Duality between the Power Minimization and Max-Min SINR Balancing Symbol-Level Precoding

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***Abstract*—This paper reveals the latent relation inherent in two typical problems in constructive interference (CI)-based symbol-level precoding (SLP). One is the power minimization (PM) problem subject to instantaneous signal-to-interference-plus-noise ratio (SINR) constraints, and the other is the weighted max-min SINR balancing (SB) problem with the symbol-level transmit power budget. In particular, we establish an explicit duality between the PM-SLP and SB-SLP problems, where we prove that one of the two problems can be uniquely mapped to the other. The proposed duality not only provides insights into the intrinsic structure of the problems and solutions but also facilitates obtaining the solution to the SB-SLP given the solution to the PM-SLP without the need for one-dimension search, and vice versa. We further propose a closed-form power scaling algorithm to solve the SB-SLP via PM-SLP, by which the separability of the PM-SLP can be leveraged to solve the two problems simultaneously. Numerical results demonstrate our derivations on the duality as well as the efficiency of the proposed power scaling algorithm.**

***Index Terms*—MU-MISO, constructive interference, symbol-level precoding, power minimization, SINR balancing, separability, duality, inverse problem.**

I. Introduction

The precoding technique that allows the transmitter to precode the data symbols onto the antenna-emitted transmit signal has been extensively investigated in academia and industry [1]. To manage interference in the downlink, channel state information (CSI) is indispensable for precoding.

Under the assumption that the data symbols are independent and identically distributed (i.i.d.), linear precoding uses CSI to eliminate or suppress interference, where transmit power and signal-to-interference-plus-noise ratio (SINR) are two frequently considered design metrics. The corresponding optimization problems account for the power minimization (PM) problem with SINR constraints and the (weighted) max-min SINR balancing (SB) problem subject to a transmit power constraint. The former is dedicated to optimizing the energy efficiency of the transmitter, and the latter focuses on fairness among users. Although nonconvex due to the SINR-related formulations, the two problems can be solved via optimization theories, such as semidefinite relaxation (SDR) [2] and second-order-cone programming (SOCP) [3]. On the other hand, solutions based on the uplink-downlink duality are viable for the two problems [4], where the uplink problems have been solved early. In addition, a more straightforward inverse relation between the two downlink problems has been proven, and the one-dimension bisection search method has also been proposed to solve the SB problem by iteratively solving different instances of PM problems [3]. The above linear precoding has a constant precoding matrix in each channel coherence time to precode a block of data symbols, also known as block-level precoding (BLP).

Contrary to the aforementioned BLP, symbol-level precoding (SLP) imposes a unique precoding matrix in each symbol slot to utilize both CSI and the deterministic information on data symbols. The intuition behind the SLP is that, with the knowledge of CSI and data symbols, the transmit signal can be predicted and designed instantaneously. As a consequence, known interference can be exploited as a source of useful power beneficial to correct detection, which is known as constructive interference (CI) [5]–[7]. Concerning symbol-level design criteria, the PM-SLP and SB-SLP adopt symbol-level transmit power and instantaneous SINR. Most solutions to the two problems are based on the Lagrangian duality. Specifically, an efficient gradient projection algorithm has been proposed in [5] for the Lagrangian dual problem of the PM-SLP problem. In [8], Lagrangian duality was exploited to derive the optimal precoding structure for the SB-SLP problem. On the other hand, the inverse relation similar to the BLP case was also explored. It has been shown in [6] that the PM-SLP problem and the SB-SLP problem are inverse problems, and a bisection search method analogous to that of the BLP case has been proposed to solve the SB-SLP problem. Nevertheless, the intensively updated precoding strategy in each symbol slot incurs an unbearing computational burden to the transmitter. A better understanding of the relation and structure of the two problems may open a new door to this.

In this paper, our goal is to inspect the PM-SLP and SB-SLP problems from a novel dual perspective, different from the uplink-downlink duality or Lagrangian duality. By reformulating the original problems, we convert them into more inspiring forms. We then prove an explicit duality between the two considered problems, which indicates that although having different objective functions and constraints, one of the two problems can be mapped to the other uniquely. More importantly, the equations of the solutions to the two dual problems are explicitly determined. A closed-form power scaling algorithm for the SB-SLP problem is subsequently developed to obtain an efficient solution leveraging the separability of the PM-SLP problem discussed in [9]. Our simulation results validate the proofs of the duality and illustrate the complexity reduction of the power scaling algorithm over compared schemes.

Notation: Scalars, vectors, and matrices, are represented by plain lowercase, boldface lowercase, and boldface capital letters, respectively. $(\cdot)^T$ denotes transpose operator. $\mathbb{C}^{M\times N}$ and $\mathbb{R}^{M\times N}$ denote the sets of $M\times N$ matrices with complex-valued and real-valued entries, respectively. $|\cdot|$ represents the absolute value of a real-valued scalar or the modulus of a complex-valued scalar. $\|\cdot\|$ denotes the Euclidean norm of a vector. $\Re\{\cdot\}$ and $\Im\{\cdot\}$ respectively denote the real part and imaginary part of a complex-valued input. $\mathbf{1}$ represents the all-ones vector of appropriate dimension.

II. System Model and Problem Formulation

In section, we present the system model and briefly review the concept of CI in the context of PSK signaling, then formulate the PM-SLP and SB-SLP problems.

A. System Model

Consider a downlink multi-user multiple-input single-output (MU-MISO) system in which a base station (BS) equipped with N_t antennas provides service to K single-antenna users in the same time-frequency resource. The data symbol vector $\tilde{\mathbf{s}} \triangleq [\tilde{s}_1, \cdots, \tilde{s}_K]^T \in \mathbb{C}^K$ contains the overall K data symbols in a symbol slot, which is mapped to the transmit signal $\tilde{\mathbf{x}} \triangleq [\tilde{x}_1, \cdots, \tilde{x}_{N_t}]^T \in \mathbb{C}^{N_t}$ at the BS via SLP. The received signal of user k in one symbol slot is expressed as

$$\tilde{y}_k = \tilde{\mathbf{h}}_k^T \tilde{\mathbf{x}} + \tilde{z}_k, \tag{1}$$

where $\tilde{\mathbf{h}}_k \in \mathbb{C}^{N_t}$ denotes the quasi-static Rayleigh flat-fading channel vector between BS and user k, and $\tilde{z}_k \sim \mathcal{CN}(0, \sigma_k^2)$ is the complex-valued additive white Gaussian noise at user k. To focus on the precoding design, perfect CSI is assumed.

B. Constructive Interference

The constructive and destructive addition of interference in the noiseless received signal $\left\{\tilde{\mathbf{h}}_k^T \tilde{\mathbf{x}}\right\}$ is jointly determined by CSI and the information on data symbols [10]. To predict and further exploit the interference, CI-SLP optimizes the transmit signal by judiciously utilizing CSI and data symbols such that all the multi-user interference can add up constructively at each receiver side [7]. Therefore, the received instantaneous SINR at user k is given by

$$\mathrm{SINR}_k = \frac{|\tilde{\mathbf{h}}_k^T \tilde{\mathbf{x}}|^2}{\sigma_k^2}. \tag{2}$$

Since all interference is exploited via CI-SLP, the instantaneous SINR is equivalent to the conventional signal-to-noise ratio (SNR).

For the sake of illustration, the geometric interpretation of CI is shown in Fig. 1. Without loss of generality, denote the symbol of interest of user k by $\tilde{s}_k$, which is an arbitrary constellation point drawn from a normalized $\mathcal{M}$-PSK

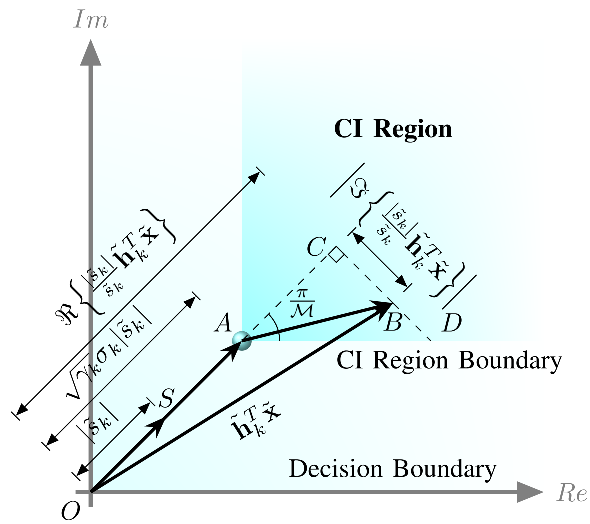

Fig. 1. Illustration of CI regions for a generic $\mathcal{M}$-PSK modulation.

constellation, corresponding to $\overrightarrow{OS}$. The received noiseless signal of user k can be expressed as $\tilde{\mathbf{h}}_k^T \tilde{\mathbf{x}}$, which is denoted by $\overrightarrow{OB}$ in Fig. 1. $\overrightarrow{OA}$ represents The nominal constellation point $\sqrt{\gamma_k}\sigma_k \tilde{s}_k$, where γ_k denotes a given instantaneous SINR threshold for user k. The CI region associated with the nominal constellation point $\overrightarrow{OA}$ is depicted as the darker-shaded area in Fig. 1, where the CI region refers to a polyhedron bounded by hyperplanes parallel to decision boundaries of the constellation point [5], [11].

From Fig. 1 we can observe that when $\overrightarrow{OB}$ is orthogonally decomposed along $\overrightarrow{OA}$, we have $\overrightarrow{OB} = \overrightarrow{OC} + \overrightarrow{CB}$, where $\overrightarrow{OC} \perp \overrightarrow{CB}$. Consequently, one of the criteria that specifies the location of $\overrightarrow{OB}$ in the CI region is $\left|\overrightarrow{CD}\right| \geq \left|\overrightarrow{CB}\right|$, where D denotes the intersection of $\overrightarrow{CB}$ and its nearest CI region boundary. The mathematical formulation of CI constraints for $\mathcal{M}$-PSK signaling can be written as [5]

$$\Re\{\hat{\mathbf{h}}_k^T \tilde{\mathbf{x}}\} - \frac{|\Im\{\hat{\mathbf{h}}_k^T \tilde{\mathbf{x}}\}|}{\tan\frac{\pi}{\mathcal{M}}} \geq \sqrt{\gamma_k}\sigma_k, \forall k, \tag{3}$$

where $\hat{\mathbf{h}}_k^T \triangleq \frac{\tilde{\mathbf{h}}_k^T}{\tilde{s}_k}$, γ_k denotes the pre-defined instantaneous SINR threshold for user k. The instantaneous SINR constraints are incorporated into the above CI constraints.

C. Problem Formulation

1) PM-SLP Problem: The PM-SLP problem aims to minimize the total transmit power subject to CI constraints. This optimization problem has the following mathematical form:

$$\begin{aligned} \min_{\tilde{\mathbf{x}}} \quad & \|\tilde{\mathbf{x}}\|^2 \\ \text{s.t.}\; & \Re\{\hat{\mathbf{h}}_k^T \tilde{\mathbf{x}}\} - \frac{|\Im\{\hat{\mathbf{h}}_k^T \tilde{\mathbf{x}}\}|}{\tan\frac{\pi}{\mathcal{M}}} \geq \sqrt{\gamma_k}\sigma_k, \forall k. \end{aligned} \tag{4}$$

The above problem is convex and can be solved via off-the-shelf solvers.

Recently, we revealed the separability of the PM-SLP problem for PSK modulation and proposed a parallelizable and inversion-free CI-SLP approach in our previous work [9]. The details are omitted here due to space limitations. Interested readers are kindly referred to [9] and the references therein.

2) SB-SLP Problem: The SB-SLP problem focuses on fairness in the system by maximizing the minimum instantaneous SINR over all users subject to a total transmit power constraint. This problem is formulated as (5) on the top of the next page, where p denotes the symbol-level transmit power budget, and, with a little abuse of notation, $\frac{1}{\sqrt{\gamma_k}}$ denotes the square root of the weight of SINR_k in the context of the SB-SLP problem.

The original max-min SB-SLP problem can be equivalently converted to a more tractable SOCP problem [5], given by

$$\begin{aligned}\max_{\tilde{\mathbf{x}},\mu}\quad & \mu \\ \text{s.t.}\ & \Re\{\hat{\mathbf{h}}_k^T\tilde{\mathbf{x}}\} - \frac{|\Im\{\hat{\mathbf{h}}_k^T\tilde{\mathbf{x}}\}|}{\tan\frac{\pi}{\mathcal{M}}} \geq \mu\sqrt{\gamma_k}\sigma_k,\ \forall k, \\ & \|\tilde{\mathbf{x}}\|^2 \leq p.\end{aligned} \tag{6}$$

Similar to the PM-SLP problem, the problem above can be solved using standard solvers for convex optimization. The SB-SLP is more complex than the linearly constrained quadratic PM-SLP problem. To solve it more efficiently, the derivation of the optimal structure is used to obtain its Lagrangian dual problem in [8], where an iterative closed-form scheme was proposed for PSK signaling.

III. PROPOSED DUALITY

In this section, we reformulate the SB-SLP and PM-SLP problems, then derive the one-to-one mapping between a pair of PM-SLP and SB-SLP problems. Based on this, a closed-form power scaling algorithm is proposed to solve the SB-SLP problem. Accordingly, the separable structure of the PM-SLP problem for PSK modulation analyzed in [9] can be employed in solving both the PM-SLP and SB-SLP problems.

A. Problem Reformulation

The real-valued equivalent of (6) can be written as

$$\begin{aligned}\max_{\mathbf{x},\mu}\quad & \mu \\ \text{s.t.}\ & \mathbf{T}\mathbf{S}_k\mathbf{H}_k\mathbf{x} \succeq \mu\sqrt{\gamma_k}\sigma_k\mathbf{1},\ \forall k, \\ & \|\mathbf{x}\|^2 \leq p,\end{aligned} \tag{7}$$

where

$$\mathbf{x} \triangleq \begin{bmatrix}\Re\{\tilde{\mathbf{x}}\} \\ \Im\{\tilde{\mathbf{x}}\}\end{bmatrix} \in \mathbb{R}^{2N_t}, \mathbf{T} \triangleq \begin{bmatrix}1 & -\frac{1}{\tan\frac{\pi}{\mathcal{M}}} \\ 1 & \frac{1}{\tan\frac{\pi}{\mathcal{M}}}\end{bmatrix} \in \mathbb{R}^{2\times 2},$$

$$\mathbf{S}_k \triangleq \begin{bmatrix}\Re\left\{\frac{1}{\tilde{s}_k}\right\} & -\Im\left\{\frac{1}{\tilde{s}_k}\right\} \\ \Im\left\{\frac{1}{\tilde{s}_k}\right\} & \Re\left\{\frac{1}{\tilde{s}_k}\right\}\end{bmatrix} \in \mathbb{R}^{2\times 2},$$

$$\mathbf{H}_k \triangleq \begin{bmatrix}\Re\{\tilde{\mathbf{h}}_k^T\} & -\Im\{\tilde{\mathbf{h}}_k^T\} \\ \Im\{\tilde{\mathbf{h}}_k^T\} & \Re\{\tilde{\mathbf{h}}_k^T\}\end{bmatrix} \in \mathbb{R}^{2\times 2N_t}.$$

We further introduce $\bar{\mathbf{A}}_k \triangleq \mathbf{T}\mathbf{S}_k\mathbf{H}_k$, and $\mathbf{b}_k \triangleq \sqrt{\gamma_k}\sigma_k\mathbf{1}$. Accordingly, the CI constraints become

$$\bar{\mathbf{A}}_k\mathbf{x} \succeq \mu\mathbf{b}_k,\ \forall k. \tag{8}$$

Stack the CI constraints over all the K users yields a compact formulation, given by

$$\mathbf{A}\mathbf{x} \succeq \mu\mathbf{b}, \tag{9}$$

where $\mathbf{A} \triangleq \left[\bar{\mathbf{A}}_1^T, \cdots, \bar{\mathbf{A}}_K^T\right]^T \in \mathbb{R}^{2K\times 2N_t}$, $\mathbf{b} \triangleq \left[\mathbf{b}_1^T, \cdots, \mathbf{b}_K^T\right]^T \in \mathbb{R}^{2K}$. It can be seen that the left-hand side of (9) can be expressed as a linear combination of the columns of $\mathbf{A}$, i.e., $\sum_{i=1}^{2N_t}\mathbf{a}_i x_i$, where $\mathbf{a}_i$ is the i-th column of $\mathbf{A}$, x_i is the i-th entry of $\mathbf{x}$. Subsequently, (7) can be rearranged as

$$\begin{aligned}\max_{\mathbf{x}_i,\mu}\quad & \mu \\ \text{s.t.}\ & \sum_{i=1}^{N}\mathbf{A}_i\mathbf{x}_i \succeq \mu\mathbf{b}, \\ & \sum_{i=1}^{N}\|\mathbf{x}_i\|^2 \leq p,\end{aligned} \tag{10}$$

where $\mathbf{x}_i \in \mathbb{R}^{n_i}$ with $\sum_{i=1}^{N} n_i = 2N_t$ and $\mathbf{A}_i \in \mathbb{R}^{2K\times n_i}$ are the i-th blocks of $\mathbf{x}$ and $\mathbf{A}$, respectively. $\mathbf{x}_i$ is composed of the adjacent and/or disadjacent elements of $\mathbf{x}$. Each column of $\mathbf{A}_i$ is uniquely taken from the columns of $\mathbf{A}$. Specifically, if the elements in $\mathbf{x}_i$ are taken from $\mathbf{x}$ continuously, we have $\mathbf{x} = \left[\mathbf{x}_1^T, \cdots, \mathbf{x}_N^T\right]^T$, $\mathbf{A} = [\mathbf{A}_1, \cdots, \mathbf{A}_N]$. On the other hand, if we want to group the disadjacent elements of $\mathbf{x}$ into one group, which can be expressed as $\mathbf{x}_i = \mathbf{E}_i^T\mathbf{x}$, $\mathbf{A}_i = \mathbf{A}\mathbf{E}_i$, where $\mathbf{E}_i \in \mathbb{R}^{2N_t\times n_i}$, and each column of $\{\mathbf{E}_i\}$ is uniquely picked from the columns of the $2N_t \times 2N_t$ identity matrix.

In accordance with the procedure formulating (10), the real-valued equivalent of the PM-SLP problem (4) can be rearranged as [9]

$$\begin{aligned}\min_{\mathbf{x}_i}\quad & \sum_{i=1}^{N}\|\mathbf{x}_i\|^2 \\ \text{s.t.}\ & \sum_{i=1}^{N}\mathbf{A}_i\mathbf{x}_i \succeq \mathbf{b}.\end{aligned} \tag{11}$$

The above formulation was first proposed in our previous work [9], where the separable structure of the PM-SLP problem for PSK modulation was proven. The separability was further utilized to decompose the original problem into multiple parallel subproblems by the proposed parallel inverse-free (PIF) algorithm.

Contrary to the separable PM-SLP problem (11), it is observed that the above SB-SLP problem (10) is not separable because of the objective function μ, which cannot be separated. Thus the PIF-SLP approach proposed in [9] is not applicable to decompose the SB-SLP problem at first glance. Fortunately, we find an explicit relation inherent in solutions of the two problems, which indicates that once the optimal solution to the PM-SLP problem is obtained via the PIF algorithm [9] or other algorithms, then finding the optimal solution to the

$$\max_{\tilde{\mathbf{x}}} \min_{k} \quad \frac{1}{\sqrt{\gamma_k}\sigma_k} \left\{ \Re\{\hat{\mathbf{h}}_k^T \tilde{\mathbf{x}}\} - \frac{\Im\{\hat{\mathbf{h}}_k^T \tilde{\mathbf{x}}\}}{\tan\frac{\pi}{\mathcal{M}}}, \Re\{\hat{\mathbf{h}}_k^T \tilde{\mathbf{x}}\} + \frac{\Im\{\hat{\mathbf{h}}_k^T \tilde{\mathbf{x}}\}}{\tan\frac{\pi}{\mathcal{M}}} \right\} \quad \text{s.t. } \|\tilde{\mathbf{x}}\|^2 \le p, \tag{5}$$

SB-SLP problem is trivial, which is termed as the duality to be presented below.

B. Duality Between the PM-SLP and SB-SLP

For the BLP that suppresses interference, it is known that the PM problem and the SB problem are a pair of inverse problems [3]. This relationship has been extended to CI-SLP by [6], which proposes to solve the SB-SLP problem via iteratively solving its inverse PM-SLP problem along with a bisection search. Unlike the high-complexity one-dimension search scheme, recently, a novel duality between the conventional multicast PM and SB problems has been revealed [12], which can explicitly determine the solution to the SB problem given the solution to the PM problem, and vice versa. Later in CI-based symbol error rate minimization precoding, a closed-form algorithm was designed to solve the detection-region-based noise uncertainty radius maximization problem under the precondition of the solved detection-region-based PM problem [13]. In this subsection, a novel duality between the PM-SLP and SB-SLP problems is established.

Let $\mathbf{x}^{PM}$ and $p^{PM} \triangleq \|\mathbf{x}^{PM}\|^2$ denote the optimal solution and objective value of the PM-SLP problem for PSK modulation (11). $\mathbf{x}^{SB}$ and $\mu^{SB} \triangleq \min\limits_i \frac{1}{b_i}\bar{\mathbf{a}}_i^T\mathbf{x}^{SB}$ are the optimal counterparts for the SB-SLP problem in (10), where $\bar{\mathbf{a}}_i$ denotes the transpose of the i-th row of $\mathbf{A}$, and $\bar{b}_i$ represents the i-th entry of $\mathbf{b}$.

Lemma 1: The PM-SLP problem (11) and the SB-SLP problem (10) are inverse problems:

$$\mathbf{x}^{PM}(\alpha\mathbf{b}) = \mathbf{x}^{SB}\left(\mathbf{b}, p^{PM}(\alpha\mathbf{b})\right), \tag{12}$$

with $\alpha = \mu^{SB}\left(\mathbf{b}, p^{PM}(\alpha\mathbf{b})\right)$. Reciprocally,

$$\mathbf{x}^{SB}(\mathbf{b}, p) = \mathbf{x}^{PM}\left(\mu^{SB}(\mathbf{b}, p)\,\mathbf{b}\right), \tag{13}$$

with $p = p^{PM}\left(\mu^{SB}(\mathbf{b}, p)\,\mathbf{b}\right)$.

Proof: Contradiction can be used to prove (12). Assume that there exists an optimal solution $\mathbf{x}^{SB}\left(\mathbf{b}, p^{PM}(\alpha\mathbf{b})\right)$ and the corresponding optimal value $\mu^{SB}\left(\mathbf{b}, p^{PM}(\alpha\mathbf{b})\right)$ for the SB-SLP problem (10) given parameters $\left(\mathbf{b}, p^{PM}(\alpha\mathbf{b})\right)$. Similarly, assume the optimal solution and the optimal value for the PM-SLP problem (11) given $\alpha\mathbf{b}$ are $\mathbf{x}^{PM}(\alpha\mathbf{b})$ and $p^{PM}(\alpha\mathbf{b})$, respectively. By definition, $\mathbf{x}^{PM}(\alpha\mathbf{b})$ is a feasible solution to the above SB-SLP problem, and the associated objective value is α. If $\alpha > \mu^{SB}\left(\mathbf{b}, p^{PM}(\alpha\mathbf{b})\right)$, then this is a contradiction for the optimality of $\mu^{SB}\left(\mathbf{b}, p^{PM}(\alpha\mathbf{b})\right)$. Otherwise, if $\alpha < \mu^{SB}\left(\mathbf{b}, p^{PM}(\alpha\mathbf{b})\right)$, then $\mathbf{x}^{SB}\left(\mathbf{b}, p^{PM}(\alpha\mathbf{b})\right)$ is also a feasible solution to the PM-SLP problem (11) given $\alpha\mathbf{b}$, for which all the CI constraints are over satisfied. Therefore, one can always find a $v \in (0,1)$ such that $v\mathbf{x}^{SB}\left(\mathbf{b}, p^{PM}(\alpha\mathbf{b})\right)$ meets all the CI constraints while providing a smaller objective value than $p^{PM}(\alpha\mathbf{b})$. This is a contradiction for the optimality of $p^{PM}(\alpha\mathbf{b})$. The above proves (12) is true with $\alpha = \mu^{SB}\left(\mathbf{b}, p^{PM}(\alpha\mathbf{b})\right)$. The proof of (13) is similar and is therefore omitted. ∎

Lemma 2: Consider the PM-SLP problem (11), for any $\alpha > 0$, we have

$$\mathbf{x}^{PM}(\alpha\mathbf{b}) = \alpha\mathbf{x}^{PM}(\mathbf{b}), \tag{14}$$

$$p^{PM}(\alpha\mathbf{b}) = \alpha^2 p^{PM}(\mathbf{b}). \tag{15}$$

For the SB-SLP problem,

$$\mathbf{x}^{SB}\left(\mathbf{b}, \alpha^2 p\right) = \alpha\mathbf{x}^{SB}(\mathbf{b}, p), \tag{16}$$

$$\mu^{SB}\left(\mathbf{b}, \alpha^2 p\right) = \alpha\mu^{SB}(\mathbf{b}, p). \tag{17}$$

Proof: Let $\mathbf{x} = \frac{\dot{\mathbf{x}}}{\alpha}$, where $\alpha > 0$, $\dot{\mathbf{x}} = \alpha\mathbf{x}$. Replacing $\mathbf{x}$ in (11) yields

$$\min_{\dot{\mathbf{x}}_i} \quad \sum_{i=1}^{N} \|\dot{\mathbf{x}}_i\|^2 \quad \text{s.t.} \sum_{i=1}^{N} \mathbf{A}_i\dot{\mathbf{x}}_i \succeq \alpha\mathbf{b}, \tag{18}$$

then (14) and (15) follow immediately.

By substituting $\mathbf{x} = \frac{\dot{\mathbf{x}}}{\alpha}$ into (10), we similarly obtain

$$\max_{\dot{\mathbf{x}}_i, \mu} \quad \alpha\mu \quad \text{s.t.} \sum_{i=1}^{N} \mathbf{A}_i\dot{\mathbf{x}}_i \succeq \alpha\mu\mathbf{b}, \quad \sum_{i=1}^{N} \|\dot{\mathbf{x}}_i\|^2 \le \alpha^2 p, \tag{19}$$

which induces (16) and (17). ∎

Theorem 1 (Duality): Let $\mathbf{x}^{PM}$ and $p^{PM} \triangleq \|\mathbf{x}^{PM}\|^2$ denote the optimal solution and the optimal value of the PM-SLP problem (11), respectively. Then the counterparts of the SB-SLP problem, $\mathbf{x}^{SB}$ and μ^{SB}, are determined as

$$\mathbf{x}^{SB}(\mathbf{b}, p) = \sqrt{\frac{p}{p^{PM}(\mathbf{b})}}\mathbf{x}^{PM}(\mathbf{b}), \tag{20}$$

$$\mu^{SB}(\mathbf{b}, p) = \sqrt{\frac{p}{p^{PM}(\mathbf{b})}}. \tag{21}$$

and vice versa as

$$\mathbf{x}^{PM}(\mathbf{b}) = \frac{1}{\mu^{SB}(\mathbf{b}, p)}\mathbf{x}^{SB}(\mathbf{b}, p), \tag{22}$$

$$p^{PM}(\mathbf{b}) = \frac{p}{\left(\mu^{SB}(\mathbf{b}, p)\right)^2}. \tag{23}$$

Proof: The optimal solution to the SB-SLP problem can be equivalently written as

$$\mathbf{x}^{SB}(\mathbf{b},p)=\mathbf{x}^{SB}\left(\mathbf{b},\frac{p}{p^{PM}(\mathbf{b})}p^{PM}(\mathbf{b})\right). \quad (24)$$

By using (15) to transfer the transmit power budget in (24), we have

$$\mathbf{x}^{SB}\left(\mathbf{b},\frac{p}{p^{PM}(\mathbf{b})}p^{PM}(\mathbf{b})\right) = \mathbf{x}^{SB}\left(\mathbf{b},p^{PM}\left(\sqrt{\frac{p}{p^{PM}(\mathbf{b})}}\mathbf{b}\right)\right). \quad (25)$$

Combining (25) with (12) yields

$$\mathbf{x}^{SB}\left(\mathbf{b},p^{PM}\left(\sqrt{\frac{p}{p^{PM}(\mathbf{b})}}\mathbf{b}\right)\right)=\mathbf{x}^{PM}\left(\sqrt{\frac{p}{p^{PM}(\mathbf{b})}}\mathbf{b}\right). \quad (26)$$

From (14) we have

$$\mathbf{x}^{PM}\left(\sqrt{\frac{p}{p^{PM}(\mathbf{b})}}\mathbf{b}\right)=\sqrt{\frac{p}{p^{PM}(\mathbf{b})}}\mathbf{x}^{PM}(\mathbf{b}). \quad (27)$$

Hence (20) is true.

We then use Lemma 1 and Lemma 2 to prove (21) and (23). It is shown in Lemma 1 that

$$p=p^{PM}\left(\mu^{SB}(\mathbf{b},p)\,\mathbf{b}\right). \quad (28)$$

Using (15), the above equality yields

$$p^{PM}\left(\mu^{SB}(\mathbf{b},p)\,\mathbf{b}\right)=\left(\mu^{SB}(\mathbf{b},p)\right)^2 p^{PM}(\mathbf{b}). \quad (29)$$

Thus (21) and (23) follow immediately.

The proof of (22) is similar to that of (20). For brevity, we give an abbreviated proof below:

$$\begin{aligned}\mathbf{x}^{PM}(\mathbf{b}) &= \mathbf{x}^{PM}\left(\frac{1}{\mu^{SB}(\mathbf{b},p)}t^{SB}(\mathbf{b},p)\,\mathbf{b}\right)\\ &\overset{(17)}{=} \mathbf{x}^{PM}\left(\mu^{SB}\left(\mathbf{b},\frac{1}{\left(\mu^{SB}(\mathbf{b},p)\right)^2}p\right)\mathbf{b}\right)\\ &\overset{(13)}{=} \mathbf{x}^{SB}\left(\mathbf{b},\frac{1}{\left(\mu^{SB}(\mathbf{b},p)\right)^2}p\right)\\ &\overset{(16)}{=} \frac{1}{\mu^{SB}(\mathbf{b},p)}\mathbf{x}^{SB}(\mathbf{b},p). \end{aligned} \quad (30)$$

■

Corollary 1: The SB-SLP problem and the PM-SLP problem can be solved simultaneously. In particular, the solution to the SB-SLP problem (10) can be obtained by first solving the PM-SLP problem (11) and then scaling the transmit power to satisfy the power budget of the SB-SLP problem, and vice versa.

Algorithm 1 Power Scaling Algorithm for the SB-SLP problem (10)

Input: $\mathbf{A}$, $\mathbf{b}$, p
Output: $\mathbf{x}$

1: Solve (11) by the PIF algorithm [9] or other algorithms, obtain $\mathbf{x}^{PM}(\mathbf{b})$ and $p^{PM}(\mathbf{b})$;
2: Compute $\mathbf{x}$ by (20).

C. Power Scaling Algorithm

Corollary 1 suggests that the SB-SLP problem for PSK modulation can be solved by a simple one-step power scaling algorithm, provided that the solution to the PM-SLP problem is available. The proposed closed-form power scaling algorithm is summarized in Algorithm 1. We point out that although the PIF algorithm cannot directly be applied to the SB-SLP problem due to the lack of separability, a power scaling PIF (SPIF) algorithm can be designed to solve the SB-SLP problem with the aid of the closed-form power scaling algorithm, which consists of two steps. In the first step, we obtain the parallelizable solution to the PM-SLP problem via the PIF algorithm proposed in [9]. Whileas in the second step, we use the closed-form power scaling algorithm to acquire the solution to the SB-SLP problem. By applying the SPIF algorithm, the separability of the PM-SLP problem can be utilized to attain a low-complexity and parallelizable solution to the SB-SLP problem.

IV. Simulation Results

In this section, we provide simulation results to validate the proposed duality and compare the SPIF algorithm with other works. The i.i.d. data symbols in $\tilde{\mathbf{s}}$ are drawn from the normalized 8PSK constellation, i.e., $\mathcal{M}=8$. We use '$K\times N_t$' to denote a downlink system with K single-antenna users and an N_t-antenna BS. The square roots of the weights $\frac{1}{\sqrt{\gamma_k}}$ and the transmit power budget p are all set to 1. We assume each random channel realization is used to transmit one frame of data symbols, where each frame contains $N_s=20$ symbol slots [14]. The benchmark schemes are selected as the conventional linear SB-BLP solved by line search [1], [3], the SB-SLP solved by interior point method (IPM) [15], and the closed-form solutions for CI-SLP (CI-CF) [8].

Fig. 2 presents the symbol error rate (SER) performance of the SPIF algorithm as a function of the number of iterations for two different MIMO configurations, the results are averaged over 10000 symbol slots, where the number of random channel realizations $N_c=500$. The required number of iterations for the BER of the SPIF algorithm converging to that of the IPM is about $T=70$ for the two considered MIMO configurations. The obtained number of iterations is used in the subsequent simulations.

Fig. 3 depicts the SER performance versus the increasing SNR for the two MIMO configurations. It can be observed

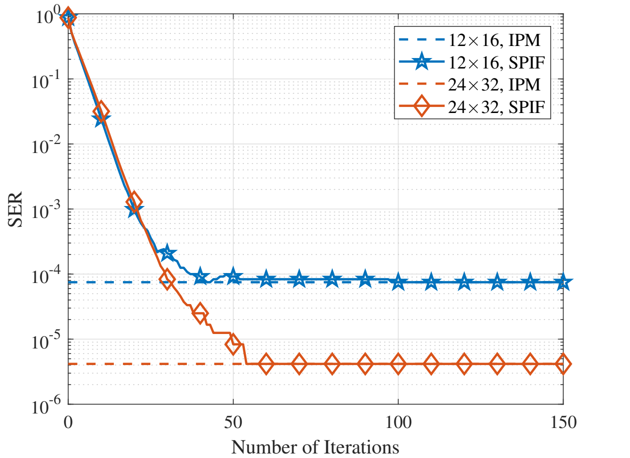

Fig. 2. SER versus number of iterations, SNR = 24dB, $N_c = 500$, $N_s = 20$, 8PSK.

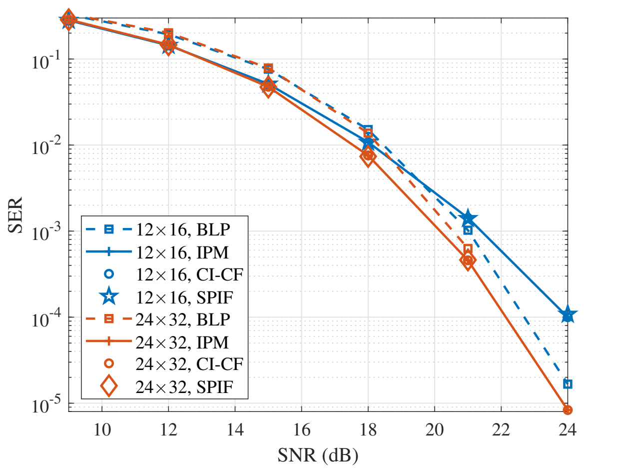

Fig. 3. SER versus SNR, $T = 70$, $N_c = 500$, $N_s = 20$, 8PSK.

that the SER performance of the proposed SPIF algorithm is almost consistent with that of the selected SLP benchmark algorithms, which validates the derivations of the proposed duality and illustrates the effectiveness of the proposed SPIF algorithm.

TABLE I
AVERAGE EXECUTION TIME PER FRAME IN SEC. SNR = 24DB, $T = 70$, $N_c = 500$, $N_s = 20$, 8PSK.

	BLP	IPM	CI-CF	SPIF
12 × 16	2.8270	4.9975	6.8639e-3	**2.7314e-3**
24 × 32	9.7372	5.3517	2.0503e-2	**9.9743e-3**

Table I lists the time complexity in terms of the average execution time per frame of the compared algorithms for the SB-SLP problem under two MIMO configurations, where the number of iterations of the SPIF algorithm is the same as in Fig. 3. The execution time of the SPIF algorithm based on the proposed duality is about 39.8% and 49.7% of that of the CI-CF algorithm in 12×16 and 24×32 MIMO configurations, respectively. The computational efficiency of the proposed SPIF algorithm is appealing in all the considered MIMO configurations, which can be further enhanced by parallel processing in practice.

V. CONCLUSION

In this work, a novel duality between the typical PM-SLP and SB-SLP problems was presented, for which we investigated the one-to-one mapping between the two considered problems. Moreover, we proved that the solutions to them admit explicit representations with determined coefficients. A closed-form power scaling algorithm for the SB-SLP problem was further developed. Numerical results were conducted to validate the proposed duality and illustrate the efficiency of the proposed algorithm.

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