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On tree decompositions whose trees are minors

Pablo Blanco[1] | **Linda Cook**[2] | **Meike Hatzel**[3] | **Claire Hilaire**[4] | **Freddie Illingworth**[5] | **Rose McCarty**[1]

[1]Department of Mathematics, Princeton University, Princeton, New Jersey, USA

[2]Discrete Math Group, Institute for Basic Science, Daejeon, Republic of Korea

[3]National Institute of Informatics, Tokyo, Japan

[4]LaBRI, Université de Bordeaux, Bordeaux, France

[5]Department of Mathematics, University College London, London, UK

Correspondence
Freddie Illingworth, Department of Mathematics, University College London, London, UK.
Email: f.illingworth@ucl.ac.uk

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Abstract

In 2019, Dvořák asked whether every connected graph G has a tree decomposition $(T, \mathcal{B})$ so that T is a subgraph of G and the width of $(T, \mathcal{B})$ is bounded by a function of the treewidth of G. We prove that this is false, even when G has treewidth 2 and T is allowed to be a minor of G.

KEYWORDS

minors, tree decomposition, treewidth

1 | INTRODUCTION

Suppose that a graph G has small treewidth, and consider all tree decompositions $(T, \mathcal{B})$ of G whose width is not too much larger than the optimum. To what extent can we choose or manipulate the "shape" of T?

For graphs with no long path, we can choose T to also have no long path [15][*]; this gives rise to the parameter called *treedepth*. Similarly, for graphs of bounded degree, we can choose T to also have bounded degree [7]; this relates to the parameters of *congestion* and *dilation*. Moreover, for graphs excluding any tree as a minor, we can choose T to just be a path [1]; this results in the parameter called *pathwidth*.

It would be wonderful if we could unify all such results into a single theorem which relates the shape of T to G. In 2019, Dvořák suggested one way of accomplishing this goal. In the question below and throughout the paper, we write $\text{tw}(G)$ for the treewidth of G.

Question 1 (Dvořák [8]). Does there exist a polynomial P such that every connected graph G has a tree decomposition $(T, \mathcal{B})$ of width at most $P(\text{tw}(G))$ such that T is a subgraph of G?

Unfortunately, we prove that the answer to Question 1 is "no" in the following strong sense.

Theorem 2. *For every positive integer k, there is a connected graph G of treewidth 2 such that if $(T, \mathcal{B})$ is a tree decomposition of G and T is a minor of G, then $(T, \mathcal{B})$ has width at least k.*

Intriguingly, in our proof of Theorem 2, it seems crucial that the constructed graphs contain all trees as minors; perhaps Question 1 could be true when $\text{tw}(G)$ is replaced by $\text{pw}(G)$, the pathwidth of G. In other words, perhaps there exists a polynomial (or even just some function) P so that every connected graph G has a tree decomposition $(T, \mathcal{B})$ of width at most $P(\text{pw}(G))$ such that T is a subgraph of G. We leave this as an open problem. Another interesting question raised by one of the reviewers is whether Question 1 can be answered in the positive if one replaces treewidth with *connected treewidth*, that is, demands the subgraphs $G[\{v \in V(G) : x \in T_v\}]$ to be connected.

There has been strong interest in obtaining good bounds for treedepth [5, 10, 13], pathwidth [9], and treewidth [3, 4] as a function of the natural obstructions (which are paths, trees, and grids, respectively[†]). These problems were in large part motivated by the desire to obtain better approximation algorithms and better win–win algorithms based on the obstructions. An affirmative answer to Question 1 would have unified these approaches, but unfortunately Theorem 2 shows that this is not possible.

There has also been recent interest in finding the 2-connected obstructions for treedepth [2] and pathwidth [6, 12] in 2-connected graphs. It seems unlikely that requiring G to be 2-connected would change the answer to Question 1, but the graphs we construct for Theorem 2 are not 2-connected, thus leaving this as an open possibility.

We present a self-contained proof of Theorem 2, however some steps were discovered independently by Hickingbotham [11]. In particular, Hickingbotham [11, Lemma 7.2.1] noticed that it is just as hard to ensure T is a subgraph of G in Question 1 as it is to ensure T is a minor of G. Thus our main contribution is Lemma 3.3, which essentially shows that we can also force each vertex of T to be in its own bag. Hickingbotham [11, Theorem 7.5.1] already proved that this stronger condition can blow up the width. Moreover, Hickingbotham proved some positive results, including that the answer to Question 1 is "yes" if G is an outerplanar graph

[*]The reference gives us an elimination tree F of G of depth k. Then we can obtain a tree decomposition $(F, \mathcal{B})$ of G of width k by letting the bag of each vertex $v \in V(F)$ be the set of all ancestors of v in F.

[†]Formally, a class of graphs has bounded treedepth/pathwidth/treewidth if and only if it does not contain all paths/trees/grids as minors, respectively. See [15, 1, 16] for the respective proofs. Note that sometimes the obstructions are considered as subgraphs or subdivisions rather than minors. This occurs when the two definitions are equivalent, for instance, when considering paths as minors (or equivalently as subgraphs).

[11, Theorem 7.3.3]. Note that outerplanar graphs are the graphs with simple treewidth at most 2 [14], and so in this sense Theorem 2 is optimal.

We outline our approach to proving Theorem 2 in more detail in the next section.

2 | PRELIMINARIES

We use the following "subtree view" of tree decompositions. Recall that a *subtree* of a graph G is any subgraph of G which is connected and acyclic.

Lemma 2.1. *Let G be a graph, let T be a tree, and let $\mathcal{B} = \{B_x : x \in V(T)\}$ be a family of subsets of $V(G)$ indexed by the vertices of T. For each vertex v of G, we define*

$$T_v := T[\{x : v \in B_x\}].$$

Then $(T, \mathcal{B})$ is a tree decomposition *of G if and only if the following conditions both hold.*

- Each T_v is a nonempty subtree of T.
- If $uv \in E(G)$, then $V(T_u) \cap V(T_v) \neq \emptyset$.

We use this notation T_v throughout the paper. When there is no chance for confusion, we refer to T_v and its vertex set $V(T_v)$ interchangeably. The *width* of $(T, \mathcal{B})$ is then the maximum, over all $x \in V(T)$, of $|\{v \in V(G) : x \in T_v\}| - 1$. The *treewidth* of G is the minimum width of a tree decomposition of G. Note that, if we are given a tree T and a collection $(T_v : v \in V(G))$ of subtrees of T which satisfy the conditions from Definition 2.1, then we can define a tree decomposition $(T, \mathcal{B})$ of G by setting $B_x := \{v \in V(G) : x \in T_v\}$ for each $x \in V(T)$.

We now outline our overall strategy for proving Theorem 2. This theorem equivalently says that Conjecture 2.2 is false, even for connected graphs of treewidth 2. We disprove Conjecture 2.2 by reducing each of the three conjectures below to the next one, and then disproving the final conjecture. Afterwards, we evaluate the treewidth of the constructed counterexamples more carefully. Note that in Conjecture 2.4, the condition "for every vertex v of G, $v \in T_v$" is equivalent to "for every vertex x of T, $x \in B_x$".

Conjecture 2.2. *There is a function f such that every connected graph G has a tree decomposition $(T, \mathcal{B})$ of width at most $f(\text{tw}(G))$ such that T is a minor of G.*

Conjecture 2.3. *There is a function f such that every connected graph G has a tree decomposition $(T, \mathcal{B})$ of width at most $f(\text{tw}(G))$ such that T is a spanning tree of G.*

Conjecture 2.4. *There is a function f such that every connected graph G has a tree decomposition $(T, \mathcal{B})$ of width at most $f(\text{tw}(G))$ such that T is a spanning tree of G and, for every vertex v of G, we have $v \in T_v$.*

Hickingbotham proved that Conjecture 2.2 implies Conjecture 2.3 in [11, Lemma 7.2.1]. In Section 3, we show that Conjecture 2.3 implies Conjecture 2.4; this crucial new step is our main contribution. Finally, in Section 4, we construct a graph that does not satisfy Conjecture 2.4.

Hickingbotham [11, Theorem 7.5.1] independently discovered a counterexample to Conjecture 2.4 which actually contains our counterexample. However, we include ours since it is slightly simpler and makes the paper self-contained. We now conclude this section by providing a short proof that Conjecture 2.2 (where T is a minor) implies Conjecture 2.3 (where T is a spanning tree), for the sake of completeness.

Lemma 2.5. *If G is a connected graph with a tree decomposition $(T, \mathcal{B})$ of width k such that T is a minor of G, then there exists a tree decomposition $(T', \mathcal{B}')$ of G of width k such that T' is a spanning tree of G.*

Proof. Since T is a minor of G, there exists a collection $(Q_x : x \in V(T))$ of pairwise disjoint nonempty subtrees of G such that, for each edge $xy \in E(T)$, there exists an edge $e_{xy} \in E(G)$ with one end in $V(Q_x)$ and the other end in $V(Q_y)$. Since G is connected, we may assume that $V(G) = \cup_{x \in V(T)} V(Q_x)$. Now, let T' be the spanning tree of G which is obtained from $\cup_{x \in V(T)} Q_x$ by adding the edge e_{xy} for all $xy \in E(T)$.

For each $v \in V(G)$, let T'_v be the subtree of T' which is induced by the union of all sets $V(Q_x)$ such that $x \in T_v$. This collection of subtrees of T' satisfies the conditions of Definition 2.1 and therefore yields a tree decomposition $(T', \mathcal{B}')$. Furthermore, this tree decomposition has the same width at $(T, \mathcal{B})$, which completes the proof. □

3 | REDUCTION TO CONJECTURE 2.4

In this section we show that Conjecture 2.3 (where T is a spanning tree) implies Conjecture 2.4 (where, additionally, for every vertex v of G, we have $v \in T_v$).

We use the following well-known fact about tree decompositions of paths. We include a proof for the sake of completeness. The bounds are not optimal; we aim for simplicity instead.

Lemma 3.1. *For any positive integers h and k, if P is a path with at least $(k+2)^h$ vertices and $(T, \mathcal{B})$ is a tree decomposition of P of width at most k, then T contains a path of length h.*

Proof. We consider a tree decomposition $(T, \mathcal{B})$ where T is rooted at an arbitrary vertex $r \in V(T)$. The *height* of T is then the maximum length of a path which has r as one of its ends. For fixed k, we prove by induction on h that, under the same hypothesis, we actually obtain the following stronger conclusion: that the height of T is at least h.

The base case of $h = 1$ holds since P has more than $k + 1$ vertices and therefore $(T, \mathcal{B})$ has more than one bag. So we may assume that $h > 1$ and the claim holds for $h - 1$. Observe that we can partition $V(P)$ into $k + 2$ sets, each of which induces in P a path with at least $(k+2)^{h-1}$ vertices. Since $(T, \mathcal{B})$ has width at most k, one of these sets is disjoint from the root bag B_r. Thus, by the inductive hypothesis, one of the components of $T - \{r\}$, when rooted at its neighbour of r, has height at least $h - 1$. So T has height at least h, as desired. □

We use the following construction to show that Conjecture 2.3 implies Conjecture 2.4. Given a positive integer k, a graph G, and an arbitrary ordering of the vertices of G, we define a new graph denoted $\widetilde{G}$. This graph $\widetilde{G}$ is obtained from G by attaching one rooted tree to each vertex of G; so $\widetilde{G}$ has the same treewidth as G (unless $E(G) = \varnothing$). Moreover, in Lemma 3.3, we prove that if $\widetilde{G}$ satisfies Conjecture 2.3 with a tree decomposition of width k, then G satisfies Conjecture 2.4 with a tree decomposition of width $k + 1$.

The attached trees are chosen such that no two have "comparable" tree decompositions. More formally, given two trees T_1 and T_2, there is no tree decomposition $(T'_2, \mathcal{B}'_2)$ of T_2 of width k such that T'_2 is a subgraph of T_1, and likewise with the roles of T_1 and T_2 reversed. We do not frame the argument in this way, but it is the underlying reason our proof works. We accomplish this condition by, up to symmetry between T_1 and T_2, making height of T_2 much larger than the height of T_1, and the width of T_1 much larger than $|V(T_2)|$. See Figure 1 for a depiction.

With this intuition, we are ready to state the main definition.

Definition 3.2. Fix a positive integer k, a graph G, and an arbitrary ordering $a_1, \ldots, a_n$ of the vertices of G. Then let $\widetilde{G}$ be the graph which is constructed from G as follows.

- First define integers $2 = h_1 \ll h_2 \ll \cdots \ll h_n$ as follows. Given h_{j-1}, we define $h_j := (k+2)^{2h_{j-1}} + 1$. Thus, by Lemma 3.1, if P is a path on at least $h_j - 1$ vertices and $(T, \mathcal{B})$ is a tree decomposition of P of width at most k, then T contains a path of length $2h_{j-1}$.
- Next define integers $(k+1)n + 1 = w_n \ll w_{n-1} \ll \cdots \ll w_1$ and corresponding rooted trees $S_n, S_{n-1}, \ldots, S_1$ as follows. Given w_j, define S_j to be the complete rooted w_j-ary tree of height h_j. Then, given $w_n, w_{n-1}, \ldots, w_{j+1}$ and $S_n, S_{n-1}, \ldots, S_{j+1}$, define

$$w_j := (k+1)\left(n + \sum_{i=j+1}^{n} |V(S_i)|\right) + 1.$$

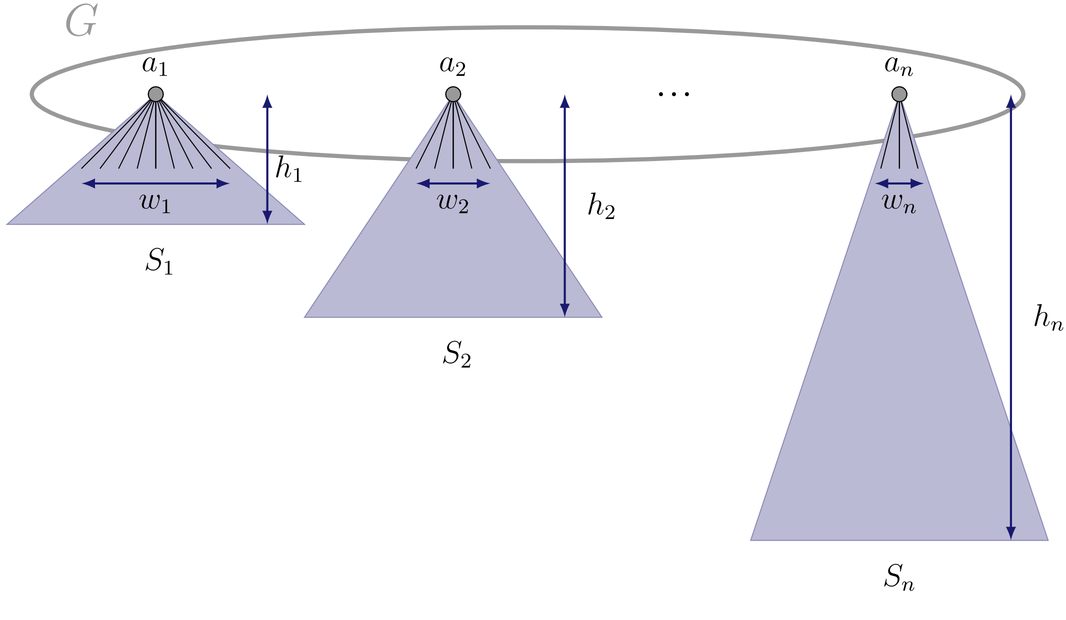

FIGURE 1 The graph $\widetilde{G}$ which is obtained from G by attaching the complete w_j-ary tree S_j of height h_j to each vertex $a_j \in V(G)$. [Color figure can be viewed at wileyonlinelibrary.com]

Finally, let $\widetilde{G}$ be the graph which is obtained from the disjoint union of $G, S_1, S_2, \ldots, S_n$ by, for each $j \in \{1, 2, \ldots, n\}$, identifying a_j with the root of S_j.

Note that the graph $\widetilde{G}$ from Definition 3.2 can be obtained from G by adding pendant vertices one at a time. It follows that $\mathrm{tw}(\widetilde{G}) = \max(\mathrm{tw}(G), 1)$. The next key lemma therefore completes the reduction from Conjecture 2.3 to Conjecture 2.4.

Lemma 3.3. *Let k be a positive integer, let G be a connected graph, let $a_1, \ldots, a_n$ be an ordering of the vertices of G, and let $\widetilde{G}$ be the resulting graph constructed using Definition 3.2. Suppose that $\widetilde{G}$ has a tree decomposition $(T, \mathcal{B})$ of width at most k such that T is a spanning tree of $\widetilde{G}$.*

Then there exists a tree decomposition $(T', \mathcal{B}')$ of G of width at most $k + 1$ such that T' is a spanning tree of G and for every $v \in V(G)$, we have $v \in T'_v$.

Proof. We use the notation introduced in Definition 3.2, except that we view each tree S_j as an induced subgraph of $\widetilde{G}$ which is rooted at the vertex $a_j \in V(G)$. For the sake of convenience, we do not distinguish between S_j and its vertex set.

In the first few claims we deduce roughly where each subtree T_v (as defined in Definition 2.1) lies. We say that two sets *meet* if their intersection is nonempty.

Claim 3.3.1. For every $j \in \{1, 2, \ldots, n\}$ and every nonleaf vertex v of S_j, the set T_v meets $(S_1 \cup \cdots \cup S_j) \backslash V(G) = (S_1 \cup \cdots \cup S_j) \backslash \{a_1, \ldots, a_n\}$.

Proof. Consider the union of the bags of $(T, \mathcal{B})$ that contain v. Each bag has size at most $k + 1$, so this union has size at most $(k + 1)|V(T_v)|$. On the other hand, this union contains every neighbour of v in $\widetilde{G}$ and so, by the choice of w_j,

$$(k + 1)|V(T_v)| \geqslant \deg_{\widetilde{G}}(v) \geqslant \deg_{S_j}(v) \geqslant w_j > (k + 1)\left(|V(G)| + \sum_{i=j+1}^{n} |V(S_i)|\right).$$

In particular, T_v is not a subgraph of $S_{j+1} \cup \cdots \cup S_n \cup G$. The claim follows. □

We say that a vertex v of $\widetilde{G}$ is *free* if T_v meets $V(G)$ or, equivalently, if $v \in B_{a_1} \cup \cdots \cup B_{a_n}$. Otherwise, we call v *constrained*. Note that if v is constrained, then T_v is a subgraph of some $S_j - a_j$ since T_v is a subtree of $\widetilde{G}$. The number of free vertices is at most

$$|B_{a_1} \cup \cdots \cup B_{a_n}| \leqslant (k + 1)n,$$

and so almost all vertices are constrained.

Claim 3.3.2. For every $j \in \{1, 2, \ldots, n\}$, the vertex a_j has a child b_j in S_j such that T_{b_j} is a subgraph of $S_j - a_j$.

Proof. As S_j is a complete w_j-ary tree of height h_j, there are w_j vertex-disjoint paths that start at the children of a_j and end at parents of leaves of S_j. Since $w_j > (k+1)n$, at least one of these paths contains no free vertices – call this path P. Let $T_P = \cup_{v \in V(P)} T_v$. So T_P is a subtree of $\widetilde{G}$. Each $v \in V(P)$ is constrained, and so for each v there is an i such that T_v is a subgraph of $S_i - a_i$. But, as T_P is connected and there is no edge between different $S_i - a_i$, this i must be the same for all $v \in V(P)$. That is, there is some i such that T_P is a subgraph of $S_i - a_i$. Let b_j be the child of a_j that is a vertex of P (since $h_1 \geqslant 2$, such a b_j exists).

By Claim 3.3.1, we have that T_{b_j} meets $(S_1 - a_1) \cup \cdots \cup (S_j - a_j)$. Since T_{b_j} is a subgraph of T_P, we have $i \leqslant j$. Next focus on the tree decomposition $(T_P, \mathcal{B}_P)$ of P where $\mathcal{B}_P$ is $\mathcal{B}$ restricted to the vertices of P. This tree decomposition has width at most k. The path P contains $h_j - 1$ vertices and so, by the choice of h_j and Lemma 3.1, the tree T_P must contain a path of length at least $2h_{j-1}$. However, T_P is a subgraph of $S_i - a_i$ whose longest paths have length less than $2h_i$. In particular, this implies that $h_i > h_{j-1}$ and so $i \geqslant j$. Thus $i = j$ and b_j is as required. □

We say that a vertex $a_j \in V(G)$ is *grounded* if T_{a_j} contains a_j.

Claim 3.3.3. If a vertex $a_j \in V(G)$ is not grounded, then T_{a_j} is a subgraph of $S_j - a_j$ and every neighbour $a_i \in V(G)$ of a_j is both grounded and satisfies $a_j \in T_{a_i}$.

Proof. Assume that $a_j \in V(G)$ is not grounded. Then $a_j \notin T_{a_j}$. Let b_j be the child of a_j given by Claim 3.3.2. As a_j and b_j are adjacent, T_{a_j} meets T_{b_j}. But T_{b_j} is a subgraph of $S_j - a_j$, and so T_{a_j} meets $S_j - a_j$. However, T_{a_j} is connected and does not contain a_j, so T_{a_j} must be a subgraph of $S_j - a_j$ as well.

Let $a_i \in V(G)$ be a neighbour of a_j. Suppose that a_i is not grounded, then T_{a_i} is a subgraph of $S_i - a_i$. But then T_{a_i} and T_{a_j} do not meet, which is impossible as a_i and a_j are adjacent. Thus a_i is grounded. Now T_{a_i} and T_{a_j} meet and so T_{a_i} meets $S_j - a_j$. So, since T_{a_i} is connected, T_{a_i} contains a_j. □

We now define a tree decomposition of G which satisfies Lemma 3.3. First, let T' be the subgraph of T induced by $V(G)$; notice that T' is a spanning tree of G since T is a spanning tree of $\widetilde{G}$. Next, delete all bags B_x where $x \notin V(T')$ and delete all vertices of $\widetilde{G}$ that are not vertices of G. Finally, if a vertex a_j is not grounded, then add a_j to the bag B_{a_j}. Call the resulting collection of bags $\mathcal{B}' = (B'_{a_j})_{1 \leqslant j \leqslant n}$. We claim that $(T', \mathcal{B}')$ is a tree decomposition of G. This completes the proof of Lemma 3.3 since $(T', \mathcal{B}')$ has width at most $k+1$, T' is a spanning tree of G, and for every $a_j \in V(G)$, we have $a_j \in T'_{a_j}$.

Notice that if a vertex $a_j \in V(G)$ is grounded, then T'_{a_j} is just the induced subgraph of T_{a_j} restricted to $V(G)$; so T'_{a_j} is still connected. Likewise, if a_j is not grounded, then by Claim 3.3.3, $T'_{a_j} = \{a_j\}$ is connected. We are left to check that for every edge $a_i a_j \in E(G)$, the subtrees T'_{a_i} and T'_{a_j} meet. First suppose that a_j is not grounded. Then, by Claim 3.3.3, T'_{a_i} and T'_{a_j} both contain the vertex a_j. The case that a_i is not grounded is symmetric, so we may assume that both a_i and a_j are grounded. As a_i and a_j are adjacent in $\widetilde{G}$, the trees T_{a_i} and T_{a_j} meet in $\widetilde{G}$. If they meet in $V(G)$, then so do T'_{a_i} and T'_{a_j} as desired. So let

$\ell \in \{1, 2, ..., n\}$ be such that T_{a_i} and T_{a_j} meet in S_ℓ. Now T_{a_i} contains a_i and is connected and T_{a_j} contains a_j and is connected, so T_{a_i} and T_{a_j} both contain a_ℓ. So both T'_{a_i} and T'_{a_j} contain a_ℓ, as required. This completes the proof of Lemma 3.3. □

4 | CONSTRUCTION

In this section we disprove Conjecture 2.4 and then combine the previous reductions to prove Theorem 2.

We begin by defining the relevant graphs. Then we prove that they are counterexamples in Lemmas 4.2 and 4.3.

Definition 4.1. The *second reflected-tree* G_2 is the cycle of length four with vertices v_1, v_2, v_3, v_4, v_1, in order. We call v_1, v_3 the *roots* of G_2. Then, for any positive integer $r \geqslant 3$, the *rth reflected-tree* G_r is constructed recursively as follows:

- Let H and H' be two disjoint copies of G_{r-1}, and let u and v be two new vertices, which we call the *root* vertices of G_r. To construct G_r, we start with H and H', then make u adjacent to a root vertex of H and a root vertex of H'. Finally, we make v adjacent to the remaining root vertex of H and the remaining root vertex of H'. See Figure 2 for a depiction.

Now we prove a lemma about the spanning trees of the reflected-tree. Whenever T is a spanning tree of a graph G, we denote the fundamental cycle of an edge $e \in E(G)\backslash E(T)$ with respect to T by C_T^e; thus C_T^e is the unique cycle in the graph obtained from T by adding e.

Lemma 4.2. *For any integer $r \geqslant 2$ and any spanning tree T of G_r, there is a matching $M \subseteq E(G_r)\backslash E(T)$ of size $r - 1$ such that*

$$\bigcap_{e \in M} V\left(C_T^e\right) \neq \varnothing.$$

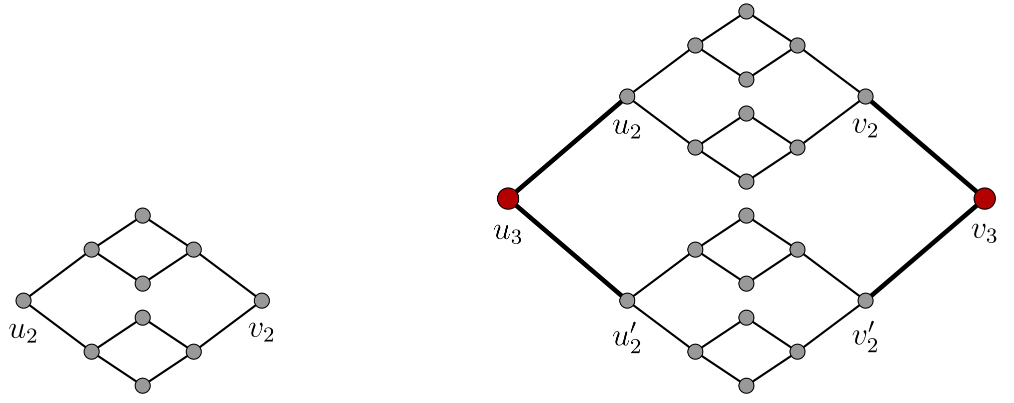

FIGURE 2 The fourth reflected-tree G_4 (right, with root vertices larger and in red) being constructed from the third reflected-tree G_3 (left). [Color figure can be viewed at wileyonlinelibrary.com]

Proof. Let u and v be the root vertices of G_r, and denote the path between them in T by P_{uv}. Under the same conditions, we prove the following stronger outcome holds by induction:

$$\bigcap_{e \in M} E\left(C_T^e\right) \cap E(P_{uv}) \neq \emptyset$$

for some matching $M \subseteq E(G_r) \backslash E(T)$ of size $r-1$. For the base case of $r=2$, the graph G_r is a cycle on four vertices; then any spanning tree T of G_2 is a path on four vertices, and we can take M to be the 1-edge matching $E(G_2) \backslash E(T)$.

Next, we may assume that $r > 2$ and the claim holds for $r-1$. By definition, $G_r - \{u, v\}$ has exactly two connected components both of which are isomorphic to G_{r-1}. We denote these components by H and H'. Exactly one of $T_H := T[V(H) \cup \{u, v\}]$ and $T_{H'} := T[V(H') \cup \{u, v\}]$ is connected in T; without loss of generality, we assume that T_H is connected in T. We can apply the inductive hypothesis on $T[V(H)]$, which is a spanning tree of H, to find a matching $M_H \subseteq E(H) \backslash E(T[V(H)])$ of size $r-2$ with $\bigcap_{e \in M_H} E(C_T^e) \cap E(P_{uv} - \{u, v\}) \neq \emptyset$. The other subgraph $T_{H'}$ is not connected. In fact, it contains exactly two components: one containing u, and the other containing v. Thus there exists an edge $e' \in E(G_r) \backslash E(T)$ with one end in each of these two components of $T_{H'}$. Observe that e' lies in $G[V(H') \cup \{u, v\}]$ which is vertex-disjoint from H. Thus $M_H \cup \{e'\}$ is a matching since $M_H \subseteq H$.

For convenience, let us define $M := M_H \cup \{e'\}$. M is a matching of size $r-1$, and we have $E(P_{uv}) \subseteq E(C_T^{e'})$. From here, it follows that

$$\bigcap_{e \in M} E\left(C_T^e\right) \cap E(P_{uv}) \neq \emptyset.$$

Thus M is our desired matching. □

We are now ready to prove the following lemma, which shows that reflected-trees are a counterexample to Conjecture 2.4.

Lemma 4.3. *For every $k \in \mathbb{N}$, if $(T, \mathcal{B})$ is a tree decomposition of G_{k+2} such that T is a spanning tree of G_{k+2} and, for every $v \in V(G_{k+2})$, we have $v \in T_v$, then the width of $(T, \mathcal{B})$ is at least k.*

Proof. We begin by finding a matching $M := \{u_1v_1, \ldots, u_{k+1}v_{k+1}\} \subseteq E(G_{k+2}) \backslash E(T)$ of size $k+1$ satisfying the properties in Lemma 4.2. Let $x \in \bigcap_{e \in M} V(C_T^e)$. By construction, x is in the path $P_{u_iv_i}$ between u_i and v_i in T for every $i \in \{1, \ldots, k+1\}$. From Definition 2.1, the trees T_{u_i} and T_{v_i} meet; furthermore, since T_{u_i} and T_{v_i} are connected in T, with $u_i \in V(T_{u_i})$ and $v_i \in V(T_{v_i})$, we find that every vertex of $P_{u_iv_i}$ is in $V(T_{u_i}) \cup V(T_{v_i})$. As a result, $x \in V(T_{u_i}) \cup V(T_{v_i})$. That is, $u_i \in B_x$ or $v_i \in B_x$ for all $i \in \{1, \ldots, k+1\}$. Since M is a matching, we have that $|B_x| \geqslant k+1$ and the width of $(T, \mathcal{B})$ is at least k. □

We are now ready to prove the main theorem, which is restated below for convenience.

Theorem 2. *For every positive integer k, there is a connected graph G of treewidth 2 such that if $(T, \mathcal{B})$ is a tree decomposition of G and T is a minor of G, then $(T, \mathcal{B})$ has width at least k.*

Proof. For convenience, we fix an integer $k \geqslant 2$. Now consider the $(k+3)$rd reflected-tree G_{k+3}. Let $a_1, \ldots, a_n$ be an arbitrary ordering of $V(G_{k+3})$, and let $\widetilde{G}_{k+3}$ be the graph obtained from the integer $k-1$, the graph G_{k+3}, and the ordering $a_1, \ldots, a_n$ by applying Definition 3.2.

We now prove that $\widetilde{G}_{k+3}$ satisfies the conditions of Theorem 2. First, recall that $\widetilde{G}_{k+3}$ has treewidth equal to $\max(\text{tw}(G_{k+3}), 1)$. Moreover, $\text{tw}(G_{k+3}) = 2$ since G_{k+3} is series parallel and not a tree. Thus $\widetilde{G}_{k+3}$ is a connected graph of treewidth 2, as desired.

Next, suppose towards a contradiction that $\widetilde{G}_{k+3}$ has a tree decomposition $(T, \mathcal{B})$ of width at most $k-1$ such that T is a minor of $\widetilde{G}_{k+3}$. Since $\widetilde{G}_{k+3}$ is connected, Lemma 2.5 says that $\widetilde{G}_{k+3}$ has a tree decomposition $(T', \mathcal{B}')$ of width at most $k-1$ such that T' is a spanning tree of $\widetilde{G}_{k+3}$. Thus, since G_{k+3} is connected, Lemma 3.3 says that G_{k+3} has a tree decomposition $(T'', \mathcal{B}'')$ of width at most k such that T'' is a spanning tree of G_{k+3} and for every $v \in V(G_{k+3})$, we have $v \in T''_v$. However, Lemma 4.3 also says that $(T'', \mathcal{B}'')$ has width at least $k+1$. This contradiction completes the proof of Theorem 2. □

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ORCID

Pablo Blanco http://orcid.org/0009-0003-9894-9942
Linda Cook https://orcid.org/0000-0002-1516-5874
Meike Hatzel http://orcid.org/0000-0003-3249-1169
Claire Hilaire http://orcid.org/0009-0006-0826-0001
Freddie Illingworth http://orcid.org/0000-0001-5350-2379
Rose McCarty http://orcid.org/0000-0002-9884-3406

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