Droplet impact onto engineered surfaces and related problems

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I, Nathaniel Isaac John Henman, confirm that the work presented in this thesis is my own. Where information has been derived from other sources, I confirm that this has been indicated in the work.
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Abstract

Droplet impact is a commonly occurring phenomenon in many industrial and natural scenarios and is the subject of a substantial amount of research. The use of surface engineering to control the dynamics of droplet impact is hugely important in today’s world. This thesis tackles various aspects of droplet impact onto two types of engineered surfaces: deformable surfaces and lubricant-infused surfaces.

First, the pre-impact phase of droplet impact is analysed by using well-established air cushioning models coupled with a general deformable surface model. This system is solved numerically, highlighting a number of key consequences of surface deformability on the pre-impact phase of droplet impact, such as reduction in pressure buildup, increased air entrapment, and considerable delay to touchdown. Connections (including subtle dependence of the size of entrapped air on the droplet velocity, reduced pressure peaks, and droplet gliding) with recent experiments and a large deformation analysis are also presented. The pre-impact behaviour of droplet impact onto lubricant-infused surfaces is also investigated by modifying the surface boundary condition to account for velocity slip.

Next, the post-impact phase of droplet impact onto lubricant-infused surfaces is investigated using direct numerical simulation. Using an idealised two-dimensional model of a surface made up of rectangular pillars with a thin layer of lubricant atop, the delicate early stages of impact are analysed, focusing mainly on the extension of the thin splash jet at impact. Results are also compared with solutions for textured, superhydrophobic surfaces, highlighting the key role surface topology plays.

Finally, two related problems to the impact scenarios are studied. The first
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is the horizontal spread of a liquid jet over a lubricant-infused surface. Using a boundary layer approximation, this scenario is modelled for two different length scales, one where the surface can be assumed flat with velocity slip at the interface, and another where the flow in the lubricant and the jet are coupled. Secondly, we study the deformation of a droplet suspended in a uniform flow using a small density ratio approximation. Two temporal regimes are identified and models pertaining to each are presented.

The results presented in this thesis will aid in the design of novel non-wetting surfaces for impact scenarios and beyond, as well as highlight possible avenues for further research.
Impact Statement

The study of droplet impact has been around since the late 19th century. Researchers from a variety of different academic and industrial disciplines have put an enormous amount of work into understanding the core dynamics of droplet impact, as well as making huge advancements in the design of new surfaces that are able to significantly alter the impact dynamics. The development of new non-wetting surfaces is particularly important in the aviation industry, where anti-icing technologies need to be implemented to guard against accruing ice on the aircraft wings after the impact of droplets suspended in clouds. Reducing the amount of ice that builds up on an aircraft has a wide range of implications, from increasing efficiency and resulting improved climate footprint, to ultimately enhancing safety and saving lives.

Over the last few decades, research methods on droplet impact have evolved into a three-pronged approach: experimental, theoretical and simulation. Experimental conclusions on droplet impact are the gold standard, however theoretical and simulation techniques are increasingly utilised to fill in the gaps to the story that experimental methods cannot access, such as certain time and length scales that elucidate vital dynamics. Analytical methods allow researchers to make informed predictions and decisions when analysing scenarios experimentally and are a cost-effective way of investigating complex geometries and large parameter spaces.

To that end, the impact that this thesis has is two-fold. Firstly, through two-dimensional modelling of complex droplet impact scenarios with engineered surfaces, it makes qualitative assessments of the effect of variations in parameters of the surface and the physical problem. Metrics such as pre-impact air entrapment and splashing are quantified and optimal parameter ranges and values are discussed
as well as their implications. Secondly, this body of work will act as a basis for further research in this area. Various results in our two-dimensional work point to interesting scenarios that may benefit greater from deeper investigation using three-dimensional modelling and/or complex analytical or computational methods.
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Chapter 1

Introduction

Extracts and some figures in this thesis have been taken from previously published work in Henman et al. [2, 3].

1.1 Background and motivation

This thesis concerns various different novel problems regarding liquid impact onto engineered surfaces, namely deformable and lubricant-infused surfaces, as well as a study of the deformation of liquid droplets when subjected to a uniform flow. Each of these problems has a wide variety of industrial applications, with the overarching theme in all of them being the aviation industry. Throughout an aircraft’s flight it is subjected to a number of different liquid impacts and resulting thin film flows across its array of different surfaces, such as the wings and the fuselage. Engineering surfaces that can benefit the durability, safety and efficiency of an aircraft is a topic of vast importance to the aviation industry. It is also important to understand the dynamics of droplets in a high speed airflow before they come into contact with an aircraft. In this section, the topics discussed above will be introduced in detail, as well as the previous literature on them.

Before we begin discussing the applications and literature of liquid impact, let us first discuss in detail what deformable and lubricant-infused surfaces are. A deformable surface is a type of surface or object that can be modified or altered by external forces or constraints. In other words, it can change its shape or configuration in response to applied forces, such as bending, stretching, twisting, or
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compressing [4]. Deformable surfaces have excellent anti-icing properties [5] and are also commonly used in microfluidic devices [6]. Lubricant-infused surfaces are porous or textured surfaces that have a low surface tension lubricant wicked into them [7, 8, 9]. Lubricant-infused surfaces became of interest from seeking alternatives to superhydrophobic surfaces, as well as through the inspiration of surfaces seen in nature. Superhydrophobic surfaces, despite all their beneficial attributes [10], do have weaknesses such as repellency only of liquids of relatively high surface tension, weak pressure stability and low mechanical stability. Methods to alleviate these drawbacks of superhydrophobic surfaces were explored by Quéré [11] and further investigated by Wong et al. [12] and Smith et al. [13], showing lubricant-infused surfaces to have excellent anti-wetting properties and pressure stability. These surfaces have also been found to have excellent anti-icing [14] and anti-biofouling [15] properties. Lubricant-infused surfaces mimic surfaces seen in nature, such as the Nepenthes pitcher plant surface [16]. This slippery surface has evolved to be able to capture insects easier as they slide into the plant’s interior.

Now, let us consider droplet impact. Droplet impact occurs in a number of natural and industrial settings, such as in foliar disease transmission [17], anti-icing [5, 14], spray coating [18], fuel combustion [19] and ink jet printing [20], to name but a very few. The research area of droplet impact has been around for over a century now. One of the very first studies of droplet impact was by Worthington [21] in 1876, who examined the different shapes and fingering patterns observed when water and mercury droplets fell vertically onto a horizontal plate. Since then, due to the vast increase in industrial applications of droplet impact during the 20th century and major advances in technologies, the field has progressed massively and still raises open questions to the present day. Present day studies typically involve either experimental [22], theoretical [23] or simulation [24] techniques, or, most powerfully, a combination [25].

When considering a liquid droplet that is surrounded by air, the droplet impact dynamics can be broadly categorised into two temporal regimes, the pre-impact and post-impact dynamics. Pre-impact dynamics refer to the stage just prior to impact
of a droplet on a surface, where the air pressures build up sufficiently in the thin air film to deform the droplet free surface and entrap air underneath the droplet upon impact. This stage of impact is vitally important to understand as it can have significant effects on the ensuing post-impact dynamics [26] and the entrapped air underneath the droplet can have detrimental effects to various industrial processes, such as ink-jet printing [20]. Early experimental work by Liow [27] was able to capture the size of the entrapped air pocket for droplet impact onto a pool of water using a drum camera and later Thoroddsen et al. [22] used high-speed photography to capture the features of the entrapped air pocket for droplet impact onto a solid surface. Early theoretical works focus on coupling droplet impact models with air cushioning models [28, 29] that were predominantly used for ship dynamics. Smith et al. [30] considered a balancing between the forces of an inviscid droplet approaching a rigid wall with a thin, lubricating air layer in between. This rational viscous-inviscid interaction work was further extended by Hicks and Purvis for three-dimensional impacts [31] and impacts with liquid layers [32]. Also, Purvis and Smith [33] considered the effect of surface tension and Mandre et al. [34], Mani et al. [35] and Hicks and Purvis [36] considered the compressibility of the air. More recently, the problem has been considered using axisymmetric direct numerical simulation (DNS), in order to have stronger connections with experiments and to explore parameter regimes and other physical effects perhaps not accessible to the inviscid/lubrication model. Wang et al. [5] performed simulations over a large range of impact parameters, identifying different outcomes of both the gas cushioning and post-impact phase of droplet impact. Similarly, Jian et al. [26] examined how variations in the liquid-gas viscosity and density ratios affect the gas cushioning and result in significantly different splashing behaviours.

Reducing the size of the entrapped air pocket is a topic of much interest. Reducing the ambient pressure of the surrounding air can reduce the entrapped air pocket size [37], but the tool of most practicality is surface engineering. A particularly popular method is the use of superhydrophobic surfaces [38, 39], which are surfaces that repel water. These types of surfaces are often created through a com-
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In terms of their application in reducing the size of entrapped air in droplet impact, it is the surface texturing that is of most importance. Surface texturing creates more avenues via which the air beneath the droplet prior to impact can escape. Hicks and Purvis [41] coupled the inviscid/lubrication model [30] to a porous surface model and showed that the introduction of porosity of the surface could greatly reduce the entrapped air size as well as the pressure peaks just prior to impact. A similar consideration to impact on superhydrophobic surfaces is impact onto lubricant-infused surfaces, which has received little consideration with respect to the pre-impact phase. This novel topic is considered in this thesis. Another type of surface engineering that is able to alter the pre-impact dynamics is deformability. Langley et al. [42] showed experimentally that surfaces coated with a layer of deformable soft material entrapped more air at impact, due to the pressure build-up before impact being sufficient enough to deform the surface. Mitra et al. [43] then showed that this increased air entrapment of soft solids, and thus extended lifetime of the air layer, facilitates bouncing behaviors of droplets at low-impact velocity and can also significantly influence the air film rupture dynamics. The role of entrapped air under a rebounding droplet impacting on a soft surface was also considered in Chen et al. [44, 45] and attributed to increasing the threshold velocity for a droplet to rebound on softer solids. Analytically and numerically, Pegg [46] showed that surface elasticity does indeed increase the amount of entrapped air, while decreases the pressure peaks just prior to impact. However, a detailed analytical and numerical consideration of the pre-impact phase of droplet impact on deformable surfaces that considers surface stiffness remains unanswered. This question is tackled in this thesis and is based on the published work in Henman et al. [2].

The post-impact dynamics have generally received a lot more attention than the pre-impact dynamics. This is due to the time scales on which the interaction takes place being much longer and easier to capture using experimental techniques. As mentioned previously, the study of post-impact dynamics is over a century old [21]. When a droplet hits a surface, depending on the droplet and surrounding
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air properties, the impact speed and the surface properties, the droplet will either deposit on the surface, bounce away from the surface or splash [47]. When a droplet does not splash, it will initially spread along the surface and reach a maximum radius. Depending on the surface wettability, the droplet will then either stay close to its maximum radius or retract. If the droplet retracts with enough energy it may even leave the surface all together and bounce away [48]. When a droplet does splash, it can either be categorised as a prompt splash, where microdrops are ejected from the tip of the advancing jet (lamella) that is ejected immediately from the point of impact, or a corona splash, where the jet remains intact as it lifts up off the surface creating a crown-like structure. According to Xu et al. [49], the prompt splashing is caused by surface roughness, while the corona splash is due to air effects on the advancing lamella. Understanding the determining features of spreading, bouncing and splashing dynamics is vitally important in industries such as ink-jet printing [20] and forensic sciences [50].

To complement the experimental work, there is a vast amount of theoretical and simulation work on droplet post-impact dynamics. Theoretical works are mostly based upon the well established asymptotic structure known as Wagner theory. Wagner [51] studied the impact of seaplanes as they landed on water and showed that the problem can be approximated as that of the impact of a flat plate whose width is changing in time on a still free surface of water. This approximation opened up a large body of theoretical work on solid body impact onto water [52, 53]. The asymptotic structure typically defines the regions of impact as the outer region, the jet root region near the contact points (two points in two dimensional studies but a ring for axisymmetric studies) and the thin jet region, with the jet emanating from the jet root region and extending along the solid body. Analogously, the same asymptotic structure can be used to study droplet impact and was first done by Howison et al. [23] who considered the impact of a droplet on a thin layer of the same fluid. This work has been further extended to account for surface deformability [54, 55, 56], non-flat surfaces [57, 58] and ice formation [59]. Simulation works on droplet impact have also come on leaps and bounds over the last
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few decades, from early work by Pasandideh-Fard et al. [24] to modern day work which utilised commercial [3, 60] and open-source software [25, 56] with advanced computational capabilities such as parallel computing.

For high-speed impacts, splashing is something that certain industries wish to control. For example, in ink-jet printing [20] splashing is undesirable as it would reduce the quality of the printed product. On the other hand, it is desirable for raindrop impacts on a car windscreen [61], to disperse water that would impede a driver’s vision, and for pesticide distribution on crops [62]. For aircraft anti-icing technologies, understanding the splash of a droplet on an aircraft is vitally important as it greatly influences the amount of water collected on the aircraft and thus the amount of ice produced [63]. Using surface engineering to control the droplet splash can be a very powerful technique. Deformable surfaces have the ability to reduce the magnitude of splashing. Pepper et al. [64] studied experimentally the impact of ethanol droplets on an elastic sheet, where the tension could be varied. Decreasing the tension in the elastic sheet lead to a suppression of the splash. Similarly, Howland et al. [65] studied the impact of ethanol droplets on soft layers of varying stiffnesses. Again, it was found that reducing the stiffness of the surface, and thus deformability, leads to a suppression of the splash. This was supported by simulations and it was proposed that the mechanism to suppress the splash was due to the pressure in the droplet being reduced due to the surface deformations. Surface texturing can also be useful in controlling the splashing behaviour. As mentioned previously, Xu et al. [49] showed that surface roughness could inhibit corona splashing, due to reduced lubrication forces below the advancing lamella, and promote prompt splashing. Latka et al. [66] drew similar conclusions and also showed that reducing the air pressure could reduce splashing.

Droplet impact studies with lubricant-infused surfaces have received little attention currently, both experimentally and numerically. Muschi et al. [67] examined the effect of varying the depth of the lubricant layer of the surface on droplet impact dynamics. A spin coating method was used to control the thickness of the lubricant and it was found that the wettability and spreading and retraction dynamics were not
greatly affected by the change in lubricant thickness, so long as the surface remained mostly homogeneous. The effect of the lubricant viscosity was investigated by Kim and Rothstein [68], where it was found that lowering the lubricant viscosity led to an increase in the spreading and retraction velocities of a droplet impact, as well as the maximum spreading diameter. The lubricant viscosity was similarly tested by Lee et al. [69], who found that the stability of the lubricant inside the nanotextured surface was improved by increasing the lubricant viscosity. While the above studies concentrated on passive methods for controlling the droplet impact dynamics, a recent study by Biroun et al. [70] focused on using surface acoustic waves as a way of actively controlling the droplet impact dynamics. They found that by applying surface acoustic waves to the droplet the contact time was reduced. Using a numerical approach, Yeganehdoust et al. [71] considered the impact of a micro-droplet onto different micro-textured surfaces. They compared results between a superhydrophobic surface and a lubricant-infused surfaces, where the textures are filled with lubricant, and found that droplet repellency and mobility on lubricant-infused surfaces were improved on low-density micro-textures surfaces compared to superhydrophobic surfaces. So far, a detailed study of the delicate early stages of a droplet impact onto a lubricant-infused surface where the length scale of the droplet is much larger than the surface asperities (which more closely mimics experimental work) has yet to be performed. This will be considered in this thesis and is based on the published work in Henman et al. [3].

When a droplet hits a surface, in many cases a thin jet is ejected along the surface. This thin, free-surface flow has analogies with that of the flow arising from a jet impact. When a jet of liquid hits a surface, it spreads radially from the point of impact. Due to the analogies with droplet impact, this topic has a number of applications that are the same as for droplet impact. Also, in aviation thin film flows are extremely common, due to the water collected by impacting droplets as well as melting ice. Thin film flows have their origin in the study of longer scale channel flows and tidal bores [72]. On a shorter length scale, such as the flow of a tap spreading along the bottom of a sink, the problem has been studied extensively
using a boundary layer approximation [73, 74]. The main metric of interest to researchers is the rate of growth of the film thickness, as well as the location of the hydraulic jump. The boundary layer approximation for the jet has been shown to agree with experimental results [75] on the two aforementioned metrics. Recent advances of this classic problem have focused mainly on modelling the effect of changing the surface topography. Saprykin et al. [76] used an integral-boundary-layer approximation to study the influence of isolated topographic changes on thin film flows, showing the existence of a capillary ridge just before the topographic change. Dressaire et al. [77] studied the influence of a more continuous, textured-like surface which has a comparable depth to the thin film. By utilising a partial-slip condition and modelling the leakage of the film in to the surface, they were able to extend the analysis of Watson [73] to account for surface texturing and conducted experiments which gave excellent agreement with the analytical work. So far, a detailed study of the influence of lubricant-infused surfaces on the dynamics of thin film flows, as well as the effect of the flow on the surface itself, has yet to be conducted. This thesis aims to explore this problem.

Now, moving away from the liquid impact process, from an aviation perspective, before impact has taken place, liquid (droplets) suspended in clouds can be subjected to impulsive, high velocity flows when an aircraft approaches. Understanding how droplets deform when suspended in a high velocity flow is therefore very important. Early experimental work in this field used shock tubes to study the deformation and breakup of droplets falling through a shock wave [78, 79, 80]. This work characterised the different breakup regimes of the droplet and secondary droplets. Present day studies predominately utilise direct numerical simulation to understand this problem in greater detail. Zhu et al. [81] performed a full three-dimensional study of the shear breakup of a droplet in very high Reynolds number ($\sim 10^5$) flows. They considered the effect of different air stream pressures on the initiation of breakup and instabilities of the free surface, as well as the secondary drop distribution and sizes after breakup. Li et al. [82] studied the deformation and acceleration of a droplet in a continuous airflow. They described the different
1.2 Aim and outline of thesis

The purpose of this thesis is to investigate the four novel questions that have arisen from considering droplet impact on engineered surfaces. Namely, that is: pre-impact dynamics of a droplet impinging on deformable and lubricant-infused surfaces, post-impact dynamics and thin film flows on lubricant-infused surfaces and droplet deformation in a continuous airflow. The applications of all four studies are wide and varied and are of significant interest to industry. The studies on impact will aid in the design of novel non-wetting surfaces and have implications in many engineering and scientific fields. Understanding thin film flows and the deformation of droplets in a uniform flow can help us to understand the larger picture of impact scenarios. The hope is that this thesis paints a picture of the droplet life cycle, from in-air, to impact and subsequent flows.

The outline of this thesis is as follows. Chapter 2 will present a study of the pre-impact phase of droplet impact onto both deformable and lubricant-infused surfaces. Chapter 3 will then utilise direct numerical simulation to study the post-impact phase for impact onto lubricant-infused surfaces. Drawing inspiration from
the post-impact dynamics, Chapter 4 will then pivot to investigate an analogous problem: thin film flows over lubricant-infused surfaces. Chapter 5 will then move away from the impact process and focus on modelling the deformation of suspended droplets in a uniform flow. Finally, Chapter 6 will present the conclusions and further avenues for research given the results presented in this thesis.
Chapter 2

Droplet impact: pre-impact

2.1 Deformable surfaces

The impact of a droplet on a deformable surface is a commonly occurring event in a number of industrial and natural settings, such as in anti-icing technologies [5], ink-jet printing [20] and rain-induced foliar disease transmission [17]. There have been a number of experimental studies into droplet impacts with flexible, or soft deformable, substrates, considering both the pre-impact [42, 43] and post-impact behaviour [65, 85, 86]. There has also been some analytical work on the post-impact behaviour [54, 56, 87] and the pre-impact dynamics of droplet settling [88], whereas the novelty in our work lies in analysing the pre-impact behaviour of a high-speed droplet impact with a deformable surface.

Liquid-elastic impacts are also the subject of a large number of studies. For example, on an inviscid basis Korobkin and Khabakhpasheva [89] studied the impact of a regular wave on an elastic plate and Khabakhpasheva and Korobkin [90] considered a liquid elastic-wedge impact. Similarly, Duchemin and Vandenberghe [91] investigated the impact of a rigid body on a floating elastic membrane. Of most relevance here are droplet-elastic impacts or droplet impacts with flexible surfaces. Pegg et al. [54] investigated the post-impact interactions of a droplet impact on an elastic plate, where it was assumed that the plate had a relatively high rigidity so that it would vibrate, rather than just be deformed by the impact. They used an axisymmetric Wagner-style model of a droplet impact, which was solved using the
method of normal modes. They found that the presence of substrate elasticity acted to slow down the velocity of the advancing contact line and that the induced oscillations of the substrate lead to the onset of splashing. Using a similar approach, the effect of surface vibrations was examined in more detail by Khabakhpasheva and Korobkin [55]. Also using post-impact axisymmetric Wagner theory, Negus et al. [56] investigated droplet impact onto a spring-supported plate, where they found solutions for the composite pressure and force on the plate, and provided an excellent comparison to results obtained via direct numerical simulation. Xiong et al. [87] performed numerical simulations of a droplet impacting a flexible surface using a Lattice-Boltzmann method, investigating the effect of bending stiffness on the contact time and wettability of the droplet on the surface.

There has also been a significant focus on experimental research on droplet impacts with deformable elastic/flexible surfaces, which has motivated the use of flexible elements in surface engineering and microfluidic devices [6]. Early work by Pepper et al. [64] examined droplet impact onto elastic membranes of variable tension. They found that by sufficiently lowering the membrane tension, splashing could be suppressed. This work was further complemented by Howland et al. [65] who investigated the impact of ethanol drops on silicone gels of different stiffnesses, where it was found that the stiffness affects the droplet splashing threshold and could also eliminate splashing altogether. In both of these works on splashing it was suggested that very early times after, even prior, to impact were critical in the overall outcome of the droplet impact. Weisensee et al. [85] considered droplet impacts with elastic superhydrophobic surfaces and found that the elasticity of the surface was an additional mechanism for reducing the contact time of a bouncing droplet. Vasileiou et al. [86] came to similar conclusions, and Vasileiou et al. [92] found, by investigating impacts with supercooled droplets, that substrate flexibility can improve icephobicity. Similarly, a detailed study of the dynamics of the flexible superhydrophobic surface by a droplet impact was performed by Kim et al. [93]. The studies discussed here are typically based on deformable substrate coatings or clamped-clamped membrane designs, however droplet impact onto cantilever
2.1. Deformable surfaces

The focus of the present study is on an analytical and numerical investigation into the pre-impact behaviour of a droplet impacting a deformable surface. Understanding the pre-impact behaviour of the droplet is vital in understanding the post-impact behaviour. Although in practice droplet impacts are three-dimensional, we will formulate a simplified two-dimensional model and we will use our study to try to gain a qualitative understanding of the effect of surface deformation on various impact quantities, such as touchdown time, contact pressure and air entrapment, which play an important role in understanding the effect on splashing, spreading and wettability of the droplet post-impact (we note a recent very interesting study by Pegg [46] on elastic surfaces which has some overlap with ours). We formally define the problem and describe an asymptotic analysis which allows us to define a reduced set of governing equations for the droplet free-surface, the pressure in the air film and the shape of the surface as they evolve and interact. We choose to model the deformable surface by the compliant surface model of Carpenter and Garrad [96] which includes rigidity, tension, stiffness, inertia and damping and so is representative of a number of different surfaces. The aim of this study is to assess the influence of each of these physical parameters.

2.1.1 Model formulation and governing equations

Suppose a two-dimensional liquid droplet of radius $R$ approaches a deformable surface with normal velocity $V$. Initially the droplet will be sufficiently far away from the surface that the pressure between the droplet and the surface is constant and the droplet remains circular. Let $(x^*, y^*)$ be the Cartesian coordinates and $t^*$ be time. Then the bottom free surface of the droplet will initially be

$$f^*(x^*, t^*) = -\sqrt{R^2 - x^*^2} - Vt^* + R.$$  \hspace{1cm} (2.1)

The deformable surface will be denoted $g^*(x^*, t^*)$ and undisturbed will lie on $y^* = 0$. Time is measured such that in the absence of air cushioning the droplet would impact the undisturbed deformable surface at $t^* = 0$. 
The aim is to derive a system of equations that govern the droplet free-surface, the air film pressure between the droplet and the deformable surface and the shape of the surface. In order to do this the fluid flow will have to be considered separately in the liquid droplet and the air film and an equation governing the shape deformations of the surface will also be considered. These three quantities will form a coupled system. The following derivation will assume that the effects of surface tension, compressibility and gravity are negligible and these assumptions will be discussed in detail later on in this section. The subsequent analysis will exploit the small density ratio $\rho_g \ll \rho_l$, of the liquid ($l$) to the gas ($g$) in order to obtain asymptotically valid equations describing the droplet free-surface, the pressure in the air gap and the shape of the surface. The small quantity used in the asymptotic analysis will be defined as the aspect ratio of the local horizontal length scale $l$, over which the pressure has a leading order effect on the droplet free surface, to the droplet radius $R$,

$$\varepsilon = \frac{l}{R}.$$  \hspace{1cm} (2.2)

Here $l$ is still to be determined. Figure 2.1 shows a schematic of the problem set-up.
2.1. Deformable surfaces

All distances are non-dimensionalised with the droplet radius $R$ and time is non-dimensionalised with $R/V$; thus $(x^*, y^*) = R(x, y)$ and $t^* = Rt/V$. Fluid velocity components in both fluids are non-dimensionalised with $V$ and pressure with $\rho l V^2$. The flow in the droplet and the air film is assumed to be governed by the incompressible Navier-Stokes equations and using the above notation these are

\[
\frac{\partial \mathbf{u}_l}{\partial t} + \mathbf{u}_l \cdot \nabla \mathbf{u}_l = -\nabla p_l + \frac{1}{Re} \nabla^2 \mathbf{u}_l, \tag{2.3a}
\]

\[
\nabla \cdot \mathbf{u}_l = 0, \tag{2.3b}
\]

for the liquid and

\[
\frac{\partial \mathbf{u}_g}{\partial t} + \mathbf{u}_g \cdot \nabla \mathbf{u}_g = -\frac{\rho_l}{\rho_g} \nabla p_g + \frac{\nu_g}{\nu_l} \frac{1}{Re} \nabla^2 \mathbf{u}_g, \tag{2.4a}
\]

\[
\nabla \cdot \mathbf{u}_g = 0 \tag{2.4b}
\]

for the air, where $\nabla = (\partial/\partial x, \partial/\partial y)$, $\nu_\alpha = \mu_\alpha/\rho_\alpha$ is the kinematic viscosity of the liquid ($\alpha = l$) or gas ($\alpha = g$), $\mathbf{u}_\alpha = (u_\alpha, v_\alpha)$ is the fluid velocity field and $Re = RV/\nu_l$ is the global Reynolds number based on the properties and initial velocity of the liquid. For the parameter regime of interest, the Reynolds number $Re$ is typically large and this will, again, be discussed at the end of this section.

The non-dimensional initial liquid free surface profile is now given by

\[
f(x, t) = -\sqrt{1 - x^2} - t + 1. \tag{2.5}
\]

To close our system we must satisfy the kinematic boundary conditions on the liquid-gas interface and the gas-surface interface. The following condition assumes that the surface is not travelling horizontally with an $O(\varepsilon^{-1})$ speed. Thus the kinematic conditions read,

\[
\frac{\partial f}{\partial t} + u_\alpha \frac{\partial f}{\partial x} = v_\alpha, \quad \text{at } y = f(x, t) \tag{2.6}
\]
and
\[
\frac{\partial g}{\partial t} + u_\alpha \frac{\partial g}{\partial x} = v_\alpha, \quad \text{at } y = g(x,t)
\] (2.7)

for \( \alpha = l \) or \( g \). Neglecting surface tension effects gives the normal stress balance on the liquid-gas interface as

\[
p_l = p_g.
\] (2.8)

Due to the disparate horizontal velocity scales between the liquid and the gas, a tangential stress balance on the liquid-gas interface is not required, as it will be seen later that a no-slip condition is applied on this boundary (this involves a thin shear layer present [30]).

To determine the behaviour of the liquid droplet free surface, close to the point of initial contact which is \( x = y = 0 \), we take the following scaling

\[
(x,y,t,f) = (\varepsilon X, \varepsilon Y, \varepsilon^2 T, \varepsilon^2 F)
\] (2.9)

where the scales of \( t \) and \( f \) come from the desire to study short time behaviour and from the form of equation (2.5), respectively. The \( O(\varepsilon^2) \) time scale is that expected for the traversing, at a unit non-dimensional velocity, of the thin air gap which has normal width of \( O(\varepsilon^2) \). The scales (2.9) lead us to take asymptotic expansions of the liquid velocity components and pressure in the following form

\[
(u_l, v_l, p_l) = (U_l, V_l, \varepsilon^{-1} P_l) + \cdots
\] (2.10)

The vertical velocity scale is order unity due to the order unity downward velocity of the droplet, then the horizontal velocity scale follows from continuity. The large pressure scale arises from seeking a balance between the liquid acceleration and the pressure gradient.

Now, for \( \text{Re} \gg 1 \), upon substitution of scales (2.9) and expansions (2.10) into the governing equations (2.3), the leading order momentum equations and continu-
2.1. Deformable surfaces

ity equation are those of unsteady potential flow

\[
\frac{\partial U_l}{\partial T} = -\frac{\partial P_l}{\partial X},
\]
(2.11a)

\[
\frac{\partial V_l}{\partial T} = -\frac{\partial P_l}{\partial Y},
\]
(2.11b)

\[
\frac{\partial U_l}{\partial X} + \frac{\partial V_l}{\partial Y} = 0.
\]
(2.11c)

The leading order kinematic condition (2.6) now reduces to

\[
V_l \to \frac{\partial F}{\partial T}, \quad \text{as } Y \to 0^+,
\]
(2.12)

while the far-field droplet behaviour reduces to

\[
F(X,T) \sim \frac{X^2}{2} - T + O(\epsilon), \quad \text{as } |X| \to \infty \text{ or } T \to -\infty,
\]
(2.13)

from (2.5).

From equations (2.11) it can be shown that \(P_l\) satisfies Laplace’s equation and due to the far-field boundedness (2.13) and condition (2.12) the pressure profile on the droplet interface \(P(X,T) = P_l(X,0,T)\) and free-surface profile \(F(X,T)\) are related by

\[
\frac{\partial^2 F}{\partial T^2} = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\partial P(\xi,T)}{\partial \xi} \frac{d\xi}{X - \xi}.
\]
(2.14)

The bar denotes the Cauchy principle value integral.

In the thin gas film between the droplet and the deformable surface, we assume that the vertical length scale is an order of magnitude smaller than the horizontal length scale to capture the slenderness of the film. Also, unlike in the droplet formulation, we now need to consider the deformable surface \(g\), which we suppose to have the same scale as the droplet free surface \(f\) in order for the surface deformation to have a leading order influence. We then take the following scaling in terms of our previously defined small parameter \(\epsilon\),

\[
(x,y,t,f,g) = (\epsilon X, \epsilon^2 Y, \epsilon^2 T, \epsilon^2 F, \epsilon^2 G).
\]
(2.15)
The scales (2.15) lead to asymptotic expansions of the form

\[(u_g, v_g, p_g) = (\varepsilon^{-1} U_g, V_g, \varepsilon^{-1} P_g) + \cdots, \quad (2.16)\]

where the horizontal velocity scale is expected to be large compared with the vertical velocity scale in order to balance the continuity equation.

When substituting scales (2.15) and expansions (2.16) into the governing equations for the gas (2.4), for \( \varepsilon \ll 1 \) the leading order equations take the form of lubrication flow,

\[
\begin{align*}
0 &= -\frac{\partial p_g}{\partial X} + \frac{\partial^2 U_g}{\partial Y^2}, \\
0 &= -\frac{\partial p_g}{\partial Y}, \\
\frac{\partial U_g}{\partial X} + \frac{\partial V_g}{\partial Y} &= 0.
\end{align*}
\tag{2.17a-2.17c}
\]

The inertial and acceleration terms do not appear in the left hand side of equations (2.17a-2.17b) because these terms are negligible compared to the pressure gradient term. This assumption requires \( \rho_g/\rho_l \ll \varepsilon \). Also, it is assumed that there is a balance in the horizontal momentum equation between the pressure gradient and the viscous terms [30], which leads to the definition

\[
\varepsilon = \left( \frac{\mu_g}{\mu_l Re} \right)^{1/3}. \tag{2.18}
\]

This, combined with the requirement \( Re \gg 1 \) in the liquid, gives the condition that \( \varepsilon \) must satisfy in order for our model to be valid, namely

\[
\frac{\rho_g}{\rho_l} \ll \varepsilon \ll \left( \frac{\mu_g}{\mu_l} \right)^{1/3}. \tag{2.19}
\]

For example, water and air give rise to the range \( 10^{-3} \ll \varepsilon \ll 0.27 \).

The leading order kinematic conditions (2.6) and (2.7) now reduce to

\[
V_g = \frac{\partial F}{\partial T}, \quad \text{at } Y = F(X, T) \quad (2.20)
\]
\begin{equation}
V_g = \frac{\partial G}{\partial T}, \quad \text{at } Y = G(X, T) \quad (2.21)
\end{equation}

respectively. The vertical momentum equation (2.17b) and the normal stress condition (2.8) imply that \( P_g(X,Y,T) = P(X,T) \), then integrating equation (2.17a) twice in \( Y \) and applying conditions (2.20-2.21) gives

\begin{equation}
\frac{\partial}{\partial X} \left( (F - G)^3 \frac{\partial P}{\partial X} \right) = 12 \frac{\partial}{\partial T} (F - G), \quad (2.22)
\end{equation}

the (Reynolds) lubrication equation that helps to link all three unknown quantities \( F, G \) and \( P \).

Finally, we require another equation linking the pressure in the air film with the deformation of the surface. The model we will use here was first described by Carpenter and Garrad [96] to model surface coatings as an inextensible elastic plate (or tensioned membrane) supported above a rigid surface by an array of springs. It is derived from a nonlinear model that includes all relevant physics which is then linearised under the assumption of longitudinal deflections being much smaller than transverse ones. Both the inextensible elastic plate and the springs it is backed on contribute to the deflection of the surface. The elastic plate is assumed to be thin such that thin plate theory applies and the deflection is proportional to the resistance to bending and the inertia of the surface. These effects introduce fourth order spatial and second order time derivative terms, respectively. Due to the inextensibility of the surface, the deflection is also resistant to tension, which introduces a second order spatial derivative term. Finally, the elastic plate is supported by springs which exhibit stiffness and damping, giving terms proportional to the deflection of the surface and its first spatial derivative, respectively. An excellent derivation of this equation can be found in Alexander et al. [97] (we will ignore viscous traction in the air film in our model). This model is also used in a number of other studies on deformable surfaces [98, 99, 100, 101, 102]. The relevant equation takes the
2.1. Deformable surfaces

non-dimensional form

\[ e_1 \frac{\partial^4 g}{\partial x^4} + e_2 \frac{\partial^2 g}{\partial x^2} + e_3 g + e_4 \frac{\partial^2 g}{\partial t^2} + e_5 \frac{\partial g}{\partial t} = p(x, g, t) - p_s(x, g, t), \]  

(2.23)

where the non-dimensional coefficients \( e_i \) are \( (e_1, e_2, e_3, e_4, e_5) = (-B^*/R^3V^2\rho_l, T^*/RV^2\rho_l, -\kappa^*/V^2\rho_l, -M^*/RP_l, -C^*/V\rho_l) \) and \( p_s \) is the non-dimensional relative base pressure, which is taken to be zero. We choose to use this thin-membrane type model over more simpler models as there is commonality in such membranes in nature and practical settings, such as droplet impact onto leaves [17], butterflies [103], umbrellas and raincoats, and relevance in many other applications of droplet impact on deformable surfaces discussed elsewhere in this paper. More simple surface deformation models can be extracted as subsets from equation (2.24) and one such, a Kelvin-Voigt model of viscoelasticity, is described in detail in Sec. 2.1.2 while the equation is considered in full generality in Sec. 2.1.3. The recent study by Pegg [46] is likewise based on the Smith et al. [30] theory but with a surface equation generally different from ours; the present work is more focused on soft deformable surfaces which are found to lead to new non-intuitive results. The constants \( B^*, T^*, \kappa^*, M^* \) and \( C^* \) correspond to the flexural rigidity, the longitudinal tension, the stiffness, the mass density and the damping constant of the surface, respectively. In order to apply equation (2.23) to our scaled liquid droplet application described above, we must apply scales for \( g, x, t \) and \( p \) given in (2.15-2.16). This gives

\[ \tilde{e}_1 \frac{\partial^4 G}{\partial \tilde{X}^4} + \tilde{e}_2 \frac{\partial^2 G}{\partial \tilde{X}^2} + \tilde{e}_3 G + \tilde{e}_4 \frac{\partial^2 G}{\partial \tilde{T}^2} + \tilde{e}_5 \frac{\partial G}{\partial \tilde{T}} = P - P_s, \]  

(2.24)

where \( (\tilde{e}_1, \tilde{e}_2, \tilde{e}_3, \tilde{e}_4, \tilde{e}_5) = (\varepsilon^{-1}e_1, \varepsilon e_2, \varepsilon^3e_3, \varepsilon^{-1}e_4, \varepsilon e_5) \) and \( P_s = \varepsilon p_s \). Therefore, the five non-dimensional parameters which control the system are the following

\[ \tilde{e}_1 = -\frac{B^*}{(\rho_l^4 \mu_g V^5 R^8)^{1/3}}, \quad \tilde{e}_2 = \frac{T^*}{(\rho_l^4 \mu_g V^7 R^4)^{1/3}}, \]

\[ \tilde{e}_3 = -\frac{\kappa^*}{\rho_l^2 \mu_g^{-1} V^3}, \quad \tilde{e}_4 = -\frac{M^*}{(\rho_l^2 \mu_g V^{-1} R^2)^{1/3}}, \]

\[ \tilde{e}_5 = -\frac{C^*}{(\rho_l^4 \mu_g^{-1} V^4 R)^{1/3}}. \]  

(2.25)
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These parameters are dependant on the structural parameters of the surface as well as the physical and fluid properties of the system through the droplet radius $R$ and velocity $V$, plus the droplet density $\rho_l$ and gas viscosity $\mu_g$. The parameter $\tilde{e}_1$ is most sensitive to variations in the droplet radius $R$ and controls the influence of the rigidity term $\tilde{e}_1 \partial^4 G / \partial X^4$. Since the droplet radius is related to the length scale of the surface, this has a key influence on the surface rigidity. Perhaps the parameter of most interest is $\tilde{e}_3$, which does not depend on the droplet radius $R$ and is most sensitive to all other important variables $V$, $\rho_l$ and $\mu_g$. This parameter controls the stiffness term $\tilde{e}_3 G$ and is a key parameter in the study that follows. In practice, each of the dimensional structural parameters $B^*$, $T_r^*$, $\kappa^*$, $M^*$ and $C^*$ can vary dramatically depending on the situation, so therefore so can each $e_i$ in (2.25). As droplet impact is a ubiquitous phenomenon, it can occur on many different surfaces of differing elastic properties, for example leaves [17], thin foils [104], skin [105] and foodstuffs [106]. It is also possible to use surface engineering to modify the elastic properties of a surface to our advantage [42, 65, 85, 86, 92]. Therefore our focus will be on studying different cases of equation (2.24), with a very wide range of value of the parameters $e_i$.

The deformable surface will be considered to be initially zero and stationary. Boundary conditions will be discussed separately for each case considered. The system to be solved is the non-linear set of governing equations (2.14), (2.22) and (2.24), subject to the boundary condition (2.13), the pressure $P$ being initially zero and decaying at infinity and the relevant boundary and initial condition on $G$. The governing equations require a numerical treatment. The numerical scheme is slightly different for each case considered and so will be discussed separately for each case.

It is useful now to mention the fluid parameters used in recent experiments. Of most relevance here, we will consider the experiments performed by Langley et al. [42] for the pre-impact behaviour of ethanol droplets impacting soft surfaces of varying stiffness. They considered a parameter regime where the Reynolds number $Re$ ranged from 1209 to 20394 and the Weber number $We$ ranged from 17 to 2825.
Hence both are typically large and in particular the assumption of a quasi-inviscid liquid droplet seems valid in this regime. Also, in the present study surface tension has been ignored and the scaled surface tension forces are given by

$$\varepsilon^2 \frac{\sigma}{\rho_l V^2 l} \frac{\partial^2 F}{\partial X^2} = \varepsilon \frac{\partial^2 F}{\partial X^2},$$  \hspace{1cm} (2.26)

where $\sigma$ is the surface tension coefficient and $\text{We} = \rho_l V^2 R/\sigma$ is the Weber number. As the parameter $\varepsilon$ is small and $\text{We}$ is typically large, this again seems a valid assumption for the vast majority of the evolution. As the droplet approaches the surface, we do still expect the free-surface to deform to the point where high curvatures are observed [30]. This would result in a large value for $\partial^2 F/\partial X^2$, and thus surface tension forces may become significant at this stage. However, for some cases in the following section the free-surface curvature will not become large at any stage and so surface tension effects remain small. The effect of surface tension is considered in Purvis and Smith [33] for droplet impacts in our parameter regime and in Vanden-Broeck and Smith [107] for a similar air-cushioning related problem for higher Reynolds numbers than in this work. In the present study, however, as has already been assumed, surface tension effects will be ignored. For all parameters considered in Langley et al. [42] the Froude number $\text{Fr} = V/\sqrt{gR}$, where $g$ is the gravitational acceleration, is also large; hence gravitational effects are ignored.

The effects of compressibility in the air film are also ignored in the present study. A scaling argument performed by Mandre et al. [34], where they balanced the gas pressure gradient with the droplet deceleration, found that compressibility effects in the air can be ignored if the compressibility parameter $\delta$ satisfies

$$\delta = \frac{p_0^*}{(R\mu_g^{-1}V^7 \rho_l^4)^{1/3}} \gg 1,$$  \hspace{1cm} (2.27)

where $p_0^*$ is the surrounding ambient pressure. In the study of Langley et al. [42] there are impacts considered both in the compressible ($\delta < 1$) and incompressible regimes ($\delta \gg 1$), so we will still attempt to draw comparisons with their work later in Sec. 2.1.4. For detailed discussions of compressible gas-cushioned droplet
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Figure 2.2: Range of parameter validity for a water droplet impact in air, with $p_0^* = 10^5$ Pa. Above the dotted line $Re \gg 1$ and the liquid droplet may be considered quasi-inviscid, below the solid line $\delta \ll 1$ and compressibility effects in the air may be ignored, above the dashed line $We \gg \varepsilon$ and surface tension forces may be ignored for the vast majority of the evolution, and below the dash-dotted line $Fr \gg 1$ and gravitational effects may be ignored. These limits are represented by the gray shaded region. The cross is the point where the droplet radius is 1 mm and the droplet velocity is 1 m s$^{-1}$, a set of values well within our regime and used commonly in this paper for dimensional calculations.

Figure 2.2 summarises the main model assumptions for a water droplet impact, when $p_0^* = 10^5$ Pa. There is a clear and rather large range of droplet velocities and radii where our model is valid and this is the gray shaded region in figure 2.2, where the cross shows where the droplet radius is 1 mm and the velocity is 1 m s$^{-1}$, a point well within our regime.

2.1.2 Impact on a viscoelastic solid

The first case that will be considered is that of an air cushioned droplet impact onto a soft viscoelastic solid. The viscoelastic material will be assumed to exhibit no rigidity, tension or inertia, such as a coating on a rigid surface of infinite horizontal extent. In this case we may remove the influence of coefficients $\tilde{e}_1$, $\tilde{e}_2$ and $\tilde{e}_4$ (this can be envisaged as removing the influence of the elastic plate, or membrane, from impacts see Mani et al. [35] and Hicks and Purvis [36] also.
2.1. Deformable surfaces

Figure 2.3: Schematic representation of the scaled, non-dimensional problem set-up of droplet impact onto a viscoelastic solid pre-impact (this is not a solution).

Equation (2.24)) and reduce our deformable surface equation (2.24) to

\[ \ddot{e}_s G + \dot{e}_s \frac{\partial G}{\partial T} = P, \]  

(2.28)

which gives the simple Kelvin-Voigt model of a solid deforming in reaction to an applied pressure. Droplet impact and dynamics on soft viscoelastic solids has received attention both experimentally [44, 45] and analytically [108]. A schematic of the problem set-up is shown in figure 2.3.

If the soft viscoelastic solid is a coating sitting on an otherwise rigid surface, the coating depth would need to be \( O(\varepsilon^2 R) \) in order for the depth to have an influence on the dynamics. If \( d^* \) is the dimensional depth of the coating, then this can be envisaged as \( d^* \ll \varepsilon R \), which for a droplet of radius 1 mm and velocity 1 m s\(^{-1}\) becomes \( d^* \ll 26 \, \mu m \). In the experiments of Howland et al. [65] they predominately considered soft substrates of depth 1 cm, well outside this limit, however they did also discuss the effect of a substrate of depth 3 \( \mu m \), which is within this limit. It was found that the splashing behaviour of the impacting droplet on this very shallow soft silicone substrate was almost identical to that of a much deeper substrate of hard acrylic. In light of this fairly uninteresting behaviour for substrates of shallow
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depth, we will limit our study to the case of surfaces of apparent infinite depth with zero under-pressure \( P_s = 0 \). The smallest depth of substrate considered in Langley et al. [42] was 1 mm, also well outside this limit.

Given the form of equation (2.28) it follows naturally that if the pressure is decaying at infinity, then so is the surface deformation. Therefore the boundary condition at infinity will be \( G \to 0 \). The viscoelastic model equation (2.28) is solved in conjunction with the free-surface equation (2.14) and the lubrication equation (2.22). The numerical method proceeds by first substituting equation (2.28) into equations (2.14) and (2.22) for \( P \). At each time step, the lubrication equation (2.22) is solved via a finite difference discretization for \( G \), which is then used to solve the free-surface equation (2.14) for \( F \), using Fast Fourier Transform algorithms. This process is repeated until successive iterates are within a convergence criteria, after which the solution proceeds to the next time step. Once we have a converged solution for \( G \), the pressure \( P \) is then readily calculated from equation (2.28). This method is akin to that used in Hicks and Purvis [41]. By eliminating the pressure from our simultaneous equations, the efficiency of the code is improved somewhat as not only do we have two equations to solve as opposed to three, we are no longer having to resolve a diverging pressure solution [30] from the lubrication equation, which can be computationally expensive. Tests were run to ensure the solutions were independent of grid size and time step size. A suitable spatial domain size here was found to be \( X \in [-20, 20] \), with the boundaries here being non-invasive and the solutions remaining unchanged for further increases to the spatial domain size. The simulations are very sensitive to the chosen start time. We performed a test on the surface with parameters \((\tilde{e}_1, \tilde{e}_2, \tilde{e}_3, \tilde{e}_4, \tilde{e}_5) = (0, 0, -0.1, 0, -0.1)\) (the most deformable surface considered in this study) with start times of \( T = -50 \) and \( T = -100 \). It was found that the difference in the size of entrapped air upon touch-down (defined later by equation (2.30)) was 2.5\% for these two start times, and so \( T = -50 \) was chosen as the start time for all simulations. A detailed description of the numerical scheme used to solve a more general version of this system, plus validation, is given in Appendix A.1. The code used to solve this sys-
Figure 2.4: Solution profiles, showing evolution of (a) the free-surface height $F$ and (b) the pressure $P$ for normal impact of a droplet on a flat rigid surface and deformable viscoelastic surfaces with $\tilde{e}_3 = \tilde{e}_5 = -1$, $-0.2$ and $-0.1$. The solutions are shown in integer time increments except for the final thick solid line, which is the solution just prior to touchdown, and the dashed line, which is the deformable surface solution $G$ just prior to touchdown. The dash-dotted line is the solution at $T = 0$ where the droplet would touchdown on an undisturbed surface in the absence of air cushioning.

The code for these simulations can be found at: https://github.com/NatHenman/PhD_thesis/blob/main/pre_impact_viscoelastic_code.m.

Figure 2.4 shows the evolution of the droplet free-surface and pressure profile for a flat rigid surface ($G = 0$) and for a viscoelastic surface for a range of values of $\tilde{e}_3 = \tilde{e}_5$. The top two panels are the solution for a flat rigid surface which has been previously reported a number of times [30, 41]. As the droplet approaches
the surface, there is an initial build-up of pressure directly beneath the droplet. The build-up of pressure acts to decelerate the falling droplet and as the gap between the droplet and the surface narrows, the pressure build-up is large enough to decelerate the droplet free-surface to rest at the center point. At this stage, the droplet begins to deform either side of the center point and eventually overtake it, resulting in an approach to touchdown in two locations. This, in turn, results in the pressure bifurcating away from the center point into two pressure peaks. The process continues until the moment of touchdown on the surface, which results in the entrapment of an air pocket, which may then subsequently form a bubble [20, 22]. For impact onto a flat rigid surface, figure 2.4 shows that as touchdown is approached the curvature of the droplet free-surface becomes large at the touchdown point. Here it is likely that significant surface tension effects would come into play, perhaps changing the dynamics at impact or delaying touchdown further [33, 41]. For impact onto viscoelastic surfaces ($\tilde{e}_3 \neq 0$), the droplet free-surface is far smoother at impact and it is unlikely that surface tension effects would be significant here.

The result for the flat rigid surface in figure 2.4 is then compared to the result for impact onto a viscoelastic surface for decreasing magnitudes of the surface parameters $\tilde{e}_3 = \tilde{e}_5$. Lowering the magnitude of the surface parameters $\tilde{e}_3$ and $\tilde{e}_5$ corresponds to lowering the surface stiffness and (viscous) damping, which results in larger surface deformations. Rows two to four in figure 2.4 show the solutions for the free-surface and the pressure profile evolution, along with the shape of the surface as touchdown is approached, for $\tilde{e}_3 = \tilde{e}_5 = -1$, -0.2 and -0.1, respectively. For a water droplet in air of radius 1 mm and impact velocity 1 m s$^{-1}$, these parameter variations correspond to a surface spring stiffness in the range $\kappa^* \sim 10^9$ - $10^{10}$ Pa m$^{-1}$ and (viscous) damping of $C^* \sim 10^3$ - $10^4$ Pa s. We note that the spring stiffness values here are larger than the stiffnesses of the soft solids considered in Langley et al. [42], however the spring stiffness defined in equation (2.24) is a stiffness per unit length and the length scales in this problem are typically very small. Smaller values of the dimensionless parameter $\tilde{e}_3$ are considered in the next set of results regarding entrapped air size.
A number of conclusions can be drawn from the influence of surface deformability in this viscoelastic model on the pre-impact dynamics of droplet impact. First of all, there is a profound delay to touchdown. In the absence of air cushioning the droplet would impact on an undisturbed surface at $X = Y = 0$ at $T = 0$. The presence of air cushioning delays this touchdown into positive time. In figure 2.4 the solution at $T = 0$ is the dash-dotted line, while the positive time solutions are the solid lines. As the droplet approaches the soft viscoelastic surface, the build-up of pressure beneath the droplet acts now to not only deform and decelerate the droplet free surface, but also push the surface away from the droplet. This results in a slower closing of the air gap between the droplet and the soft viscoelastic surface, with lower magnitudes of surface parameters $\tilde{e}_3$ and $\tilde{e}_5$ resulting in a slower closing of the air gap. In consequence there is a larger delay to touchdown as illustrated by the thick solid lines in figure 2.4, corresponding to the solution as touchdown is approached, where the time has increased from $T = 5.84$ for a flat rigid surface to $T = 22.3$ for the soft viscoelastic surface with $\tilde{e}_3 = \tilde{e}_5 = -0.1$. The dimensional time scale is given by

$$t^* = \frac{\varepsilon^2 R}{V T} = \frac{\mu g^{2/3} R^{1/3}}{\rho_l^{2/3} V^{5/3} T}. \quad (2.29)$$

For a 1 mm water droplet in air with impact velocity $1 \text{ m s}^{-1}$, the touchdown delay incurred here is $11.4 \text{ } \mu s$.

The slower closing of the air gap also influences the build-up of pressure beneath the droplet. In figure 2.4 as the parameters $\tilde{e}_3$ and $\tilde{e}_5$ are decreased in magnitude, the bifurcating behaviour of the pressure profiles is still present. However, the pressure peak amplitude as touchdown is approached is lowered by the introduction of surface deformability, with larger surface deformations resulting in lower pressures.

Figure 2.5 highlights the link between the rate at which the air gap closes and the maximum pressure. The lower pressures seen for more deformable surfaces are found to be present at all time.

It is also interesting to note the shape of the pressure solution as touchdown is
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Figure 2.5: The (a) minimum air film thickness and the (b) maximum pressure as a function of time $T$ for viscoelastic surfaces with coefficients $\dot{\varepsilon}_3 = \dot{\varepsilon}_5$ ranging from -1 to -0.1 in intervals of 0.1. The dashed line in figure (a) corresponds to the minimum film thickness in the absence of air cushioning.

Figure 2.6: As figure 2.5, except showing (a) the air film thickness $H = F - G$ and (b) the pressure near touchdown.
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Figure 2.7: As figure 2.5, except showing the positive touchdown point $X_d$ as a function of shifted time $T - T_0$.

approached. Figure 2.6 shows the air film thickness $H = F - G$ and pressure profile solution for a range of values $\tilde{\epsilon}_3 = \tilde{\epsilon}_5$. It can be seen that for surface parameters of sufficiently high magnitude, the pressure profile solution near touchdown is virtually identical to that of the flat surface solution [30, 41], with a very sharp pressure peak located at the cusp of the air film thickness $H$. As the magnitude of the surface parameters is decreased the sharp pressure peak near touchdown still exists up until around $\tilde{\epsilon}_3 = \tilde{\epsilon}_5 = -0.2$ (this solution is also shown in figure 2.4) where a rounder pressure peak located just behind the cusp of the air film thickness $H$ is seen. As the magnitude of the surface parameters is decreased further to $\tilde{\epsilon}_3 = \tilde{\epsilon}_5 = -0.1$, the rounder pressure peak overtakes the sharp one near touchdown. The location of the lower, rounder pressure peak just behind the cusp of the air film thickness, as opposed to at it, could help explain the extended air gliding behaviour for droplet impacts on to softer solids seen in experiments [42]. Air gliding is when a droplet skids on a thin air film as opposed to touching down; the numerical solutions of the air film thickness in figure 2.6(a) appear to show the onset of such behaviour.

The positive location of the minimum air film thickness, or touchdown point, $X_d$ moves outwards away from zero after the pressure bifurcates. The location and speed of this point is of great importance to splashing [26]. Figure 2.7 shows the positive touchdown point $X_d$ as a function of shifted time $T - T_0$, where $T_0$ is the time at which air cushioning in the film begins and the pressure bifurcates, for vari-
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ous values of $\tilde{e}_3 = \tilde{e}$. Figure 2.7 shows that the touchdown points move away from zero slower on more deformable surfaces. The implications of this are that once the droplet fluid reaches touchdown with the surface, the fluid at the touchdown location is travelling horizontally at a lower speed on surfaces that are more deformable. This would result in the jet of fluid ejected at impact [23] being of a lower velocity on more deformable surfaces, which could result in splash suppression [65].

A vitally important outcome of the dynamics described here is the area of entrapped air underneath the droplet at impact. Trapped air at impact can cause potential problems when it comes to using droplets to spray coat materials or in cooling processes. From the numerical results of pre-impact air-water-surface interaction in figure 2.4 it is clear that air entrapment still occurs. What is perhaps more clear in figure 2.6 is that the presence of surface deformability in the viscoelastic model leads to an increase in the area of air entrapped at impact. The framework of our model is in two dimensions, and it is realised that droplet impact is clearly a three dimensional phenomenon. Despite this, we are able to use our model to make a qualitative assessment of the size of entrapped air for variations in the surface stiffness and (viscous) damping.

The $X$-wise symmetry of the solutions allows us to calculate the dimensionless area of entrapped air $B$ as

$$B = 2 \int_0^{X_d} H(X, T_d) dX,$$

where $T_d$ is the touchdown time. It should be stressed that, numerically, touchdown where $H = 0$ is never quite realised due to the parabolic degeneracy of the lubrication equation (2.22). Therefore, the time of touchdown has to be pre-determined as a point when the air film thickness reduces to a very small positive value. Numerically, the smaller the grid size and time step in the numerical scheme, the smaller we can make this value. However, this has to be balanced with the increased computational cost due to the huge amount of simulations needed to be run to build a parametric picture. In the present section we therefore set this pre-determined value of air film thickness, where the droplet is considered to have reached touchdown, as
$H_{\text{min}} = 0.1$ (the solutions in figures 2.4 and 2.6 are plotted up until $H_{\text{min}} = 0.2$, for comparison).

Figure 2.8 shows how the dimensionless area of entrapped air $B$ depends on the surface parameters $\tilde{e}_3$ and $\tilde{e}_5$, for the viscoelastic surface model. Clearly, for any decrease in magnitude of $\tilde{e}_3$ or $\tilde{e}_5$ there is an increase in the area of entrapped air. The dimensional entrapped air area $b^*$ can be related to the dimensional area $B$ by applying the vertical and horizontal length scales, which yields

$$b^* = \varepsilon^3 R^2 B = \frac{H_g R}{\rho V} B,$$  \hspace{1cm} (2.31)

where the pre-factor $B$ is, in the viscoelastic model, a function of the parameters $\tilde{e}_3$ and $\tilde{e}_5$ and its dependency is shown in figure 2.8 for parameters in the range $[-10,-0.1]$. For parameter values of magnitude lower than 0.1, the area of entrapped air continues to increase and for parameters of magnitude higher than 10 the behaviour of the droplet is identical to that of a flat rigid surface.

Equation (2.31) for the dimensional entrapped air area highlights the importance of the droplet velocity and radius on the size of entrapped air at impact. For a flat rigid surface, where $B$ is a given constant, droplets with a larger radius will
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entrap more air due to having a larger free-surface to interact with the air prior to impact and droplets with a larger velocity will entrap less air due to the droplet having less time to deform prior to impact. In the viscoelastic model, the numerical pre-factor $B$ depends on $\tilde{e}_3$ and $\tilde{e}_5$ and formulas for these parameters are given in (2.25). It is interesting to note that the parameter $\tilde{e}_3$ does not depend on the droplet radius $R$. It is therefore of interest to see how the size of entrapped air varies with droplet velocity and radius for given fluid and structural parameters (the structural parameters of concern here are the stiffness $\kappa^*$ and the damping $C^*$).

Figure 2.9 gives a comparison produced by results for the dimensional entrapped air area of a water droplet in air over wide ranges of the droplet radius $R$ and velocity $V$, for a flat rigid surface and soft viscoelastic surfaces. We chose to vary the droplet velocity from $0.1 \text{ m s}^{-1}$ to $2 \text{ m s}^{-1}$ and the radius from $10^{-4}$ m to $10^{-2}$ m in order to keep the parameter regime mostly contained in the valid model parameter regime given in figure 2.2. Figure 2.9(a) shows the variations of the entrapped air area $b^*$ as the droplet velocity and radius vary, comparing a flat rigid surface to a soft viscoelastic surface with spring stiffness $\kappa^* = 6 \text{ MPa m}^{-1}$ and (viscous) damping $C^* = 20 \text{ kPa s}$. For all combinations of droplet velocity and droplet radius, more air is entrapped by the soft viscoelastic surface, as is to be expected. For small droplet impact velocities, the effect of the soft solid is minimal and it behaves much like a rigid surface. As the droplet velocity increases, the effect of the soft viscoelastic surface is far more profound. From equation (2.31) we can see that for a flat rigid surface the area of entrapped air decreases with increased droplet velocity, whereas for the soft viscoelastic surface this decrease in entrapped air area is delayed and even halted for increased droplet velocity, for the parameters under consideration here. This can be seen more clearly in figure 2.9(c) where the entrapped air area is given as a function of droplet velocity for a 1 mm water droplet in air, for a variety of spring stiffness $\kappa^*$ and (viscous) damping $C^*$. This is due to higher air film pressures, induced by higher droplet velocities, causing larger surface deformation prior to impact. By considering the shape of the contour lines in figure 2.9(a), variations of the droplet radius have a less profound influence on the
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Figure 2.9: For impact of a water droplet in air, (a) variations in the entrapped air area $b^\ast(\mu m^2)$ as the droplet velocity $V(m s^{-1})$ and radius $R(m)$ vary for a flat rigid surface (solid line) and a soft viscoelastic surface with spring stiffness $\kappa^\ast = 6$ MPa m$^{-1}$ and (viscous) damping $C^\ast = 20$ kPa s (dashed line), (b) entrapped air area for a droplet velocity of 1 m s$^{-1}$ and variations in droplet radius and (c) entrapped air area for a droplet of radius 1 mm and variations in droplet velocity. In figures (b–c) results are presented for a variety of spring stiffness $\kappa^\ast$ and (viscous) damping $C^\ast$. 
increase in entrapped air on a soft viscoelastic solid compared to a flat rigid surface. This is shown more clearly in figure 2.9(b), where the entrapped air area is plotted as a function of droplet radius for an impact velocity of 1 m s\(^{-1}\). On a logarithmic scale, the increased amount of entrapped air due to impact onto a soft viscoelastic surface is almost independent of droplet radius.

### 2.1.3 Impact on a flexible surface

We now turn our attention to analysing the air cushioning effect in droplet impact onto a more general flexible surface, where all terms except for the damping term in equation (2.24) are retained. The model here is an elastic plate, or membrane, sitting atop a bed of springs, which is akin to many previous studies on fluid-flexible surface interactions [96, 97, 98, 99, 100, 101, 102]. The depth of the bed of springs is assumed to be much larger than \(O(\varepsilon^2 R)\), and so the influence of the rigid base on which the springs are based can be ignored. Thus, this rather general model is a semi-infinite flexible surface which includes all relevant physics in two-dimensional deformation. This case is similar to 2.1.2, however now the influence of the elastic plate is considered and so physical parameters to account for surface rigidity, tension and inertia are now also included and examined. A schematic of the problem
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In practise, the magnitude of the dimensionless parameters $\tilde{e}_i$ can vary dramatically and over a number of orders of magnitude. By considering figure 2.2 we can see the range of droplet velocities and radii where our model is applicable, and it is the structural parameters that will require special attention. Explicit formulae for the parameters $\tilde{e}_i$ are given in (2.25) and if we consider a water droplet of radius 1 mm and impact velocity 1 m s$^{-1}$ we are able to obtain an order of magnitude estimate of the dimensionless parameters $\tilde{e}_i$ in terms of their corresponding structural coefficient,

$$\left(|\tilde{e}_1|, |\tilde{e}_2|, |\tilde{e}_3|, |\tilde{e}_4|\right) \sim (10^7B^*, 10^{-2}T^*_i, 10^{-11}\kappa^*, 10^1M^*).$$  \hspace{1cm} (2.32)

Therefore, if we wish for the dimensionless surface parameters $\tilde{e}_i$ to be of order one, plus or minus a few orders of magnitude, then we now know what order of magnitude the dimensional structural parameters need to be in our model. If we wish to consider the rather large range of parameter magnitudes $10^{-4} < |\tilde{e}_i| < 10^4$, then this implies our structural parameters are of the order $10^{-11} < B^* < 10^{-3}$, $10^{-2} < T^*_i < 10^6$, $10^7 < \kappa^* < 10^{15}$ and $10^{-5} < M^* < 10^3$. Rather unsurprisingly, we require a flexural rigidity $B^*$ of small magnitude. The flexural rigidity can be defined by the formula

$$B^* = \frac{Eh^3}{12(1-\nu^2)},$$  \hspace{1cm} (2.33)

where $E$ is the Young’s Modulus of the material, $h$ is the elastic thickness and $\nu$ is the Poisson ratio. Therefore, for the flexural rigidity $B^*$ to be of a suitably small magnitude, we would require a combination of a relatively low Young’s Modulus combined with thin plates. Examples of impacts involving such a combination exist in nature, such as impact onto leaves [17], and for impact onto thin foils of certain materials [104].

The surface shape equation (2.24) is then solved numerically, again coupled with equation (2.14) and (2.22). The boundary conditions and the numerical treatment are slightly different from 2.1.2, due to the spatial derivatives now present in
the surface equation. Because of the spatial derivatives, boundary conditions for
the surface shape $G$ are required now at fixed positions. We choose clamped plate
boundary conditions, which are

$$G = \frac{\partial G}{\partial X} = 0, \quad \text{at } G = X_1, X_2,$$

(2.34)

where $X_1, X_2$ are given $X$ stations and will be stated in each case. Typically
we take $X_1, X_2$ as the end points of the computational domain. The same ef-
fficient numerical scheme applied in 2.1.2 cannot practically be used here be-
cause of the increased numerical complexity due to sixth order derivatives oc-
curring when substituting the surface equation (2.24) into the lubrication equa-
tion (2.22). We therefore adopt a more standard approach of iterating between
each equation (2.14), (2.22) and (2.24) and solving for $P, F$ and $G$, respec-
tively, until convergence, using the same algorithms outlined in Sec. 2.1.2. A
detailed description of this method is given in Appendix A.1 and the code can
be found at: https://github.com/NatHenman/PhD_thesis/blob/
main/pre_impact_flexible_code.m.

Figure 2.11 shows the solution profiles for the free-surface $F$ and the pressure
$P$ for three different values of the stiffness parameter $\tilde{e}_3$, with the other surface
parameters fixed. In practice zero stiffness is not possible, hence, while all the
other parameters are fixed, $\tilde{e}_3 \to 0^-$ is a limiting scenario. The dashed line in each
plot of figure 2.11(a) corresponds to the solution of the flexible surface shape $G$ as
touchdown is approached. The results illustrate the outcomes of variations in the
stiffness parameter $\tilde{e}_3$, however variations in any of the other surface parameters
yield qualitatively similar results. The solution in the flat surface case is not shown
here for brevity, see figure 2.4 for comparisons. As in the flat surface case, the
droplet free-surface curvature is large at the touchdown point in all cases here, and
so surface tension effects would become significant here again.

The aim of this section is to highlight the effect surface flexibility can have
on the pre-impact phase of droplet impact, as opposed to the effect of a soft de-
formable surface considered in 2.1.2. Figure 2.11 shows that much of the same
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Figure 2.11: Solution profiles, showing evolution of (a) the free-surface height \( F \) and (b) the pressure \( P \) for normal impact of a droplet onto a flexible surface with parameters \((\tilde{e}_1, \tilde{e}_2, \tilde{e}_3, \tilde{e}_4, \tilde{e}_5) = (-1, 1, \tilde{e}_3, -1, 0)\) and \( \tilde{e}_3 = -1, \tilde{e}_3 = -0.1 \) and \( \tilde{e}_3 \to 0^- \). The boundary of the flexible surface is at \([X_1, X_2] = [-10, 10]\). The solutions are shown in integer time increments except for the final thick solid line, which is the solution just prior to touchdown, and the dashed line, which is the flexible surface solution \( G \) just prior to touchdown. The dash-dotted line is the solution at \( T = 0 \) where the droplet would touchdown on an undisturbed surface in the absence of air cushioning.
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Figure 2.12: The (a) minimum air film thickness and the (b) maximum pressure as a function of time $T$ for a flat rigid surface and flexible surfaces with coefficients $(\tilde{e}_1, \tilde{e}_2, \tilde{e}_3, \tilde{e}_4, \tilde{e}_5) = (-1, 1, \tilde{e}_3, -1, 0)$, a range of values of $\tilde{e}_3$ and boundary $[X_1, X_2] = [-10, 10]$. In (a), the dashed line corresponds to the solution in the absence of air cushioning.

Figure 2.13: As figure 2.12, except showing (a) air film thickness $H$ and (b) pressure $P$ near touchdown.
conclusions can be drawn. First, increased surface flexibility leads to a delay to touchdown. This delay to touchdown acts to further decelerate the droplet free surface and results in a lower pressure build-up underneath the droplet. The pressure peak amplitude is again lower near touchdown for more flexible surfaces. Figure 2.12 shows the minimum air film thickness and the maximum pressure as functions of time, where it can be seen that this lower pressure buildup is present for all time. Figure 2.13 compares the solutions of the air film thickness and the pressure near touchdown, showing clearly the reduced pressure peak and increased air entrapment near touchdown for reductions in magnitude of the stiffness parameter $\tilde{e}_3$.

The main difference to highlight when comparing air cushioning behaviour of droplet impact on a flexible surface to impact on a soft viscoelastic surface is the shape of the pressure profile upon touchdown. Here, in figure 2.11, reductions in the magnitude of the parameter $\tilde{e}_3$ lead to reductions in the amplitude of the pressure peak near touchdown, but the peak remains sharp. This is unlike the finding for the viscoelastic surface, where increased deformability leads to a decrease in the pressure peak amplitude and also a rounder, softer peak. This could be due, in part, to the more abrupt approach to touchdown seen in figure 2.12, for a flexible surface, compared with the gentler approach seen in figure 2.5, for a soft viscoelastic surface. Also, the droplet free surface cusp remains relatively sharp in the flexible surface case as deformability increases, while in the viscoelastic case it becomes rounded, which is likely to play a key role in the shape of the pressure peak. However, as the droplet free-surface becomes sharp, surface tension effects will become significant and would need to be considered here in the flexible surface case. At leading order, the local touchdown behaviour here is identical to that described in Smith, Li and Wu [30].

Defined in the same way as in Sec. 2.1.2, figure 2.14 shows the positive touchdown point $X_d$ as a function of shifted time $T - T_0$. Again, it is clear that a more deformable flexible surface results in the positive touchdown point moving outwards at a slower speed, a possible mechanism for reduced splashing.

Using equation (2.30) we are again able to make a qualitative assessment of
how variations in the surface properties affect the size of entrapped air. Here, we perform a study of how individual variations in the parameters $\tilde{e}_1$, $\tilde{e}_2$, $\tilde{e}_3$ and $\tilde{e}_4$ (with $\tilde{e}_5 = 0$) alter the area of entrapped air at touchdown. The size of the air gap at which the calculation of equation (2.30) is performed is 0.26 for this model, which is the smallest achievable for all compared results and larger than that considered in 2.1.2 due to the more difficult numerical task.

Figure 2.15 shows how individual variations in the surface parameters $\tilde{e}_1$, $\tilde{e}_2$, $\tilde{e}_3$ and $\tilde{e}_4$ affect the entrapped air area at touchdown. Logarithmic variations of each parameter lead to a similar dependency on the entrapped air area. Each figure exhibits plateauing behaviour to the flat rigid surface value for large magnitude parameters and to the value for the solution as the parameter tends to zero, except for figure 2.15(d), where, in the current setting, variations of the magnitude of the mass density parameter $\tilde{e}_4$ much lower than 1 lead to an unbounded surface velocity $\partial G/\partial T$ at some point. From figure 2.15 it is clear that any increase in flexibility of the surface leads to an increase in entrapped air.

### 2.1.4 Connections with experiments

In 2.1.2 and 2.1.3 we performed a numerical study into the air cushioning phase of droplet impact onto deformable surfaces. Here, we now attempt to make connections to recent experiments. Due to our work being two dimensional and approxi-
Figure 2.15: Area of entrapped air $B$ for droplet impact onto flexible surfaces, with a boundary of $[X_1, X_2] = [-10, 10]$ and surface parameters

(a) $(\tilde{e}_1, \tilde{e}_2, \tilde{e}_3, \tilde{e}_4, \tilde{e}_5) = (\tilde{e}_1, 1, -0.1, -1, 0)$ for variations in $\tilde{e}_1$, (b) $(-1, \tilde{e}_2, -0.1, -1, 0)$ for variations in $\tilde{e}_2$, (c) $(-1, 1, \tilde{e}_3, -1, 0)$ for variations in $\tilde{e}_3$ and (d) $(-1, 1, -0.1, \tilde{e}_4, 0)$ for variations in $\tilde{e}_4$. In each figure, the dash-dotted line corresponds to the value for a flat rigid surface and the dashed line corresponds to the limiting value as the relevant parameter tends to zero. In (d) the limit as the mass density parameter $\tilde{e}_4$ tends to zero cannot be found, due to an unbounded surface velocity.
mate, using a number of assumptions on the fluid and structural parameters, these connections are tentative, but they seem encouraging nonetheless.

In Langley et al. [42] it was found, experimentally, that droplet impacts onto softer, more deformable, solids would entrap more air. They performed experiments using ultra-high-speed interferometry to capture the droplet free surface at impact, and varied the droplet velocity and the surface stiffness. We have qualitative agreement with these findings in our analytical results. We found in Sec. 2.1.2 and 2.1.3 that reductions in surface stiffness resulted in an increase in entrapped air. A more subtle point to make from our analytical work is the non-intuitive dependency of the size of entrapped air on the droplet velocity. On a rigid flat surface, the size of entrapped air will decrease for higher velocities, whereas for impact onto a soft viscoelastic surface this reduction is delayed and even halted for increased impact velocities. This is alluded to in Langley et al. [42] also and, although described in terms of gas compression while our work is incompressible, they show that entrapped air can increase with increased droplet velocity for impact onto soft solids. The increase in entrapped air on more deformable surfaces is also likely to influence the overall outcome of the droplet impact, with implications on rebounding [43, 44, 45] and splashing [26] droplets

Howland et al. [65] focused their study on how soft deformable surfaces affect droplet splashing. They performed experiments by impacting ethanol droplets on silicone or acrylic substrates of varying stiffnesses, and found that the lower the stiffness, the less likely the droplet was to splash. This was found to be most likely due to higher stiffness substrate having sheet ejection of higher velocity, resulting in the sheet leaving the surface and breaking up, forming a corona splash. By contrast, for less stiff surfaces, the ejected sheet is of lower velocity and does initially leave the substrate, but can fall back down onto the substrate and slow down, suppressing the splash. They were unable to investigate the sheet ejection experimentally, because of the small time and length scales associated with it. Hence, they investigated it using numerical simulation, and found that lowering the surface stiffness reduced the contact pressure just before the sheet is ejected. Our results in Sec.
2.1.2 and 2.1.3 show clearly that reducing the surface stiffness leads to a reduction in the pressure peak just prior to impact. In particular, the results in Sec. 2.1.2 for droplet impact onto a viscoelastic solid show a softening of the pressure peak prior to touchdown. Also, our results have highlighted that a decrease in surface stiffness results in the touchdown points prior to impact moving outwards at slower speeds. Although our study is solely focused on pre-impact behaviour, this reduction in pressure and lower touchdown point speed appear important in understanding the reduced sheet ejection speed mentioned in Howland et al. [65], which results in potential splash suppression.

2.1.5 Large surface deformation

In this section we examine the effect of large surface deformations on the system. Here, we will consider the system in the limit $\tilde{e}_1 \to 0$, $\tilde{e}_2 \to 0$, $\tilde{e}_3 \to 0$ and $\tilde{e}_5 \to 0$, where equation (2.24) may be written simply as $\tilde{e}_4 \partial^2 G / \partial T^2 = P$. What this allows us to do is reduce our previous system of three equations to two, by writing $H = F - G$, the air film thickness. The new system is now

$$\frac{\partial^2 H}{\partial T^2} = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\partial P(\zeta, T)}{\partial \zeta}(\zeta, T) \frac{d\zeta}{X - \zeta} - \frac{P}{\tilde{e}_4}, \quad (2.35a)$$

and

$$\frac{\partial}{\partial X} \left(H^3 \frac{\partial P}{\partial X}\right) = 12 \frac{\partial H}{\partial T}. \quad (2.35b)$$

Equations (2.35) can be further simplified by rescaling all the variables to account for $|\tilde{e}_4|$. We are interested in the case of $\tilde{e}_4$ being small (and negative) and the time scale being large, as for large surface deformations we expect touchdown to be further delayed. Hence we take the following rescaling,

$$(H, P, X, T) = (|\tilde{e}_4|^{-1} \tilde{H}, |\tilde{e}_4|^2 \tilde{P}, \tilde{e}_4^{-1/2} \tilde{X}, |\tilde{e}_4|^{-1} \tilde{T}); \quad (2.36)$$

then equations (2.35) reduce to, dropping the overbar notation,

$$\frac{\partial^2 H}{\partial T^2} = P, \quad (2.37a)$$
\[
\frac{\partial}{\partial X} \left( H^3 \frac{\partial P}{\partial X} \right) = 12 \frac{\partial H}{\partial T},
\]
(2.37b)

where the relative error in ignoring the Cauchy integral is \( O(|\tilde{e}|^{3/2}) \), which is suitably small. We impose at \( |X| \to \infty \) and \( T \to -\infty \)

\[
H \sim \frac{X^2}{2} - T
\]
(2.38a)

\[
P \sim 0
\]
(2.38b)
as the far-field conditions. Ignoring the Cauchy integral and the scales defined in (2.36) imply that the deformable surface is deforming at an air pressure much lower than that of the droplet free surface. So, in essence the scaled system here describes a solid body impacting a highly deformable surface that deforms due to air pressures prior to impact. The film thickness \( H = \frac{X^2}{2} - T - G \), where \( G \) is the scaled deformable surface shape. For brevity, the following analysis will proceed with the film thickness \( H \).

As was first remarked in Korobkin et al. [109] for a different model, the pressure-shape law (2.37a) can be viewed as an alternative to the Cauchy-Hilbert law in Smith et al. [30], such as in pressure-displacement laws in interacting boundary layers [110, 111]. The new coupled system (2.37) allows us to make far more analytical progress than for the previous system examined computationally in 2.1.2 and 2.1.3.

The coupled system (2.37) was solved numerically using a scheme identical to that outlined in 2.1.2, with the new local pressure-shape relationship (2.37a). As is to be expected, this scheme is far faster than the one involving the Cauchy-Hilbert integral. An appropriate spatial domain here was found to be \([-20, 20]\).

Figure 2.16 shows the time-marched solutions of \( H \) and \( P \). The mechanisms are essentially the same as before. The droplet is released at some suitably large negative time and begins to approach the surface. The air film thickness is initially parabolic and as the air film thickness begins to decrease, the pressure begins to rise in a single peak and results in the air film thickness deviating from the parabolic
shape. The pressure then starts to have two peaks as the air film thickness approaches zero in two different locations equidistant from the center line.

The solutions of $H$ and $P$ in Figure 2.16 show clearly two traits of droplet impact on a very deformable surface that can be intuitively expected from the results in 2.1.2 and 2.1.3. Figure 2.16 shows solutions up until $T = 60$. The longer time scales associated here are clear to see and it is expected that touchdown is not reached in finite time. We can also see that the horizontal bubble extent has significantly increased.

The computational results given in figure 2.16 show that the time scales involved in this system are large and suggest that perhaps touchdown in finite time is not reached on this time scale. Hence, we seek a solution at large positive time. Suppose that the scaled air thickness and pressure are rising relatively fast in spatial terms, near a region where $X = cT^\alpha$ say, with $\alpha$ a positive constant. The length scaling in this region then takes the form

$$X = cT^\alpha + T^m \xi, \quad \text{as } T \to \infty,$$

(2.39)

where $\xi$ is order unity, the constants $\alpha$ and $m$ are unknown and in order for the region to be local we require that $m < \alpha$. Expansions of $H$ and $P$ are then taken in the rather general form

$$H \sim T^\lambda \hat{H}(\xi),$$

(2.40a)
\[ P \sim T^{\lambda + 2\alpha - 2m - 2} \hat{P}(\xi) \]  

(2.40b)

where \( \lambda \) is an unknown constant. It is to be expected that \( \lambda \leq 0 \) as \( H \) is not growing in time, and the expansion of \( P \) is inferred from seeking a balance in equation (2.37a). Substitution of the expansions (2.40) into the governing equations (2.37) leads to the system

\[ \alpha^2 c^2 \frac{d^2 \hat{H}}{d\xi^2} = \hat{P}, \]  

(2.41a)

\[ \frac{d}{d\xi} \left( \hat{H}^3 \frac{d\hat{P}}{d\xi} \right) = -12\alpha c \frac{d\hat{H}}{d\xi}, \]  

(2.41b)

to leading order, subject to

\[ m = \lambda + \frac{\alpha - 1}{3} \]  

(2.42)

and also \( 3\lambda < 2\alpha + 1 \). The system (2.41) can be further reduced to an ordinary differential equation in \( \hat{H} \) alone,

\[ \hat{H}^3 \frac{d^3 \hat{H}}{d\xi^3} = -\frac{12}{\alpha c} \hat{H} + k, \]  

(2.43)

where \( k \) is an integration constant and must be positive in order to keep \( \hat{H} \) positive. Equations of the form (2.43) occur commonly in a number of applied mathematics problems [33, 112, 113] and it is expected that the solution \( \hat{H} \) will tend to a non-zero constant \( \alpha ck/12 \) as \( \xi \to -\infty \). Perturbations to this constant value occur, such that

\[ \hat{H} \sim \frac{\alpha ck}{12} + \Re[\gamma_1 \exp(\gamma_2 \xi)] \quad \text{as} \quad \xi \to -\infty, \]  

(2.44)

where \( \Re \) denotes the real part and \( \gamma_1 \) and \( \gamma_2 \) are complex constants. Note, if we were to take the perturbation in the form \( \gamma_1 \exp(\gamma_2 \xi) \), where \( \gamma_1 \) and \( \gamma_2 \) are real constants, then that would lead to \( \gamma_2 < 0 \) and thus an exponentially growing perturbation. The complex constant \( \gamma_2 \) satisfies \( \gamma_2^3 = -q \), where \( q = (12/\alpha ck^{3/4})^4 \), and the complex constant \( \gamma_1 \) remains arbitrary. There are then three possible solutions for \( \gamma_2 \), and we wish to choose the ones such that \( \Re[\gamma_2] > 0 \) so that we have a decaying perturbation. There are two such solutions, \( \gamma_2 = q^{1/3}(1 \pm i\sqrt{3})/2 \), which leads to the large
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Figure 2.17: Solution of equation (a) (2.43) for $\hat{H}$, and then (b) (2.41a) for $\hat{P}$, given $\hat{H}$, in normalised form for $\gamma_1 = \gamma_2 = 1$. The inset of both figures shows the corresponding local solution of the full system (2.37) at $T = 60$ for (a) $H$ and (b) $P$.

negative $\xi$ asymptote taking the form

$$
\hat{H} \sim \frac{\alpha ck}{12} \left\{ \gamma_{1,1} \cos \left( \frac{\sqrt{3}}{2} q^{1/3} \xi \right) + \gamma_{1,2} \sin \left( \frac{\sqrt{3}}{2} q^{1/3} \xi \right) \right\}
$$

as $\xi \to -\infty$, (2.45)

where $\gamma_{1,1}$ and $\gamma_{1,2}$ remain arbitrary, still. For large positive $\xi$ it can be readily shown that

$$
\hat{H} \sim \lambda_1 \xi (3 \ln \xi)^{1/3} \quad \text{as} \quad \xi \to \infty, \quad (2.46)
$$

where $\lambda_1 = (12/\alpha c)^{1/3}$. From equation (2.41a) we can also show that the corresponding $\hat{P}$ asymptote is

$$
\hat{P} \sim \lambda_2 \xi^{-1} (3 \ln \xi)^{-2/3} \quad \text{as} \quad \xi \to \infty, \quad (2.47)
$$

where $\lambda_2 = \alpha^2 c^2 \lambda_1$.

Equation (2.43) was then solved numerically as a boundary value problem using Newton-Raphson iterations, with the asymptotes for $\hat{H}$ (2.45-2.46) imposed at suitable large negative and positive $\xi$ values, respectively. The corresponding solution for $\hat{P}$ is then calculated from equation (2.41a) using central finite differences.
Without loss of generality, the constants $\alpha^2 c^2$, $12/(\alpha c)$ and $k$ can be normalised to unity by a division of $\hat{H}$, $\xi$, $\hat{P}$ by $\alpha c k/12$, $(\alpha c/12)^{4/3} k$, $(12^5 \alpha c)^{1/3} k$ in turn. Figure 2.17 shows the solution of $\hat{H}$ and $\hat{P}$ for $\gamma_{1,1} = \gamma_{1,2} = 1$. Variations in the value, even sign, of the parameters $\gamma_{1,1}$ and $\gamma_{1,2}$ yield identical results to that presented in figure 2.17.

The solutions given in figure 2.17 show excellent agreement locally with the solutions of the full system (2.37), solved numerically and presented in figure 2.16, at large time, especially the solution found for $\hat{P}$ which is qualitatively identical to the solution of $\bar{P}$ at large times (the large time solution to the full system is shown in the inset of figure 2.17, for comparison).

Now, let us move further rightwards into positive $\xi$ by considering $\xi = \mathcal{D} \bar{X}$, where $\mathcal{D} \gg 1$ and $\bar{X}$ is of order unity. We can infer expansions of $\hat{H}$ and $\hat{P}$ from their large positive $\xi$ asymptotes (2.46-2.47),

\[
\hat{H} = \mathcal{D}(\hat{H}_0(3 \ln \mathcal{D})^{1/3} + \hat{H}_1(3 \ln \mathcal{D})^{-2/3} + \cdots),
\]

(2.48a)

\[
\hat{P} = \mathcal{D}^{-1}(\hat{P}_0(3 \ln \mathcal{D})^{-2/3} + \hat{P}_1(3 \ln \mathcal{D})^{-5/3} + \cdots).
\]

(2.48b)

To leading order, equation (2.43) then yields $d^3\hat{H}_0/d\bar{X}^3 = 0$ and matching requires $\hat{H}_0 \sim \lambda_1 \bar{X}$ as $\bar{X} \to 0^+$. Hence

\[
\hat{H}_0 = \lambda_1 \bar{X} + \mu_1 \bar{X}^2,
\]

(2.49)

where $\mu_1$ is a positive constant. Now we can see that, at leading order for both $\hat{H}$ and $\hat{P}$,

\[
\hat{H} \sim \mu_1 X^2,
\]

(2.50a)

\[
\hat{P} \sim 0
\]

(2.50b)

as $\bar{X} \to \infty$, which is a form resembling the far field condition (2.38). In light of (2.39), (2.40a) and (2.50a), $H$ emerges as

\[
H \sim \mu_1 \mathcal{D}^{-2} T^{-2(1-\alpha)/3-\lambda}(X - cT^\alpha)^2
\]

(2.51)
just to the right of the local zone, which, as far as the $X^2$ requirement in (2.38a) is concerned, indicates that the constants must satisfy

$$\alpha = 1 - \frac{3\lambda}{2}, \quad m = \frac{\lambda}{2};$$  \hspace{1cm} (2.52)

hence any $\lambda \leq 0$ works here. To the right of the above local region we have the balance $H_{TT} \sim 0$, which when matching to (2.38a) at infinity gives the solution

$$H \sim \frac{X^2}{2} - T,$$  \hspace{1cm} (2.53)

which in turn matches with the far-field solution.

The most acceptable looking solution to arise from this analysis would be if $\lambda = 0$, resulting in $\alpha = 1$ and $m = 0$, with $\xi = X - cT$ and $H$ and $P$ depending on $\xi$ alone. This nonlinear travelling wave form is described fully by (2.39-2.53), capturing the large time features of the full system (2.37) successfully and confirming the absence of touchdown on this time scale.

The analysis here is a similar one to that seen in Purvis and Smith [33] and is analogous to a finite-time break up formulation [30, 114]. A physical interpretation of this large time analysis of the large surface deformation system could well be the increased gliding extent observed in Langley et al. [42] for droplet impacts onto soft solids. Gliding occurs when the droplet does not make contact with the substrate at the kink of the dimple, and instead glides on a thin air layer. In Langley et al. [42] this is seen to occur more frequently and to a greater extent for more deformable surfaces. This could be interpreted as what is occurring in the numerical results for the reduced system in figure 2.16, with the increased time scales and horizontal extent of the kink being potential traits of droplet gliding in the absence of defects and localised wetting [115].

### 2.2 Lubricant-infused surfaces

For the same reasons as the deformable surface scenario, understanding the pre-impact dynamics of droplet impact on lubricant-infused surfaces is also vitally im-
2.2. Lubricant-infused surfaces

Important. To recap, a lubricant-infused surface is a textured or porous surface that has a thin layer of lubricant wicked into it. These surfaces have solely received attention regarding the post-impact dynamics [67, 68, 69, 71] and no significant modelling has been performed for the pre-impact phase. Modelling of the lubricant-infused surface using the same inviscid-lubrication model of Sec. 2.1 is challenging however, due to the complexity of the surface and the introduction of a third phase: the lubricant. We therefore make an approximation of this surface to be a flat surface with a modification of the boundary condition. A partial-slip boundary condition will be applied to model the surface, a condition which has been used before in many studies regarding lubricant-infused surfaces [9, 116]. This model is similar to that of Hicks and Purvis [41], who modelled the pre-impact dynamics of droplet impact on porous surfaces, however it is unique in the sense that it will only allow slip flow across the interface of the surface, and no vertical penetration. This is a key distinction between lubricant-infused surface and textured, superhydrophobic surfaces.

2.2.1 Model formulation and governing equations

Here, we shall assume exactly the same model set-up as Sec. 2.1 but with an alternative representation of the conditions at the impact surface. The surface is now assumed to be a lubricant-infused surface, as opposed to a deformable surface. The lubricant-infused surface is assumed to be a two-dimensional array of rectangular pillars with a fluid different from the surrounding gas filling the space between the pillars. The top of the pillars is located at $y^* = 0$ and the pillars are assumed to have typical horizontal and vertical length scales of $W$ and $H$, respectively. The fluid between the pillars is assumed to be initially flat and level with the top of the pillars. Figure 2.18 shows a schematic of the problem set-up.

The non-dimensionalisation process is the same as in Sec. 2.1, with all lengths scaled with $R$, velocities with $V$ and time with $R/V$. The assumptions on the scale of Re and $\varepsilon$ that lead to potential flow in the droplet and lubrication flow in the gas also remain the same.

We will assume that the vertical and horizontal length scales of the lubricant-
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Figure 2.18: Schematic of the model problem of a two-dimensional droplet of radius $R$ approaching a lubricant-infused surface with velocity $V$, with an air film in-between. The horizontal and vertical length scales of the lubricant-infused surface are $W$ and $H$, respectively. The impact is cushioned by the air film between the droplet and the deformable surface. This schematic is not drawn to scale and the surface asperities continue infinitely in the positive and negative horizontal directions.

Figure 2.19: Schematic representation of the scaled, non-dimensional problem set-up of droplet impact onto a lubricant-infused surface pre-impact (this is not a solution).
infused surface pillars are both $O(\varepsilon^2R)$ and that any deformation of the lubricant meniscus [117] is much smaller than $\varepsilon^2R$. Therefore the surface is flat to leading order, located at $y = 0$. However, since the depth of the surface asperities is of the same order as the gas film thickness prior to impact, it is likely to have a discernible influence on the dynamics. This effect will be due to slippage occurring on the interface between the gas and the lubricant. Hicks and Purvis [41] considered a similar problem of gas cushioning in droplet impacts with porous surfaces, which were modelled as a rectangular array of pillars but with the same fluid as the surrounding gas being contained in between the pillars. Their model allowed gas to escape vertically into the pillars, as well as slip on the interface, and the flow in the gas needed to be coupled with the flow in the pillars. Our model is somewhat simpler, since gas cannot escape vertically into the pillars due to the presence of the lubricant. Returning to dimensional variables, the influence of the lubricant-infused surface is modelled as a partial-slip condition [9, 116] at the interface $y^* = 0$

$$u^*_g = \lambda^*(x^*) \frac{\partial u^*_g}{\partial y^*}$$

(2.54)

where $\lambda^*(x^*)$ is the slip length, which is usually interpreted as the depth below the surface at which the velocity would extrapolate to zero. When the slip length $\lambda^* = 0$ we have the no-slip condition and when $\lambda^* \to \infty$ we have the no-shear condition. Re-writing the boundary condition (2.54) in the non-dimensional and scaled formulation as in Sec. 2.1 gives at $Y = 0$

$$U_g = \Lambda \frac{\partial U_g}{\partial Y},$$

(2.55)

where $\Lambda = \lambda^*/\varepsilon^2R = O(1)$. From here onwards we will only consider cases where $\Lambda$ is constant. Figure 2.19 shows a schematic of the scaled, non-dimensional problem.

Now, solving the lubrication equations (2.17) subject to the boundary condition
(2.55) yields the horizontal velocity component in the gas film as

\[ U_g = \frac{1}{2} \left( \frac{Y^2(F + \Lambda) - F^2(Y + \Lambda)}{F + \Lambda} \right) \frac{\partial P}{\partial X}. \] (2.56)

Here, since no gas is able to escape into the lubricant-infused surface and the free surface of the lubricant is assumed to be flat at leading order, we apply the no-penetration condition \( V = 0 \) on the vertical velocity component at \( Y = 0 \). This, together with the continuity equation (2.17c), allows us to derive a lubrication equation of the form

\[ \frac{\partial F}{\partial T} = \frac{1}{12} \frac{\partial}{\partial X} \left( \frac{F^3(F + 4\Lambda)}{F + \Lambda} \frac{\partial P}{\partial X} \right). \] (2.57)

It can be seen here that for \( \Lambda = 0 \) we are left with the original lubrication equation for the gas film in gas cushioning droplet impacts with a flat plate [30]. For an infinite slip length \( \Lambda \to \infty \) we obtain a similar equation to the flat plate case [30] but with a 1/3 factor instead of the classic 1/12 factor.

The solution here is again sought numerically. Equation (2.57) is solved in conjunction with the Cauchy integral equation (2.14) with identical boundary conditions on \( F \) and \( P \) as in Sec. 2.1. The numerical method to solve the system is also identical to that in Sec. 2.1, with a spatial range of \( X \in [-10, 10] \) being suitable and sufficient here as increasing the slip length \( \Lambda \) does not increase the horizontal extent of the pressure peaks. The code used to solve this system can be found at: https://github.com/NatHenman/PhD_thesis/blob/main/pre_impact_LIS_code.m.

### 2.2.2 Effect of the slip length

The only parameter that needs to be considered and investigated here is the slip length \( \Lambda \). Figure 2.20 shows the time evolution of the free-surface \( F \) and the pressure \( P \) for \( \Lambda = 0, 1, 10 \) and \( \Lambda \to \infty \). As time advances, the same process as in Sec. 2.1 occurs, with the pressure building as the air gap closes at \( X = 0 \) until the point at which it is sufficient to halt the free-surface, after which the free-surface begins to approach the solid surface either side of the center and the pressure bifurcates into two peaks as the air gap closes in two places.
Figure 2.20: Solution profiles, showing evolution of (a) the free-surface height $F$ and (b) the pressure $P$ for normal impact of a droplet onto a lubricant-infused surface with a slip length of $\Lambda = 0, 1, 10$ and $\Lambda \to \infty$. The solutions are shown in integer time increments except for the final thick solid line, which is the solution just prior to touchdown. The dash-dotted line is the solution at $T = 0$ where the droplet would touchdown on an undisturbed surface in the absence of air cushioning.
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Figure 2.21: (a) The air film thickness $F$ and (b) the pressure $P$ at touchdown for $\Lambda = 0, 1, 10$ and $\Lambda \to \infty$.

Figure 2.22: The trapped air area $B$ as a function of the slip length $\Lambda$. The solid line corresponds to the result when $\Lambda = 0$ and the dashed line for $\Lambda \to \infty$. 
2.2. Lubricant-infused surfaces

Figure 2.23: The (a) minimum air film thickness and the (b) maximum pressure as a function of time $T$ for $\Lambda = 0, 1, 10$ and $\Lambda \to \infty$. In (a), the dashed line corresponds to the solution in the absence of air cushioning.

It can clearly be seen from figure 2.20 that increasing the slip length $\Lambda$ results in a smaller pocket of trapped air prior to impact. However, unlike in the case of a porous surface [41], the trapped air cannot be completely suppressed and is still present as $\Lambda \to \infty$. The free-surface and pressure profiles at touchdown (in this section this is taken to be when the thickness of the air film reaches 0.2) are compared in figure 2.21. The reduced size of the trapped air coincides with a reduced horizontal extent of the pressure peaks as the slip length $\Lambda$ is increased. We can also notice a decrease in the time taken to reach touchdown, which could be a possible explanation for the reduced air entrapment due to there being less time prior to impact for the air pocket to build up. Figure 2.22 shows the dependency of the trapped air area at impact, given by equation (2.30), on the slip length $\Lambda$ over a large range. The area of trapped air decreases monotonically for increasing values of $\Lambda$, from a value of 21.9 when $\Lambda = 0$ down to 5.65 when $\Lambda \to \infty$, which corresponds to a possible reduction of up to 74%.

Figures 2.20 and 2.21 also suggest an interesting effect of the slip length $\Lambda$ on the pressure $P$. As the slip length $\Lambda$ is increased from zero, the horizontal extent of the pressure peaks decreases, but the magnitude is non-monotonic, initially decreasing from $\Lambda = 0$ to 1, then increasing until $\Lambda \to \infty$. Figure 2.23 plots the minimum film thickness and maximum pressure against time $T$. It can be seen clearly that the film thickness decreases faster on surfaces with a larger slip length, as the air is
more readily drained from beneath the impacting drop. Considering the maximum pressure evolution, at large negative times, it is smaller on surfaces with a larger slip length. However, as the air film thickness decreases faster on surfaces with a larger slip length, once the cushioning effect begins to become significant, the pressure begins to spike on surfaces with a larger slip length. This effect would appear to be in competition with the size of the maximum pressure at large negative times, resulting in a non-monotonic relationship between the maximum pressure and the slip length. To highlight this effect more clearly, figure 2.24 shows the dependency of the maximum pressure at touchdown on $\Lambda$ over a large range. This graph clearly exhibits a trough at $\Lambda = 0.58$. The apparent scatter of the data in this graph is due to the evaluation of a diverging peak on a discrete spatial grid. As the slip length varies, the location of the peak moves, and so evaluating this continuously results in slight noise being added to the results. In order to have a smooth graph this would require a prohibitively small spatial discretisation step. Despite this, there is an obvious trend in figure 2.24. In terms of practicality, in order to minimise the pressure peak at impact, this would require a slip length of around $\Lambda = 0.58$ and
corresponds to a reduction of 19% from the maximum pressure exerted on a no-slip surface ($\Lambda = 0$).

To summarise, this chapter modelled the pre-impact dynamics of droplet impact onto two types of engineered surfaces, utilising an analytical and numerical approach. A coupled potential flow and lubrication flow modelling the droplet and the air film between the droplet and the surface is combined with an imposed model of the surface in question. The two types of surfaces modelled are deformable and lubricant-infused surfaces. The deformable surface is modelled using an equation that incorporates all relevant physical parameters, namely surface rigidity, tension, stiffness, damping and mass density. Two cases were considered in this study: a viscoelastic model, considering only surface stiffness and damping, and a general flexible surface considering all surface parameters. In both cases it is found that increased surface deformability leads to increased air entrapment, reduced touchdown point speed and decreased pressure peaks, while for the viscoelastic model effects such as increased air gliding and rounder pressure peaks are found and investigated. The lubricant-infused surface is modelled as a flat surface with a partial-slip condition that allows air to slip along the flat surface. In this scenario, increased slippage at the surface leads to decreased air entrapment but a non-monotonic effect on the pressure peaks, with no-slip and no-shear surfaces exhibiting larger pressure peaks than an intermediate value.
Chapter 3

Droplet impact on textured and lubricant-infused surfaces: post impact

Next we move on to examine the post-impact dynamics of droplet impact on textured and lubricant-infused surfaces. Due to the complexity of the surface, we now change tactic from the previous Chapter and tackle the problem using direct numerical simulation.

A small number of experimental studies of droplet impact on lubricant-infused surfaces [67, 68, 69] have motivated the use of direct numerical simulation to understand in greater detail the mechanisms at play, as well as accessing timescales not available to experimental methods. Yeganehdoust et al. [71] performed simulations of micro-droplet impact onto lubricant-infused surfaces with micrometer sized asperities. This study, on the other hand, aims to more closely mimic the experimental work by simulating droplet impact on lubricant-infused surfaces where the length scale of the droplet is much larger than that of the surface asperities.

In this study, we aim to numerically examine the delicate early stages of droplet impact, with emphasis on the splash jet that is emitted at the point of impact. This splash jet is observed experimentally [1, 65] and also predicted by the well-established asymptotic structure known as Wagner theory [51, 118]. Wagner theory has been shown to accurately predict the radius of the wetted region for droplet impacts.
(or, the location of the jet root) as well as exhibiting excellent agreement with numerical simulations [119]. Extensions to Wagner theory include considerations of surface deformability [54, 55, 56] and roughness elements [57] as well as surface water [23] and ice formation [59]. Recently, Hicks [58] performed a study on a wide variety of continuous non-flat surfaces, using Wagner theory to predict the leading order free-surface shapes near impact. His model is able to go close to replicating the influence of a textured surface on droplet impacts, with the caveat that the surface shape must be continuous, without vertical surface elements. The use of analytical methods such as Wagner theory has so far been unable to model the leading order solution or the thin splash jet for droplet impacts onto significantly textured, superhydrophobic surfaces (and, extending further, lubricant-infused surfaces). To that end, we will utilise direct numerical simulation to study the delicate early stages of droplet impact on lubricant-infused surfaces in order to access parameters unavailable to Wagner theory. So far a detailed study of the early stages of impact when the depth of the surface asperities are of the same order as the thickness of the splash jet has not been performed. Our results will complement previous experimental results, which are commonly performed as a millimeter sized droplet impinging on a surface with micrometer sized asperities [67, 68], a situation in which the comparable scales of the splash jet thickness and the surface asperities are likely to hold [23].

Two-dimensional studies of droplet impact and splashing on flat surfaces have been shown to be a reasonable approximation of the very much three-dimensional problem [60]. Despite the clear geometrical discrepancies, two-dimensional droplet impact dynamics on flat surfaces are qualitatively very similar to experimental studies, a matter which will be discussed below. The extension of this to textured surfaces remains an open question, due to the lack of axisymmetry exhibited by the surface. However, using a two-dimensional many-body dissipative particle dynamics method, Wang et al. [120] were able to show close agreement on splashing with experimental results. As a first approximation, a two-dimensional study of the delicate early stages of droplet impact on textured and lubricant-infused surfaces is
hugely important to the initial understanding of this problem and has not been re-
ported previously in the literature. We make approximate, qualitative conclusions 
on the impact of our results in reality, and do believe our results build a basis for 
new research in the area.

3.1 Model setup and governing equations

Suppose a two-dimensional Cartesian circular liquid droplet of radius $R$ approaches 
normally a rectangular textured surface with a downward velocity $V$. The geometry 
of the textured surface is characterised by three lengths; the height $H$, the width $W$ 
and the gap $G$. The textures are filled up to a depth $\delta$ with a lubricant. Figure 3.1 
shows a schematic of the problem, with the physical parameters labelled.

We are considering here a three phase fluid flow problem. The three phases to 
consider are the liquid droplet, the surrounding air and the lubricant. Each phase 
will be considered as an incompressible Newtonian fluid with density $\rho_i$ and dy-
namic viscosity $\mu_i$ for $i = l$ (liquid droplet), $i = g$ (surrounding air) and $i = LIS$ 
(lubricant), where LIS stands for "lubricant-infused surface". The surface tension 
between the liquid droplet and the air will be written $\sigma_{l,g}$, between the liquid droplet 
and the lubricant will be $\sigma_{l,LIS}$ and between the air and the lubricant will be $\sigma_{g,LIS}$. 
The effects of gravity are assumed to be negligible and are not included in the sim-
ulations.
The fluid flow is modelled by the Navier-Stokes equations

\[ \frac{\partial}{\partial t} (\rho \mathbf{u}) + \nabla \cdot (\rho \mathbf{u} \mathbf{u}) = -\nabla p + \nabla \cdot (\mu (\nabla \mathbf{u} + \nabla \mathbf{u}^T)) + \mathbf{F}, \]
\[ \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0, \]

where \( t \) is time, \( \nabla = (\partial / \partial x, \partial / \partial y) \) is the two-dimensional gradient operator, \( p \) is the pressure, \( \mathbf{u} = (u, v) \) is the velocity vector, where \( u \) is the horizontal and \( v \) the vertical components, and \( \mathbf{F} \) is the surface tension force. In order to accurately capture the free-surface of both the droplet and the lubricant, equations (3.1) are solved in conjunction with a volume-of-fluid (VOF) method. The VOF method defines a volume fraction \( \alpha_i \) of each of the three phases at each point in the domain and calculates the properties of the fluid at each point based on this volume fraction, as follows. The sum of the volume fractions of all three phases is equal to one everywhere,

\[ \alpha_l + \alpha_g + \alpha_{LIS} = 1. \]

Now, the density \( \rho \) and viscosity \( \mu \) in the Navier-Stokes equations (3.1) are defined everywhere by

\[ \rho = \alpha_l \rho_l + \alpha_g \rho_g + \alpha_{LIS} \rho_{LIS}, \]
\[ \mu = \alpha_l \mu_l + \alpha_g \mu_g + \alpha_{LIS} \mu_{LIS}. \]

The values of density \( \rho \) and viscosity \( \mu \) vary across regions in which the fluid interfaces pass through. Hence, gradients of each need to be considered and so the Navier-Stokes equations (3.1) are solved to account for non-constant \( \rho \) and \( \mu \). The evolution of the free surfaces of the droplet and the lubricant are calculated using the volume fraction equation

\[ \frac{\partial \alpha_i}{\partial t} + \nabla \cdot (\alpha_i \mathbf{u}) = 0 \]

for \( i = l, LIS \). Finally here, the surface tension force \( \mathbf{F} \) is calculated using the con-
continuum surface force model [121], which, for a three phase system, is formulated as

\[ F = \sum_{\text{pairs } i,j, i \neq j} \sigma_{i,j} \alpha_i \rho_i \kappa_j \nabla \alpha_j + \alpha_j \rho_j \kappa_i \nabla \alpha_i \frac{1}{2} (\rho_i + \rho_j), \tag{3.5} \]

where the pairs \( i,j \) are given by the three distinct pairs of the different interfaces and \( \kappa_i \) is the curvature of the interface, given by

\[ \kappa_i = -\nabla \cdot \left( \frac{\nabla \alpha_i}{|\nabla \alpha_i|} \right). \tag{3.6} \]

### 3.2 Computational setup

The present study considers a water droplet (\( \rho_l = 10^3 \text{ kg m}^{-3}, \mu_l = 10^{-3} \text{ Pa s} \)) in air (\( \rho_g = 1.2 \text{ kg m}^{-3}, \mu_g = 1.8 \times 10^{-5} \text{ Pa s} \)) of radius \( R = 1 \text{ mm} \). It is assumed that the lubricant is a member of the Krytox series of lubricants, with a density of \( \rho_{LIS} = 1.85 \times 10^3 \text{ kg m}^{-3} \) and a vast range of viscosities [122]. The surface tension coefficients between the interfaces are taken as \( \sigma_{l,g} = 7.3 \times 10^{-2} \text{ N m}^{-1} \), \( \sigma_{l,LIS} = 5.3 \times 10^{-2} \text{ N m}^{-1} \) and \( \sigma_{g,LIS} = 1.7 \times 10^{-2} \text{ N m}^{-1} \). For the Krytox series of lubricant, despite vast varieties of viscosities, the surface tension between the lubricant and the air/water remains roughly constant [122].

We exploit the symmetry of the problem and only calculate the right half of the droplet. The numerical domain on which our calculations are performed is a two-dimensional square box with sides of length \( 4R = 4 \text{ mm} \). The surface texturing is achieved by having small rectangular domains appended onto the bottom of the domain at \( y = 0 \), protruding into \( y < 0 \), resulting in \( y = 0 \) being the top of the textures. The boundary conditions on the top and right side of the domain are set as pressure outlets, the left side of the domain is a symmetry boundary and the bottom, including all the sides of the appended rectangles, is set to an impenetrable wall condition, with no slip. To ensure consistency within our results and to keep the symmetry, \( x = 0 \) is always located in the middle of a pillar. A schematic showing the computational domain and boundary conditions is shown in Figure 3.2.

In order to more closely mimic previous experimental studies [67, 68], we
3.2. Computational setup

Figure 3.2: Schematic showing the numerical domain and the boundary conditions.

wish to consider cases where $W, H, G \ll R$. For brevity, we only vary the distance between the pillars $G$, and henceforth fix $W = H = 10 \, \mu m$.

The three phase fluid flow problem, governed by equations (3.1-3.6), is solved in the described computational domain using the CFD code ANSYS Fluent v19.5. The pressure-velocity coupling is performed using the PISO algorithm and spatial discretization of the Navier-Stokes equations is done using the QUICK scheme. For pressure discretization the PRESTO scheme is used. The spatial discretization of the volume fraction equation is done using the geometrical reconstruction scheme. A first order implicit method is used for the unsteady terms in the Navier-Stokes equations, with a variable time step. The variable time step is based on the Courant number, which is kept below 0.2 for all simulations. The code advances to the next time step when the scaled residuals of the continuity and momentum equations are below $10^{-4}$. The effect of dynamic wetting is not considered in the study, thus a static contact angle between each of the three phases is set as $\pi/2$. During the spreading phase of droplet impact, when inertial effects are dominant, various authors have reported that dynamic wetting is unimportant [24, 123, 124]. In cells
3.2. Computational setup

Square mesh
Refined mesh

Figure 3.3: An example of the mesh used in the simulations, including initial droplet shape and position.

<table>
<thead>
<tr>
<th>Mesh</th>
<th>Refined mesh size</th>
<th>Total cells</th>
</tr>
</thead>
<tbody>
<tr>
<td>Very coarse</td>
<td>0.625 $\mu$m</td>
<td>$1.77 \times 10^6$</td>
</tr>
<tr>
<td>Coarse</td>
<td>0.556 $\mu$m</td>
<td>$2.34 \times 10^6$</td>
</tr>
<tr>
<td>Standard</td>
<td>0.5 $\mu$m</td>
<td>$2.72 \times 10^6$</td>
</tr>
<tr>
<td>Fine</td>
<td>0.417 $\mu$m</td>
<td>$4.03 \times 10^6$</td>
</tr>
<tr>
<td>Very fine</td>
<td>0.357 $\mu$m</td>
<td>$5.22 \times 10^6$</td>
</tr>
</tbody>
</table>

Table 3.1: Details of the five different meshes used in the mesh independence study.

where all three phases are present, the angles between the interfaces of each phase are determined through the continuum surface force model [121].

The square domain box is divided into three regions: a triangular mesh region, a square mesh region and a refined mesh region which encapsulates the subsequent splash jet after impact, where the square mesh cells are decreased in size by a factor of two. Figure 3.3 gives an example of the mesh used in the simulations, including the initial droplet shape and position, as well as an enlarged image of the mesh in the refined region near the impact sight. A mesh independence study was carried out on five different mesh sizes, as well as a case where the domain sides are doubled to $8R$ (calculated on the standard mesh size), as summarized in Table 3.1, for a test case with pillar spacing $G/R = 0.01$, initial lubricant depth $\delta/H = 1$ and lubri-
Figure 3.4: The non-dimensional variation in time of the horizontal extent of the splash jet $J_x/R$ for pillar spacing $G/R = 0.01$, lubricant depth of $\delta/H = 1$ and lubricant viscosity $\mu_{LS}/\mu_l = 30$ at $We = 70$, calculated on the five different meshes listed in Table 3.1.

Figure 3.5: Comparison of two-dimensional and three-dimensional simulation results for the jet horizontal extent $J_x/R$ to experimental results from Riboux and Gordillo [1] for a water droplet impact on a flat surface at $We = 98$ and $Re = 3462$. The inset shows the two-dimensional simulation and the experimental data but on a log-log scale.
3.3 Results and discussion

The results section is split into two sections. The first is an investigation into the parameters present in a lubricant-infused surface, namely the lubricant viscosity, the gap between the pillars and the depth of the lubricant. Lubricant-infused surfaces have many desirable features and are preferred to standard superhydrophobic, textured surfaces in many situations [125]. Hence, it is important that an isolated study of the parameters pertaining to a lubricant-infused surface are investigated...
and results compared. Next, drawing inspiration from Yeganehdoust et al. [71], we will compare and contrast the results against those for a superhydrophobic, textured surface to try and shed light on the key differences of the splash jet dynamics.

In order to improve the scope of our findings, results are presented in non-dimensional variables, with the liquid droplet properties as reference values. Time is also scaled such that the time \( t = 0 \) corresponds to the time that the droplet bottom free-surface would reach \( y = 0 \) in the absence of air cushioning. The presence of the surrounding air results in the impact being delayed slightly into positive time [2, 31].

An example solution is shown in figure 3.6. This solution shows the full free-surface of the droplet in the impact. We see that the code is able to resolve the small scale features of the splash, most importantly the splash jet, which can be seen being emitted from the bottom of the droplet from time \( \frac{Vt}{R} = 0.05 \). A representation of the splash jet at time \( \frac{Vt}{R} = 0.16 \) on a scale where the surface textures can be seen is shown in figure 3.7(a), together with the velocity magnitude \( \frac{|u|}{V} \), where \( |u| = \sqrt{u^2 + v^2} \). In what follows, a vitally important point on the splash jet is the tip, which is defined as the point of the splash jet with maximum horizontal extent. The coordinates \((J_x, J_y)\) of this point are shown in figure 3.7(b) in non-dimensional form. Also shown is the location of the jet root \( J_r \), a representation of the leading order wetted area of the surface, which will be examined briefly also. This point is defined as the turnover point in the free surface from the jet to the main body of the droplet.

3.3.1 Lubricant-infused surfaces parameter investigation

First, we investigate the parameters which describe a lubricant-infused surface. Namely, they are the gap between the pillars \( G \), the depth of the lubricant \( \delta \) and the lubricant viscosity \( \mu_{LIS} \). Outcomes of the droplet impact are illustrated for various impact velocities. All other parameters describing the fluid flow and the geometry are held constant at the values given in Sec. 3.2.

We consider now the effect of varying the pillar spacing \( G \). Physically, increasing the gap between the pillars decreases the solid-fraction of the topmost surface,
Figure 3.6: Full free-surface solution for a droplet impact onto a surface with $G/R = 0.01$, $\mu_{LJS}/\mu_l = 30$ and $\delta/H = 1$ at six different time instants. Impact is at $We = 90$. The blue line represent the solution in the refined mesh region. The surface textures and the lubricant free-surface solutions are too small to display on this scale.
3.3. Results and discussion

Figure 3.7: As in figure 3.6, but at $V t / R = 0.16$ only and enlarged around the splash jet. (a) shows the droplet free-surface (white), the lubricant free-surface (yellow) and the velocity magnitude field (colour). (b) shows the droplet free-surface (blue), the lubricant free-surface (red) and the surface textures (black), along with the jet root location and splash jet tip coordinates.

which for asperities filled with air can reduce the contact time of bouncing droplets [120, 126]. Here, in this section, we aim to analyse the effect pillar gap variation has on the dynamics of the splash jet when the asperities are filled with a lubricant. The lubricant viscosity and depth are kept constant at $\mu_{LIS} / \mu_l = 30$ and $\delta / H = 1$, respectively. The viscosity ratio here is chosen such that our parameters are similar to Yeganehdoust et al. [71], who model the lubricant in their simulations on Krytox GPL 101, a commonly used lubricant in lubricant-infused surfaces [7].

Figure 3.8 shows free-surface solutions for both the droplet and the lubricant at $We = 90$ and four different values of $G / R$ from 0.01 to 0.04. In all cases here we have a distinct jet emerging from the droplet impact and this jet detaches from the surface each time. The ejection of microdrops is seen to occur at all values of $G / R$ except for 0.01. The horizontal extent of the splash jet decreases for increased values of $G / R$, and this difference increases as time goes on.

The jet root location $J_r / R$ remains unchanged with variations of $G / R$, which is shown in Figure 3.9 for $We = 90$. The jet root location compares well with the
3.3. Results and discussion

**Figure 3.8:** Free-surface solutions of the droplet (blue) and the lubricant (red) at 4 equal time intervals $V_t/R = 0.01, 0.08, 0.15, 0.22$, with $\mu_{LJS}/\mu_l = 30, \delta/H = 1$ and $We = 90$. The pillar gap is varied, with (a) $G/R = 0.01$, (b) $G/R = 0.02$, (c) $G/R = 0.03$ and (d) $G/R = 0.04$.

**Figure 3.9:** The jet root location $J_r/R$ for $\mu_{LJS}/\mu_l = 30, \delta/H = 1$ and $We = 90$, for $G/R = 0.01$ (black), 0.02 (blue), 0.03 (red) and 0.04 (green). The dashed line shows the early-time Wagner theory prediction for the jet root location $2\sqrt{V_t/R}$ for impact on a flat plate.
3.3. Results and discussion

Figure 3.10: Time series data for the jet horizontal extent (left) and the jet tip horizontal velocity (middle) as well as the jet tip coordinates (right), with $\mu_{LIS}/\mu = 30$, $\delta/H = 1$ and $G/R = 0.01$ (black), 0.02 (blue), 0.03 (red), 0.04 (green) for row (a) $We = 45$, (b) $We = 90$ and (c) $We = 180$.

early-time Wagner theory prediction of $2\sqrt{Vt/R}$ [23] for impact on a flat plate. The same was found to be true for $We = 45$ and 180, for brevity this is not shown.

Time series data is shown in figure 3.10 for the aforementioned horizontal extent of the splash jet, as well as the horizontal velocity of the splash jet tip, $J_v/V = (dJ_x/dt)/V$, and the coordinates of the jet tip for $G/R = 0.01$ to 0.04, at $We = 45$, 90 and 180. At $We = 45$ it can be seen that the effect of pillar gap variations is minimal. There is only a small reduction in the horizontal velocity of the jet tip as the pillar gap is increased. The jet tip coordinates show that there is an effect on the vertical position of the jet tip, but we note here that these vertical coordinates are small, and in fact the jet is attached to the surface in each case here; differences in the vertical coordinates are perhaps due to changes of curvature of the jet tip due to the definition of the jet tip in figure 3.7(b). As the Weber number is increased to $We = 90$ (the free-surface solutions of this case are shown in figure 3.8) the effect of pillar gap variations is much more distinct. There is a clear and significant decrease in the horizontal extent of the jet, as well as the velocity and the coordinate of the splash jet tip as the pillar gap is increased from $G/R = 0.01$ to $G/R = 0.04$. This
is again seen for $We = 180$. Although a larger scatter of the data is seen in the horizontal velocity, which is due to emission of micro droplets from the jet and the subsequent retraction of the jet due to surface tension [127, 128], as the jet forms and extends the horizontal velocity of the jet tip is significantly lower for $G/R = 0.04$ than for $G/R = 0.01$. Interestingly, we see that the vertical position of the jet tip at $We = 180$ remains the same for $G/R = 0.02$ to 0.04 but is far larger for $G/R = 0.01$. This is, however, in keeping with the relation of larger pillar gaps resulting in a lower vertical position of the splash jet tip (see Figure 3.8).

The early-time splashing behaviour is characterised into four categories: either a microdrop ejection splash (where small drops are ejected from the impact sight or from the advancing splash jet, for an example see Figure 3.13(c)); a jet detachment splash (where a thin splash jet with a round rim is ejected away from the surface, see Figure 3.8(a)); both microdrop ejection and jet detachment (see Figure 3.8(b)) or no splash (see Figure 3.13(a)). The microdrop ejection characterisation is only considered if at least one of the ejected microdrops has a diameter of at least ten grid cells (in the standard mesh this is $5 \mu m$). Figure 3.11 shows a summary of all early time splashing behaviour (up to $V_t/R \approx 0.35$ in each simulation) of droplet impacts in the parameter range considered, with $\delta/H = 1$ and $\mu_{LIS}/\mu_l = 30$ fixed and $G/R$ and $We$ varied. Below $We \approx 79$, no splash is observed for all surfaces considered.
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Figure 3.12: Free-surface solutions of the droplet (blue) and the lubricant (red) at 4 equal time intervals $Vt/R = 0.1, 0.21, 0.33, 0.45$, with $G/R = 0.04$, $\mu_{LS}/\mu_l = 30$, $\delta/H = 1$ and $We = 225$. 
3.3. Results and discussion

As $We$ increases to 135, the behaviour transitions to a splash with both microdrop ejection and jet detachment characteristics, with this splashing behaviour seen first at larger values of $G/R$, suggesting that despite larger values of $G/R$ having a suppressing effect on the horizontal extent of the splash jet, it may promote microdrop ejection. This behaviour then continues up until $We \approx 202$, where at $G/R = 0.4$ the presence of a distinct jet detachment ceases to be seen, perhaps due to larger velocities and asperities allowing more lubricant to leave the surface and influence the advancing of the splash jet. Figure 3.12 shows the solution at four time intervals in this case.

The results here show that the pillar gap $G$ can be an important quantity in reducing the splash jet’s horizontal extent and the velocity and coordinate of the jet tip, but has no effect on the jet root location. These three metrics suggest that the magnitude of a jet detachment splash can be suppressed by increasing the gaps between pillars on a lubricant-infused surface. There is a limited number of experimental studies on high-speed droplet impact on lubricant-infused surfaces, and an absence of such that consider the effect of varying the distance between the asperities of the surface. One possible mechanism of the suppressed splash dynamics for larger values of $G$ could be the effect the lubricant has in making the surface effectively rough. Larger fluid filled cavities are susceptible to greater deformation from flow across them [117], thus surfaces with larger values of $G$ deform more as the droplet impacts and subsequently spreads over it. Significant surface roughness is known to inhibit corona splashing [66], which is related to jet detachment, and so is then consistent with our results.

Next we turn our attention to the initial depth $\delta$ of the lubricant present. The initial depth of the lubricant may not necessarily be something that is designed on purpose, but instead something that happens over time [122]. For example, a lubricant-infused surface may begin life with a lubricant level that is slightly above the asperities ($\delta/H > 1$), but after successive liquid impacts or droplet sliding, lubricant depletion may occur [13, 129], eventually resulting in a lubricant level below the asperities ($\delta/H < 1$). The lubricant depth can be engineered however, by us-
3.3. Results and discussion

Figure 3.13: Free-surface solutions of the droplet (blue) and the lubricant (red) at 4 equal time intervals $Vt/R = 0.01, 0.08, 0.15, 0.22$, with $\mu_{LIS}/\mu_i = 30$, $G/R = 0.01$ and $We = 90$. The lubricant depth is varied, with (a) $\delta/H = 0$, (b) $\delta/H = 0.5$, (c) $\delta/H = 1$, (d) $\delta/H = 1.5$ and (e) $\delta/H = 2$.

In this section, we aim to investigate the dynamics of a droplet impact for different lubricant levels $\delta$. Again, we will keep the viscosity of the lubricant constant at $\mu_{LIS}/\mu_i = 30$ and we will choose the surface textures to be all equal such that $G/R = 0.01$.

Figure 3.13 shows free-surface solutions for both the droplet and the lubricant at $We = 90$ and five different values of $\delta/H$ from 0 to 2. The droplet impact dynamics illustrated by figure 3.13 are similar to that seen in figure 3.10, the splash jet clearly emerges and extends horizontally as time goes on for all values of $\delta/H$; however there are some key and interesting outcomes as $\delta/H$ is varied. When $\delta/H = 0$, the surface is no longer a lubricant-infused surface and is in fact a textured surface. The splash jet spreads along the surface without detaching, exhibiting no splashing behaviour whatsoever. Almost identical behaviour is seen for $\delta/H = 0.5$, showing that a small amount of lubricant present has little effect. The solution for $\delta/H = 1$ is the same as in figure 3.10(a), showing that once the lubricant is at a depth large enough to affect the droplet impact it results in the splash jet detaching from the surface and having a larger horizontal extent. Here, a small increase in
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Figure 3.14: The jet root location $J_r/R$ for $G/R = 0.01$, $\mu_{LS}/\mu_l = 30$ and $We = 90$, for $\delta/H = 0$ (black), $\delta/H = 0.5$ (blue), $\delta/H = 1$ (red), $\delta/H = 1.5$ (green) and $\delta/H = 2$ (magenta). The dashed line shows the early-time Wagner theory prediction for the jet root location $2\sqrt{Vt/R}$ for impact on a flat plate.

the depth to $\delta/H = 1.5$ has little effect. When $\delta/H = 2$, the splash jet is clearly suppressed, being expelled along the liquid surface as opposed to up and away from it.

As in the previous section, the jet root location is virtually unaffected by variations in the initial lubricant depth, as shown by figure 3.14. Again, the results agree well with Wagner theory at early times. The same is true at $We = 45$ and 180.

The time series data for the splash jet horizontal extent and tip velocity, as well as the splash jet tip coordinates, for variations in $\delta/H$ is shown in figure 3.15. As in the previous section, at $We = 45$, varying $\delta/H$ has a minimal effect on the three splash metrics considered here. At $We = 90$ we have the dynamics that are explained above. The horizontal extent of the splash jet and velocity of the tip are significantly reduced for $\delta/H = 0$ and 0.5 compared to larger values of $\delta/H$. The vertical coordinate of the splash jet tip behaves similarly, except for when $\delta/H = 2$, where the viscous film has a suppressing effect. At a higher Weber number, $We = 180$, the dynamics are slightly different: specifically the suppressing effect on the splash jet for surfaces with $\delta/H > 1$ is much more profound. The horizontal extent
3.3. Results and discussion

![Figure 3.15](image1)

Figure 3.15: Time series data for the jet horizontal extent (left), the jet tip horizontal velocity (middle), as well as the jet tip coordinates (right), with $\mu_{LJS}/\mu_l = 30$, $G/R = 0.01$ and $\delta/H = 0$ (black), 0.5 (blue), 1 (red), 1.5 (green), 2 (purple) for row (a) $We = 45$, (b) $We = 90$ and (c) $We = 180$.

![Figure 3.16](image2)

Figure 3.16: Early time splashing behaviour of droplet impacts onto surfaces with $G/R = 0.01$ and $\mu_{LJS}/\mu_l = 30$. The different behaviours are defined as either no splash (NS), microdrop ejection (ME) or jet detachment (JD).

of the splash jet is lower for all time for the $\delta/H > 1$ cases than it is for the $\delta/H < 1$ cases, unlike for $We = 90$. The horizontal velocities and vertical positions of the jet tip are also reduced to a similar level. Again, we see greater horizontal extent of the jet and velocity and vertical positions of the jet tip for the level case of $\delta/H = 1$.

As in the previous section, a parametric diagram (Figure 3.16) summarises the
early time splashing behaviour for various values of $\delta/H$ and We, with $G/R = 0.01$ and $\mu_{LIS}/\mu_l = 30$. Again, below $We \approx 79$, no splashing is observed. However, now, no splashing is observed up until $We = 90$ for $\delta/H < 1$. After $We \approx 90$, the different outcomes are split above and below $\delta/H = 1$. At $\delta/H = 1$, the droplet impact exhibits both microdrop ejection and jet detachment splashing past $We \approx 90$, whereas above and below $\delta/H = 1$, the droplet impact exhibits only microdrop ejection splashing behaviour, except for higher We where we begin to see jet detachment splashing occurring at $\delta/H = 1.5$.

Muschi et al. [67] considered the effect of varying the lubricant thickness in a lubricant-infused surface. In their study, at $We = 66$, they show that in the early stages of droplet impact the droplet spreads slower on a porous surface with no lubricant present than it does on a surface with the lubricant slightly above the asperities, which agrees with our results at $We = 45$ and 90 somewhat over the small time scales considered here. Our results point towards an interesting transition as the depth of the lubricant $\delta$ is increased from zero. When the surface of the lubricant is below the level of the asperities of the solid surface, the latter surface behaves like a textured surface, with the droplet able to penetrate much further into the surface and more space is available for the air underneath the tip of the advancing jet to drain into, suppressing the jet detachment splash [1, 49, 66]. As the depth is increased however to the level of the asperities the splash magnitude becomes rather large (only lubricant depths which are above the asperities are considered in Muschi et al. [67]). In this case, the air is no longer able to drain from underneath the advancing jet tip and the disturbances from the deformation in the lubricant result in the splash jet being ejected away from the surface. As the depth is increased further, above the asperities, the splashing can be seen to become suppressed again. Thin layers of viscous fluid could have connection to soft solids, which are known to have a suppressing effect on droplet splashing [65]. This points towards a non-monotonic relationship in the level of splashing seen as the depth of the lubricant is varied. The magnitude of the splash is also seen to be dependent on the Weber number, by figure 3.15, with the magnitude of the splash particularly affected by the Weber
number when the lubricant surface is above the asperities. Of course, interesting cases where the lubricant level is a small amount below or above the asperities would ideally be considered here also, but to ensure numerical accuracy the mesh would need to be refined to a level beyond our computational capabilities. Our results here nevertheless do suggest a broad and general trend.

Our final parameter of interest will be the viscosity of the lubricant. This is perhaps the most commonly investigated parameter in experimental studies [7, 68]. In practise, the viscosity of the lubricant used varies drastically [122], with magnitudes ranging from \( O(10^{-2})-O(1) \) Pa s for the Krytox series of lubricants, which for a water droplet gives viscosity ratios in the range \( \mu_{\text{LIS}}/\mu_l = O(1)-O(10^3) \). As well as the previous case of \( \mu_{\text{LIS}}/\mu_l = 30 \), which was based on the Krytox 101 lubricant, we will now consider other values of the viscosity ratio, choosing a range which represents the orders of magnitude range above. Here, the depth of the lubricant and the gap between the pillars will be held constant, with \( \delta/H = 1 \) and \( G/R = 0.01 \).

Free surface solutions for the droplet and the lubricant at \( \text{We} = 90 \) are shown in figure 3.17, for \( \mu_{\text{LIS}}/\mu_l = 2, 10, 30, 100, 500 \) and 1500. Again we see a number of different outcomes of the splash jet. Starting with \( \mu_{\text{LIS}}/\mu_l = 1500 \), we can see from the solution at \( V_1/R = 0.08 \) that the splash jet is ejected at a small angle, almost along the surface. As time increases the jet continues to extend along the surface and does make contact with the surface again, resulting in a break-up of the jet (although this separated fluid is then absorbed by the spreading jet at later times). Decreasing the viscosity ratio to \( \mu_{\text{LIS}}/\mu_l = 500 \) and \( \mu_{\text{LIS}}/\mu_l = 100 \) results in the jet being ejected at a much larger angle, with results that are almost identical to that of the previous results with viscosity ratio of \( \mu_{\text{LIS}}/\mu_l = 30 \), which is repeated here for clarity. Decreasing the viscosity ratio further to \( \mu_{\text{LIS}}/\mu_l = 10 \) we can see again at early times that the jet is ejected at an angle similar to those seen in the previous two cases, but this time much more of the jet is still attached to the surface, with atomisation also occurring. Both these attributes contribute to the horizontal extent of the jet being reduced compared to previous cases. Finally, for a relatively low
3.3. Results and discussion

Figure 3.17: Free-surface solutions of the droplet (blue) and the lubricant (red) at 4 equal time intervals \( Vt/R = 0.01, 0.08, 0.15, 0.22 \), with \( \delta/H = 1, G/R = 0.01 \) and \( \text{We} = 90 \). The lubricant viscosity is varied, with (a) \( \mu_{LIS}/\mu_l = 1500 \), (b) \( \mu_{LIS}/\mu_l = 500 \), (c) \( \mu_{LIS}/\mu_l = 100 \) (d) \( \mu_{LIS}/\mu_l = 30 \), (e) \( \mu_{LIS}/\mu_l = 10 \) and (f) \( \mu_{LIS}/\mu_l = 2 \).

As in the previous two sections, the jet root location is virtually unaffected by variations in the viscosity ratio and agrees well with Wagner theory. This is shown in figure 3.18. The same is true at \( \text{We} = 45 \) and 180.

Time series data for horizontal extend of the splash jet and the velocity of the jet tip, as well as the jet tip coordinates are shown in figure 3.19 and compared to the same data found from impacts at \( \text{We} = 45 \) and \( \text{We} = 180 \). At \( \text{We} = 45 \), as for the previous two parameters we studied, the changes in the horizontal extent of the jet and velocity of the tip are relatively small as we vary the viscosity ratio. There would appear to be some difference here in the jet tip coordinates, with a gradual viscosity ratio of \( \mu_{LIS}/\mu_l = 2 \), the lubricant is able to deform more and allows the droplet to penetrate deeper during impact. This results in the jet being ejected at a small angle and being of comparatively small extent. As time advances, the jet remains comparatively small in both horizontal and vertical extent.
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Figure 3.18: The jet root location $J_r/R$ for $G/R = 0.01$, $\mu_{LIS}/\mu_l = 30$ and $We = 90$, for $\mu_{LIS}/\mu_l = 2$ (black), $\mu_{LIS}/\mu_l = 10$ (blue), $\mu_{LIS}/\mu_l = 30$ (red), $\mu_{LIS}/\mu_l = 100$ (green), $\mu_{LIS}/\mu_l = 500$ (magenta) and $\mu_{LIS}/\mu_l = 1500$ (yellow). The dashed line shows the early-time Wagner theory prediction for the jet root location $2\sqrt{VT/R}$ for impact on a flat plate.

Figure 3.19: Time series data for the jet horizontal extent (left) and the jet tip horizontal velocity (middle), as well as the jet tip coordinates (right), with $\delta/H = 1$, $G/R = 0.01$ and $\mu_{LIS}/\mu_l = 2$ (black), 10 (blue), 30 (red), 100 (green), 500 (purple), 1500 (orange) for row (a) $We = 45$, (b) $We = 90$ and (c) $We = 180$. 
3.3. Results and discussion

decrease in vertical extent seen as $\mu_{LIS}/\mu_l$ decreases from 500 to 2, and then also a decrease as we increase it from 500 to 1500. Time series data for the free-surface results at $We = 90$ discussed above are also shown, and highlight similar trends to the $We = 45$ case for the three metrics considered, a gradual decrease in magnitude as the viscosity ratio decreases from 500 to 2 and then a decrease also as it is increased from 500 to 1500. However this time the results for $\mu_{LIS}/\mu_l = 500, 100$ and 30 are very similar and the decrease in magnitude seen from the increase of the viscosity ratio from 500 to 1500 is much more profound now, compared with that at $We = 45$. At $We = 180$, we can see that the results for $\mu_{LIS}/\mu_l = 500$ are now below those of $\mu_{LIS}/\mu_l = 30$ and 100, which are again almost identical, for all three metrics. Hence, these time series data shown in figure 3.19 would point again towards a non-monotonic relationship between the viscosity ratio and the jet dynamics and subsequent splashing. For a larger viscosity ratio, the fluid between the asperities is less able to deform, thus the surface itself would behave more like a flat rigid surface, on which it is harder for a droplet to splash than is the case on a rough surface [49]. However, as the viscosity ratio is reduced further, we observe a transition of the dynamics and the magnitude of the splash starts to decrease. As the viscosity ratio decreases, the droplet fluid is more able to penetrate deeper into the asperities of the surface, which results in the surface behaving more like a textured surface (in which the asperities would be filled with air, which obviously has a much smaller viscosity than the droplet fluid), which also has splash suppressant qualities. These dynamics are also dependent on the impact Weber number. More specifically, it would appear that the decrease in splash magnitude seen for very large viscosity ratios is more profound at higher Weber numbers.

Figure 3.20 summarises the early time splashing behaviour for these parameters. Interestingly, unlike in the previous two investigated parameters, for values above $\mu_{LIS}/\mu_l \approx 100$, a jet detachment splash is observed for low $We$. The phase diagram here is, mainly, split into four regions. At low $We$ and below $\mu_{LIS}/\mu_l \approx 500$, no splashing behaviour is exhibited. At high viscosity ratios, below $We \approx 135$ only jet detachment splashing occurs and at $\mu_{LIS}/\mu_l = 2$, above $We \approx 45$, only microdrop
Figure 3.20: Early time splashing behaviour of droplet impacts onto surfaces with $G/R = 0.01$ and $\delta/H = 1$. The different behaviours are defined as either no splash (NS), microdrop ejection (ME) or jet detachment (JD).

Spreading and retracting dynamics of droplet impacts with lubricant-infused surfaces of varying viscosities were examined experimentally in Kim and Rothstein [68], with viscosity ratios ranging from 0.8 to 16.7. They found that, in contrast to our results, decreasing the surface lubricant viscosity led to a faster spreading and retracting rate of droplet impacts. However, it should be noted that in their study only droplet impacts of relatively low velocity were considered (up to $1.9 \text{ ms}^{-1}$) and that the overall spreading and retracting dynamics happen over a much longer time scale than that considered in our study. In our study, a 1 mm radius water droplet impact at $\text{We} = 45$ corresponds to an impact velocity of $V = 1.81 \text{ ms}^{-1}$, at the upper end of the experimental range in Kim and Rothstein [68], and figure 3.19 shows minimal variation in the horizontal extent of the splash jet. At higher We, the horizontal extent of the splash jet decreases noticeably (over the range of viscosity ratios in the experimental study [68]). And so we conclude that at lower impact velocities, where the deformation of the lubricant itself is minimal, droplets will spread faster for lower viscosity lubricants, due to larger slippage occurring at the lubricant-droplet interface. But at higher impact velocity, where the lubricant can deform and allow the droplet to penetrate into the asperities, the splash jet that is
emitted extends at a lower velocity for lower lubricant viscosities. As the viscosity ratio becomes $O(100)$, this trend does begin to change somewhat, suggesting new dynamics that need to be considered.

### 3.3.2 Comparison of lubricant-infused surfaces and textured, superhydrophobic surfaces

Finally, we wish to compare the results for lubricant-infused surfaces, given above, to those of textured, superhydrophobic surfaces, where there is no lubricant present and the surface cavities are filled with air instead (this is equivalent to setting $\delta/H = 0$ in the modelling). While lubricant-infused surfaces have benefits in their own right, textured, superhydrophobic surfaces also have many benefits, some of which are the same as lubricant-infused surfaces, some not [130, 131, 132].

In this section we compare the splash jet dynamics of droplet impacts onto a lubricant-infused surface with viscosity ratio $\mu_{\text{LIS}}/\mu_l = 30$ and a lubricant depth of equal size to the asperities of the surface ($\delta/H = 1$), to that of a textured, superhydrophobic surface ($\delta/H = 0$). We consider and compare different impact velocities and variations in the pillar gap $G/R$.

Figure 3.21 shows a comparison of the droplet free-surface solutions for impacts onto a lubricant-infused surface and a textured surface, both with a pillar gap of $G/R = 0.01$, at $\text{We} = 45, 90$ and $180$. At $\text{We} = 45$, although marginal, it is quite clear that the horizontal extent of the splash jet is reduced on a textured surface compared to a lubricant-infused surface. This is also seen at $\text{We} = 90$, and much more profound this time. Not only is the horizontal extent less for a textured surface, the vertical extent is also, with the jet remaining attached to the substrate whereas it is detached on a lubricant-infused surface. At $\text{We} = 180$, we begin to see a significant amount of atomisation and breakup of the jet, and at early times ($Vt/R = 0.07$) the jet is detached from the surface in both cases. However, as time advances the jet on the textured surface is attached to the surface and has a horizontal and vertical extent significantly less than for the lubricant-infused surface.

In order to explore the effect of different pillar gap sizes and the influence of the impact velocity, for brevity a single metric will be considered, which is the time
3.3. Results and discussion

Figure 3.21: Comparison of free-surface solutions for droplet impacts onto lubricant-infused surfaces (blue) and textured, superhydrophobic surfaces (red). Results are shown for $\mu_{LIS}/\mu_i = 30$ and $\delta/H = 1$ (lubricant-infused surface) and $\delta/H = 0$ (textured surface), with $G/R = 0.01$ and $We = 45$ (column (a)), $We = 90$ (b) and $We = 180$ (c). Results are shown for $y/R > 0$ only, and free-surface solutions for the lubricant are not shown.

taken for tip of the splash jet to reach a specified point on the $x$-axis. Explicitly, this is given in non-dimensional terms as

$$T_i = \left\{ \frac{V_i}{R}, \text{ such that } \frac{J_x}{R} = c \right\}$$

for some value $c$ and $i = LIS$ for a lubricant-infused surface whereas $i = SHS$ for a textured, superhydrophobic surface. For our study we will consider $c = 1.6$, which corresponds to a point towards the end of the refined mesh region in the simulations.

The parameter $T_i$ will be evaluated at a number of different pillar gaps $G/R$ and Weber numbers. Figure 3.22 shows the difference between the time parameters for a lubricant-infused surface and a textured, superhydrophobic surface $T_{SHS} - T_{LIS}$ plotted against pillar gap $G/R$. This figure shows that, as shown in figure 3.21, the jet tip advances slower on the textured surface compared to the lubricant-infused surface for $G/R = 0.01$, and so the difference $T_{SHS} - T_{LIS} > 0$ for all three Weber numbers considered. At each Weber number, when the pillar gap $G/R$ is increased,
the difference $T_{SHS} - T_{LIS}$ decreases (except for $We = 45$ from $G/R = 0.01$ to 0.02, where it increases slightly), where eventually for $G/R = 0.04$ we have $T_{SHS} - T_{LIS} < 0$ or similar, for all three Weber numbers, illustrating that when the pillar gap is raised sufficiently, the jet extends slower on a lubricant-infused surface than on a textured one.

To summarise, this chapter investigates the problem of droplet post-impact dynamics on textured and lubricant-infused surfaces. This complex fluid dynamics problem is tackled computationally utilising direct numerical simulation with detailed small scale geometry and fine meshing. To closely mimic the length scales of experimental work that usually studies millimeter sized droplets impacting onto micro or nanometer sized surface textures, a two-dimensional framework is used as a three-dimensional framework is computationally unfeasible. The surface is de-
signed as an array of rectangular pillars with lubricant filled up to a certain depth. Three parameters pertaining the surface, namely the surface topology, the depth of the lubricant and the viscosity of the lubricant, are investigated against the velocity of the droplet. The metric of interest is the thin splash jet ejected at impact, where the time evolution of the tip is studied for different parameters. Broadly speaking, the wider the spacing of the pillars, the jet tip advances at a lower speed and for both the depth of the lubricant and the viscosity of the lubricant, the relationship is non-monotonic. The results for the lubricant-infused surface are then compared to that of a superhydrophobic textured surface, with no lubricant present, where it is seen that the jet tip advances slower on a textured surface for closely spaced pillars but faster for wider spaced pillars, compared with a lubricant-infused surface.
Chapter 4

The horizontal spread of a jet on a lubricant-infused surface

The previous chapter motivates a study of thin film flows on lubricant-infused surfaces. In general, high-speed liquid impact on any type of surface results in jetting of some kind, when liquid is displaced perpendicularly to the direction of impact.

This chapter will attempt to model a related, canonical problem regarding a boundary layer jet atop of a lubricant-infused surface. This has applications similar to that of droplet impact, but some in it’s own right, such as spin coating [133]. The study of flow across lubricant-infused surfaces has received a large amount of attention, due to the drag reducing ability of lubricant-infused surfaces (see Hardt and McHale [134] for a review on this topic). As such, the focus of this work is just as much on the effect the fluid flow has on the surface as vice versa, from a desire for the surface to remain stable and in a lubricant-infused state. Improving the stability of a lubricant-infused surface will benefit industrial users of such surfaces. Despite the large number of studies on flow across lubricant-infused surfaces, there is a distinct lack of studies that consider the effect of the fluid atop the lubricant-infused surface being of a finite depth, with a free surface.

The work in this chapter will describe two models. The first model will be a simple extension of the work by Watson [73], who studies the radial spread of a boundary layer jet on a flat plate, to account for the influence of fluid slippage. This model assumes that the horizontal length scale of the surface asperities is much
4.1 Model formulation

Suppose a steady two-dimensional liquid jet is incident normal to a lubricant-infused surface and is horizontally spreading across the surface. Figure 4.1 shows a diagram of this problem. The main focus of this study is the horizontally spread-

This model is similar to that in Dressaire et al. [77], who modelled thin film flows over microdecorated surfaces, but with novelty arising from having no leakage into the surface and comparing analysis with numerical solutions. Next, we consider a case where the horizontal length scale of the surface asperities is comparable to that of the jet, resulting in a coupling of the flow in the jet to the flow in the lubricant as well as the deformation of the interface between the two fluids. The approximate model is based on the idea in Asmolov et al. [117], who studies Stokes flow above a confined domain of lubricant, but is extended to include the effects of inertia and a finite thickness of the fluid atop the surface.
Figure 4.2: A schematic highlighting the flow region of interest, as well as the four main length scales: the jet horizontal length scale $L$, the jet vertical length scale $l$, the surface asperities horizontal length scale $G$ and the surface asperities vertical length scale $D$.

ing thin film of liquid as well as the influence and reaction of the lubricant-infused surface. The thin liquid film is assumed to be a finite depth boundary layer jet, as in the classical formulation of Watson [73] and, subsequently, others [74, 77] and is assumed steady throughout. The study here considers only the relatively short scale interactions and not the longer ones, such as hydraulic jumps. The lubricant-infused surface is made up of rectangular pillars with the space in between the pillars filled with a fluid that is different from the jet liquid atop of it. The lubricant and the jet are assumed to remain in contact, with no air pockets trapped in between.

Figure 4.2 shows a schematic of the flow region of interest, that is from the point downstream of the jet where the viscous boundary layer has grown to encapsulate the whole jet. There are four length scales that need to be considered in this problem: the jet horizontal length scale $L$, the jet vertical length scale $l$, the surface asperities horizontal length scale $G$ and the surface asperities vertical length scale $D$. Two cases will be considered based on the relative size of the length scales, varying the size of the surface asperities while always keeping the aspect ratio of the jet $l/L$ small and the vertical length scales of the jet and the surface asperities comparable, in order to have a strong interaction. The first case to be considered is that of relatively short lubricant elements with comparable vertical and horizontal length scales, such that

$$G \sim D \sim l \ll L$$  \hspace{1cm} (4.1)
4.2. Short lubricant elements

In this scenario, since the horizontal length scale of the lubricant is much smaller than the jet, we need not consider the flow of the lubricant in order to study the flow in the jet. The next case that will be considered is that of long/shallow lubricant elements, such that

\[ D \sim l \ll G \sim L. \]  \hspace{1cm} (4.2)

Here, the flow of the jet and the lubricant will be coupled and the meniscus of the lubricant, the interface between the jet and the lubricant, will come into effect.

Each of the scaling assumptions above will be considered in turn, with models derived for each case. Both cases will assume the jet to be a boundary layer jet, with the lubricant being a different fluid to the jet fluid but with comparable viscosity and density, as is commonly the case with lubricant infused surfaces [7, 122].

\section*{4.2 Short lubricant elements}

In this section we will assume the scalings given by (4.1) apply to the model in figure 4.2. Hence, consider a steady, two-dimensional jet of typical velocity \( U_0 \), thickness \( l \) and length \( L \) travelling atop a lubricant-infused surface. The jet has viscosity \( \mu \) and density \( \rho \). The problem is defined on a Cartesian coordinate system \((x^*, y^*)\), where the asterisk superscript denotes dimensional variables or coordinates. The flow is considered from the point \( x_0^* \), where the boundary layer has grown to encapsulate the whole jet. The lubricant-infused surface is assumed to be of a length scale much smaller than the jet, so only influences the jet via velocity slippage at the interface, which is significant since the vertical length scales of the jet and the

Figure 4.3: Schematic of the model problem of a boundary layer jet atop a lubricant-infused surface with relatively short lubricant elements.
4.2. Short lubricant elements

lubricant-infused surface are comparable. The boundary condition on the lubricant-infused surface is modelled as a partial slip condition \([9, 116]\), which in dimensional form is

\[ u^* = \lambda^*(x^*) \frac{\partial u^*}{\partial y^*} \]  

applied at \( y^* = 0 \), where \( u^* \) is the horizontal velocity component and \( \lambda^*(x^*) \) is the slip length. The slip length is commonly interpreted as the distance below the surface at which the horizontal velocity component will extrapolate to zero. The interface of the lubricant-infused surface and the jet is assumed to remain flat on these scales and the unknown free surface of the jet is given by \( \gamma^* = h^*(x^*) \). Figure 4.3 shows a schematic of this model problem.

All lengths are non-dimensionalised by \( L \), hence \((x^* - x^*_0, y^*) = L(\tilde{x}, \tilde{y})\) and \( h^* = L\tilde{h} \), all velocities by \( U_0 \) and pressure by the inertial scale \( \rho U_0^2 \). The flow in the jet is assumed to be governed by the steady Navier-Stokes equations and the effects of gravity and compressibility are considered negligible. The steady Navier-Stokes equations in non-dimensional form are

\[ \tilde{\mathbf{u}} \cdot \tilde{\nabla} \tilde{\mathbf{u}} = -\tilde{\nabla} \tilde{p} + \frac{1}{\text{Re}} \tilde{\nabla}^2 \tilde{\mathbf{u}} \]  

\[ \tilde{\nabla} \cdot \tilde{\mathbf{u}} = 0 \]

where \( \tilde{\mathbf{u}} = (\tilde{u}, \tilde{v}) \) is the velocity field, \( \tilde{p} \) is the pressure, \( \tilde{\nabla} = (\partial / \partial \tilde{x}, \partial / \partial \tilde{y}) \) is the two-dimensional gradient operator and \( \text{Re} = \rho U_0 L / \mu \) is the Reynolds number.

At the interface between the lubricant-infused surface and the jet, the partial-slip condition is now

\[ \tilde{u} = \tilde{\lambda}(\tilde{x}) \frac{\partial \tilde{u}}{\partial \tilde{y}}, \quad \text{at} \ \tilde{y} = 0 \]

where \( \tilde{\lambda} = \lambda^*/L \). No penetration of fluid into the surface is also assumed, hence \( \tilde{v} = 0 \) at \( \tilde{y} = 0 \). At the interface between the top of the jet and the air \( \tilde{y} = \tilde{h} \), air effects are assumed negligible, hence we apply a zero shear condition \( \partial \tilde{u} / \partial \tilde{y} = 0 \). Also, neglecting surface tension yields \( \tilde{p} = 0 \) (atmospheric) from the normal stress condition. Finally, the kinematic condition at \( \tilde{y} = \tilde{h} \) is \( \tilde{v} = \tilde{u}(d\tilde{h}/d\tilde{x}) \).
4.2. Short lubricant elements

Now, we scale the system by defining the small parameter

\[ \varepsilon = \frac{l}{L} \ll 1, \quad (4.6) \]

which implies a coordinate scaling of \( (\tilde{x}, \tilde{y}) = (x, \varepsilon y) \). First order expansions of the fluid variables are then taken as

\[ (\tilde{u}, \tilde{v}, \tilde{p}, \tilde{h}) = (u, \varepsilon v, p, \varepsilon h) + \cdots \quad (4.7) \]

where the scaling of the vertical velocity component arises from balancing the continuity equation with an order one horizontal velocity. Now, at leading order we have the following system on the \( x = O(1) \) scale

\[ \begin{align*}
\frac{u}{\varepsilon} \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} &= \frac{1}{\bar{\text{Re}}} \frac{\partial^2 u}{\partial y^2} \quad (4.8a) \\
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} &= 0 \quad (4.8b) \\
\frac{\partial u}{\partial y} &= 0, \quad v = u \frac{dh}{dx} \quad \text{at } y = h(x) \quad (4.8c) \\
u &= \lambda(x) \frac{d u}{d x}, \quad v = 0 \quad \text{at } y = 0 \quad (4.8d)
\end{align*} \]

where \( \bar{\text{Re}} = \varepsilon^2 \text{Re} = O(1) \) and \( \lambda = \tilde{\lambda}/\varepsilon = O(1) \). The pressure term is no longer present here as the vertical component of the momentum equation \( (4.4a) \) gives \( \partial p/\partial y = 0 \) to leading order. Application of the normal stress condition at the free surface then yields \( p = 0 \) everywhere. At \( x = 0 \) we impose a velocity profile that satisfies the partial-slip condition, the zero shear condition and has a unit average velocity. Hence, the conditions at \( x = 0 \) are

\[ h = 1, \quad u = \frac{3}{2 + 6\lambda(0)} \left( 2\lambda(0) + 2y - y^2 \right), \quad v = 0 \quad \text{at } x = 0. \quad (4.9) \]

A constant mass flow rate in the jet through each \( x \)-station needs to also be imposed,
and given the above condition this implies

\[ \int_0^{h(x)} u \, dy = 1 \quad (4.10) \]

The system (4.8-4.10) forms a fully closed system defining the flow of a thin boundary layer jet atop a lubricant-infused surface modelled by a partial-slip boundary condition. Solutions will be sought in the proceeding sections via a similarity solution which will then also be used to guide numerical solutions.

### 4.2.1 Similarity solution

Watson [73] showed that a similarity solution exists for the boundary layer jet alone spreading across a solid surface, with a no-slip condition applied at the solid surface. It is assumed that a corresponding solution exists in the current scenario, the main difference being the partial-slip condition now being applied at the solid surface. The following analysis is very similar to that of Dressaire et al. [77], who performed a similar analysis to Watson [73] but with a partial-slip condition applied at the solid surface to model thin film flows over textured surfaces, whereas our analysis is for lubricant-infused surfaces which does have the slight difference that it is assumed that the lubricant prevents any leakage flow into the surface, whereas leakage is present in Dressaire et al. [77].

First, assume that the horizontal velocity component can be represented as

\[ u = U(x)f(\eta) \quad (4.11) \]

where \( \eta = y/h(x) \) and \( U(x) \) is the velocity at the free surface \( y = h(x) \). This representation of \( u \) immediately implies the following boundary conditions on \( f \) at \( \eta = 1 \)

\[ f(1) = 1, \quad \frac{df}{d\eta}(1) = 0 \quad (4.12) \]

by (4.8c). Now in order for the similarity solution to exist, it must be assumed that the slip length can be defined as \( \lambda(x) = \Lambda h(x) \), where \( \Lambda \) is a constant which represents the magnitude of slippage and \( h(x) \) gives its spatial variation. This then
gives the boundary condition at $\eta = 0$ as

$$f(0) = \Lambda \frac{df}{d\eta}(0)$$ (4.13)

by (4.8d). The mass flow rate condition (4.10) gives

$$hU \int_{0}^{1} f \, d\eta = 1$$ (4.14)

which implies that $hU$ is a constant, which in turn implies that

$$h \frac{dU}{dx} = - \frac{dh}{dx} U.$$ (4.15)

Combining the continuity equation (4.8b) and (4.14-4.15) then gives the vertical velocity component as

$$v = \frac{dh}{dx} U \eta f$$ (4.16)

Next, substituting the expressions (4.11) and (4.16) into the boundary layer equation (4.8a) yields

$$h^2 \frac{dU}{dx} f^2 = \frac{1}{Re} \frac{d^2 f}{d\eta^2}$$ (4.17)

which implies that $h^2 dU/dx$ is a constant. In fact, since the shear stress is greatest at $y = 0$ ([73] and this is also confirmed later by the numerical solution), $d^2 f/d\eta^2 \leq 0$ and so it is convenient to write

$$h^2 \frac{dU}{dx} = - \frac{3c^2}{2Re}.$$ (4.18)

Thus, $f$ now satisfies the ordinary differential equation

$$2 \frac{d^2 f}{d\eta^2} = -3c^2 f^2$$ (4.19)

which can be integrated using the boundary conditions (4.12) to yield

$$\left( \frac{df}{d\eta} \right)^2 = c^2 (1 - f^3).$$ (4.20)
4.2. Short lubricant elements

Figure 4.4: (a) $f(0)$ as a function of the slip length parameter $\Lambda$ and (b) the corresponding value of the constant $c$.

It is assumed that no regions of reversed flow exist in the jet, so $df/d\eta \geq 0$ and the positive root of the right-hand-side of equation (4.20) can be taken. Separating variables yields an expression for the constant $c$

$$c = \int_{f(0)}^1 \frac{1}{\sqrt{1 - z^3}} \, dz = c_W - f(0) \, _2F_1 \left( \frac{1}{3}, \frac{1}{2}; \frac{4}{3}; f(0)^3 \right)$$

(4.21)

where $_2F_1$ is the hypergeometric function and $c_W = _2F_1(1/3, 1/2; 4/3; 1) = 1.402182\ldots$ is a constant (equivalent to the constant $c$ found by the same method in Watson [73] for a no-slip surface). The partial-slip boundary condition (4.13) in conjunction with equation (4.20) defines an equation which can be solved for $f(0)$ in terms of $\Lambda$,

$$\frac{f(0)}{\Lambda \sqrt{1 - f(0)^3}} + f(0) \, _2F_1 \left( \frac{1}{3}, \frac{1}{2}; \frac{4}{3}; f(0)^3 \right) = c_W.$$  

(4.22)

Once $f(0)$ has been found, $c$ can be subsequently calculated. This is done numerically and $f(0)$ and $c$ as functions of $\Lambda$ are shown in figure 4.4. As the slip length parameter $\Lambda$ increases, $f(0)$ increases towards unity, implying the slip velocity grows towards the free surface velocity and variation of the horizontal velocity across the jet decreases. As $\Lambda$ increases the corresponding value of $c$ decreases from the value given in Watson [73] $c = 1.402182\ldots$ towards zero.

From equation (4.20), the following quantity can also be calculated
4.2. Short lubricant elements

Figure 4.5: The quantity \( Q_f = \int_0^1 f \, d\eta \) as a function of the slip parameter \( \Lambda \).

\[
Q_f = \int_0^1 f \, d\eta = \frac{1}{c} \int_{f(0)}^{1} \frac{z}{\sqrt{1-z^3}} \, dz = \frac{1}{2c} \left( c_Q - f(0)^2 \, _2F_1 \left( \frac{1}{2}, \frac{2}{3}; \frac{5}{3}; f(0)^3 \right) \right) \tag{4.23}
\]

where \( c_Q = _2F_1(1/2, 2/3; 5/3, 1) = 1.724740 \ldots \) is a constant.

Finally, all that is required to find \( h \) and \( U \) is to solve the equations (4.14) and (4.18), which yields

\[
U = \frac{2\dot{\text{Re}}}{3c^2 Q_f^2} \frac{1}{x+k} \tag{4.24a}
\]

\[
h = \frac{3c^2 Q_f}{2\dot{\text{Re}}} (x+k) \tag{4.24b}
\]

where \( k \) is an integration constant. As in Watson [73], it is again found that the thickness of the jet grows linearly in \( x \) and the velocity decays like \( x^{-1} \). The integration constant \( k \) arises from solving an ordinary differential equation for \( U \), and so in order to find \( k \) we can apply the \( x = 0 \) condition (4.9), which gives

\[
k = \frac{4\dot{\text{Re}}(1+3\Lambda)}{9c^2 Q_f^2(1+2\Lambda)}. \tag{4.25}
\]

The full similarity solution for \( U \) and \( h \) is subsequently given as
Figure 4.6: (a) The free surface velocity $U$, given by equation (4.26a), and (b) the jet thickness $h$, given by equation (4.26b), for $\Lambda = 1$ and $\tilde{\text{Re}} = 1, 2, 3, 4$ and $5$.

Figure 4.7: As in figure 4.6, but with $\tilde{\text{Re}} = 1$ and $\Lambda = 0, 0.1, 1$ and $10$.

$$U = \frac{6\text{Re}(1 + 2\Lambda)}{9c^2Q_f(1 + 2\Lambda)x + 4\text{Re}(1 + 3\Lambda)}$$ \hspace{1cm} (4.26a)

$$h = \frac{3c^2Q_f}{2\text{Re}}x + \frac{2(1 + 3\Lambda)}{3(1 + 2\Lambda)}Q_f$$ \hspace{1cm} (4.26b)

One family of solutions is shown in figure 4.6 for $\Lambda = 1$ and different values of $\tilde{\text{Re}}$. Clearly, increasing the value of the (reduced) Reynolds number $\tilde{\text{Re}}$ results in an increased free surface velocity and a reduced jet thickness. It is also reassuring to see $h(0)$ very close to 1 (as in (4.9)), despite not ever being imposed on the similarity solution. Another family of solutions is shown in figure 4.7 for $\tilde{\text{Re}} = 1$ and different values of $\Lambda$. The effect of increasing the slip parameter $\Lambda$ is to increase the free surface velocity, downstream, while decreasing the jet thickness. This is to be expected, since increasing the slip parameter of the surface increases the velocity...
slippage at the interface, resulting in the velocity across the jet decreasing less as it travels downstream. By conservation of mass the jet thickness must decrease for higher $\Lambda$.

The slip velocity along the surface $y = 0$ can be found via $u(x, 0) = U(x)f(0)$, where $U$ is given by equation (4.26a) and $f(0)$ is a function of $\Lambda$, the solution of equation (4.40) and given graphically in figure 4.4(a). Figure 4.8 shows the slip velocity corresponding to the same two family of solutions as in figures 4.6-4.7. Similar trends can be seen here for increasing $\tilde{\text{Re}}$ as in the free surface velocity, as the slip velocity also increases. The slip velocity when $\Lambda = 0$ is zero, and then is increasing for larger values of $\Lambda$.

From equation (4.26b), $h$ is a linear function of $x$ of the form $h = Ax + B$, where $A = 3c^2Q_f/2\tilde{\text{Re}}$ and $B \approx 1$. Therefore, the jet thickness linear gradient $A$ is the only indicator of the thickness of the jet and depends on both $\tilde{\text{Re}}$ and $\Lambda$ (through $c$ and $Q_f$). Figure 4.9 shows a contour map of $A$ for variations in $\tilde{\text{Re}}$, which indicates any increase in either $\tilde{\text{Re}}$ or $\Lambda$ will always lead to a decrease in the jet thickness. This makes sense since increasing the Reynolds number (by reducing the viscous effects) and the slip parameter (by increasing velocity slippage at the interface) both increase the velocity across the jet. By conservation of mass this results in a decrease in the jet thickness.

**Figure 4.8:** The slip velocity along the surface $y = 0$ for (a) $\Lambda = 1$ and $\tilde{\text{Re}} = 1, 2, 3, 4$ and $5$ and (b) $\tilde{\text{Re}} = 1$ and $\Lambda = 0, 0.1, 1$ and $10$. 

4.2. Short lubricant elements
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\[ A = 3c^2Q_f/2\tilde{Re} \]

Figure 4.9: Contour map of the jet thickness linear gradient \( A = 3c^2Q_f/2\tilde{Re} \) for variations in \( \tilde{Re} \) and \( \Lambda \).

4.2.2 Full numerical solution

In this section, a numerical solution of the system (4.8-4.10) is sought. The same expression for the slip length used in the similarity solution will also be used, \( \lambda(x) = \Lambda h(x) \). The numerical solution is found by first applying Prandtl’s transposition and a mapping of the flow region to the unit strip via \( Y = y/h \). The scheme then marches forward from \( x = 0 \) and approximates the solution at each \( x \)-station using second-order central finite differences for the \( Y \) derivatives and first-order implicit backward finite differences for the \( x \) derivatives. At each \( x \)-station the scheme iterates until the horizontal velocity component converges below a given tolerance, usually \( 10^{-6} \), with each iteration calculating the horizontal velocity component from the boundary layer equation (4.8a) via Newton iterations to account for the non-linearity, the vertical velocity from the continuity equation (4.8b) and the jet thickness \( h \) from the mass flow rate condition (4.10). A description and validation of a more complicated version of this system is given in Appendix A.2 and the code used for this problem can be found at: https://github.com/NatHenman/PhD_thesis/blob/main/BL_flat_LIS_code.m.

In figure 4.10 the numerical solution for the slip velocity on the surface \( u(x,0) \)
4.3. Long/shallow lubricant elements

and the jet thickness $h$ is compared with the corresponding similarity solution for $\Lambda = 1$ and various $\tilde{Re}$ values. The same is plotted in figure 4.11 for $\tilde{Re} = 1$ and various $\Lambda$ values. The numerical solution exhibits excellent agreement with the similarity solution and thus the same conclusions can be drawn. In fact, the numerical solution can act as validation for the similarity solution which gives explicit formulae for the flow quantities of interest.

4.3 Long/shallow lubricant elements

Now we turn our attention to the problem with the scaling assumption (4.2). In this scenario the lubricant-infused surface has long/shallow lubricant elements and as such coupled flow models in both the jet and the lubricant need to be considered.
For simplicity, our study will be restricted to the flow above one such lubricant element and assume that the flow over other elements is quantitatively similar but just re-scaled. Hence, now consider a two-dimensional jet with typical velocity $U_0$, thickness $l$ and length $L$ travelling atop a confined rectangular fluid domain of height $D$ and length $G$. The jet and lubricant have viscosities and densities of $\mu_l$, $\mu_{LIS}$ and $\rho_l$, $\rho_{LIS}$, respectively. The flow is considered in the Cartesian coordinate system $(x^*, y^*)$ between the $x^*$-stations of $x^*_0$ and $x^*_0 + G$, where $x^*_0$ is the leading edge of the confined lubricant which is assumed to be past the point where the boundary layer has filled the whole jet, see figure 4.2. The flow in the lubricant is confined between $-D < y^* < H^*(x^*)$ and the flow in the jet is confined between $H^*(x^*) < y^* < H^*(x^*) + h^*(x^*)$, where $H^*(x^*)$ is the position of the lubricant meniscus and $h^*(x^*)$ is the thickness of the liquid jet. Figure 4.3 shows a schematic of this problem.

The governing equations here are derived in a very similar fashion to the previous section. First, all lengths are non-dimensionalised with $G$, hence $(x^*, y^*) = G(\tilde{x}, \tilde{y})$, $h^* = G\tilde{h}$ and $H^* = G\tilde{H}$, and all velocities with $U_0$. The pressure scale in the jet is taken as the inertial scale $\rho_l U_0^2$ while the viscous pressure scale is taken in the lubricant $\mu_{LIS}U_0/G$ since we require the pressure gradient and the viscous term

Figure 4.12: A schematic showing the flow region of interest: a thin boundary layer jet atop a similarly thin, confined lubricant element.
to remain comparable even if the viscosity ratio between the lubricant and the jet becomes large. The flow is assumed to be governed by the steady Navier-Stokes equations in both the jet and the lubricant. Flow variables in the jet will be denoted with lowercase letters, while in the lubricant they will be denoted with capital letters. In the jet they are

\[
\vec{u} \cdot \vec{\nabla} \vec{u} = -\vec{\nabla} \bar{p} + \frac{1}{Re} \vec{\nabla}^2 \vec{u}
\]

(4.27a)

\[
\vec{\nabla} \cdot \vec{u} = 0
\]

(4.27b)

where \( \vec{u} = (\vec{u}, \vec{v}) \) is the velocity field and \( \bar{p} \) is the pressure. In the lubricant they are

\[
\vec{U} \cdot \vec{\nabla} \vec{U} = -\frac{\mu}{\rho Re} \vec{\nabla} \bar{p} + \frac{\mu}{\rho Re} \vec{\nabla}^2 \vec{U}
\]

(4.28a)

\[
\vec{\nabla} \cdot \vec{U} = 0
\]

(4.28b)

where \( \vec{U} = (\vec{U}, \vec{V}) \) is the velocity field and \( \bar{P} \) is the pressure. Here, \( Re = \rho_i U_0 G / \mu_i \) is a Reynolds number based on the jet properties, \( \mu = \mu_{LIS} / \mu_i \) is the viscosity ratio and \( \rho = \rho_{LIS} / \rho_i \) is the density ratio. Currently, it is assumed that the viscosity and density ratios are order one, which is supported by fluid properties typically used in these types of surfaces [7, 122].

At the lubricant meniscus \( \vec{y} = \vec{H} \) the continuity of shear stress, normal stress and velocity and the kinematic condition read

\[
\frac{\partial \vec{u}}{\partial \vec{y}} = \mu \frac{\partial \vec{U}}{\partial \vec{y}}, \quad \frac{\rho Re}{\mu} \bar{p} - \bar{P} = \frac{1}{\mu Ca} \kappa(\vec{x}), \quad \vec{u} = \vec{U}, \quad \vec{v} = \vec{u} \frac{d\vec{H}}{dx},
\]

(4.29)

respectively, where \( Ca = \mu_i U_0 / \sigma \) is the Capillary number, with \( \sigma \) being the surface tension coefficient between the lubricant and the jet liquid, and \( \kappa(\vec{x}) \) is the curvature of the lubricant meniscus, given by

\[
\kappa(\vec{x}) = \frac{d^2 \vec{H}}{dx^2} \left( 1 + \left( \frac{d\vec{H}}{dx} \right)^2 \right)^{-3/2}
\]

(4.30)
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At the free surface of the jet \( \bar{y} = \bar{H} + \bar{h} \) it is assumed that the fluid above the jet is air and can subsequently be ignored. Also ignoring the effects of surface tension on this interface gives the continuity of shear stress and normal stress and the kinematic condition as

\[
\frac{\partial \bar{u}}{\partial \bar{y}} = 0, \quad \bar{p} = 0, \quad \bar{v} = \bar{u} \frac{d}{d\bar{x}}(\bar{H} + \bar{h})
\]

In addition, we require a no-slip and no-penetration condition on the walls bounding the lubricant. Thus at \( \bar{x} = 0, 1 \) and \( \bar{y} = -\frac{D}{G} \) the model requires

\[
\bar{U} = 0.
\]

Now, the system is to be scaled. The small parameter that is used in this section shall be

\[
\varepsilon = \frac{D}{G} \ll 1,
\]

(not to be confused with the small parameter used in the model of Sec. 4.2). The coordinate scaling is \((\bar{x}, \bar{y}) = (x, \varepsilon y)\) and leading order expansions of the fluid variables are

\[
(\bar{u}, \bar{v}, \bar{p}, \bar{h}) = (u, \varepsilon v, p, \varepsilon h) + \cdots
\]

in the jet and

\[
(U, \bar{V}, \bar{P}, \bar{H}) = (U, \varepsilon V, \varepsilon^{-2} P, \varepsilon H) + \cdots
\]

in the lubricant. The large pressure scale in the lubricant is taken in order to balance the pressure gradient term with the viscous terms. The governing equations at leading order are now

\[
u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = \frac{1}{\varepsilon^2 \text{Re}} \frac{\partial^2 u}{\partial y^2}
\]

\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0
\]

in the jet, where again the pressure terms drop out due to \( p \) being independent of \( y \) and the normal stress condition at the jet free surface. The small parameter \( \varepsilon \) is, unusually, retained in the problem as opposed to scaling out in a reduced Reynolds number as in the previous section. This is due to, in what will transpire in the
proceeding sections, the unscaled angle the lubricant meniscus makes with the solid surface needing to be calculated. As such, the small parameter $\varepsilon$ remains as a, albeit small, parameter in the problem which represents the geometry. Larger values of $\varepsilon$ correspond to shorter lubricant elements and vice versa. The boundary conditions at the lubricant meniscus $y = H$ now reduce to

$$
\frac{\partial u}{\partial y} = \mu \frac{\partial U}{\partial y}, \quad P = -\frac{\varepsilon^3}{\mu Ca} \frac{d^2 H}{dx^2}, \quad \mathbf{u} = \mathbf{U}, \quad v = u \frac{dH}{dx}, \quad (4.37)
$$

while at the jet free surface $y = H + h$ they are

$$
\frac{\partial u}{\partial y} = 0, \quad v = u \frac{d}{dx}(H + h). \quad (4.38)
$$

The no-slip and no-penetration conditions in the lubricant are now applied to the sides and bottom of a unit box, $x = 0, 1$ and $y = -1$, where

$$
\mathbf{U} = 0. \quad (4.39)
$$

Now, the flow in the lubricant is confined between the regions $x = 0$ and $1$, and therefore re-circulation will be present throughout. In order to make progress in modelling a reduced system for this model which is capable of capturing the meniscus deformation, an approximation is made to the velocity profile, which will be supported by Navier-Stokes simulations of the unscaled problem with a flat meniscus. A similar approach was taken in Asmolov et al. [117], who assumed a uni-directional, parabolic velocity profile with zero flow rate driven by the slip velocity at the lubricant meniscus. The approach here is similar, except now it would seem apt to attempt to model the edge effects of the lubricant, since the vertical scales of both flows of interest (the jet and the lubricant) are comparable. This is done by removing the strictly uni-directional assumption and imposing an extra condition to the horizontal velocity component to the ones above. We assume that, along with the bottom of the lubricant where the no-slip condition applies, the horizontal velocity is also zero along a given curve $\eta = f_0(x)$, where $\eta = (y + 1)/(H + 1)$ is a
normalised vertical coordinate in the lubricant with $0 \leq \eta \leq 1$. From solutions of the Navier-Stokes equations in a thin cavity driven by a given velocity, it can be shown that this curve is approximately independent of the given velocity profile at the interface and Reynolds number. It is given by

$$f_0(x) = \frac{1}{3} \left( 4 - \text{erf} \left( \frac{2}{\varepsilon}x \right) + \text{erf} \left( \frac{2}{\varepsilon}(x - 1) \right) \right). \quad (4.40)$$

where

$$\text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} \, dt \quad (4.41)$$

is the error function. Previous studies on lid driven cavities and slip lengths on surfaces of a wide variety of geometries discuss how the centre line of the main vortex in the cavity fluid and the slip length grow like the error function near the edges of the cavity [135, 136, 137]. Figure 4.13 gives a comparison for the approximated curve on a flat meniscus ($y = f_0(x) - 1$) against the curve found from numerical solutions, calculated using standard relaxation methods, to the Navier-Stokes equations in the unscaled system (4.28) calculated on the thin domain $0 \leq \tilde{x} \leq 1$, $-\varepsilon \leq \tilde{y} \leq 0$ with a flat meniscus, an interface velocity of $\overline{U}(x, 0) = 1$ and $\mu \text{Re}/\rho = 100$. 

![Comparison of the approximated curve](image)

**Figure 4.13:** Comparison of the approximated curve for the interior position of zero horizontal velocity to the position found from solutions of the Navier-Stokes equations (4.28) in the lubricant calculated on the thin domain $0 < \tilde{x} < 1$, $-\varepsilon \leq \tilde{y} \leq 0$ with a flat meniscus, an interface velocity of $\overline{U}(x, 0) = 1$ and $\mu \text{Re}/\rho = 100$. 

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\[-\varepsilon \leq \bar{y} \leq 0\] with \(\bar{U}(x,0) = 1\) and \(\mu \text{Re}/\rho = 100\). For the vast majority of the domain, the approximation and the full solution are both constant around \(y = -1/3\), while both move towards zero at the end regions with close agreement with one another. Further validation and investigation of this approximation is given in Appendix B.

In the lubricant we therefore approximate the velocity profile as

\[
U \approx u_s(x)(A(x)\eta^3 + B(x)\eta^2 + C(x)\eta),
\]

(4.42a)

where \(u_s(x) = u(x,H)\) is the slip velocity along the lubricant meniscus. This velocity profile has zero flow rate and satisfies \(U(x,\eta = 0) = U(x,\eta = f_0) = 0\). The functions \(A, B\) and \(C\) are defined in terms of the function \(f_0\) and it is worth noting that for the vast majority of the flow region \(f_0 = 2/3\) and so \(A = 0, B = 3\) and \(C = -2\), which results in a parabolic velocity profile akin to that in Asmolov et al. [117].

No approximation to the vertical velocity is made, nor, as will become evident, is necessary, other than it satisfying the kinematic condition in (4.37), \(V = u_s dH/dx\), at \(y = H\).

Given the velocity profile (4.42a), a number of simplifications can be made. Firstly, the shear at the lubricant meniscus \((\eta = 1)\) can be calculated as

\[
\frac{\partial U}{\partial y} = \frac{u_s(3A + 2B + C)}{H + 1}
\]

(4.43)

which can be used to formulate the continuity of shear stress and velocity into one boundary condition

\[
u = \frac{(H + 1)}{\mu(3A + 2B + C)} \frac{\partial u}{\partial y}
\]

(4.44)

which is a partial-slip condition with slip length \((H + 1)/\mu(3A + 2B + C)\). Also, at leading order in \(\varepsilon\), the horizontal pressure gradient can be estimated as

\[
\frac{\partial P}{\partial x} \approx \frac{\partial^2 U}{\partial y^2} - \frac{\varepsilon^2 \rho \text{Re}}{\mu} \left( \frac{U}{\partial x} + V \frac{\partial U}{\partial y} \right)
\]

(4.45)
which when evaluated at $y = H$ and combined with the $x$ derivative of the normal stress condition in (4.37) yields an equation governing the meniscus deformation

$$\frac{d^3 H}{dx^3} = - \left( \frac{2\mu Ca u_s (3A + B)}{\varepsilon^3} \right) + \frac{\rho Re Ca}{\varepsilon} \frac{d}{dx} (u_s(A + B + C)).$$

(4.46)

The three boundary conditions applied to this equation are that the two ends of the meniscus are pinned to the corner of the lubricant cavity and conservation of mass, giving

$$H(0) = H(1) = \int_0^1 H \, dx = 0.$$

(4.47)

The assumption and potential break down of the pinning condition is discussed in Sec. 4.3.3.

To summarise, a coupled system between the boundary layer jet and the meniscus deformation has been derived. In the region $H \leq y \leq H + h$ the flow in the jet is governed by the boundary layer equations (4.36), with boundary conditions at the free surface $y = h + H$ given by (4.38) and at the lubricant meniscus $y = H$ given by

$$u = \frac{(H + 1)}{\mu (3A + 2B + C)} \frac{\partial u}{\partial y}, \quad v = u \frac{dH}{dx}.$$

(4.48)

The position of the lubricant meniscus is given by equation (4.46) with conditions (4.47) applied. The functions $A, B$ and $C$ are given by the expressions (4.42b) which are in terms of $f_0$, given by (4.40). At $x = 0$, the jet thickness is specified and the velocity profile is assumed to be parabolic with zero shear at the jet free surface and a unit average velocity

$$h(0) = h_0, \quad u(0, y) = \frac{3y}{2h_0} (2h_0 - y).$$

(4.49)

A conservation of mass flow rate condition is also applied here, which is

$$\int_H^{H+h} u \, dy = h_0.$$

(4.50)
The model here depends on six non-dimensional parameters

$$\varepsilon = \frac{D}{G}, \quad \text{Re} = \frac{\rho_l G U_0}{\mu_l}, \quad \text{Ca} = \frac{U_0 \mu_l}{\sigma}, \quad \mu = \frac{\mu_{LIS}}{\mu_l}, \quad \rho = \frac{\rho_{LIS}}{\rho_l}, \quad h_0 = \frac{l}{D}. \quad (4.51)$$

### 4.3.1 Large lubricant viscosity limit

A practically important limit to consider here is one of large lubricant viscosity, such that $\mu \gg 1$. In this limit, the boundary condition at the lubricant meniscus $y = H$ becomes the no-slip condition (to leading order)

$$u = v = 0. \quad (4.52)$$

The slip velocity in this scenario, which is now $O(\mu^{-1})$, can be expressed as

$$u_s = \frac{(H + 1)}{\mu (3A + 2B + C)} u_y^{NS}, \quad (4.53)$$

where $u_y^{NS} = \frac{\partial u}{\partial y}|_{y = H}$ is the velocity shear exhibited on the lubricant meniscus when the no-slip condition is applied. The two terms on the right-hand-side of the meniscus equation (4.46) are now $O(1)$ and $O(\mu^{-2})$, respectively. Thus the second term may now be neglected, resulting in a meniscus equation reading

$$\frac{d^3 H}{dx^3} = \frac{2Ca}{\varepsilon^3 (H + 1)(3A + 2B + C)} u_y^{NS} (3A + B). \quad (4.54)$$

In this limit the inertial terms have become negligible and there is only a balance between the pressure gradient and the viscous terms in (4.28). This limit reduces the number of parameters from six to four.

### 4.3.2 Numerical solutions

The system (4.36),(4.38) and (4.46-4.50) requires a numerical treatment. The boundary layer equations (4.36) are solved in a similar manner as in Sec. 4.2.2 except now the new vertical variable in the Prandtl transposition is $Y = (y - H)/h$. The meniscus equation is then solved by successively integrating three times a semi-implicit discrete version of equation (4.46), with the slip velocity $u_s$ taken at the
current iteration and the meniscus shape $H$ on the right-hand-side taken from the previous. The solution is then found by applying the conditions (4.47). The scheme then iterates between the boundary layer equations in the jet (4.36) and the meniscus equation (4.46) until the residual between two successive iterates of both the slip velocity $u_s$ and the meniscus shape $H$ fall below a threshold, which was $10^{-6}$. Grid stretching is introduced at either end of the horizontal domain via a transformation $x \rightarrow f(x)$ which exhibits behaviour $f(x) \sim x^{1/2}$ as $x \to 0^+$ and $f(x) \sim (1-x)^{1/2} + 1$ as $x \to 1^-$. This was to improve accuracy at the end regions where large gradients of the slip velocity are present. This method and it’s validation is given in detail in Appendix A.2, and the code used can be found here: https://github.com/NatHenman/PhD_thesis/blob/main/BL_meniscus_LIS_code.m.

The system here depends on a number of non-dimensional parameters (4.51). In order to better understand the parameters and their influence, our focus will be on variations of parameters of physical and practical importance. As discussed in Chapter 3, the lubricants used in these types of surfaces can have a vast range of viscosities, but a more or less constant density and surface tension [122]. For that reason the density ratio is fixed at $\rho = 1$ from here onwards. The dimensional parameters of interest here are the jet velocity, the lubricant viscosity, the surface geometry and the thickness of the jet.

Figure 4.14 shows the solutions for the relative jet thickness $h/h_0$, the slip velocity $u_s$ and the meniscus shape $H$ for $\mu = 1$, $h_0 = 1$ and $\text{Re} = 10^4 \text{Ca}$ from 100
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Figure 4.15: As in figure 4.14 but for $\text{Re} = 100$, $\text{Ca} = 0.01$, $\varepsilon = 0.1$, $h_0 = 1$ and $\mu = 10^{-1}, 10^{-1/2}, 10^0, 10^{1/2}$ and $10^1$. The black dashed line corresponds to the large viscosity limit calculated using equation (4.54).

This particular variation in parameters corresponds to a linear variation in the jet typical velocity. As $\text{Re} = 10^4\text{Ca}$ is increased, the amount of deformation of the meniscus increases with all cases exhibiting a trough to the left and a peak to the right of the center point of the spatial domain. In all cases the slip velocity increases sharply as the jet begins to travel over the lubricant, due to the introduction of slip at the interface. The slip velocity reaches a peak and then, for the cases $\text{Re} = 10^4\text{Ca} = 100$ and 150, starts to gradually decrease towards zero with a sharp decrease near $x = 1$ as the edges of the cavity come into effect. For the other cases, the positive meniscus deformation on the right of the domain is sufficient for the slip velocity to start increasing again, before returning to zero. This is because the slip length is proportional to the depth of the lubricant $H + 1$. As $\text{Re} = 10^4\text{Ca}$ increases, the relative jet thickness decreases since the conservation of mass flow rate condition (4.50) implies that the jet thickness will be smaller for larger horizontal velocities, as is the case for larger values of $\text{Re} = 10^4\text{Ca}$.

Next we examine what effect different lubricant viscosities will have. Figure 4.15 again shows the relative jet thickness, the slip velocity and the meniscus shape, this time for $\text{Re} = 100$, $\text{Ca} = 0.01$, $h_0 = 1$ and a range of viscosity ratios from $\mu = 10^{-1}$ to 10. The figure also gives the corresponding solution in the large lubricant viscosity limit using equation (4.54). As the viscosity ratio $\mu$ is increased, the slip velocity decreases towards zero, which is to be expected as the slip length is inversely proportional to the viscosity ratio and thus the boundary condition would
4.3. Long/shallow lubricant elements

The minimum value of the meniscus shape $H$ as a function of the viscosity ratio $\mu$ for $\varepsilon = 0.1$, $h_0 = 1$ and $(\text{Re}, \text{Ca}) = (100, 0.01), (200, 0.02)$ and $(300, 0.03)$. The dashed line corresponds to the large lubricant viscosity limit calculated using equation (4.54).

Figure 4.16: The minimum value of the meniscus shape $H$ as a function of the viscosity ratio $\mu$ for $\varepsilon = 0.1$, $h_0 = 1$ and $(\text{Re}, \text{Ca}) = (100, 0.01), (200, 0.02)$ and $(300, 0.03)$. The dashed line corresponds to the large lubricant viscosity limit calculated using equation (4.54).

tend towards the no-slip condition. The relative jet thickness increases, since for lower slip lengths, the slower the jet horizontal velocity is, and so the jet thickness must increase accordingly. The jet thickness tends towards the large lubricant viscosity limit and by $\mu = 10^1$ the two results are identical. Interestingly, the maximum deformation of the meniscus shape is slightly non-monotonic in this case for variations in the viscosity ratio. To attempt to examine this effect in greater detail, figure 4.16 shows the minimum value of the meniscus shape $H$ as a function of the viscosity ratio $\mu$ for three different values of $\text{Re} = 10^4 \text{Ca}$. For $\text{Re} = 10^4 \text{Ca} = 100$, the non-monotonic behaviour seen in figure 4.15 is evident, with the minimum value also tending towards the large viscosity limit calculated using equation (4.54). However, as $\text{Re} = 10^4 \text{Ca}$ is increased, the curve becomes monotonically decreasing. The change in monotonicity is perhaps due to the competing effects between the viscous and the inertial terms in equation (4.46). The intriguing result of larger viscosity ratio leading to, for the most part, larger meniscus deformation is surprising, but is also seen in a similar problem [117]. The converse is true for deeper/shorter lubricant elements [138].

Next, figure 4.17 shows the solutions for $\text{Re} = 100$, $\text{Ca} = 0.01$, $\mu = 1$ and $h_0 = 1/(10\varepsilon)$ ranging from 1 to 2. This particular variation of parameters corresponds to a linear decrease in the depth of the lubricant cavity. Here it is evident that shallower cavities result in greater (relative to the cavity depth, it should be stressed) deformation of the meniscus. This is accompanied by a decrease in the slip velocity...
4.3. Long/shallow lubricant elements

Figure 4.17: As in figure 4.14 but for $Re = 100$, $Ca = 0.01$, $\mu = 1$ and $(\varepsilon, h_0) = (0.1, 1), (0.08, 1.25), (0.0667, 1.5), (0.0571, 1.75)$ and $(0.05, 2)$.

Figure 4.18: As in figure 4.14 but for $Re = 100$, $Ca = 0.01$, $\varepsilon = 0.1$, $\mu = 1$ and $h_0 = 1, 2, 3, 4$ and 5.

due to the reduction in slip length, as well as a slight increase in the relative jet thickness.

The final situation we consider in this study is shown in figure 4.18, where $h_0$ varies from 1 to 5 while $Re = 100$, $Ca = 0.01$, $\varepsilon = 0.1$ and $\mu = 1$. This represents a linear increase in the thickness of the jet from equal to the depth of the lubricant cavity to five times the depth of the cavity. The effect of the jet thickness has no influence on the pressure gradient along the meniscus interface other than through the slip velocity. Large values of the incident jet thickness clearly reduce the slip velocity, with the effect of comparable thicknesses apparent in the slip velocity profiles. The slip velocity for $h_0 = 1$ has a large peak towards the front edge of the cavity and is very skewed to the left of the cavity. As the jet thickness increases, this peak reduces and the slip velocity profile moves towards a symmetric profile. This symmetric profile is similar to those seen in studies involving larger domains.
of fluid above the lubricant-infused surface [117, 137]. In this case the reduced slip velocity corresponds to a reduced amount of deformation of the meniscus. The relative jet thickness also decreases with increasing value of \( h_0 \), since the average incident velocity remains unity throughout and so the thinner the jet, the more it is able to slow down which increases its relative change in thickness.

### 4.3.3 Collapse of the surface

In this section we will consider when the surface potentially becomes unstable and collapses, either by the lubricant/jet interface becoming unpinned from the cavity corners, as previously assumed by condition (4.47), or by deforming enough to touch the bottom wall of the cavity. To do this we follow a similar procedure as in Asmolov et al. [117] where the angle the meniscus shape makes with the vertical at each edge of the cavity is compared with the solid surface’s liquid contact angle \( \Theta \). The value of \( \Theta \) must be greater than \( \pi/2 \) in order for the surface to remain in a lubricant-infused state. Figure 4.19 shows a schematic of the two angles \( \theta_1 \) and \( \theta_2 \) the meniscus makes with the vertical axis in the unscaled, dimensional coordinate system, as well as a representation of the liquid contact angle \( \Theta \) of the solid surface. In this particular scenario, multiple authors [139, 140, 141] have reported that the angle the meniscus can make with the vertical axis when pinned to a right-angled edge must be between \( \Theta - \pi/2 \) and \( \Theta \), otherwise depinning would occur. For \( \Theta \geq \pi/2 \) and the meniscus shape shown in figure 4.19 (as well as in all figures in Sec. 4.3.2) the depinning would clearly always occur first at the leading (left) edge.
To calculate \( \theta_1 \) and \( \theta_2 \), we must work in the unscaled, dimensional system, hence
\[
\theta_1 = \frac{\pi}{2} - \tan^{-1} \left( \frac{dH^*}{dx^*}(x_0^*) \right) = \frac{\pi}{2} - \tan^{-1} \left( \varepsilon \frac{dH}{dx}(0) \right), \tag{4.55a}
\]
\[
\theta_2 = \frac{\pi}{2} + \tan^{-1} \left( \frac{dH^*}{dx^*}(x_0^* + G^*) \right) = \frac{\pi}{2} + \tan^{-1} \left( \varepsilon \frac{dH}{dx}(1) \right). \tag{4.55b}
\]
It is clear now why the parameter \( \varepsilon \) needs to be explicitly stated in the problem. As mentioned before, however, it is only necessary to calculate \( \theta_1 \) usually, as \( \theta_1 \) will exceed \( \Theta \) before \( \theta_2 \) is less than \( \Theta - \pi/2 \).

The liquid contact angle \( \Theta \) of the solid surface can be engineered by changing its material properties \([142]\), but cannot exceed \( 2\pi/3 \) in most cases \([142, 143, 144]\). Therefore, we use \( \Theta = 2\pi/3 \) to lead to the conclusion that if \( \theta_{1,2} < \pi/6 \) or \( \theta_{1,2} > 2\pi/3 \) then the surface is unstable and depinning will occur. Again, the only criterion for depinning in this scenario will be \( \theta_1 > 2\pi/3 \).

Figure 4.20 shows the critical curves in the \( (\varepsilon, \text{Re}) \) space and the \( (\mu, \text{Re}) \) parameter space, for \( h_0 = 1 \) and various values of the Capillary number \( \text{Ca} \).
In Figure 4.20(a) the surface is unstable and will collapse for parameters on the left of the critical curve. The black circles mark events where the meniscus depins at the leading edge. For a fixed $\varepsilon$ and a Reynolds number $Re$ below the red circles on the curve, any increase in $Re$ results in an increase in meniscus deformation (a decrease in lubricant thickness at its minimum), right up until the red circles on each curve, while the angle $\theta_1$ remains below the critical value $2\pi/3$. Figure 4.21 shows the minimum lubricant thickness as a function of the Reynolds number for the case $\varepsilon = 0.05$ in figure 4.20. Branching behaviour in the minimum lubricant thickness is seen as the Reynolds number approaches the critical value. For Reynolds numbers $Re$ above the red circles in figure 4.20, no stationary solution exists, and it would seem sensible to assume this is when the meniscus touches the bottom wall of the cavity [117]. Interestingly, these two scenarios intersect smoothly on these critical curves, with the depinning occurring at $\varepsilon \approx 0.05375$ for $Ca = 0.01$ and 0.02, and at $\varepsilon = 0.04875$ for $Ca = 0.03$. In this parameter space, increasing the Capillary number $Ca$ results in the critical curve moving to the right and increasing the size of the unstable parameter regime, suggesting that higher $Ca$ results is a more unstable surface. These curves do appear to (most obviously for $Ca = 0.03$) bend...
4.3. Long/shallow lubricant elements

Figure 4.22: Critical curves in the (a) \((\varepsilon, \text{Ca})\) space with \(\mu = 1\) and \(h_0 = 1\) and the (b) \((\mu, \text{Ca})\) space with \(\varepsilon = 0.1\) and \(h_0 = 1\). The surface is unstable above the curves. The red circles in the curves correspond to events where the meniscus touches the bottom wall of the cavity and the black circles correspond to events where \(\theta_1 > 2\pi/3\) and depinning occurs. In (a), on the right of the black vertical lines corresponds to regions where \(\varepsilon^2\text{Re} = \mathcal{O}(1)\) for \(\text{Re} = 100\) (dotted), 200 (dashed) and 300 (dash-dotted), and in (b) the blue dashed line corresponds to the critical value in the large lubricant viscosity limit, calculated using (4.54).

back on themselves, hence for a fixed \(\varepsilon\) the deformation of the meniscus depends non-monotonically on the Reynolds number \(\text{Re}\).

In figure 4.20(b) the surface in unstable to the right of the critical curves. In all cases here the event which leads to the collapse of the surface is the depinning of the meniscus at the leading edge. Each curve asymptotes exactly towards the large viscosity limit and interestingly for \(\text{Ca} = 0.04\) this appears to reach its large viscosity limit before even \(\mu = 1\). Clearly there is minimal variation past \(\mu = 10\), highlighting that, despite this very much being a possible scenario [122], considering this parameter range is a fruitless task for this model. Increasing the Capillary number \(\text{Ca}\) results in the critical curve moving towards the left of the parameter space, which, again, results in larger unstable parameter regimes, suggesting that increasing \(\text{Ca}\) makes the surface more unstable. It is also seen again that the critical curve bends back on itself again, so for a fixed \(\mu\) the deformation depends non-monotonically on the Reynolds number \(\text{Re}\), suggests the deformation of the meniscus is more generally non-monotonic to variations in \(\text{Re}\).

Figure 4.22 shows the same critical curves but now in the \((\varepsilon, \text{Ca})\) and \((\mu, \text{Ca})\)
4.3. Long/shallow lubricant elements

Figure 4.23: The minimum lubricant thickness $\min(H + 1)$ as a function of the Capillary number $Ca$, for $Re = 100$, $\varepsilon = 0.05$, $\mu = 1$ and $h_0 = 1$. The vertical dashed line is the critical Capillary number $Ca_{crit} = 0.024672965$ and the inset shows sample results for $(Ca_{crit} - Ca)/Ca_{crit} \sim 10^{-6}$ on a log-log scale.

parameter space, with $h_0 = 1$ and various values of the Reynold’s number $Re$. In all these curves, the parameter space above the curves is unstable.

Figure 4.22(a) shows the critical curves in the $(\varepsilon, Ca)$ parameter space with $\mu = 1$, $h_0 = 1$ and $Re = 100, 200$ and 300. The two collapsing events are again present here, with the depinning occurring here at $\varepsilon \approx 0.05125$ for $Re = 100$ and at $\varepsilon \approx 0.055$ for $Re = 200$ and 300. Figure 4.23 shows how the lubricant thickness behaves as the Capillary number $Ca$ approaches the critical value, for the $Re = 100$ and $\varepsilon = 0.05$ case in figure 4.22. The same branching behaviour as in figure 4.21 is exhibited here as $Ca$ approaches the critical value, so we again assume this scenario corresponds to the meniscus touching the bottom wall of the cavity. Increasing the Reynold’s number $Re$, for the most part, moves the curve downwards in this parameter space, resulting in a larger regime which is unstable. However, towards the top end of the $\varepsilon$ range (ie what can still constitute ‘small’), we see curve for $Re = 300$ is above the $Re = 200$ curve. This perhaps is due to the non-monotonic effects $Re$ has on this system, as highlighted in figure 4.20.

Figure 4.22(b) gives the critical curves in the $(\mu, Ca)$ space with $\varepsilon = 0.1$, $h_0 = 1$ and $Re = 100, 200$ and 300. Interestingly, here both collapsing events are
4.3. Long/shallow lubricant elements

Figure 4.24: Solutions for (a) the meniscus shape $H$ and (b) the pressure at the lubricant/jet interface $P(x,H)$ for $Re = 300$, $\varepsilon = 0.1$, $h_0 = 1$ and $(\mu, Ca) = (10^{-1}, 0.136), (10^{-0.84}, 0.106), (10^{-0.68}, 0.0747), (10^{-0.52}, 0.0518)$ and $(10^{-0.36}, 0.0376)$.

now present, unlike in figure 4.22. For the $Re = 100$ curve, only depinning at the leading edge occurs, while for $Re = 200$, depinning begins at $\mu \approx 0.158$, and for $Re = 300$ at $\mu \approx 0.263$. All three curves converge rather rapidly towards their large viscosity ratio limit. For small $\mu$, increasing the Reynold’s number $Re$ acts to move the critical curve upwards, while the converse is true for large viscosity. It is also interesting to note here the slight change in the gradient of the critical curve exhibited for the $Re = 300$ curve. To try to understand what is happening here, we plot five different results for $\mu = 10^{-1}, 10^{-0.84}, 10^{-0.68}, 10^{-0.52}$ and $10^{-0.36}$ and the corresponding value of $Ca$ in figure 4.22(b) which is just below the critical curve, for $Re = 300$. This is shown in figure 4.24, where the meniscus shape $H$ and the pressure at the lubricant/jet interface $P(x,H)$, calculated using the normal stress condition in (4.37), are plotted for the aforementioned parameters. As is the basis of our model derivation, the meniscus shape is driven by the pressure gradient $\partial P/\partial x$ at the lubricant/jet interface, which is proportional to $d^3H/dx^3$. If the pressure gradient is always positive, the meniscus shape will always resemble an almost antisymmetric profile, with trough near the leading edge and peak near the trailing edge. For the $(\mu, Ca) = (10^{-0.36}, 0.0376)$ case in figure 4.24, the pressure gradient is always positive. However, for the other cases at lower $\mu$ and larger $Ca$, the pressure gradient near the leading edge begins to become negative and the effect this has
on the meniscus shape is to begin to decrease the angle it makes to the vertical axis at the leading edge. Eventually, the angle becomes smaller than $\pi/2$ and the shape exhibits two peaks. This is what is causing the collapsing events in figure 4.22(b) to be when the meniscus touches the bottom wall of the cavity. It should be noted here that despite figure 4.24 suggesting that the amount of deformation is decreasing as the viscosity ratio $\mu$ decreases, increases in the Capillary number $Ca$ towards the critical curve are still associated with a decrease in the lubricant thickness and the angles the meniscus makes at the leading and trailing edge remain within the stable range discussed earlier. Therefore it still remains reasonable to assume this scenario is associated with the lubricant touching the bottom wall of the cavity. The scenario that has been discovered here where two peaks in the meniscus shape are exhibited is an interesting result not previously seen in other studies [117] and is likely to be due to the introduction of the inertial effects into the modelling.

To summarise, this chapter has considered the canonical problem arising from jet impact onto a lubricant-infused surface. Two scenarios were considered based on the length scales of the jet and the surface asperities. First, we considered a situation where the horizontal length scale of the surface asperities was much smaller than the jet’s. The resulting boundary layer jet with a partial slip boundary condition applied at the bottom was analysed using a similarity solution and compared to full numerical solutions, with excellent agreement. The effect of increasing the slip length of the surface was found to reduces the thickness of the jet downstream. Next, a case was considered where the horizontal length scale of the jet was much larger than it’s vertical scale and comparable with the jet’s horizontal length scale. An approximate model was derived and the multi parameter problem was analysed numerically to understand the influence of each physical parameter on both the jet and the deformation of the meniscus of the jet/lubricant interface. Parameter regimes of unstable surfaces were found and also discussed.
Chapter 5

Droplet distortion in uniform flow

The previous three Chapters have all been on impact and related flows. Now, in this final topic of consideration, we move away from impact to study the deformation of droplets suspended in a uniform flow. This topic is important for the aviation industry, as before droplets impact onto aircraft, they can be subjected to high velocity, impulsive flow when the aircraft approaches. Understanding how droplets deform and possibly breakup prior to coming close to impact with an aircraft is important as it can give us an idea of the size and distribution of droplets coming into contact with an aircraft.

Previous experimental studies on this topic [78, 79, 80] have been complemented recently by direct numerical simulations [81, 82, 145]. To build a fuller picture of this problem, it would be beneficial to support these findings with theoretical work. This Chapter will present a preliminary study of the problem, building on previous work by Smith and Purvis [146], Smith et al. [147] and Fry [84], who proposed an asymptotic model based on a small density ratio assumption. This work will present detailed numerical results of the reduced models and will account for high Reynolds numbers flows, which are of most practical interest. The work in this Chapter will hopefully build a basis for further research in this topic.

5.1 Model formulation

Suppose initially a two-dimensional circular droplet of radius $R$ is centered in a Cartesian coordinate system $(x^*, y^*)$. At time $t^* = 0$, the fluid surrounding the
droplet is impulsively subjected to a uniform, horizontal flow of speed $U_0$. The * superscript denotes dimensional variables. The droplet fluid is assumed to be water, with density $\rho_l$ and viscosity $\mu_l$, while the surrounding fluid is assumed to be air, with density $\rho_g$ and viscosity $\mu_g$. A schematic of this problem is given in figure 5.1.

All lengths are non-dimensionalised by the droplet radius $R$, hence $(x^*, y^*) = R(x, y)$, all velocities by $U_0$ and pressures by the inertial scale $\rho_g U_0^2$. The time scale is also taken as the inertial one, such that $t^* = R \bar{t} / U_0$. The flow in both the droplet and the air is assumed to be governed by the incompressible Navier-Stokes equations. That is,

\begin{align}
\frac{\partial \tilde{u}_g}{\partial \bar{t}} + \tilde{u}_g \cdot \nabla \tilde{u}_g &= -\nabla \tilde{p}_g + \frac{1}{Re} \nabla^2 \tilde{u}_g \quad (5.1a) \\
\nabla \cdot \tilde{u}_g &= 0 \quad (5.1b)
\end{align}

in the air and

\begin{align}
\frac{\partial \tilde{u}_l}{\partial \bar{t}} + \tilde{u}_l \cdot \nabla \tilde{u}_l &= -\rho \nabla \tilde{p}_l + \frac{\rho}{\mu Re} \nabla^2 \tilde{u}_l \\
\nabla \cdot \tilde{u}_l &= 0 \quad (5.2b)
\end{align}

in the droplet, where $\tilde{u}_l = (\tilde{u}_l, \tilde{v}_l)$ is the velocity field, $\tilde{p}_l$ is the pressure, $\nabla = (\partial / \partial x, \partial / \partial y)$ is the two-dimensional gradient operator, $\text{Re} = \rho_g RU_0 / \mu_g$ is the Reynolds number, $\mu = \mu_g / \mu_l$ is the air/water viscosity ratio and $\rho = \rho_g / \rho_l$ is the

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure5.1.png}
\caption{Schematic of the model problem of a circular droplet of radius $R$ in a uniform flow of speed $U_0$.}
\end{figure}
5.2 Small density ratio scaling

As it is the most common scenario, and one with multiple practical applications [63], the droplet is assumed to have the properties of water and the surrounding fluid to be air. Under these assumptions, the viscosity and density ratios are approximately

\[ \mu \approx 1.8 \times 10^{-2}, \quad \rho \approx 10^{-3}. \]  

(5.5)
Therefore, a small parameter is defined as

\[ \varepsilon = \rho \ll 1. \]  
(5.6)

The viscosity ratio is assumed to be of the same order, via \( \mu = \varepsilon \mu_1 \), where \( \mu_1 = O(1) \).

The small parameter \( \varepsilon \) will now be used to perform an asymptotic analysis of the system (5.1-5.2).

### 5.3 First temporal stage

In the first temporal stage, where \( \tilde{t} = t = O(1) \), it is assumed that the deformation of the air/droplet interface is small. Suppose that the interface is located at \( F(r, \theta, t) = r - 1 - f(\theta, t) = 0 \), where \( r = \sqrt{x^2 + y^2} \) and \( \theta = \tan^{-1}(y/x) \) are polar coordinates. An expansion of the interface function is taken as

\[ r - 1 - f(\theta, t) = r - 1 - (\varepsilon f_1(\theta, t) + \cdots) \]  
(5.7)

where \( f_1 \) is to be determined. The small deformation assumption is valid until \( f_1 = O(\varepsilon^{-1}) \).

Based on the previously discussed assumptions and the fact that we wish to study the impulsive motion of airflow past an initially stationary droplet, leading order expansions of the flow variables in the air are taken as

\[ (\tilde{u}_g, \tilde{p}_g) = (u_g, p_g) + \cdots \]  
(5.8)

and in the droplet as

\[ (\tilde{u}_l, \tilde{p}_l) = (\varepsilon u_l, p_l) + \cdots. \]  
(5.9)

As such, the leading order governing equations are now the unsteady Navier-Stokes equations

\[ \frac{\partial u_g}{\partial t} + u_g \cdot \nabla u_g = -\nabla p_g + \frac{1}{Re} \nabla^2 u_g \]  
(5.10a)

\[ \nabla \cdot u_g = 0 \]  
(5.10b)
in the air, with the polar velocity field \( \mathbf{u}_g = (u_{gr}, u_{g\theta}) \), and the unsteady Stokes equations

\[
\frac{\partial \mathbf{u}_l}{\partial t} = -\nabla p_l + \frac{1}{\mu_1 \text{Re}} \nabla^2 \mathbf{u}_l \quad (5.11a)
\]

\[
\nabla \cdot \mathbf{u}_l = 0 \quad (5.11b)
\]

in the droplet, with \( \mathbf{u}_l = (u_{lr}, u_{l\theta}) \). To leading order, the interface between the air and the droplet is located at \( r = 1 \). Since there is a disparity in the velocity scales, the continuity of velocity condition at \( r = 1 \) is simply the no-slip condition in the air

\[
u_{gr} = u_{g\theta} = 0 \quad (5.12)
\]

while the kinematic condition is now reduced to

\[
\frac{\partial f_1}{\partial t} = u_{lr}. \quad (5.13)
\]

Ignoring the velocity gradient terms in the normal stress condition (\( \text{Re} \gg 1 \)) yields the pressure jump equation at \( r = 1 \)

\[
 p_l - p_g = \frac{1}{\text{We}} \quad (5.14)
\]

while the shear stress condition now reads

\[
 \frac{\partial u_{lr}}{\partial \theta} + \frac{\partial u_{l\theta}}{\partial r} = \mu_1 \frac{\partial u_{g\theta}}{\partial r}. \quad (5.15)
\]

For these scales, the flow in the droplet is coupled to the flow in the air via the shear and normal stress condition, but the flow in the air is not coupled to the flow in the droplet. The flow in the air amounts to the flow past a solid cylinder, with the no-slip and no-penetration condition applied at the cylinder surface, which is a classical problem [148, 149, 150]. The shear stresses and pressure at the surface of the cylinder then drive the flow inside the droplet.
5.3. First temporal stage

5.3.1 Flow in the air

The flow in the air is solved in a polar coordinate domain of \(1 \leq r < \infty\) and \(0 \leq \theta \leq \pi\) (with symmetry assumed along the line \(\theta = 0\) and \(\theta = \pi\)). The governing equations in the air (5.10) are solved numerically using a streamfunction-vorticity formulation, which in polar coordinates, with radial velocity \(u_{gr}\) and azimuthal velocity \(u_{g\theta}\), is

\[
\nabla^2 \psi_g = -\zeta_g
\]

\[
\nabla^2 \zeta_g = \text{Re} \left( \frac{1}{r} \frac{\partial}{\partial \theta} \frac{\partial \zeta_g}{\partial r} - \frac{1}{r} \frac{\partial}{\partial r} \frac{\partial \psi_g}{\partial \theta} + \frac{\partial \zeta_g}{\partial t} \right)
\]

where \(\psi_g\) is the streamfunction in the air, defined via

\[
u_{gr} = \frac{1}{r} \frac{\partial \psi_g}{\partial \theta}, \quad \nu_{g\theta} = -\frac{\partial \psi_g}{\partial r},
\]

and \(\zeta_g\) is the vorticity in the air, defined by

\[
\zeta_g = \frac{\partial u_{g\theta}}{\partial r} + \frac{u_{g\theta}}{r} - \frac{1}{r} \frac{\partial u_{gr}}{\partial \theta}.
\]

The Laplace operator in polar coordinates is

\[
\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2}
\]

Vortex shedding is not considered in this study, thus along the lines \(\theta = 0\) and \(\theta = \pi\) symmetry is assumed by imposing

\[
\psi_g = \zeta_g = 0.
\]

This, combined with the no-slip condition, allows us to simplify the boundary condition on the cylinder \(r = 1\) to

\[
\psi_g = \frac{\partial \psi_g}{\partial r} = 0.
\]

As \(r \to \infty\) (numerically this is taken as \(r \approx 150\)), in order to aid numerical solutions
we apply the potential flow solution [151] for a cylinder in a uniform stream, as well as zero vorticity

\[ \psi_g \rightarrow \left( \frac{1}{r} - r \right) \sin \theta, \quad \zeta_g \rightarrow 0 \]  

(5.22)

This far field solution is also imposed everywhere at \( t = 0 \). Ideally, to model impulsive motion a uniform stream \( \psi_g = -r \sin \theta \) would be imposed everywhere at \( t = 0 \). However, this introduces a numerical complexity with a thin boundary layer present on the surface of the cylinder, which for our purpose is a relatively unimportant consideration. Imposing a potential flow at the start of the calculation has good agreement with experimentally observed flows of the impulsive motion of a cylinder at high Reynolds number [150].

The system (5.16-5.22) needs to be solved numerically. We desire to study scenarios where \( \text{Re} \gg 1 \), since this is of most practical interest [63], hence we need to utilise a suitable numerical scheme. The numerical scheme used in this work is based on that of Hakizumwami [150]. First, a transformation is made to the radial coordinate \( r = e^\xi \), where \( 0 \leq \xi \leq \infty \), which allows greater allocation of grid points closer to the cylinder as well as simplifying the equations somewhat. The vorticity is advanced in time via a second-order Adams-Bashford discretisation of the temporal derivative in equation (5.16b), combined with second-order central finite difference approximations for the spatial derivatives. The streamfunction is found by assuming a Fourier series solution of \( \psi_g \) which solves equation (5.16a) and numerically calculating the Fourier coefficients. The scheme then advances to the next time step. This numerical scheme is given in Appendix A.3.1 along with a validation study. The code used can be found at: https://github.com/NatHenman/PhD_thesis/blob/main/droplet_air_code.m.

The streamlines (contours of the streamfunction \( \psi_g \)) calculated for \( \text{Re} = 100 \) are shown in figure 5.2, for time intervals up until \( t = 10 \). The contour values of \( \psi_g \) are not equally spaced and are chosen simply to represent the flow structure. As time advances, a single vortex forms behind the cylinder, which grows in length and stays relatively thin.
5.3. First temporal stage

![Figure 5.2: Contours of the streamfunction $\psi_g$ for $Re = 100$ at $t = 0, 2, 4, 6, 8$ and 10.](image)

Figure 5.2: Contours of the streamfunction $\psi_g$ for $Re = 100$ at $t = 0, 2, 4, 6, 8$ and 10.

![Figure 5.3: Solutions at $Re = 100$ for (a) the vorticity $\zeta_g$ and (b) the pressure $p_g$ on the cylinder $r = 1$, at $t = 0, 2, 4, 6, 8$ and 10.](image)

Figure 5.3: Solutions at $Re = 100$ for (a) the vorticity $\zeta_g$ and (b) the pressure $p_g$ on the cylinder $r = 1$, at $t = 0, 2, 4, 6, 8$ and 10.

Our main interest here is calculating the vorticity and pressure on the cylinder $r = 1$. On the cylinder, the pressure satisfies the equation

$$\frac{\partial p_g}{\partial \theta} = \frac{1}{Re} \frac{\partial \zeta_g}{\partial r}, \quad (5.23)$$

which can be integrated to yield an equation for $p_g$ on the surface of the cylinder, using $p_g(r = 1, \theta = 0, t) = 0$ as a reference pressure,

$$p_g = \frac{1}{Re} \int_0^\theta \frac{\partial \zeta_g}{\partial r} \, d\theta'. \quad (5.24)$$

Figure 5.3 shows the solutions for the vorticity and the pressure evaluated on the
5.3. First temporal stage

Figure 5.4: As in figure 5.2, but for $Re = 1000$. 

Figure 5.5: As in figure 5.3, but for $Re = 1000$. 

cylinder for $Re = 100$, up until $t = 10$. The vorticity profiles are typically negative at the leading edge of the cylinder ($\theta = \pi$) while positive and of a lower magnitude towards the trailing edge ($\theta = 0$), while the pressure is positive at the leading edge and negative at the trailing. As time advances, the variation in time of both the vorticity and the pressure on the cylinder begins to decrease. The pressure and vorticity on the cylinder become almost steady (although we are aware that in reality the flow outside the cylinder will never become steady due to the high Reynolds number and associated instability and turbulence).

For comparison, a second case is examined with $Re = 1000$. Figure 5.4 show the streamlines around the cylinder at $Re = 1000$. The vortex structure behind the cylinder is now more complex, with secondary and tertiary vortices forming close
to the cylinder. The primary vortex is also now shorter and wider in the horizontal and vertical directions, respectively, compared to the Re = 100 case. The implications of the secondary and tertiary vorticies on the vorticity and pressure exerted on the cylinder surface are shown in figure 5.5. The profiles now fluctuate much more than in the Re = 100 case, however, qualitatively they remain similar.

5.3.2 Flow in the droplet

The flow in the droplet is governed by the Stokes equations (5.11), driven by the normal and shear stress conditions (5.14-5.15) at the droplet interface. Again, this problem is solved using polar coordinates in the domain $0 \leq r \leq 1$, $0 \leq \theta \leq \pi$, with symmetry assumed along the lines $\theta = 0$ and $\theta = \pi$.

The governing equations are expressed in velocity-pressure form as

\begin{align}
\frac{\partial u_{lr}}{\partial t} & = -\frac{\partial p_l}{\partial r} + \frac{1}{\mu_1 \text{Re}} \left( \nabla^2 u_{lr} - \frac{u_{lr}}{r^2} - \frac{2}{r^2} \frac{\partial u_l}{\partial \theta} \right) \quad (5.25a) \\
\frac{\partial u_{l\theta}}{\partial t} & = -\frac{1}{r} \frac{\partial p_l}{\partial \theta} + \frac{1}{\mu_1 \text{Re}} \left( \nabla^2 u_{l\theta} - \frac{u_{l\theta}}{r^2} + \frac{2}{r^2} \frac{\partial u_{lr}}{\partial \theta} \right) \quad (5.25b) \\
\frac{\partial u_{lr}}{\partial r} + \frac{u_{lr}}{r} + \frac{1}{r} \frac{\partial u_{l\theta}}{\partial \theta} & = 0, \quad (5.25c)
\end{align}

where $u_{lr}$ and $u_{l\theta}$ are the radial and azimuthal velocity components in the droplet. The symmetry condition at $\theta = 0, \pi$ is

\begin{equation}
\frac{\partial u_{lr}}{\partial \theta} = u_{l\theta} = \frac{\partial p_l}{\partial \theta} = 0. \quad (5.26)
\end{equation}

Given the pressure $p_\theta(1, \theta, t)$ and vorticity $\zeta_\theta(1, \theta, r)$ on $r = 1$ found from the solutions in the air, the normal (5.14) and shear (5.15) stress conditions act as boundary conditions on $r = 1$ in the droplet

\begin{align}
p_l & = p_\theta(1, \theta, t) + \frac{1}{\text{We}} \quad (5.27a) \\
\frac{\partial u_{lr}}{\partial \theta} + \frac{\partial u_{l\theta}}{\partial r} & = \mu_1 \zeta_\theta(1, \theta, t) \quad (5.27b)
\end{align}

where the velocity gradients on the cylinder $r = 1$ in the air can be expressed as the
vorticity due to the no-slip condition.

For Stokes flow, the pressure is harmonic and satisfies Laplace’s equation,

$$\nabla^2 p_l = 0,$$

which, combined with the symmetry conditions (5.26), the normal stress condition (5.14) and ensuring regularity of the solution at \( r = 0 \), can be solved analytically using separation of variables to yield

$$p_l = \frac{1}{We} + \frac{1}{\pi} \int_0^\pi p_g(1, \theta, t) d\theta + \frac{2}{\pi} \sum_{n=1}^{\infty} r^n \cos(n\theta) \int_0^\pi p_g(1, \theta, t) \cos(n\theta) d\theta.$$  \tag{5.29}

This highlights that the effect of surface tension is to simply vary the pressure everywhere by a constant \( 1/We \). The velocities depend only on the pressure gradients, thus are independent of surface tension effects in the first temporal stage.

The system (5.25) is solved numerically using standard relaxation methods. The system is singular at the origin \( r = 0 \); so to remove this from our calculation the variables are transformed using \( P = rp, U = ru_\rho \) and \( V = ru_\theta \), which forces the variables to be zero at the origin. It is computationally cheaper to solve a discretised version of equation (5.28) for \( p_l \) than evaluate the infinite sum (5.29) at each point in space. Hence, at each time step, the numerical procedure solves for \( P \) from a transformed version of (5.28), then iterates between the two momentum equations until convergence to find \( U \) and \( V \). The variables are all zero at \( r = 0 \) and the symmetry conditions (5.26) are applied at \( \theta = 0 \) and \( \pi \). At \( r = 1 \), the shear stress condition (5.27b) is used to apply a Neumann boundary condition on \( V \) while the continuity equation (5.25c) is used to apply a Neumann condition on \( U \). These two boundary conditions at \( r = 1 \) are coupled, so need to be solved as part of the iterations. A detailed description of this method and its validation can be found in Appendix A.3.2. The code can be viewed at: https://github.com/NatHenman/PhD_thesis/blob/main/droplet_inside_code.m.

Figure 5.2(a) shows the pressure solution as a colour map for Re = 100 and
5.3. First temporal stage

Figure 5.6: Solutions inside the droplet at Re = 100 for, column (a), the pressure $p_l$ shown as a colour map with $We = 1$, column (b), the streamlines for $\mu_1 = 1$ and column (c), the streamlines for $\mu_1 = 20$. Solutions shown at four different time intervals, $t = 2, 4, 6$ and $8$.

We = 1 up until $t = 8$. The pressure evolves from a solution which has a high pressure region near the leading edge of the droplet and low pressure regions on the sides of the droplet, to one with just a high pressure region as the pressure begins to increase in the low pressure region. The corresponding streamlines are shown in Figure 5.2(b-c) for $\mu_1 = 1$ and $\mu_1 = 20$ (the pressure is unaffected by variations in the viscosity ratio $\mu_1$). For an air/water scenario $\mu_1 = \mu_g \rho_l / \mu_l \rho_g \approx 20$ and it is worth comparing it to another value to understand its effect. In both cases the streamlines are qualitatively very similar, predominately moving from left to right from the high pressure region and deviating away from the trailing edge due to the recirculation region behind the droplet. The effect of raising the viscosity ratio from 1 to 20 is a slight bending of the streamlines near the air/droplet interface, which is due to the stronger influence of the shear stress condition at the boundary for higher $\mu_1$. Nonetheless, the effect is rather small.
5.3. First temporal stage

Figure 5.7: As in figure 5.6, but with Re = 1000.

Figure 5.7 shows the same results as figure 5.6, but for Re = 1000. The pressure evolves very similarly to the Re = 100 case, except here at early times the pressure is negative at the sides of the droplet and as time evolves the high pressure region at the leading edge of the droplet is slightly smaller. The streamlines are also similar, however the effect of the recirculation region behind the droplet is now more profound here. Again, the effect of raising the viscosity ratio is the slight bending of the streamlines near the air/droplet interface.

The key finding of this study is how the flow inside the droplet, driven by the flow in the air outside of the droplet, effects the distortion of the droplet interface. In this first temporal stage, it is assumed that the interface expands as a small perturbation around \( r = 1 \), as shown in (5.7). The first order correction \( f_1 \) is obtained from the leading order kinematic condition (5.13), which requires an integration in time of the radial velocity on the air/droplet interface. Figure 5.8 shows the two aforementioned quantities, \( u_{lr} \) and corresponding \( f_1 \), at time intervals up to \( t = 8 \) for the four previous cases of Reynold number/viscosity ratio considered previously.
5.3. First temporal stage

Figure 5.8: Column (a), the radial velocity $u_{r_r}$ evaluated on the air/droplet interface $r = 1$ and, column (b), the corresponding first order correction to the interface shape $f_1$. 
Figure 5.9: The droplet interface distortion \( r = 1 + 10^{-3} f_1 \) when \( \max |f_1| = 100 \), for (a) \( \text{Re} = 100, \mu_1 = 1 \), (b) \( \text{Re} = 100, \mu_1 = 20 \), (c) \( \text{Re} = 1000, \mu_1 = 1 \) and (d) \( \text{Re} = 1000, \mu_1 = 20 \). The time values are the corresponding times when \( \max |f_1| = 100 \) and the dashed lines are the undisturbed circular droplet shape.
5.3. First temporal stage

For the Re = 100 cases, the radial velocity exhibits two main peaks and troughs at the front of the droplet, incident to the flow, near $\theta = \pi$, which transpires to the maximum deformation $f_1$ being present at the front of the droplet also. When the Reynolds number is increased to 1000, as seen in Sec. 5.3.1, the vorticity behind the droplet is much larger than for Re = 100, hence not only do we see significant radial velocity and, correspondingly, deformation at the front of the droplet, we also see significant radial velocity and deformation at the back of the droplet near $\theta = 0$. Here, it is also clear that the effect of increasing the viscosity ratio $\mu_1$ is to increase the radial velocity and deformation everywhere, for both Reynolds numbers. Figure 5.9 shows illustrative examples of the full droplet shapes for $\epsilon = 10^{-3}$ (air/water scenario) when $\max |f_1| = 100$ (at this point the $\epsilon f_1$ starts to be $O(1)$ so is the limit of the first temporal stage model). The incident uniform flow acts to elongate the droplet slightly in the vertical direction. Stronger vortex structures behind the droplet, as in the Re = 1000 case, act to deform the interface inwards behind the droplet.

It is also significant to understand how both the velocity in the droplet and the deformation of the interface evolve with time, as this will give us the appropriate scales to consider when the droplet enters the second, order one deformation, temporal stage. Figure 5.10 shows the maximum radial velocity at the air/droplet interface and the first order correction to the deformation as functions of time. It appears here that the way that the velocity evolves with time is very much depen-
5.4. Second temporal stage

The second temporal stage begins when the interface perturbation $\varepsilon f_1$ becomes $O(1)$. As discussed above, this introduces new temporal and velocity scales based on the Reynolds number and the viscosity ratio. For $Re = 100$ and $\mu_1 = 1$, $u_{lr} \sim t^{1/2}$ and, correspondingly, $f_1 \sim t^{3/2}$, while for $Re = 1000$ and $\mu_1 = 20$, $u_{lr} \sim t$ and $f_1 \sim t^2$. The droplet enters the second temporal stage when $f_1 \sim \varepsilon^{-1}$, thus the scales as we enter the second temporal stage are (or, perhaps most likely, somewhere in between!) either

$$t \sim \varepsilon^{-2/3}, \quad u_l \sim \varepsilon^{-1/3},$$

or,

$$t \sim \varepsilon^{-1/2}, \quad u_l \sim \varepsilon^{-1/2}.$$  

(5.30)

(5.31)

Models pertaining to both scenarios here will be considered in the proceeding section.

Our $f_1 \sim t^2$ compares similarly with that found in Fry [84], who considered shear flow past a wall mounted droplet at moderate Reynolds numbers. However, it should be noted that the full shear stress condition was not applied in Fry [84] at the air/droplet interface (only continuity of vorticity was assumed). Our study appears to show that for Reynolds number of 100, $f_1 \sim t^{3/2}$ and could perhaps decrease further for moderate Reynolds number. Smith and Purvis [146] (see also Smith et al. [147]) proposed a $f_1 \sim t$ scaling, however this is based on the velocities in the droplet eventually balancing with the order one velocities in the air, resulting in $f_1 \sim t$ due to the kinematic condition. For the Reynolds number considered in our study, there is no evidence to suggest the velocities are growing to a comparable scale to the air velocities, however a scenario where $u_l \sim 1$ and $f_1 \sim t$ could be possible for moderate Reynolds numbers based on the relation of the relative scales to the Reynolds number seen in our study. Overall, our results suggest a one-size-fits-all approach to the scalings in the next temporal stage is not appropriate, and it very much depends on the parameters of the problem, such as the Reynolds number and the viscosity ratio.

5.4 Second temporal stage

The second temporal stage begins when the interface perturbation $\varepsilon f_1$ becomes $O(1)$. As discussed above, this introduces new temporal and velocity scales based
on (5.30-5.31). The second stage models are introduced here, but no solutions are presented due to the complexity and time constraints of the project, despite many attempts to find one. It is hoped that the models here are a basis for further research in this topic.

For order one deformations to the air/droplet interface, the interface is now located at \( r = 1 + f(\theta, t) \). To derive the stress boundary conditions at the interface, unit normal and tangent vectors to the air/droplet interface, pointing into the air, need to be defined, in the polar space \((r, \theta)\), as

\[
\hat{n} = \frac{1}{\left( (1+f)^2 + \left( \frac{\partial f}{\partial \theta} \right)^2 \right)^{1/2}} \left( 1 + f, -\frac{\partial f}{\partial \theta} \right), \quad (5.32a)
\]

\[
\hat{t} = \frac{1}{\left( (1+f)^2 + \left( \frac{\partial f}{\partial \theta} \right)^2 \right)^{1/2}} \left( \frac{\partial f}{\partial \theta}, 1 + f \right). \quad (5.32b)
\]

The curvature of the surface can now also be defined as

\[
\nabla \cdot \hat{n} = \frac{(1+f) \left( 1 + f \epsilon + \frac{\partial^2 f}{\partial \theta^2} \right)}{\left( (1+f)^2 + \left( \frac{\partial f}{\partial \theta} \right)^2 \right)^{3/2}}. \quad (5.33)
\]

The governing equations are derived based on the scales (5.30-5.31), but it should be noted that these scalings are based on the first temporal stage variables, in which time and velocity (in the droplet) were originally scaled with 1 and \( \epsilon \), respectively.

### 5.4.1 \( \tilde{t} \sim \epsilon^{-2/3} \) scale

The first case we consider is the \( \tilde{t} \sim \epsilon^{-2/3} \) scale. Here, reverting back to the original unscaled system, time is scaled via \( \tilde{t} = \epsilon^{-2/3} t \) and leading order expansions are taken as

\[
(\tilde{u}_l, \tilde{p}_l) = (\epsilon^{2/3} u_l, p_l) \quad (5.34)
\]
in the droplet, while the same order one expansion as in the first temporal stage (5.8) is taken in the air.

The leading order governing equations in the air are now the steady Navier-Stokes equations

\[ \mathbf{u}_g \cdot \nabla \mathbf{u}_g = - \nabla p_g + \frac{1}{\text{Re}} \nabla^2 \mathbf{u}_g \]  
\[ \nabla \cdot \mathbf{u}_g = 0 \]  

while in the droplet they reduce to

\[ 0 = - \nabla p_l + \frac{1}{\mu_l \text{Re}} \nabla^2 \mathbf{u}_l \]  
\[ \nabla \cdot \mathbf{u}_l = 0 \]

where $\text{Re} = \epsilon^{1/3} \text{Re}$. The Reynolds number in the droplet is rescaled as without a rescaling the viscous terms would be dominant with respect to $\epsilon$. Since $\text{Re} \gg 1$, this does not make sense physically, so the Reynolds number is rescaled so that the viscous terms can balance with the next largest term in $\epsilon$, which was the pressure gradient. The unsteady and inertial terms are $O(\epsilon^{1/3})$ smaller than the pressure gradient term, so can be neglected. For $\epsilon = 10^{-3}$, $\text{Re} = 100 - 1000$ corresponds to $\tilde{\text{Re}} = 10 - 100$, so the viscous terms are of the same, if not smaller, order as the pressure gradient terms in the droplet.

In the far field the velocity in the air remains as a uniform stream and symmetry conditions can be applied along $\theta = 0$ and $\theta = \pi$, while due to the velocity imbalance, the no-slip condition again applies,

\[ u_{gr} = u_{g\theta} = 0, \]  

at $r = 1 + f$. Now, since the velocity scale in the droplet has increased from that in the first stage, there is an imbalance in the shear stress condition at the interface $r = 1 + f$, with the droplet terms dominating. As such, the leading order shear stress
condition at \( r = 1 + f \) is
\[
2(1+f) \frac{\partial f}{\partial \theta} \left( (1+f) \frac{\partial u_{lr}}{\partial r} - \frac{\partial u_{l\theta}}{\partial r} \right) \\
+ \left( (1+f)^2 - \left( \frac{\partial f}{\partial \theta} \right)^2 \right) \left( (1+f) \frac{\partial u_{l\theta}}{\partial r} + \frac{\partial u_{lr}}{\partial \theta} \right) = 0.
\]
(5.38)

The normal stress condition at \( r = 1 + f \) can be expressed as
\[
p_l - p_g = \frac{(1+f) \left( 1 + f + \frac{\partial^2 f}{\partial \theta^2} \right)}{\text{We} \left( (1+f)^2 + \left( \frac{\partial f}{\partial \theta} \right)^2 \right)^{3/2}}
\]
(5.39)
where again the velocity gradient terms are ignored as \( \text{Re} \gg 1 \). Finally, at \( r = 1 + f \) the kinematic condition now contains all terms as the velocities and time derivatives of the interface now balance, giving
\[
\frac{\partial f}{\partial t} = u_{lr} - u_{l\theta} \frac{\partial f}{\partial \theta}
\]
(5.40)
at \( r = 1 + f \).

The system (5.35-5.40) is a closed one. For a numerical solution, the interface boundary \( r = 1 + f \) would need to be mapped to a constant one, \( r = 1 \) say, through a transformation such as \( r \to r/(1+f) \), so that a discrete numerical approximation could be found. However, this throws up the major caveat that the Laplacean operator is no longer separable, so that the Fourier series approximation for the flow in the air, vital for large Reynolds number flows [148, 150], is no longer applicable. Other, more complex, numerical methods would likely need to be applied here.

### 5.4.2 \( \tilde{t} \sim \varepsilon^{-1/2} \) scale

In this case, time is scaled such that \( \tilde{t} = \varepsilon^{-1/2} t \) and the expansions in the droplet are
\[
(\tilde{u}_l, \tilde{p}_l) = (\varepsilon^{1/2} u_l, p_l) + \cdots
\]
(5.41)
and the same order one expansions are taken in the air as before.
The governing equations in the air (5.35) and the boundary conditions at the air/droplet interface \( r = 1 + f \) (5.37-5.40) remain the same as the \( t \sim \varepsilon^{-2/3} \) scale. However, the governing equations in the droplet are now the unsteady Navier-Stokes equations

\[
\frac{\partial \mathbf{u}_l}{\partial t} + \mathbf{u}_l \cdot \nabla \mathbf{u}_l = -\nabla p_l + \frac{1}{\mu_1 \tilde{Re}} \nabla^2 \mathbf{u}_l \quad (5.42a)
\]

\[
\nabla \cdot \mathbf{u}_l = 0 \quad (5.42b)
\]

where \( \tilde{Re} = \varepsilon^{1/2} \text{Re} \). The Reynolds number is again rescaled to avoid the viscous terms dominating while \( \text{Re} \gg 1 \). For \( \varepsilon = 10^{-3} \), \( \text{Re} = 100 - 1000 \) corresponds to \( \tilde{Re} = 3.2 - 32 \).

The same problems as those described in the previous case remain here and are accentuated by the addition of the inertial and unsteady terms in the droplet. This case is perhaps a more general version of the previous one due to the additional terms in the governing equations in the droplet. Hence, a solution to this problem would be much more valuable.

### 5.5 Comparison with direct numerical simulations

Direct comparisons of our results to experimental work is challenging due to the physical parameters involved as well as the time scales. However, some promising comparisons can be made to direct numerical simulations. Both Bian et al. [145] and Li et al. [82] considered identical problems, albeit in an axisymmetric formulation, using direct numerical simulation and we shall compare our results, for the first temporal stage, with these papers.

First of all, it is worth verifying that the deformation of a droplet in a continuous uniform flow is relatively small over the same time scales that are considered in our study. Both Bian et al. [145] and Li et al. [82] give results for the deformation of the droplet, defined as the ratio of maximum axial extent to radial extent in Li et al. [82] and vice versa in Bian et al. [145], as time advances. Bian et al. [145] showed results for Reynolds numbers (expressed in our formulation based on the air properties) in the range 77-387 and Li et al. [82] in the range 26-193. Both studies show that the deformation exceeds 10% around the non-dimensional
5.5. Comparison with direct numerical simulations

time (converted to our time formulation, recall in the first temporal stage the non-dimensional time is defined as \( t = \bar{t} = U_0 t^*/R \) of 10, for all cases. See figures 11, 12 and 14 in Bian et al. [145] and 8 and 9 in Li et al. [82]. This value is in agreement with our assumption that on an order unity time scale, the deformation of the droplet is small. Quantitatively, this value is also similar to the time values that arose from our solutions at which we deemed the deformation to be significant. The shape of the deformed droplet found in the first temporal stage in our study also agrees well with the simulation results [82, 145]. Despite studying a far longer time scale than us, the simulation results agree with an initial elongation in the direction perpendicular to the oncoming flow, with the flattening of the leading side of the droplet being more profound than the trailing side. See the first two time instances in Figures 9(c-d) in Li et al. [82] compared with our results in Figure 5.9.

Finally, regarding the streamlines of the flow, both simulation studies [82, 145] also exhibit a recirculation region behind the droplet of increasing complexity with increasing Reynolds number. Also, a detailed depiction of the streamlines inside the droplet in Bian et al. [145] (see Figure 22), albeit for a non-circular droplet, does appear to show the streamlines responding to an external area of recirculation in a similar manner as our results seen in Figures 5.6-5.7.

To summarise, in this chapter we have described the early stages of droplet deformation when subjected to a uniform flow at high Reynolds number. Models pertaining to larger deformations are described and it is hoped that a numerical solution could be found to this problem, despite not being done so here. This work builds on previous work by Smith and Purvis [146], Smith et al. [147] and Fry [84] and offers a detailed insight into the early stages of droplet deformation at high Reynolds number, something which has previously not been seen using analytical methods. Qualitative connections are also made to available direct numerical simulations of the same problem.
Chapter 6

Conclusions

This thesis has explored a variety of problems related to droplet impact onto engineered surfaces. Through a variety of analytical, numerical and simulation methods, we have investigated two different types of surfaces, deformable and lubricant-infused surfaces. This thesis studied four main problems: the pre-impact phase of droplet impact onto both deformable and lubricant-infused surfaces, the post-impact phase of droplet impact onto lubricant-infused surfaces, thin film flows across lubricant-infused surfaces and finally, a consideration of the deformation of a droplet suspended in a uniform flow. The conclusions and implications of the results of each problem will now be discussed in detail.

In Chapter 2, we considered the pre-impact phase of droplet impact. A fluid–structure interaction model describing the pre-impact air cushioning behavior of a droplet impacting a deformable surface has been developed. It rationally couples the thin film lubrication flow of the air to an approaching quasi-inviscid droplet approaching from above and a membrane-type model of the deformable surface below. Building on previous work by Smith et al. [30], the model assumes a deformable surface deflection of the same order as the droplet free surface deformation, which allows us to couple the surface deflection with the air film pressure and free-surface deflection.

The deflection of the surface depends on the parameters describing the surface properties, which are incorporated into the membrane type deformable surface equation. These parameters correspond to the surface rigidity, tension, spring stiff-
ness, mass density and damping. We considered two separate major cases of the surface equation. The first, a viscoelastic model, only considered the surface stiffness and the (viscous) damping. Numerical solutions to this system were presented and a number of conclusions were drawn. It was found that by lowering the magnitude of the surface parameters and increasing deformability, the approach to touchdown is considerably delayed, the speed of the touchdown point is lowered, pressure buildup is decreased and more air is entrapped as touchdown is approached. For sufficiently low magnitude parameters, the pressure peak as touchdown is approached is a round one, as opposed to a sharp peak seen before on a rigid surface. The pressure peak is also seen to be located just behind the advancing cusp of the air film thickness, resulting in an increased extent of the air film as touchdown is approached. A numerical analysis was also conducted on the effect of variations of droplet radius and impact velocity on the size of entrapped air at impact, comparing a flat rigid surface to soft viscoelastic surfaces. It was found that the increase in entrapped air due to a soft viscoelastic surface was almost independent of droplet radius, on a logarithmic scale, while it is strongly dependent on the droplet velocity. For a flat rigid surface, increased droplet velocity results in decreased air entrapment, while for an impact on a soft solid, this decrease in air entrapment is delayed and even halted for increased droplet velocity. A more general, flexible surface was then considered and broadly the same conclusions can be drawn here as for the viscoelastic surface case. A substantial difference, however, is that for a flexible surface, the pressure peak as touchdown is approached remains sharp, despite reductions in amplitude, for increased flexibility. It is shown that reductions in magnitude of each parameter corresponding to the flexural rigidity, tension, stiffness and mass density (for an undamped flexible surface) resulted in an increased air entrapment. Qualitative connections of these findings were made to recent experimental work by Howland et al. [65] and Langley et al. [42]

A case was then considered where we sought a solution in the limit of large surface deformations, for which the air film thickness is dominated by the surface deformation rather than by the free-surface deformation. This leads to a new sys-
tem of governing equations which is similar to that of the flat rigid surface impact, albeit with a new pressure-shape relationship. This new system, solved computationally, showed the results highlighting the significant increase in horizontal bubble extent and delay to touchdown. A large time analysis was found to give excellent agreement with the full numerical results and confirmed the apparent absence of touchdown (thus hinting at so called gliding) for that particular system.

The same modelling was then applied to analyse the pre-impact phase of droplet impact onto a lubricant-infused surface. The surface was approximated by assuming that velocity slippage could occur at the, now flat, surface by applying a partial slip condition there. A similar inviscid-lubrication model was derived which depended only on one parameter, the slip length. The system was solved numerically, with the main implication being found was that increasing the slip length reduced the size of the entrapped air. Another interesting conclusion was that the peak pressure at impact varied non-monotonically with the slip length, suggesting that a particular slip length lead to optimal pressure peaks.

When considering the pre-impact phase of droplet impact, the results presented in this thesis have broad and important implications. Reductions in the pressure peaks forming underneath the droplet are important when considering impact led damage and ensuing, post-impact dynamics. For example, rain droplet impact onto wind turbines can, over time, lead to significant erosion of the wind turbine blades [152]. Reduced pressure peaks appear to also be significant in reducing splashing [65], which in turn has a number of vital implications, such as in aircraft safety [5, 63], pesticide application [62], forensic science [153] and many others. Our results should also help engineer novel non-wetting and protective surfaces where impact pressure plays an important role. Our results also point towards a consequence of increased air entrapment, which is undesirable in inkjet printing applications [20], for example.

In Chapter 3, we considered the post-impact dynamics of droplet impact onto textured and lubricant-infused surfaces. In this chapter we have explored the solutions to a computational model of droplet impact on to textured and lubricant-
infused surfaces. The textured surface was modelled as an array of rectangular asperities in an otherwise flat, rigid surface, with the asperities filled up to a certain depth with lubricant. We systematically investigated parameters of the surface, namely the distance between the asperities, the depth of the lubricant and the lubricant viscosity.

We first compared results for variations in the parameters pertaining to a lubricant-infused surface. For variations of any of the surface parameters there was no change in the jet root location. All variations in the early-time dynamics were seen in the thin splash jet ejected at impact. It was found that on increasing the distance between the surface asperities, the splash jet extent and tip velocity were reduced, effects which are more profound at higher impact velocity. The depth of the lubricant was identified to have a suppressing effect on the splash jet extent and velocity when the depth of the lubricant is above or below the asperities, compared to when the lubricant depth is level with the asperities. When varying the viscosity of the lubricant, a non-monotonic relationship is also apparent. As the viscosity ratio is increased from an order unity value, the extent and the tip velocity of the splash jet are increased, due to less penetration of the droplet into the asperities. Then, as the viscosity ratio becomes very large, the extent and tip velocity of the splash jet start to decrease again, as the surface becomes flatter due to less deformability of the lubricant. The suppressing effect on the splash jet for large viscosity ratios is more profound at higher impact velocity. For each parameter, a rough phase field diagram of different early time splashing behaviours (jet detachment or micro-drop ejection splashing) was presented for variations in the parameter of interest and the Weber number. When considering the pillar spacing, it was seen that the splashing behaviour transitioned from no splash to both jet detachment and micro-drop ejection splashing relatively quickly when increasing the Weber number, with both behaviours being seen at a lower Weber number when the pillar spacing was larger, suggesting that larger pillar spacing promotes microdrop ejection splashing while suppressing jet detachment splashing. When the pillar spacing was larger and the Weber number was sufficiently high, the jet detachment splash was completely
suppressed. For variations in the lubricant depth it was found that depths of lubricant above and below the asperities had a suppressing effect on the jet detachment splash, which eventually waned slightly for lubricant levels above the asperities as the Weber number increased. When considering the viscosity ratio, we broadly saw four regions. At low Weber number and viscosity ratio, no splash was observed, while for low Weber numbers and large viscosity ratios, a jet detachment splash was observed. At very low viscosity ratio, microdrop ejection splashing was observed for most Weber numbers considered. Finally for large Weber numbers and large viscosity ratios, both jet detachment and microdrop ejection splashing occurred.

We then compared the results for lubricant-infused surfaces to those of textured, superhydrophobic surfaces, where no lubricant is present. It was found that as the distance between the surface asperities was increased, the extent of the splash jet eventually went from being larger on the lubricant-infused surfaces, compared to the textured, superhydrophobic surfaces, to smaller (or similar) values.

The aim of this numerical study was to explore parameters and scales unavailable to analytical methods and which have received only a small amount of interest experimentally. We hope that our idealised two-dimensional model can be used as a qualitative measure of the parameters pertaining to lubricant-infused surfaces and hopefully suggest further avenues for experimental and analytical work.

Chapter 4 considered a related flow to that of droplet impact, thin film flows. This study focused on a model of the horizontal spread of a liquid jet over a lubricant-infused surface. This model considered a viscous boundary layer jet travelling atop a textured surface filled with lubricant. Two length scales were considered, based on the relative horizontal length scale of the surface asperities. The first case considered the horizontal length scale of the surface to be much smaller than that of the jet, so could be considered flat to leading order. The effect that the lubricant-infused surface had on the jet was modelled using a partial-slip condition at the bottom of the jet. This system was solved using a similarity solution, and was then compared with full numerical solutions, with excellent agreement. The solutions showed that increasing the slip length resulted in a thinner jet downstream.
After that, a case was considered where the horizontal length scales of the lubricant and the jet were comparable, and as such the dynamics coupled. An approximate solution for the flow in the lubricant was used to define a partial slip condition at the interface, which was in turn allowed to deform via surface tension. This system had many parameters, which were systematically investigated to understand the effect of various physical factors. It was found that increasing the typical velocity in the jet lead to the jet being thinner downstream and increased meniscus deformation. Increasing the viscosity of the lubricant resulted in an increased jet thickness and, predominately, an increased meniscus deformation. Decreasing the depth of the lubricant cavity lead to minimal changes in the jet thickness, with an increase in the meniscus deformation. Also, increasing the thickness of the incoming jet lead to decreased relative jet thickness and meniscus deformation. Scenarios whereby the surface may become unstable and collapse, either through the meniscus depinning from the cavity edges or touching the bottom wall, were then considered. Parameter spaces were considered based on the cavity geometry or lubricant viscosity with the Reynolds or Capillary numbers. Critical curves were found, defining regions where the surface will be unstable.

Upon impacting on a surface, a droplet often ejects thin jets along the surface. Studying jets themselves in isolation is a very useful task for the overall understanding of droplet impact in general. To that end, our idealised steady jet study has many of the same application listed already. It does also have applications in it’s own right, such as spin coating [133]. Lubricant-infused surfaces are known to have excellent drag reduction capabilities [9, 122] and have received a lot of attention for relatively slow channel flow problems [116, 117], but none in terms of fast free-surface flows. Understanding this problem in greater detail can allow as to assess the suitability for these types of surfaces in industries where thin free-surface flows occur, such as in aviation.

Finally, in Chapter 5 the focus moved away from impact scenarios to study the deformation of a droplet suspended in a uniform flow. The purpose of this Chapter was to build on the previous work of Fry [84] (and others [146, 147]), who studied
the deformation of semi-circular wall mounted droplet in shear flow at moderate Reynolds number. Our focus was on suspended droplets in a uniform flow at high Reynolds number, which is a common situation in aviation settings and has been the subject of recent work utilising direct numerical simulation [82]. An asymptotic analysis was presented based on a small droplet to air density ratio, where two temporal regimes were defined based on an assumption that the initial deformation is small. In the first temporal regime, where the deformation of the droplet is small, a reduced system was derived where the flow in the air surrounding the droplet was governed by the unsteady Navier-Stokes equations with no-slip conditions at the droplet interface. The flow in the droplet was governed by the unsteady Stokes equations which were driven by the pressure and vorticity in the air evaluated at the interface. Through the kinematic condition, we were then able to evaluate the first order correction to the free surface shape of the droplet, with all cases showing an initial elongation of the droplet in the direction perpendicular to the incoming flow. Using the growth of the velocity inside the droplet in this first temporal stage allows us to define scales for the next stage. In general, the next stage is governed by the steady Navier-Stokes equations in the air and unsteady Navier-Stokes in the droplet, with the deformation of the droplet now order unity. The non-circular droplet domain makes this new system extremely complicated to solve and, as such, due to the time constraints of the project this system is not solved. The connections to direct numerical simulation results [82, 145], while limited, are also discussed in this chapter and are promising. The simulation results support the modelling assumptions in this chapter and agree with the initial shape deformation of the droplet and the corresponding streamlines both inside and outside the droplet. This Chapter provides an important insight into some of the dynamics involved in the deformation of droplet in uniform flow. In aviation, it is vitally important to understand the shape and distribution of the droplets before they come into contact with a aircraft. It is hoped that this preliminary study helps to build a basis for further analytical work in this area.

To summarise, this thesis has studied a number of problems related to the im-
6.1 Future work

There are a number of possible avenues for further research on the topics presented in this thesis, which will now be discussed.

A common theme in each Chapter is the two-dimensional modelling. Droplet dynamics are very clearly three-dimensional, and it would be very useful to be able to study all our problems in a three-dimensional framework. Using the method of Hicks and Purvis [31], three-dimensional modelling of the pre-impact phase of droplet impact is possible, and could be applied to the viscoelastic surface and lubricant-infused surface models. However, the membrane model is based on two-dimensional modelling and its extension to three-dimensions is very complicated. For the direct numerical simulation of droplet impact onto lubricant-infused surfaces, the restriction to two-dimensional was solely down to computational resources. Further advancements of computational capabilities and efficient meshing solutions should lead to a fully resolved three-dimensional simulation of droplet impact onto lubricant-infused surfaces at some stage. The study of droplet deformation in a uniform flow could also have been calculated in a three-dimensional axisymmetric framework [154], however this is a significantly more difficult task than the two-dimensional model. The flow in the air may be able to be calculated, but then using this solution to drive the flow in the droplet is not straightforward and would be a far more computationally expensive task. Due the time constraints, this was not attempted but would be the natural next step in this study.

In Chapter 2, there are a number of physical influences that have been ignored, such as surface tension and gas compressibility. It would be interesting to see how these effects interact with the surface properties. It is also assumed that the de-
formable surface is allowed to deform freely and is not influenced by a solid base. In many scenarios, a deformable coating is used and may have a depth of the same order as the air cushioning deformations, so may influence the dynamics.

Due to mesh and computational constraints, the simulations presented in Chapter 3 are limited to a certain time frame. It would be beneficial to be able to run the simulations longer to build a fuller picture of the splashing dynamics. Other physical factors such as oblique impacts, liquid compressibility and thermal effects would also be hugely interesting to study.

In Chapter 4, the boundary layer analysis for relatively short surface asperities could be readily extended to three-dimensions, but is kept in two-dimensions so that it is in line with the rest of the work in the thesis and, most importantly, the subsequent section of Chapter 4. So far, when considering the longer scale surface asperities, we have considered one lubricant element in isolation. Although it is thought likely that any further elements downstream would have similar dynamics but rescaled, it would be worth doing a longer calculation downstream, with more lubricant elements, to see if this is true. Also, and this applies to both cases considered in this Chapter, to tie this work more closely with droplet impacts, unsteady effects and inviscid jets would be intriguing to study as well.

For Chapter 5, to complete this work we would have ideally solved the problem derived in the second temporal stage. The main problem here is the non-circular droplet domain, which complicates standard techniques, but a more complex numerical method may yield results. A one-to-one quantitative comparison of the theoretical work to direct numerical simulations would be very powerful here also.
Appendix A

Numerical methods

This appendix will describe the main numerical methods and algorithms used in this thesis, as well as provide validation of their accuracy. All code used in this thesis can be found at: https://github.com/NatHenman/PhD_thesis.

A.1 Inviscid-lubrication model coupled with deformable surface

First, let's consider the system derived in Chapter 2 which is an coupled system of three integro-differential equations which govern the pre-impact phase of droplet impact with a deformable surface. To recap, the system governing the droplet free surface position $F(X, T)$, the deformable surface shape $G(X, T)$ and the pressure in the air film $P(X, T)$ is

\[
\frac{\partial}{\partial X} \left( (F - G)^3 \frac{\partial P}{\partial X} \right) = 12 \frac{\partial}{\partial T} (F - G), \quad (A.1a)
\]

\[
\frac{\partial^2 F}{\partial T^2} = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\partial P}{\partial \zeta} (\zeta, T) \frac{d\zeta}{X - \zeta}, \quad (A.1b)
\]

\[
\tilde{e}_1 \frac{\partial^4 G}{\partial X^4} + \tilde{e}_2 \frac{\partial^2 G}{\partial X^2} + \tilde{e}_3 G + \tilde{e}_4 \frac{\partial^2 G}{\partial T^2} + \tilde{e}_5 \frac{\partial G}{\partial T} = P, \quad (A.1c)
\]

\[
F \rightarrow \frac{X^2}{2} - T, \quad P \rightarrow 0, \quad \text{as } |X| \rightarrow \infty \text{ and } T \rightarrow -\infty, \quad (A.1d)
\]

\[
G = \frac{\partial G}{\partial X} = 0, \quad \text{at } X = X_1, X_2. \quad (A.1e)
\]
A.1. Inviscid-lubrication model coupled with deformable surface

This is the most general form of the problem. When considering impact onto viscoelastic surfaces (where $\tilde{e}_1 = \tilde{e}_2 = \tilde{e}_4 = 0$), the pressure-shape relationship is substituted into the lubrication equation to reduce the number of equations to two. The method to solve this system is the same as when considering all three equations, so will not be described in detail here.

The infinite spatial and temporal domains are truncated to suitable finite values. The system (A.1) is then solved on a discrete uniform grid $X_i = X_0 + i\Delta X$, $i = 0, \ldots, N$, at each time step $T_m = T_0 + m\Delta T$, with $X_0$ chosen as the finite value to represent the boundary condition at negative infinity and $T_0$ the start time. The spatial domain is symmetric and centered at the origin, hence $N\Delta X = -2X_0$. The function to be found are written as $F_i^m = F(X_i, T_m)$, $G_i^m = G(X_i, T_m)$ and $P_i^m = P(X_i, T_m)$.

The numerical scheme works by solving the discrete version of equation (A.1a) for $P$, equation (A.1b) for $F$ and equation (A.1c) for $G$, taking the terms of each equation as the latest known value. All partial derivative terms are approximated using central, second order finite differences for spatial derivatives and first order, backward finite differences for the time derivatives. The algorithm iterates between all three equations until a convergence criteria has been met. To initialise the solution, since the problem is second order in time, we need to impose the solution at the first two time steps $T_0$ and $T_1$. Since the initial condition is at infinity, we can do this by simply applying the far field condition at $T_0$ and $T_1$.

Equation (A.1a) can be written as

$$3(F_i^m - G_i^m)^2 \left( \frac{F_{i+1}^m - G_{i+1}^m - F_{i-1}^m + G_{i-1}^m}{2\Delta X} \right) \left( \frac{P_{i+1}^m - P_{i-1}^m}{2\Delta X} \right) + (F_i^m - G_i^m)^3 \frac{P_{i+1}^m - 2P_i^m + P_{i-1}^m}{\Delta X^2} = 12 \left( \frac{F_i^m - G_i^m - F_{i-1}^m + G_{i-1}^m}{\Delta T} \right).$$

This equation is solved at each interior point $i = 1, \ldots, N - 1$, with the far field condition being imposed with $P_0^m = P_N^m = 0$. Equation (A.2) is linear in the terms
A.1. Inviscid-lubrication model coupled with deformable surface

$P_{i-1}^m, P_i^m$ and $P_{i+1}^m$, so can be expressed as

$$a_i P_{i-1}^m + b_i P_i^m + c_i P_{i+1}^m = f_i, \quad (A.3)$$

which when evaluated at each interior point $i = 1, \ldots, N - 1$ sets up a sparse matrix system

$$\mathbf{J}_P = \mathbf{f}, \quad (A.4)$$

where $\mathbf{P} = (P_1^m, \ldots, P_{N-1}^m)$, $\mathbf{f} = (f_1, \ldots, f_{N-1})$ and

$$J_{i,i-1} = a_i, \quad J_{ii} = b_i, \quad J_{i,i+1} = c_i$$

$$J_{ij} = 0, \quad \text{if } |j - i| > 1 \quad (A.5)$$

The solution for $\mathbf{P}$ is then found from inverting $\mathbf{J}$

$$\mathbf{P} = \mathbf{J}^{-1}\mathbf{f}, \quad (A.6)$$

which is most efficiently done using the backslash command in MATLAB. Once a new pressure solution is found, to improve stability it is then relaxed with the pressure solution from the previous time step

$$P_i^m \rightarrow \frac{1}{2}(P_i^m + P_i^{m-1}), \quad (A.7)$$

for $i = 1, \ldots, N - 1$.

Next, the solution for $F$ is found from equation (A.1b). The far field condition is used to evaluate $F_0^m = X_0/2 - T_m$ and $F_N^m = X_N/2 - T_m$, then the interior points are found through evaluating

$$\frac{F_i^m - 2F_i^{m-1} + F_i^{m-2}}{\Delta T^2} = \mathcal{H} \left( \frac{\partial P}{\partial X} \right)_i \quad (A.8)$$

at each $i = 1, \ldots, N - 1$ to find $F_i^m$, where $\mathcal{H}$ is the Hilbert transform. Using Fast Fourier Transforms, the Hilbert transform can be efficiently calculated in MATLAB using the `hilbert()` command, making equation (A.8) straightforward to evaluate.
A.1. Inviscid-lubrication model coupled with deformable surface

with $\partial P/\partial X$ calculated with second order central finite differences.

Finally, equation (A.1c) can be expressed using finite differences as

$$\tilde{e}_1 \frac{1}{\Delta X^4} (G_{i+2}^m - 4G_{i+1}^m + 6G_i^m - 4G_{i-1}^m + G_{i-2}^m) + \tilde{e}_2 \frac{1}{\Delta X^2} (G_{i+1}^m - 2G_i^m + G_{i-1}^m)$$

$$+ \tilde{e}_3 G_i^m + \tilde{e}_4 \frac{1}{\Delta T^2} (G_i^m - 2G_i^{m-1} + G_i^{m-2}) + \tilde{e}_5 \frac{1}{\Delta T} (G_i^m - G_i^{m-1}) = P_i^m. \quad (A.9)$$

This is, again, linear in the terms $G_{i-2}^m$, $G_{i-1}^m$, $G_i^m$, $G_{i+1}^m$ and $G_{i+2}^m$ hence can be expressed as

$$A_i G_{i-2}^m + B_i G_{i-1}^m + C_i G_i^m + D_i G_{i+1}^m + E_i G_{i+2}^m = q_i. \quad (A.10)$$

The boundary conditions (A.1d-A.1e) applied at the end points of the spatial domain yield

$$G_0^m = G_N^m = 0, \quad (A.11a)$$

$$4G_1^m - G_2^m = 0, \quad G_{N-2}^m - 4G_{N-1}^m = 0, \quad (A.11b)$$

where second order one-sided differences have been used for the Neumann condition $\partial G/\partial X = 0$. The system of equations (A.10) when evaluated at $i = 2, \ldots, N-2$ combined with (A.11b) yield another sparse matrix system

$$M G = q. \quad (A.12)$$

where $G = (G_1^m, \ldots, G_{N-1}^m)$, $q = (0, q_2, \ldots, q_{N-2}, 0)$ and

$$M_{11} = 4, \quad M_{12} = -1, \quad M_{N-1,N-2} = 1, \quad M_{N-1,N-1} = -4, \quad$$

$$M_{i,i-2} = A_i, \quad M_{i,i-1} = B_i, \quad M_{ii} = C_i, \quad M_{i,i+1} = D_i, \quad M_{i,i+2} = E_i, \quad (A.13)$$

$$M_{ij} = 0, \quad \text{if } |j - i| > 2.$$

The system (A.12) is solved in an identical way as (A.4).

The scheme iterates between each equation until successive iterates are below
A.1. Inviscid-lubrication model coupled with deformable surface

a certain threshold. This criteria is defined by

\[ \frac{N \max_i |F_i^m - F_i^\text{old}|}{\sum_i |F_i^m|} < 10^{-6}, \]  

(A.14)

where \( F_i^\text{old} \) are the values of \( F_i^m \) from the previous iteration, and similarly for \( G \) and \( P \). Once this criteria has been met for all variables \( F, G \) and \( P \), the solution advances to the next time step. The code terminates when the air film thickness \( H = F - G \) reaches a predefined small value \( H_{\text{min}} \).

The successive steps this algorithm takes can be summarised as

1. Initialise the solutions with \( F_0^0 = X_0^2 / 2 - T_0 \), \( F_1^1 = X_1^2 / 2 - T_1 \), \( G_0^0 = G_1^1 = 0 \) and \( P_0^0 = P_1^1 = 0 \) \( \forall i \).

2. Start time loop at \( m = 2 \) and \( T_m = T_0 + m\Delta T \).

3. Set the boundary conditions \( F_0^m = X_0^2 / 2 - T_m \), \( F_N^m = X_N^2 / 2 - T_m \), \( G_0^m = G_N^m = 0 \) and \( P_0^m = P_N^m = 0 \).

4. Solve for \( P_i^m \) from (A.4) for \( i = 1, \ldots, N - 1 \).

5. Solve for \( F_i^m \) from (A.8) for \( i = 1, \ldots, N - 1 \).

6. Solve for \( G_i^m \) from (A.12) for \( i = 1, \ldots, N - 1 \).

7. Check convergence criteria (A.14) for \( F \), similarly for \( G \) and \( P \). If all three are satisfied, move to the next step. Otherwise, go back to step 4.

8. Check if \( H < H_{\text{min}} \). If it is, terminate code. If not, \( m \to m + 1 \), \( T_m \to T_{m-1} + \Delta T \) and go back to step 3.

To validate our code, the different sources of error need to be tested against to ensure the error is sufficiently small. The code will be tested against spatial discretisation step \( \Delta X \), time step \( \Delta T \) and start time \( T_0 \). The length of the spatial domain, which is infinite, also needs to be approximated by a finite domain and ensure that this does not affect the results. However, the end points of the domain are assumed to be the boundaries of the deformable surface, so are essentially a
Inviscid-lubrication model coupled with deformable surface

<table>
<thead>
<tr>
<th>Case</th>
<th>$\Delta X$</th>
<th>$\Delta T$</th>
<th>$T_0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>0.05</td>
<td>0.005</td>
<td>-50</td>
</tr>
<tr>
<td>B</td>
<td>0.025</td>
<td>0.005</td>
<td>-50</td>
</tr>
<tr>
<td>C</td>
<td>0.05</td>
<td>0.002</td>
<td>-50</td>
</tr>
<tr>
<td>D</td>
<td>0.05</td>
<td>0.005</td>
<td>-100</td>
</tr>
</tbody>
</table>

Table A.1: Details of the parameters used in the validation study for the inviscid-lubrication modelled coupled with a deformable surface.

Figure A.1: Solutions of (a) the free surface height $F$, (b) the deformable surface shape $G$ and (c) the pressure $P$ when $H = 0.5$, for the test parameters detailed in table A.1 and surface parameters $(\tilde{e}_1, \tilde{e}_2, \tilde{e}_3, \tilde{e}_4, \tilde{e}_5) = (-1, 1, 0, -1, 0)$ with boundaries $[X_1, X_2] = [-10, 10]$.

The four different test cases are shown in table A.1. These parameters are run on a surface with parameters $(\tilde{e}_1, \tilde{e}_2, \tilde{e}_3, \tilde{e}_4, \tilde{e}_5) = (-1, 1, 0, -1, 0)$ with end boundaries $[X_1, X_2] = [-10, 10]$ up until $H = 0.5$, with the solutions for $F$, $G$ and $P$ shown in figure A.1. Excellent agreement is found for cases A, B and D, and satisfactory agreement with case C. The largest source of error seen in case C is from the time step size. However, steps have been taken to alleviate the large source of error that comes from the start time $T_0$ [46] by having very large negative start times. This results in the code being run for longer in time, and allowing temporal errors to be larger. A balance has to be struck between accuracy and the time taken for the code to run, as such the validation presented in figure A.1 is as good as it can be, we believe. All results presented in the main body of this thesis are calculated using the parameters in case C.
A.2 Boundary layer equations coupled with deformable meniscus

This Appendix concerns the coupled set of partial differential equations derived in Chapter 4 which govern the integration of the steady boundary layer equations with an approximate model of a deformable meniscus. The governing set of equations and conditions are

\[ \frac{u}{\partial x} + v \frac{\partial u}{\partial y} = \frac{1}{\varepsilon^2 \text{Re}} \frac{\partial^2 u}{\partial y^2}, \quad (A.15a) \]

\[ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \quad (A.15b) \]

\[ \frac{d^3 H}{dx^3} = -\frac{2\mu \text{Ca} \, u_s (3A + B)}{\varepsilon^3 (H + 1)^2} + \frac{\rho \text{ReCa}}{\varepsilon} u_s \frac{d}{dx} (u_s (A + B + C)), \quad (A.15c) \]

\[ \frac{\partial u}{\partial y} = 0, \quad v = u \frac{d}{dx} (H + h), \quad \text{at } y = H + h, \quad (A.15d) \]

\[ u = \frac{H + 1}{\mu (3A + 2B + C)} \frac{\partial u}{\partial y}, \quad v = u \frac{dH}{dx}, \quad \text{at } y = H, \quad (A.15e) \]

\[ h = h_0, \quad u = \frac{3y}{2h_0^2} (2h_0 - y), \quad \text{at } x = 0, \quad (A.15f) \]

\[ \int_H^{H+h} u \, dy = h_0. \quad (A.15g) \]

\[ H(0) = H(1) = \int_0^1 H \, dx = 0. \quad (A.15h) \]

The system here needs to solved in the region \(0 < x < 1\) and \(H < y < H + h\), with \(H\) and \(h\) being solved as part of the problem. The non-uniform domain in \(y\) requires a normalisation. This is achieved through Prandtl’s transposition [155], which defined a new normalised vertical coordinate \(Y\) through \(y = hY + H\) and a transformed vertical velocity component

\[ \tilde{v} = v - \left( \frac{dh}{dx} Y + \frac{dH}{dx} \right) u. \quad (A.16) \]

In order to improve accuracy near the points \(x = 0\) and \(x = 1\), a grid stretching transformation in the horizontal direction \(x\) is applied to increase the number of grid
A.2. Boundary layer equations coupled with deformable meniscus points at the end regions. This is done by defining a new transformed horizontal position as

\[ X = \frac{(x + \delta)^{1/2} - (1 - x + \delta)^{1/2}}{2((1 + \delta)^{1/2} - \delta^{1/2})} + \frac{1}{2} \]  

(A.17)

where \( \delta \ll 1 \). This transformation has \( X \sim x^{1/2} \) near \( x = 0 \) and \( X \sim 1 - (1 - x)^{1/2} \) near \( x = 1 \). The introduction of \( \delta \) is necessary to allow the calculation of \( x \) derivatives at the end points \( x = 0 \) and \( x = 1 \) (they would be singular otherwise). The value used in all results was \( \delta = 0.0001 \). This transformation is applied only when solving the boundary layer equation, and not the meniscus equation, so only first order \( x \) derivatives need to be transformed as

\[ \frac{\partial}{\partial x} = \frac{\partial X}{\partial x} \frac{\partial}{\partial X} = X(x) \frac{\partial}{\partial X} \]  

(A.18)

where

\[ X(x) = \frac{(x + \delta)^{-1/2} + (1 - x + \delta)^{-1/2}}{4((1 + \delta)^{1/2} - \delta^{1/2})}. \]  

(A.19)

Hence, the new transformed problem in the jet is

\[
\begin{align*}
    h^2 X_x u \frac{\partial u}{\partial X} + h \bar{v} \frac{\partial u}{\partial Y} &= \frac{1}{\varepsilon^2 \text{Re}} \frac{\partial^2 u}{\partial Y^2}, \\
    X_x \frac{\partial}{\partial X} (hu) + \frac{\partial \bar{v}}{\partial Y} &= 0, \\
    \frac{\partial u}{\partial Y} &= 0, \quad \bar{v} = 0, \quad \text{at } Y = 1, \\
    u &= \frac{H + 1}{\mu h(3A + 2B + C)} \frac{\partial u}{\partial Y}, \quad \bar{v} = 0, \quad \text{at } Y = 0, \\
    u &= \frac{3Y}{2} (2 - Y), \quad \text{at } X = 0, \\
    h \int_0^1 u \, dY &= h_0,
\end{align*}
\]  

(A.20a) (A.20b) (A.20c) (A.20d) (A.20e) (A.20f)

while the meniscus shape equation (A.15c) and conditions (A.15h) remain unchanged as they are calculated with respect to the original horizontal position \( x \).

The transformed vertical \( 0 \leq Y \leq 1 \) and horizontal \( 0 \leq X \leq 1 \) domains are discretised into uniform grids \( X_i = i\Delta X, i = 0, \ldots, N \) and \( Y_j = j\Delta Y, j = 0, \ldots, M \), with
A.2. Boundary layer equations coupled with deformable meniscus

\[ N\Delta X = 1 \text{ and } M\Delta Y = 1. \]  
Once the transformed horizontal grid has been defined, equation (A.17) can be numerically inverted to find the non-uniform grid \( x_i, i = 0, \ldots, N. \) All functions are expressed as \( u(x_i, Y_j) = u(X_i, Y_j) = u_{ij}, \) and similarly for \( \tilde{v}, \) and \( h(x_i) = h(X_i) = h_i, \) and similarly for all other functions of \( x \) present in the problem.

The solution for \( u \) is sought by marching forward in \( X \) and solving as a boundary value problem in \( Y, \) with boundary conditions given by (A.20c-A.20d). Using forward first order finite differences for the \( X \) derivative and second order central differences for the \( Y \) derivatives, equation (A.20) can be expressed in the form

\[
f_j = h_i^2 X_i(x_i) u_{ij} \frac{(u_{ij} - u_{i-1,j})}{\Delta X} + h_i \tilde{v}_{ij} \frac{(u_{i,j+1} - u_{i,j-1})}{2\Delta Y} - \frac{1}{\varepsilon^2 \text{Re}} \frac{(u_{i,j+1} - 2u_{ij} + u_{i,j-1})}{\Delta Y^2} = 0 \tag{A.21}
\]

for \( j = 1, \ldots, M - 1. \) When \( j = 1, \) the velocity component at the interface \( Y = 0 \) is evaluated as

\[
u_{i0} = \frac{(H + 1)}{2\mu h_i(3A_i + 2B_i + C_i)\Delta Y + 3(H_i + 1)}(4u_{i1} - u_{i2}) \tag{A.22}
\]

which is substituted into equation (A.21) when \( j = 1. \) Similarly when \( j = M - 1 \) the no-shear condition is evaluated at \( Y = 1 \) as

\[
u_{iM} = \frac{4u_{i,M-1} - u_{i,M-2}}{3}. \tag{A.23}
\]

The equation \( f_j = 0 \) is non-linear in the terms \( u_{i,j-1}, u_{ij} \) and \( u_{i,j+1}, \) so needs to be solved iteratively using a Newton algorithm. The Newton algorithm works by successively updating the horizontal velocity \( u \rightarrow u + d, \) where \( u = (u_{i1}, \ldots, u_{i,M-1}) \) and \( d \) is the solution to the matrix equation

\[
Jd = -f, \tag{A.24}
\]
A.2. Boundary layer equations coupled with deformable meniscus

where \( f = (f_1, \ldots, f_{M-1}) \) and \( J \) is the Jacobian matrix made up of entries

\[
J_{jk} = \begin{cases} \frac{\partial f_j}{\partial u_{i,j-1}}, & \text{if } |k-i| = 1, \\ \frac{\partial f_j}{\partial u_{i,j}}, & \text{for } j = 1, \ldots, M-1 \\ \frac{\partial f_j}{\partial u_{i,j+1}}, & \text{for } j = 1, \ldots, M-1 \\ 0, & \text{if } |k-i| > 1. \end{cases}
\] (A.25)

This procedure is repeated, using updated values of \( u \) each time in the Jacobian and in \( f \), until a criterion based on the size of \( \mathbf{d} \) is satisfied

\[
\frac{M \max_j |d_j|}{\sum_j |u_{ij}|} < 10^{-6}.
\] (A.26)

The matrix equation (A.24) is solved using the same method outlined in Appendix A.1.

In order to satisfy the zero boundary conditions for \( \tilde{v} \) (A.20c-A.20d), a second order version of the continuity equation is solved

\[
\frac{\partial^2 \tilde{v}}{\partial Y^2} = -X \frac{\partial}{\partial X} \left( h \frac{\partial u}{\partial Y} \right),
\] (A.27)

as a boundary value problem. Equation (A.27) can be expressed discretely as

\[
\tilde{v}_{i,j+1} - 2\tilde{v}_{ij} + \tilde{v}_{i,j-1} = -X_i(x_i) \left( \frac{(h_i - h_{i-1}) (u_{i,j+1} - u_{i,j-1})}{\Delta X} \right) \frac{2\Delta Y}{\Delta X} \\
+ h_i \left( u_{i,j+1} - u_{i-1,j+1} - u_{i,j-1} + u_{i-1,j-1} \right) \frac{2\Delta X \Delta Y}{\Delta Y}
\] (A.28)

which can be solved for \( \tilde{v}_{i1}, \ldots, \tilde{v}_{i,M-1} \), where \( \tilde{v}_{i0} = \tilde{v}_{iM} = 0 \), using the standard matrix techniques already described.

The jet thickness \( h \) is then readily calculated from the flow rate condition (A.20f)

\[
h = \frac{h_0}{\int_0^1 u \, dY},
\] (A.29)

where the integration is performed numerically using a trapezium rule.
A.2. Boundary layer equations coupled with deformable meniscus

The code marches forward in $X$, iterating at each $X$-station between the updated solutions for $u$, $\tilde{v}$ and $h$ until the solutions converge locally in $Y$. The solution is deemed to have converged when successive iterates are sufficiently close together for all three variables. This criteria is calculated for $u$ and $\tilde{v}$ in a similar manner as equation (A.14), while the criteria for $h$ is

$$\frac{|h_i - h^{\text{old}}|}{|h_i|} < 10^{-6}.\quad (A.30)$$

The code marches forward until $X = 1$.

Once the flow has been calculated in the jet, given the latest known value of $H$, the meniscus shape $H$ can be updated through integration of equation (A.15c) on the non-uniform grid $x_i$. The slip velocity $u_s$ is the horizontal velocity of the jet evaluated at the interface $Y = 0$. This can be evaluated discretely at every $x$-station as $u_{i0}$. By considering the right-hand-side of equation (A.15c) as known from the most recent iteration, $H$ is then updated by

$$H = G(x) + \alpha x^2 + \beta x, \quad (A.31)$$

where

$$G(x) = \int_0^x \int_0^{x'} \int_0^{x''} g(x''') \, dx''', \quad (A.32a)$$

$$g(x) = -\frac{2\mu Ca}{\epsilon^3} u_s (3A + B) \frac{1}{(H^{\text{old}} + 1)^2} + \frac{\rho Re Ca}{\epsilon} u_s \frac{d}{dx} (u_s (A + B + C)), \quad (A.32b)$$

$$\alpha = 3 \left( 2 \int_0^1 G \, dx - G(1) \right), \quad (A.32c)$$

$$\beta = 2 \left( G(1) - 3 \int_0^1 G \, dx \right). \quad (A.32d)$$

The integration steps are done successively using a trapezium rule and the derivatives in equation (A.32b) are calculated using central finite differences. Once an updated version of $H$ has been calculated, the code then calculated the updated flow properties in the jet. This process is continued until successive solutions for the slip velocity $u_s$ from marching forward in $X$ in the jet and solutions from the meniscus
A.2. Boundary layer equations coupled with deformable meniscus

<table>
<thead>
<tr>
<th>Case</th>
<th>$\Delta X$</th>
<th>$\Delta Y$</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>0.005</td>
<td>0.01</td>
</tr>
<tr>
<td>B</td>
<td>0.01</td>
<td>0.01</td>
</tr>
<tr>
<td>C</td>
<td>0.005</td>
<td>0.005</td>
</tr>
</tbody>
</table>

Table A.2: Details of the parameters used in the validation study for the boundary layer equations coupled with a deformable meniscus equation.

shape $H$ calculation fall below a certain threshold. The calculation for convergence is done in an identical manner as previously mentioned.

The steps taken in this algorithm can be summarised as follows:

1. Set the the horizontal velocity solution at $X = 0$ as $u_{0j} = 3Y_j(2 - Y_j)/2$, for $j = 0, \ldots, M$. For the first march, assume that $H = 0$. Then $i = 1$.

2. Solve for $u_{ij}$, $j = 1, \ldots, M - 1$, from applying a Newton algorithm on (A.21-A.26). $u_{i0}$ and $u_{iM}$ are then sought from equations (A.22-A.23), respectively.

3. Solve for $\tilde{v}_{ij}$, for $j = 1, \ldots, M - 1$, from (A.28), with $\tilde{v}_{i0} = \tilde{v}_{iM} = 0$.

4. Solve for $h_i$ from equation (A.29).

5. Check convergence of $u$, $\tilde{v}$ and $h$. If convergence not satisfied, move back to step 2. If it is, move to the next step

6. Repeat steps 2-5 for $i = 2, \ldots, N$.


8. Check convergence on $H$ and $u_s$. If convergence not satisfied, move back to step 2. If it is satisfied, terminate code.

The sources of error that need to be checked for accuracy are the spatial discretisation step sized $\Delta X$ and $\Delta Y$. The values used in the validation study are given in table A.2. Solutions for the jet thickness $h$, meniscus shape $H$ and slip velocity $u_s$ are shown in figure A.2 for the discretisation steps listen in table A.2, for $Re = 100$, $Ca = 0.01$, $\varepsilon = 0.1$, $\mu = 1$ and $h_0 = 1$. Clearly, these results agree excellently with each other and we can be very confident of the accuracy of our code.
A.3. Droplet deformation: first temporal stage

Figure A.2: Solutions for the (a) jet thickness $h$, (b) meniscus shape $H$ and (c) slip velocity $u_s$, calculated with the parameters listed in Table A.2, for $Re = 100$, $Ca = 0.01$, $\epsilon = 0.1$, $\mu = 1$ and $h_0 = 1$.

The results presented in the main body of this thesis were calculated using the Case A parameters.

A.3 Droplet deformation: first temporal stage

A.3.1 Flow in the air

Here, the numerical scheme used to calculate solutions to the unsteady Navier-Stokes equations in the air surrounding the droplet, which in the first temporal stage can be assumed solid, used in Chapter 5 will be described. The system was formulated in a streamfunction-vorticity form in polar coordinates $(r, \theta)$. The subscript $g$ referring to the air quantities will be dropped here to facilitate more detailed indexing. The system to be solved is

$$\nabla^2 \psi = -\zeta, \quad \text{(A.33a)}$$

$$\nabla^2 \zeta = Re \left( \frac{1}{r} \frac{\partial \psi}{\partial \theta} \frac{\partial \zeta}{\partial r} - \frac{1}{r} \frac{\partial \psi}{\partial r} \frac{\partial \zeta}{\partial \theta} + \frac{\partial \zeta}{\partial t} \right) \quad \text{(A.33b)}$$

$$\psi = \frac{\partial \psi}{\partial r} = 0, \quad \text{at } r = 1, \quad \text{(A.33c)}$$

$$\psi \to \left( \frac{1}{r} - r \right) \sin \theta, \quad \zeta \to 0, \quad \text{as } r \to \infty, \quad \text{(A.33d)}$$

$$\psi = \zeta = 0, \quad \text{at } \theta = 0, \pi, \quad \text{(A.33e)}$$

where initially $\zeta = 0$ The system is transformed using $r = e^\xi$, $0 \leq \xi < \infty$, which helps simplify the equations as well as increase accuracy near the droplet surface.
A.3. Droplet deformation: first temporal stage

The system then becomes

\[
\frac{\partial^2 \psi}{\partial \xi^2} + \frac{\partial^2 \psi}{\partial \theta^2} = -e^{2\xi} \zeta \quad (A.34a)
\]

\[
\frac{\partial^2 \zeta}{\partial \xi^2} + \frac{\partial^2 \zeta}{\partial \theta^2} = \text{Re} \left( \frac{\partial \psi}{\partial \theta} \frac{\partial \zeta}{\partial \xi} - \frac{\partial \psi}{\partial \xi} \frac{\partial \zeta}{\partial \theta} + e^{2\xi} \frac{\partial \zeta}{\partial t} \right) \quad (A.34b)
\]

\[
\psi = \frac{\partial \psi}{\partial \xi} = 0, \quad \text{at} \ \xi = 0, \quad (A.34c)
\]

\[
\psi \to 2 \sinh \xi \sin \theta, \quad \zeta \to 0, \quad \text{as} \ \xi \to \infty, \quad (A.34d)
\]

\[
\psi = \zeta = 0, \quad \text{at} \ \theta = 0, \pi, \quad (A.34e)
\]

A discrete grid is defined on the transformed polar space as \(\xi_i = i\Delta \xi, \ i = 0, \ldots, N\) and \(\theta = j\Delta \theta, \ j = 0, \ldots, M\), with \(N\Delta \xi = \xi_\infty\) and \(M\Delta \theta = \pi\) where \(\xi_\infty\) is the finite value chosen to represent the infinite boundary condition. At each time step \(t_k = k\Delta t, \ k = 0, 1, \ldots\), the variables are represented as \(\psi(\xi_i, \theta_j, t_k) = \psi_{ij}^k\) and similarly for \(\zeta\).

The solution for \(\zeta\) is found from a Adams-Bashford discretisation of the temporal derivative in equation (A.34b)

\[
\zeta_{ij}^k = \zeta_{ij}^{k-1} + \Delta t G_{ij}^{k-1}, \quad k = 1, \quad (A.35a)
\]

\[
\zeta_{ij}^k = \zeta_{ij}^{k-1} + \frac{\Delta t}{2} (3G_{ij}^{k-1} - G_{ij}^{k-2}), \quad k = 2, 3, \ldots \quad (A.35b)
\]

where

\[
G_{ij}^k = e^{\xi_i} \left\{ \frac{1}{\text{Re}} \left( \frac{\zeta_{i+1,j}^k - 2\zeta_{ij}^k + \zeta_{i-1,j}^k}{\Delta \xi^2} + \frac{\zeta_{i,j+1}^k - 2\zeta_{ij}^k + \zeta_{i,j-1}^k}{\Delta \theta^2} \right) \right.
\]

\[
- \frac{(\psi_{i,j+1}^k - \psi_{i,j-1}^k)}{2\Delta \theta} \left( \frac{\zeta_{i+1,j}^k - \zeta_{i-1,j}^k}{2\Delta \xi} \right)
\]

\[
+ \frac{(\psi_{i+1,j}^k - \psi_{i-1,j}^k)}{2\Delta \xi} \left( \frac{\zeta_{i,j+1}^k - \zeta_{i,j-1}^k}{2\Delta \theta} \right) \right\}. \quad (A.36)
\]

The boundary conditions imply that \(\zeta_{N,j} = \zeta_{0,j} = \zeta_{iM} = 0\), while Taylor series expansions in \(\psi\) around the two \(\xi\) values closest to \(\xi = 0\) results in an expression for
A.3. Droplet deformation: first temporal stage

\[ \zeta \text{ at the boundary } \xi = 0, \]

\[ \zeta_0^{i,j} = \frac{\psi_{2j}^{k-1} - 8\psi_{1j}^{k-1}}{2\Delta\xi^2}. \tag{A.37} \]

which has been evaluated explicitly using values from the previous time step.

The streamfunction \( \psi \) is sought through a truncated Fourier series representation

\[ \psi = \sum_{n=1}^{N_f} f_n(\xi) \sin(n\theta), \tag{A.38} \]

where from equation (A.33a) and (A.33c-A.33d), the function \( f_n \) can be shown to satisfy

\[ \frac{d^2 f_n}{d\xi^2} - n^2 f_n = -\frac{2e^{2\xi}}{\pi} \int_0^{\pi} \zeta \sin(n\theta) d\theta, \tag{A.39a} \]

\[ f_n(0) = 0, \quad n = 1, \ldots, N_f, \tag{A.39b} \]

\[ f_1(\xi_\infty) = 2\sinh \xi_\infty, \quad f_n(\xi_\infty) = 0, \quad n = 2, \ldots, N_f. \tag{A.39c} \]

The system (A.39) can be solved numerically at each \( n \), using standard matrix methods, for \( f_n \), which can then be used to calculate \( \psi \) from equation (A.38).

Since \( \zeta \) is calculated explicitly from solutions from the previous time steps, no iteration is needed in this scheme. The algorithm used here is as follows:

1. Initially, apply \( \zeta_0^{i,j} = 0 \) everywhere, which implies that the potential flow solution for the streamfunction needs to be applied everywhere, \( \psi_{i,j}^0 = 2\sinh \xi_i \sin \theta_j \). Set \( k = 1 \).

2. Calculate \( \zeta_{i,j}^{k} \) from (A.35-A.37)

3. Calculate \( f_n \) for \( n = 1, \ldots, N_f \) from (A.39), then use to calculate \( \psi_{i,j}^{k} \) from (A.38).

4. \( k \to k + 1 \). Go back to step 2.

5. Repeat steps 2-4 for as long as necessary.

As in the previous sections, we will validate this code to ensure that the sources of error are sufficiently small. Sources of error arise from the discretisation steps.
A.3. Droplet deformation: first temporal stage

<table>
<thead>
<tr>
<th>Case</th>
<th>$\Delta \xi = \Delta \theta$</th>
<th>$\Delta t$</th>
<th>$N_f$</th>
<th>$\xi_\infty$</th>
</tr>
</thead>
<tbody>
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<td>5</td>
</tr>
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<td>B</td>
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<td>C</td>
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<td>80</td>
<td>5</td>
</tr>
<tr>
<td>D</td>
<td>$\pi/150$</td>
<td>0.005</td>
<td>40</td>
<td>10</td>
</tr>
</tbody>
</table>

Table A.3: Details of the parameters used in the validation study for the unsteady Navier-Stokes equations past a solid droplet.

Figure A.3: The vorticity $\zeta$ on the droplet surface $r = 1$ when $t = 3$, for $Re = 100$. The parameters used in the calculations are listed in table A.3.

$\Delta \xi$ and $\Delta \theta$, the time step $\Delta t$, the finite value used for the infinite boundary $\xi_\infty$ and the number of terms $N_f$ used in the Fourier series representation of $\psi$. The values used for the validation in this case are shown in table A.3. The vorticity calculated on the droplet surface $r = 1$ at $t = 3$ for $Re = 100$ is shown in figure A.3 for the four different cases of parameters listed in table A.3. All results exhibit excellent agreement with each other, so we are very satisfied with the accuracy of our code. The work in the main body of the thesis uses Case A parameters.

A.3.2 Flow in the droplet

The flow inside the droplet was found to be governed by the unsteady Stokes equations in the first temporal stage. In order to drive the flow inside the droplet, the
A.3. Droplet deformation: first temporal stage

pressure and the vorticity from the air need to be evaluated at the interface \( r = 1 \). For this purpose we will reintroduce the subscript \( g \) to refer to quantities evaluated in the air. The pressure on the interface \( r = 1 \) is calculated using

\[
p_g = \frac{1}{\text{Re}} \int_0^\theta \frac{\partial \zeta_g}{\partial r} \bigg|_{r=1} \, d\theta',
\]

which is calculated numerically using a trapezium rule and one sided second order finite differences for the derivative in the integrand. The flow inside the droplet is governed by, dropping the \( l \) subscript for quantities inside the droplet,

\[
\frac{\partial u_r}{\partial t} = -\frac{\partial p}{\partial r} + \frac{1}{\mu_1 \text{Re}} \left( \nabla^2 u_r - \frac{u_r}{r^2} - \frac{2}{r^2} \frac{\partial u_\theta}{\partial \theta} \right),
\]

(A.41a)

\[
\frac{\partial u_\theta}{\partial t} = -\frac{1}{r} \frac{\partial p}{\partial \theta} + \frac{1}{\mu_1 \text{Re}} \left( \nabla^2 u_\theta - \frac{u_\theta}{r^2} + \frac{2}{r^2} \frac{\partial u_r}{\partial \theta} \right),
\]

(A.41b)

\[
\frac{\partial u_r}{\partial r} + \frac{u_r}{r} + \frac{1}{\mu_1 \text{Re}} \frac{\partial u_\theta}{\partial \theta} = 0,
\]

(A.41c)

\[
p = p_g|_{r=1} + \frac{1}{\text{We}}, \quad \frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} = \mu_1 \zeta_g|_{r=1}, \quad \text{at } r = 1,
\]

(A.41d)

\[
\frac{\partial u_r}{\partial \theta} = u_\theta = \frac{\partial p}{\partial \theta} = 0, \quad \text{at } \theta = 0, \pi.
\]

(A.41e)

Rearranging the governing equations also lead to the pressure satisfying the Laplace equation

\[
\nabla^2 p = 0,
\]

(A.42)

which can be solved analytically, however it is more efficient to solve numerically since the analytical solution is in the form of infinite sums.

This system is singular at \( r = 0 \) in its current form. To remove this singularity, transformations to the variables are made of the form

\[
U = ru_r, \quad V = ru_\theta, \quad P = rp.
\]

(A.43)
which results in the system

\[
\begin{align*}
\frac{r^2 \partial U}{\partial t} &= -r^2 \frac{\partial P}{\partial r} + rP + \frac{1}{\mu_1 \text{Re}} \left( r^2 \frac{\partial^2 U}{\partial r^2} - r \frac{\partial U}{\partial r} + \frac{\partial^2 U}{\partial \theta^2} - 2 \frac{\partial V}{\partial \theta} \right) \quad (A.44a) \\
\frac{r^2 \partial V}{\partial t} &= -r^2 \frac{\partial P}{\partial \theta} + \frac{1}{\mu_1 \text{Re}} \left( r^2 \frac{\partial^2 V}{\partial r^2} - r \frac{\partial V}{\partial r} + \frac{\partial^2 V}{\partial \theta^2} + 2 \frac{\partial U}{\partial \theta} \right) \quad (A.44b) \\
\frac{\partial U}{\partial r} + \frac{1}{r} \frac{\partial V}{\partial \theta} &= 0 \quad (A.44c) \\
U &= V = P = 0, \quad \text{at } r = 0, \quad (A.44d) \\
P &= p_g |_{r=1} + \frac{1}{\text{We}}, \quad \frac{\partial U}{\partial \theta} + \frac{\partial V}{\partial r} - V = \mu_1 \zeta_g |_{r=1}, \quad \text{at } r = 1, \quad (A.44e) \\
\frac{\partial U}{\partial \theta} = V = \frac{\partial P}{\partial \theta} = 0, \quad \text{at } \theta = 0, \pi, \quad (A.44f)
\end{align*}
\]

while the transformed pressure satisfies

\[
\frac{r^2 \partial^2 P}{\partial r^2} - r \frac{\partial P}{\partial r} + P + \frac{\partial^2 P}{\partial \theta^2} = 0. \quad (A.45)
\]

The problem is now a second order boundary value problem on the "box" \(0 \leq r \leq 1\) and \(0 \leq \theta \leq 1\), which is solved using standard relaxation methods.

The radial domain is discretised as \(r_i = i \Delta r, \; i = 0, \ldots, N\) and the azimuthal as \(\theta_j = j \Delta \theta, \; j = 0, \ldots, M\), with \(N \Delta r = 1\) and \(M \Delta \theta = \pi\). At each time step \(t_k = k \Delta t\), the variables are expressed as \(U(i \Delta r, j \Delta \theta, k \Delta t) = U_{ij}^k\), and similarly for \(V\) and \(P\).

The pressure \(P\) is uncoupled from the velocity components, so at each time step, the code solves for \(P\) first. The pressure at \(r = 1\) is set using the boundary condition (A.44e), giving \(P_{Nj}^k = p_g(1, \theta_j, t_k) + 1/\text{We}\), and also \(P_{0j}^k = 0\) for \(j = 0, \ldots, M\). Then, at all other points the pressure is updated iteratively using

\[
P_{ij}^k = \alpha_P P_{ij}^* + (1 - \alpha_P) P_{ij}^{\text{old}}, \quad (A.46)
\]

where \(\alpha_P\) is the relaxation parameter, \(P_{ij}^{\text{old}}\) is the value of the pressure from the
previous iteration and $P^*_ij$ is found from

$$P^*_ij = \left( r_i^2 \frac{(p^{k}_{i+1,j} + p^{k}_{i-1,j})}{\Delta r^2} - r_i \frac{(p^{k}_{i+1,j} - p^{k}_{i-1,j})}{2\Delta r} \right. $$
$$
+ \left. \frac{p^{k}_{i,j+1} + p^{k}_{i,j-1}}{\Delta \theta^2} \right) \left( \frac{2r_i^2}{\Delta r^2} - 1 + \frac{2}{\Delta \theta^2} \right),$$

(A.47)

for $i = 1, \ldots, N - 1$ and $j = 1, \ldots, M - 1$, with

$$P^*_{i0} = \frac{4p^{k}_{i1} - p^{k}_{i2}}{3}, \quad P^*_{iM} = \frac{4p^{k}_{i,M-1} - p^{k}_{i,M-2}}{3},$$

(A.48)

for $i = 1, \ldots, N - 1$. The pressure is successively updated everywhere until consecutive iterates are sufficiently close together, which is defined as

$$NM \max_{ij} \left| \frac{p^{k}_{ij} - P^{old}_{ij}}{\sum_{ij} |p^{k}_{ij}|} \right| < 10^{-5}.$$  \hspace{1cm} (A.49)

The value of the relaxation parameter that was found to provide the fastest convergence was $\alpha_p = 1.9$.

Once the pressure has been calculated, it can be used in the momentum equations to find the velocity components. The momentum equations (A.44a-A.44b) and their boundary conditions are coupled, so the relaxation method will need to iterate between solutions for $U$ and $V$ until both converge. To find $U$, the boundary condition at $r = 0$ is applied so that $U_{0j}^k = 0$ for $j = 0, \ldots, M$. Then, as before, the solution is updated using

$$U_{ij}^k = \alpha_U U_{ij}^* + (1 - \alpha_U) U_{ij}^{old}$$

(A.50)
where $\alpha_U$ is the relaxation parameter and

$$
U_{ij}^* = \left\{ \frac{r_i^2 U_{ij}^{k-1}}{\Delta t} - r_i \frac{(P_{i+1,j}^k - P_{i-1,j}^k)}{2\Delta r} + r_i p_i^k 
+ \frac{1}{\mu_1 \text{Re}} \left( \frac{r_i^2 (U_{i+1,j}^k + U_{i-1,j}^k)}{\Delta r^2} - r_i \frac{(U_{i+1,j}^k - U_{i-1,j}^k)}{2\Delta r} + \frac{U_{i,j+1}^k + U_{i,j-1}^k}{\Delta \theta^2} 
- 2 \frac{(V_{i,j+1}^k - V_{i,j-1}^k)}{2\Delta \theta} \right) \right\} / \left( \frac{r_i^2}{\Delta t} - \frac{1}{\mu_1 \text{Re}} \left( \frac{2r_i^2}{\Delta r} + \frac{2}{\Delta \theta^2} \right) \right)
$$

(A.51)

for $i = 1, \ldots, N - 1$ and $j = 1, \ldots, M - 1$, the symmetry condition (A.44f) gives

$$
U_{i0}^* = \frac{4U_{i1}^k - U_{i2}^k}{3}, \quad U_{iM}^* = \frac{4U_{iM-1}^k - U_{iM-2}^k}{3},
$$

(A.52)

for $i = 1, \ldots, N - 1$ and the continuity equation (A.44c) is used to define a Neumann boundary condition at $r = 1$, which gives

$$
U_{Nj}^* = \frac{1}{3} \left( 4U_{N-1,j}^k - U_{N-2,j}^k - \frac{\Delta r}{\Delta \theta} (V_{N,j+1}^k - V_{N,j-1}^k) \right)
$$

(A.53)

for $j = 0, \ldots, M$. The value of the relaxation parameter that resulting in the fastest convergence was $\alpha_U = 1.6$. Once the horizontal velocity has been updated, it is then used to update the vertical velocity component in the same manner. The boundary conditions for the vertical velocity imply that $V_{0j}^k = V_{i0} = V_{iM} = 0$. Then, the vertical velocity is updated everywhere else using

$$
V_{ij}^k = \alpha_V V_{ij}^* + (1 - \alpha_V) V_{ij}^{\text{old}},
$$

(A.54)
A.3. Droplet deformation: first temporal stage

where $\alpha_V$ is the relaxation parameter and

$$
V_{ij}^* = \left\{ \begin{array}{l}
\frac{r_i^2 V_{i,j-1}^{k-1}}{\Delta t} - r_i^2 \frac{(P_{i,j+1}^k - P_{i,j-1}^k)}{2 \Delta \theta} \\
+ \frac{1}{\mu_1 \text{Re}} \left( \frac{r_i^2}{\Delta r^2} \frac{(V_{i+1,j}^k + V_{i-1,j}^k) - r_i^2}{2 \Delta r} \right) + \frac{V_{i,j+1}^k + V_{i,j-1}^k}{\Delta \theta^2}
\end{array} \right. \\
\frac{2}{\mu_1 \text{Re}} \left( \frac{r_i^2}{\Delta r} - \frac{1}{\mu_1 \text{Re}} \left( \frac{2 r_i^2}{\Delta r} + \frac{2}{\Delta \theta^2} \right) \right)
$$

(A.55)

for $i = 1, \ldots, N - 1$ and $j = 1, \ldots, M - 1$ and the shear stress condition in (A.44e) gives

$$
V_{N,j}^* = \left( \frac{4 V_{N-1,j}^k - V_{N-2,j}^k}{2 \Delta r} - \frac{U_{N,j+1}^k - U_{N,j-1}^k}{2 \Delta \theta} \right) + \mu_1 \xi_\theta(1, \theta, t_k) \left( \frac{3}{2 \Delta r} - 1 \right)
$$

(A.56)

for $j = 1, \ldots, M - 1$. The relaxation parameter that resulted in the fastest convergence was $\alpha_V = 1.4$. Once the vertical velocity has been updated, convergence is checked using a similar condition as (A.49) for both $U$ and $V$. If both are satisfied, the code moves to the next time step.

The steps the algorithm takes to solve this system can be summarised as

1. Initialise the solution with $U_{ij}^0 = V_{ij}^0 = P_{ij}^0 = 0$ everywhere. Set $k = 1$.

2. Calculate the pressure field $P_{ij}^k$ using (A.46-A.49).

3. Update the horizontal velocity $U_{ij}^k$ using (A.50-A.53).

4. Update the vertical velocity $V_{ij}^k$ using (A.54-A.56).

5. Check convergence of $U$ and $V$. If convergence is not satisfied, go back to step 3. If it is satisfied, $k \rightarrow k + 1$ and go back to step 2.

6. Repeat steps 2-5 for as long as necessary.

7. Calculate the original variables, with $u_r = U/r$, $u_\theta = V/r$ and $p = P/r$.
A.3. Droplet deformation: first temporal stage

<table>
<thead>
<tr>
<th>Case</th>
<th>∆r</th>
<th>∆θ</th>
<th>∆t</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>0.01</td>
<td>π/150</td>
<td>0.005</td>
</tr>
<tr>
<td>B</td>
<td>0.005</td>
<td>π/250</td>
<td>0.002</td>
</tr>
</tbody>
</table>

Table A.4: Details of the parameters used in the validation study for the unsteady Stokes equations inside a droplet.

Figure A.4: The (a) radial and (b) azimuthal velocity evaluated at the droplet interface \( r = 1 \), as well as the pressure evaluated along \( \theta = \pi/2 \), when \( t = 3 \) for the parameters listen in table A.4.

The sources of error in this numerical scheme are the spatial discretisation and time steps. We validate our code by testing against a fine grid computation, with the details of both cases given in table A.4. Figure A.4 shows the radial and azimuthal velocity evaluated at the droplet interface \( r = 1 \), as well as the pressure evaluated along \( \theta = \pi/2 \), at \( t = 3 \), for both cases in table A.4. The results for both cases show excellent agreement with one another. The results presented in the main body of the thesis are evaluated using case A.
Appendix B

Validation of thin cavity velocity approximation

In this Appendix, the support is provided for the velocity approximation for the thin cavity flow in Chapter 4. This approximation assumed a cubic form of the horizontal velocity in the lubricant driven by the slip velocity at the lubricant/jet interface, which satisfies a zero flow rate at every point and is equal to zero on a given curve. This form of the horizontal velocity component was imposed using a normalised vertical coordinate, taking into account the shape of the lubricant/jet meniscus shape, and used to approximate the partial slip condition. To support this approximation, we will compare it when evaluated using a given slip velocity on a flat meniscus to the full solutions of the steady Navier-Stokes equations in a thin cavity, driven by the same slip velocity.

For this task, we solve the steady Navier-Stokes equations on a Cartesian grid in a thin cavity \(0 \leq x \leq 1, -\varepsilon \leq y \leq 0\), driven by a given slip velocity \(u_s(x)\) at \(y = 0\)

\[
\begin{align*}
\frac{u}{\partial x} + \frac{v}{\partial y} &= \frac{-\partial p}{\partial x} \frac{1}{Re} \nabla^2 u & (B.1a) \\
\frac{u}{\partial x} + \frac{v}{\partial y} &= \frac{-\partial p}{\partial y} \frac{1}{Re} \nabla^2 v & (B.1b) \\
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} &= 0 & (B.1c) \\
u = u_s(x), \quad v = 0, \quad \text{at} \quad y = 0, & (B.1d)
\end{align*}
\]
Figure B.1: Slip velocities $u_s^{(1)}$, $u_s^{(2)}$ and $u_s^{(3)}$ and the corresponding streamlines in the thin cavity, for Re = 100 and 500.

\begin{equation}
\begin{array}{c}
u = v = 0, \quad \text{at } y = -\varepsilon, \quad x = 0, 1 \\
\end{array}
\end{equation}

where $\varepsilon \ll 1$ and $\nabla = \partial^2 / \partial x^2 + \partial^2 / \partial y^2$.

Our first task is to support the approximation that the horizontal velocity is
Figure B.2: Comparison of the vertical position of zero horizontal velocity for \( u_s^{(1)} \), \( u_s^{(2)} \) and \( u_s^{(3)} \) at \( \text{Re} = 100 \) and 500, when \( \varepsilon = 0.1 \) (top) and 0.05 (bottom). The black solid line represents the approximated curve.

zero along the curve \( \eta = f_0(x) \), where in this case \( \eta = (y + \varepsilon)/\varepsilon \), and \( f_0 \) is given by

\[
f_0 = \frac{1}{3} \left( \frac{4}{3} - \text{erf} \left( \frac{2x}{\varepsilon} \right) + \text{erf} \left( \frac{2(x-1)}{\varepsilon} \right) \right). \tag{B.2}
\]

The system (B.1) is solved in a streamfunction-vorticity formulation (using the same method as in Appendix A.3.2). Three different slip velocities will be considered:

\[
u_s^{(1)} = 2 \sqrt{x(1-x)}, \tag{B.3a}
\]

\[
u_s^{(2)} = \text{erf} \left( \frac{2x}{\varepsilon} \right) - \text{erf} \left( \frac{2(x-1)}{\varepsilon} \right) - 1, \tag{B.3b}
\]

\[
u_s^{(3)} = \frac{1}{5x+1} \left( \text{erf} \left( \frac{2x}{\varepsilon} \right) - \text{erf} \left( \frac{2(x-1)}{\varepsilon} \right) - 1 \right), \tag{B.3c}
\]
Figure B.3: Comparison of the horizontal velocity $u$ from the full solution (solid) and the approximation (dashed) at $x = 0.05, 0.25, 0.75$ and $0.95$ for $u_{s}^{(3)}$, $\varepsilon = 0.1$ and $Re = 100$.

where $u_{s}^{(1)}$ is a parabolic symmetric slip velocity, $u_{s}^{(2)}$ is a symmetric slip velocity that has boundary layers at the end regions and $u_{s}^{(3)}$ is a non-symmetric slip velocity that has boundary layers at the end regions. These slip velocities and their corresponding streamlines in the thin cavity are shown in figure B.1, for $Re = 100$ and 500. As can be seen in the streamlines, there is one primary vortex, roughly centered at $y/\varepsilon = -1/3$. Figure B.2 compares the vertical position of zero horizontal velocity to the approximated curve for the three slip velocities considered at $Re = 100$ and 500, when $\varepsilon = 0.1$ and 0.05. For $\varepsilon = 0.1$ the approximation is good for solutions at $Re = 100$ and satisfactory when $Re = 500$, and typically better at the leading edge then the trailing edge. When $\varepsilon = 0.05$, the approximation improves.

Next, we need to compare the horizontal velocity from the full solution to the
approximated solution

\[ u \approx u_s(x)(A(x)\eta^3 + B(x)\eta^2 + C(x)\eta), \]

(B.4)

where \( A, B \) and \( C \) are functions of \( x \) defined in terms of \( f_0 \) and are given in Chapter 4. Figure B.3 shows a comparison of the horizontal velocity from the full solution to the approximation, for \( u_s^{(3)} \) at \( \varepsilon = 0.1 \) and \( \text{Re} = 100 \). At the points \( x = 0.25 \) and \( 0.75 \), the approximation is almost identical to the full solution. When we go closer to the end points at \( x = 0.05 \) and \( 0.95 \) (these points are within the typical boundary layer that is \( O(\varepsilon) = O(0.1) \)) the quality of the approximation starts to wane, but it is still very good, and it remains an excellent approximation of the velocity shear at the interface, which is the main component of the approximation that is used in the modelling to define the partial slip condition.
Bibliography


