Fluid Antenna Systems with Outdated Channel Estimates

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***Abstract*—A desired characteristic of future communication networks is the notion of reconfigurability. For a wireless device, this can be realized through the employment of the so-called fluid antennas (FAs). An FA consists of a dielectric holder, in which a radiating liquid moves between pre-defined locations (called ports) that serve as the device's antennas. Therefore, due to the nature of liquids, an FA can essentially take any size and shape, making them both flexible and reconfigurable. In this paper, we study the outage probability of FAs where the scheduled port, based on selection combining, is subject to scheduling delays. An analytical framework is provided for the performance with and without estimation errors, as a result of post-scheduling delays. We show that even though FAs achieve maximum channel (spatial) diversity, this cannot be attained in the presence of delays.**

***Index Terms*—Fluid antennas, outage probability, outdated channels, diversity.**

I. Introduction

The various advances in antenna technology and antenna arrays have been important in the evolution of communication systems towards 5G and beyond 5G networks [1]. Indeed, the implementation of multiple-input multiple-output (MIMO) antenna architectures has been an essential element in wireless networks for the realization of high data rates and spectral efficiency due to beamforming and spatial multiplexing. Such antennas are usually made of metal and are designed in such a way so as to meet specific network requirements. Furthermore, their design is subject to physical constraints, the most significant being the spacing between two antennas, which needs to be at least as half as the carrier's wavelength to avoid electromagnetic coupling [2]. Naturally, this metallic structure makes them static (i.e., inflexible), impractical and too costly for very small devices to have many antennas.

Recently, there has been several efforts to introduce reconfigurability in wireless networks. On one hand, reconfigurable intelligent surfaces were proposed to control the propagation environment via software-controlled metasurfaces [3]. On the other hand, from a device's point-of-view, the concept of fluid antennas (FAs) has been recently proposed in order to add both flexibility and reconfigurability at the radio frequency (RF) front-end [4]. Specifically, FAs consist of radiating liquid elements (e.g. Mercury, eutectic gallium-indium (EGaIn)) in a dielectric holder [5]. The holder contains several pre-defined positions, known as ports, where the employed liquid can be moved towards a selected port in a programmable and controllable manner. Moreover, the FA uses just one RF chain and so the spacing constraint does not apply in this case, making it a suitable technology for very small devices [6]. Therefore, compared to conventional static metallic antennas, FAs can adjust their physical configuration (e.g., shape, feeding) but also their electrical properties (e.g., resonant frequency, radiation pattern) [7]. It follows that this technology provides a new degree of freedom in the design of wireless communication systems.

Despite the fact that FAs have been studied from an RF/microwave engineering perspective and several early prototypes exist in the literature, see for example [4], [5], [8], the theoretical foundations of FAs and the investigation of communication techniques that unlock their liquid dimension are still not understood. Indeed, the exploitation of the liquid dimension associated with the FAs will open new design opportunities and establish a new communication paradigm [6], [9], [10]. In [6], the authors study the performance of an FA system in terms of outage probability and show that an FA can outperform a maximum ratio combining (MRC) system with conventional antennas, when the number of ports is sufficiently large. The work in [9] extends the study with respect to the ergodic capacity, where it is demonstrated that FAs can match the capacity of MRC systems. Finally, the performance of FAs with multiuser interference is studied in [10]; it is shown that with a large enough number of ports, the FA attains a relatively low outage probability.

The aforementioned studies assume that the performance is not affected by any imperfections as a result of the channel estimation and selection process. However, due to the sequential nature of FAs, estimation and selection will be affected by delays between the pre-scheduling and the post-scheduling of a port, especially for a large number of ports. Therefore, a proper analysis considering processing imperfections is of great importance. Motivated by this, in this paper, we study

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the outage probability of an FA system, where the channel at the selected port is subject to practical delays. Specifically, the contributions of this paper are as follows. We study the performance of FA systems by taking into account estimation errors due to delays and we present an analytical framework for the outage probability of outdated channels. Moreover, asymptotic expressions are provided that quantify the system's diversity and outage gain. We show that an FA achieves full channel (spatial) diversity when no estimation errors exist, which is independent of correlation, whereas its outage gain depends on the correlation pattern. However, as a result of the post-scheduling errors, the performance deteriorates significantly both in diversity as well as in outage gain; indeed, in some cases, the performance is inversely proportional to the number of ports.

Notation: $\mathbb{P}\{X\}$ and $\mathbb{E}\{X\}$ represent the probability and expectation of X, respectively; $\Gamma(\cdot)$, $\Gamma(\cdot,\cdot)$ and $\gamma(\cdot,\cdot)$ denote the complete, upper, and lower incomplete gamma function, respectively; $Q_1(\cdot,\cdot)$ denotes the Marcum-Q function of the first order; $J_n(\cdot)$ and $I_n(\cdot)$ are the Bessel function and the modified Bessel function, respectively, of the first kind and order n [11]; $\jmath=\sqrt{-1}$ is the imaginary unit; $\mathbb{1}_X$ is the indicator function of X with $\mathbb{1}_X = 1$, if X is true, and $\mathbb{1}_X = 0$, otherwise.

II. System Model

A. Network Model

Consider a simple point-to-point network, consisting of a conventional single-antenna transmitter and a single-FA receiver. The transmitter utilizes a fixed transmission power P. The receiver's FA is connected to a single RF chain and consists of N ports, evenly distributed over a dielectric holder of linear space, i.e., a uniform linear array [2], [6], as illustrated in Fig. 1.

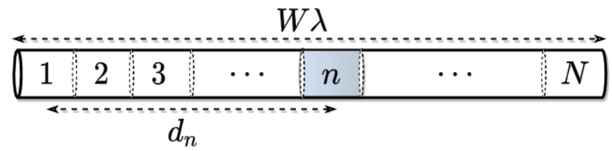

Fig. 1: The FA receiver with N ports.

The length of the FA is characterized by the value $W\lambda$, where λ is the wavelength of the transmitted signals and $W > 0$. We assume that the FA can switch on a single port by displacing the employed liquid to its location with a mechanical pump [4]. Therefore, the displacement (i.e. the Euclidean distance) between the n-th port and the first one can be written as [6]

$$d_n = \frac{n-1}{N-1} W\lambda. \tag{1}$$

Clearly, the displacement increases with n and decreases with the number of ports N. Moreover, all N displacements are distinct, i.e. $d_1 \neq d_2 \neq \cdots \neq d_N$ and the maximum displacement is unique and equal to d_N (see Fig. 1).

B. Channel Model

All wireless links are assumed to exhibit Rayleigh fading and we model the correlated fading channels at the N ports as follows [6]

$$g_1 = \sigma(x_1 + \jmath y_1), \tag{2}$$

$$g_n = \sigma\Big(\sqrt{1-\rho_n^2}x_n + \rho_n x_1 + \jmath\big(\sqrt{1-\rho_n^2}y_n + \rho_n y_1\big)\Big), \tag{3}$$

for $n = 2, \ldots, N$, $\sigma > 0$, where x_n and y_n are independent Gaussian random variables with zero mean and variance $1/2$, i.e. $x_n, y_n \sim \mathcal{N}(0, 1/2)$; throughout this paper, we will consider $\sigma = 1$ for simplicity but the generalization to any σ is straightforward.

Therefore, by having the first port as the reference point, the correlation coefficient between h_n and h_1 can be modeled by[1]

$$\rho_n = J_0\left(2\pi\frac{d_n}{\lambda}\right), \tag{4}$$

where d_n is defined above. Finally, we assume that the channel coefficients are known to the FA receiver, but not the transmitter.

III. FAs with Outdated Channel Estimates

In this section, we study the performance of the FA system with port selection. We assume that the FA employs a selection combiner and thus chooses the port with the strongest received signal, that is, the port with

$$h = \max\{h_1, h_2, \ldots, h_N\}, \tag{5}$$

where $h_i = |g_i|^2$. We consider the case where the estimated (pre-scheduling) channel at any port is subject to delays [13]. Since the estimation is done sequentially at each port, these delays correspond to the duration needed for the liquid to be displaced at each port and estimate the received channels. As such, by the time the port with the best estimate is scheduled (i.e. switched on), that estimation could be outdated. We first focus on the performance of the estimated channels and then the one of the outdated channels.

A. Estimated Channels

To facilitate the analysis, we let $\hat{h}_n$ denote the estimated received channel at the n-th port and $\hat{h} = \max\{\hat{h}_1, \hat{h}_2, \ldots, \hat{h}_N\}$. The theorem below provides the cumulative distribution function (CDF) of the estimated $\hat{h}$.

Theorem 1 ([6])**.** *The CDF of the estimated $\hat{h}$ is given by*

$$F_{\hat{h}}(x) = \int_0^x \exp(-z) \prod_{n=2}^{N} \phi_n(z, x) dz, \tag{6}$$

[1] Note that an alternative spatial correlation model was recently proposed in [12] which differs slightly from the one used in this paper. It would be an interesting future work to extend the work of this paper using the new model.

where

$$\phi_n(z,x) = 1 - Q_1\left(\sqrt{\frac{2z\rho_n^2}{1-\rho_n^2}}, \sqrt{\frac{2x}{1-\rho_n^2}}\right), \tag{7}$$

and ρ_n *is given by* (4).

It is important to point out that the performance of the FA improves as the correlation coefficients get smaller. To better show this behavior, we take an asymptotic approach and derive the achieved diversity order and outage gain [14]. Firstly, we provide a closed-form series representation of the CDF, to assist with the analysis. For the sake of convenience, we will denote

$$S \triangleq 1 + \sum_{n=2}^{N} \frac{\rho_n^2}{1-\rho_n^2}. \tag{8}$$

Proposition 1. *A series representation of the CDF of the estimated* $\hat{h}$ *can be written as*

$$F_{\hat{h}}(x) = \sum_{k=0}^{\infty} \frac{c_k}{S^{k+1}} \gamma\left(k+1, Sx\right), \tag{9}$$

where

$$c_k = \sum_{\substack{l_2,l_3,\ldots,l_N \geq 0 \\ l_2+l_3+\cdots+l_N = k}} \alpha_{2,l_2}\alpha_{3,l_3}\cdots\alpha_{N,l_N}, \tag{10}$$

and

$$\alpha_{n,l} = \left(\frac{\rho_n^2}{1-\rho_n^2}\right)^l \frac{\gamma\left(l+1, \frac{x}{1-\rho_n^2}\right)}{\Gamma(l+1)^2}. \tag{11}$$

Proof. See Appendix A. □

Obviously, for $\rho_n = 0$, $\forall n$, (9) reduces to the independent case as

$$F_{\hat{h}}(x) = \gamma\left(1, x\right)^N = (1 - \exp(-x))^N, \tag{12}$$

by keeping only the $k = 0$ term of the series and since $S = 1$. Now, the estimated signal-to-noise ratio (SNR) at the n-th port is given by

$$\hat{\eta}_n = \frac{P}{\nu^2}\hat{h}_n, \tag{13}$$

with average SNR $\bar{\eta} = \mathbb{E}[\eta_n] = P/\nu^2$, where ν^2 is the variance of the additive white Gaussian noise (AWGN). Let $\hat{\eta}$ be the largest estimated SNR, i.e. $\hat{\eta} = \max\{\hat{\eta}_1, \ldots, \hat{\eta}_n\}$. Then, the outage probability can be written as

$$P_{\mathrm{o}}(\theta) = \mathbb{P}\{\log_2(1+\hat{\eta}) < \theta\} = F_{\hat{h}}\left(\frac{1}{\bar{\eta}}(2^\theta - 1)\right), \tag{14}$$

where θ is a pre-defined rate threshold. Therefore, from Proposition 1, one can easily obtain an asymptotic expression for the outage probability.

Corollary 1. *For high SNR values, the outage probability of the estimated* $\hat{\eta}$ *is approximated by*

$$\lim_{\bar{\eta}\to\infty} P_{\mathrm{o}}(\theta) \approx \left(\frac{1}{\bar{\eta}}(2^\theta - 1)\right)^N \frac{1}{\prod_{n=2}^{N}(1-\rho_n^2)}. \tag{15}$$

Proof. See Appendix B. □

Based on Corollary 1, the diversity order of the considered FA system is $G_d = N$ and the outage gain is equal to

$$G_o = \frac{(2^\theta - 1)^N}{\prod_{n=2}^{N}(1-\rho_n^2)}. \tag{16}$$

It is well-known that the diversity defines the slope of the outage probability's curve and the outage gain represents the "distance" to the vertical axis [14], i.e., the smaller the value G_o, the better. From (16), we can observe that correlation negatively effects the outage gain. As such, the minimum outage gain is obtained by the independent case, that is, when $\rho_n = 0$, $\forall n$.

B. Outdated Channels

In what follows, we turn our attention to the outdated scenario. Thus, we will focus on the performance of h, given by (5), conditioned on the outdated estimation $\hat{h}$. In this case, the instantaneous outdated fading channels can be written as

$$g_1 = \sqrt{1-\mu_1^2}\,q_1 + \mu_1\hat{g}_1, \tag{17}$$

$$g_n = \sqrt{1-\mu_n^2}\,q_n + \mu_n\hat{g}_n, \tag{18}$$

where $\hat{g}_1 = \hat{x}_1 + \jmath\hat{y}_1$ and $\hat{g}_n = \sqrt{1-\rho_n^2}(\hat{x}_n + \jmath\hat{y}_n) + \rho_n\hat{g}_1$ are the estimated channels with $\hat{x}_n, \hat{y}_n \sim \mathcal{N}(0, \sigma^2/2)$ and $q_n \sim \mathcal{CN}(0, \sigma^2)$ for $n = 1, \ldots, N$; as before, we assume $\sigma = 1$. As such, the estimation error between $\hat{h}_n$ and h_n is captured by the correlation parameter μ_n modeled by [13]

$$\mu_n = J_0(2\pi f T_n), \tag{19}$$

where f is the Doppler frequency and T_n is the delay between the estimation and the activation of the n-th port. Obviously, the delay T_n is proportional to the time it needs to estimate the channel at a port [13], the size of the topology, the liquid's chemical properties but also the efficiency of the employed pump mechanism [4], [5]. In this work, we will focus on the first two even though the model could essentially capture all deficiencies. Let τ_e be the duration for estimating the channel at a port. Then, we assume that

$$T_n = \left(\frac{N-n+1}{N}\right)\tau_e W. \tag{20}$$

Now, let E_n denote the event that the n-th port has been activated. Then, the CDF of the outdated channel h is given by

$$F_h(x) = \sum_{n=1}^{N} \int_0^\infty F_{h_n|\hat{h}_n}(x|\hat{h}_n) f_{\hat{h}_n|E_n}(y|E_n)dy, \tag{21}$$

where $F_{h|\hat{h}}(\cdot|\hat{h})$ is the conditional CDF and $f_{\hat{h}_n|E_n}(\cdot|E_n)$ is the probability distribution function (PDF) of $\hat{h}$ conditioned on the n-th port being selected, given in the following proposition.

Proposition 2. *The conditional PDF of the estimated* $\hat{h}_n$ *is*

$$f_{\hat{h}_1|E_1}(x|E_1) = \exp(-x)\prod_{k=2}^{N}\phi_k(x,x)dz, \tag{22}$$

for $n=1$, and

$$f_{\hat{h}_n|E_n}(x|E_n)=\frac{1}{1-\rho_n^2}\int_0^x \exp\left(-\frac{x+z}{1-\rho_n^2}\right)I_0\left(\frac{2\sqrt{xz\rho_n^2}}{1-\rho_n^2}\right)\times\prod_{\substack{k=2\\k\neq n}}^{N}\phi_k(z,x)dz, \tag{23}$$

for $n=2,3,\ldots,N$.

Proof. See Appendix C. □

The result in Proposition 2 quantifies the effect of each port on the FA's overall performance. Indeed, one can easily evaluate the CDF of the estimated $\hat{h}$ by taking the sum of the conditional CDFs, that is,

$$F_{\hat{h}}(x)=\sum_{n=1}^{N}F_{\hat{h}_n|E_n}(x|E_n), \tag{24}$$

where $F_{\hat{h}_n|E_n}(x|E_n)$ are given in Appendix C; so (24) corresponds to Theorem 1, albeit in a more complex form. We can now state the final result.

Theorem 2. *The CDF of the outdated h can be written as*

$$F_h(x)=1-\sum_{n=1}^{N}\int_0^\infty Q_1\left(\sqrt{\frac{2y\mu_n^2}{1-\mu_n^2}},\sqrt{\frac{2x}{1-\mu_n^2}}\right)\times f_{y|E_n}(y|E_n)dy, \tag{25}$$

where $f_{y|E_n}(y|E_n)$ is given by Proposition 2.

Proof. By substituting the expressions of Proposition 2 in (21) and since $h_n\,|\,\hat{h}_n$ is a non-central chi-square random variable with 2 degrees of freedom and non-centrality parameter $2\hat{h}_n\mu_n^2/(1-\mu_n^2)$, the result follows. □

Note that the above expression is valid for $0\leq\mu_n<1$. If for a specific n we have $\mu_n=1$ (no delays), then the integral for the n-th term in (25) is reduced to (36) or (40), accordingly. Obviously, if $\mu_n=1$, $\forall n$, the performance is given by Theorem 1 (or by (24)). On the other hand, if $\mu_n=0$, $\forall n$, i.e. when the estimates are completely outdated (independent), $F_h(x)$ gives the performance of a randomly selected port, that is,

$$F_h(x)=1-\Gamma(1,x)=1-\exp(-x), \tag{26}$$

since $Q_1(0,z)=\Gamma(1,z^2/2)$. We should also remark that a closed-form series representation of Theorem 2 could be derived by following the same methodology as in Proposition 1, but we omit it for the sake of brevity. Nevertheless, we will provide a simplified asymptotic expression for high SNRs. Let η denote the outdated SNR, so the outage probability is $P_{\rm o}(\theta)=F_h\left(\frac{1}{\bar{\eta}}(2^\theta-1)\right)$.

Corollary 2. *For high SNR values, the outage probability for the outdated η simplifies to*

$$\lim_{\bar{\eta}\to\infty}P_{\rm o}(\theta)\approx\frac{(2^\theta-1)\Gamma(N)}{\bar{\eta}\prod_{n=2}^{N}(1-\rho_n^2)}\left(\frac{(1-\mu_1^2)^{N-1}}{(\mu_1^2+(1-\mu_1^2)S)^N}+\sum_{n=2}^{N}(1-\mu_n^2)^{N-1}\left(\frac{1-\rho_n^2}{1-\mu_n^2\rho_n^2}\right)^N\right). \tag{27}$$

Proof. See Appendix D. □

As expected, when the channels are outdated, the system is dispossessed of the channel diversity and the achieved diversity is reduced to one. Attaining full channel diversity would require $\mu_n\to 1$, which could be realized with extremely small FAs, i.e. $W\to 0$. Still, this may be impractical and so, in most cases, post-scheduling error due to delays is an inherent characteristic of FAs.

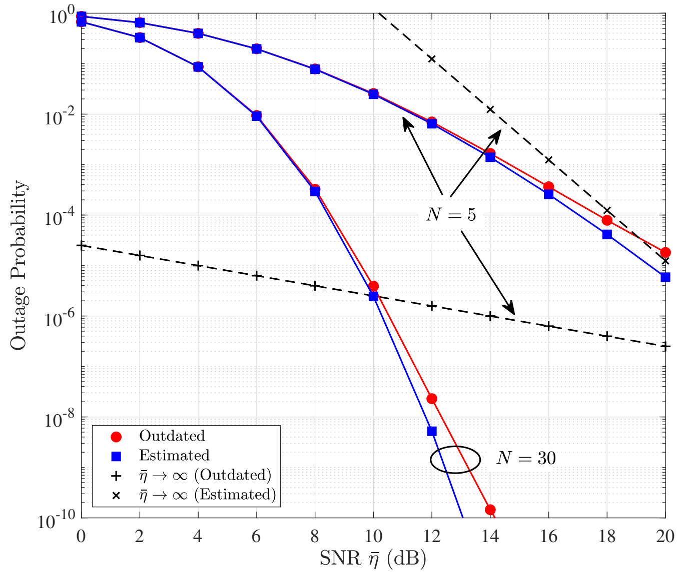

Fig. 2: Outage probability versus SNR for $W=0.2$.

IV. Numerical Results

We now validate our theoretical analysis with computer simulations. For the sake of presentation and, unless otherwise stated, we consider the following parameters: $\theta=2$ bps, $f=100$ Hz, and $\tau_e=1/(10f)$ s.

Fig. 2 illustrates the outage probability achieved by the considered FA system in terms of the SNR for $W=0.2$ and $N=5,30$. The performance of both estimated and outdated channels is depicted. It is obvious that at low SNR, both cases achieve the same performance. However, as the SNR increases, the performance of the outdated channels deteriorates. As shown in Corollary 1, FA realizes full channel diversity gains with the estimated channels, i.e., when there are no delays. On the other hand, in the outdated scenario, the channel diversity is lost and the achieved diversity becomes one. The change in diversity is obvious from the asymptotic curves ($\bar{\eta}\to\infty$) for $N=5$.

In Fig. 3, we look at the performance for a larger FA ($W=0.8$). Similar observations to Fig. 2 can be deduced. However, the negative effect of delays is more severe in this case, due to the FA's larger size. Indeed, the performance gap between the two scenarios exists even at low SNR. Moreover,

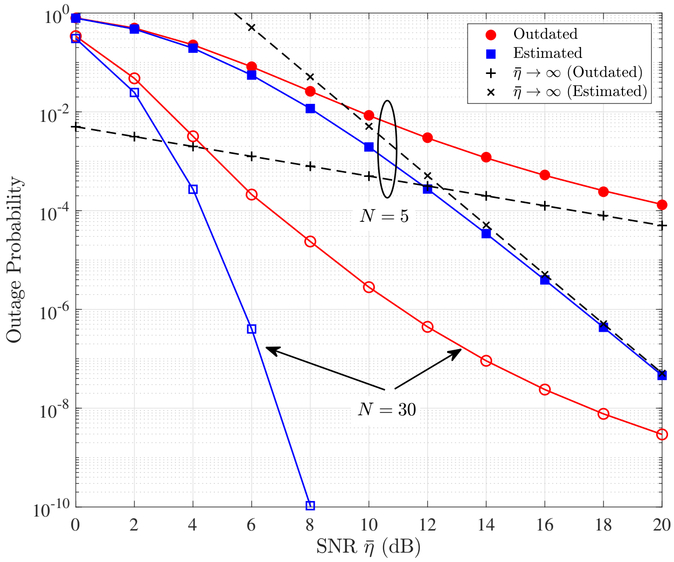

Fig. 3: Outage probability versus SNR for $W = 0.8$.

the "shift" towards the y-axis (i.e. the outage gain), differs due to the fact that it depends on the spatial correlation. Specifically, for a small FA size ($W = 0.2$), the spatial correlation is relatively larger, which in turn gives a larger outage gain (as can be evaluated by (16)). Finally, in both figures, our analytical expressions (lines) perfectly match the simulation results (markers), which validates our theoretical methodology.

V. Conclusions

In this paper, we focused on the outage probability and diversity of FA systems subject to scheduling delays. Analytical expressions were provided for the performance with and without post-scheduling errors. It was shown that despite FAs achieving maximum channel diversity, equal to the number of ports, this diversity is dispossessed due to scheduling delays. Our result exhibit the potentials of FA in communication systems but also their limitations. Future directions include the consideration of prediction methods in order to avoid scheduling delays but also other more intelligent port combining techniques.

Appendix

A. Proof of Proposition 1

In order to simplify (6), we use the following series representation of the Marcum-Q function

$$Q_1(a,b) = 1 - \exp\left(-\frac{a^2}{2}\right)\sum_{k=0}^{\infty}\frac{1}{k!}\frac{\gamma(k+1, b^2/2)}{\Gamma(k+1)}\left(\frac{a^2}{2}\right)^k. \tag{28}$$

Therefore, we can write

$$F_{\hat{h}}(x) = \int_0^x \exp(-z)\prod_{n=2}^{N}\exp\left(-\frac{z\rho_n^2}{1-\rho_n^2}\right) \times \sum_{k=0}^{\infty}\frac{1}{k!}\frac{\gamma\left(k+1, \frac{x}{1-\rho_n^2}\right)}{\Gamma(k+1)}\left(\frac{z\rho_n^2}{1-\rho_n^2}\right)^k dz$$
$$= \int_0^x \exp(-Sz)\prod_{n=2}^{N}\sum_{k=0}^{\infty}\frac{1}{k!}\frac{\gamma\left(k+1, \frac{x}{1-\rho_n^2}\right)}{\Gamma(k+1)} \times \left(\frac{z\rho_n^2}{1-\rho_n^2}\right)^k dz, \tag{29}$$

where S has been defined in (8). The above expression involves the Cauchy product of $N-1$ power series. Hence, it follows that

$$F_{\hat{h}}(x) = \int_0^x \exp(-Sz)\sum_{k=0}^{\infty}c_k z^k dz = \sum_{k=0}^{\infty}c_k\int_0^x \exp(-Sz)\, z^k dz, \tag{30}$$

where the coefficients c_k are given by (10). Finally, the proposition is proven by using the transformation $z \to S/t$ and the fact that $\int_0^b \exp(-t)t^{a-1}dt = \gamma(a,b)$ [11].

B. Proof of Corollary 1

By using (9), we can write the outage probability, defined in (14), as

$$P_{\rm o}(\theta) = \sum_{k=0}^{\infty}\frac{c_k}{S^{k+1}}\gamma\left(k+1, \frac{S}{\bar{\eta}}(2^\theta - 1)\right). \tag{31}$$

Thus, for $\bar{\eta} \to \infty$,

$$\lim_{\bar{\eta}\to\infty} P_{\rm o}(\theta) \to \sum_{k=0}^{\infty}\frac{c_k}{(k+1)S^{k+1}}\left(\frac{S}{\bar{\eta}}(2^\theta - 1)\right)^{k+1}, \tag{32}$$

which follows from the fact that $\gamma(a,b) \to b^a/a$ for $x \to 0$. In this case, the term $k=0$ dominates and so

$$\lim_{\bar{\eta}\to\infty} P_{\rm o}(\theta) \to \frac{c_0}{\bar{\eta}}(2^\theta - 1), \tag{33}$$

with

$$c_0 = \prod_{n=2}^{N}\frac{2^\theta - 1}{\bar{\eta}(1-\rho_n^2)}, \tag{34}$$

and the result follows after several algebraic manipulations.

C. Proof of Proposition 2

We start by deriving the conditional CDF $F_{\hat{h}_n|E_n}(x|E_n)$. Therefore, the CDF given that the first port is selected is given by

$$F_{\hat{h}_1|E_1}(x|E_1) = \mathbb{P}\{h_1 < x | h_1 > \max\{h_2, \dots, h_N\}\}$$
$$= \mathbb{P}\{h_1 < x | h_1 > h_2, \dots, h_1 > h_N\}. \tag{35}$$

By fixing h_1, the events above are independent and so we can write

$$F_{\hat{h}_1|E_1}(x|E_1) = \mathbb{E}_{h_1}\left\{\prod_{n=2}^{N}\mathbb{P}\{h_n < h_1 | h_1\}\,\middle|\, h_1 < x\right\}$$

$$= \mathbb{E}_{h_1}\left\{\prod_{n=2}^{N}\phi_n(h_1,h_1)\,\middle|\, h_1 < x\right\},$$

$$= \int_0^x \exp(-z)\prod_{k=2}^{N}\phi_k(z,z)dz, \tag{36}$$

which follows as h_1 is a central chi-square random variable with 2 degrees of freedom. Similarly, for the n-th port, $1 < n \le N$, we have

$$F_{\hat{h}_n|E_n}(x|E_n) = \mathbb{P}\{h_n < x|h_n > h_1,\ldots,h_n > h_{n-1}, \\ h_n > h_{n+1},\ldots,h_n > h_N\}. \tag{37}$$

In this case, conditioning on both h_1 and h_n, we get

$$F_{\hat{h}_n|E_n}(x|E_n) = \mathbb{E}_{h_1,h_n}\left\{\prod_{\substack{k=2\\k\neq n}}^{N}\phi_k(h_1,h_n)\,\middle|\, h_1 < h_n < x\right\} \\ = \int_0^x\int_0^y f_{h_1,h_n}(z,y)\prod_{\substack{k=2\\k\neq n}}^{N}\phi_k(z,y)dzdy, \tag{38}$$

where $f_{h_1,h_n}(\cdot,\cdot)$ is the joint PDF of h_1 and h_n, which can be obtained with Bayes's rule as

$$f_{h_1,h_n}(z,y) = f_{h_n|h_1}(z|y)f_{h_1}(y) \\ = \frac{1}{1-\rho_n^2}\exp\left(-\frac{y+z}{1-\rho_n^2}\right)I_0\left(\frac{2\sqrt{yz\rho_n^2}}{1-\rho_n^2}\right), \tag{39}$$

where $f_{h_n|h_1}(z|y)$ is the conditional PDF of a non-central chi-square random variable and $f_{h_1}(y)$ is the PDF of a central chi-square random, both of 2 degrees of freedom. Then, we can write

$$F_{\hat{h}_n|E_n}(x|E_n) = \frac{1}{1-\rho_n^2}\int_0^x\int_0^y\exp\left(-\frac{y+z}{1-\rho_n^2}\right) \\ \times I_0\left(\frac{2\sqrt{yz\rho_n^2}}{1-\rho_n^2}\right)\prod_{\substack{k=2\\k\neq n}}^{N}\phi_k(z,y)dzdy. \tag{40}$$

Finally, the proposition is proven by taking the derivative of the CDFs with respect to x.

D. Proof of Corollary 2

To assist with the simplification of the outage probability, we first apply the transformation $y \to t/\bar{\eta}$ to (25). In other words, we obtain the CDF in terms of SNRs. Therefore, we can simplify the PDF $f_{\hat{h}_n|E_n}(y|E_n)$, $n \neq 1$ (Eq. (23)), as follows

$$\lim_{\bar{\eta}\to\infty} f_{t|E_n}(t|E_n) \overset{(a)}{\approx} \frac{1}{\prod_{k=2}^{N}(1-\rho_k^2)}\exp\left(-\frac{t}{\bar{\eta}(1-\rho_n^2)}\right) \\ \times\left(\frac{t}{\bar{\eta}}\right)^{N-2}\int_0^{\frac{t}{\bar{\eta}}}\exp(-Sz)dz$$

$$\overset{(b)}{=} \frac{1}{\prod_{k=2}^{N}(1-\rho_k^2)}\exp\left(-\frac{t}{\bar{\eta}(1-\rho_n^2)}\right) \\ \times\left(\frac{t}{\bar{\eta}}\right)^{N-2}\frac{1}{S}\gamma\left(1,\frac{S}{\bar{\eta}}t\right), \tag{41}$$

where (a) follows by using (28) and keeping the sum's first term as well as from the fact $I_0(x) \approx (x/2)^2$ for $x \approx 0$; (b) follows from $\int_0^b \exp(-t)t^{a-1}dt = \gamma(a,b)$ [11]. Finally, as $\gamma(a,b) \approx b^a/a$ for $x \approx 0$, we end up with

$$\lim_{\bar{\eta}\to\infty} f_{t|E_n}(t|E_n) \approx \frac{1}{\prod_{k=2}^{N}(1-\rho_k^2)}\exp\left(-\frac{t}{\bar{\eta}(1-\rho_n^2)}\right) \\ \times\left(\frac{t}{\bar{\eta}}\right)^{N-1}. \tag{42}$$

The approximation for $n = 1$ (Eq. (22)) can be derived in a similar manner. Then, by approximating the Marcum-Q function in (25) as before and substituting the PDFs, the final expression follows after several algebraic operations.

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