theorem 21.8:

Lemma 21.10 (Monotone*). *If $\Gamma \vDash_{\mathscr{B}} \varphi$, then $\Gamma \vDash_{\mathscr{C}} \varphi$ for any $\mathscr{C} \supseteq \mathscr{B}$.*

Proof. By (Inf), the conclusion $\Gamma \vDash_{\mathscr{C}} \varphi$ means: for every $\mathscr{D} \supseteq C$, if $\vDash_{\mathscr{D}} \gamma$ for every $\gamma \in \Gamma$, then $\vDash_{\mathscr{D}} \varphi$. Since $\mathscr{D} \supseteq C \supseteq B$, this follows by (Inf) on the hypothesis $\Gamma \vDash_{\mathscr{B}} \varphi$. □

Lemma 21.11 (AndCut). *If $\vDash_{\mathscr{B}} \varphi \wedge \psi$ and $\varphi, \psi \vDash_{\mathscr{B}} \chi$, then $\vDash_{\mathscr{B}} \chi$.*

Proof. We proceed by induction on the structure of χ:

- $\chi = \mathrm{p} \in \mathbb{A}$. This follows immediately by expanding the hypotheses with $(\wedge)$ and (Inf), choosing the atom to be χ.

- $\chi = \chi_1 \to \chi_2$. By $(\to)$, the conclusion is equivalent to $\sigma \Vdash_{\mathscr{B}} \tau$. By (Inf), this is equivalent to the following: for any $\mathscr{C} \supseteq B$, if $\Vdash_{\mathscr{C}} \chi_1$, then $\Vdash_{\mathscr{C}} \chi_2$. Therefore, fix an arbitrary $\mathscr{C} \supseteq B$ such that $\Vdash_{\mathscr{C}} \chi_1$. By the induction hypothesis (IH), it suffices to show: (1) $\Vdash_{\mathscr{C}} \varphi \wedge \psi$ and (2) for any $\mathscr{D} \supseteq C$, if $\Vdash_{\mathscr{D}} \varphi$ and $\Vdash_{\mathscr{D}} \psi$, then $\Vdash_{\mathscr{D}} \chi_2$. By **Monotone*** on the first hypothesis we immediately get (1). For (2), fix an arbitrary base $\mathscr{D} \supseteq C$ such that $\Vdash_{\mathscr{D}} \varphi$, and $\Vdash_{\mathscr{D}} \psi$. By the second hypothesis, we obtain $\Vdash_{\mathscr{D}} \chi_1 \to \chi_2$ — that is, $\chi_1 \Vdash_{\mathscr{D}} \chi_2$. Hence, by (Inf) and **Monotone*** (since $\mathscr{D} \supseteq \mathscr{B}$) we have $\Vdash_{\mathscr{D}} \chi_2$, as required.

- $\chi = \chi_1 \wedge \chi_2$. By $(\wedge^*)$, the conclusion is equivalent to the following: for any $\mathscr{C} \supseteq B$ and atomic p, if $\chi_1, \chi_2 \Vdash_{\mathscr{C}} \mathrm{p}$, then $\Vdash_{\mathscr{C}} \mathrm{p}$. Therefore, fix arbitrary $\mathscr{C} \supseteq B$ and p such that $\chi_1, \chi_2 \Vdash_{\mathscr{C}} \mathrm{p}$. By (Inf), for any $\mathscr{D} \supseteq C$, if $\Vdash_{\mathscr{D}} \chi_1$ and $\Vdash_{\mathscr{D}} \chi_2$, then $\Vdash_{\mathscr{Y}} \mathrm{p}$. We require to show $\Vdash_{\mathscr{C}} \mathrm{p}$. By the IH, it suffices to show the following: (1) $\Vdash_{\mathscr{C}} \varphi \wedge \psi$ and (2), for any $\mathscr{E} \supseteq C$, if $\Vdash_{\mathscr{E}} \varphi$ and $\Vdash_{\mathscr{E}} \psi$, then $\Vdash_{\mathscr{E}} \mathrm{p}$. Since $\mathscr{B} \subseteq C$, By **Monotone*** on the first hypothesis we immediately get (1). For (2), fix an arbitrary base $\mathscr{E} \supseteq C$ such that $\Vdash_{\mathscr{E}} \varphi$ and $\Vdash_{\mathscr{E}} \psi$. By the second hypothesis, we obtain $\Vdash_{\mathscr{D}} \mathrm{p}$, as required.

- $\chi = \chi_1 \vee \chi_2$. By $(\vee)$, the conclusion is equivalent to the following: for any $\mathscr{C} \supseteq B$ and atomic p, if $\chi_1 \Vdash_{\mathscr{C}} \mathrm{p}$ and $\chi_2 \Vdash_{\mathscr{C}} \mathrm{p}$, then $\Vdash_{\mathscr{C}} \mathrm{p}$. Therefore, fix an arbitrary base $\mathscr{C} \supseteq B$ and atomic p such that $\chi_1 \Vdash_{\mathscr{C}} \mathrm{p}$ and $\chi_2 \Vdash_{\mathscr{C}} \mathrm{p}$. By the IH, it suffices to prove the following: (1) $\Vdash_{\mathscr{C}} \varphi \wedge \psi$ and (2). for any $\mathscr{D} \supseteq C$, if $\Vdash_{\mathscr{D}} \varphi$ and $\Vdash_{\mathscr{D}} \psi$, then $\Vdash_{\mathscr{D}} \mathrm{p}$. By **Monotone*** on the first hypothesis we immediately get (1). For (2), fix an arbitrary $\mathscr{D} \supseteq C$ such that $\Vdash_{\mathscr{D}} \varphi$ and $\Vdash_{\mathscr{D}} \psi$. Since $\mathscr{D} \supseteq B$, we obtain $\Vdash_{\mathscr{D}} \chi_1 \vee \chi_2$ by the second hypothesis. By $(\vee)$, we obtain $\Vdash_{\mathscr{D}} \mathrm{p}$, as required.

- $\chi = \bot$. By $(\bot)$, the conclusion is equivalent to the following: $\Vdash_{\mathscr{B}} \mathrm{r}$ for all atomic r. By the IH, it suffices to prove the following: (1) $\Vdash_{\mathscr{B}} \varphi \wedge \psi$ and (2), for any $\mathscr{C} \supseteq B$, if $\Vdash_{\mathscr{C}} \varphi$ and $\Vdash_{\mathscr{C}} \psi$, then $\Vdash_{\mathscr{C}} \mathrm{r}$. By the first hypothesis we have

(1). For (2), fix an arbitrary $\mathscr{C} \supseteq B$ such that $\Vdash_{\mathscr{C}} \varphi$ and $\Vdash_{\mathscr{C}} \psi$. By the second hypothesis, $\Vdash_{\mathscr{C}} \bot$ obtains. By $(\bot)$, we obtain $\Vdash_{\mathscr{C}}$ r, as required.

This completes the induction. □

This concludes the presentation of the B-eS for IPL by Sandqvist [183]. In Chapter 22, we show that this B-eS contains the declarative content of the version of P-tV in Chapter 20.

Chapter 22

From Basic Proof-theoretic Validity to Base-extension Semantics

What does P-tV tell us about the logical constants? That is, we desire to understand the following validity judgement relation in terms of the logical constants:

$$\Gamma \vDash \varphi \qquad \text{iff} \qquad \text{there is a valid argument for } \Gamma \rhd \varphi$$

By Definition 14.16, the relation factors through a *relative* validity judgement,

$$\Gamma \vDash_{\mathscr{B}} \varphi \qquad \text{iff} \qquad \text{there is a } \mathscr{B}\text{-valid argument for } \Gamma \rhd \varphi$$

We desire to explicate this in terms of the logical constants. Such attempts have been made before — see, for example, Peicha et al. [157]. However, they invariably give an interpretation of the situation in which $\Gamma \neq \varnothing$ that is, perhaps, not justified according to the constructiveness of Definition 20.10. This is where the work in this chapter departs from the earlier work.

Recall in Chapter 20 that P-tV is the generalization of a more limited notion of validity introduced by Prawitz [164], which we call *basic* P-tV. This chapter concentrates on the declarative content of *basic P-tV* and shows that it is precisely Sandqvist's B-eS for IPL [183]. While restricting attention in this way is somewhat limited with respect to ideas of P-tS, the completeness of IPL with respect to even this special case is not yet known.

First, in Section 22.1, we analyze the constructiveness of basic P-tV and contrast it to the classical approach used by Piecha et al. [157, 156, 159]. Second, in Section 22.2, we demonstrate that the declarative content of basic P-tV is precisely Sandqvist's B-eS for IPL [183] (Chapter 21). Finally, in Section 22.3, we discuss the meaning of negation, which is subtle in P-tS, relative to these results. As in Chapter 20, we take it that some set of atomic systems is fixed as the notion of *base*. It will be restricted as appropriate. For the remainder of this chapter, when we say P-tV, we mean *basic* P-tV.

22.1 Proof-theoretic Semantics, Constructively

The universal validity relation is obtained from the relative validity judgement by quantifying over bases (see Definition 20.10):

$$\Gamma \vDash \varphi \qquad \text{iff} \qquad \forall \mathscr{B}, \Gamma \vDash_{\mathscr{B}} \varphi$$

It remains to characterize $\vDash$ according to the structure of sequents. By Definition 20.9, we distinguish the cases of $\Gamma \triangleright \varphi$ when $\Gamma = \varnothing$ and $\Gamma \neq \varnothing$.

For $\Gamma = \varnothing$, using Proposition 20.12 (or Definition 20.9), we may characterize P-tV as in Figure 22.1.

Example 22.1. *By Proposition 20.12, $\mathcal{A} : \varnothing \Vdash_{\varnothing} \varphi \wedge \psi$ iff $\mathcal{A} \rightsquigarrow^{*} \bar{\mathcal{A}} \in \wedge_{\mathsf{I}}(\mathcal{A}_1, \mathcal{A}_2)$ such that $\mathcal{A}_1 : \varnothing \Vdash_{\varnothing} \varphi$ and $\mathcal{A}_1 : \varnothing \Vdash_{\varnothing} \varphi$. Hence, there is a valid argument $\mathcal{A}$ witnessing $\varphi \wedge \psi$ iff there are valid argument $\mathcal{A}_1$ and $\mathcal{A}_2$ witnessing φ and ψ, respectively. This is expressed by the clause-$\wedge$ in Figure 22.1.* ■

It remains to consider the case $\Gamma \neq \varnothing$. Using Proposition 20.12 (or Definition 20.9),

$$\Gamma \vDash_{\mathscr{B}} \varphi \quad \text{iff} \quad \exists \mathcal{A} \text{ st. } \forall \mathscr{C} \supseteq \mathscr{B}\, \forall \mathcal{B}, \text{if } \mathcal{B} : \varnothing \Vdash_{\mathscr{C}} \Gamma, \text{ then } \mathsf{cut}(\mathcal{B}, \mathcal{A}) : \varnothing \Vdash_{\mathscr{C}} \varphi \qquad (\dagger)$$

That $\mathcal{B}$ appears in the judgement for φ explains that the arguments that witness the validity of φ are (pointwise) sensitive to the arguments that (hypothetically) witness

$$\begin{array}{lll}
\vDash_{\mathscr{B}} \mathrm{p} & \text{iff} & \vdash_{\mathscr{B}} \mathrm{p} \\
\vDash_{\mathscr{B}} \varphi \wedge \psi & \text{iff} & \vDash_{\mathscr{B}} \varphi \text{ and } \vDash_{\mathscr{B}} \psi \\
\vDash_{\mathscr{B}} \varphi \vee \psi & \text{iff} & \vDash_{\mathscr{B}} \varphi \text{ or } \vDash_{\mathscr{B}} \psi \\
\vDash_{\mathscr{B}} \varphi \to \psi & \text{iff} & \varphi \vDash_{\mathscr{B}} \psi \\
\vDash_{\mathscr{B}} \bot & & \text{never}
\end{array}$$

Figure 22.1: Proof-theoretic Validity (empty context)

the validity of Γ. This is subtle. This is a *constructive* condition on the arguments witnessing φ relative to the arguments witnessing Γ.

Example 22.2. *That $\varphi \vee \psi \Vdash^{*}_{\varnothing} \psi \vee \varphi$ obtains is witnessed by the argument $\mathcal{A}$ defined as follows:*

$$\dfrac{\varphi \vee \psi \qquad \dfrac{[\varphi]}{\varphi \vee \psi}\,\vee_{\mathsf{I}} \qquad \dfrac{[\psi]}{\psi \vee \varphi}\,\vee_{\mathsf{I}}}{\psi \vee \varphi}\,\vee_{\mathsf{E}}$$

That $\mathcal{A}$ is valid requires considering an arbitrary base $\mathscr{B}$ and argument $\mathcal{B}$ such that $\mathcal{B} : \varnothing \Vdash_{\mathscr{B}} \varphi \vee \psi$. Without loss of generality, take $\mathcal{B}$ to be a canonical proof. The validity of $\mathcal{A}$ follows from the assertion $\mathsf{cut}(\mathcal{B}, \mathcal{A}) : \varnothing \Vdash_{\mathscr{B}} \varphi \vee \psi$*. The argument* $\mathsf{cut}(\mathcal{B}, \mathcal{A})$ *is the following:*

$$\dfrac{\overset{\mathcal{B}}{\varphi \vee \psi} \qquad \dfrac{[\varphi]}{\varphi \vee \psi}\,\vee_{\mathsf{I}} \qquad \dfrac{[\psi]}{\psi \vee \varphi}\,\vee_{\mathsf{I}}}{\psi \vee \varphi}\,\vee_{\mathsf{E}}$$

Since $\mathcal{B}$ is a canonical proof, it concludes by $\vee_{\mathsf{I}}$, and therefore the immediate subproof $\mathcal{C}$ is a $\mathscr{B}$-valid argument for either φ or for ψ. Whatever the case, $\mathsf{cut}(\mathcal{A}, \mathcal{B})$ *reduces to the following, where $\chi \in \{\varphi, \psi\}$:*

$$\dfrac{\overset{\mathcal{C}}{\chi}}{\psi \vee \varphi}\,\vee_{\mathsf{I}}$$

It is important to note that all this work takes place with a hypothetical *$\mathcal{B}$, not a fixed one, that depends on the base $\mathscr{B}$. Hence, what makes $\mathcal{A}$ valid depends on a* *case analysis of its hypotheses.* ∎

To see the constructiveness of ($\dagger$), contrast it with the following:

$$\Gamma \vDash_{\mathscr{B}} \varphi \qquad \text{iff} \qquad \forall \mathscr{C} \supseteq \mathscr{B}, \text{if} \vDash_{\mathscr{C}} \Gamma, \text{then} \vDash_{\mathscr{C}} \varphi \tag{$\dagger\dagger$}$$

Acquiescing to ($\dagger\dagger$) recovers the B-eS studied by Piecha et al. [157, 156, 159]. Thus, we distinguish two relations:

- Let $\vDash^1$ denote $\vDash$ with bases as properly first-level atomic systems and ($\dagger\dagger$) as the defining condition for $\Gamma \neq \varnothing$.
- Let $\vDash^2$ denote $\vDash$ with bases as properly second-level atomic systems and ($\dagger$) as the defining condition for $\Gamma \neq \varnothing$.

To see that these relations are not the same, consider the validity of *Harrop's Law*,

$$\frac{(\varphi \to \bot) \to (\psi_1 \vee \psi_2)}{((\varphi \to \bot) \to \psi_1) \vee ((\varphi \to \bot) \to \psi_2)}$$

Notably, this is one of the standard examples of a rule that is admissible for IPL but not derivable.

Example 22.3. *Let* $\gamma := (\mathrm{a} \to \bot) \to (\mathrm{b} \vee \mathrm{c})$ *and* $\theta := ((\mathrm{a} \to \bot) \to \mathrm{b}) \vee ((\mathrm{a} \to \bot) \to \mathrm{c})$. *Piecha et al. [157, 156, 159] have shown that* $\gamma \vDash^1 \theta$. *We show that* $\gamma \nvDash^2 \theta$. *The demonstration merits comparison with the standard rejection of (the formula translation of) Harrop's Law in the BHK interpretation of intuitionistic logic.*

By definition, $\gamma \vDash^2 \theta$ *iff* $\gamma \vDash^2_{\mathscr{B}} \theta$ *for arbitrary* $\mathscr{B}$. *By* ($\dagger$), $\gamma \vDash^2_{\mathscr{B}} \theta$ *is equivalent to the following:*

$$\exists \mathcal{A} \textit{ st. } \forall \mathscr{C} \supseteq \mathscr{B}\, \forall \mathcal{B}, \textit{if } \mathcal{B} : \varnothing \Vdash_{\mathscr{C}} \gamma, \textit{ then } \mathsf{cut}(\mathcal{B}, \mathcal{A}) : \varnothing \Vdash_{\mathscr{C}} \theta$$

We argue by contradiction that no such $\mathcal{A}$ *exists.*

Suppose $\mathcal{A}$ *exists. Without loss of generality, it begins by eliminating the logical structure of* γ *— that is, it contains one or more sub-proofs of the following* *form, for possibly different χ, where* $\mathrm{a} \to \bot$ *is eventually discharged,*

$$\dfrac{\dfrac{\gamma \quad (\mathrm{a} \to \bot)}{\mathrm{b} \lor \mathrm{c}}\to_{\mathsf{E}} \qquad \begin{matrix}\gamma, [\mathrm{b}], (\mathrm{a}\to\bot) \\ \vdots \\ \chi\end{matrix} \qquad \begin{matrix}\gamma, [\mathrm{c}], (\mathrm{a}\to\bot) \\ \vdots \\ \chi\end{matrix}}{\begin{matrix}\chi \\ \vdots\end{matrix}}\lor_{\mathsf{E}}$$

Fix one such χ, call one of its sub-derivations $\mathcal{X}$, and note that it is $\mathscr{B}$-valid. Recall that except when the principal connective of the conclusion is an implication, $\mathscr{B}$-valid arguments may be taken to conclude with an introduction rule. Thus, there is a $\mathscr{B}$-valid argument of the following form:

$$\dfrac{\dfrac{\begin{matrix}\gamma, \chi, [(\mathrm{a}\to\bot)] \\ \vdots \\ \mathrm{b}\end{matrix}}{(\mathrm{a}\to\bot)\to\mathrm{b}}\to_{\mathsf{I}} \qquad \dfrac{\begin{matrix}\gamma, \chi, [(\mathrm{a}\to\bot)] \\ \vdots \\ \mathrm{c}\end{matrix}}{(\mathrm{a}\to\bot)\to\mathrm{c}}\to_{\mathsf{I}}}{\theta}\lor_{\mathsf{I}}$$

In particular, there are $\mathscr{B}$-valid arguments,

$$\begin{matrix}\gamma, \chi, (\mathrm{a}\to\bot) \\ \vdots \\ \mathrm{b}\end{matrix} \qquad \textit{and} \qquad \begin{matrix}\gamma, \chi, (\mathrm{a}\to\bot) \\ \vdots \\ \mathrm{c}\end{matrix}$$

Therefore, χ may be replaced by b *and* c *in $\mathcal{X}$. That is, there are $\mathscr{B}$-valid arguments of the following form:*

$$\dfrac{\dfrac{\gamma \quad (\mathrm{a} \to \bot)}{\mathrm{b} \lor \mathrm{c}}\to_{\mathsf{E}} \qquad \begin{matrix}\gamma, [\mathrm{b}], (\mathrm{a}\to\bot) \\ \vdots \\ \mathrm{b}\end{matrix} \qquad \begin{matrix}\gamma, [\mathrm{c}], (\mathrm{a}\to\bot) \\ \vdots \\ \mathrm{b}\end{matrix}}{\mathrm{b}}\lor_{\mathsf{E}}$$

$$\dfrac{\dfrac{\gamma \quad (\mathrm{a} \to \bot)}{\mathrm{b} \lor \mathrm{c}}\to_{\mathsf{E}} \qquad \begin{matrix}\gamma, [\mathrm{b}], (\mathrm{a}\to\bot) \\ \vdots \\ \mathrm{c}\end{matrix} \qquad \begin{matrix}\gamma, [\mathrm{c}], (\mathrm{a}\to\bot) \\ \vdots \\ \mathrm{c}\end{matrix}}{\mathrm{c}}\lor_{\mathsf{E}}$$

Since $\mathscr{B}$ is arbitrary, let it contain the following rules:

$$\begin{array}{c}[\mathrm{b}]\\ \mathrm{c}\\ \hline \mathrm{a}\end{array} \qquad \begin{array}{c}[\mathrm{c}]\\ \mathrm{b}\\ \hline \mathrm{a}\end{array}$$

Hence, we witness the following $\mathscr{B}$-valid arguments, where $x, y \in \{\mathrm{b},\mathrm{c}\}$ and $x \neq y$:

$$\dfrac{\mathrm{a} \to \bot \qquad \dfrac{\begin{array}{c}\gamma, [y], (\mathrm{a} \to \bot)\\ \vdots\\ \mathrm{x}\end{array}}{\mathrm{a}}}{\bot}$$

It follows that $\gamma \wedge (\mathrm{a} \to \bot) \vDash^2_{\mathscr{B}} \bot$. Hence, since (without loss of generality) a witnessing argument begins by eliminating the logical structure from the context, $\mathrm{b} \vee \mathrm{c} \vDash^2_{\mathscr{B}} \bot$. This is impossible as there is no rule introducing $\bot$. We conclude, therefore, that $\mathcal{A}$ does not exist. ■

This shows that accounting for the constructiveness of P-tV results in a different entailment relation than previously studied. In the next section, we show that it exactly corresponds to the Sandqvist's [183] B-eS for IPL ($\Vdash_{\_}$) by Sandqvist [183] (see Chapter 21). Henceforth, $\vDash$ and $\vDash_{\mathscr{B}}$ mean $\vDash^2$ and $\vDash^2_{\mathscr{B}}$, respectively.

22.2 From Satisfaction to Support

We require some technical results for the induction. None of them are surprising as they reflect standard results in proof theory for IPL, which is the basis of P-tV — see, for example, Troelstra and Schwichtenberg [207], Negri and von Plato [149], and Dyckhoff [53, 54].

Proposition 22.4. *The following hold for arbitrary φ_1, $\varphi_2 \in \mathbb{F}$, sets of formulae Γ, and bases $\mathscr{B}$:*

A) $\Gamma \vDash_{\mathscr{B}} \varphi_1 \wedge \varphi_2$ *iff* $\Gamma \vDash_{\mathscr{B}} \varphi_1$ *and* $\Gamma \vDash_{\mathscr{B}} \varphi_2$

B) $\Gamma \vDash_{\mathscr{B}} \varphi_1 \to \varphi_2$ *iff* $\Gamma, \varphi_1 \vDash_{\mathscr{B}} \varphi_2$

C) $\Gamma, \varphi_1 \wedge \varphi_2 \vDash_{\mathscr{B}} \chi$ *iff* $\Gamma, \varphi_1, \varphi_2 \vDash_{\mathscr{B}} \chi$

D) $\Gamma, \varphi_1 \vee \varphi_2 \vDash_{\mathscr{B}} \chi$ *iff* $\Gamma, \varphi_1 \vDash_{\mathscr{B}} \chi$ *and* $\Gamma, \varphi_2 \vDash_{\mathscr{B}} \chi$

All the claims follow from appropriate use of the introduction and elimination rules. We illustrate c), the others being similar.

Proof of Proposition 22.4 c). By Proposition 20.12, for any $\mathscr{C}$, in the presence of a $\mathscr{C}$-valid argument for φ_1, there is a $\mathscr{C}$-valid argument $\varphi_1 \to \varphi_2$ iff there is a $\mathscr{C}$-valid argument for φ_2. Therefore, by Definition 20.9, there is a $\mathscr{B}$-valid argument witnessing $\varphi_1, \varphi_2, \Gamma \Vdash \chi$ iff there is a $\mathscr{B}$-valid argument witnessing $\varphi_1 \wedge \varphi_2, \Gamma, \vDash_{\mathscr{B}} \chi$, as required. □

Proposition 22.5 (Monotonicity of Bases). *If* $\Gamma \vDash_{\mathscr{B}} \chi$ *and* $\mathscr{C} \supseteq \mathscr{B}$*, then* $\Gamma \vDash_{\mathscr{C}} \chi$.

Proof. Follows immediately from Definition 20.9 by the monotonicity of derivability in a base — that is, $\vdash_{\mathscr{B}}$p implies $\vdash_{\mathscr{C}}$p for any $\mathscr{C} \supseteq \mathscr{B}$. □

Proposition 22.6 (Cut). *If* $\Gamma \vDash_{\mathscr{B}} \chi$ *and* $\chi, \Delta \vDash_{\mathscr{B}} \varphi$*, then* $\Gamma, \Delta \vDash_{\mathscr{B}} \varphi$.

Proof. Follows immediately from Definition 20.9 by the composition of witnessing arguments. □

We may prove the main result of the chapter:

Theorem 22.7. $\Gamma \vDash_{\mathscr{B}} \varphi$ *iff* $\Gamma \Vdash_{\mathscr{B}} \varphi$ *(*$\Gamma \neq \varnothing$*)*

Proof. The direction $\Gamma \Vdash_{\mathscr{B}} \varphi$ implies $\Gamma \vDash_{\mathscr{B}} \varphi$ follows immediately from Theorem 21.5, as $\mathsf{NJ} \cup \mathscr{B}$-derivations are valid arguments.

It remains to show the other direction; that is, $\Gamma \vDash_{\mathscr{B}} \varphi$ implies $\Gamma \Vdash_{\mathscr{B}} \varphi$. We identify a sequent $\Gamma \triangleright \varphi$ with a multiset of the elements of Γ together with φ and proceed by induction on the multiset ordering induced by the ordering on the size of formulae (i.e., the number of binary connectives they contain). Recall the following abbreviation from Chapter 2:

$$\hat{\Gamma} := \bigwedge_{\psi \in \Gamma} \psi$$

- Base Case. We take Γ and φ to be composed of formulas of minimal weight; that is, $\Gamma \cup \{\varphi\} \subseteq \mathbb{A} \cup \{\bot\}$. Three cases require distinct consideration:

  - $\bot \in \Gamma$. By $\wedge$-clause, for any $\mathscr{C} \supseteq \mathscr{B}$: if $\Vdash_{\mathscr{C}} \hat{\Gamma}$, then $\Vdash_{\mathscr{C}} \bot$; and, if $\Vdash_{\mathscr{C}} \bot$, then $\Vdash_{\mathscr{C}}$ p for any p $\in \mathbb{A}$. Thus, $\Vdash_{\mathscr{C}} \hat{\Gamma}$ implies $\Vdash_{\mathscr{C}} \varphi$. Therefore, $\Gamma \Vdash_{\mathscr{B}} \varphi$.

  - $\bot \notin \Gamma$ and $\varphi = \bot$. This case is impossible since there is no rule in any base of in NJ concluding $\bot$ without it occurring as a sub-formula of the premisses.

  - $\bot \notin \Gamma$ and $\varphi \neq \bot$. From $\Gamma \vDash_{\mathscr{B}} \varphi$ we infer that, for any $\mathscr{C} \supseteq \mathscr{B}$, either $\vdash_{\mathscr{C}} \varphi$ or there is an argument $\mathcal{A}$ such that $\mathcal{A} : \Gamma \Vdash_{\mathscr{C}} \varphi$. By composing arguments, it follows that, if $\vdash_{\mathscr{C}}$q for q $\in \Gamma$, then $\vdash_{\mathscr{C}} \varphi$. Hence, $\Vdash_{\mathscr{C}} \hat{\Gamma}$ implies $\Vdash_{\mathscr{C}} \varphi$. That is, $\Gamma \Vdash_{\mathscr{B}} \varphi$.

- Inductive Step. There is $\chi \in \Gamma \cup \{\varphi\}$ such that $\chi \notin \mathbb{A} \cup \bot$. We distinguish two cases, $\chi = \varphi$ and $\chi \neq \varphi$.

  Let $\chi = \varphi$. We proceed by case analysis on the structure of φ:

  - $\varphi = \varphi_1 \wedge \varphi_2$. By Proposition 22.4, $\Gamma \vDash_{\mathscr{B}} \varphi_1$ and $\Gamma \vDash_{\mathscr{B}} \varphi_2$. By the induction hypothesis (IH), $\Gamma \Vdash_{\mathscr{B}} \varphi_1$ and $\Gamma \Vdash_{\mathscr{B}} \varphi_2$. By $\wedge$-clause, $\Gamma \Vdash_{\mathscr{B}} \varphi_1 \wedge \varphi_2$ follows.

  - $\varphi = \varphi_1 \vee \varphi_2$. By Proposition 22.6 and Proposition 22.5, using the assumption $\Gamma \vDash_{\mathscr{B}} \varphi \vee \psi$, we have the following: for any $\mathscr{C} \supseteq \mathscr{B}$ and any p,

$$\Gamma, \varphi \vee \psi \vDash_{\mathscr{C}} \text{p implies } \Gamma \vDash_{\mathscr{C}} \text{p}$$

    By Proposition 22.4,

$$\Gamma, \varphi_1 \vDash_{\mathscr{C}} \text{p and } \Gamma, \varphi_2 \vDash_{\mathscr{C}} \text{p implies } \Gamma \vDash_{\mathscr{C}} \text{p}$$

    Using the already established direction of the theorem together with the IH yields the following: for all $\mathscr{C} \supseteq \mathscr{B}$ and for all p,

$$\Gamma, \varphi_1 \Vdash_{\mathscr{C}} \text{p and } \Gamma, \varphi_2 \Vdash_{\mathscr{C}} \text{p implies } \Gamma \Vdash_{\mathscr{C}} \text{p}$$

    By $\vee$-clause, $\Gamma \Vdash_{\mathscr{C}} \varphi_1 \vee \varphi_2$, as required.

- $\varphi = \varphi_1 \to \varphi_2$. By Proposition 22.4, $\Gamma \vDash_{\mathscr{B}} \varphi_1 \to \varphi_2$ implies $\Gamma, \varphi_1 \vDash_{\mathscr{B}} \varphi_2$. By the IH, $\Gamma, \varphi_1 \Vdash_{\mathscr{B}} \varphi_2$. By the $\to$-clause, $\Gamma \vDash_{\mathscr{B}} \varphi_1 \to \varphi_2$.

This completes the case analysis.

Let $\chi \neq \varphi$. It must be that $\chi \in \Gamma$. That is, we have $\chi, \Delta \vDash_{\mathscr{B}} \varphi$ for some set Δ. We proceed by case analysis on the structure of χ:

- $\chi = \chi_1 \wedge \chi_2$. By Proposition 22.4, $\chi_1, \chi_2, \Delta \vDash_{\mathscr{B}} \varphi$. Hence, by the IH, $\chi_1, \chi_2, \Delta \Vdash_{\mathscr{B}} \varphi$. Therefore, by Definition 21.3, $\chi_1 \wedge \chi_2, \Delta \Vdash_{\mathscr{B}} \varphi$.
- $\chi = \chi_1 \vee \chi_2$. By Proposition 22.4, $\chi_1, \Delta \vDash_{\mathscr{B}} \varphi$ and $\chi_2, \Delta \vDash_{\mathscr{B}} \varphi$. Hence, by the IH, $\chi_1, \Delta \Vdash_{\mathscr{B}} \varphi$ and $\chi_2, \Delta \Vdash_{\mathscr{B}} \varphi$. Therefore, by Theorem 21.5 (using $\vee_{\mathsf{I}}$), we have $\chi_1 \vee \chi_2, \Delta \Vdash_{\mathscr{B}} \varphi$.
- $\chi = \chi_1 \to \chi_2$. Assume (i) $\chi_1 \to \chi_2, \Delta \vDash_{\mathscr{B}} \varphi$. We require to show, $\chi_1 \to \chi_2, \Delta \Vdash_{\mathscr{B}} \varphi$. By Definition 21.3, for arbitrary $\mathscr{C} \supseteq \mathscr{B}$, also assuming (ii) $\chi_1 \Vdash_{\mathscr{C}} \chi_2$, we require to conclude $\Delta \Vdash_{\mathscr{C}} \varphi$. By Proposition 22.5, from (i), infer $\chi_1 \to \chi_2, \Delta \vDash_{\mathscr{C}} \varphi$. By Theorem 21.5, from (ii), infer $\chi_1 \vDash_{\mathscr{C}} \chi_2$. By the $\to_{\mathsf{I}}$-rule,it follows that $\vDash_{\mathscr{C}} \chi_1 \to \chi_2$. By Proposition 22.6, infer $\Delta \vDash_{\mathscr{C}} \varphi$. By the IH, $\Delta \Vdash_{\mathscr{C}} \varphi$, as required.

This completes the case analysis on the structure of χ.

This completes the induction. □

A corollary is an affirmative answer to Prawitz's Conjecture for *basic* P-tV with bases understood as properly second-level atomic systems. Of course, this is a minor completeness result as basic P-tV is very limited, but it is missing in the literature. The techniques used in this chapter may yield a more general account in the future.

Corollary 22.8 (Conjecture 22.8)**.** *Let bases be properly second-level systems,*

$$\Gamma \vDash \varphi \qquad \textit{implies} \qquad \Gamma \vdash \varphi$$

Proof. There are two cases to consider: $\Gamma \neq \varnothing$ and $\Gamma = \varnothing$.

Let $\Gamma \neq \varnothing$. We have the following:

$$\begin{array}{llll}\Gamma \vDash \varphi & \text{implies} & \Gamma \Vdash^{*}_{\varnothing} \varphi & \text{(Figure 22.1)} \\ & \text{implies} & \Gamma \Vdash_{\varnothing} \varphi & \text{(Theorem 22.7)} \\ & \text{implies} & \Gamma \vdash \varphi & \text{(Theorem 21.5)}\end{array}$$

The appeal to Theorem 21.5 requires the condition that bases are properly second-level atomic systems.

Let $\Gamma = \varnothing$. Since it is not the case that $\vDash \alpha$ for any $\alpha \in \mathbb{A} \cup \{\bot\}$, it suffices to consider $\varphi = \psi \circ \chi$ for $\circ \in \{\to, \wedge, \vee\}$. Define the complexity κ of φ as follows:

$$\kappa(\varphi) := \begin{cases} 0 & \text{if } \varphi = \psi \to \chi \\ \max\{\kappa(\psi), \kappa(\chi)\} + 1 & \text{if } \varphi = \psi \wedge \chi \\ \max\{\kappa(\psi), \kappa(\chi)\} + 1 & \text{if } \varphi = \psi \vee \chi \end{cases}$$

We proceed by induction on $\kappa(\varphi)$.

- BASE CASE. $\kappa = 0$. It must be that $\varphi = \varphi_1 \to \varphi_2$, for some φ_1 and φ_2. Therefore, $\varphi \vDash \varphi_2$ — see Figure 22.1. Hence, by the case for $\Gamma \neq \varnothing$, conclude $\varphi_1 \vdash \varphi_2$. Applying $\to_{\mathsf{I}}$, we have $\vdash \varphi_1 \to \varphi_2$ — namely, $\vdash \varphi$ — as required.

- INDUCTIVE STEP. $\kappa > 0$. Either $\varphi = \varphi_1 \wedge \varphi_2$ or $\varphi = \varphi_1 \vee \varphi_2$, for some φ_1 and φ_2. We consider each case separately:

  - $\wedge$. Since $\vDash \varphi \wedge \psi$, both $\vDash \psi$ and $\vDash \varphi$ — see Figure 22.1. Therefore, by the induction hypothesis (IH), $\vdash \varphi_1$ and $\vdash \varphi_2$. Applying $\wedge_{\mathsf{I}}$, we have $\vdash \varphi \wedge \psi$ — namely, $\vdash \varphi$ — as required.

  - $\vee$. Since $\vDash \varphi_1 \vee \varphi_2$, either $\vDash \varphi_1$ or $\vDash \varphi_2$. By the IH, either $\vdash \varphi_1$ or $\vdash \varphi_2$. Applying $\vee_{\mathsf{I}}$, we have $\vdash \varphi_1 \vee \varphi_2$ — namely, $\vdash \varphi$ — as required.

This completes the induction. □

In Section 22.3, we reflect on the position of *ex falso quodlibet* relative to this finding.

22.3 Ex Falso Quodlibet?

There is an apparent mismatch between the treatment of $\bot$ for $\vDash$ and for $\Vdash$ — that is,

$$\vDash_{\mathscr{B}} \bot \qquad \text{never}$$

but

$$\Vdash_{\mathscr{B}} \bot \qquad \text{iff} \qquad \Vdash_{\mathscr{B}} \mathsf{p} \text{ for every } \mathsf{p} \in \mathbb{A}$$

This is possible as Theorem 22.7 only applies for $\Gamma \neq \varnothing$. It illustrates that $\vDash_{\mathscr{B}}$ and $\Vdash_{\mathscr{B}}$ do not coincide otherwise; indeed, the latter may be interpreted as the former under the assumption of some hypothesis — see Example 22.3 for how under certain assumptions one could conclude bottom. We briefly analyze this situation and what it says about P-tV. Indeed, the meaning of negation is a subtle issue in P-tS — see, for example, Kürbis [126].

Heuristically, the reason is that *satisfaction* is bivalent while *support* is constructive. This is exposed through the *ex falso quodlibet* rule (henceforth, EFQ),

$$\frac{\bot}{\varphi}\,\bot_{\mathsf{E}}$$

Despite not being constructive, EFQ makes sense from the perspective of P-tV. Assuming a $\mathscr{B}$-valid argument of the hypotheses, one must be able to *construct* an $\mathscr{B}$-valid argument for the conclusion, whether such arguments exist or not. For example, if one has $\mathscr{B}$-valid arguments for $\varphi \vee \psi$ and $\neg\varphi$, then one must have a $\mathscr{B}$-valid argument for ψ. As above, we reason by definitional reflection, which is captured by the following construction:

$$\dfrac{\begin{matrix}\mathcal{D}_1\\ \varphi \vee \psi\end{matrix} \qquad \dfrac{\dfrac{[\varphi] \quad \begin{matrix}\mathcal{D}_2\\ \neg\varphi\end{matrix}}{\bot}\to_{\mathsf{I}}}{\psi}\,\mathsf{EFQ} \qquad [\psi]}{\psi}\,\vee_{\mathsf{E}}$$

Here EFQ captures the contradiction at the meta-level, which is classical and therefore has no problem with EFQ, between assuming simultaneously that there is a $\mathscr{B}$-valid argument for φ and that there is not. This explains why EFQ is appropriate as a rule in NJ according to P-tV, and it indicates that the semantics is fundamentally based on hypothetical reasoning — see Schroeder-Heister [192].

Nonetheless, something is not quite right about the current setup. Like EFQ, the disjunctive syllogism used above is non-constructive, yet it is part of constructive reasoning. Bennett has defended its position within constructive logic [17], but in terms of reasoning, it leaves something to be desired. Tennant [204] provides a pertinent and compelling reflection on EFQ:

> In general, a proof of Ψ from Δ is suasively appropriate only if a person who believes Δ can reasonably decide, on the basis of the proof, to believe Ψ. But if the proof shows his belief set Δ to be inconsistent "on the way to proving" Ψ from Δ, then the reasonable reaction is to *suspend* belief in Δ rather than acquiesce in the doxatic inflation administered by the absurdity rule.

This warrants replacing EFQ by something more 'suasively appropriate' to handle reasoning such as disjunctive syllogism. For example, we may take the *liberalization* of $\vee_{\mathsf{E}}$ by Tennant [204, 205] as a rule of inference; that is, the figure

$$\frac{\varphi \vee \psi \quad \begin{array}{c}[\varphi]\\ \bot/\chi\end{array} \quad \begin{array}{c}[\psi]\\ \bot/\chi\end{array}}{\chi}$$

which is understood as saying that if both subordinate conclusions are of the same form, then the rule behaves as usual, but if precisely one of them is $\bot$, then one brings down the other as the main conclusion of the inference. Taking this rule leads to Core Logic introduced by Tennant [205], perhaps suggesting that, to the extent that P-tS concerns suasive content, Core Logic is a valuable logic to study.

Chapter 23

Definite Formulae, Negation-as-Failure, and the Base-extension Semantics for Intuitionistic Propositional Logic

There is an intuitive encoding of atomic rules as formulae; more precisely, as *definite* formulae in the sense of Chapter 2. Under this encoding, atomic systems live within the hereditary Harrop fragment of IPL. The latter has a simple operational reading via proof-search for uniform proofs (see Chapter 2) that enables a proof-theoretic denotational semantics — the least fixed point construction. We use this well-understood phenomenon to deliver the completeness of IPL with respect to Sandqvist's B-eS [183] — see Chapter 21. This chapter is based on the following paper:

> Gheorghiu, A. V., and Pym, D. J. Definite Formulae, Negation-as-Failure, and the Base-extension Semantics of Intuitionistic Propositional Logic. *Bulletin of the Section of Logic* (2023)

The essential idea of the proof does not change, only the techniques involved. Indeed, one may already view the completeness proof by Sandqvist [183] (Chapter 21) as proceeding through LP — namely, the proof-theoretic approach by Schroeder-Heister and Hallnäs [94, 95] using higher-level rules.

The point of studying the B-eS for IPL in terms of definite formulae is that it enables an interpretation of negation in terms of the *negation-as-failure* (NAF) protocol — the denial of a formula is the failure to find a proof of it. This is valuable as negation is a subtle issue in P-tS — see, for example, Kürbis [126].

23.1 Atomic Systems vs. Programs

Intuitively, atomic systems are definitional in precisely the same way as programs in the hereditary Harrop fragment of IPL are definitional. To illustrate this, we must systematically move between them, which we do by encoding atomic systems as programs.

Let $\lfloor - \rfloor$ be as follows:

- The encoding of a zeroth-level rule is as follows:

$$\left\lfloor \frac{}{\mathrm{c}} \right\rfloor := \mathrm{c}$$

- The encoding of a first-level rule is as follows:

$$\left\lfloor \frac{\mathrm{p}_1 \quad \cdots \quad \mathrm{p}_n}{\mathrm{c}} \right\rfloor := (\mathrm{p}_1 \land \ldots \land \mathrm{p}_n) \to \mathrm{c}$$

- The encoding of an *n*th-level rule is as follows:

$$\left\lfloor \frac{\begin{matrix} [\Sigma_1] & & [\Sigma_n] \\ \mathrm{p}_1 & \ldots & \mathrm{p}_n \end{matrix}}{\mathrm{c}} \right\rfloor := \big((\lfloor \Sigma_1 \rfloor \to \mathrm{p}_1) \land \ldots \land (\lfloor \Sigma_n \rfloor \to \mathrm{p}_n) \big) \to \mathrm{c}$$

For example, apply $\lfloor - \rfloor$ to the $(\to_\mathrm{I})^\flat$-rule in Figure 21.2 yields the following:

$$(\varphi^\flat \to \psi^\flat) \to (\varphi \to \psi)^\flat$$

The hierarchy of atomic system provided by Definition 20.1 precisely corresponds to the inductive depth of the grammar for hereditary Harrop formulae —

that is, if $\mathscr{A}$ is an n-th level atomic system, then

$$\vdash_{\mathscr{A}} \mathrm{p} \qquad \text{iff} \qquad \lfloor \mathscr{A} \rfloor \vdash \mathrm{p}$$

Therefore, we may suppress the encoding function and henceforth use atomic systems and programs interchangeably — that is, we may write $\mathscr{A} \vdash \mathrm{p}$ to denote $\lfloor \mathscr{A} \rfloor \vdash \mathrm{p}$.

Of course, for the B-eS for IPL in Chapter 21, we are limited to properly second-level atomic systems, but the grammar of definite clauses can handle considerably more.

Formally, to say that bases are definitional in the sense of programs, we mean the following:

$$\Vdash_{\mathscr{B}} \varphi \qquad \text{iff} \qquad \mathscr{N} \cup \mathscr{B} \vdash \varphi^{\flat} \tag{$*$}$$

Here $\mathscr{N}$ contains rules governing φ when the formula is complex — that is, φ is a sub-formula of a sequent $\Gamma \triangleright \gamma$ which generates $\mathscr{N}$ — and arbitrary otherwise.

It is important that we use $\varphi^{\flat}$ rather than φ in $(*)$. It is certainly *not* the case that bases behave exactly as contexts; that is, we do *not* have the following equivalence:

$$\Vdash_{\mathscr{B}} \varphi \qquad \text{iff} \qquad \mathscr{B} \vdash \varphi \tag{$**$}$$

That this generalization fails is shown by the following counter-example:

Example 23.1. *Consider the following formula:*

$$\varphi := (\mathrm{a} \to \mathrm{b} \vee \mathrm{c}) \to \big((\mathrm{a} \to \mathrm{b}) \vee (\mathrm{a} \to \mathrm{c})\big)$$

The formula φ is not a consequence of IPL; hence, by completeness of IPL with respect to the B-eS we have $\Vdash_{\mathscr{B}} (\mathrm{a} \to \mathrm{b} \vee \mathrm{c})$ and $\nVdash_{\mathscr{B}} (\mathrm{a} \to \mathrm{b}) \vee (\mathrm{a} \to \mathrm{c})$, for some $\mathscr{B}$. However, assuming $(**)$, the second judgment obtains whenever the the first obtains — that is, $\Vdash_{\mathscr{B}} (\mathrm{a} \to \mathrm{b} \vee \mathrm{c})$ implies *$\Vdash_{\mathscr{B}} (\mathrm{a} \to \mathrm{b}) \vee (\mathrm{a} \to \mathrm{c})$, for any $\mathscr{B}$! The* *following computation in hHLP witnesses this:*

$$
\begin{array}{llr}
\Vdash_{\mathscr{B}} \mathrm{a} \to \mathrm{b} \lor \mathrm{c} \quad \textit{implies} & \mathscr{B} \vdash \mathrm{a} \to \mathrm{b} \lor \mathrm{c} & (**) \\
\textit{implies} & \mathscr{B} \cup \{\mathrm{a}\} \vdash \mathrm{b} \lor \mathrm{c} & (\mathsf{LOAD}) \\
\textit{implies} & \mathscr{B} \cup \{\mathrm{a}\} \vdash \mathrm{b} \textit{ or } \mathscr{B} \cup \{\mathrm{a}\} \vdash \mathrm{c} & (\mathsf{OR}) \\
\textit{implies} & \mathscr{B} \vdash \mathrm{a} \to \mathrm{b} \textit{ or } \mathscr{B} \vdash \mathrm{a} \to \mathrm{c} & (\mathsf{LOAD}) \\
\textit{implies} & \mathscr{B} \vdash (\mathrm{a} \to \mathrm{b}) \lor (\mathrm{a} \to \mathrm{c}) & (\mathsf{OR}) \\
\textit{implies} & \Vdash_{\mathscr{B}} (\mathrm{a} \to \mathrm{b}) \lor (\mathrm{a} \to \mathrm{c}) & (**)
\end{array}
$$

That LOAD *and* OR *may be used invertibly is justified by case-analysis on the structure of the goal formula with respect to the operational semantics (Figure 2.4) — it can also be seen by Lemma 8.15.* ∎

To see how $(*)$ works in contrast to the failure of $(**)$, it is instructive to consider an example that explicitly uses the proof-search for the definite formulae as a meta-calculus for derivability in a base.

Example 23.2. *By Theorem 21.5, we have* $\Vdash_{\varnothing} \mathrm{a} \lor \mathrm{b} \to \mathrm{b} \lor \mathrm{a}$. *That* $\mathscr{N} \vdash (\mathrm{a} \lor \mathrm{b} \to \mathrm{b} \lor \mathrm{a})^\flat$ *indeed obtains is witnessed by the following computation:*

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\mathscr{N}, (\mathrm{a} \lor \mathrm{b})^\flat \vdash (\mathrm{a} \lor \mathrm{b})^\flat} \Uparrow \mathsf{IN} \qquad \mathcal{R}_a \quad \mathcal{R}_b}{\mathscr{N}, (\mathrm{a} \lor \mathrm{b})^\flat \vdash (\mathrm{a} \lor \mathrm{b})^\flat \land (\mathrm{a} \to (\mathrm{b} \lor \mathrm{a})^\flat) \land (\mathrm{b} \to (\mathrm{b} \lor \mathrm{a})^\flat)} \Uparrow \mathsf{AND}}{\mathscr{N}, (\mathrm{a} \lor \mathrm{b})^\flat \vdash (\mathrm{b} \lor \mathrm{a})^\flat} \Uparrow \mathsf{CLAUSE}(\lor_\mathsf{E})^\flat}{\mathscr{N}, (\mathrm{a} \lor \mathrm{b})^\flat \vdash (\mathrm{b} \lor \mathrm{a})^\flat} \Uparrow \mathsf{LOAD}}{\mathscr{N} \vdash (\mathrm{a} \lor \mathrm{b} \to \mathrm{b} \lor \mathrm{a})^\flat} \Uparrow \mathsf{CLAUSE}(\to_\mathsf{I})^\flat
$$

in which $\mathcal{R}_x$ *for* $x \in \{a, b\}$ *is*

$$
\dfrac{\dfrac{\dfrac{}{\mathscr{N}, (\mathrm{b} \lor \mathrm{a})^\flat, \mathrm{x} \vdash \mathrm{x}} \Uparrow \mathsf{IN}}{\mathscr{N}, (\mathrm{b} \lor \mathrm{a})^\flat, \mathrm{x} \vdash (\mathrm{b} \lor \mathrm{a})^\flat} \Uparrow \mathsf{CLAUSE}(\lor_\mathsf{I})^\flat}{\mathscr{N}, (\mathrm{b} \lor \mathrm{a})^\flat \vdash \mathrm{x} \to (\mathrm{b} \lor \mathrm{a})^\flat} \Uparrow \mathsf{LOAD}
$$

∎

In the next section, we use the relationship between atomic systems and programs to prove the completeness of IPL with respect to the B-eS.

23.2 Completeness of IPL via Logic Programming

We may prove the completeness of IPL with respect to the B-eS by passing through hHLP as follows:

$$\begin{tikzcd}
T^{\omega}I_{\bot}, \mathscr{N} \vDash \varphi^{\flat} \arrow[r, leftrightarrow] & \mathscr{N} \vdash \varphi^{\flat} \arrow[d] \\
\Vdash_{\mathscr{N}} \varphi \arrow[u] & \vdash \varphi
\end{tikzcd}$$

The diagram requires three claims, the middle of which is Lemma 8.15. The other two are Lemma 23.3 and Lemma 23.4 (below), respectively, reading in the direction of the arrows.

The intuition of the completeness argument is two-fold: firstly, that $\mathscr{N}$ is to $\varphi^{\flat}$ as NJ is to φ; secondly, the use of a rule in a base corresponds to the use of a clause in the corresponding program; thirdly, execution in $\mathscr{N}$ corresponds to proof(-search) in NJ. In this setup, the T^{ω} construction captures the construction of a proof: the application of a rule corresponds to a use of T, and the iterative application of rules corresponds to the iterative application of T — that is, to T^{ω}.

It remains to prove the claims and completeness. Fix a sequent $\Gamma \triangleright \varphi$ and let $-^{\flat}$ and $\mathscr{N}$ be constructed as in Chapter 21 for this sequent. Let Δ be an arbitrary set of sub-formulae of the sequent and ψ an arbitrary subformula of the sequent.

Proposition 23.3 (Emulation)**.** *If* $\Vdash_{\mathscr{N}} \psi$*, then* $T^{\omega}I_{\bot}, \mathscr{N} \vDash \psi^{\flat}$.

Proof. We prove a stronger proposition: for any $\mathscr{N}' \supseteq \mathscr{N}$, if $\Vdash_{\mathscr{N}'} \psi$, then $T^{\omega}I_{\bot}, \mathscr{N}' \vDash \psi^{\flat}$. We proceed by induction on support in a base according to the various cases of Figure 21.1. For the sake of economy, we combine the clauses $\Rightarrow$ and $\to$.

- $\psi \in \mathbb{A}$. Note $\psi^{\flat} = \psi$, by definition. Therefore, if $\Vdash_{\mathscr{N}'} \psi$, then $\vdash_{\mathscr{N}'} \psi$, but this is precisely emulated by application of T. Hence, $T^{\omega}I_{\bot}, \mathscr{N}' \Vdash \psi$.

- $\psi = \bot$. If $\Vdash_{\mathscr{N}'} \bot$, then $\Vdash_{\mathscr{N}'} \mathrm{p}$, for every $\mathrm{p} \in \mathbb{A}$. By the induction hypothesis (IH), $T^{\omega}I_{\bot}, \mathscr{N}' \Vdash \mathrm{p}$ for every $\mathrm{p} \in \mathbb{A}$. It follows that $T^{\omega}I_{\bot}, \mathscr{N}' \Vdash \bot^{\flat}$.

- $\psi := \psi_1 \wedge \psi_2$. By the $\wedge$-clause for support, $\Vdash_{\mathscr{N}'} \psi_1$ and $\Vdash_{\mathscr{N}'} \psi_2$. Hence, by the IH, $T^{\omega}I_{\bot}, \mathscr{N}' \Vdash \psi_1^{\flat}$ and $T^{\omega}I_{\bot}, \mathscr{N}' \Vdash \psi_2^{\flat}$. By $\wedge$-clause for satisfaction, $T^{\omega}I_{\bot}, \mathscr{N}' \Vdash \psi_1^{\flat} \wedge \psi_2^{\flat}$. The result follows by $\wedge_{\mathsf{I}}{}^{\flat}$-schema.

- $\psi := \psi_1 \vee \psi_2$. By Lemma **Flat** (in proof of Theorem 21.5), $\psi_1 \Vdash_{\mathscr{N}'} \psi_1^{\flat}$ and $\psi_2 \Vdash_{\mathscr{N}'} \psi_2^{\flat}$. By the $\vee_{\mathsf{I}}$-scheme in $\mathscr{N}'$, both $\psi_1^{\flat} \Vdash_{(} \psi_1 \vee \psi_2)^{\flat}$ and $\psi_2^{\flat} \Vdash_{(} \psi_1 \vee \psi_2)^{\flat}$. Therefore, by $\Rightarrow$-clause for support, we have $\psi_1 \Vdash_{\mathscr{N}'} (\psi_1 \vee \psi_2)^{\flat}$ and $\psi_2 \Vdash_{\mathscr{N}'} (\psi_1 \vee \psi_2)^{\flat}$. Using the $\vee$-clause for support on the assumption $\Vdash_{\mathscr{N}'} \psi_1 \vee \psi_2$ with these results means that $\Vdash_{\mathscr{N}'} (\psi_1 \vee \psi_2)^{\flat}$. That is, $T^{\omega}, \mathscr{N}' \Vdash (\psi_1 \vee \psi_2)^{\flat}$, as required.

- $\psi := \psi_1 \to \psi_2$. By the $\to$-clause for satisfaction, $\psi_1 \Vdash_{\mathscr{N}'} \psi_2$. So, by the $\Rightarrow$-clause for satisfaction, $\Vdash_{\mathscr{N}''} \psi_1$ implies $\Vdash_{\mathscr{N}''} \psi_2$ for any $\mathscr{N}'' \supseteq \mathscr{N}'$. Let $\mathscr{N}'' := \mathscr{N}' \cup \{\psi_1^{\flat}\}$. Since $\Vdash_{\mathscr{N}', \psi_1^{\flat}} \psi_2^{\flat}$, by Lemma **Flat** (in proof of Theorem 21.5), we have $\Vdash_{\mathscr{N}', \psi_1^{\flat}} \psi_2$, hence we infer $\Vdash_{\mathscr{N}', \psi_1^{\flat}} \psi_2$. By the IH, $T^{\omega}I_{\bot}, \mathscr{N}' \cup \{\psi_1^{\flat}\} \Vdash \psi_2^{\flat}$. Hence, $T^{\omega}I_{\bot}, \mathscr{N}' \Vdash \psi_1^{\flat} \to \psi_2^{\flat}$. By the $\to_{\mathsf{I}}{}^{\flat}$-scheme, $T^{\omega}I_{\bot}\mathscr{N}' \Vdash (\psi_1 \to \psi_2)^{\flat}$, as required.

This completes the induction. □

Proposition 23.4 (Simulation)**.** *If $\mathscr{N} \cup \Delta^{\flat} \vdash \psi^{\flat}$, then $\Delta \vdash \psi$.*

Proof. We proceed by induction on the length of execution. Intuitively, the execution of $\mathscr{N} \cup \Delta^{\flat} \vdash \psi^{\flat}$ simulates the reductive construction of a proof of ψ from Δ in NJ — that is, a proof-search. We proceed by induction on the length of the execution.

BASE CASE: It must be that $\psi \in \Delta$, so $\Delta \vdash \psi$ is immediate.

INDUCTIVE STEP: By construction of $\mathscr{N}$, the execution concludes by CLAUSE applied to a definite clause ρ simulating a rule $\mathrm{r} \in \mathsf{NJ}$; that is, $\mathscr{N} \cup \Delta^{\flat} \vdash \psi_i^{\flat}$ for ψ_i such that $\psi_1^{\flat} \wedge .... \wedge \psi_n^{\flat} \to \psi^{\flat}$. By the induction hypothesis (IH), $\Delta \vdash \psi_i$ for $1 \leq i \leq n$. It follows that $\Delta \vdash \psi$ by applying $r \in \mathsf{NJ}$.

For example, if the execution concludes by CLAUSE applied to the clause for

$\wedge$-introduction (i.e., $\psi^\flat \wedge \psi^\flat \to (\psi \wedge \psi)^\flat$), then the trace is as follows:

$$\dfrac{\dfrac{\overset{\vdots}{\mathcal{N} \cup \Delta^\flat \vdash \psi^\flat} \quad \overset{\vdots}{\mathcal{N} \cup \Delta^\flat \vdash \psi^\flat}}{\mathcal{N} \cup \Delta^\flat \vdash \psi^\flat \wedge \psi^\flat}}{\mathcal{N} \cup \Delta^\flat \vdash (\psi \wedge \psi)^\flat}$$

By the induction hypothesis, we have proofs witnessing $\Delta \vdash \psi$ and $\Delta \vdash \psi$, and by $\wedge$-introduction:

$$\dfrac{\overset{\vdots}{\psi} \quad \overset{\vdots}{\psi}}{\psi \wedge \psi}$$

This completes the induction. □

Following the diagram, we have the completeness of IPL with respect to the B-eS:

Proof. Theorem 21.5 — Completeness. We require to show that $\Vdash_\varphi$ implies $\Vdash_{\mathcal{N}} \varphi$ for arbitrary φ. To this end, assume $\Vdash_\varphi$. Let $\mathcal{N}$ be the natural base generated by φ. By definition, from the assumption, we have $\Vdash_{\mathcal{N}} \varphi$. Hence, by Lemma 23.3, it follows that $T^\omega I_\bot, \mathcal{N} \Vdash \varphi^\flat$. Whence, by Lemma 8.15, we obtain $\mathcal{N} \vdash \varphi^\flat$. Thus, by Lemma 23.4, $\vdash \varphi$, as required. □

The following section discusses how reductive logic delivers the completeness proof above and the essential role played by both proofs and refutations.

23.3 Negation-as-Failure

A reduction in a proof system is constructed co-recursively by applying the rules of inference backwards. Even though each step corresponds to the application of a rule, the reduction can fail to be a proof as the computation arrives at an irreducible sequent that is not an instance of an axiom in the logic. For example, in hHLP, one may compute the following:

$$\dfrac{\dfrac{\mathrm{p} \triangleright \mathrm{q}}{\mathrm{p} \triangleright \mathrm{p} \vee \mathrm{q}} \Uparrow \mathsf{OR}}{\varnothing \triangleright \mathrm{p} \to (\mathrm{p} \vee \mathrm{q})} \Uparrow \mathsf{Load}$$

This reduction fails to be a proof, despite every step being a valid inference, since $\mathrm{p} \triangleright \mathrm{q}$ is not an instance of IN. In reductive logic, such failed attempts at constructing proofs are not meaningless: Pym and Ritter [173] have provided a semantics of the reductive logic of IPL in which such reductions are given meaning by using hypothetical rules — that is, the construction would succeed in the presence of the following rule:

$$\frac{\mathrm{p}}{\mathrm{q}}$$

The categorical treatment of this semantics has them as *indeterminates* in a polynomial category — this adumbrates current work by Pym et al. [175], who have shown that the B-eS is entirely natural from the perspective of categorical logic. The use of such additional rules to give semantics to constructions that are not proofs directly corresponds to the use of atomic systems in the B-eS for IPL; for example, let $\mathscr{A}$ be the atomic system containing the rule above, then the judgement $p \Vdash_{\mathscr{A}} q$ obtains. This suggests a close relationship between B-eS and reductive logic, which manifests with the operational reading of definite clauses and their relationship to atomic rules in Section 23.1.

Within P-tS, negation is a subtle issue — see Kürbis [126]. We may use the perspective of LP developed herein to review the meaning of absurdity ($\bot$).

There is no introduction rule for $\bot$ in NJ. One may not construct a proof of absurdity without it already being, *in some sense*, assumed; for example, $\varphi, \varphi \to \bot \vdash \bot$ obtains because the context $\{\varphi, \varphi \to \bot\}$ is already, in some sense, absurd. We may use LP to understand what that sense is. To simplify matters, observe that the judgement $\Gamma \vdash \bot$ is equivalent to $\vdash \varphi \to \bot$ for some formula φ. Therefore, we may restrict attention to negations of this kind to understand the meaning of absurdity.

By Theorem 21.5 (Soundness) and Lemma 23.4 (Simulation), we see that the converse of Theorem 23.3 holds. Therefore,

$$\Vdash_{\neg} \varphi \qquad \text{iff} \qquad T^{\omega} I_{\bot}, \mathscr{N} \vdash (\neg\varphi)^{\flat}$$

Unfolding the semantics, this is equivalent to $T^{\omega}I_{\perp}, \mathscr{N} \cup \{\varphi^{\flat}\} \vdash \perp^{\flat}$. Thus, the sense in which φ is absurd is that its interpretation under $T^{\omega}I_{\perp}$ contains absurdity; that is, φ is absurd iff $\perp^{\flat} \in T^{\omega}I_{\perp}(\varphi)$. What does this tell us about the meaning of $\neg\varphi$? Since there s no proof of $\perp^{\flat}$, we have that the meaning of $\neg\varphi$ is that there is no proof of $(\varphi)^{\flat}$ in $\mathscr{N}$. This is the *negation-as-failure* principle. How does it yield the clause for $\perp$ in Figure 21.1?

Passing through $(*)$ in Section 23.1,

$$\Vdash_{\mathscr{B}} \perp \qquad \text{iff} \qquad \mathscr{N} \cup \mathscr{B} \vdash \perp^{\flat}$$

Since there is no introduction rule for $\perp^{\flat}$ in $\mathscr{N}$, it must be that $\mathscr{B}$ derives it. Thus, there is rule in $\mathscr{B}$ of the following form:

$$\frac{\begin{matrix}[\Sigma_1] & & [\Sigma_n] \\ \mathrm{p}_1 & \cdots & \mathrm{p}_n\end{matrix}}{\perp^{\flat}}$$

To simplify matters, we introduce alien q and $\bar{\mathrm{q}}$ as 'conjunctions' of some subset $q_1,...,q_k$ and $q_{k+1},...,q_n$ of $\mathrm{p}_1,...,\mathrm{p}_n$ in the inferentialist sense. That is, we introduce the following, where $\Pi_i = \Sigma_j$ iff $\mathrm{q}_i = \mathrm{p}_i$ for $i, j \in \{1,...,n\}$:

$$\frac{\begin{matrix}[\Pi_1] & & [\Pi_n] \\ \mathrm{q}_1 & \cdots & \mathrm{q}_n\end{matrix}}{\mathrm{q}} \qquad \frac{\begin{matrix}[\Pi_{k+1}] & & [\Pi_n] \\ \mathrm{q}_{k+1} & \cdots & \mathrm{q}_n\end{matrix}}{\bar{\mathrm{q}}}$$

Doing this allows us to replace the above rule with the following:

$$\frac{\mathrm{q} \quad \bar{\mathrm{q}}}{\perp^{\flat}}$$

In this case, the inferential behaviour of q and $\bar{\mathrm{q}}$ is that they are contradictory propositions; that is, together they infer absurdity.

What is significant from this analysis is that the semantics of $\perp$ requires us to observe that there is no proof of it and thus extend the space with proofs of contradictory q and $\bar{\mathrm{q}}$. If they are proved in $\mathscr{B}$, then one has proved absurdity; if $\mathscr{B}$

has proved absurdity, then one has proofs for each of these. The subtlety is that since we do not have negation *explicit* in our atoms, we only admit the principle that some atoms are contradictory. If we prove all atoms, then we prove these contradictory atoms; if we prove these contradictory atoms, then we have proved absurdity. This justifies the clause for $\bot$,

$$\Vdash_{\mathscr{B}} \bot \qquad \text{iff} \qquad \Vdash_{\mathscr{B}} \mathrm{p} \text{ for any } \mathrm{p} \in \mathbb{A}$$

Piecha and Schroeder-Heister [194, 158] have argued that there are two views on atomic systems: knowledge and definitional. This becomes clear according to various ways in which a program may be regarded in LP. The negation-as-failure protocol uses the definitional perspective; its analogue in terms of knowledge is the *closed-world assumption*. In this case, a knowledge base treats everything that is not known to be valid as invalid. There is significant literature about the closed-world assumption that may be useful for understanding P-tS and what it tells us about reasoning — see, for example, Clark [40], Reiter [177], and Kowalski [121, 118], and Harland [96, 97].

Chapter 24

Conclusion to Part III

Proof-theoretic semantics (P-tS) is disciplin in which the central notion in terms of which meanings are assigned to certain expressions of our language, in particular to logical constants, is that of proof — understood as objects denoting collections of acceptable inferences from accepted premisses — rather than truth. This is subtle. It is not that one desires a proof system that precisely characterizes a logic of interest, but rather that one desires to define aspects of the logic in terms of proofs and provability. Therefore, importantly, as Schroeder-Heister [191] observes, since no formal system is fixed (only notions of inference), the relationship between semantics and provability remains the same as it has always been: soundness and completeness are desirable features of formal systems with respect to the semantics. What differs is that *proofs* serve the role of *truth* in model-theoretic semantics. The semantic paradigm supporting P-tS is *inferentialism* — the view that meaning (or validity) arises from rules of inference (see Brandom [26]).

Chapter 20 presents three version of proof-theoretic validity (P-tV) in the Dummett-Prawitz tradition: first, it gives a *basic* P-tV that applied only to NJ-derivations; second, it gives a general version of P-tV (following Schroeder-Heister [190]); third it gives an abstract version of P-tV relative to the theory of tactical proof. The basic and general versions are close to the BHK interpretation of IPL and to the semantics of reductions by Pym and Ritter [173]. In particular, the use of atomic systems to give validity to unfinished reduction, in the sense that there are unproved statements in P-tV, is symmetric to the use of indiscernible to complete reductions in the work on Reductive Logic. This suggests that versions of P-tV are the appropriate way of giving semantics to reductions in various formal systems, which is justified by the abstract version of P-tV relative to the theory of tactical proof.

Chapter 21 presents the base-extension semantics (B-eS) of intuitionistic propositional logic (IPL) by Sandqvist [183]. This is a semantics of IPL in terms of proofs. The chapter is only background. It is a subtle subject, however, as the literature on P-tV contains numerous incompleteness results for various intuitive notions of B-eS with respect to intuitionistic logics — see, for example, Piecha et al. [157, 156, 159], Goldfarb [85], Sandqvist [181, 182, 184, 183], Stafford [197].

Chapter 22 demonstrates that the B-eS for IPL in Chapter 21 is the declarative counterpart to the basic P-tV in Chapter 20. It is not as simple as observing that the inductive validity conditions of the version of P-tV amounts to the unfolding of the B-eS. One has to account for the constructiveness of P-tV wherein the validity of an argument with open assumptions depends on how the open assumptions are validated. This is subtle, but it explains the clause for disjunction in the B-eS of IPL, which is distinctive since it takes the form of the elimination rule for disjunction. A corollary is an affirmative answer to Prawitz's Conjecture for basic P-tV.

Chapter 23 studies the B-eS for IPL in terms of the operational reading of definite formulae given by Miller et al. [139, 140]. Importantly, this operational reading is in terms of proof-search; that is, it is a reading that takes place from the point of view of Reductive Logic. The main idea of the chapter is that atomic systems — that is, bases — may be canonically read as collections of definite formulae, which is adumbrated by Hallnäs and Schroeder-Heister [94, 95]. Accordingly, the chapter presents proof of the completeness of the B-eS for IPL in terms of the unfolding of the operational semantics of definite formulae. This perspective enables the meaning of negation, which is a subtle issue in P-tS (see K urbis [126]) to be given in terms of the negation-as-failure (NAF) protocol, which is a long-standing and well-developed approach to it in various areas of Reductive Logic such as logic programming.

Overall, we have presented P-tS as an approach to semantics that arises naturally from Reductive Logic. In contrast to model-theoretic semantics (M-tS), P-tS has received relatively little mathematical development, rendering these investigations limited to IPL. Nonetheless, the major themes and ideas can be applied to other logics — for example, there is no moral limitation, though there are several technical challenges, in giving a P-tS of BI. This suggests a future general approach to the semantics of logics and constructions in Reductive Logic. Moreover, there is an opportunity to relate P-tS to other semantics paradigm through Reductive Logic. For example, the handling of M-tS in Part II in terms of correctness conditions on constructions in calculi for a meta-logic is intuitively related. Finally, one expects the application, the purpose, of P-tS (beyond philosophical considerations) to give a mathematical account of *reasoning* in practice. This remains to be done but is a problem of Reductive Logic as this is the paradigm through which one typically solves problems using formal representations.

Chapter 25

General Conclusion

This monograph investigates the interplay between semantics and proof from the perspective of Reductive Logic. It comprises three parts, each of which has its conclusion. Therefore, we do not discuss the details of their contributions here but rather present the overall picture this monograph delivers.

Reductive Logic is the approach to logic in which one begins with a putative conclusion and uses inference rules backward to derive sufficient premisses for that conclusion. This stands in contrast to the traditional approach to logic known as Deductive Logic, in which one begins with established premisses and derives conclusions using inference rules. The interest in Reductive Logic stems from the fact that it is the approach to logic used for practical reasoning problems. That is, while Deductive Logic describes the valid forms of inference, Reductive Logic explains how they are used during problem-solving. This is justified in Chapter 1. One desires to investigate the interplay between semantics and proof from this perspective to understand how they inform each other by understanding whence they come; for example, in Chapter 17, we show that the exploration of the proof-search space for IPL, suitably encoded within FOL, is the same as the unfolding of the logic's traditional semantics by Kripke [125]. Thus, one can derive one from the other.

For Part I and Part II, semantics is understood in terms of the model-theoretic reading of consequence given by Tarski [201, 203]. Briefly, it proceeds by interpreting propositions within algebraic structures and expressing logical constants in terms of the dynamics of elements of the algebraic structures. The central result is soundness and completeness, which shows that a collection of algebraic structures, under a certain interpretation, capture a logic — or, reciprocally, that a logic correctly describes properties of the algebraic structures in which one is interested. This is the dominant approach to meaning in logic, and it is both intuitive and valuable for deploying logic in practical reasoning problems.

While Part I largely considers the proof theory of BI, concentrating on questions about proof-search, it culminates by encoding the model-theoretic semantics of BI proof-theoretically and thereby delivers a novel approach to soundness and completeness that bypasses term- and counter-model constructions. The underlying idea is that one can relate proof in a logic of interest to the logic's model-theoretic semantics through proof-search in FOL. This inspires Part II, in which these ideas are developed.

Part II introduces a paradigm of proof systems called *alegbraic constraint system* (ACSs). These systems give a systematic account of how proof in one logic may be expressed in terms of proofs in another logic together with algebraic constraints. The method generalizes earlier work by Harland and Pym [99, 98], who developed the *resource-distribution via Boolean constraints* to study proof-search in sub-structural logics. Therefore, there are two primary uses for ACSs: first, to study proof-search, and second, to study semantics in Reductive Logic. The first use is not developed in any generality within this monograph. Instead, the monograph concentrates on the second use and indeed provides a general account of the aforementioned method to soundness and completeness employed for BI in Part I. Moreover, their combined use delivers Chapter 17, mentioned above as illustrating the thesis of this monograph — namely, that through the lens of Reductive Logic, one can see explicitly how semantics and proof inform each other.

Part III departs from model-theoretic semantics and instead studies proof-theoretic semantics. There has been some work relating the two paradigms of meaning — see, for example, Goldfarb [85] — but further investigations are required. This field is closely related to other work in Reductive Logic but was developed separately and independently. Therefore, in contrast to the treatment of model-theoretic semantics in Part I and Part II, this part is not about relating proof-theoretic semantics and proof-search but instead studies the field. The significant contributions are to relate the dominant branches — proof-theoretic validity, in the Dummett-Prawitz tradition and base-extension semantics — and to give an account of negation in terms of the negation-as-failure protocol.

Ultimately, this monograph illustrates that there are deep connections between semantics and proof and that they can be witnessed through Reductive Logic. Nevertheless, it is an investigation rather than a complete theory. While the monograph contains several contributions to logic across mathematics, informatics, and philosophy, its fundamental contribution is demonstrating the viability and merit of studying semantics from Reductive Logic and giving methods, techniques, and tools for a systematic theory to be developed.

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Appendix

Appendix A

Coalgebra

Endofunctors on the category of sets and functions are a suitable mathematical framework for an abstract notion of structure, and throughout we will use the word *functor* exclusively for such mappings. We may suppress the composition symbol and simply write $\mathcal{G}\mathcal{F}$ for the mapping which first applies $\mathcal{F}$ and then applies $\mathcal{G}$; similarly, we may write $\mathcal{F}X$ for the application of functor $\mathcal{F}$ on a set X.

There are numerous functors used throughout mathematics and computer science, for example elements of the *flat polynomial* functors,

$$\mathcal{F} ::= \mathcal{I} \mid \mathcal{K}_A \mid \mathcal{F} \times \mathcal{F} \mid \mathcal{F} + \mathcal{F}$$

Here $\mathcal{I}$ is the identity functor (i.e., the mapping fixing both objects and functions); $\mathcal{K}_A$ is the constant functor for a given set A, which is defined by mapping any set to the set A and any arrow to the identity function on A; $\mathcal{F} \times \mathcal{G}$ is the cartesian product of $\mathcal{F}$ and $\mathcal{G}$; and, $\mathcal{F} + \mathcal{G}$ is the disjoint union of $\mathcal{F}$ and $\mathcal{G}$.

Occasionally one can transform one functor into another uniformly. That is, one can make the transformation componentwise, so that the actions on sets and function cohere.

Definition A.1 (Natural Transformation)**.** *A collection of functions indexed by sets* $\mathfrak{n} := (\mathfrak{n}_X)$ *is a natural transformation between functors* $\mathcal{F}$ *and* $\mathcal{G}$ *if and only if*

$\mathfrak{n}_X : \mathcal{F}X \to \mathcal{G}X$ and if $f : X \to Y$ then then the following diagram commutes:

$$\begin{tikzcd}
\mathcal{F}X \arrow[r, "\mathfrak{n}_X"] \arrow[d, "\mathcal{F}(f)"'] & \mathcal{G}X \arrow[d, "\mathcal{G}(f)"] \\
\mathcal{F}(Y) \arrow[r, "\mathfrak{n}_Y"'] & \mathcal{G}(Y)
\end{tikzcd}$$

Every functor $\mathcal{F}$ admits at least one natural transformation called the identity: $\mathfrak{i}_X := I_{\mathcal{F}X}$, where $I_{\mathcal{F}X}$ is the identify function on $\mathcal{F}X$. As an abuse of notation, we use the notation of function types when speaking about natural transformation; that is, we may write $\mathfrak{n} : \mathcal{F} \to \mathcal{G}$ to denote that $\mathfrak{n}$ is a natural transformation between $\mathcal{F}$ and $\mathcal{G}$.

There are two particularly well-behaved classes of functors, called *monads* and *comonads*, that are useful abstractions of data-type and behaviour-type when modelling computation.

Definition A.2 (Monad and Comonad)**.** *Let $\mathcal{T}$ be a functor. It is a (co)monad if there are natural transformations $\mathfrak{u} : \mathcal{I} \to \mathcal{T}$ and $\mathfrak{m} : \mathcal{T}^2 \to \mathcal{T}$ (resp. $\mathfrak{u} : \mathcal{T} \to \mathcal{I}$ and $\mathfrak{m} : \mathcal{T} \to \mathcal{T}^2$) satisfying the following commutative diagrams:*

$$\begin{tikzcd}
(\mathcal{T}\mathcal{T}\mathcal{T})X \arrow[r, "\mathcal{T}\mathfrak{m}_X"] \arrow[d, "\mathfrak{m}_{\mathcal{T}X}"'] & (\mathcal{T}\mathcal{T})X \arrow[d, "\mathfrak{m}_X"] \\
(\mathcal{T}\mathcal{T})X \arrow[r, "\mathfrak{m}_X"'] & \mathcal{T}X
\end{tikzcd}
\qquad
\left(\text{resp.}\ \begin{tikzcd}
(\mathcal{T}\mathcal{T}\mathcal{T})X & (\mathcal{T}\mathcal{T})X \arrow[l, "\mathcal{T}\mathfrak{m}_X"'] \\
(\mathcal{T}\mathcal{T})X \arrow[u, "\mathfrak{m}_{\mathcal{T}X}"] & \mathcal{T}X \arrow[l, "\mathfrak{m}_X"] \arrow[u, "\mathfrak{m}_X"']
\end{tikzcd} \right)$$

$$\begin{tikzcd}
\mathcal{T}X \arrow[r, "\mathfrak{u}_{\mathcal{T}X}"] \arrow[d, "\mathcal{T}\mathfrak{u}_X"'] \arrow[dr, "\mathcal{I}"] & (\mathcal{T}\mathcal{T})X \arrow[d, "\mathfrak{m}_X"] \\
(\mathcal{T}\mathcal{T})X \arrow[r, "\mathfrak{m}_X"'] & \mathcal{T}X
\end{tikzcd}
\qquad
\left(\text{resp.}\ \begin{tikzcd}
\mathcal{T}X & (\mathcal{T}\mathcal{T})X \arrow[l, "\mathfrak{u}_{\mathcal{T}X}"'] \\
(\mathcal{T}\mathcal{T})X \arrow[u, "\mathcal{T}\mathfrak{u}_X"] & \mathcal{T}X \arrow[l, "\mathfrak{m}_X"] \arrow[u, "\mathfrak{m}_X"'] \arrow[ul, "\mathcal{I}"']
\end{tikzcd} \right)$$

The natural transformations $\mathfrak{u}$ and $\mathfrak{m}$ are often called the (co)unit and (co)multiplication of the (co)monad. There is an abundance of examples of monads; for example, the powerset functor $\mathscr{P}$, which takes sets to their powersets and functions to their direct image fuctions, is a monad whose unit is the singleton function and whose multiplication is the union operator. Simple examples of comonads are less common. However, since they are used to define behaviour-type, modelling operational semantics will involve defining one: the proof-search comonad.

Under relatively mild conditions, there is a canonical way to construct a (co)monad from a functor: the (co)free construction. Heuristically, this is the indefinite application of the functor structure until a fixed-point is reached. The cofree construction is analogous.

Example A.3. *Consider the functor $\mathcal{F}_A : X \mapsto \mathsf{nil} + A \times X$, where nil is the emptyset. One can generate the least fixed point $\mathcal{L}_A$ for $\mathcal{F}_A$ by the ω-chain in the following diagram:*

$$\mathsf{nil} \longrightarrow \mathsf{nil} + A \times \mathsf{nil} \longrightarrow \mathsf{nil} + A \times (\mathsf{nil} + A \times \mathsf{nil}) \longrightarrow \ldots$$

The arrows are inductively defined by extending with the unique function out of the emptyset. The mapping $A \mapsto L_A$ defines the free functor $\mathcal{L}_A$, which can be understood as structuring elements of A as lists (identified with products). It is a monad whose unit is the single-element list constructor $a \mapsto a :: (\mathsf{nil} :: \mathsf{nil}\ldots)$ and whose multiplication concatenation. ■

Given a space structured by a functor $\mathcal{F}$, one can define actions which respect the structure. There are two directions: either one wants the domain to be structured, in which case one has an $\mathcal{F}$-algebra, or the codomain, in which case one has a $\mathcal{F}$-coalgebra. When structure represents a data-type (resp. a behaviour-type) given by a (co)monad, one may add extra conditions on the (co)algebra that make it a co(module). These functions are used to give abstract models of data and behaviour.

Definition A.4 (Algebra and Coalgebra)**.** *Let $\mathcal{T}$ be a functor, then a function $\alpha : \mathcal{T}X \to X$ is called an $\mathcal{T}$-algebra and any function $\beta : X \to \mathcal{T}X$ is a $\mathcal{T}$-coalgebra. If $\langle \mathcal{T}, \mathfrak{u}, \mathfrak{m} \rangle$ is a monad (resp. comonad), then α (resp. β) is a module (resp. comodule) when the following diagrams commute:*

$$\begin{tikzcd}
X \arrow[r, "\mathrm{u}_X"] \arrow[dr, "\mathrm{i}_X"'] & \mathcal{T}X \arrow[d, "\alpha"] \\
 & X
\end{tikzcd}
\qquad
\left(\textit{resp.} \quad \begin{tikzcd}
X & \mathcal{T}X \arrow[l, "\mathrm{u}_X"'] \\
 & X \arrow[ul, "\mathrm{i}_X"] \arrow[u, "\alpha"']
\end{tikzcd}\right)$$

$$\begin{tikzcd}
\mathcal{T}\mathcal{T}X \arrow[r, "\mathcal{T}\alpha"] \arrow[d, "\mathrm{m}_A"'] & \mathcal{T}X \arrow[d, "\alpha"] \\
\mathcal{T}X \arrow[r, "\alpha"'] & X
\end{tikzcd}
\qquad
\left(\textit{resp.} \quad \begin{tikzcd}
\mathcal{T}\mathcal{T}X & \mathcal{T}X \arrow[l, "\mathcal{T}\alpha"'] \\
\mathcal{T}X \arrow[u, "\mathrm{m}_A"] & X \arrow[l, "\alpha"] \arrow[u, "\alpha"']
\end{tikzcd}\right)$$

The abstract modelling of operational semantics witnesses both the use of algebra and coalgebra: the former for specifying the constructs, and the latter for specifying the transitions. In the best case the two structures cohere, captured mathematically by the mediation of a natural transformation called a distributive law, and form a bialgebra.

Definition A.5 (Distributive Law). *A distributive law for a functor $\mathcal{G}$ over a functor $\mathcal{F}$ is a natural transformation $\partial : \mathcal{G}\mathcal{F} \rightarrow \mathcal{F}\mathcal{G}$.*

Definition A.6 (Bialgebra). *Let $\partial : \mathcal{G}\mathcal{F} \rightarrow \mathcal{F}\mathcal{G}$ be a distributive law, and let $\alpha : \mathcal{G}X \rightarrow X$ be an algebra and $\beta : X \rightarrow \mathcal{F}X$ be a coalgebra. The triple (X, α, β) is a ∂-bialgebra when the following diagram commutes:*

$$\begin{tikzcd}
\mathcal{G}X \arrow[r, "\alpha"] \arrow[d, "\mathcal{G}\beta"'] & X \arrow[r, "\beta"] & \mathcal{F}X \\
\mathcal{G}\mathcal{F}X \arrow[rr, "\partial_X"'] & & \mathcal{F}\mathcal{G}X \arrow[u, "\mathcal{F}\alpha"']
\end{tikzcd}$$

There are additional coherence condition which may be applied for when one has a monad or a comonad structure.

In Turi's and Plotkin's bialgebraic models of operational semantics [209], the algebra supplies the structure of the syntax and the coalgebra supplies the behaviour of execution, and under relatively mild conditions (i.e. the coalgebra structure preserves weak pullbacks) forms are even a *full* abstractions (with respect to bisimulation).

Appendix B

What is a Valid Argument?

The central subject of logic is *consequence* — that is, a judgement that says that a proposition follows logically from a theory (a collection of propositions). Traditionally, consequence is explained truth-theoretically: a proposition is a consequence of a theory if and only if the proposition *holds* whenever the theory *holds* —see Tarski [201, 202]. This perspective gives rise to model-theoretic semantics, which concerns explicating what it means to 'hold' using abstract mathematical structures. The subject of *proof theory*, which concerns mathematical objects with static and dynamic structure that witness consequence, is relegated to a technology for establishing consequence — that is, for understanding when consequence judgements obtain rather than what they are. This essay concerns an alternative *inferential* (or *proof-theoretic*) reading of consequence in which a proposition follows from a theory if and only if the proposition is yielded by the *suasive potential* of that theory. Rather than considering models in which propositions hold, one considers systems of rules of inference that *support* propositions — that is, systems of rules that capture the suasive potential of the propositions. These systems of rules are called *bases*. The inferential reading of consequence is as follows: a proposition is consequence of a theory if and only if the proposition is supported by any base supporting the theory. In the process of the analysis, the essay recovers various logics — in particular, minimal, intuitionistic, classical, and core logic — according to naturally arising design choices on the 'suasive potential' of propositions; and it also recovers proof-theoretic validity in the Dummett-Prawitz tradition — see, for example Schroeder-Heister [190] — as an answer to how suasive potential delivers consequence.

A *proposition* is a complete idea. For example, ‘Tammy is a vixen’ is a proposition, and so is ‘ “Tammy is a vixen” is a proposition.’ The details of the form and nature of propositions are not important here. A collection of propositions constitutes a *theory*. They are used in intellectual activities to explain, characterize, or define; for example, *the theory of general relativity* (GR) explains that space and time morph according to energy and mass.

To better understand the idea of consequence in logic, it is instructive to briefly consider its deployment in theoretical work. After a theory has been proposed, one tests it by considering its consequences. Suppose a proposition φ is a consequence of a theory Γ: if φ is acceptable, then φ is evidence *for* Γ; but, if φ is unacceptable, then φ is evidence *against* Γ.

For example, Newton’s theory of gravity postulates the existence of a planet Vulcan somewhere between Mercury and the Sun. However, despite efforts to find it, Vulcan has never been observed, rendering its existence an unacceptable proposition. Therefore, it is evidence *against* the theory as an accurate explanation of gravity. Meanwhile, it follows from GR that light is subject to gravity and, therefore, that its path bends around massive objects. Hence, if GR were a correct explanation of the world, the position of stars near the Sun would shift during a solar eclipse in South America in 1919. The phenomenon was indeed observed by British expeditions, yielding evidence for GR.

The same principle applies in settings that are not phenomenological such as mathematics and philosophy. For example, *utilitarianism* is the theory that the morality is determined by optimizing a utility function, typically with respect to harm caused. A consequence of (certain forms of) utilitarianism is that a surgeon should kill a healthy person to save five unhealthy ones — see [206]. If one believes such an action is immoral, then the conclusion represents evidence against (this form of) utilitarianism as an explanation of morality.

Evidently, understanding consequence is essential to all forms of intellectual work. What does it actually mean? A model is an abstract mathematical structure in which propositions can be interpreted. In the standard reading given by Tarski [201, 202], a proposition φ is a *consequence* of a theory Γ — denoted $\Gamma \vdash \varphi$ — if and only if every model of Γ is also a model of φ. This reading is the traditional and dominant approach to logic, with widespread application in mathematics, informatics, and philosophy. This essay considers an alternate reading of consequence $\Gamma \vdash \varphi$ in which on arrives at φ by reasoning from Γ. The term 'entailed' is used in place of 'consequence' to distinguish the inferential and model-theoretic readings.

In the reading given Tennant [204], a proposition φ is *entailed* by a theory Γ — denoted $\Gamma \vDash \varphi$ — if and only if there is a *valid argument* for φ from Γ. Hence, the central question of the essay is, 'What is a valid argument?' Of course, entailment and consequence (in the traditional reading) are certainly related: if φ is entailed by Γ, but φ is not a consequence of Γ (i.e., φ is not the case whenever Γ is the case), the reasoning must surely be faulty! Hence, that φ is a consequence of Γ in the traditional sense is a necessary condition for φ to be entailed by Γ — cf. the approach to arguments satirized by Schopenhauer [185]. The reasoning power afforded by a proposition is its *suasive potential*, which requires analysis to answer the central question.

Intuitively, an argument comprises individual reasoning steps called *inferences*. This suggests a quick answer to the central question of the essay:

> An argument is valid if and only if it is comprised of *accepted* inferences.

What determines what the accepted inferences are? This problem bridges the gap between the central question on the validity of arguments and the objects to which they pertain — namely, propositions.

An inference is warranted by a *rule*. This choice of form of rules fixed here is arbitrary to enable some concrete analysis — Piecha and Schroeder-Heister [194] have analyzed various possible choices according to philosophical and mathematical desiderata. In this essay, a rule is defined by a rule figure,

$$\frac{\begin{matrix}[\Gamma_1] & & [\Gamma_n] \\ P_1 & \dots & P_n\end{matrix}}{C}$$

— $C, P_1, \dots, P_n$ are propositions, and Γ_1,...,Γ_n are theories. Note, the rule is taken as written and not closed under substitution; that is, it applies to the particular propositions stated in the rule figure. The proposition C is called the *conclusion* of the rule, the propositions P_1,...,P_n are the *premisses* of the rule, and the theories Γ_1,...,Γ_n are the hypotheses of rule. A rule may contain no hypotheses (i.e., Γ_1,...,Γ_n may be empty). Moreover, it may have no premisses, in which case the rule is an *axiom* that allows one to assert the conclusion,

$$\frac{}{C}$$

The following is an example of a rule:

$$\frac{\text{John is a man} \quad \text{John is unmarried}}{\text{John is a bachelor}}$$

An inference concluding a proposition C from some given propositions $P_1, \dots, P_n$ may be denoted by a horizontal line adorned with $\Downarrow$ with the $P_1, \dots, P_n$ above it and C below,

$$\frac{P_1 \quad \dots \quad P_n}{C} \Downarrow$$

A rule warrants an inference from the premisses to the conclusion whenever the premisses have been *established* relative to their hypotheses. That is, the rule

$$\frac{\begin{matrix}[\Gamma_1] & & [\Gamma_n] \\ P_1 & \dots & P_n\end{matrix}}{C}$$

justifies the inference above if one has valid arguments for P_i from Γ_i for $i = 1, \dots, n$. An argument completely regulated by some rules is a *proof* relative to those rules.

For example, relative to the rules

$$\frac{\text{John is a man} \quad \text{John is unmarried}}{\text{John is a bachelor}} \qquad \frac{}{\text{John is a man}} \qquad \frac{}{\text{John is unmarried}}$$

one has the following proof:

$$\frac{\frac{}{\text{John is unmarried}}\Downarrow \quad \frac{}{\text{John is male}}\Downarrow}{\text{John is a bachelor}}\Downarrow$$

A declared collection of rules is called a *base* — a basis of reasoning. This refines the analysis of a valid argument to *valid argument relative to a base*. That a theory Γ entails φ *in the base* $\mathscr{B}$ is denoted $\Gamma \vDash_{\mathscr{B}} \varphi$.

Typically, rules are not explicitly provided, but implicit in the meaning of the propositions involved. For example, the proposition 'Tammy is a vixen' may be inferred from the propositions 'Tammy is female' and 'Tammy is a fox' because of their meaning. As far as this essay is concerned, the meaning of a proposition *is* its inferential behaviour. For example, 'Tammy is a vixen' may be *defined* by the fact that is inferred from 'Tammy is female' and 'Tammy is a fox,'

$$\frac{\text{Tammy is female} \quad \text{Tammy is a fox}}{\text{Tammy is a vixen}}$$

That the meaning of a proposition is determined by its inferential behaviour is known as *inferentialism* — see Brandom [26] — which is a particular instantiation of the 'meaning as use' principle advanced by Wittgenstein [215].

The scope of inferential behaviours expressible is determined by the form of rules admitted. In short, a inferential behaviour is described by a *rule schema*, which has the form of a rule but not particular content —- that is, does not pertain to any particular propositions. In other words, a rule schema defines a *logical structure* such that if a proposition has that logical structure its inferential behaviour is given by the defining rule schema. This is illustrated presently. Significantly, this approach to logical structure departs from the traditional one; in particular, it is not that logical structures provide a grammar of propositions — see, for example, van Dalen [211] — but instead they are used to describe the meaning of propositions (on inferentialism).

A rule schema is presented as a rule figure with the propositions replaced by meta-variables. For example, the following is a rule schema:

$$\frac{\varphi \quad \psi}{\varphi \land \psi}$$

The logical structure ($\land$) defined by this schema is called *conjunction*. The above analysis of the proposition 'Tammy is a vixen' amounts to saying that the proposition is a conjunction of 'Tammy is female' and 'Tammy is a fox' — hence, its inferential behaviour is described by the rule schema for conjunction. This is unlike the situation for 'John is a bachelor' above as there the rules were explicit while here they are implicit in the meaning (on inferentialism) of the proposition. Notably, a proposition can only have one logical structure — namely, the one defined by all the rule schemas describing its meaning (on infernetialism).

A proposition that has a logical structure is a *complex proposition*, and a proposition that does not have a logical structure is an *atomic proposition*. Henceforth, the term *formula* may be used to denote a proposition that is either complex or atomic. Since the inferential behaviour of a complex proposition is entirely determined by its logical structure, if it were to appear in a base then its meaning would become garbled, so that inference warranted by its intended logical structure is not longer permitted (i.e., invalid). For example, suppose that one were to declare by fiat the following rule:

$$\frac{\text{John is a bachelor}}{\text{Tammy is a vixen}}$$

The meaning (on inferentialism) of 'Tammy is a vixen' has thus changed; that is, it is no longer a conjunction of 'Tammy is female' and 'Tammy is a fox'. Hence, the inferential behaviour afforded it by that logical structure is not longer applicable and, therefore, such inferences are no longer acceptable. To guard against such semantic disestablishment, the propositions occurring bases must henceforth all be atomic. Of course, an atomic proposition can behave in a particular base precisely as a formula behaves according to its logical structure — this is elaborated upon later because it requires first explicating precisely what it means for a rule schema to *define* a logical structure.

This approach to logical structure departs from the traditional presentation of logic in which logical structures are used to *define* a class of propositions rather than define inferential behaviour — see, for example, van Dalen [211]. That is, in this essay propositions as conceptionally prior to logical structure. That is not to say that this essay contradicts the traditional approach, rather that it explicates the relationship between 'logic' and 'thought.' This is adumbrated by Popper [163] — see Binder et al. [21]. The distinction manifests in the handling of the deployment of 'elimination' rule schemas below in the analysis of valid arguments.

The format of rules fixed above allows only three dimensions to inferential behaviour: justification from rules with multiple premisses, justification from multiple rules, justification from rules with hypotheses. These are captured by the logical structure of *conjunction*, *disjunction*, and *implication*, respectively. The conjunction $\varphi \land \psi$ of formulae φ and ψ and has already been defined — it concerns inference from multiple premisses. The disjunction $\varphi \lor \psi$ of formulae φ and ψ concerns inference from multiple rules; it is defined by the following rule schemas:

$$\frac{\varphi}{\varphi \lor \psi} \qquad \frac{\psi}{\varphi \lor \psi}$$

For example, 'It is raining or the King is dead' is the disjunction $R \lor K$ in which R is 'It is raining,' and K is 'The King is dead.' The implication $\varphi \to \psi$ of φ to ψ concerns inference from hypotheses; it is defined by the following rule schema:

$$\frac{\begin{array}{c}[\varphi]\\ \psi\end{array}}{\varphi \to \psi}$$

For example, 'If it rains, then the harvest will be ruined' is the implication $R \to H$ in which R is as above, and H is 'The harvest is ruined.'

Since these logical structures describe all the components of the justification of inferences, as afforded by the forms of rule schemas in this essay, all other logical structures can be expressed in terms of them. That is, conjunction, disjunction, and implication collectively express all the inferential behaviours a proposition may have. For example, suppose a logical structure $\circ$ is determined by the following rules:

$$\frac{\varphi_1 \qquad \overset{[\varphi_3,\varphi_4]}{\varphi_2}}{\circ(\varphi_1,\varphi_2,\varphi_3,\varphi_4,\varphi_5)} \qquad \frac{\varphi_5}{\circ(\varphi_1,\varphi_2,\varphi_3,\varphi_4,\varphi_5)}$$

This is already captured since $\circ(\varphi_1,\varphi_2,\varphi_3,\varphi_4,\varphi_5)$ has the same inferential behaviour as $(\varphi_1 \wedge ((\varphi_3 \wedge \varphi_4) \to \varphi_2)) \vee \varphi_5$.

The analysis of logical structures so far explains when a complex proposition may be inferred, but not what may be inferred from it. This is important as complex propositions may appear in the theory from which one is arguing. Surely, from the proposition 'Tammy is a vixen', one should be able to infer 'Tammy is female' and 'Tammy is a fox' precisely by its meaning. The justification is that the latter are implicit in the assertion of the former because it is defined by them having been established. In general, in saying that a rule schema *defines* a logical structure means that the *suasive potential* of a complex proposition with that logical structure is the same as that of the premisses of the rule schema. Halläs and Schroeder-Heister [92, 188] have called this *principle of definitional reflection* (DR):

> Whatever follows from all the defining conditions of an assertion also follows from the assertion itself

This determines how complex propositions appearing as assumptions behave.

According to DR, if a proposition χ is inferred from φ and ψ, then it is also inferred from $\varphi \wedge \psi$ — that is, DR justifies the following elimination rule schema:

$$\frac{\varphi \wedge \psi \qquad \overset{[\varphi,\psi]}{\chi}}{\chi}$$

Since the argument for χ proceeds from the hypotheses φ and ψ only, this rule schema has the same expressive power as the following:

$$\frac{\varphi \wedge \psi}{\varphi} \qquad \frac{\varphi \wedge \psi}{\psi}$$

This recovers the intuition on ‘Tammy is a vixen’ above. Analogously, one has the following elimination rule schemas for disjunction and implication:

$$\frac{\varphi \vee \psi \quad \begin{matrix}[\varphi] \\ \chi\end{matrix} \quad \begin{matrix}[\psi] \\ \chi\end{matrix}}{\chi} \qquad \frac{\varphi \to \psi \quad \varphi}{\psi}$$

Indeed, DR enables an explicit characterization of *suasive potential.* A base $\mathscr{B}$ *supports* a proposition φ when it has the suasive potential of φ — denoted $\Vdash_{\mathscr{B}} \varphi$. This means that whatever may inferred from φ admits a valid argument in $\mathscr{B}$. According to DR, support satisfies the following conditions:

$$\begin{array}{lll}
\Vdash_{\mathscr{B}} P & \text{if and only if} & \text{the proposition } P \text{ is atomic and provable in } \mathscr{B} \\
\Vdash_{\mathscr{B}} \varphi \wedge \psi & \text{if and only if} & \text{for any } \mathscr{C} \text{ extending } \mathscr{B} \text{ and formula } \chi, \\
 & & \text{if } \varphi, \psi \Vdash_{\mathscr{C}} \chi, \text{ then } \Vdash_{\mathscr{C}} \chi \\
\Vdash_{\mathscr{B}} \varphi \vee \psi & \text{if and only if} & \text{for any } \mathscr{C} \text{ extending } \mathscr{B} \text{ and formula } \chi, \\
 & & \text{if } \varphi, \Vdash_{\mathscr{C}} \chi \text{ and } \psi, \Vdash_{\mathscr{C}} \chi, \text{ then } \Vdash_{\mathscr{C}} \chi \\
\Vdash_{\mathscr{B}} \varphi \to \psi & \text{if and only if} & \varphi \vDash_{\mathscr{B}} \psi
\end{array}$$

Intuitively, a proposition is entailed by a theory in a particular base if and only if it is entailed in any base in which the propositions comprising the theory are supported — that is, for non-empty Γ,

$$\begin{array}{lll}
\Gamma \vDash_{\mathscr{B}} \varphi & \text{if and only if} & \text{for any } \mathscr{C} \text{ extending } \mathscr{B}, \\
 & & \text{if } \Vdash_{\mathscr{C}} \psi \text{ for any } \psi \text{ in } \Gamma, \text{ then } \vDash_{\mathscr{C}} \varphi.
\end{array}$$

For example, the proposition ‘Socrates is mortal’ is entailed from a theory containing the propositions ‘Socrates is a man’ and ‘All men are mortal’ in any base containing the following rule

$$\frac{\text{Scorates is a man} \quad \text{All men are mortal}}{\text{Socrates is mortal}}$$

because any base supporting the theory (i.e., admitting proofs of the propositions comprising the theory) admits a proof of the desired proposition.

Let an argument be *open* if it proceeds from assumptions (i.e., unjustified propositions), and *closed* otherwise. For example, the argument

$$\frac{\text{John is a man} \quad \text{John is unmarried}}{\text{John is a bachelor}}\Downarrow$$

is open, but the argument

$$\frac{\dfrac{}{\text{John is a man}}\Downarrow \quad \dfrac{}{\text{John is unmarried}}\Downarrow}{\text{John is a bachelor}}\Downarrow$$

is closed. The above statement on support amounts to the following condition on the validity of arguments:

> An open argument $\mathcal{A}$ is $\mathscr{B}$-valid if and only if, for any $\mathscr{C}$ extending $\mathscr{B}$, any result of supplying $\mathscr{C}$-valid arguments for the assumptions of $\mathcal{A}$ yields an overall $\mathscr{C}$-valid argument.

It is now possible to explicate under what conditions an atomic proposition behaves as though it has a logical structure. This occurs whenever one is in a base containing rules for the atomic proposition that mimic the introduction and elimination rules of the logical structure. For example, the atomic proposition ‘Tammy is a female fox’ behaves in the following base precisely as ‘Tammy is a vixen’ does according to its logical structure:

$$\frac{\text{Tammy is female} \quad \text{Tammy is a fox}}{\text{Tammy is a female fox}}$$

$$\frac{\text{Tammy is a female fox}}{\text{Tammy is female}} \qquad \frac{\text{Tammy is a female fox}}{\text{Tammy is a fox}}$$

That is, for any argument in this base (or any extension of this base), any occurrence of ‘Tammy is a vixen’ may be substituted for ‘Tammy is a female fox’ without affecting the validity of the argument.

To summarize, propositions may have a logical structure, according to which they have certain inferential behaviours. These are complex propositions. A logical structure is understood as being *defined* by a rule schema. Initially, such rule schemas describe under what conditions the complex propositions may be asserted; for example, having established the propositions ‘Tammy is female’ and ‘Tammy is a fox’ one may infer ‘Tammy is a vixen’ on the understanding that the latter is a conjunction of the former. Since these rule schemas describe how the complex proposition may be inferred, they are called *introduction rules*. That the introduction rule schemas *define* logical structure invokes DR. Importantly, DR explains what may be inferred from complex propositions according to their logical structure — that is, it defines their *suasive potential*. For example, having established the proposition ‘Tammy is a vixen’ one should be able to infer the propositions ‘Tammy is female’ and ‘Tammy is a fox’ for the former is defined by the latter being established. This behaviour is guaranteed by acquiescing to additional rules that warrant inferences from complex propositions. Since these rules describe what may be inferred from a complex proposition, they are called *elimination rules*.

A collection of logical structures is called a *logic*. The introduction rule schemas closed under DR (e.g., together with their corresponding elimination rule schemas) above comprise *minimal logic* — see Johansson [110].

The analysis of inference so far only concerned the *assertion* of propositions; for example, under what conditions one may assert ‘John is a bachelor’ or ‘Tammy is a vixen’ or ‘Socrates is mortal’. However, following Frege [62] — see also Geach and Black [75] — one may distinguish between the *content* of a speech act (i.e., the occurrence of a proposition) and its *force*: for example, ‘It is raining’ and ‘Is it raining?’ share a content but differ in their forces. The force dual to assertion is *denial*, which merits consideration. Recall, in the myth, Odysseus tricks the cyclops Polyphemus into believing that his name is ‘Nobody,’ so that, when blinded by the hero and his men, the monster exclaims: ‘Nobody is killing me!’ The other cyclopes, who misunderstood the force of his cry, provide no aid for their fellow monster.

The traditional account of denial advanced by Frege (*ibid.*) — and Geach [76] states that denying a proposition φ is to assert its negation $\neg\varphi$ — for example, denying ‘Tammy is a vixen’ is the same as asserting ‘Tammy is not a vixen’. That $\neg\varphi$ is a negation of φ can be interpreted in two ways:

- Weak. For any theory, if φ is a consequence of the theory, then $\neg\varphi$ is not a consequence of that theory; and conversely, if $\neg\varphi$ is a consequence of the theory, then φ is not a consequence of the theory.

- Strong. For any theory, precisely one of φ and $\neg\varphi$ is a consequence.

Of course, the strong reading of negation implies the condition defining the weak reading, justifying the taxonomy. Of course, if one assumed both φ and $\neg\varphi$, one has an absurd situation according to the weak reading. Hence, $\neg\varphi$ has the logical structure $\varphi \to \bot$ in which $\bot$ is the proposition, ‘The assumptions of this argument are absurd!’ More precisely, $\bot$ is any proposition asserting that some contradictory propositions have been asserted. The strong reading of negation also warrants the following rule known as *tertium non datur* (TND):

$$\frac{}{\varphi \vee \neg\varphi}$$

This reading of denial suggests the *disjunctive syllogism*, $\varphi \vee \psi, \neg\psi \vDash_{\mathscr{B}} \varphi$ — for example, if the proposition ‘Tammy is either a fox or a woman’ is asserted, but the proposition ‘Tammy is a woman’ is denied at the same time, then one may assert the proposition ‘Tammy is a fox’. The justification is that $\varphi \vee \psi$ means that either φ or ψ has been established, so if it cannot have been ψ (because $\neg\psi$ has been established), then it must have been φ. Of course, by the use of the introduction rule for disjunction, this suggests the following behaviour $\psi, \neg\psi \vDash_{\mathscr{B}} \varphi$. Since $\neg\varphi$ has the logical structure $\varphi \to \bot$ this justifies the following rule know as *ex falso* *quodlibet* (EFQ) — from absurdity, anything:

$$\frac{\bot}{\varphi}$$

That EFQ enables the disjunctive syllogism is witnessed by the following:

$$\dfrac{\varphi \vee \psi \quad [\varphi] \quad \dfrac{\dfrac{[\psi] \quad \neg\psi}{\bot}\Downarrow}{\varphi}\Downarrow}{\varphi}\Downarrow$$

One may think of $\bot$ as a logical structure which has the elimination rule schema of EFQ but for which there is no defining introduction rule schema. This stands in contrast with the other logical structures.

Gentzen [200] showed that minimal logic with EFQ is *intuitionistic propositional logic* (IPL), while minimal logic with EFQ and TND is *classical propositional logic* (CPL). That EFQ describes the suasive content of $\bot$ yields the following condition on support:

$$\Vdash_{\mathscr{B}} \bot \qquad \text{if and only if} \qquad \Vdash_{\mathscr{B}} \chi \text{ for any formula } \chi$$

Following Sandqvist [183], Gheorghiu et al. [78] proved that a support judgement satisfying all of the above conditions characterizes IPL in the following sense: there is a derivation for φ in IPL — that is, an argument entirely regulated by inferences warranted by the introduction and elimination rule schemas of IPL — if and only if φ is supported in the empty base. The idea that atomic propositions behave in a particular base as complex propositions do according to their logical structures is key in delivering the result.

Despite being well-motivated, this account of denial leaves something to be desired. In particular, it is subject to the Regicide Paradox — see Bennett [17]:

One evening, a man consults the oracle, who declares, 'Either there will be rain this month, or the King will die!' $(R \vee K)$. On his way home, he visits his farmer friend to relay the portentous news. The farmer is concerned by the prophecy and reminds the man of the well-known fact: ‘If there is rain this month, then the harvest will be ruined!’ $(R \to H)$. The man leaves to go home before it gets too late, but makes sure to pass by the temple to make a small offering, as is customary. At the temple, he relays his concerns to the priest, who is infallible on religious matters. The priest warns him, ‘If the harvest is ruined, then the sky gods will be angry!’ $(H \to S)$. When the man returns home, he tells his wife all about his day. She observes that since raining will make the sky gods angry, ‘Either the sky gods will be angry or the King will die!’ $(S \vee K)$ — presumably reasoning as follows:

$$\dfrac{R \vee K \qquad \dfrac{\dfrac{\dfrac{[R] \quad R \to H}{H}\Downarrow \quad H \to S}{S}\Downarrow}{S \vee K}\Downarrow \qquad \dfrac{[K]}{S \vee K}\Downarrow}{S \vee K}\Downarrow$$

After a sleepless night worried about this dilemma, the man returns to the priest for guidance. The priest tells him that if he sacrifices a goat, then ‘The sky gods will not be angry’ $(\neg S)$. The man is relieved and quickly organizes the sacrifice. At precisely the moment that the goat is killed, the sky breaks open: it begins to rain... Lo! The oracle’s prophecy has come true — that is, ‘It is raining’ (R) is true , and therefore ‘It is raining or the King will die’ $(R \vee K)$ is true. Thus, by the man’s wife’s reasoning (contingent on $R \vee K$), either the sky gods will be angry, or the King will die $(S \vee K)$. Alas! The man accuses himself of regicide: in appeasing the gods, he has forced the hands of fate since, by the disjunctive syllogism, the King must die — that is,

$$\dfrac{S \vee K \quad \neg S}{K}\Downarrow$$

Does it make sense for the man’s action to lead to the King dying? It is not the strength of the Oracle’s prophecy, which apparently ties the King’s destiny to the weather, that forces the hands of fate. The problem is that the story concerns a contradictory collection of propositions R, $R \to H$, $H \to S$, $\neg S$. Hence, by EFQ, anything can be inferred. Indeed, K may equally have been the proposition ‘The King is immortal.’ The moral of the story is that negation and denial are subtle.

Recall that $\bot$ is not like the other logical structures but rather like a punctuation mark denoting that the argument has absurd assumptions. Accordingly, Tennant [204] provides a compelling reflection on EFQ:

> In general, a proof of Ψ from Δ is suasively appropriate only if a person who believes Δ can reasonably decide, on the basis of the proof, to believe Ψ. But if the proof shows his belief set Δ to be inconsistent "on the way to proving" Ψ from Δ, then the reasonable reaction is to *suspend* belief in Δ rather than acquiesce in the doxastic inflation administered by the absurdity rule [EFQ].

This reveals that it is, perhaps, a mistake to reason forward from something that has been observed to be contradictory, which is implicit in the justification of EFQ above. Tennant (*ibid.*) suggests the following rules to more closely resemble the idea behind the disjunctive syllogism:

$$\frac{\varphi \vee \psi \quad \begin{matrix}[\varphi] \\ \bot\end{matrix} \quad \begin{matrix}[\psi] \\ \chi\end{matrix}}{\chi} \qquad \frac{\varphi \vee \psi \quad \begin{matrix}[\varphi] \\ \chi\end{matrix} \quad \begin{matrix}[\psi] \\ \bot\end{matrix}}{\chi}$$

Indeed, this says that if φ (resp. ψ) is shown to be absurd, then it cannot have been the reason that $\varphi \vee \psi$ obtained, so the reason must be ψ (resp. φ); thus (by DR), whatever follows from ψ (resp. φ) follows from $\varphi \vee \psi$. This is precisely the justification of the disjunctive syllogism above. Replacing EFQ by these rules in IPL and CPL yields *intuitionistic core logic* and *classical core logic*, respectively —see Tennant [205].

Other treatments of denial differ from the traditional view presented above — see Kürbis [126]. For example, there is the bilateralist approach advanced by Rumfitt [179] in which assertion and denial are both primitive concepts. Similarly, other readings of negation exist; in particular, Berto et al. [18] have suggested that negation is properly thought of as a *modality*, which is consistent with the idea of denial as the force of a speech act. These choices are details in the general analysis of reasoning that this essay concerns and are left behind to proceed with the thesis.

Recall that arguments are valid when they are regulated by accepted inferences. Those inferences are determined by rules, which may be given explicitly in a base or implicitly in the logical structure of the propositions involved. However, an undesirable behaviour has been introduced by admitting elimination rules for complex propositions. Recall that the intended use of the elimination is only when the proposition occurs as a hypothesis or in the theory, but such a restriction has yet to be mandated. For example, consider the following argument:

$$\dfrac{\dfrac{\text{Tammy is female} \quad \text{Tammy is a fox}}{\text{Tammy is a vixen}}\Downarrow}{\text{Tammy is a fox}}\Downarrow$$

The proposition ‘Tammy is a vixen’ is spurious because the suasive content of the elimination rule has already been given, *ipso facto*, in the proposition being established by an introduction rule. The purpose of the elimination rule is to satisfy the definitionality of the corresponding introduction rule according to DR. Therefore, they apply only when the propositions they concern appear as assumptions, and not in general. Hence, though the argument is regulated by accepted inferences, its validity comes from the fact that it represents a *direct* argument that encapsulates its actual suasive content.

An argument is *indirect* if it has a proposition in it that is concluded by an instance of an introduction rule schema and then appears in the corresponding elimination rule schema; otherwise, it is *direct*. The intended use of elimination rules described above amounts to the following principle:

> An indirect argument is valid in a base $\mathscr{B}$ if it *represents* a direct argument in the base $\mathscr{B}$

The significance is that it avoids detours (i.e., inferences devoid of suasive content); for example, $\varphi, \psi, \psi \to \chi \vDash_{\mathscr{B}} \chi$ is witnessed by the following:

$$\dfrac{\dfrac{\varphi \qquad \dfrac{\psi \to \chi \quad \psi}{\chi}\Downarrow}{\varphi \land \chi}\Downarrow}{\chi}\Downarrow$$

The conjunction is a detour because it is eliminated after having been introduced according to its logical structure. It may be removed without loss of suasive content to yield the following direct argument:

$$\frac{\psi \to \chi \quad \psi}{\chi} \Downarrow$$

When does one argument represent another? One may impose various conditions, but they are not the subject of this essay. Instead, take some set of operator $\mathcal{J}$ to be fixed, allowing one to map arguments to others, such that when a mapping takes place, the source 'represents' the target. For example, Prawitz [168] provides a collection of 'reductions' that remove detours in the above sense (i.e., when an elimination rule is applied to a complex proposition asserted by an introduction rule).

The ideas presented so far comprise an inductive definition of valid argument. Let $\mathscr{B}$ be a base and $\mathcal{J}$ a collection of operators on arguments. An argument $\mathcal{A}$ is $(\mathscr{B},\mathcal{J})$-valid if and only if it satisfies one of the following:

- the argument $\mathcal{A}$ is a proof in $\mathscr{B}$
- the argument $\mathcal{A}$ is closed and reduces according to $\mathcal{J}$ to a direct $(\mathscr{B},\mathcal{J})$-valid argument
- the argument $\mathcal{A}$ is open with assumptions $P_1,...,P_n$ and, for any $\mathscr{C}$ extending $\mathscr{B}$, the result of composing $\mathcal{A}$ with arbitrary $(\mathscr{C},\mathcal{J})$-valid arguments $\mathcal{A}_1$,...,$\mathcal{A}_n$ for $P_1,...,P_n$, respectively, results in a $(\mathscr{C},\mathcal{J})$-valid argument.

The definition of valid was first given by Dummett [51], based on the technical achievements and ideas of Prawitz [168, 164], and is thus known as the Dummett-Prawitz proof-theoretic validity — see Schroeder-Heister [190]. Significantly, restricting attention to IPL, Prawitz [166] has conjectured the following: if there is a valid argument for φ from Γ in an arbitrary base (relative to some notion of reduction), then there is a derivation of $\gamma \to \varphi$ in IPL, where γ is the conjunction of all the propositions in Γ. This remains an open problem.

This essay concerns the question ‘What is a valid argument?’ and presents it as a foundation of logic. The typical logics of interest — IPL and CPL — are derived according to principled design choices. Significantly, this essay differs from the traditional approach to logic in that logical structures (e.g., conjunction, disjunction, implication, absurdity) are understood as meta-level expressions of the inferential behaviour of propositions instead of grammatical constructions. The main idea is that logical structures are determined by stating that a proposition is ‘defined’ according to a certain inferential behaviour — that is, its functions in inferences — thereby setting this approach in the philosophical paradigm of inferentialism. The analysis of this essay recovers proof-theoretic validity in Dummett-Prawtiz tradition. It also explicates how the *base-extension semantics* for IPL given by Sandqvist [183] naturally arises according to the central question. While only a limited selection of logics are treated, this results from the limited form of rules chosen. Indeed, if the notion of the rule were generalized, one could analogously recover modal, substructural, first-, second-, and higher-order logics. The overall thesis is that the validity of arguments is the underlying question of logic, so logic may properly be called the study of reasoning.