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Doctoral Thesis

Finite Representations in Relation Algebra

Jas Semrl
Primary Supervisor: Robin Hirsch
Secondary Supervisor: James Brotherston

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This work was submitted in fulfilment of the requirements for the degree of Doctor of Philosophy.

I, Jas Semrl, confirm that the work presented in my thesis is my own. Where information has been derived from other sources, I confirm that this has been indicated in the thesis.

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Abstract

Binary relations provide a great abstraction for a number of concepts, both in theoretical and applied topics. This is why structures of binary relations have found applications in formal verification, temporal and spatial reasoning in AI, regular language equivalence, sequent calculi, and more. In general, a finite relation algebra cannot be finitely represented. This negatively impacts the feasibility of implementing any of the aforementioned applications based on these structures. Our work focuses on finding large classes of relational structures for which the finite representation property either holds or fails. Furthermore, we examine related topics such as the decidability of the [finite] representation problem and finite axiomatisability. Moreover, we examine the relationship between these properties. We refine Hirsch's conjecture on which relation algebra reduct signatures have the finite representation property and prove the negative implication of it. Furthermore, we provide a number of results that reveal a possible direction for proving the positive side. We present the first known signature that fails to have a finitely axiomatisable representation class but has the finite representation property. We generalise the results for the undecidability of the representation decision problem and show that semigroups with the Heyting implication fail to have the said problem decidable. We prove and disprove a number of properties for the structures of binary relations with combined operators, motivated by various topics in computer science. Finally, we show a number of results in the area of weakening relation algebras and show the finite weakening representation property for some signatures with their finite representation property open.

Impact Statement

This chapter outlines the impact of the work presented — both academic and non-academic. While the work is highly theoretical in nature and its impact may appear primarily academic, we demonstrate in this section that this is not the case. The results and techniques developed during our investigations have already shown the potential to have a measurable real-world impact in the future. This is in addition to all the potential future investigations into the potential applications of our findings.

We begin with the assessment of the academic impact of the thesis. The work presented has resulted in five publications in peer-reviewed journals and conference proceedings, including a top-ranked conference in theoretical computer science. Furthermore, the results have been communicated through a number of research talks given, some of which resulted in collaboration with other researchers beyond UCL and the UK. Besides the results published, a number of manuscripts are at various stages of preparation for submission. Furthermore, future ideas for collaborations with experts on the topics are being discussed.

While the research has been well-received within the relation-algebraic community, it is important to mention that it has been communicated with the larger scientific community and the lay public at various poster sessions and talks. This extended the outreach of the research to a larger audience, including the wider academia and the public.

Finally, we assess the potential applications of this research in the future. While we acknowledge that the research is not applied in its current form, we show that it has potential applications, for example in Chapter 4 where we show that relation algebra can be used to reason about program correctness. In addition, in our currently ongoing investigations, we research the behaviour of relational structures with applications in natural language processing. While these future applications are not yet at the implementation stage, we lay the theoretical foundation for their future development.

In conclusion, the research presented in this document has, over the duration of the studentship, amounted to a strong publication record and laid theoretical foundations for future applications in artificial intelligence and verification.

Contents

Chapter 1

Introduction

Binary relations are a great abstract tool for reasoning about various concepts. This is why relation algebra has found applications in various areas of computer science, ranging from theoretical topics like formal verification [MDM87] and regular language equivalence [KN12], to applied topics like spatial reasoning in artificial intelligence [Dün05].

Given a base set, we define a relation as a collection of ordered pairs of elements in the base. For example, we take the base to be the users of some social network and define the relation "follows" as the set of pairs of users where the first person follows the second person in the pair. Note that the term ordered pair is important as, for example, a voter may follow their local MP, but the MP may not follow all of their voters.

Observe how in our social network example a user may follow no people, one person or many people. This is the key distinction between relations and functions. If "follows" was a function, each user would have to follow exactly one person. This generalisation is why relations can be used to reason about certain concepts more broadly than functions. In fact, functions are just special cases of relations.

We may define a number of operations for relations, for example, converse and composition. Intuitively, a converse of "follows" would be the "is followed by" relation. Or, continuing with the social network analogy, the composition of the "friend" relation with the "follows" relation would result in the "friend follows" relation, a useful relation when the social network would like to suggest its users more people to follow. Furthermore, we can define predicates for relations. For example, we can define inclusion, e.g. the "friends" relation is included in the "follows" relation.

Given a collection of constants, predicates and operations, we can define a structure with binary relations over some base set as its elements. This structure must be closed under the given operations and include the definition of predicates and constants. Such a structure is called a proper or a concrete structure. In contrast, if the elements of a structure are not binary relations, we call it an abstract structure. This structure is representable if and only if there exists a one-to-one mapping to a concrete structure of binary relations over some base set. The mapping is called a representation and the representation is finite if the concrete structure consists of binary relations over a finite set.

Our research focuses on the nature of finite representability. This is a neat property as it provides two very useful guarantees. First, it implies the decidability of the representation decision problem, as well as a number of other relevant decision problems. Second, it often explicitly defines a tool for easily switching between reasoning about the abstract to the concrete.

The guarantee of finite representability may come in different flavours. For example, the finite representation property for finite structures fails for the full relation algebra signature, but some less expressive languages do have it. Furthermore, there may exist a property (like the conjectured inclusion of a flexible atom) that guarantees finite representability. In either case, the goal is to outline a broad class of finitely representable structures with easily verifiable membership conditions.

Currently, the results in the literature answer these questions case by case. This may include showing a signature has got the finite representation property for finite members of its representation class. Or perhaps a result presented may determine whether finite representability is possible for a finite list of relation algebra algebras.

These are all useful results, however, they are sparse and very specific. Our goal is to expand on that knowledge base and provide new and more general answers towards finite representability guarantees. In the long term, we aspire to provide neat properties of relation algebras or signatures that guarantee finite representability. These would have to be easily checkable (like the inclusion of a flexible atom in the algebra or the inclusion of operations in the signature) and broad enough to cover a significant portion of structures.

1.1 Preliminaries

In this section, we familiarise the reader with relation algebras. We define the concepts, touched on in the introduction, in a more rigorous way. These definitions will lead to the problem statement. We start by defining binary relations and proper relation algebras.

Definition 1. *Let X be a set. We define a* binary relation *$R \subseteq X \times X$ as a set of ordered pairs with both elements from X. We call X the* base set*, or simply the base of R.*

We continue by defining some relevant concepts from model theory and universal algebra. We assume, however, that the reader is familiar with first-order logic.

Definition 2. *A* signature *τ is a tuple of predicates and operations that define a first-order language.*

Remark 3. *In this document, we consider first order logic to include equality, that is, there exists a logical symbol = that is always interpreted as the identity predicate. We do not include = in τ.*

Definition 4. *A* τ-structure *is a tuple $\mathcal{A} = (A, \langle \sigma^{\mathcal{A}} \mid \sigma \in \tau \rangle)$ where A is a non-empty* carrier *or* underlying set *and $\sigma^{\mathcal{A}}$ is the interpretation of the symbol over A. In this document, we abbreviate $(A, \langle \sigma^{\mathcal{A}} \mid \sigma \in \tau \rangle)$ to (A, τ).*

Remark 5. *If τ consists of operations only, then $\mathcal{A} = (A, \tau)$ is an* algebra.

Definition 6. *Let K be a class of τ'-structures and let $\tau \subseteq \tau'$. We say an n-ary predicate p is* atom-definable *in K from τ if and only if there exists a atomic τ-formula ϕ with all variables in $\{a_1, a_2, \ldots, a_n\}$ such that $\mathsf{K} \models p(a_1, a_2, \ldots, a_n) \Leftrightarrow \phi$. Similarly, an n-ary operation f is* term-definable *in K from τ if and only if there exists a τ term t with all variables in $\{a_1, a_2, \ldots, a_n\}$ such that $\mathsf{K} \models f(a_1, a_2, \ldots, a_n) = t$.*

Remark 7. *In this document, we often omit the class in which a predicate is atom-definable or an operation is term-definable. In those cases, the class is assumed to be* RRA*, defined later in this section.*

Definition 8. *We say that τ is an* expansion *of τ' — or equivalently τ' is a* reduction *of τ — denoted $\tau' \subseteq \tau$ if and only if every predicate and operation in τ' is either in τ or is term/atom-definable from τ.*

Definition 9. *A τ-structure $\mathcal{A}' = (A', \tau)$ is a* substructure *of $\mathcal{A} = (A, \tau)$, denoted $\mathcal{A}' \subseteq \mathcal{A}$, if $A' \subseteq A$ and the interpretation of all operations for $\mathcal{A}'$ is the A'-restriction of the interpretation for the said operation in $\mathcal{A}$. Conversely, $\mathcal{A}$ is said to be an* extension *of $\mathcal{A}'$.*

Definition 10. *Let $\tau' \subseteq \tau$. We say that $\mathcal{A}' = (A', \tau')$ is a* τ'-reduct *of $\mathcal{A} = (A, \tau)$ if $A' = A$ and the interpretation of all operations in τ' is the same in both $\mathcal{A}'$ and $\mathcal{A}$.*

Definition 11. *Let $\tau' \subseteq \tau$. We say that $\mathcal{A}' = (A', \tau')$ is a* τ'-subreduct *of $\mathcal{A} = (A, \tau)$ if there exists an $\mathcal{A}'' \supseteq \mathcal{A}'$ such that $\mathcal{A}''$ is a τ'-reduct of $\mathcal{A}$.*

We now define the class of proper relation algebras.

Definition 12. *The class of* proper relation algebras (PRA) *is the class of all structures $\mathcal{A} = (A, 1, -, +, 1', \smile, ;)$ where there exists some base set X, some binary relation $\top \subseteq X \times X$ such that $A \subseteq \wp(\top)$ and for all $R, S \in A$*

(1) top is interpreted as $1 = \top$,

(2) join is interpreted as $R + S = R \cup S$,

(3) negation is interpreted as $-R = \top \backslash R$,

(4) identity is interpreted as $1' = \{(x, x) \mid x \in X\}$,

(5) converse is interpreted as $\breve{R} = \{(y, x) \mid (x, y) \in R\}$,

(6) composition is interpreted as $R;S = \{(x, y) \mid \exists z : ((x, z) \in R \wedge (z, y) \in S)\}$.

Remark 13. *The interpretation of* 1 *will be an equivalence relation. This is because the structure is closed under composition, converse, and identity.*

Remark 14. *One may also interpret the top as the Cartesian product of the base with itself. We call* $1 = X \times X$ *the* square *top. Similarly, we define the* square *negation of a binary relation* R *over* X *as* $-R = (X \times X)\backslash R$.

A number of operations, constants, and predicates, constructed from those defined for proper relation algebras, are also of interest to the research community. For $R, S \in A$, some example constructed operations include

(1) order: $R \leq S \iff R + S = S$,

(2) meet: $R \cdot S = -((-R) + (-S))$,

(3) domain: $\mathrm{D}(R) = 1' \cdot (R;\breve{R})$,

(4) range: $\mathrm{R}(R) = 1' \cdot (\breve{R};R)$,

(5) bottom: $0 = -1$,

(6) diversity: $0' = -1'$,

(7) left residual: $R\backslash S = -(\breve{R};-S)$,

(8) right residual: $R/S = -(-R;\breve{S})$.

These operations are defined here as terms in the language of relation algebra, but they also have equivalent set-theoretic definitions

(1) order: $R \leq S \iff R \subseteq S$,

(2) meet: $R \cdot S = R \cap S$,

(3) domain: $\mathrm{D}(R) = \{(x, x) \mid \exists y : (x, y) \in R\}$,

(4) range: $\mathrm{R}(R) = \{(x, x) \mid \exists y : (y, x) \in R\}$,

(5) bottom: $0 = \emptyset$,

(6) diversity: $0' = \{(x, y) \in \top \mid x \neq y\}$,

(7) left residual: $R\backslash S = \{(x, y) \mid \forall z : ((z, x) \in R \Rightarrow (z, y) \in S)\}$,

(8) right residual: $R/S = \{(x, y) \mid \forall z : ((y, z) \in S \Rightarrow (x, z) \in R)\}$.

The lists above are not exhaustive. Furthermore, some non-term-definable operations, like the Kleene star, were of interest in the literature, however, they are not considered here. We denote the signature of proper relation algebras τ_{RA}. We can now abstract away from proper relation algebras and define representable relation algebras.

Definition 15. *A* representation *is an isomorphism* $h : \mathcal{A} \to \mathcal{A}'$ *where* $\mathcal{A}$ *is a* τ_{RA}*-structure and* $\mathcal{A}' \in \mathsf{PRA}$*. If the carrier of* $\mathcal{A}'$ *consists of relations over a finite base, then the representation is* finite*. The* class of representable relation algebras *is denoted* RRA*.*

In a similar fashion to the classes of proper and representable relation algebras, we can define similar classes for the reductions of τ_{RA}.

Definition 16. *Let* $\tau \subseteq \tau_{\mathsf{RA}}$*. The* class of proper τ-structures *is the class* $\mathsf{P}(\tau)$ *of all structures* $\mathcal{A} = (A, \tau)$ *where there exists some* $\top \subseteq X \times X$ *and* $A \subseteq \wp(\top)$ *and all members of* τ *are interpreted as described in Definition 12. The closure of* $\mathsf{P}(\tau)$ *under isomorphic copies is called the* class of representable τ-structures $\mathsf{R}(\tau)$*. Again, an isomorphism mapping a structure* $\mathcal{A} \in \mathsf{R}(\tau)$ *to* $\mathcal{A}' \in \mathsf{P}(\tau)$ *is called a* representation*.*

Remark 17. *In most cases, respectively, the classes* $\mathsf{P}(\tau)$*,* $\mathsf{R}(\tau)$ *can also be defined as the closure of classes* PRA*,* RRA *under subreducts. However, this is not always the case. Take, for example, the structure* $\mathcal{A} = (\{0,1\}, 0, 1, ;)$ *where all compositions result in* 0*. The structure is clearly representable via* h *over* $\{x, y\}$ *where* $h(0) = \emptyset, h(1) = \{(x, y)\}$*. However, there exists no nontrivial relation algebra where* $1;1 = 0$*.*

Although the class of representable relation algebras is *elementary*, i.e. it can be characterised by first-order formulas, it is not finitely axiomatisable. A finite number of axioms was presented by Tarski in [Tar41] that characterise a class of structures by finitely many first-order formulas, all sound for representable relation algebras.

Definition 18. *A* relation algebra *is a structure* $\mathcal{A} = (A, 1, -, +, 1', \smile, ;)$ *that for all* $s, t, u \in A$

(B1) $s + s = s$,

(B2) $s + t = t + s$,

(B3) $s + (t + u) = (s + t) + u$,

(B4) $s + (-s) = 1$,

(B5) $- - s = s$

(B6) $s + 0 = s$,

(B7) $s \cdot (t + u) = (s \cdot t) + (s \cdot u)$,

(R1) $s;1' = s$,

(R2) $\breve{\breve{s}} = s$,

(R3) $\overbrace{(s + t)} = \breve{s} + \breve{t}$,

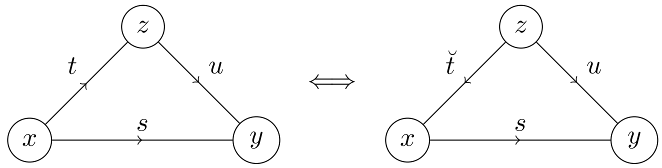

Figure 1.1: The soundness of the triangle equivalence for relation algebras.

(R4) $s;(t+u) = s;t + s;u$,

(R5) $\smile{(s;t)} = \breve{t};\breve{s}$,

(R6) $s;(t;u) = (s;t);u$,

(R7) $\breve{s};-(s;t) \leq -t$,

where the meet and the ordering are term/atom-definable in the signature as described earlier. The class of relation algebras is denoted RA*.*

The axioms (B1)–(B7) ensure that the $(1, -, +)$-reduct of a relation algebra is a Boolean algebra. The remaining axioms (R1)–(R7) are relation algebra specific. The reader can check that the axioms (R1)–(R6) are sound, but we examine (R7) more closely. Observe that all axioms stated are equations. A more natural restatement of (R7) is the two-way implication

$$\text{(R8a)} \quad s \cdot (t;u) = 0 \iff u \cdot (\breve{t};s) = 0.$$

Proposition 19. *The theories (B1)–(R7) and (B1)–(R6),(R8a) axiomatise the same elementary classes.*

Proof. To check that (B1)–(R7) suffice to conclude (R8a), observe that $s \cdot (t;u) = 0$ implies $s \leq -(t;u)$ and thus $\breve{t};s \leq \breve{t};-(t;u) \leq -u$ and we conclude $u \leq -\breve{t};s$, or equivalently $u \cdot (\breve{t};s) = 0$. By a similar argument, we prove the right-to-left implication, but we must use the equation $-(s;t);\breve{t} \leq -s$, which we can prove from (R7),(R5). To show one can prove (R7) from (B1)–(R6),(R8a), see that $-(s;t) \cdot (s;t) = 0$, so by (R8a) $(\breve{s};-(s;t)) \cdot t = 0$, or equivalently, $\breve{s};-(s;t) \leq -t$. ☐

Proposition 20. *The axiom (R8a) is sound for representable relation algebras.*

Proof. If an algebra is representable, say via some representation h over X then $s \cdot (t;u) \neq 0$ if and only if there exist $x, y \in X$ such that $(x, y) \in h(s \cdot (t;u))$. Because h represents $\cdot, ;$ correctly, there exists $z \in X$ such that $(x, z) \in h(t), (z, y) \in h(u)$, and $(x, y) \in h(s)$ — see Figure 1.1. Therefore, $(z, x) \in h(\breve{t})$ as h represents $\smile$ correctly and we can see that there exists $z, y \in X$ such that $(z, y) \in h(u \cdot (\breve{t};s))$. We conclude that $u \cdot (\breve{t};s) \neq 0$. By a similar argument, we show the other implication required. ☐

As the $(-, +, 1)$-reduct of any relation algebra is a Boolean algebra, it will be the case that any finite relation algebra will have atoms.

Definition 21. *Let* $\mathcal{A} = (A, 1, -, +, 1', \smile, ;)$ *be a relation algebra. An element* $a \in A$ *is an* atom *if and only if for all* $s \in A$ *if* $s \leq a$ *then either* $s = a$ *or* $s = 0$ *(but not both).*

Any finite Boolean algebra is uniquely defined up to an isomorphism by its atoms. Furthermore, while not every infinite Boolean algebra is atomic, it is the case that every infinite Boolean algebra embeds into an atomic one. We now look at how one can present a relation algebra in terms of its atom structure [Lyn50].

Definition 22. *An* atom structure *is a structure* $\mathcal{A} = (A, \mathrm{I}, \smile, \mathrm{T})$ *where* A *is the carrier set of atoms,* I *is a unary predicate,* $\smile$ *is a unary operation, and* T *is a ternary predicate such that for all* a, b, c, d, e

(T1) $\breve{\breve{a}} = a$,

(T2) $\mathrm{T}(a, b, c) \Leftrightarrow \mathrm{T}(\breve{a}, \breve{c}, \breve{b})$,

(T3) $\mathrm{T}(a, b, c) \Leftrightarrow \mathrm{T}(b, a, \breve{c})$,

(T4) $(\mathrm{I}(a) \wedge \mathrm{T}(a, b, c)) \Rightarrow b = \breve{c}$,

(T5) $\mathrm{I}(a) \Rightarrow \breve{a} = a$,

(T6) $\exists! f : (\mathrm{I}(f) \wedge \mathrm{T}(f, a, \breve{a}))$,

(T7) $(\mathrm{T}(a, b, c) \wedge \mathrm{T}(a, d, e)) \Rightarrow \exists f : (\mathrm{T}(d, b, f) \wedge \mathrm{T}(c, f, e))$.

We say that an atomic relation algebra $\mathcal{A} = (A, 1, -, +, 1', \smile, ;)$ has a *corresponding atom structure* $\mathcal{A}' = (\mathbf{At}(A), \mathrm{I}, \smile\restriction_{\mathbf{At}(A)}, \mathrm{T})$ where $\mathbf{At}(A) = \{a \in A \setminus \{0\} \mid \forall s \in A \setminus \{0\} : (s \leq a \Rightarrow s = a)\}$ is the set of atoms, I is the set of sub-1' atoms, $\smile\restriction_{\mathbf{At}(A)}$ is interpreted as the $\mathbf{At}(A)$-restriction of $\smile$ and for all atoms a, b, c we have $\mathrm{T}(a, b, c)$ if and only if $a \leq b;c$. If $\mathcal{A}$ is finite, it is isomorphic to the complex algebra over $\mathcal{A}'$.

The reader can check that every atom structure has a unique complex relation algebra. Namely, from (T1),(T2) we get the converse laws, (T3) ensures the triangle law, (T4),(T6) entail the identity, and (T7) ensures associativity. Each triangle (a, b, c) included in T implies five additional inclusions by (T2)(T3). All these triangles are *Peircean transforms* of each other. Finally, we state without proof that all finite relation algebras have a unique corresponding atom structure and that any relation algebra (if not atomic itself) embeds into its *canonical extension*, which is atomic [JT51]. Furthermore, a relation algebra is in RRA if and only if its canonical extension is in RRA. The latter is an unpublished result by Monk, first announced in [McK66] and proved in [Mad83].

Finally, we generalise these abstract classes of relation algebras in terms of n-dimensional bases. These were introduced in [Mad83] for $3 \leq n \leq \omega$. We start by defining *involutive unital Boolean magmas* as the class of all $(1, -, +, 1', \smile, ;)$-structures that obey (B1)–(B7) as well as (R1)–(R4). We acknowledge that these structures are Boolean algebras and thus are either atomic or have an atomic canonical extension.

Definition 23. *An atomic involutive unital Boolean magma $\mathcal{A} = (A, 1, -, +, 1', \smile, ;)$ has an n-dimensional basis for some $3 \leq n \leq \omega$ if and only if there exists such an $M \subseteq (\mathbf{At}(A))^{n \times n}$ such that:*

(1) for all $a \in \mathbf{At}(a)$, there exists $m \in M$ such that $m_{01} = a$,

(2) for all $m \in M$, $i, j, k < n$ we have $m_{ii} \leq 1'$, $m_{ij} = \breve{m_{ji}}$, and $m_{ij} \leq m_{ik};m_{kj}$,

(3) for all $m \in M, \forall i, j, k < n$, and $a, b \in \mathbf{At}(A)$ if $m_{ij} \leq a;b$ and $i \neq k \neq j$ then there exists such an $m' \in M$ that $m'_{ik} = a, m'_{kj} = b$ and for all $i' \neq k \neq j'$, $m_{i'j'} = m'_{i'j'}$.

We call M a basis *for $\mathcal{A}$.*

The notion of an n-dimensional basis enables us to define for $3 \leq n \leq \omega$ the class RA_n as follows.

Definition 24. *Let $3 \leq n \leq \omega$. The class of* relation algebras with an n-dimensional basis RA_n *is the class of all atomic involutive Boolean magmas that have an n-dimensional basis as well as the non-atomic involutive Boolean magmas whose canonical extension has an n-dimensional basis.*

Without proof, we state the following facts. These classes form a proper chain of varieties. The class RA_3 is axiomatised by the axioms for RA, except associativity, which is replaced by semi-associativity. The class of semi-associative relation algebras is denoted SA [Mad83]. The classes $\mathsf{RA}_4, \mathsf{RA}_\omega$ are equal to RA and RRA respectively [Mad83]. For $5 \leq n \leq \omega$, RA_n is countably, but not finitely axiomatisable [HH00].

$$\begin{array}{ccccccc} \mathsf{SA} & & \mathsf{RA} & & & & \mathsf{RRA} \\ \parallel & & \parallel & & & & \parallel \\ \mathsf{RA}_3 & \supset & \mathsf{RA}_4 & \supset & \dots & \supset & \mathsf{RA}_\omega \end{array}$$

1.2 Problems

The class of representable relation algebras is badly behaved. Therefore, our search for good behaviour continues by looking at subclasses, reduct classes, subreduct classes of RRA, and more. In doing so, we sacrifice some generality in the ability to reason about binary relations, but we gain nicer behaviour. This leads to a few sets of problems to be answered.

These problems, as a whole, revolve around finite representability, however, we consider a few closely related properties as well. We begin by defining the finite representation property.

Definition 25. *Let $\tau \subseteq \tau_{\mathsf{RA}}$. The class $\mathsf{R}(\tau)$ has the* finite representation property *if and only if all finite structures in $\mathsf{R}(\tau)$ have a finite representation.*

So far, determining the finite representation property (FRP) has been largely done on a case-by-case basis. We tackle the problem from a more general perspective.

Problem 26. *Given a Relation Algebra reduct signature τ, does there exist an easily verifiable condition that is both necessary and sufficient for $\mathsf{R}(\tau)$ to have Finite Representation Property?*

The initial conjecture was set by Hirsch, stating that $\mathsf{R}(\tau)$ has the finite representation property if and only if $(\cdot, ;) \not\subseteq \tau \subseteq \tau_{\mathsf{RA}}$. However, recent results from the literature have shown a counterexample as $R(\leq, -, ;)$ has no finite representation property [Neu16], if we require that the representation is square. We expand on this result in later sections. Furthermore, in light of some further consideration of individual signatures and properties, we refine the hypothesis. We begin by stating it for the signatures without some term-definable operators. Namely, we say that τ in the following conjecture can consist of any operations or predicates we have defined thus far, except residuals.

Conjecture 27. $\mathsf{R}(\tau)$ *will have the finite representation property if and only if* $(-, ;) \not\subseteq \tau \not\supseteq (\cdot, ;)$.

When it comes to the signatures conjectured to have the finite representation property above, we only consider signatures with term-definable operations that are monotone with respect to $\subseteq$. This immediately excludes operations like residuals, which on their own are not enough to define negation, but in conjunction with 0', $\smile$, it is possible to define $-a = \breve{a}\backslash 0'$.

A neat consequence of the finite representation property is the decidability of the representation problem, defined below. We show its decidability is a consequence of the finite representation property in later sections.

Problem 28. *Let $\tau \subseteq \tau_{\mathsf{RA}}$. Does there exist an algorithm that takes a finite τ-structure $\mathcal{A}$ as input and determines in a finite amount of time whether or not $\mathcal{A} \in \mathsf{R}(\tau)$?*

Another guarantee of decidability is the finite axiomatisability (FA) of the representation class. Furthermore, first-order formula checking is in $LOGSPACE$, which is a much better computational guarantee than the mere decidability we get from FRP. We will see later that FA and FRP are independent of each other.

Other questions for relation algebra reducts include whether or not the representation class is a variety, whether or not the equational theory of the class is finitely axiomatisable, and whether or not the finite representability decision problem is decidable.

Another set of problems is regarding the subclasses of RRA. One of the big focuses here is finding easily verifiable properties for relation algebras that provide the guarantee of finite representability. One of the major open problems concerns the class of relation algebras with a flexible atom.

Definition 29. *A relation algebra $\mathcal{A} = (A, 1, -, +, 1', \smile, ;)$ is* integral *if* $1' \in \mathbf{At}(A)$.

Definition 30. *Let $\mathcal{A} = (A, 1, -, +, 1', \smile, ;)$ be an integral relation algebra. An atom $a \in \mathbf{At}(A)$ is flexible if and only if for all $b, c \in \mathbf{At}(A) \setminus \{1'\}$ we have $a \leq b;c$.*

Problem 31. *Does every finite integral relation algebra with a flexible atom have a finite representation?*

It has been shown that an algebra with a flexible atom is always representable, however, the proofs do not yield finite representations. The generally accepted conjecture [Mad94] is stated below.

Conjecture 32 (Maddux)**.** *Any finite integral relation algebra with a flexible atom is representable over a finite base.*

We acknowledge that solving either of the two problems stated in this section is a large undertaking. Finite representability (or the failure of it) is not known for some four-atom relation algebras, as well as many algebras with five atoms. In other words, proving the finite representability of a single algebra is not a trivial task. The difficulty of proving finite representability for large classes of relational structures is therefore even greater. Therefore, we rely on case-by-case analysis to examine some of the aspects of both conjectures.

All of the problems raised above can be posed for different variations of relational calculus, motivated by various concepts in computer science. These include structures of binary relations with demonic operators, motivated by the behaviour of nondeterministic Turing machines where the demon is in control of the nondeterminism. Another example is the weakening relation algebras, motivated by the weakening rule in sequent calculi and Hoare logic.

1.3 Result Summary

In this section, we summarise the theorems proven in this thesis and outline the structure of the remainder of the document. As suggested in the problem statement, the thesis is largely concerned with Conjecture 27 and related topics like finite axiomatisability. We present the results, discuss their dependencies and significance, and contextualise them with regards to this main conjecture. We present this narrative of the thesis chapter-by-chapter.

We begin with Chapter 2, where we present a literature review. This consists of stating and proving the theorems we use in later chapters as well as surveying the known results in the area of finite representability. Specifically, we outline how some special cases of Conjecture 27 have been shown. This develops a toolbox for further novel results presented in the thesis.

Chapter 3 is the first results chapter. It begins with showing the below theorem. This, together with the results from the literature review chapter, proves one side of the two-way implication in Conjecture 27. In addition, we show that the class $\mathsf{R}(-, ;)$ is not finitely axiomatisable, a closely related but (as we show later) completely independent property. We additionally show an interesting connection between these problems and Conjecture 32.

Theorem 80. *Take a signature τ such that $(-, ;) \subseteq \tau \subseteq \tau_{\mathsf{RA}}$. The finite representation property fails for finite structures in* $\mathsf{R}(\tau)$*.*

Theorem 87. *The class* $\mathsf{R}(-, ;)$ *is not finitely axiomatisable.*

The above results are novel and the result of sole authorship, however, we hope to prove further results before these are submitted for publication. We continue the chapter by showing some undecidability results, resulting from joint work with Andrew Lewis-Smith and published in the proceedings of RAMICS 2023 [LSŠ23]. The arrow denotes the Heyting implication, see Section 3.3 for more detail. While the $(-, ;)$ case of Conjecture 27 is answered by this, the decidability of membership in $\mathsf{R}(-, ;)$ remains open for finite structures. However, the following related results are proved in the chapter.

Theorem 94. *The [finite] representation problem is undecidable for any signature* $(\rightarrow, ;) \subseteq \tau \subseteq \tau_{\mathsf{RA}} \setminus (\smile)$*.*

Here we use the square brackets to indicate both the representation and the finite representation problem are undecidable. We will use square brackets in a similar fashion throughout this document.

Theorem 99. *The [finite] square-representation problem is undecidable for any signature* $(\rightarrow, ;) \subseteq \tau \subseteq \tau_{\mathsf{RA}} \setminus (\smile)$*.*

Before we conclude the chapter, we present a discussion with speculations about the positive side (the remaining open part) of Conjecture 27 and speculate on the steps required to prove it.

In Chapter 4, we examine some classes of structures that became of interest more recently. We show a number of results about signatures where demonic lattice operations and predicates are mixed with ordinary relational composition. This is the result of joint work with Robin Hirsch, published in Algebra Universalis [HŠ21b] and the proceedings of LICS 2021 [HŠ21a]. We begin by showing the computer science motivation for studying these algebras, followed by proving the following theorems.

Theorem 110. $\mathsf{R}(\sqsubseteq, ;)$ *is axiomatised by partial order, associativity and* $\{\sigma_n \mid n < \omega\}$*. Finite structures* $\mathcal{A} = (A, \sqsubseteq, ;) \in \mathsf{R}(\sqsubseteq, ;)$ *are representable over a finite base X with* $|X| \leq (1 + |A|)^2 + 2 \cdot (1 + |A|)$*.*

Theorem 113. *The class* $\mathsf{R}(\sqsubseteq, ;)$ *cannot be axiomatised by finitely many axioms.*

The formulas σ_n in Theorem 110 above are defined in the section where the proof is provided. The signature $(\sqsubseteq, :)$ is the first known signature to have the finite representation property, but not a finitely axiomatisable representation class. This means that we now have an example reduction signatures of relation algebra for all combinations of finite axiomatisability and the finite representation property, showing that the two properties are independent.

Theorem 124. $\mathsf{R}(\sqcup, ;)$ *is not finitely axiomatisable.*

Theorem 125. *The finite representation property fails for finite structures in* $\mathsf{R}(\sqcap, ;)$.

Theorem 127. *The [finite] representation problem is undecidable for finite* $(\sqcup, \sqcap, ;)$*-structures.*

The above results are in line with Conjecture 27 and none of them are stronger or weaker than one another. Further, they lay ground for further research questions to be addressed, see Chapter 7.

In Chapter 5, we examine the results regarding the representation classes of structures with domain and range in their signature. Much like the results in the previous chapter, these are motivated by program behaviour and correctness. These results are the result of sole authorship and were published at RAMICS 2021 [Šem21].

Theorem 134. $\mathsf{R}(\mathrm{D}, \mathrm{R}, *)$ *is not finitely axiomatisable.*

Theorem 137. *For any signature* $(\smile, \mathrm{D}, ;) \subseteq \tau \subseteq (\leq, 0, 1, 1', \mathrm{D}, \mathrm{R}, \smile, ;)$*, finite members of* $\mathsf{R}(\tau)$ *will have a finite representation.*

While former theorem shows a negative result about the signature of demonic domain-range semigroups, it does not contradict Conjecture 27 — as discussed, the two properties (FA of the representation class and FRP) are independent. However, the latter theorem does show a special case of Conjecture 27. Before the chapter is concluded, we present some preliminary results and speculations about finitely representing the converse-free signatures with domain and range.

The final results chapter — Chapter 6 — is about weakening relation algebras (not to be confused with weakly representable relation algebras). These are yet another iteration of relation algebra variations motivated by their applications. In this case, the interpretation of the identity element is motivated by the weakening rules in formal systems. We define the class of weakening relation algebras as well as define weakening representability (and hence the weakening representation class wkR for signatures τ). We show a number of novel results about these, stated below. Some of the work in this chapter was done in collaboration with Peter Jipsen and was published at RAMICS 2023 [JŠ23].

We begin by showing some results about the class of representable weakening relation algebras RwkRA.

Theorem 152. *The representation problem is undecidable for finite [diagonally] representable weakening relation algebras.*

Theorem 158. *A bounded cyclic unital dℓ magama* $\mathcal{A}$ *is representable if and only if* $\exists$ *has a winning strategy for* $\Gamma_\omega(\mathcal{A})$.

Theorem 159. *The class* RwkRA *has a recursively enumerable axiomatisation.*

In Theorem 158, Γ_ω refers to an infinite length representation game we define, akin to those in Section 2.2 and [HH02]. Using games, we can define the hierarchy of abstract classes of weakening relation algebras, akin to $\mathsf{RA}_3, \mathsf{RA}_4, \ldots$ and a finite first-order theory Φ_3 for which we show the below result.

The proofs of theorems above are based on their classical relation algebra counterparts, but do not follow directly from them. Further, they indicate some level of similarity in the behaviour of the classes of representable weakening relation algebras and representable relation algebra, as expected.

Theorem 182. wkRA_3 *is axiomatised by the basic axioms for bounded cyclic involutive unital $d\ell$-magmas and Φ_3.*

Furthermore, diagonally representable weakening relation algebras are equivalent to $\mathsf{R}(\tau_{\mathsf{wkRA}})$ where τ_{wkRA} is the signature of weakening relation algebras. Unlike the class RwkRA, the following is true for DRwkRA.

Theorem 185. *Simple diagonally representable weakening relation algebras have a discriminator term.*

Corollary 186. *Diagonally representable weakening relation algebras form a discriminator variety.*

Again, these results are completely independent and do not follow directly from any of the results in the classical relation algebraic setting.

Finally, we look at the weakening representation classes for two signatures and examine how the finite representation property, conjectured on in Conjecture 27, behaves in the weakening setting. The $\triangleleft$ predicate in Theorem 189 is defined by infinitary $(\mathrm{D}, \mathrm{R}, ;)$-formulas.

Theorem 188. *The class $\mathsf{wkR}(\leq, 1', ;)$ is axiomatised by associativity, partial order, identity, and monotonicity. Its finite members admit finite representations.*

Theorem 189. *The class $\mathsf{wkR}(\mathrm{D}, \mathrm{R}, ;)$ is axiomatised by the following axioms:*

(1) the operation ; is associative,

(2) ; acts as a meet operation for the domain-range elements,

(3) $\mathrm{D}(s) \triangleleft \mathrm{D}(t) \Rightarrow \mathrm{D}(s);\mathrm{D}(t) = \mathrm{D}(t)$,

(4) $s \triangleleft t \Rightarrow \mathrm{D}(s) \triangleleft \mathrm{D}(t)$,

(5) $\mathrm{D}(\mathrm{D}(s)) = \mathrm{R}(\mathrm{D}(s)) = \mathrm{D}(s)$ *and* $\mathrm{D}(\mathrm{R}(s)) = \mathrm{R}(\mathrm{R}(s)) = \mathrm{R}(s)$,

(6) $\mathrm{D}(s);s = s;\mathrm{R}(s) = s$,

(7) $\triangleleft$ is antisymmetric,

for all $s \in A$. Furthermore, finite members of $\mathsf{wkR}(\mathrm{D}, \mathrm{R}, ;)$ admit finite representations.

Theorem 190. *The class $\mathsf{wkR}(\mathrm{D}, \mathrm{R}, ;)$ is not finitely axiomatisable.*

The theorems above clearly show that if we extend Conjecture 27 to the weakening setting, it holds in certain cases that remain open in the classical setting. Further, the finite axiomatisability property disagrees between the classical and the weakening in the former case, which further indicates that none of the results in the chapter easily follow from the known results for relation algebra reductions.

In the final chapter, we outline some open problems. These are all closely related to the work presented in this document and indicate a possible direction for future work.

Chapter 2

Related Work

In this chapter, we survey the results, relevant to the results, we present in the later chapters of in this thesis. We begin with a brief history of relation algebra and present the key general results and theorems. We then examine how tools from universal algebra, model theory, and game theory have been utilised to reason about relation algebras. These tools will be used to show novel results later in the thesis. We conclude the chapter with a literature review of the existing results in the area of the finite representation property for τ_{RA} reduction languages as well as in the area of the flexible atom conjecture.

2.1 Towards Relation Algebra

This section outlines some of the theorems most relevant to our research. A more detailed survey into the history of Relation Algebra is provided by Maddux in [Mad91].

It is difficult to pinpoint the exact moment in history one may dub the beginning of the relation-algebraic era. Perhaps it is best to begin by acknowledging Boole's investigations in the calculus of logic [Boo48] that (through later research in the twentieth century) led to the modern-day algebraic definition of Boolean algebras as axiomatised by (B1)–(B7) in the previous section.

While these axioms are sound and complete for the calculus of unary relations (or sets) they provide us little to no insight into how binary relations behave. The origins of researching the behaviour of binary relations and how they interact may perhaps lie in De Morgan's [DM64] and Pierce's [Pei83] studies in the late nineteenth century. These led to the discovery of the axiom (R8a) in its non-equational form below. For the proof of soudness, refer to Proposition 20.

$$s \cdot (t;u) = 0 \Longleftarrow u \cdot (\breve{t};s) = 0$$

As a consequence of this proposition, we can prove the axiom (T3) for the atom structures.

Corollary 33. *If a, b, c are atoms, we have $a \leq b;c \Leftrightarrow b \leq a;\breve{c}$*

Proof. By triangle law, we have $a \cdot b;c \neq 0 \Leftrightarrow b \cdot a;\breve{c} \neq 0$. Since a and b are atoms, we know that $a \cdot b;c \neq 0$ if and only if $a \leq b;c$ and $b \cdot a;\breve{c} \neq 0$ if and only if $b \leq a;\breve{c}$. □

Lyndon [Lyn50] proves an important theorem.

Theorem 34. *Tarski's axiomatisation of* RA *is not complete for* RRA*.*

Finally, we will state another important theorem (without proof), proved by Monk in [Mon64].

Theorem 35. *The class* RRA *cannot be axiomatised by a finite number of first-order formulas.*

2.2 Model Theory, Universal Algebra, Games

In this section, we outline some of the step-by-step techniques in the literature for reasoning about relation algebra. One may choose to think of these techniques as two-player games with a challenger and a builder player. These techniques date back to the mid-twentieth century [Lyn50] and they have been developed throughout the decades to follow. The most notable contribution and a near-complete reference for relation algebra by games is the 2002 book by Hirsch and Hodkinson [HH02].

As a prerequisite for the step-by-step game-theoretic constructions, we begin the section by stating some theorems from model theory and universal algebra relevant to relation algebras by games and then examine the step-by-step construction for atomic relation algebras. In the later chapters, we then adapt this technique to prove some of the novel contributions. We assume the reader is familiar with the fundamental definitions and theorems of first-order logic, direct products, ultraproducts, and König's lemma.

Theorem 36 (Löwenheim–Skolem)**.** *Take a finite signature τ, an infinite τ-structure $\mathcal{A}$ and an infinite cardinal number κ. There exists a τ-structure $\mathcal{A}'$ of size κ such that*

(LS1) if $|\mathcal{A}| < \kappa$ then $\mathcal{A}$ is an elementary substructure of $\mathcal{A}'$,

(LS2) if $\kappa \leq |\mathcal{A}|$ then $\mathcal{A}'$ is an elementary substructure of $\mathcal{A}$.

Remark 37. *The case (LS1) where $\mathcal{A}'$ is an elementary extension of $\mathcal{A}$ is referred to as the* upward Löwenheim–Skolem theorem *and the case (LS2) where $\mathcal{A}'$ is an elementary substructure of $\mathcal{A}$ is referred to at the* downward Löwenheim–Skolem theorem*.*

The proofs for the above theorem can be found in [CK90, Theorem 1.3.21 and its corollaries]. We introduced the idea of an elementary class in the previous section. We now examine some properties of those. The below theorem is stated without proof, one may refer to [CK90, Theorem 4.1.12] for more detail.

Theorem 38. *A class* K *is closed under ultraproducts and elementary equivalence if and only if it is elementary. That is, there exists a first-order theory* Σ *such that* $\mathsf{K} = \{\mathcal{A} \mid \mathcal{A} \models \Sigma\}$*. We say* K *is defined by* Σ*.*

A class being elementary is a strong property, however, requiring the defining theory to be of a certain form leads to stronger properties. We list a few below, along with the key results.

Definition 39. *An elementary class is* universal *if it can be defined by a theory of universally quantified formulas.*

Theorem 40 (Łoś–Tarski)**.** *Universal classes are closed under substructures. If an elementary class is closed under substructures, it is universal.*

The proof for the above theorem can be found in [CK90, Theorem 3.2.2]. We continue by examining special types of universal classes.

Definition 41. *A* universal Horn clause *is a universally quantified disjunction of atomic formulas and negations of atomic formulas with at most one positive disjunct. A Horn clause, consisting of only equations and inequations as its disjuncts, is called a* quasi-equation*.*

Definition 42. *A universal class is a* Horn class *if it can be defined by a theory consisting of universal Horn clauses.*

Horn clauses were first introduced by McKinsey [McK43] and subsequently studied algebraically by Horn [Hor51] and Maltsev [Mal71]. The below theorem is proved in [CK90, Chapters 6.2, 6.3, Theorem 6.2.5'].

Theorem 43 (Horn–Maltsev–McKinsey)**.** *A class of structures is a Horn class if and only if it is closed under isomorphisms, substructures, direct products, and ultraproducts.*

Remark 44. *We may refer to the property above as having* $\mathbb{SP}$-closure *and the theorem as the* $\mathbb{SP}$-closure theorem*.*

Remark 45. *If the class of algebras satisfies the* $\mathbb{SP}$*-closure property, then it has a quasi-equational defining theory. Such a class is called a* quasivariety*.*

Finally, we define varieties and state Birkhoff's theorem.

Definition 46. *A universal class is a* variety *if it can be defined by an equational theory.*

One of the fundamental theorems in universal algebra is proven in [Bir35].

Theorem 47 (Birkhoff)**.** *A class of structures is a variety if and only if it is closed under homomorphisms, substructures, direct products.*

Remark 48. *We may refer to the property above as having* $\mathbb{HSP}$-closure *and the theorem as the* $\mathbb{HSP}$-closure theorem*.*

To employ the step-by-step constructions to prove that certain properties hold for a class of structures of binary relations, it is necessary to show that it is elementary. In order to achieve that, we may first opt to show that a weaker property holds.

Definition 49. *A* pseudo-elementary class *is a class* K *of* τ*-structures such that there exists an elementary* τ'*-class* K' *where* τ' *is a two-sorted language with two disjoint sorts* r, a *(representation and algebra respectively) where all symbols in* τ *is included in* τ' *as* a*-sort symbols and* K *is the* τ*-reduct of the* a*-restriction of* K'*.*

Trivially, all elementary classes are also pseudo-elementary. Furthermore, pseudo-elementary classes are still closed under ultraproducts [Ekl77]. Showing the class of representable relational structures is pseudo-elementary — a much easier task than showing its elementary nature — is, therefore, sufficient to show that the class satisfies the ultraproduct closure condition.

To show that the class of representable relation algebras is pseudo-elementary, we show that there exists a two-sorted theory, containing all symbols in τ_{RA} as operations of type a, that defines an elementary class K such that RRA is a reduct of its a-restriction. We do that by defining an additional ternary predicate *holds* of sort a, r, r. The theory below defines the class of representable relation algebras.

(2R1) $\forall s : \big(s \neq 0 \iff \exists x, y : \mathit{holds}(s, x, y)\big)$,

(2R2) $\forall s, x, y : \big(\mathit{holds}(s, x, y) \iff (\mathit{holds}(1, x, y) \wedge \neg \mathit{holds}(-s, x, y))\big)$,

(2R3) $\forall s, t, x, y : \big(\mathit{holds}(s + t, x, y) \iff (\mathit{holds}(s, x, y) \vee \mathit{holds}(t, x, y))\big)$,

(2R4) $\forall x, y : (\mathit{holds}(1', x, y) \iff x = y)$,

(2R5) $\forall s, x, y : (\mathit{holds}(s, x, y) \iff \mathit{holds}(\breve{s}, y, x))$,

(2R6) $\forall s, t, x, y : \big(\mathit{holds}(s;t, x, y) \iff \exists z : (\mathit{holds}(s, x, z) \wedge \mathit{holds}(t, z, y))\big)$.

Additionally, the reader can check that by restricting the representation of an extension structure to the elements of a substructure, one can show closure under substructures and disjoint unions of representations suffice to represent direct products of structures in the representation class. The pseudo-elementary class argument can, with very minor modifications, even be extended to all τ_{RA}-reduct languages to prove the following lemma.

Lemma 50. *For any* $\tau \subseteq \tau_{\mathsf{RA}}$*, the class* $\mathsf{R}(\tau)$ *is a Horn class. If* τ *is an algebraic signature, then* $\mathsf{R}(\tau)$ *is a quasi-variety.*

This is a well known strong general result about relation algebras. However, for this document, the most important part is the fact that these classes are elementary. Due to this, we can prove certain properties using relation algebras by games. To do that, we start by defining networks.

Definition 51. *Let $\mathcal{A}$ be a Boolean unital involutive magma. Define a* network *as $\mathcal{N} = (N, \lambda)$ where N is the set of nodes and $\lambda : N^2 \to \wp(\mathcal{A})$ is a labelling function such that for all $x, y \in N$, we have $1' \in \lambda(x, x)$ and if $a \in \lambda(x, y)$ then $-a \notin \lambda(x, y)$.*

Definition 52. *A network $\mathcal{N} = (N, \lambda)$ is a* prenetwork *of $\mathcal{N}' = (N', \lambda')$, denoted $\mathcal{N} \subseteq \mathcal{N}'$, if and only if $N \subseteq N'$ and for all $x, y \in N$ we have $\lambda(x, y) \subseteq \lambda'(x, y)$.*

We can now define a two-player representation game for a structure $\mathcal{A}$. In the game, the challenger player, called Abelard ($\forall$), will challenge the builder player Heloïse ($\exists$) to build a representation over a series of steps (or moves).

Definition 53. *Let $\mathcal{A}$ be a Boolean unital involutive magma. Define, for $0 < n \leq \omega$, an n-round two-player game $\Gamma_n(\mathcal{A})$. On the ith round, for $0 \leq i < n$, the challenger $\forall$ requests the builder $\exists$ returns a network $\mathcal{N}_i$ such that $\mathcal{N}_j \subseteq \mathcal{N}_k$ for all $0 \leq j \leq k < n$. The game is won by $\forall$ if and only if $\exists$ at any point cannot return a network. The zeroth move is called the initialisation and $\forall$ must play the below move.*

Initialisation *$\forall$ picks $s \not\leq t \in \mathcal{A}$ and $\exists$ must return a network $\mathcal{N}_0 = (N_0, \lambda_0)$ such that there exist $x, y \in N_0$ with $s, -t \in \lambda_0(x, y)$.*

Given the network $\mathcal{N}_{i-1} = (N_{i-1}, \lambda_{i-1})$ was played on the $i-1$st move, $\forall$ can choose from one of the following moves for the ith move where $0 < i < n$.

Choice *$\forall$ picks $x, y \in N_{i-1}$ and $s, t, u \in \mathcal{A}$ such that $s \in \lambda_{i-1}(x, y)$ and $s \leq t + u$. $\exists$ must return a network $\mathcal{N}_i = (N_i, \lambda_i)$ such that $t \in \lambda_i(x, y)$ or $u \in \lambda_i(x, y)$.*

Complement *$\forall$ picks $x, y \in N_{i-1}$ such that $\lambda_{i-1}(x, y) \neq \emptyset$ as well as an $s \in \mathcal{A}$. $\exists$ must return a network $\mathcal{N}_i = (N_i, \lambda_i)$ such that $s \in \lambda_i(x, y)$ or $-s \in \lambda_i(x, y)$.*

Converse *$\forall$ picks $x, y \in N_{i-1}$ and $s \in \mathcal{A}$ such that $s \in \lambda_{i-1}(x, y)$. $\exists$ must return a network $\mathcal{N}_i = (N_i, \lambda_i)$ such that $\breve{s} \in \lambda_i(y, x)$.*

Composition *$\forall$ picks $x, y, z \in N_{i-1}$ and $s, t \in \mathcal{A}$ such that $s \in \lambda_{i-1}(x, y), t \in \lambda_{i-1}(y, z)$. $\exists$ must return a network $\mathcal{N}_i = (N_i, \lambda_i)$ such that $s;t \in \lambda_i(x, z)$.*

Witness *$\forall$ picks $x, y \in N_{i-1}$ and $s, t \in \mathcal{A}$ such that $s;t \in \lambda_{i-1}(x, y)$. $\exists$ must return a network $\mathcal{N}_i = (N_i, \lambda_i)$ such that there exists a $z \in N_i$ where $s \in \lambda_i(x, z), t \in \lambda_i(z, y)$.*

The moves of the game are visualised in Figure 2.1

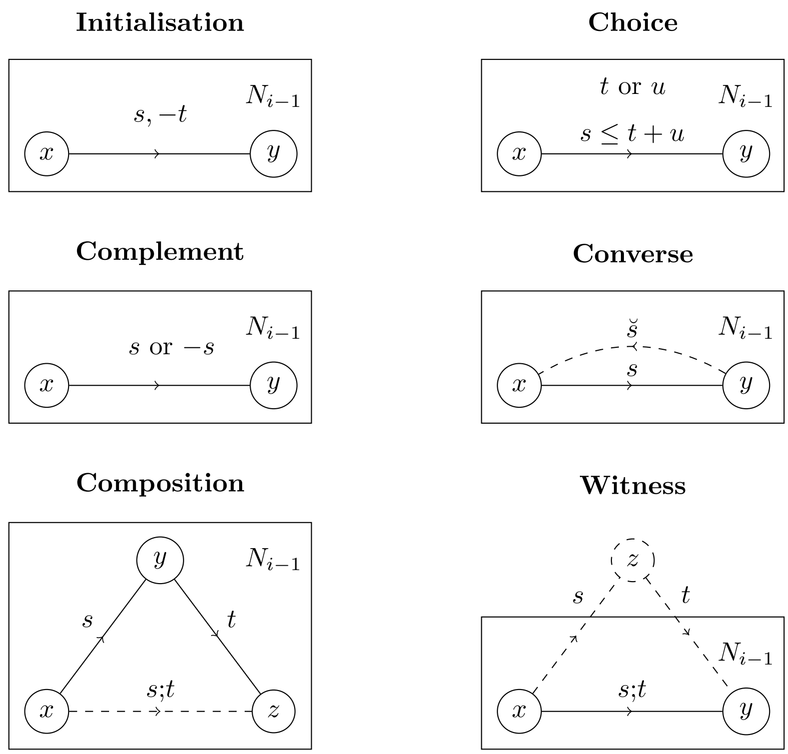

Figure 2.1: The moves of $\Gamma_n(\mathcal{A})$, visualised.

A number of neat properties of games are shown below. The proofs for these claims presented here are outlines. For more details, refer to [HH02]. We begin with some definitions.

Definition 54. *A* term network *is a tuple $\mathcal{N} = (N, \lambda)$ where N is a set of nodes and λ is a labelling function mapping pairs of nodes to finite sets of terms in the language τ_{RA}.*

Definition 55. *Let $\uplus$ denote the disjoint union. We now define for a term network $\mathcal{N}$, $x \in N \uplus \{x^+\}$ (where x^+ is a new node not yet included in the network), $y \in N$, $t \in terms(\tau_{\mathsf{RA}})$* label addition *as*

$$+[\mathcal{N}, x, y, t] = (N \cup \{x\}, \lambda^+)$$

where

$$\lambda^+(z, w) = \begin{cases} \lambda(z, w) \cup \{t \mid z = x, w = y\} & \text{if } z, w \in N \\ \{1' \mid z = w\} \cup \{t \mid z = x, w = y\} & \text{otherwise} \end{cases}.$$

Lemma 56. *There exists a first-order formula σ_n for every $0 < n < \omega$ such that $\exists$ has a winning strategy for $\Gamma_n(\mathcal{A})$ if and only if $\mathcal{A} \models \sigma_n$.*

Proof. To show that $\exists$ has a winning strategy, it suffices to show that she has a winning strategy when she opts to play *conservatively* and only adds the minimum amount of nodes and labels required to the network. We can then define a formula $\psi_n[\mathcal{N}]$ recursively for a term network $\mathcal{N}$, denoting that $\exists$ can survive n rounds of the representation game. Let

$$\psi_0[\mathcal{N}] = \bigwedge_{x,y \in N} \bigwedge_{t_1, t_2 \in \lambda(x,y)} t_1 \neq -t_2$$

and for $0 < i < \omega$

$$\begin{aligned} \psi_{i+1}[\mathcal{N}] = & \bigwedge_{x,y \in N} \bigwedge_{t \in \lambda(x,y)} \forall s, u : \left(t \leq s + u \Rightarrow \left(\begin{array}{c} \psi_i[+[\mathcal{A}, x, y, s]] \\ \vee \\ \psi_i[+[\mathcal{A}, x, y, u]] \end{array} \right) \right) \\ & \wedge \bigwedge_{x,y \in N, \lambda(x,y) \neq \emptyset} \forall s : \Big(\psi_i[+[\mathcal{A}, x, y, s]] \vee \psi_i[+[\mathcal{A}, x, y, -s]] \Big) \\ & \wedge \bigwedge_{x,y \in N} \bigwedge_{t \in \lambda(x,y)} \psi_i[+[\mathcal{N}, y, x, \breve{t}]] \\ & \wedge \bigwedge_{x,y,z \in N} \bigwedge_{t_1 \in \lambda(x,y), t_2 \in \lambda(y,z)} \psi_i[+[\mathcal{N}, x, z, t_1 ; t_2]] \\ & \wedge \bigwedge_{x,y \in N} \bigwedge_{t \in \lambda(x,y)} \forall s, u : \left(t = s ; u \Rightarrow \bigvee_{z \in N \uplus \{x^+\}} \psi_i[+[+[\mathcal{N}, z, y, u], x, z, s]] \right). \end{aligned}$$

Below, we can define, for $1 < n < \omega$ the formula σ_n such that for a Boolean unital involutive magma $\mathcal{A}$, $\exists$ will have a winning strategy for $\Gamma_n(\mathcal{A})$ if and only if $\mathcal{A} \models \sigma_n$.

$$\forall s, u : s \not\leq u \Rightarrow \psi_{n-1}[\{x, y\}, \{(x, x) \mapsto \{1'\}, (y, y) \mapsto \{1'\}, (x, y) \mapsto \{s, -u\}\}]$$

□

Theorem 57 (Hirsch–Hodkinson)**.** *A Boolean unital involutive magma $\mathcal{A}$ is representable if and only if $\exists$ has a winning strategy for $\Gamma_\omega(\mathcal{A})$.*

Proof. If $\mathcal{A}$ is representable, then $\exists$ can trivially play the game by mapping her moves from a representation. For the converse, we begin by examining the case of countable algebras. If $\mathcal{A}$ is countable, then $\forall$ can schedule his moves so that every move is called eventually. This ensures that in the limit, all moves have been called and when the limit network after a winning play is quotiented by 1' — which will be an equivalence relation — all the operations are represented correctly. In addition, there exists a pair of points that discriminates the two elements played on initialisation. Therefore, the disjoint union of these limit networks for all initialisation pairs is a representation.

To extend the result to uncountable structures, observe that $\exists$ only has a finite number of choices at each turn of a well-scheduled game. Therefore, by König's lemma, having a winning strategy for an infinite-length game is equivalent to having a winning strategy for all finite-length games. Thus a countable τ_{RA}-structure $\mathcal{A}$ is representable if and only if it obeys the axioms of Boolean unital involutive magmas as well as σ_i for all $0 < i < \omega$. We have established that the class RRA is a quasivariety and thus elementary. Therefore, if the result does not extend to uncountable structures, then there must exist an unrepresentable uncountable structure $\mathcal{A}$ that is not a model for a theory that defines RRA but $\exists$ has a winning strategy for $\Gamma_\omega(\mathcal{A})$. By the downwards Löwenheim–Skolem theorem, there must exist a countable elementary substructure $\mathcal{A}' \subseteq \mathcal{A}$. The structure $\mathcal{A}'$ is countable, but by elementary equivalence, it is not representable, however, $\exists$ has a winning strategy for $\Gamma_\omega(\mathcal{A}')$. □

A neat consequence of this game-theoretic approach is shown below. We will use this to show that $\mathsf{R}(\tau)$ cannot be axiomatised finitely, for some $\tau \subseteq \tau_{\mathsf{RA}}$.

Corollary 58. *To show that the class of relation algebras is not finitely axiomatisable, it suffices to show that for every $0 < n < \omega$ there exists an unrepresentable structure $\mathcal{A}_n$ for which $\exists$ has a winning strategy for $\Gamma_n(\mathcal{A}_n)$.*

Proof. Let Σ be the theory consisting of the axioms for Boolean unital involutive magmas together with $\{\sigma_n \mid 0 < n < \omega\}$. Σ defines the class RRA. Now suppose this theory were axiomatised finitely — or equivalently by a single axiom ϕ. $\Sigma \cup \{\neg\phi\}$ is clearly not consistent. But every finite subtheory of it contains axioms up to some maximal σ_n and thus $\mathcal{A}_n$ is a model for it. By compactness, $\Sigma \cup \{\neg\phi\}$ is therefore consistent and we have reached a contradiction. □

We have now looked at the representation game for the Signature of relation algebras and the results related to it. It can be very easily modified to accommodate for the reductions of τ_{RA}. However, the atomic game can also be defined for atom structures as well. In order to do that, we first recognise that an $(A, \mathrm{I}, \smile, \mathrm{T})$-structure will be the atom structure, corresponding to an atomic Boolean unital involutive magma if and only if it obeys the axioms (T1) and (T4)–(T6) from Definition 22. We begin with a slightly different definition of a network.

Definition 59. *Let $\mathcal{A}$ be such the atom structure of a Boolean unital involutive magma. Define an* atomic network *to be a pair $\mathcal{N} = (N, \lambda)$ where $\lambda : N^2 \to A$ such that for all $x, y, z \in N$ we have $\mathrm{I}(\lambda(x, x))$ and $\mathrm{T}(\lambda(x, y), \lambda(x, z), \lambda(z, y))$.*

Definition 60. *We say that an atomic network $\mathcal{N} = (N, \lambda)$ is a* prenetwork *of $\mathcal{N} = (N', \lambda')$, denoted $\mathcal{N} \subseteq \mathcal{N}'$, if $N \subseteq N'$ and λ is a N-restriction of λ'.*

Definition 61. *For an atom structure $\mathcal{A}$ of a Boolean unital involutive magma, define an infinite-length two-player game $\mathcal{G}(\mathcal{A})$. On the ith round, for $0 \leq i < \omega$, the challenger $\forall$ will request the builder $\exists$ returns the network $\mathcal{N}_i$.*

Initialisation *On the zeroth move $\forall$ picks an atom $a \in \mathcal{A}$ and $\exists$ must return a network $\mathcal{N}_0 = (N_0, \lambda_0)$ with two nodes x, y with $\lambda_0(x, y) = a$.*

Witness *For the subsequent moves, given the network played at the previous move was $\mathcal{N}_{i-1} = (N_{i-1}, \lambda_{i-1})$, $\forall$ will pick some $x, y \in N_{i-1}$ and some a, b such that $T(\lambda_{i-1}(x, y), a, b)$ and there does not exist a $z \in N_{i-1}$ such that $\lambda_{i-1}(x, z) = a, \lambda_{i-1}(z, y) = b$. $\exists$ must respond by returning a network $\mathcal{N}_{i-1} \subset \mathcal{N}_i = (N_{i-1} \uplus \{z\}, \lambda_i)$ such that $\lambda_i(x, z) = a, \lambda_i(z, y) = b$.*

$\forall$ wins if $\exists$ cannot respond to one of his requests. $\exists$ wins otherwise. Note that unlike with the algebraic game, $\forall$ may not always be able to call a move. In that case, $\exists$ wins.

Without proof, we state the theorem below.

Theorem 62 (Hirsch–Hodkinson)**.** *A Boolean unital involutive magma is representable if and only if for its corresponding atom structure (or the atom structure of its canonical extension) $\mathcal{A}$, $\exists$ will have a winning strategy for $\mathcal{G}(\mathcal{A})$.*

Finally, we generalise the representation game slightly.

Definition 63. *For an atom structure $\mathcal{A}$ of a Boolean unital involutive magma and $3 \leq m \leq \omega$, define the m-pebble infinite-length game $\mathcal{G}^m(\mathcal{A})$ that is played the same as $\mathcal{G}(\mathcal{A})$, except that at witness move $\forall$ picks a prenetwork $\mathcal{N}'_{i-1} \subseteq \mathcal{N}_{i-1}$ such that $|N'_{i-1}| < m$ and requests a witness for a composition $\mathcal{N}'_{i-1}$.*

We conclude the section by stating the below theorem.

Theorem 64 (Hirsch–Hodkinson)**.** *A Boolean unital involutive magma is a member of* RA_n*, for* $3 \leq n \leq \omega$ *if and only if for its corresponding atom structure (or the atom structure of its canonical extension)* $\mathcal{A}$*,* $\exists$ *has a winning strategy for* $\mathcal{G}^n(\mathcal{A})$*.*

2.3 Results Regarding the FRP

We now look at the existing results shown in the literature about how finite structures behave for different τ_{RA}-reductions. More specifically, we are interested in which $\tau \subseteq \tau_{\mathsf{RA}}$ have been shown to have the finite representation property for $\mathsf{R}(\tau)$.

The signature in question may contain the operations and predicates described in Section 1.1. However, more recently, with the introduction of the demonic calculus [Ngu91] to reason about Dijkstra's demonic computational model [DS90] (see Chapter 4 for detail), the search has also widened to include demonic operations and predicates. We define three operations and predicates below but motivate them in later sections.

Definition 65. *Demonic join* ($\sqcup$)*, refinement* ($\sqsubseteq$) *and composition* ($*$) *are defined as*

$$R \sqcup S = \{(x, y) \mid (x, y) \in R \cup S \wedge (x, x) \in \mathrm{D}(R) \cap \mathrm{D}(S)\}$$
$$R \sqsubseteq S \Leftrightarrow R \sqcup S = S$$
$$R * S = \{(x, y) \in R;S \mid \forall z : (x, z) \in R \Rightarrow (z, z) \in \mathrm{D}(S)\}$$

for some binary relations $R, S \subseteq X \times X$*.*

The reader can check that $*, \sqcup$ are associative and $\sqcup$ is also commutative. Furthermore, $\sqsubseteq$ is a partial order with the top element $\emptyset$ and $*$ is monotone with respect to $\sqsubseteq$.

Positive results in the area of finite representability of both 'angelic' reductions of τ_{RA} and their demonic counterparts are sparse and they usually rely on an explicit construction of a representation from a structure. Most of these constructions follow the same basic principles as the Cayley Representation — we shall call it h^{Cay} — of a Group $\mathcal{A} = (A, 1', ^{-1}, ;)$ as permutations (which are really just a special case of binary relations).

Definition 66 (Cayley Representation)**.** *Given a group* $\mathcal{A} = (A, 1', ^{-1}, ;)$*, we define a representation* $h^{Cay} : A \to \wp(A \times A)$ *where for* $s, t, u \in A$ *we have* $(s, t) \in h^{Cay}(u)$ *if and only if* $s;u = t$ *as shown in Figure 2.2 left.*

The reader can check that the representation is faithful (that is, the mapping is injective) and it represents composition, the inverse, and the identity correctly. Additionally, the same basic idea will work for representing monoids in $\mathsf{R}(1', ;)$ as binary relations. By amending a representable semigroup $\mathcal{A} \in \mathsf{R}(;)$ with an element e such that $s;e = e;s = s$, for all $s \in A \cup \{e\}$, to ensure faithfulness, the Cayley representation will work for this signature as well. In [Zar59], Zareckiĭ expands the idea to accommodate for partial ordering as well.

Figure 2.2: Cayley representation of a group (left) and Zareckiĭ representation of an ordered semigroup (right).

Theorem 67 (Zareckiĭ)**.** $\mathsf{R}(\leq, ;)$ *is axiomatised by partial order, associativity, and monotonicity. Additionally, the class has the finite representation property.*

Proof. Let $\mathcal{A} = (A, \leq, ;)$ be a structure where $\leq$ is a partial order, ; is associative and order-preserving. These axioms are trivially sound for $\mathsf{R}(\leq, ;)$. For completeness, amend $\mathcal{A}$ with the element e with only the mandatory reflexive pair in $\leq$ as well as $e;s = s;e = s$, for all $s \in A \cup \{e\}$, to obtain $\mathcal{A}'$. Define $h^{Zar} : A \to \wp(A' \times A')$ where for $s, t \in A'$ and $u \in \mathcal{A}$, we have $(s, t) \in h^{Zar}(u)$ if and only if $t \leq s;u$, see Figure 2.2 right. Without loss, take $s \not\leq t \in A$ and we see that $(e, s) \in h^{Zar}(s)$ but not in $h^{Zar}(t)$, so the representation is faithful. Using associativity and monotonicity, one can show that both $\leq, ;$ are represented correctly. If A is finite, then so is A', and thus the representation h^{Zar} is over a finite base. □

Neatly, this result works for its demonic dual $\mathsf{R}(\sqsubseteq, *)$, but cannot be trivially expanded to the class of representable ordered monoids $\mathsf{R}(\leq, 1', ;)$, ordered convoluted semigroups $\mathsf{R}(\leq, \smile, ;)$ or ordered involutive monoids $\mathsf{R}(\leq, \smile, 1', ;)$. Indeed, the finite representation property remains open for all of these signatures.

However, Egrot, Hirsch, and Mikulás [HE13, HM13] provide us with an elegant extension of the Zareckiĭ representation to finitely represent ordered convoluted monoids with domain and range. The main idea is to represent the structure over closed sets — rather than single elements of the structure. We present the outline of the argument below.

Theorem 68 (Egrot–Hirsch–Mikulás)**.** *The representation class* $\mathsf{R}(\leq, 0, \mathrm{D}, \mathrm{R}, 1', \smile, ;)$ *has the finite representation property.*

Proof. Take a finite representable structure $\mathcal{A} = (A, \leq, 0, \mathrm{D}, \mathrm{R}, 1', \smile, ;)$. This structure may be represented over the set of closed sets — we will call it $\mathcal{C} \subseteq \wp(A)$. Semantically speaking, these correspond to sets $S \subseteq A$ such that in a representation $h : A \to \wp(X \times X)$, there exist $x, y \in X$ such that $S = \{s \mid (x, y) \in h(s)\}$. Syntactically, these sets will be all sets S that are upwardly closed with respect to $\leq$ and with $d, r \in A$ such that $\mathrm{D}(d) = d, \mathrm{D}(r) = r$ and for all $s \in S$, $\mathrm{D}(d;s;r) = d, \mathrm{R}(d;s;r) = r$, and $d;s;r \in S$. A mapping $h : A \to \wp(\mathcal{C} \times \mathcal{C})$ is then defined as

$$(S, T) \in h(u) \iff (S;u \subseteq T \wedge T;\breve{u} \subseteq S)$$

where for some $S \subseteq A$ and $t \in A$, we define $S;t = \{s;t \mid s \in S\}$. The finite axiomatisation of $\mathsf{R}(\leq, 0, \mathrm{D}, \mathrm{R}, 1', \smile, ;)$ presented in [HE13] suffices to show that h is indeed a representation. The upper bound on the size of the smallest representation of $\mathcal{A}$ is, therefore, $2^{|A|}$. □

These are the only positive results in the literature, prior to our investigations. In addition, the finite representation property has been studied for representing these algebras as algebras of unary partial functions [JS13, HJM16]. These are yet another special case of binary relations. Although the results do not easily translate for algebras of binary relations, they reveal some techniques that could be utilised when proving the finite representation property of representation classes that are of interest to us in our future investigations.

In contrast to the sparse positive results, there seems to be an abundance of negative results in the area. They are roughly divided into two types. The first branch takes a finite representable algebra and shows that it cannot be represented over a finite base. The second is to show the undecidability of the representation problem. The most common algebra used for the former is the point algebra, introduced in [MT44].

Definition 69. *The* point algebra *— denoted $\mathcal{P} = (P, 0, 1, -, +, 1', \smile, ;)$ — is the relation algebra, defined by the atom structure $\mathcal{A} = (A, \mathrm{I}, \smile, \mathrm{T})$ with three atoms $A = \{1', l, g\}$. The converse is defined as $\breve{1'} = 1'$ and $\breve{g} = l$ and $\mathrm{I} = \{1'\}$. The predicate* T *consists of all mandatory identity triangles as well as all Peircean transforms of $(l, l, l), (l, g, l)$ as defined in Chapter 1.*

This algebra is representable via $h : P \to \wp(\mathbb{Q}\times\mathbb{Q})$ where $h(1'), h(l), h(g)$ are represented as $=, <, >$ respectively. Andréka shows in [AGN94] that the algebra is not finitely representable. This result has been extended recently to a larger class of signatures, independently by Maddux and Neuzerling [Mad16, Neu17]. We present our simplified version of the argument below.

Theorem 70 (Maddux–Neuzerling)**.** *The finite representation property fails for any $(\cdot, ;) \subseteq \tau \subseteq \tau_{\mathsf{RA}}$.*

Proof. For the failure of the finite representation property, it suffices to show that some finite τ-structure does not have a finite representation. Suppose that the τ-reduct of the point algebra had a finite representation $h : P \to \wp(X \times X)$ where $|X| = n < \omega$. Because $0 \cdot g = 0$, and consequently $g \nleq 0$, there must exist $x_0, x_n \in X$ such that $(x_0, x_n) \in h(g)$ and $(x_0, x_n) \notin h(0)$. Observe that $g = g;g = g;g;g = \cdots = g^n$. Thus there must exist $x_1, x_2, \ldots, x_{n-1}$ to witness this composition, that is, for $0 \leq i < j \leq n$, we have $(x_i, x_j) \in h(g)$, see Figure 2.3 left. Additionally, we have $(x_i, x_j) \notin h(0)$ to prevent the compositions with 0 resulting in $(x_0, x_n) \in h(0)$. By the pigeonhole principle, there must exist $0 \leq i < j \leq n$ such that $x_i = x_j$ and thus there exists x that $(x, x) \in h(g)$ and $(x, x) \notin h(0)$. Because $g = g;1'$ there must exist a y such that $(x, y) \in h(g), (y, x) \in h(1')$, see Figure 2.3 right — note that because 1' is not necessarily in the signature, it may be the case that $x \neq y$. Because $(y, x) \in h(1')$, $(x, x) \in h(g)$, and $g = 1';g$, we get $(y, x) \in h(g)$. Because $(x, y) \in h(g)$,

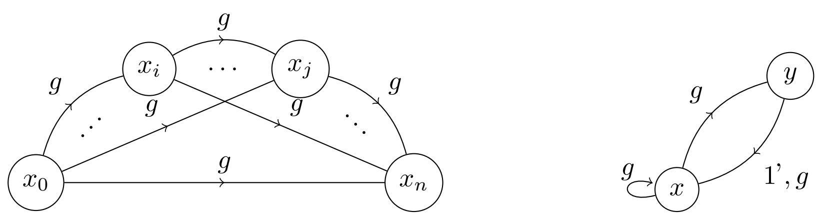

Figure 2.3: The proof of Theorem 70, visualised.

$(y, x) \in h(g) \cap h(1')$, and $g;(g \cdot 1') = 0$, it must hold that $(x, x) \in h(0)$ and we have reached a contradiction. □

As mentioned, the second technique used in the literature to show that the finite representation property fails is by showing that the representation decision problem is undecidable. Recall that for a signature τ, the representation problem is the decision problem of determining membership in $\mathsf{R}(\tau)$ for finite τ-structures.

Proposition 71. *If a language $\tau \subseteq \tau_{\mathsf{RA}}$ has the finite representation property, then the representation decision problem for τ is decidable.*

Proof. We will show that the representation decision problem is decidable by showing that it is semidecidable and complement-semidecidable. We do that by explicitly defining the procedures for semideciding the problem and its complement, given some finite input structure $\mathcal{A} = (A, \tau)$.

For the semidecidability of the representation problem, see that the class of relational τ-structures over a finite base set X is recursively enumerable. Take the procedure where we enumerate all proper τ-structures over a finite base set and check for isomorphism with $\mathcal{A}$. If $\mathcal{A}$ is representable, this procedure will eventually terminate and accept because τ has the FRP and because isomorphism checking is decidable for finite structures.

For the complement-semidecidability, we use arguments analogous to those in Lemma 50, Lemma 56, and Theorem 57 to show that $\mathsf{R}(\tau)$ has a recursively enumerable axiomatisation. As first-order formula checking in a model is decidable and because if $\mathcal{A} \notin \mathsf{R}(\tau)$ there will exist an axiom for $\mathsf{R}(\tau)$ that $\mathcal{A}$ is not a model for. We know that the procedure where we enumerate the axioms and check whether $\mathcal{A}$ is a model for them semidecides the complement of the representation problem. □

This means that if one can show that the representation problem is undecidable for a signature τ, then the finite representation property fails. Undecidability also implies there is no finite axiomatisability (as for finitely axiomatisable classes, the problem is well known to be in $LOGSPACE$). In literature, the undecidability of the representation problem is usually proven by reducing the tiling problem or the group embedding problem to the representation problem for τ. This has been done extensively, much like the study of finite axiomatisability utilising the model-theoretic approaches and representations by games. The results are summarised in Table 2.1

We note a few things about this table. The blank entries are open problems. The references are either in square brackets and refer to a paper (or an unpublished manuscript) or bold, with a parentheses reference to a theorem number in this thesis where the results are novel. Where $(\cdot, ;) \subseteq \tau \subseteq \tau_{\mathsf{RA}}$, we say that the finite axiomatisability depends as a large number of such τ do not have $\mathsf{R}(\tau)$ finitely axiomatisable [And91, HM07, Hai91], but $\mathsf{R}(\cdot, ;)$ turns out to be finitely axiomatisable [BS78] as noted in the entry below it.

The result by [Neu16] stating that for signatures containing $(\leq, -, ;)$, the representability problem is undecidable, however, this only applies to square representability. This is why we expand on these results in Section 3.1 and generalise some of them without requiring square representability.

The reference given for the failure of the FRP for signatures containing $(\cdot, ;)$ is by Maddux [Mad16] although the result also independently appears in Neuzerling's PhD thesis [Neu17]. Similarly, the reference for the failure of the FRP for $(\leq, \backslash, /, ;)$ is [Mik22], however, a joint unpublished manuscript by Mikulás and Hodkinson also proves the result.

The table also omits results regarding signatures without composition — demonic or 'angelic'. This is because these signatures all have the finite representation property and the representation problem decidable, a direct consequence of Stone's representation theorem. There are, however, a few notable results in the area, you may refer to [Mik04] for a survey of those.

In addition to the finite axiomatisability of the representation class, the finite representation property, and the decidability of the representation problem, other properties about these τ_{RA}-reduction languages are considered in the literature. These include (but are not limited to) the decidability of the finite representability problem, the decidability of the equational theory, having an equational axiomatisation, and the finite axiomatisability of the equational theory. Furthermore, there are results in the area of slightly relaxed definitions of representations — such as weak or feeble representations.

2.4 Results Regarding the Flexible Atom Conjecture

In this section, we examine the results in the area of the flexible atom conjecture that have been shown in the literature. Recall that an atomic relation algebra is integral if the identity is one of its atoms and that a flexible atom is a diversity atom f in an integral relation algebra such that for all diversity atoms a, b we have $(f, a, b) \in \mathrm{T}$.

Comer shows in [Com84] that every countable integral relation algebra with a flexible atom has a countable representation. We present a different argument (using the step-by-step constructions from Section 2.2) to prove the theorem

Signature (τ)	$\mathsf{R}(\tau)$ **FA**	**FRP**	**RP Dec.**
$(\leq, -, ;) \subseteq \tau \subseteq \tau_{\mathsf{RA}}$	No*	No*	No* [Neu16]
$(-, ;) \subseteq \tau \subseteq \tau_{\mathsf{RA}}$		**No (80)**	
$(-, ;)$	**No (87)**	**No**	
$(\cdot, ;) \subseteq \tau \subseteq \tau_{\mathsf{RA}}$	Depends	No [Mad16]	Depends
$(\cdot, ;)$	Yes [BS78]	No	Yes [BS78]
$(\cdot, +, ;) \subseteq \tau \subseteq \tau_{\mathsf{RA}}$	No [And91]	No	No [Neu16]
$(\cdot, +, 1', ;) \subseteq \tau \subseteq \tau_{\mathsf{RA}}$	No	No	No [HJ00]
$(\cdot, +, ;) \subseteq \tau \subseteq \tau_{\mathsf{RA}}$	No	No	No [HHJ21]
$(\cdot, \smile, ;) \subseteq \tau \subseteq \tau_{\mathsf{RA}}$	No [Hai91]	No	No [HJ00]
$(\mathrm{D}, ;) \subseteq \tau \subseteq (\mathrm{D}, \mathrm{R}, 0, 1', ;)$	No [HM11]		
$(\leq, \smile, \mathrm{D}, \mathrm{R}, 1', 0, ;)$	Yes [HM13]	Yes [HM13]	Yes [HM13]
$(\smile, \mathrm{D}, ;) \subseteq \tau \not\supseteq (-, +, \cdot)$		**Yes (137)**	**Yes**
$(\leq, 1', ;)$	No [Hir05]		
$(\cdot, 1', ;)$	No [HM07]		
$(+, ;)$	No [And88]		
$(\smile, ;)$	No [Bre77]		
$(\leq, \backslash, /, ;)$	Yes [AM94]	No [Mik22]	Yes
$(\rightarrow, ;) \subseteq \tau \subseteq \tau_{\mathsf{RA}} \setminus (\smile)$	**No**	**No**	**No (94)**
$(\leq, ;), (\sqsubseteq, ;)$	Yes [Zar59]	Yes [Zar59]	Yes [Zar59]
$(\leq, *)$	No [HMS20]		
$(\sqsubseteq, ;)$	**No (113)**	**Yes (110)**	**Yes**
$(\sqcup, ;)$	**No (124)**		
$(\sqcap, ;)$		**No (125)**	
$(\sqcap, \sqcup, ;)$	**No**	**No**	**No (127)**
$(\mathrm{D}, \mathrm{R}, *)$	**No (134)**		

Table 2.1: Known results regarding the finite axiomatisability of the representation class, the finite representation property, and the decidability of the representation problem for different signatures $\tau \subseteq \tau_{\mathsf{RA}}$.

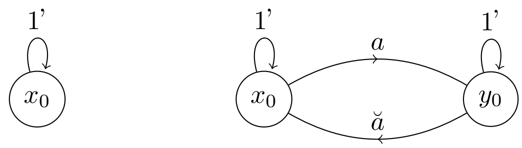

Figure 2.4: The initial move for a game $\mathcal{G}(\mathcal{A})$ where $\mathcal{A}$ is a countable integral atom structure with a flexible atom.

below. Later in [Mad94], Maddux conjectures that every finite relation algebra with a flexible atom will have a finite representation, see Problem 31, Conjecture 32.

Theorem 72 (Comer)**.** *Every countable relation algebra with a flexible atom has got a countable representation.*

Proof. Let $\mathcal{A} = (A, \mathrm{I}, \smile, \mathrm{T})$ be the atom structure of a countable relation algebra with a flexible atom. Let $f \in A$ be flexible. This means that for all diversity atoms b, c we have all six Peircean transforms of (f, a, b) in T. We have examined in Section 2.2 how to show that a countable relation algebra has a representation, it suffices to show that $\exists$ has a winning strategy for $\mathcal{G}(\mathcal{A})$. We show that by induction below.

Base Case On the initialisation move, $\exists$ returns the trivial reflexive network if the atom played is the identity 1' – see Figure 2.4 left — and a non-reflexive network with two nodes x, y with $\lambda(x, x) = \lambda(y, y) = 1'$, $\lambda(x, y) = a$, and $\lambda(y, x) = \breve{a}$ if some diversity atom a is played – see Figure 2.4 right. Trivially both of these are networks.

Induction Case Suppose $\exists$ was able to play a network $\mathcal{N}_{i-1}$ at the $i-1$st move for $0 < i < \omega$. Now $\forall$ demands a witness for $\mathrm{T}(c, a, b)$ for some $x, y \in N_{i-1}$. Because the identity triangles are all witnessed in $\mathcal{N}_{i-1}$, a, b must be diversity atoms. $\exists$ has to add a new node z, label the mandatory $\lambda(z, z) = 1', \lambda(x, z) = a, \lambda(z, x) = \breve{a}, \lambda(z, y) = b, \lambda(y, z) = \breve{b}$, and label the remaining pairs of nodes f, see Figure 2.5. The reader can check that this is indeed a network, bearing in mind that if $c = 1'$ that $a = \breve{b}$. □

Although the flexible atom conjecture has not been proved in general, there exist a number of results in the literature that construct finite representations either for individual finite relation algebras with a flexible atom or for classes of such algebras. We now direct our attention to some of those. We begin with a theorem by Jipsen, Maddux, and Tuza [JMT95].

Theorem 73. *Let* $0 < n < \omega$. *Take the atom structure* $\mathcal{A} = (A, \mathrm{I}, \smile, \mathrm{T})$ *where* $A = \{1'\} \cup \{f_i \mid 0 \leq i < n\}$, $\mathrm{I} = \{1'\}$, $\breve{a} = a$ *for all* $a \in A$, *and* T *consisting of all mandatory identity cycles as well as all diversity cycles. The complex algebra of* $\mathcal{A}$ *is finitely representable.*

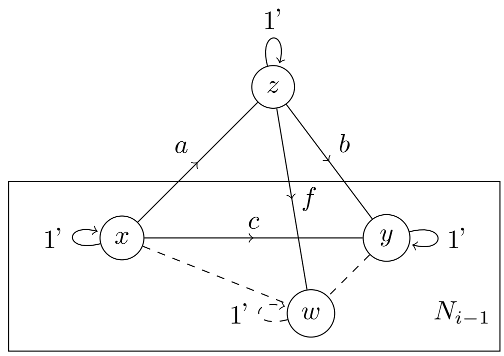

Figure 2.5: The initial move for a game $\mathcal{G}(\mathcal{A})$ where $\mathcal{A}$ is a countable integral atom structure with a flexible atom.

Proof. The paper provides two proofs, one with an explicit construction and another one that is probabilistic in the style of Erdős' random graphs [ER59]. We look at the latter. If we label all the reflexive nodes of a random graph of size $|N|$ with 1' and all other edges uniformly at random with one of the n flexible atoms, we can see that we will indeed obtain a network. With the probability $p = 1 - \frac{1}{n}$, we will have that a single edge is not in f_i for any $0 \leq i < n$. Hence, we can see that the probability that a single edge (x, y) does not have a given witness $(f_i;f_j)$ via some specific node z is p^2. The probability this witness is not in any of the edges z is, therefore, $p^{2|N|}$. Now, by using the inequality $P(\phi \vee \psi) \leq P(\phi) + P(\psi)$, we can conclude that the probability for any pair of nodes not being witnessed by any pair of atoms f_i, f_j is at most $|N|^2 n^2 p^{2|N|}$ and hence the probability that the said random graph is a representation is bigger than $1 - |N|^2 n^2 p^{2|N|}$. We can see that the probability is above 0 for a large enough $|N|$ and so there exists a finite random graph that is also a representation of the algebra. It can be shown that the upper bound on the minimal size to guarantee representability is in $O(n^2)$. □

Remark 74. *The proof above describes the technique from [AMN91] called splitting. We have taken an algebra with two atoms* $1', f$ *and then split the atom* f *into smaller atoms* $f_0, f_1, ..., f_{n-1}$.

Above we see how we can use probabilistic splitting to show the finite representability of 'maximally flexible' algebras. Alm, Maddux, and Manske show in [AMM08] that probabilistic splitting can be used to show the finite representability of 'minimally flexible' algebras.

Theorem 75. *Let* $0 < n < \omega$. *There exists a finite representation for every relation algebra whose corresponding atom structure is* $\mathcal{A} = (A, \mathrm{I}, \smile, \mathrm{T})$ *with* $A = \{1', f, a_0, a_1, ..., a_{n-1}\}$ *consisting of all self-converse atoms,* $\mathrm{I} = \{1'\}$, f *is the flexible atom, and the only other cycles in* T *are the mandatory identity cycles.*

Proof. First, we will show that we can make an arbitrarily large representation of the pre-splitting subalgebra with atoms $\{1', f, a\}$. Let $[m]$ denote the set $\{1, 2, ..., m\}$ and $[m]_k$ the set of all subsets of $[m]$ of size k. Let us define a graph with nodes $[3k-4]_k$ where

$$\lambda(S,T) = \begin{cases} 1' & \text{if } |S \cap T| = k \\ a & \text{if } |S \cap T| \leq 1 \\ f & \text{otherwise} \end{cases} .$$

See how no forbidden triangles arise as that would imply that the three nodes with the forbidden cycle S, T, U had $|S \cap T| \leq 1$, meaning $|S \cup T| \geq 2k - 1$ and $|S \cap U| \leq 1$, $|T \cap U| \leq 1$ further implying $|S \cup T \cup U| \geq 3k - 3$ and we have reached a contradiction.

To prove the theorem, one can show using the probabilistic splitting technique, similar to Theorem 73 to show $\mathcal{A}$ has a finite representation. □

The upper bound on the minimal representation size we get from the proof above is very high. In [DH13], a slightly more elaborate technique is introduced to improve the bound. Another technique to show finite representability for algebras with a flexible atom is called merging. The idea is to find a representable extension of the algebra and merge atoms. Usually, this is done starting from a cyclic group. These constructions were first introduced in [Com83], but recently a number of algebras have been shown to have a finite representation using these [AL19, AAL23].

The techniques first described in [Com83] partition cyclic groups into triangle-free partitions. Although they are not named so in the original paper, we will call these Comer schemes. One may then embed the algebras into these by 'merging' these partitions. In [AY19], Alm and Ylvisaker describe a fast algorithm for generating those.

We conclude this chapter with a conjecture from [AAL23].

Conjecture 76 (Strong flexible atom conjecture)**.** *Every finite relation algebra with a flexible atom has a representation via an embedding into a Comer scheme.*

Chapter 3

The FRP Conjecture

This is the first result chapter in this thesis. It presents results most closely related to the problems outlined in Chapter 1 in the classical relation algebraic setting. More specifically, we look at a large number of cases of Conjecture 27. The results presented in this chapter are the proof of the negative implication of the main conjecture, related nonfinite axiomatisability and undecidability results, as well as an interesting connection between these results and the flexible atom conjecture — the full detailed list of results will follow in this section. Although the results in this chapter make significant progress in proving Conjecture 27 — especially the negative side of it —, it remains open. Besides presenting the results, we speculate on the possible direction of solving the remaining open cases of Conjecture 27.

As we have discussed in the previous chapters, the full signature of Relation Algebra behaves quite badly. This is why we look for nicer behaviour by restricting our search to its reductions. Intuitively, the full Relation Algebra signature is really good at capturing the behaviour of relations but shows little to no neat properties like the finite axiomatisability of the representation class or the finite representation property for finite structures. This is why we sacrifice some of that ability to encapsulate the relational behaviour, by simply dropping a few members of the signature in order to obtain these properties.

The property of most interest to us is the finite representation property for these reduction signatures. As we have discussed, not many positive results exist in the area. Furthermore, they are largely done on a case by case for particular signatures. This chapter outlines our progress in proving Conjecture 27. Recall that it states that a signature $\tau \subseteq \tau_{\mathsf{RA}}$ will have the finite representation property if and only if it is not an expansion of $(-, ;)$ or $(\cdot, ;)$. We split the conjecture into two implications. The *negative* implication of the conjecture is the implication stating that a sufficient condition for the finite representation property to fail is the inclusion of either negation and composition or meet and composition in the signature. The other implication is called the *positive* implication.

We note that the positive implication of the conjecture only considers the signatures with term-definable operations that do not require meet or negation for their definition. This excludes signatures containing such operations as the domain and range, the residuals, or the demonic operations. These will be considered in the separate chapters of the thesis. On the other hand, we extend the negative side of the conjecture to some of the negation-like operations like the Heyting implication, which we define later in this chapter.

We begin this chapter by showing that the inclusion of the negation and composition in the signature is a sufficient condition for the FRP to fail. This, together with the results from [Mad16, Neu17] proves the negative implication of the conjecture. We then present an interesting connection between the signature of complemented semigroups and the flexible atom conjecture. We extend the results from [HHJ21] to show the undecidability of semigroups of binary relations with the Heyting arrow. We conclude by presenting some preliminary results and speculations about the positive side of the conjecture.

3.1 The Negative Implication of the FRP Conjecture

In this section, we prove that the inclusion of negation and composition in the signature is sufficient for the finite representation property to fail. We then also show the non-finite axiomatisability results. Both of these results are novel and have been raised as open problems in [Neu16, Problem 3.10]. Jackson shows in a forthcoming manuscript that the square representation problem is also undecidable for the signature $(-, ;)$, however, the decidability remains open for the representation problem without the square-representability requirement.

We begin by proving the negative side of Conjecture 27. As mentioned, the fact that the finite representation property fails for signatures including $(\cdot, ;)$ was never published, but shown independently by Maddux and Neuzerling [Mad16, Neu17]. Therefore we only have to prove that the finite representation property fails if negation and composition are both in the signature — or are term-definable from the signature. We break the proof down into a few lemmas. In our proof we use the point algebra $\mathcal{P} = (P, \tau_{\mathsf{RA}})$, recall that it is representable over the base set $\mathbb{Q}$ with the atom g represented as $<$ and its negation $-g$ represented as $\geq$.

Lemma 77. *If h is a faithful composition-preserving mapping from the point algebra carrier set P to $\wp(X \times X)$, for some X, then there exist $x, y \in X$ such that $(x, y) \in h(1)$.*

Proof. Because h is faithful and $0 \neq 1$, there must either exist some $x, y \in X$ such that $(x, y) \in h(1)$ (see Figure 3.1 left) or some $x, z \in X$ such that $(x, z) \in h(0)$. In the latter case — because h preserves composition — there must exist some $y \in X$ to witness the composition of $0 = 1;0$ (see Figure 3.1 right). Therefore, we have $(x, y) \in h(1)$. □

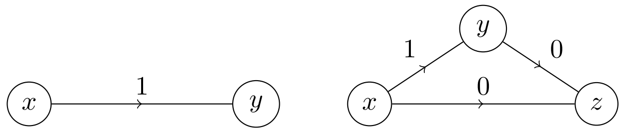

Figure 3.1: The proof of Lemma 77, visualised.

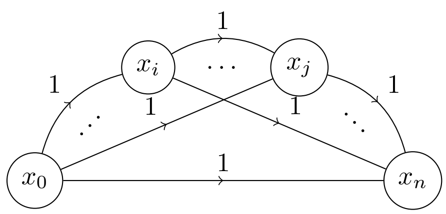

Figure 3.2: The proof of Lemma 78, visualised.

Lemma 78. *If h is a faithful composition-preserving mapping from the point algebra carrier set P to $\wp(X \times X)$, for some finite X, then there exists some $x \in X$ such that $(x, x) \in h(1)$.*

Proof. Let $|X| = n < \omega$. By Lemma 77, there exists $x_0, x_n \in X$ such that $h(1)$. Recall that $1 = 1;1 = 1;1;1 = \cdots = 1^n$. To witness the composition $1 = 1^n$, we require points $x_1, x_2, \ldots, x_{n-1} \in X$. Note that — owning to h preserving composition — this results in $(x_i, x_j) \in h(1)$ for $0 \leq i < j \leq n$, see Figure 3.1. See that there are n points in X and $n + 1$ points $x_i, 0 \leq i \leq n$. By the pigeonhole principle, there must exist $0 \leq i < j \leq n$ such that $x_i = x_j$. Thus we fix $x = x_i$ to prove the lemma. □

Lemma 79. *Let h be a mapping from P to $\wp(X \times X)$ that preserves composition and negation. Assume there exists a point $x_1 \in X$ such that $(x_1, x_1) \in h(1)$. Then the base set X is not finite.*

Proof. We show by induction that there must exist, for every $0 < n < \omega$ at least n points, call them $x_1, x_2, \ldots, x_n$ such that for all $0 < i, j \leq n$, we have $(x_i, x_i) \in h(1)$, $(x_i, x_j) \in h(g)$ if $i < j$, and $(x_i, x_j) \in h(-g)$ if $i > j$.

Base Case For $n = 1$, we trivially have this because there exists a point $x_1 \in X$ such that $(x_1, x_1) \in h(1)$, by the assumption in the lemma.

Induction Case Assume that for $0 < n < \omega$ we have $x_1, x_2, \ldots, x_n$ such that for all $0 < i, j \leq n$, we have $(x_i, x_i) \in h(1)$, $(x_i, x_j) \in h(g)$ if $i < j$, and $(x_i, x_j) \in h(-g)$ if $i > j$ — see Figure 3.3. See how $(x_n, x_n) \in h(1)$ and recall that $1 = g;(-g)$. Because h represents composition correctly, this has to be witnessed by some point x_{n+1}. Observe how if this were some x_i where $0 < i < n$, that would mean that $(x_n, x_{n+1}) \in h(g)$ (as it witnesses

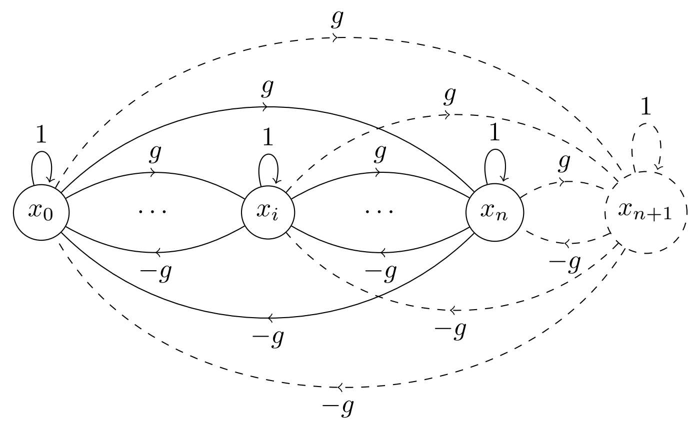

Figure 3.3: The proof of Lemma 79, visualised.

the composition) but also $(x_n, x_{n+1}) \in h(-g)$ (from the induction hypothesis) and if $i = n$ then by witnessing the composition we would get (x_n, x_n) in both $g, -g$. As h represents negation correctly, this is not the case and x_{n+1} must be new. To preserve the induction hypothesis, see that $(x_{n+1}, x_{n+1}) \in h(1)$ due to the composition of $(-g);g = 1$ via x_n and that for all $0 < i < n$, we have $(x_i, x_{n+1}) \in h(g)$ due to the composition of $g;g = g$ via x_n and similarly $(x_{n+1}, x_i) \in h(-g)$ due to the composition of $(-g);(-g) = -g$ via x_n, see Figure 3.3. □

Theorem 80. *Take a signature τ such that $(-, ;) \subseteq \tau \subseteq \tau_{\mathsf{RA}}$. The finite representation property fails for finite structures in $\mathsf{R}(\tau)$.*

Proof. To prove the claim, it suffices to show that a finite structure fails to have a finite representation if $-$ and ; are correctly represented. We use the τ-reduct of the point algebra $\mathcal{P}$. Now assume that the τ-reduct of $\mathcal{P}$ was representable via some h over a finite set X. By Lemma 78, we have that there exists a point $x_0 \in X$ such that $(x_0, x_0) \in h(1)$. By Lemma 79 we then have that X must be infinite and we have reached a contradiction. □

We now proceed to prove the failure of finite axiomatisation of $\mathsf{R}(-, ;)$. It is shown in [HJ12] that converse-free signatures containing $(-, 1', ;)$ fail to have the representation decidable, and thus their representation class is not finitely axiomatisable. The paper, however, stops short of proving the non-finite axiomatisability of $\mathsf{R}(-, ;)$ and the problem is later raised as an open problem in [Neu16, Problem ADD]. We employ the tools and techniques described in Section 2.2. The game will need to be modified slightly to accommodate for the specific signature in question.

Definition 81. *A* negated semigroup *is a structure* $\mathcal{A} = (A, -, ;)$ *where* ; *is associative and* $--s = s$*, for all* $s \in A$.

Definition 82. *Define, for a negated semigroup* $\mathcal{A}$ *and an* $0 < n \leq \omega$*, an* n*-round two player game* $\Gamma_n^{NS}(\mathcal{A})$*. The game is played in the same way as* Γ_n*, as defined in Definition 53, except*

(1) The identity condition is omitted for networks,

(2) On initialisation, Abelard picks a pair $s \neq t$ *and Héloïse returns a network* $\mathcal{N}_0 = (N_0, \lambda_0)$ *such that there exists* $x, y \in N_0$ *such that either* $s, -t \in \lambda_0(x, y)$ *or* $t, -s \in \lambda_0(x, y)$*,*

(3) After initialisation, Abelard may only choose to play complement, composition, or witness moves.

Lemma 83. *To show non-finite axiomatisability of* $\mathsf{R}(-, ;)$*, it suffices to show that there exists a set of non-representable negated semigroups* A *such that for every* $0 < n < \omega$ *there exists a structure* $\mathcal{A} \in \mathsf{A}$ *for which* $\exists$ *has a winning strategy for* $\Gamma_n^{NS}(\mathcal{A})$.

Proof. The argument follows closely the argument in Section 2.2, leading up to Corollary 58. In order to show that $\mathsf{R}(-, ;)$ is pseudo-elementary, we may use the composition axiom (2R6) from Section 2.2, whereas the faithfulness and negation conditions (2R1),(2R2) have to be modified slightly to obtain

$$\text{(2S1')}\ \ \forall a, b : \left((a \neq b) \Rightarrow \left(\exists x, y : \left(\begin{array}{c} (holds(a, x, y) \wedge holds(-b, x, y)) \\ \vee \\ (holds(-a, x, y) \wedge holds(b, x, y)) \end{array} \right) \right) \right),$$

$$\text{(2S2')}\ \ \forall x, y : \big(\big(\exists a : holds(a, x, y)\big) \Leftrightarrow \big(\forall b : (holds(b, x, y) \Leftrightarrow \neg holds(-b, x, y))\big)\big).$$

To replicate the argument from Lemma 56, we only use the conjuncts of ψ_{i+1}, pertinent to the complement, composition, and witness moves. Furthermore, we then slightly modify the definition of σ_n to

$$\forall s, u : s \neq u \Rightarrow \left(\begin{array}{c} \psi_{n-1}[(\{x, y\}, \{(x, y) \mapsto \{s, -u\}\})] \\ \vee \\ \psi_{n-1}[(\{x, y\}, \{(x, y) \mapsto \{-u, s\}\})] \end{array} \right).$$

The argument for Theorem 57 can be replicated without the need for 1' to be the equivalence relation and without quotienting the limit network by 1'. We conclude the proof for the lemma by iterating on the argument from Corollary 58. □

We will now define the set A_{NS} to be the set of structures $\mathcal{A}_n = (A_n, -, ;)$, for $0 < n < \omega$. Let $N = 2^n + 1$. We define A_n as the carrier set with $4N + 4$ elements, defined below.

$$A_n = \{0, 1, c, -c\} \cup \{a_i, -a_i, b_i, -b_i \mid 0 \leq i < N\}$$

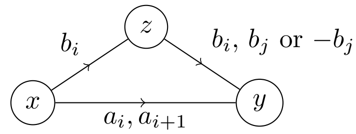

Figure 3.4: The proof of Lemma 84, visualised.

The negation pairs are defined in the definition of A_n, together with $0 = -1$. Let $0 \leq i < N$ and let $+$ denote arithmetic addition modulo N. We define the compositions

$$b_i;s = \begin{cases} a_i & \text{if } s = b_i \\ a_{i+1} & \text{otherwise} \end{cases}$$

for $s \in \{1, -c, -a_j, b_j, -b_j \mid 0 \leq j < N\}$ and

$$s;0 = 0;s = c;s = s;c = a_i;s = s;a_i = 0$$

for $s \in A$. All other compositions, including those with 1 as the operand, result in c.

Lemma 84. *For any $0 < n < \omega$, $\mathcal{A}_n$ is not representable.*

Proof. Suppose $\mathcal{A}_n$ was representable via h over X. We know that for some $0 \leq i \neq j < N$, we have $a_i \neq a_j$ and so, without loss, there exist $x, y \in X$ such that $(x, y) \in h(a_i), (x, y) \in h(-a_j)$. Because $b_i;b_i = a_i$, the composition must be witnessed via some z so that $(x, z) \in h(b_i), (z, y) \in h(b_i)$, see Figure 3.4. The pair (z, y) must therefore either be in $h(b_j)$ or $h(-b_j)$. In either case, composing b_i with $-b_j$ or b_j via z will result in $(x, y) \in h(a_{i+1})$. By iterating on this argument, we eventually get that $(x, y) \in h(a_j)$ and we will have reached a contradiction. □

Lemma 85. *If $\forall$ plays any initialisation pair but $a_i \neq a_j$ for some $0 \leq i \neq j < N$, then $\exists$ will have a winning strategy for $\Gamma^{NS}_\omega(\mathcal{A}_n)$.*

Proof. It suffices to show that there exists a network that $\exists$ can keep returning in response to any move. That is, there exists a network that discriminates all non $a_i \neq a_j$ pairs, all non-empty label sets will contain for all $s \in A_n$ either s or $-s$ but not both, and all compositions are both closed and witnessed. We start by defining $C(\mathcal{A}_n)$ to be the set of all sets $S \subseteq A_n$ such that $0 \notin S$ and for all $s \in A_n$ either $s \in S$ or $-s \in S$ but not both. We define a network $\mathcal{N} = (N, \lambda)$ to have the following nodes

$$\begin{aligned} N = &\{x, z, v, q\} \\ &\cup \{y_{ST} \mid \{-c, -a_i \mid 0 \leq i < N\} \subseteq S, T \in C(\mathcal{A}_n)\} \\ &\cup \{w_V \mid \{-c, -a_i \mid 0 \leq i < N\} \subseteq V \in C(\mathcal{A}_n)\} \end{aligned}$$

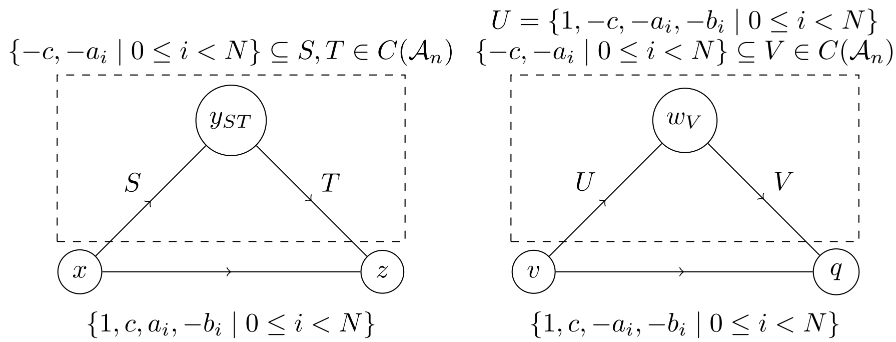

Figure 3.5: The proof of Lemma 85, visualised.

and the following labels

$$\lambda(x, z) = \{1, c, a_i, -b_i \mid 0 \leq i < N\} \qquad \lambda(x, y_{ST}) = S$$
$$\lambda(v, w_V) = \{1, -c, -a_i, -b_i \mid 0 \leq i < N\} \qquad \lambda(w_V, q) = V$$
$$\lambda(v, q) = \{1, c, -a_i, -b_i \mid 0 \leq i < N\} \qquad \lambda(y_{ST}, z) = T$$

with all other pairs of nodes mapping to $\emptyset$. This network is visualised in Figure 3.5.

Observe that all labels are in $C(\mathcal{A}_n)$ and thus negation is preserved and no label contains 0. We now check that all pairs of non equal elements except $a_i \neq a_j$ are discriminated. From the preservation of $-$, it suffices to only check that either s or $-s$ is correctly discriminated against other elements. For 0, observe that every non-zero element appears in at least one label set. Let $0 \leq i, j < N$. For $a_i \neq b_i$ or $a_i \neq -c$, take (x, z) and for $a_i \neq -b_i$ or $a_i \neq c$ take (v, q). Finally, for $b_i \neq c$ take (x, y_{ST}), using suitable S, T, and for $-b_i \neq c$, take any (v, w_V). For composition, we first let the reader check coherence, that is, if $d \in \lambda(x, y), e \in \lambda(y, z)$ then $d;e \in \lambda(x, z)$. Namely, due to U not including b_i, we allow for $-a_i$ in the label of (v, q) and compositions resulting in 0 do not appear as $0, a_i, c$ do not exist outside the labels of (x, z) and (v, q). We now check that all compositions are witnessed. Compositions resulting in 0 are all vacuously witnessed as 0 does not appear on any label. For any composition $b_i;s$ for $0 \leq i < N, s \neq 0, a_j, c, 0 \leq j < N$, see that they occur precisely on x, z via some y_{ST}. It is true that every composition is witnessed by a y_{ST} or a w_V, namely by some y_{UT} or some w_T such that the second element in the composition is in T. □

Lemma 86. *Suppose $\forall$ plays $a_i \neq a_j$ at initialisation for some $0 \leq i \neq j < N$. then $\exists$ will have a winning strategy for $\Gamma_n^{NS}(\mathcal{A}_n)$.*

Proof. Without loss, say $\forall$ plays $a_0 \neq a_{(N-1)/2}$ at initialisation and $\exists$ returns a network with nodes x_0, y_0. $\exists$ may deploy a strategy where after every turn she returns a network, closed under composition, witness, and negation moves, except for $\lambda(x_0, y_0)$ where she will maintain a $0 \leq \iota^+ \leq \iota^- \leq (N-1)/2$ such that

$$\lambda(x_0, y_0) = \{1, c, -b_i, a_j, -a_k \mid 0 \leq j \leq \iota^+, \iota^- < k < N, i < N\}.$$

We show by induction that the network will reach an inconsistency only when $\iota^+ = \iota^-$ and that $\iota^- - \iota^+ \geq \lfloor (N-1)/2^{k+1} \rfloor$ after kth turn.

Base Case For $k = 0$, she assigns $\iota^+ = 0, \iota^- = (N-1)/2$, yielding a consistent and closed network.

Induction Case Due to closure and the rest of the network being saturated, $\forall$ may only ask for witness and complement moves for $\lambda(x_0, y_0)$. In case where the witness move is for c, $\exists$ adds a node z with $\lambda(x_0, z) = \{1, -c, -a_i, -b_i \mid i < N\}$ and the appropriate $\lambda(z, y_0) \in C(\mathcal{A}_n)$ to maintain the induction hypothesis. In case the witness is for some $a_i, i \leq \iota^+$, she adds a node z with appropriate choice sets in $\lambda(x_0, z), \lambda(z, y_0)$ with minimal b_i inclusion in $\lambda(x_0, z)$ to ensure that ι^+ increases by at most one, maintaining the induction hypothesis. Finally, for a complement move between a_k and $-a_k$, she can set ι^+ or ι^- to k in a way that at most halves $\iota^- - \iota^+$, maintaining the induction hypothesis. □

Theorem 87. *The class* $\mathsf{R}(-, ;)$ *is not finitely axiomatisable.*

Proof. By Lemma 85 and Lemma 86, we know that for every $n < \omega$, there will exist a structure in $\mathcal{A} \in \mathsf{A}_{NS}$ such that $\exists$ has a winning strategy for $\Gamma_n^{NS}(\mathcal{A})$, for example, $\mathcal{A}_n$. By Lemma 84, all structures in A_{NS} are also not representable. Thus, by Lemma 83, we conclude that the class is not finitely axiomatisable. □

3.2 Negated Semigroups and the Flexible Atom Conjecture

In Section 3.1, we have seen how we can use the point algebra to show that the finite representation property fails for negated semigroups. Another example of an algebra that fails to have a finite representation is the anti-Monk algebra, we will define it in this section. When studying whether $(-, ;)$-subreducts of this algebra admit finite representations, we encounter an interesting connection to the flexible atom conjecture by proposing a possible counterexample. After failing to find finite representations, we turned to Jeremy Alm, who in [AY19] proposes efficient algorithms for finding representations. In [Alm23], he proves the algebra in question admits finite representations.

Definition 88. *The* anti-Monk algebra $\mathcal{AM}$ *is the algebra with the atom structure* $(A, \mathrm{I}, \smile, \mathrm{T})$ *where the set of self-converse atoms is* $A = \{1', r, g, b\}$*, the predicate* I *consists of* 1' *only, and* T *contains all mandatory identity triangles, as well as all diversity triangles, except* (r, g, b) *and its Peircean transforms.*

Much like the point algebra, the anti-Monk algebra is well known to be representable, but not finitely representable. We now observe that in proving that the $(-, ;)$-reduct of the point algebra does not have the finite representation property, see the proof of Theorem 80, we did not have to use the full $(-, ;)$-reduct of $\mathcal{P}$. Instead, we could have used the subreduct of the $\mathcal{P}$ with the carrier set only including $0, 1$, as well as all the diversity atoms and their negations. In the case of the anti-Monk algebra, we observe a similar structure $\mathcal{AM}' = (AM', -, ;)$ of with its eight-element carrier set defined below

$$AM' = \{0, 1, r, -r, g, -g, b, -b\}$$

with all compositions with 0 resulting in 0, $r;g = g;r = -b, r;b = b;r = -g, b;g = g;b = -r$, and all other compositions resulting in 1. This is not a $(-, ;)$-subreduct of $\mathcal{AM}$, but rather a similarly-behaved structure containing only $0, 1$ and all the diversity atoms with their negations in the carrier.

Now let $\mathcal{AM}^f$ be the algebra we obtain if we amend the atom structure of $\mathcal{AM}$ with a flexible diversity atom f. In Maddux's enumeration of small relation algebras [Mad06], this algebra is called 1896_{3013} and its finite representability was previously unknown.

Proposition 89. *If $\mathcal{AM}^f$ is finitely representable, then $\mathcal{AM}'$ is representable too.*

Proof. Suppose $\mathcal{AM}^f$ is representable via some $h : AM^f \to \wp(X \times X)$. Define $h : AM' \to \wp(X \times X)$ as

$$h'(a) = h(a)$$
$$h'(-a) = h(b) \cup h(c) \cup h(f)$$

for all $a \neq b \neq c \in \{r, g, b\}$ and

$$h'(1) = h(f) \cup h(r) \cup h(g) \cup h(b).$$

This mapping is faithful, namely from the definition of $h'(a)$ for $a \in \{r, g, b\}$. Furthermore, observe that because $r;g = -b$ (and similarly for all permutations of r, g, b), we have that the label set $\{1, -r, -g, -b\}$ is 'flexible', that is, in a network, it can be in any triangle. Furthermore, one may check that all compositions are indeed witnessed. □

Because of the above proposition and its similarity to the point algebra argument, it was not unreasonable to suspect that this algebra could have been a potential counterexample to the flexible atom conjecture. However, Alm [Alm23] shows that the algebra does indeed have a finite representation of size 1531. This is done by embedding into a cyclic group of order 1531, namely $\mathbb{Z}/1531\mathbb{Z}$. Note that 1531 is a prime and 1530 is divisible by 15. This justifies the choices made for the embedding, see [Alm23] for more details.

Below, see the definition of the embedding $f : AM^f \to \wp(\mathbb{Z}/1531\mathbb{Z}))$ for each of the atoms of the algebra. The image of f for the remaining elements of $\mathcal{A}M^F$ can be inferred.

$$f(1') = \{0\}$$
$$f(r) = \{x^{15} \mod 1531 \mid 0 < x < 1531\}$$
$$f(g) = \{32 \cdot x^{15} \mod 1531 \mid 0 < x < 1531\}$$
$$f(b) = \{1024 \cdot x^{15} \mod 1531 \mid 0 < x < 1531\}$$

The reader may check that f is indeed an embedding into $\mathbb{Z}/1531\mathbb{Z}$, see [Alm23] for more detail. Because complex algebras of groups are members of RRA, their substructures are also representable over the base set of the size of their atom structures. Thus $\mathcal{A}M^f$ is representable — with the upper bound on the size of the smallest representation at 1531 — and not a counterexample to the flexible atom conjecture.

3.3 Semigroups with Heyting Implication

The results presented in this section were shown in collaboration with Andrew Lewis-Smith. The author thanks him for agreeing to present these results in this thesis.

In this section, we examine another novel result, closely related to the negative side of the finite representation property conjecture. Although we were able to show that $\mathsf{R}(-, ;)$ is not finitely axiomatisable and that the finite representation property fails for its finite members, it remains open whether the representation problem is decidable for the signature. In his forthcoming work, Jackson shows that the representation problem is undecidable if we require square representability. However, the argument relies on this requirement and cannot be trivially extended to arbitrary representations.

Here we show the undecidability of the (finite) representation problem for another small fragment of τ_{RA} with a negation-like operation, the Heyting arrow. The Heyting arrow is term-definable from relation algebra as

$$R \to S = (-R) + S$$

or alternatively as

$$R \to S = \{(x, y) \in \top \mid (x, y) \in R \Rightarrow (x, y) \in S\}$$

where $\top$ is some maximal relation over X, as discussed in Definition 12. Because the signature is closed under composition, the relation $\top$ must be transitive, however, unlike with the class PRA, we do not require it to be reflexive or symmetric. We also consider the case of square representability, that is, when we require $\top = X \times X$.

The signature $\tau = (\to)$ — or equivalently, $\tau = (\to, 1)$ as 1 is term definable as the constant $s \to s$ for all s — has been studied by Abbott [Abb67], Rasiowa

[Ras74], Diego [Die65], and their students. They independently show that the class $\mathsf{R}(\to)$ is axiomatised by the axioms below, as well as a Stone-like representation theorem for the class, implying $\mathsf{R}(\to)$ has the finite representation property.

(IA1) $(s \to t) \to s = s$ (contraction),

(IA2) $(s \to t) \to t = (t \to s) \to s$ (quasi-commutativity),

(IA3) $s \to (t \to u) = t \to (s \to u)$ (exchange).

In their work, they also show important properties about the operation $\to$. We prove them below.

Proposition 90. *Let* $\to \in \tau$ *and let* $\mathcal{A} = (A, \tau) \in \mathsf{R}(\tau)$ *via some representation* h*. For any* $s, t \in A$ *we have that*

(1) $h(s \to s) = \top$,

(2) $h((s \to t) \to t) = h(s) \cup h(t)$.

As a consequence, the symbols $\leq, +, 1$ *are atom/term-definable from* $\to$.

Proof. For (1), see that for any $(x, y) \in \top$ if $(x, y) \in h(s)$ then $(x, y) \in h(s)$ is a tautology and thus all $(x, y) \in \top$ are also in $h(s \to s)$. For (2), see that if $(x, y) \in h((s \to t) \to t)$ then it is either in $h(t)$ or not in $h(s \to t)$ — or equivalently in $h(s)$ and not $h(t)$. Thus (x, y) is in $h(t)$ or $h(s)$ if and only if it is in $h((s \to t) \to t)$. □

Proposition 91. *Let* $\to, 0 \in \tau$ *and let* $\mathcal{A} = (A, \tau) \in \mathsf{R}(\tau)$ *via some representation* h*. For any* $s \in A$ *we have that*

$$h(s \to 0) = -h(s) = \top \setminus h(s).$$

As a consequence, $-$ *is term-definable from* $\to, 0$.

Proof. The image $h(s \to 0)$ — as $\to$ is correctly represented — is equal to $-h(s) \cup h(0) = (\top \setminus h(s)) \cup h(0)$. As 0 is correctly represented as $\emptyset$ by h, this proves the proposition. □

The signatures containing $\to, ;$ have been studied in [HJ12], more specifically, they show that the signature $(\to, 1', ;)$ has the [finite] representation problem undecidable — that is, given a finite $(\to, 1', ;)$-structure, it is not algorithmically decidable in a finite amount of time whether the structure is representable [over a finite base].

We use the subsequent undecidability results from [HJ12, HHJ21, Neu16] to extend the results to all converse-free signatures containing $\to, ;$, dropping the 1' inclusion requirement. The proofs are split into two cases. In the first case, we show the decidability failure for the conventional version of the [finite] representation problem and, in the second case, we show the same failure for the [finite] square-representation problem.

The reduction in the first case comes from the undecidability of the [finite] representation problem for $\mathsf{R}(0, 1, -, +, ;)$ from [HHJ21].

Theorem 92 (Hirsch–Hodkinson–Jackson)**.** *Any signature* $(\cdot, +, ;) \subseteq \tau \subseteq (0, 1, -, +, 1', ;)$ *has the [finite] representation decision problem undecidable.*

Lemma 93. *Let* $\mathcal{A} = (A, \tau)$ *be a* τ*-structure for some* $(\to, ;) \subseteq \tau \subseteq \tau_{\mathsf{RA}}$ *that contains some element* 0 *such that for all* $s \in A$ *we have* $0 \le s$ *and* $0;s = s;0 = 0$*. If* $\mathcal{A}$ *is representable via some representation* h *then there exists a representation* h' *of* $\mathcal{A}$ *where* $h'(0) = \emptyset$*. If* h *is over a finite base, then so is* h'*.*

Proof. Let $h(\mathcal{A})$ be the proper image of h for some $\top \subseteq X \times X$. As h is a representation, there exists for every pair $s \not\le t \in A$ a *discriminator pair* $(\iota, o) \in \top$ such that $(\iota, o) \in h(s) \setminus h(t)$. Fix any such s, t and also an ι, o for them.

Define $X^{\iota,o}$ as

$$X^{\iota,o} = \left\{ x \in X \;\middle|\; \Big(x = \iota \vee (\iota, x) \in \top\Big) \wedge \Big(x = o \vee (x, o) \in \top\Big) \right\},$$

$\top^{\iota,o}$ as $\top \cap (X^{\iota,o} \times X^{\iota,o})$, and a mapping $h^{\iota,o} : A \to \wp(\top^{\iota,o})$ where $h^{\iota,o}(s) = h(s) \cap \top^{\iota,o}$.

First, observe that $h^{\iota,o}(0) = \emptyset$. Suppose that there was a pair $(x, y) \in h^{\iota,o}(0)$. If $x = \iota$, we have $(\iota, y) \in h^{\iota,o}(0)$, else $(\iota, x) \in h(1) = \top$ and thus $(\iota, y) \in h(0)$ since $1;0 = 0$ and h preserves composition. Similarly if $y = o$ we get $(\iota, o) \in h(0)$, else by composing $(\iota, y) \in h(0)$ with $(y, o) \in h(1)$ we get $(\iota, o) \in h(0)$. Since $t \ge 0$ that would mean $(\iota, o) \in h(t)$ that contradicts the fact that (ι, o) is a discriminator pair for $s \not\le t$.

Now let us check that $h^{\iota,o}$ preserves composition. Suppose $(x, y) \in h^{\iota,o}(u;v)$. This means that there exists $z \in X$ such that $(x, z) \in h(u)$ and $(z, y) \in h(v)$. If $x = \iota$, we trivially have $(x, z) \in h(1) = \top$. Else, by composing $(\iota, x) \in h(1)$ and $(x, z) \in h(u)$ we get $(\iota, z) \in h(1) = \top$ as $1;u \le 1$. Similarly $(z, o) \in \top$ and thus $z \in X^{\iota,o}$. Thus we have $(x, z) \in h^{\iota,o}(u), (z, y) \in h^{\iota,o}(v)$ and we have shown $h^{\iota,o}(u;v) \subseteq h^{\iota,o}(u);h^{\iota,o}(v)$. The fact that $h^{\iota,o}(u;v) \supseteq h^{\iota,o}(u); h^{\iota,o}(v)$ follows from $(x, y), (y, z) \in \top^{\iota,o}$ then $x, z \in X^{\iota,o}$ and we have $(x, z) \in \top^{\iota,o}$. Thus $h^{\iota,o}$ preserves composition.

All other operations, including $\to, +, -, 1, \smile, 1'$ are defined on at most two points and are thus trivially preserved by $h^{\iota,o}$. Thus we conclude that $h^{\iota,o}$ is a homomorphism that discriminates the pair $s \not\le t$.

Now let us pick for every $s \not\le t$ some pair of points $\delta(s, t) = (\iota, o)$ such that $(\iota, o) \in \top$ is a discriminator pair for $s \not\le t$ and let $\uplus$ denote a disjoint union. A mapping

$$h' : A \to \wp\left(\biguplus_{s \not\le t \in A} \top^{\delta(s,t)} \right)$$

$$h'(u) = \biguplus_{s \not\le t \in A} h^{\delta(s,t)}(u)$$

still represents all the operations correctly, as well as discriminates all pairs $s \not\le t$ — it is injective. That makes it a representation. Furthermore, $h'(0) = \emptyset$ and the size of its base is bounded by $|A|^2|X|$. ☐

With the help of the above lemma, we conclude.

Theorem 94. *The [finite] representation problem is undecidable for any signature* $(\to, ;) \subseteq \tau \subseteq \tau_{\mathsf{RA}} \setminus (\smile)$.

Proof. Let $\tau' = (\tau \setminus (\to)) \cup (0, 1, +, -)$. Take a finite τ'-structure $\mathcal{A}'$ and its τ-reduct $\mathcal{A}$. We show that the following two statements are equivalent:

(1) $\mathcal{A}'$ has a [finite] representation,

(2) $\mathcal{A}$ has a [finite] representation and $\mathcal{A}' \models \forall s : 0 \leq s \wedge s;0 = 0;s = 0$.

See that (1)$\Rightarrow$(2) follows from $s;0 = 0;s = 0, 0 \leq s$ being valid for all structures in $\mathsf{R}(\tau')$. The converse follows from Lemma 93. We know that (1) is undecidable as Hirsch, Hodkinson, and Jackson show (see Theorem 92). Further, the right hand side of (2) is decidable. As determining τ-reduct of a finite τ'-structure is a computable function, we conclude that the left hand side of (2) is undecidable. As this is a special case of the decision problem in question, we conclude it is undecidable as well. □

We now consider the square-representation problem and the finite square-representation problem for these signatures. The argument presented above does not trivially translate into this setting due to the use of disjoint unions. However, the undecidability can be shown via an easier route from the following theorem from [HJ12].

Theorem 95 (Hirsch–Jackson)**.** *The [finite] square-representation problem is undecidable for* $\tau_{\mathsf{RA}} \setminus (\smile)$.

We break our reduction into a few lemmas. These proofs are closely based on those presented in [Neu16] to show the undecidability of the square-representation problem for $(\leq, -, ;)$.

Lemma 96. *Let* $\mathcal{A} = (A, \tau)$ *be a* τ*-structure for some* $(\leq, ;) \subseteq \tau \subseteq \tau_{\mathsf{RA}}$ *that contains some element* 0 *such that for all* $s \in A$ *we have* $0 \leq s$ *and* $0;s = s;0 = 0$. *In any square* τ*-representation* h *of* $\mathcal{A}$, $h(0) = \emptyset$.

Proof. Suppose there existed a pair $(x, y) \in h(0)$. As h preserves composition and for any pair of points (z, w), we have (due to square-representability) some $s, t \in A$ such that $(z, x) \in h(s), (y, w) \in h(t)$ and so $(z, w) \in h(s;0;t) = h(0)$. Because 0 is below all other elements in A and h represents $\leq$ correctly, this means that for all (z, w) are in $h(s)$, for all s and h is not injective. Thus we have reached a contradiction and $h(0)$ must be equal to $\emptyset$. □

Lemma 97. *Let* $\mathcal{A} = (A, \tau)$ *be a* τ*-structure for some* $(-, ;) \subseteq \tau \subseteq \tau_{\mathsf{RA}}$ *with some element* $1'$ *such that for all* $s \in A$, $s;1' = 1';s = s$. *In any square representation* h *of* $\mathcal{A}$, *we will have that* $h(1')$ *is an equivalence relation.*

Proof. As h represents ; and 1';1' = 1', we have that the image $h(1')$ is transitive. Suppose $h(1')$ was not reflexive. Then there would exist an x such that $(x,x) \in h(-1')$ as h preserves $-$. Furthermore, as $-1' = (-1');1'$, there must exist some y such that $(y,x) \in h(1')$ to witness this composition. As $-1' = 1';(-1')$ and $(y,x) \in h(1'), (x,x) \in h(-1')$ we must have that $(y,x) \in h(-1')$. This would imply that $-$ is not correctly represented and thus $h(1')$ must be reflexive. For symmetry, assume there existed a pair $(x,y) \in h(1'), (y,x) \in h(-1')$. By composing $1';(-1') = -1'$ via y, we have that $(x,x) \in h(-1')$ and thus by composing $(-1');1' = -1'$ via x, we have that $(x,y) \in h(-1')$ and we have reached a contradiction. Thus $h(1')$ is symmetric, reflexive, and transitive. □

Lemma 98. *Let $\mathcal{A} = (A, \tau)$ be a τ-structure for some $(-, ;) \subseteq \tau \subseteq \tau_{\mathsf{RA}}$ with some element 1' such that for all $s \in A$, $s;1' = 1';s = s$. If $\mathcal{A}$ is square representable via some h over X, then it is square-representable via some h' over X' such that $h'(1') = \{(x,x) \mid x \in X'\}$. If X is finite, X' is as well.*

Proof. By the previous lemma, $h(1')$ is an equivalence relation over X. Let $X' = X/h(1')$. Observe that for all $s \in \mathcal{A}$ and $x, x', y, y' \in X$ such that $x' \in [x]_{h(1')}, y' \in [y]_{h(1')}$, we have the below equivalence.

$$(x,y) \in h(s) \Longleftrightarrow (x',y') \in h(s)$$

This is because for all s we have $(x,y) \in h(s)$ if and only if $(x',y') \in h(1');h(s);h(1') = h(1';s;1') = h(s)$.

Thus we can define $h : \mathcal{A} \to \wp(X' \times X')$ as

$$h'(a) = \{([x]_{h(1')}, [y]_{h(1')}) \mid (x,y) \in h(a)\}.$$

h' is trivially still faithful and represents all operations correctly. Furthermore, if X is finite, so is X'. □

We can now conclude the below theorem.

Theorem 99. *The [finite] square-representation problem is undecidable for any signature $(\to, ;) \subseteq \tau \subseteq \tau_{\mathsf{RA}} \setminus (\smile)$.*

Proof. Take a finite $(\tau_{\mathsf{RA}} \setminus (\smile))$-structure $\mathcal{A}'$ and its τ-reduct $\mathcal{A}$. We show the following two statements are equivalent.

(1) $\mathcal{A}'$ is [finitely] representable,

(2) $\mathcal{A}$ is [finitely] squarely representable and $\mathcal{A}' \models \forall s : 1';s = s;1' = s \wedge 0 \leq s \wedge s;0 = 0;s = 0$.

Again (1)⇒(2) follows from the fact that $\forall s : 1';s = s;1' = s \wedge 0 \leq s \wedge s;0 = 0;s = 0$ is valid in $\mathsf{R}(\tau_{\mathsf{RA}} \setminus (\smile))$. The converse follows from Lemmas 96, 98. We know that (1) is undecidable as Hirsch and Jackson show (see Theorem 95). Further, the right hand side of (2) is decidable. As determining τ-reduct of a finite τ'-structure is a computable function, we conclude that the left hand side of (2) is undecidable. As this is a special case of the decision problem in question, we conclude it must be undecidable as well. □

3.4 The Positive Implication of the FRP Conjecture

In this section, we discuss how the point algebra or the anti-Monk algebra reducts admit finite representations in the signatures conjectured to have the said property. In fact, any relation algebra — that is members of RA, not necessarily RRA — will have its reduct finitely representable in all of these signatures. While this does not prove the positive direction of the FRP conjecture, it is a good indication of why it may in fact be true. We begin this section by stating a theorem by Jonsson and Tarski and then continue to speculate how it could help us prove the positive direction of the FRP conjecture.

We begin the section by outlining the Tarski-Jonsson construction for representing the $(0, 1, +, 1', \smile, ;)$-reduct of relation algebras, yielding a finite representation for finite structures. By extension, this means that any subreduct of a finite relation algebra in any of the conjectures signatures is finitely representable. Therefore, the problem of [finite] representability for structures in these signatures can be restated to the embedding into a [finite] relation algebra.

The theorem below was proven in [JT52, Theorem 4.22].

Theorem 100 (Jónsson–Tarski)**.** *Take a relation algebra* $\mathcal{A} \in \mathsf{RA}$*. Its* $(0, 1, +, 1', \smile, ;)$*-reduct is representable.*

The construction notes that without loss we may assume that the algebra is atomic and defines a representation $h : A \to \wp(\mathbf{At}(\mathcal{A}) \times \mathbf{At}(\mathcal{A}))$ where $(a, b) \in h(s)$ if and only if $a \leq s;b$. A neat consequence of this is that a finite relation algebra (which is also atomic) will be finitely $(0, 1, +, 1', \smile, ;)$-representable.

This on its own is a strong indication towards the positive side of the conjecture, however, it does come with its caveats. As structures in the representation classes are trivially closed under substructures, we get the following corollary.

Corollary 101. *A finite structure* $\mathcal{A} \in \mathsf{R}(\tau)$ *for some* $\tau \subseteq \tau_{\mathsf{RA}} \setminus (-)$ *is finitely representable if and only if it embeds into a finite member of* $\mathcal{A}' \in \mathsf{RA}$ *via some embedding where if* $1 \in \tau$*,* $1 \in A$ *is not necessarily preserved as* $1 \in A'$*, however, all elements in* A *are below* 1*.*

Proof. From Theorem 100 and the fact that representation classes are closed under substructures (even without preserving 1) we show that if such an embedding exists, then $\mathcal{A}$ is finitely representable. For the converse, take a finitely representable structure $\mathcal{A}$ via some h over some finite X. Take the proper algebra $\mathcal{A}' = (\wp(X \times X), 0, 1, +, -, 1', \smile, ;)$. Because $\mathcal{A}'$ is a proper relation algebra, it clearly has a four-dimensional basis and as such we have $\mathcal{A}' \in \mathsf{RA}$. The structure $\mathcal{A}$ embeds into $\mathcal{A}'$ trivially. □

Remark 102. *From now on, we will refer to such an embedding into a relation algebra that doesn't necessarily preserve* 1*, but ensures that it remains the top element of the substructure, simply as an embedding.*

Therefore, our search focuses on defining an embedding into a finite relation algebra for all finite structures in $\mathsf{R}(\tau)$ where $\tau \in \tau_{\mathsf{RA}} \setminus (-)$. We sketch a possible way of showing such an embedding exists. Let τ be a signature conjectured to have the FRP and $\mathcal{A}$ a representable τ-structure.

We start by defining an atom structure $\mathcal{A}_1 = (A_1, \mathrm{I}_1, \smile_1, \mathrm{T}_1)$ as

(1) A_1 contains the following atoms:

- $d(S)$ for all $S \subseteq A$ such that there exists a representation h of $\mathcal{A}$ over X with $x \in X$ such that for all $s \in A$ we have $(x, x) \in h(s) \Leftrightarrow s \in S$,
- $a(s), b(S)$ for all $S \subseteq A$ such that there exists a representation h of $\mathcal{A}$ over X with $x \neq y \in X$ such that for all $s \in A$ we have $(x, y) \in h(s) \Leftrightarrow s \in S$,

(2) I_1 contains all atoms $d(S)$,

(3) $\smile_1$ is defined as all corresponding $a(S), b(S)$ and is the identity for $d(S)$,

(4) T_1 is defined as all the triplets of atoms $(a, b, c) \in A_1^3$ such that there exists a representation h over X and points (x, y, z) such that the points (x, z) are an example for a, (x, y) are an example for b, and (y, z) are an example c.

Observe that this atom structure has at most $3 \times 2^{|A|}$ atoms. Furthermore, we will say that a set $S \subseteq A$ *defines* an atom, whereas a pair of points in some representation is an *example for* that atom. Furthermore, it is easy to see how elements of $\mathcal{A}$ embed into this atom structure via some f, where we say that $f(s)$ is the join of all atoms $d(S), a(S)$ if $s \in S$ as well as all $b(S)$ such that for all examples (y, x) for $b(S)$ we have that $(x, y) \in h(s)$.

The reader can check that $\mathcal{A}_1$ obeys all relation algebra axioms for atom structures, except not necessarily (T6) and (T7). This is why further iterations of splitting are needed. Given, for some $1 \leq i < \omega$, an atom structure $\mathcal{A}_i = (A_i, \mathrm{I}_i, \smile_i, \mathrm{T}_i)$ where every atom in A_i has an example pair of points in some representation, we can define an atom structure $\mathcal{A}_{i+1} = (A_{i+1}, \mathrm{I}_{i+1}, \smile_{i+1}, \mathrm{T}_{i+1})$ as

(1) A_{i+1} is the set of all $t(a, S)$ where $a \in A_i$ and $S \subseteq \wp(A_i \times A_i)$ such that there exists a representation h in which there exists a pair of points x, y that are an example for a and a pair (b, c) is included in S if and only if there exists a point z with (x, z) an example for b and (z, y) an example for c,

(2) I_{i+1} is the set of all $t(a, S)$ where $a \in \mathrm{I}_i$,

(3) $\smile_{i+1}$ for $t(a, S)$ is defined as $t(a', S')$ where a, a' are a $\smile_i$ pair and S' is the set of all (c', b') where $(b, c) \in S$ and b, b' as well as c, c' are $\smile_i$ pairs,

(4) T_{i+1} is defined as all the triplets of atoms $(a, b, c) \in A_{i+1}^3$ such that there exists a representation h over X and points x, y, z such that the points

(x, z) are an example for a, (x, y) are an example for b, and (y, z) are an example c.

Again, $\mathcal{A}_i$ embeds into $\mathcal{A}_{i+1}$ via some f where $f(a)$ is the join of all $d(a, S) \in A_{i+1}$. Furthermore, for all i, because $\mathcal{A}_i$ is finite, so is $\mathcal{A}_{i+1}$. Finally, $\mathcal{A}_{i+1}$ still obeys all axioms for relation algebra atom structures, except perhaps (T6) and (T7).

Now we have defined for some structure $\mathcal{A}$ and every $1 \leq i < \omega$, an atom structure $\mathcal{A}_i$ such that $\mathcal{A}$ embeds into its corresponding Boolean unital involutive magma. However, whether any of these $\mathcal{A}_i$ is a Relation Algebra for any of these structures remains open, namely whether there is a guarantee that as i grows, this structure is likely to become associative.

We speculate that, using these structures obtained by iterative splitting, the conjecture could be proven in one of the following ways.

(1) Showing that for every finite representable structure, $\mathcal{A}$, there will exist an $i < \omega$ for which $\mathcal{A}_i$ is an atom structure corresponding to a member of RA,

(2) Showing that one can quotient the cannonical extension of the limit structure $\mathcal{A}_\omega$ via an equivalence relation with finitely many classes to obtain a finite structure in which $\mathcal{A}$ embeds,

(3) Using step-by-step techniques to show that the ultraproduct of these structures is associative and by Łoś' theorem show that a large number of structures $\mathcal{A}_i$ are associative for $i < \omega$,

(4) Using probabilistic arguments (such as Erdös' random graphs) to show that eventually $\mathcal{A}_i$ (or their subalgebras into which $\mathcal{A}$ embeds) are in fact relation algebras.

3.5 Conclusion

In conclusion, this chapter makes significant progress in proving the main conjecture discussed in this thesis, however, in its most general case, it remains open. Namely, we have proved the negative implication of Conjecture 27 but left the positive implication open. We then continued by showing that some other related (but not weaker) properties also fail for the signature of negated semigroups and their closely related cousins — implication semigroups. This means that as a result of our work, it suffices to only show the positive implication if one wants to prove Conjecture 27. We also speculate on possible directions of solving the remaining open cases of this main conjecture. Further, we have had a look at an interesting connection between two open conjectures in the area of relation algebra. In the chapters that follow, we move away from the classical relation algebraic setting, but continue to examine the open part of Conjecture 27 and related properties and show some novel results in the area.

Chapter 4

Demonic Lattice Semigroups

The results presented in this chapter were shown in collaboration with Robin Hirsch. The author thanks him for agreeing to present these results in this thesis.

Now that we have examined the problems from the classical setting, we continue by looking at finite representability and related topics in a non-classical setting. More specifically, this chapter is concerned with the behaviour of certain signatures that contain both Tarskian relation algebra operations, as well as their demonic duals. Namely, we are interested in fragments with ordinary (or 'angelic') relational composition and the demonic lattice operations. This mixing and matching may appear somewhat counterproductive or unintuitive, however, it is well-motivated. We show that it provides us with a neat trick in modelling termination and total correctness in formal verification.

In the first section, we motivate the definitions of demonic operations and outline the framework for equationally expressing the total correctness of non-deterministic programs. Furthermore, we motivate and define the demonic meet ($\sqcap$) for binary relations — an operation that had not been studied previously.

We then show that the representation class $\mathsf{R}(\sqsubseteq, ;)$ is not finitely axiomatisable, but has the FRP for its finite members. This appears to be the first such result for a reduction of relation algebra. We continue by showing that $\mathsf{R}(\sqcup, ;)$ is not finitely axiomatisable and that FRP fails for finite members of $\mathsf{R}(\sqcap, ;)$. We conclude with a result showing the undecidability of the representation problem for finite $(\sqcap, \sqcup, ;)$-structures.

None of these results contradict the main conjecture, in fact all results regarding the finite representation property in this chapter prove a special case of it. Further, the signature $(\sqsubseteq, ;)$ is the first known reduction signature of relation algebra that fails to have a finitely axiomatisable representation class, but finite representation property. This shows, together with other examples in literature, that the two properties, while related, are not dependent on each other.

4.1 Relational Modelling of Total Correctness

In this section, we formalise the framework for relational reasoning about termination and total correctness of nondeterministic programs using demonic lattice and ordinary composition. This is based on the refinement algebra [vW04] (an extension of Kleene algebra), but adapted for relational reasoning about correctness as described in [MDM87].

We begin by defining the configuration space C. For Turing machines, a *configuration* is a tuple that contains the current state, the position of the head, and the contents of the tape. In software engineering, configurations are often thought of as assignments of variables (including those yet to be assigned) and the instruction pointer for some imperative program. The configuration space is the collection of all possible configurations for a Turing machine (or equivalently, a program).

A deterministic program fragment can then be modelled as a partial function from the space of configurations C of the machine to itself where the image of $c_1 \in C$ is the $c_2 \in C$ where the program terminates after its execution starts from c_1 if such c_2 exists and undefined otherwise. For nondeterministic programs this c_2 may not be unique, thus we model a nondeterministic program fragment as a relation.

To define it formally, given the configuration space C, we say that a nondeterministic program fragment A is modelled as a relation over the base C where $(c_1, c_2) \in A$ if and only if there exists a possible run of A from c_1 that terminates in c_2.

Using this type of modelling, we can use relational calculus to model the behaviour of program fragments. Some examples include

(1) Relational composition (;) to model sequential runs of two program fragments,

(2) Join (+) as the nondeterministic choice between two program fragments,

(3) The identity relation (1') to model the 'skip' program fragment,

(4) The empty relation (0) to model an infinitely looping program fragment.

Furthermore, reflexive relations over the same configuration space may be used to model conditions, for tests. We say that a condition P is modelled by the set of all pairs $(c, c) \in C \times C$ such that $c \models P$. These come in useful in providing relational semantics for Hoare logic [Hoa69]. A Hoare triple is a triple (P, A, Q) where P, Q are conditions, called the pre- and the postcondition and A is a program fragment.

For deterministic program fragments A, a Hoare triple (P, A, Q) is said to be *partially correct* if and only if the program fragment A, when run from a configuration satisfying P, either terminates in a configuration satisfying Q or loops forever. It is said to be *totally correct* if and only if it is partially correct and every execution of the program fragment A from a configuration satisfying P terminates.

This is generalised to nondeterministic programs as follows. Let A be a nondeterministic program fragment. We say that a Hoare triple (P, A, Q) is partially [totally] correct if and only if for any well-scheduled deterministic run A' of A, the triple (P, A', Q) is partially [totally] correct. By a well-scheduled run, we mean a run where every computational step is scheduled to be executed eventually. We note that this is not the only generalisation of Hoare logic semantics to nondeterministic programs, however, it appears to be a widely accepted generalisation and it is the definition we will use in this section.

The reader can now check that relationally, the partial correctness of a Hoare triple (P, A, Q) can be expressed equationally in two ways

(1) $P;A;Q = P;A$,

(2) $P;A;(\neg Q) = P;0 = 0$.

Note that the negation of the condition Q is expressed as $\neg$ and not as $-$. This is because the conditions are complemented with respect to the maximal reflexive relation and not the top relation — that is, $\neg Q = 1' \cdot (-Q)$.

The two statements are equivalent when 0 is the bottom element of the 'angelic' lattice, that is, an empty set. We will define the relational interpretation of the bottom of the demonic lattice later in this section where the two equations will no longer be equivalent.

Before we see that, let us first define and motivate the demonic operations for binary relations. Recall from Definition 65 that the demonic join ($\sqcup$), refinement ($\sqsubseteq$), and composition ($*$) are defined for relation algebra in terms of first-order logic as

$$\begin{aligned} R \sqcup S =& \Big\{(x,y) \;\Big|\; (x,y) \in R \cup S, \\ & \exists z : (x,z) \in R, \exists z : (x,z) \in S\Big\}, \\ R \sqsubseteq S \Leftrightarrow& \Big(\{x \mid \exists y : (x,y) \in S\} \subseteq \{x \mid \exists y : (x,y) \in R\} \\ & \wedge \{(x,y) \in R \mid \exists z : (x,z) \in S\} \subseteq S\Big), \\ R * S =& \Big\{(x,y) \;\Big|\; \exists z : (x,z) \in R \wedge (z,y) \in S, \\ & \forall z : ((x,z) \in R \Rightarrow \exists w : (z,w) \in S)\Big\}. \end{aligned}$$

Note that we have previously defined as $R \sqsubseteq S \Leftrightarrow S = R \sqcup S$, however, the reader may check the two are equivalent. Both operations and the predicate are also term/atom-definable from relation algebra as

$$\begin{aligned} R \sqcup S &= (1' \cdot R;\breve{R} \cdot S;\breve{S}); (R + S), \\ R \sqsubseteq S &\Leftrightarrow (1' \cdot R;\breve{R} \cdot S;\breve{S}); (R + S) = S, \\ R * S &= (1' \cdot -\mathrm{D}(R;(1' \cdot -\mathrm{D}(S))));R;S. \end{aligned}$$

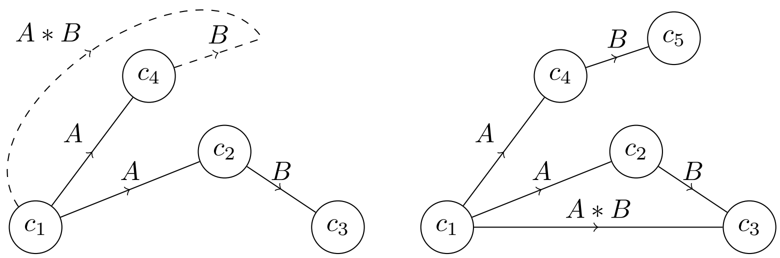

Figure 4.1: The demonic composition motivation, visualised.

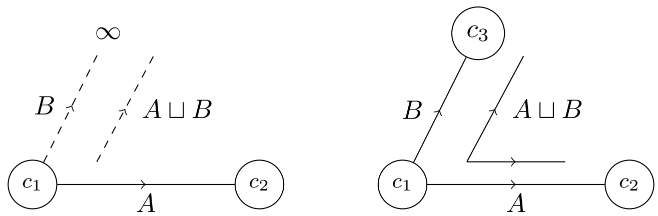

Figure 4.2: The demonic join motivation, visualised.

Note that we use the domain operation D in the last definition, however, as D is term-definable, so is $*$.

We now motivate the demonic composition. Imagine the demon was in control of the nondeterminism in the machine. His goal is to abort or loop indefinitely if possible, otherwise, maximise the opportunities to establish the wrong postcondition. So given two program fragments A, B, modelled as binary relations and some $(c_1, c_2) \in A, (c_2, c_3) \in B$, the demon would not include $(c_1, c_3) \in A * B$ if there existed some $c_4, (c_1, c_4) \in A$ with no c_5 such that $(c_4, c_5) \in B$ (see the left part of Figure 4.1). This is because running B from c_4 results in an infinite loop and so he picks the run of A that takes it to c_4 when it is succeeded by B, resulting in an infinite loop. Alternatively, if for all $c_4, (c_1, c_4) \in A$ there exists a c_5 such that $(c_4, c_5) \in B$ (see the right part of Figure 4.1), then (c_1, c_3) should be included in $A * B$ as a run to c_3 may establish an undesirable postcondition while other runs do not.

Similarly, we motivate demonic join with choice in the demonic nondeterminism. Consider again two program fragments A, B and, without loss, for a pair of configurations $(c_1, c_2) \in A$ we have $(c_1, c_2) \in A \cup B$. However, if the demon is in control of the machine and there is no c_3 such that $(c_1, c_3) \in B$, this means that B loops infinitely from c_1. Thus the demon will choose to run B from c_1 and hence $(c_1, c_3) \notin A \sqcup B$. Otherwise, if c_1 is in the domain of B, the demon will try to maximise the possibility of the program potentially reaching an undesirable postcondition, so $(c_1, c_2) \in A \sqcup B$. See Figure 4.2.

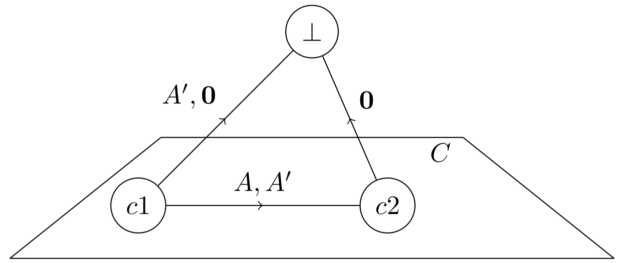

Figure 4.3: The 'magic' program, visualised.

The definition of the demonic join enables us to talk about the demonic semilattice. It also enables us to talk about the demonic refinement $\sqsubseteq$. It becomes apparent that the empty relation 0 is the top element of this lattice. Intuitively speaking, it is easy to see why. As discussed, 0 models the aborting and non-terminating programs. Thus for any program A, when the demon is given a choice between 0 and A, he will choose 0, i.e. $A \sqcup 0 = 0$, $A \sqsubseteq 0$. This is why we define the demonic top $\mathbf{1}$ as the empty relation 0.

While the demonic top and join were straightforward to define, the same cannot be said for the demonic bottom and meet. Two arbitrary binary relations may not have an infimum with respect to $\sqsubseteq$. For example, $R = \{(x, y)\}$ and $S = \{(x, z)\}$ both have the domain of (x, x) and thus any relation that demonically refines R and S will need to have x in its domain. However, there is no pair (x, w) such that $(x, w) \in R, (x, w) \in S$. Furthermore, this highlights the fact that there is no program that the demon would always avoid.

To address that, we introduce a program $\mathbf{0}$, that the demon would avoid at all costs. In fact, this program is akin to magic, it should both terminate and establish any postcondition, even $\bot$. To model this, we extend the space C to $C' = C \cup \{\bot\}$ and let

$$\mathbf{0} = \{(x, \bot) \mid x \in C\}.$$

For all conditions P, including contradictions, we let $P' = P \cup \{(\bot, \bot)\}$ and for all programs A, we let $A' = A \sqcup \mathbf{0} = A \cup \{(x, \bot) \mid (x, x) \in \mathrm{D}(A)\}$, as illustrated in Figure 4.3.

Now that we have defined the demonic bottom $\mathbf{0}$, we can define the meet as the infimum with respect to $\sqsubseteq$ for all binary relations above $\mathbf{0}$. We say that if we have some $S, R \subseteq X \times X$ and we appropriately amend the set X to get X' and relations S', T', we define the meet in terms of first-order logic as

$$\begin{aligned} S' \sqcap T' = &\{(x, y) \in S' \mid \neg\exists z : (x, z) \in T'\} \\ &\cup \{(x, y) \in S' \mid (x, y) \in T'\} \\ &\cup \{(x, y) \in T' \mid \neg\exists z : (x, z) \in S'\} \end{aligned}$$

or as terms in τ_{RA} as

$$S' \sqcap T' = S' \cdot T' + (1' \cdot -\mathrm{D}(S'));T' + (1' \cdot -\mathrm{D}(T'));S'.$$

Observe how (P', A', Q') is still correct if and only if (P, A, Q) is correct — both partially and totally. Intuitively, this is because if the program A' is defined as $A \sqcup \mathbf{0}$, i.e. the demon's choice between A and 'magic', the behaviour of the program will not change. However, observe that the following equations for characterising the correctness of (P', A', Q') are no longer equivalent.

(1) $P';A';Q' = P';A'$,

(2) $P';A';(\neg Q') = P';\mathbf{0}$.

In fact, (1) still expresses partial correctness, that is, all terminating runs of A' from P' establish the postcondition Q'. However, (2) now characterises total correctness, that is, all terminating runs of A' from P' establish Q' and there exists a terminating run of A' from P'. This equational characterisation of mixing the bottom element of the demonic lattice with the ordinary 'angelic' composition provides a neat motivation for studying the algebras and structures examined in this section.

4.2 Demonic Refinement Semigroups

In this section, we focus on the signature $(\sqsubseteq, ;)$, that is, the signature of partially ordered semigroups where the refinement is demonic and the composition is 'angelic'. Zareckiĭ [Zar59] provides a finite axiomatisation for $\mathsf{R}(\leq, ;)$, which axiomatises $\mathsf{R}(\sqsubseteq, *)$ as well. Recently, it has been shown [HMS20] that the $\mathsf{R}(\leq, *)$ is not finitely axiomatisable. However, the problem of the finite axiomatisability of $\mathsf{R}(\sqsubseteq, ;)$ remained open.

In this section, we first define a recursively enumerable, complete axiomatisation of $\mathsf{R}(\sqsubseteq, ;)$. While this axiomatisation is not finite, it is much simpler than that obtained from relation algebra by games as shown in Section 2.2. We then explicitly define a representation for all members of the class. As the base of this representation is finite for finite structures, we show the finite representation property for the signature. Finally, we show that the class cannot be axiomatised by a finite number of first-order formulas. This novel result also presents the first known reduction signature of τ_{RA} that does not have a finitely axiomatisable representation class but the finite representation property. However, this is not the first class of algebras of binary relations more generally to have this combination of properties, for example Crs_n for $3 \leq n < \omega$, see [Mon00].

The signature does not include the domain operation, nor does it include the 'angelic' ordering predicate. However, we will define via infinitary $(\sqsubseteq, ;)$-formulas, the predicates $\blacktriangleleft, \lhd^s$ for some s in the structure to signify the domain inclusion and inclusion of the restriction to the domain of s respectively, see Lemma 104 below.

Let

$$a \blacktriangleleft b \Leftrightarrow \bigvee_{n<\omega} a \blacktriangleleft_n b$$

$$a \triangleleft^s b \Leftrightarrow \bigvee_{n<\omega} a \triangleleft^s_n b$$

where

$$a \blacktriangleleft_0 b \Leftrightarrow (a \sqsupseteq b \vee \exists c(a \sqsupseteq b;c))$$
$$a \triangleleft^s_0 b \Leftrightarrow ((a \sqsubseteq b \wedge s = b) \vee a = b)$$
$$a \blacktriangleleft_{n+1} b \Leftrightarrow \left\{ \begin{array}{l} (a \triangleleft^a_n b) \vee \\ \exists c\ (a \blacktriangleleft_n c \wedge c \blacktriangleleft_n b) \vee \\ \exists d, f, f'\ (a = d; f \wedge f \blacktriangleleft_n f' \wedge b = d; f')) \vee \\ \exists c\ (a \blacktriangleleft_n c \wedge c \triangleleft^a_n b) \end{array} \right\}$$
$$a \triangleleft^s_{n+1} b \Leftrightarrow \left\{ \begin{array}{l} (\exists c\ (a \triangleleft^s_n c \wedge c \triangleleft^s_n b)) \vee \\ \exists c, c', d, d'\ (a = c; d \wedge c \triangleleft^s_n c' \wedge d \triangleleft^d_n d' \wedge b = c'; d') \vee \\ \exists s'(a \triangleleft^{s'}_n b \wedge s \blacktriangleleft_n s') \end{array} \right\}$$

Now that we have defined $\blacktriangleleft, \triangleleft^s$, we prove the following statements about these predicates.

Lemma 103. *Let $\mathcal{A} = (A, \sqsubseteq, ;)$ with $\sqsubseteq$ a partial order and ; associative. For all $a, b, c \in A$ we have*

(1) Reflexivity holds for both $\blacktriangleleft, \triangleleft^s$ for any $s \in A$,

(2) $a \blacktriangleleft b \wedge b \blacktriangleleft c \to a \blacktriangleleft c$, $a \triangleleft^s b \wedge b \triangleleft^s c \to a \triangleleft^s c$, so $\blacktriangleleft$ and $\triangleleft^s$ are transitive, for each $s \in A$,

(3) $a \triangleleft^s b;c$, $b \triangleleft^s b'$, $c \triangleleft^c c'$ implies $a \triangleleft^s b';c'$,

(4) $d \blacktriangleleft a;c$, $a \sqsubseteq a'$, $d \blacktriangleleft a'$, $c \blacktriangleleft c'$ implies $d \blacktriangleleft a';c'$,

(5) $s \blacktriangleleft s'$, $a \triangleleft^{s'} b$ implies $a \triangleleft^s b$,

(6) $\blacktriangleleft_i \subseteq \blacktriangleleft_j$ and $\triangleleft^s_i \subseteq \triangleleft^s_j$, for all $i \leq j < \omega, s \in A$.

Proof. The statements (1), (2), (3), and (5) follow directly from the definitions of $\blacktriangleleft, \triangleleft$. For (4), see that as $a \sqsubseteq a'$ we have $a \triangleleft^{a'} a'$ and by (5) and $d \blacktriangleleft a'$ we have $a \triangleleft^d a'$. By the second disjunct in the definition of $\triangleleft_{n+1}$, we have that $a;c \triangleleft^d a';c$. This, together with $d \blacktriangleleft a;c$, gives us $d \blacktriangleleft a';c$ by the fourth disjunct of the $\blacktriangleleft_{n+1}$ definition. Further, as $c \blacktriangleleft c'$, we get $a';c \blacktriangleleft a';c'$ from the third clause of the $\blacktriangleleft_{n+1}$ definition. By (2) we get $d \blacktriangleleft a';c'$ from $d \blacktriangleleft a';c \blacktriangleleft a';c'$. For (6), see that $\blacktriangleleft_0$ is trivially reflexive. In the base case, if $a \blacktriangleleft_0 b$, we know that because $b \blacktriangleleft_0 b$, we have $a \blacktriangleleft_1 b$ by the second disjunct (and similarly for $\triangleleft^s$ by the first disjunct). In the induction case, we may safely assume by the induction hypothesis that $\blacktriangleleft_n$ is also reflexive, so by the second disjunct we have that if $a \blacktriangleleft_n b$, we also get $a \blacktriangleleft_{n+1} b$. A similar argument can be constructed for $\triangleleft^s$, using the first disjunct of the $\triangleleft^s_{n+1}$ definition. ∎

Lemma 104. *Let* $\mathcal{A} \in \mathsf{R}(\sqsubseteq, ;)$ *and let* h *be a representation of* $\mathcal{A}$*. For all* $a, b, c \in A$ *we have*

$$a \blacktriangleleft b \Rightarrow \mathrm{D}(h(a)) \leq \mathrm{D}(h(b)) \text{ and}$$
$$a \lhd^s b \Rightarrow \mathrm{D}(h(s));h(a) \leq h(b).$$

Proof. We prove both claims by a single induction over $0 \leq n < \omega$.

Base Case If $a \blacktriangleleft_0 b$ then either $h(a) \sqsupseteq h(b)$ or $h(a) \sqsupseteq h(b);h(c)$ for some c. Hence $\mathrm{D}(h(a)) \leq \mathrm{D}(h(b))$. And if $a \lhd^s_0 b$ then $s = b$, $a \sqsubseteq b$, so $\mathrm{D}(h(s));h(a) = \mathrm{D}(h(b));h(a) \leq h(b)$. Alternatively $a = b$ and so $\mathrm{D}(h(s));(h(a)) = \mathrm{D}(h(s));(h(b))$.

Induction Case Suppose $a \blacktriangleleft_{n+1} b$, from the recursive definition, there are four alternatives. In the first case, $a \lhd^a_n b$ then inductively $h(a) = \mathrm{D}(h(a));h(a) \leq h(b)$ so $\mathrm{D}(h(a)) \leq \mathrm{D}(h(b))$. In the second case, inductively $\mathrm{D}(h(a)) \leq \mathrm{D}(h(c)) \leq \mathrm{D}(h(b))$. In the third case, there are d, f, f' where $a = d;f$, $f \blacktriangleleft_n f'$ and $b = d;f'$. For any $(x, x) \in \mathrm{D}(h(a))$, there is y such that $(x, y) \in h(a)$ and there is z such that $(x, z) \in h(d)$, $(z, y) \in h(f)$. Inductively, $(z, z) \in \mathrm{D}(h(f)) \leq \mathrm{D}(h(f'))$ so there is w such that $(z, w) \in h(f')$, hence $(x, w) \in h(d);h(f') = h(b)$, so $(x, x) \in \mathrm{D}(h(b))$, proving $\mathrm{D}(h(a)) \leq \mathrm{D}(h(b))$. Finally, in the fourth case, $\mathrm{D}(h(a)) \leq \mathrm{D}(h(c))$, so for all $(x, x) \in \mathrm{D}(h(a))$, there will exist a y such that $(x, y) \in h(c)$. $(x, x) \in \mathrm{D}(h(a))$ so by $c \lhd^a_n b$ the induction hypothesis, we know $(x, y) \in h(b)$. So every $(x, x) \in \mathrm{D}(h(a))$ is also in $\mathrm{D}(h(b))$.

Now suppose $a \lhd^s_{n+1} b$. There are three alternatives in the recursive definition. In the first case, inductively $\mathrm{D}(h(s));h(a) \leq h(c)$ and $\mathrm{D}(h(s));h(c) \leq h(b)$, so $\mathrm{D}(h(s));h(a) \leq h(b)$. In the second case, there are c, c', d, d' as in the definition. If $(x, x) \in \mathrm{D}(h(s))$ and $(x, y) \in h(a)$ then there is z such that $(x, z) \in h(c)$, $(z, y) \in h(d)$. Inductively, $(x, z) \in h(c')$ and $(z, y) \in h(d')$, hence $(x, y) \in h(c';d') = h(b)$. In the third case, $\mathrm{D}(h(s)) \leq \mathrm{D}(h(s'))$, so $\mathrm{D}(h(s));h(a) \leq \mathrm{D}(h(s'));h(a) \subseteq h(b)$. This proves $\mathrm{D}(h(s));h(a) \leq h(b)$, as required. □

Now, let

$$\sigma_n = ((b \blacktriangleleft_n a \wedge a \lhd^b_n b) \rightarrow a \sqsubseteq b),$$
$$\sigma = ((b \blacktriangleleft a \wedge a \lhd^b b) \rightarrow a \sqsubseteq b).$$

For finite n, σ_n is a first-order formula, while σ is infinitary and is equivalent to $\bigwedge_{n<\omega} \sigma_n$.

Lemma 105.

$$\mathsf{R}(\sqsubseteq, ;) \models \sigma.$$

Proof. Let $\mathcal{A} \in \mathsf{R}(\sqsubseteq, ;)$ and let h be a representation of $\mathcal{A}$. Assume the premise of σ, $\mathcal{A} \models (b \blacktriangleleft a \wedge a \lhd^b b)$. By the previous Lemma, $\mathrm{D}(h(b)) \leq \mathrm{D}(h(a))$ and $\mathrm{D}(h(b));h(a) \leq h(b)$, that is, $h(a) \sqsubseteq h(b)$. Since θ represents $\sqsubseteq$ correctly, we must have $\mathcal{A} \models a \sqsubseteq b$. Thus $\mathcal{A} \models \sigma$. □

We now define an explicit construction of a representation to prove the completeness of the axiomatisation of $\mathsf{R}(\sqsubseteq, ;)$ consisting of partial order axioms, associativity, and σ_n for all $n < \omega$. It will be loosely based on representation for ordered semigroups h^{Zar}, but where at each point we also record information about the domain of outgoing labels. In our proofs, we will assume that the structures in question have already been amended with an identity element e, much like in the construction of h^{Zar}. We begin by defining the base X.

Definition 106. *Let $\mathcal{A} = (A, \sqsubseteq, ;)$. Consider the base set*

$$X = X_i \uplus X_f \uplus X_b$$

where $\uplus$ denotes disjoint union and

$$\begin{aligned} X_i &= \{(a, b) : a, b \in A,\ a \blacktriangleleft b\}, \\ X_f &= \{b : b \in A\}, \\ X_b &= \{a' : a \in A\}. \end{aligned}$$

We may use a prime symbol $'$ for points in X_b in order to distinguish them from points in X_f. For $x = (a, b) \in X_i$ let $\lambda(x) = b,\ \delta(x) = a$, for $b \in X_f$ let $\lambda(b) = \delta(b) = b$, and for $a' \in X_b$ let $\lambda(a')$ be undefined and $\delta(a') = a$.

We refer to the points in X_i, X_f, X_b as *initial points*, *follow points* and *branch points*, respectively. For $x \in X$ we may refer to $\delta(x)$ as the *domain* of x and for $x \in X_i \uplus X_f$, $\lambda(x)$ is the *label* of x. Suppose $\mathcal{A}$ contains a left and right identity e for ;. In Definition 107 below, we will define a representation where for $x \in X_i \uplus X_f$, $\lambda(x)$ will label the edge from x to the fixed point $e \in X_f$, and for $x \in X$, $\delta(x)$ will be a tight lower $\blacktriangleleft$-bound for the label of any outgoing edge from x. Note that the label of a follow point equals the domain of that point, the label of a branch point is undefined. An example of an initial, follow and a branch point is visualised in Figure 4.4.

Definition 107. *Let $\mathcal{A} = (A, \sqsubseteq, ;)$ where $\sqsubseteq$ is a partial order, ; associative, and $\mathcal{A} \models \sigma$. Define $h : A \to \wp(X \times X)$ where for each $a \in A$, $h(a) \subseteq X \times X$ is a binary relation such that $(x, y) \in h(a)$ if and only if*

(1) $y \notin X_i$,

(2) $x \in X_b \Rightarrow y \in X_b$,

(3) $\delta(x) \blacktriangleleft a;\delta(y)$,

(4) $x \in X_i \uplus X_f,\ y \in X_f \Rightarrow \lambda(x) \triangleleft^{\delta(x)} a;\lambda(y)$.

In Figure 4.5, see $(x, y) \in h(a)$ with (i) $x \in X_i \uplus X_f$ and $y \in X_f$, (ii) $x \in X_i \uplus X_f$ and $y \in X_b$, and (iii) $x \in X_b,\ y \in X_b$. It is required in each case that $\delta(x) \blacktriangleleft a;\delta(y)$, and in case (i) additionally that $\lambda(x) \triangleleft^{\delta(x)} a;\lambda(y)$. In (i) and (ii) if $x \in X_f$ then the $\delta(x)$ and $\lambda(x)$ arrows coincide, see Definitions 106, 107.

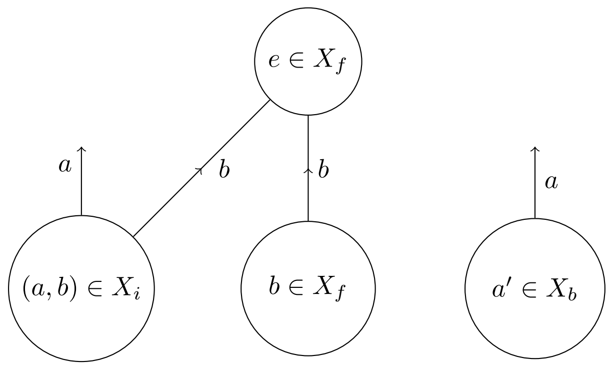

Figure 4.4: Example points in the base set of a representation of a structure in $\mathsf{R}(\sqsubseteq, ;)$.

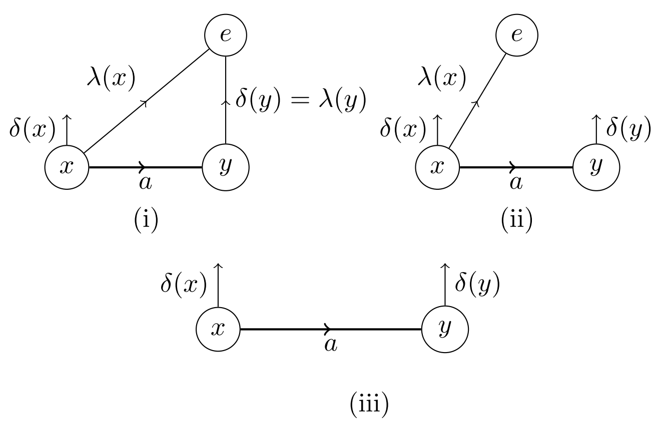

Figure 4.5: The different cases of inclusions of point pairs in $h(a)$.

Lemma 108. *Let $\mathcal{A} = (A, \sqsubseteq, ;)$ where ; is associative, $\sqsubseteq$ is a partial order, for all $n < \omega$, $\mathcal{A} \models \sigma_n$, and suppose there is an identity $e \in \mathcal{A}$. Let h be from Definition 107. Then for all $a, b \in A$ we have $a \sqsubseteq b$ if and only if $h(a) \sqsubseteq h(b)$.*

Proof. Assume $a \not\sqsubseteq b$, so either $\neg b \blacktriangleleft a$ or $\neg a \lhd^b b$, by σ. In the former case, consider $b' \in X_b$, and recall that $\delta(b') = b$. Then $(b', b') \in \mathrm{D}(h(b))$ (since $(b', e') \in h(b)$). Suppose for contradiction that $(b', b') \in \mathrm{D}(h(a))$. So there exists a $c \in A$ such that $(b', c') \in h(a)$. Take any such c and observe that (since $\delta(b') = b$ and $\delta(c') = c$) we get $b \blacktriangleleft a;c$ which, together with $a;c \blacktriangleleft_0 a$ implies $b \blacktriangleleft a$ which yields a contradiction. Hence $(b', b') \in \mathrm{D}(h(b)) \setminus \mathrm{D}(h(a))$. In the second case, $b \blacktriangleleft a$ and $\neg a \lhd^b b$, but then define $x = (b, a) \in X_i$ and observe that (x, e) is in $h(a)$ but not in $h(b)$, yet $(x, x) \in \mathrm{D}(h(b))$ (since $(x, e') \in h(b)$). Either way, $h(a) \not\sqsubseteq h(b)$.

Now suppose $a \sqsubseteq b$. First we check that $\mathrm{D}(h(b)) \leq \mathrm{D}(h(a))$. If $(x, x) \in \mathrm{D}(h(b))$ there is $y \in X$ where $(x, y) \in h(b)$. It follows that $\delta(x) \blacktriangleleft b \blacktriangleleft a$, so $(x, e') \in h(a)$ and $(x, x) \in \mathrm{D}(h(a))$. Secondly, if $(x, x) \in \mathrm{D}(h(b))$ (so $\delta(x) \blacktriangleleft b$) and $(x, y) \in h(a)$ we know that (3)–(4) hold for a, in particular $\delta(x) \blacktriangleleft a;\delta(y)$. It follows that $\delta(x) \blacktriangleleft b;\delta(y)$, by Lemma 103 (4), as required by (3). Conditions (1),(2) remain true for $h(b)$. For (4) if $x \in X_f$ then $\lambda(x) \lhd^{\delta(x)} a;\lambda(y) \lhd^{\delta(x)} b;\lambda(y)$, by Lemma 103 (3). Hence $(x, y) \in h(b)$ and $h(a) \sqsubseteq h(b)$. □

Lemma 109. *Let $\mathcal{A} = (A, \sqsubseteq, ;)$ where ; is associative, $\sqsubseteq$ is a partial order, for all $n < \omega$, $\mathcal{A} \models \sigma_n$, and let h be a mapping as defined in Definition 107. For any $a, b \in \mathcal{A}$, we have $h(a;b) = h(a);h(b)$.*

Proof. First, let's show that $h(a);h(b) \subseteq h(a;b)$. Take any $(x, y) \in h(a)$ and $(y, z) \in h(b)$. We have $\delta(y) \blacktriangleleft b;\delta(z)$, so by the definition of $\blacktriangleleft_{n+1}$ we get $a;\delta(y) \blacktriangleleft a;b;\delta(z)$, which, together with $\delta(x) \blacktriangleleft a;\delta y$, gets us to $\delta(x) \blacktriangleleft a;b;\delta(z)$. If $x \in X_i \uplus X_f$ then $y, z \in X_f \uplus X_b$, by (1), (2), so $\lambda(y) = \delta(y)$, $\lambda(x) \lhd^{\delta(x)} a;\lambda(y)$, $\lambda(y) \lhd^{\lambda(y)} b;\lambda(z)$, and then $\lambda(x) \lhd^{\delta(x)} a;b;\lambda(z)$, by Lemma 103 (3). Hence $(x, z) \in h(a;b)$.

Conversely, to show that $h(a;b) \subseteq h(a);h(b)$, take any $(x, y) \in h(a;b)$. By (1) $y \notin X_i$. If $y \in X_b$ let $z = (b;\delta(y))' \in X_b$ (see Figure 4.6 right), otherwise $y \in X_f$ and we let $z = (b;\lambda(y)) \in X_f$ (see Figure 4.6 left). In each case $\delta(z) = b;\delta(y)$, in the latter case $\lambda(z) = (b;\lambda(y))$, so $(x, z) \in h(a)$, $(z, y) \in h(b)$, as required. □

Theorem 110. $\mathsf{R}(\sqsubseteq, ;)$ *is axiomatised by partial order, associativity and $\{\sigma_n \mid n < \omega\}$. Finite structures $\mathcal{A} = (A, \sqsubseteq, ;) \in \mathsf{R}(\sqsubseteq, ;)$ are representable over a finite base X with $|X| \leq (1 + |A|)^2 + 2 \cdot (1 + |A|)$.*

Proof. The soundness of partial order and associativity is clear. The soundness of σ_n is proved Lemma 105. For completeness, take any associative, partially ordered structure $\mathcal{A} = (A, \sqsubseteq, ;)$ with $\mathcal{A} \models \{\sigma_n \mid n < \omega\}$. We may define $\mathcal{A}' = (A', \sqsubseteq, ;)$ by adding a new ;-identity e to A unordered with other elements. From that, we add for all s, $s \blacktriangleleft e$, $e \lhd^s e$, and all $a \lhd^s b$ triplets with $a \lhd^a b$. The predicate $\lhd^s$ must also be closed under transitivity, for all s. The reader can then check by induction that this is an exhaustive enumeration of all added

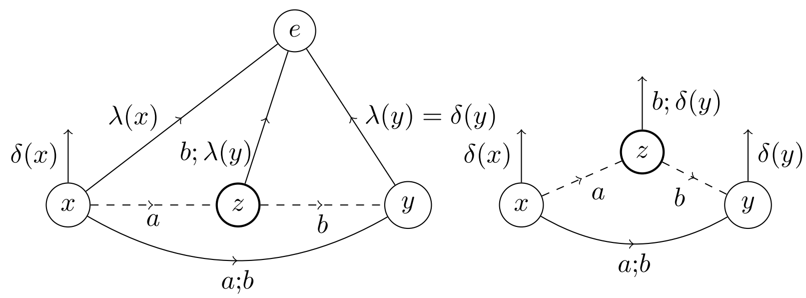

Figure 4.6: The composition preserving property of h, visualised.

$\blacktriangleleft$ pairs and $\triangleleft$ triplets. This requires no additional $\sqsubseteq$ inclusions, forced by σ, namely because for every $a \neq b$ where $a \triangleleft^s b$ was added will either fail to have $b \blacktriangleleft a$ or $s \blacktriangleleft b$ (by the last disjunct of the $\triangleleft^s_{n+1}$ definition). Thus for all $i < \omega$ we have $\mathcal{A}' \models \sigma_i$. By Lemmas 108 and 109, the map h of Definition 107 is a $(\sqsubseteq, ;)$-representation of $\mathcal{A}'$, hence it restricts to a $(\sqsubseteq, ;)$-representation of $\mathcal{A}$. The representation h has its base contained in a disjoint union of a copy of $(A')^2$ and two copies of A' and thus contains at most $(1 + |A|)^2 + 2 \cdot (1 + |A|)$ elements. ☐

Now, we prove that the class $\mathsf{R}(\sqsubseteq, ;)$ is not finitely axiomatisable. We do that by defining an infinite set of unrepresentable structures with a representable ultraproduct.

Definition 111. *Let $1 < n < \omega$, $N = 1 + 2^n$ and let $\mathcal{A}_n = (A_n, \sqsubseteq, ;)$ with the carrier set A_n has $3 + 3N$ elements*

$$A_n = \{0, b, c\} \cup \{a_i, a_ib, a_ic \mid i < N\}$$

where composition ; is defined by $a_i;b = a_ib$, $a_i;c = a_ic$ (all $i < N$) and all other compositions result in 0, and the refinement operation $\sqsubseteq$ is defined as the reflexive and transitive closure of

$$\{(s, 0) \mid s \in A_n\} \cup \{(a_{i+1}b, a_i), (a_i, a_{i+1}c), (a_ib, a_ic) \mid i < N\}$$

where here and below the symbol + denotes arithmetic addition modulo N.

The reader can check that ; is associative and $\sqsubseteq$ is a partial order.

Lemma 112. *For $n \geq 2$, $\mathcal{A}_n$ is not representable, but $\mathcal{A}_n \models \sigma_k$ for $k < n$.*

Proof. First, we show that the structure is not representable. This is because for $i < N$ (since $a_{i+1};b \sqsubseteq a_i$) we get $a_i \blacktriangleleft a_{i+1}b \blacktriangleleft a_{i+1}$, $a_i \blacktriangleleft a_{i+1}$, so all a_i, a_j, a_ib, a_jb, for all i, j, form a $\blacktriangleleft$-clique using Lemma 103 (2). Additionally, all pairs $(a_ic, a_j), (a_ic, a_jb)$ are included in $\blacktriangleleft$ for $i, j < N$ from the definition of $\blacktriangleleft_0$, together with transitivity of $\blacktriangleleft$. Finally, see that for $i < N$ we have $a_i \sqsubseteq a_{i+1}c$ and thus $a_i \triangleleft^{a_{i+1}c} a_{i+1}c$. By the second alternative in the definition of $\triangleleft^s_{n+1}$ and the reflexivity of $\lhd$, we have $a_i b \lhd^{a_{i+1}c} 0$. Similarly, we have $a_{i+2}b \lhd^{a_{i+1}} a_{i+1}$ and thus $0 \lhd^{a_{i+1}} a_{i+1}c$ and because $a_{i+1}c \blacktriangleleft a_{i+1}$ also $0 \lhd^{a_{i+1}c} a_{i+1}c$. By transitivity, we get $a_i b \lhd^{a_{i+1}c} a_{i+1}c$ and because $a_{i+1}c \blacktriangleleft a_i b$, we should have that $a_i b \sqsubseteq a_{i+1}c$. As this is not the case, the structure is not representable.

Let $k < n, i < N, a \in \{a_{i+1}c, a_{i+1}b, a_i\}, a' \in \{a_{i+j+1}b, a_{i+j+1}, a_i \mid j < 2^k\}$, and $s, t \in A_n$. We show, by induction, that if $u \blacktriangleleft_k v$ then (u, v) is one of the below pairs

$$(s,\ s) \quad (0,\ s) \quad (a,\ a')$$

and similarly if $u \lhd_k^v w$ then (u, v, w) is one of the following triplets

$$\begin{array}{cccc} (s,\ t,\ s) & (s,\ 0,\ t) & (0,\ s,\ t) & (a_{i+1}b,\ a'',\ a_i) \\ (a_i,\ a_{i+1}c,\ a_{i+1}c) & (a_i b,\ a_i c,\ a_i c) & (a_i b,\ a_{i+1}c,\ s) & (a_i c,\ a_{i+1}c,\ s) \end{array}$$

where $a'' \in A \setminus \{b, c\}$. Note that this is an upper bound (the reverse implications do not hold) and not a characterisation of the predicates. In fact, some triples above are never included in $\lhd$.

Base Case In the base case, see that the reverse of all pairs in $\sqsubseteq$ is included and so are the compositional cases. Namely this consists of the reflexive pairs as, $(0, s)$, $(a_i c, a_i), (a_i b, a_i), (a_{i+1}c, a_{i+1}b)$, and $(a_i, a_{i+1}b)$. Similarly all $\lhd$-triplets are included, namely the reflexive triplets and $a_{i+1}b \lhd^{a_i} a_i, a_i \lhd^{a_{i+1}c} a_{i+1}c$, and $a_i b \lhd^{a_i c} a_i c$.

Induction Case In the induction case, we look at the definitions of $\blacktriangleleft_{n+1}$ $, \lhd_{n+1}$. For the first alternative in the definition of $\blacktriangleleft$, see that we have all reflexive pairs, as well as $0 \blacktriangleleft s$ and $a_{i+1}b \blacktriangleleft a_i$. For the second alternative, observe that the only gap in transitivity is in some of the $a \blacktriangleleft a'$, but notice that the distance in the indices can at most double in the inductive step. For the third alternative, all compositions of d with some $f \blacktriangleleft f'$ respectively end up in reflexive pairs, namely $0 \blacktriangleleft 0, a_i b \blacktriangleleft a_i b, a_i c \blacktriangleleft a_i c$. In the fourth alternative, there are only three possible nontrivial inclusions, based on the upper bound of triplets in $\lhd$. First, we know that if $a'' \blacktriangleleft a_{i+1}b$ then we also have $a'' \blacktriangleleft a_i$. The other two possibilities result $a_{i+1}c \blacktriangleleft s$, however, that would require $a_{i+1}c \blacktriangleleft a_i c$ or $a_{i+1}c \blacktriangleleft a_i b$, neither of which is the case as $2^k < N$. For $\lhd_{n+1}$, see that the definition is transitive. For the second alternative, see that the only non-0 compositions that can result from right composition are considered, namely by the $(0, s, t)$ and the $(a_i b, a_{i+1}c, s), (a_i c, a_{i+1}c, s)$ triples. Finally, by the induction hypothesis, a'' is downward closed with respect to $\blacktriangleleft_k$.

By the second claim, we can see that the upper bound on $\blacktriangleleft_k, \lhd_k$ contains no pairs that would contradict σ. Namely, this is because if $s \neq 0$ we have $s \not\blacktriangleleft 0$ and because neither $a_{i+1}b \blacktriangleleft_k a_i b$ nor $a_{i+1}c \blacktriangleleft_k a_i b$. □

Theorem 113. *The class* $\mathsf{R}(\sqsubseteq, ;)$ *cannot be axiomatised by finitely many axioms.*

	Rep. Class FA	**Rep. Class NFA**
FRP	$(\leq, ;)$[Zar59]	$(\sqsubseteq, ;)$
No FRP	$(\cdot, ;)$[BS78, Mad16]	τ_{RA} [HH02]

Table 4.1: An example signature for all possible combinations of [not] having the representation class finitely axiomatisable and [not] having the finite representation property for finite structures.

Proof. By Theorem 110, the class is axiomatised by partial order, associativity, and σ_n for $n < \omega$. Now assume it was possible to axiomatise it finitely, or equivalently, by a single formula ϕ. We know that the theory of partial order, associativity, $\sigma_n \mid n < \omega$ and $\neg\phi$ is not consistent, but by Lemma 112 every finite subset of it is. By compactness of first-order logic, we reach a contradiction. □

As mentioned in the introduction section, this is the first known relation algebra reduction signature — if we allow term/atom-definable symbols to be included in reduction signatures — with a non-finitely axiomatisable representation class but the finite representation property for finite structures. This proves that the two properties are independent, with an example signature having both, neither, and both cases of exactly one summarised in Table 4.1

4.3 Demonic Join Semigroups

In this section, we look at the signature $(\sqcup, ;)$, analogous to the angelic $(+, ;)$. The representation class $\mathsf{R}(+, ;)$ of the corresponding angelic signature was proven non-finitely axiomatisable in [And88, AM11], however, it remains unknown whether finite algebras in $\mathsf{R}(+, ;)$ admit finite representations. Our proof of non-finite axiomatisabililty of $\mathsf{R}(\sqcup, ;)$ shows some affinity with the proof in [AM11].

We use representation games based on [HH02] and described in Section 2.2 to show that the representation class $\mathsf{R}(\sqcup, ;)$ has a recursively enumerable axiomatisation but is not finitely axiomatisable. Although our arguments are based on those from that section, we will require a slight redefinition of networks that will now include two types of labelling functions, one for nodes and one for pairs of nodes. This is to accommodate for the domain conditions, imposed by the demonic operations.

A network $\mathcal{N} = (N, \top, \bot)$ is defined for a structure $\mathcal{A} = (A, \sqcup, ;)$ as the set of nodes N, an edge labelling function $\top : (N \times N) \to \wp(A)$ and a node labelling function $\bot : N \to \wp(A)$ (identifying elements forbidden on outgoing edges). Similarly to networks for relation algebra, if $\mathcal{N}' = (N', \top', \bot')$ is another network, we write $\mathcal{N} \subseteq \mathcal{N}'$ and say $\mathcal{N}'$ *extends* $\mathcal{N}$ if $N \subseteq N'$ and for all $x, y \in N$ we have $\top(x, y) \subseteq \top'(x, y)$, $\bot(x) \subseteq \bot'(x)$. We say $\mathcal{N}$ is *consistent* if and only if, for all $x, y \in N$

$$\top(x,y) \cap \bot(x) = \emptyset,$$

closed if for all $x, y, z \in \mathcal{N}$,

$$(a \in \top(x,y) \wedge b \in A) \Rightarrow ((a \sqcup b) \in \top(x,y) \vee b \in \bot(x)),$$
$$(a \in \top(x,y) \wedge b \in \top(y,z)) \Rightarrow (a;b) \in \top(x,z),$$

and *saturated* if it is consistent, closed, and for all $x, y \in N$,

$$(a \sqcup b) \in \top(x,y) \Rightarrow ((a \in \top(x,y) \vee b \in \top(x,y)) \wedge \exists z a \in \top(x,z) \wedge \exists z : b \in \top(x,z)),$$
$$(a;b) \in \top(x,y) \Rightarrow \exists z (a \in \top(x,z) \wedge b \in \top(z,y)).$$

We now define the representation game similarly to the game in Section 2.2. The game $\Gamma_n(\mathcal{A})$ has $n \leq \omega$, played by two players, Abelard ($\forall$) and Héloïse ($\exists$). A play of the game consists of a sequence of networks $\mathcal{N}_i$ (for $0 \leq i < n$) with $\mathcal{N}_i \subseteq \mathcal{N}_j$ if $i \leq j$, together with $x_0, y_0 \in \mathcal{N}_0$ and $s_\bot \in A$. As all of the moves had to be slightly adapted to networks with two labelling functions so we define all of them below.

Initialisation Move $\forall$ picks $a \neq b \in A$ and $\exists$ plays a network $\mathcal{N}_0 = (N_0, \top_0, \bot_0)$, picking a pair of nodes $x_0, y_0 \in N_0$. Then she either ensures $a \in \top_0(x_0, y_0)$ and $s_\bot = b$ or $b \in \top_0(x_0, y_0)$ and $s_\bot = a$.

Choice Move $\forall$ picks a pair of nodes x, y in the network $\mathcal{N}_i = (N_i, \top_i, \bot_i)$ and an $a \sqcup b \in \top_i(x,y)$. $\exists$ must pick a node z (existing or new) and return $\mathcal{N}_{i+1} = (N_{i+1}, \top_{i+1}, \bot_{i+1})$ extending $\mathcal{N}_i$ such that either $a \in \top_{i+1}(x,y), b \in \top_{i+1}(x,z)$ or $b \in \top_{i+1}(x,y), a \in \top_{i+1}(x,z)$.

Join Move $\forall$ picks a pair of nodes x, y in the current network $\mathcal{N}_i = (N_i, \top_i, \bot_i)$, and some $a \in \top_i(x,y)$ as well as some $b \in A$. $\exists$ returns a network $\mathcal{N}_{i+1} = (N_{i+1}, \top_{i+1}, \bot_{i+1}) \supseteq \mathcal{N}_i$ such that either $a \sqcup b \in \top_{i+1}(x,y)$ or $b \in \bot_{i+1}(x)$.

Witness Move $\forall$ picks a pair of nodes x, y in the current network $\mathcal{N}_i = (N_i, \top_i, \bot_i)$ and some $a;b \in \top_i(x,y)$. $\exists$ picks a (potentially new) node z and returns $\mathcal{N}_{i+1} = (N_{i+1}, \top_{i+1}, \bot_{i+1}) \supseteq \mathcal{N}_i$, such that $a \in \top_{i+1}(x,z), b \in \top_{i+1}(z,y)$.

Composition Move $\forall$ picks some nodes $x, y, z \in N_i$ where $\mathcal{N}_i = (n_i, \top_i, \bot_i)$ is the current network. He then picks some $a \in \top_i(x,y)$, $b \in \top_i(y,z)$ and $\exists$ must respond with a network $\mathcal{N}_{i+1} = (N_{i+1}, \top_{i+1}, \bot_{i+1}) \supseteq \mathcal{N}_i$ with $a;b \in \top_{i+1}(x,z)$.

Let $n < \omega$ be finite. If after n moves, the network $\mathcal{N}_n$ is still consistent and $s_\bot \notin \top_n(x_0, y_0)$, we say $\exists$ has won $\Gamma_n(\mathcal{A})$. Otherwise, $\forall$ has won. For the infinite game $\Gamma_\omega(\mathcal{A})$, if an inconsistent network is played or $s_\bot \in \top(x_0, y_0)$ occurs in any round then $\forall$ has won, if this never happens then $\exists$ is the winner. We additionally define a variant of the game $\Gamma_n(\mathcal{A}, \mathcal{N}, s_\bot, x_0, y_0)$, where the initialisation move is skipped and $\mathcal{N}_0$ is set to $\mathcal{N}$ with $x_0, y_0 \in N$ and $s_\bot \in A$. By a similar argument to those presented in Section 2.2, we can prove the lemmas we state without proof below.

Lemma 114. *For* $(\sqcup, ;)$*-structures* $\mathcal{A}$ *with* $(A, \sqcup)$ *a join semilattice and* $;$ *associative, the following are equivalent:*

(1) $\exists$ *has a winning strategy for* $\Gamma_n(\mathcal{A})$*, for all* $n < \omega$*,*

(2) $\exists$ *has a winning strategy for* $\Gamma_\omega(\mathcal{A})$*,*

(3) $\mathcal{A}$ *is representable.*

Lemma 115. *Let* $n < \omega$*. There exists a formula* σ_n *such that* $\mathcal{A} \models \sigma_n$ *if and only if* $\exists$ *has a winning strategy for* $\Gamma_n(\mathcal{A})$*. Thus* $\{\sigma_n \mid n < \omega\}$*, together with the join semilattice and associativity axioms, axiomatises* $\mathsf{R}(\sqcup, ;)$*.*

Lemma 116. *If for every* $n < \omega$ *there exists a structure* $\mathcal{A}_n$ *such that* $\exists$ *will have winning strategy for* $\Gamma_n(\mathcal{A}_n)$ *but not for* $\Gamma_{n+1}(\mathcal{A}_n)$*, then* $\mathsf{R}(\sqcup, ;)$ *is not finitely axiomatisable.*

Remark 117. *We note that the lemmas above cannot be proved in sequence, however, one must first prove Lemma 114 for countable structures, then show the first part of Lemma 115, and then prove the class* $\mathsf{R}(\sqcup, ;)$ *is a quasivariety. This then helps us generalise Lemma 114 to uncountable structures by the Lowenheim–Skölem theorem and show that the theory* $\{\sigma_n \mid n < \omega\}$ *does indeed axiomatise* $\mathsf{R}(\sqcup, ;)$*.*

Now, we define for each $1 \leq n < \omega$ a structure $\mathcal{A}_n = (A_n, \sqcup, ;)$ that is not representable, but $\exists$ has a winning strategy for at least $\Gamma_n(\mathcal{A}_n)$. Let $N = 2^n + 1$ and let $\mathcal{X} = L \cup R \cup P \cup D$ where

$$\begin{aligned} L &= \{a^l, b^l_i, c^l_i \mid 0 \leq i < N\}, \\ R &= \{a^r, b^r_i, c^r_i \mid 0 \leq i < N\}, \\ P &= \{p, p'\}, \\ D &= \{d_1, d_2, d_3\}. \end{aligned}$$

We define a partial binary function $\bullet$ over $\mathcal{X}$ as follows. Over $L \times R$ let

$$\begin{array}{lll} b^l_i \bullet c^r_i = p' & b^l_i \bullet b^r_{i+1} = p' & b^l_{i+1} \bullet b^r_i = p' \\ c^l_i \bullet b^r_i = p' & c^l_i \bullet c^r_{i+1} = p' & c^l_{i+1} \bullet c^r_i = p' \end{array}$$

for all $0 \leq i < N$ where addition of indices is modulo N, and for any other pair of elements $l \in L, r \in R$ we let $l \bullet r = p$. For pairs off $L \times R$, let

$$l \bullet d_2 = d_1, \quad r \bullet d_3 = d_2, \quad p \bullet d_3 = p' \bullet d_3 = d_1$$

for $l \in L,\ r \in R$, and let $\bullet$ be undefined on all other pairs. For any subsets S, T of $\mathcal{X}$ let

$$S \bullet T = \{s \bullet t \mid s \in S,\ t \in T, s \bullet t \text{ is defined}\}.$$

Now we define for all $A \subseteq \mathcal{X}$ the closure (denoted $\widehat{A}$) as the limit of repeating the following steps:

(1) if $|\{d_1, d_2, d_3\} \cap A| > 1$, set $A = \{\}$,

(2) if $A \cap (L \cup P) \neq \{\}$, add d_1 to A,

(3) if $R \cap A \neq \{\}$, add d_2 to A,

(4) for each $0 \leq i < N$,

  (a) if $\{b_i^l, c_i^l\} \subseteq A$, add a^l to A,
  (b) if $\{b_i^r, c_i^r\} \subseteq A$, add a^r to A,

Since each iteration either expands A or replaces it by $\{\}$, this limit is well-defined. Also note we use $\{\}$ instead of $\emptyset$. This is to prevent $\{\}$ being confused with the empty relation $\emptyset$. Let the carrier set A_n of $\mathcal{A}_n = (A_n, \sqcup, ;)$ be defined as

$$A_n = \{\widehat{A} \mid A \subseteq \mathcal{X}\}$$

and to define $\sqcup, ;$ for $\mathcal{A}_n$, let

$$\begin{aligned} S \sqcup \{\} = \{\} \sqcup S &= \{\}, \\ S \sqcup T &= \widehat{S \cup T} && \text{if } S, T \neq \{\}, \\ S; T &= \widehat{S \bullet T}, \end{aligned}$$

for all $S, T \in A_n$.

The top element of this join-semilattice is $\{\}$ and steps (1), (2), (3) of $\widehat{\ }$ ensure that each non-empty set $\widehat{S}$ is contained in exactly one of $L \cup P \cup \{d_1\}$, $R \cup \{d_2\}$ or $\{d_3\}$ and includes a unique domain element, which we denote as $\delta(\widehat{S}) \in \{d_1, d_2, d_3\}$, as visualised in Figure 4.7.

The $\sqcup$ defined for the structures above is *nondistributive*, i.e. it does not hold for every a, b, c that if $a \sqsubseteq b \sqcup c$ there exist some b', c' such that $b' \sqsubseteq b,\ c' \sqsubseteq c$ and $b' \sqcup c' = a$ where $\sqsubseteq$ is defined by $\sqcup$. To see this, take $b = \{d_1, b_i^l\},\ c = \{d_1, c_i^l\},\ b \sqcup c = \{d_1, b_i^l, c_i^l, a^l\},\ a = \{d_1, a^l\}$, for any $i < N$ and similar with $(r, 2)$ in place of $(l, 1)$.

Since the structures are nondistributive, we cannot assume that irreducible elements are prime. By an *irreducible* element we mean an element a such that for all s, t if $a = s \sqcup t$ then $a = s$ or $a = t$ and by a *prime* element we mean an element a such that if $a \sqsubseteq s \sqcup t$ we have $a \sqsubseteq s$ or $a \sqsubseteq t$.

The irreducible elements are singletons $\{d_1\},\ \{d_2\},\ \{d_3\}$ and doubletons $\{d_1, s\},\ \{d_2, r\}$ where $s \in L \cup P,\ r \in R$. All of these are prime, except $\{d_1, a^l\}$ and $\{d_2, a^r\}$ which fail to be prime, as we saw.

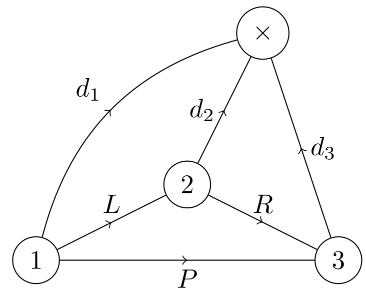

Figure 4.7: Elements of A_n and their unique domain elements, visualised.

Our argument, as well as that in [AM11], for non-finite axiomatisability heavily relies on the failure of distributivity. However, our structures differ from those in [AM11] because of the differences in the join and the demonic-join moves. While the equivalent of the below lemma is trivial in $\mathsf{R}(+, ;)$, the 'domain elements' $\{d_1\}, \{d_2, \}, \{d_3\}$ are needed here.

Lemma 118. *Suppose $\mathcal{N}$ occurs in a play of the game, $S \in \top(x, y)$, $T \in \bot(x)$ where $\{\} \neq S \subseteq T \in A_n$. Then $\exists$ will lose the game in no more than two moves.*

Proof. We may assume $S \neq T$ else $\mathcal{N}$ is already inconsistent. Let $\delta(S) = \delta(T) = d_i$, where $i = 1, 2$ or 3. Since $T \supsetneq \{d_i\}$ we know $i \neq 3$ and T has non-empty intersection with either L, R or P. Hence, there is $j > i$ such that $T;\{d_j\} = \{d_i\}$.

$\forall$ can play a choice move $S \sqcup \{d_i\} = S$ to force an outgoing edge (x, z) with $\{d_i\} \in \top(x, z)$. But then, a witness move for $T;\{d_j\} = \{d_i\}$ forces an edge (x, w) with $T \in \top(x, w)$, yielding an inconsistent network. □

A consequence is stated below.

Corollary 119. *Let $\{\} \neq S \subseteq T \in A_n$. If a network $\mathcal{N}$ is played and $S \in \top(x, y)$, then $\forall$ has a way of forcing $T \in \top(x, y)$.*

Proof. Assume $S \in \top(x, y)$ and $\forall$ plays the join move for S, T over (x, y). By Lemma 118, if $\exists$ adds T to $\bot(x)$, she will lose. Therefore, she must add $S \sqcup T = T$ to $\top(x, y)$. □

We now have all the tools to show the below lemma.

Lemma 120. *The structure $\mathcal{A}_n$ is not in $\mathsf{R}(\sqcup, ;)$ for any $1 < n < \omega$.*

Proof. We show this by showing that $\forall$ has a winning strategy in $\Gamma_{2N+1}(\mathcal{A}_n)$. Assume $\forall$ picks $\{p', d_1\} \neq \{p, p', d_1\}$. If $\exists$ returns the network where $\{p', d_1\} \in \top(x_0, y_0)$ and $s_\bot = \{p, p', d_1\}$, $\forall$ wins by Lemma 118, Corollary 119.

Now, let us have a look at the case where $\exists$ puts $\{p, p', d_1\}$ in $\top(x_0, y_0)$ and $s_\bot = \{p', d\}$. When faced with the choice move $\{p, p', d_1\} = \{p, d_1\} \sqcup \{p', d_1\}$ over x_0, y_0, she must put $\{p, d_1\} \in \top(x_0, y_0)$ as $s_\bot = \{p', d_1\}$. $\forall$ may request a

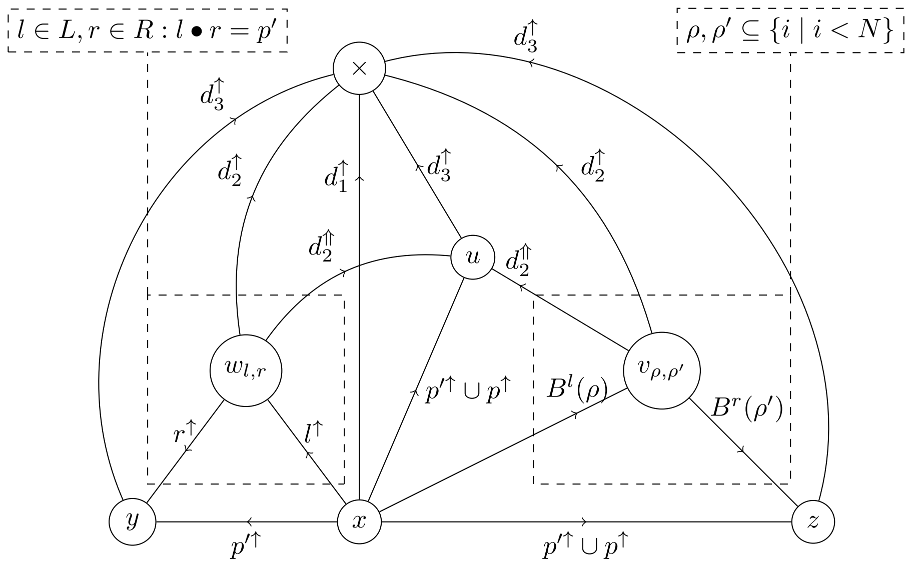

Figure 4.8: The proof of Lemma 121, visualised.

witness over x_0, y_0 for $\{p, d_1\} = \{a^l, d_1\};\{a^r, d_2\}$, let us call the node $\exists$ chooses to witness this composition z.

If $\forall$ plays a join move $\{a^l, d_1\} \sqcup \{a^l, d_1, b_i^l, c_i^l\}$ over (x_0, z), by Lemma 118, $\exists$ will have to add $\{b_i^l, c_i^l, a^l, d_1\}$ to $\top(x_0, z)$ or lose the game, and similarly she can be forced to add $\{b_i^r, c_i^r, a^r, d_2\}$ to $\top(z, y_0)$. Observe that $\{b_i^l, c_i^l, a^l, d_1\} = \{b_i^l, d_1\} \sqcup \{c_i^l, d_1\}$, so over a series of $2N$ choice moves, she must add either $\{b_i^l, d_1\}$ or $\{c_i^l, d_1\}$ to $\top(x_0, z)$ and either $\{b_i^r, d_2\}$ or $\{c_i^r, d_2\}$ to $\top(z, y_0)$, for $i < N$.

Without loss, assume she picks $\{b_0^l, d_1\}$ for (x_0, z). To avoid a composition resulting in $\{p', d_1\}$ she must also pick $\{b_0^r, d_1\}$ for (z, y_0). Again to avoid the undesired composition, she must also pick $\{c_1^l, d_1\}, \{c_1^r, d_2\}$ for $\top(x_0, z), \top(z, y_0)$ respectively. More generally, she must add $\{c_i^l, d_1\}, \{c_i^r, d_2\}$ if i is odd and $\{b_i^l, d_1\}, \{b_i^r, d_2\}$ if i is even. Since $N-1$ is even, she must choose $\{b_{N-1}^l, d_1\}, \{b_{N-1}^r, d_2\}$ and this results in a composition $\{b_{N-1}^l, d_1\}; \{b_0^r, d_2\} = \{p', d_1\} = s_\perp$, so the network is inconsistent. □

Lemma 121. $\exists$ *can win the game* $\Gamma_\omega(\mathcal{A}_n)$ *if* $\forall$ *does not use the initialisation pair* $\{p', d_1\} \neq \{p, p', d_1\}$.

Proof. We show this by explicitly defining a saturated network $\mathcal{N}$, see Figure 4.8 below, discriminating all pairs except $\{p', d_1\} \neq \{p, p', d_1\}$. Since $\exists$ is not required to play conservatively, she may play $\mathcal{N}$ in the initial round, by selecting $x_0, y_0 \in N$, $s_\perp \in A_n$ appropriately.

First, some notation and a definition. For any $S \subseteq \mathcal{X}$ let

$$S^{\uparrow} = \{T \in A_n \mid S \subseteq T\}$$
$$S^{\Uparrow} = S^{\uparrow} \setminus \{S\}$$

and for $s \in \mathcal{X}$ write $s^{\uparrow}, s^{\Uparrow}$ for $\{s\}^{\uparrow}, \{s\}^{\Uparrow}$. Observe that $(b_i^l)^{\uparrow} = \{S \subseteq \mathcal{X} \mid \{b_i^l, d_1\} \subseteq S \subseteq L \cup P \cup \{d_1\}\}$, and so on.

While the $S^{\uparrow}, S^{\Uparrow}$ may seem similar in notation, we will use them differently. $S^{\uparrow}$ will be used as analogous to an upwardly closed set in the classical $\mathsf{R}(+,;)$ setting. $S^{\Uparrow}$, on the other hand will be mainly used for the domain elements d_i where $d_i^{\Uparrow}$ to denote all elements where $\{d_i\}$ can be used to enforce Lemma 118.

Let $S \subseteq A_n$ be upward closed, that is,

$$(A \in S \wedge A \subseteq B \in A_n) \Rightarrow B \in S.$$

In view of Lemma 118 we may assume that $\exists$ chooses upward closed sets for each label $\top(x, y)$. We say that S is *prime* if it is upward closed and for all $A, B \in A_n$ we have $(A \sqcup B) \in S \Rightarrow (A \in S \vee B \in S)$. The edge labels $\top(x, y)$ of a saturated network must be prime.

Note that $s^{\uparrow}$ is prime, for $s \in \mathcal{X} \setminus \{a^l, a^r\}$, but $(a^l)^{\uparrow}, (a^r)^{\uparrow}$ are not. Also note that $d_2^{\Uparrow} = \{\{d_2\} \cup R_0 \in A_n \mid \{\} \neq R_0 \subseteq R\}$ is prime. A prime set including $S \ni a^l$ must include an element including either b_i^l or c_i^l, for each $i < N$. So, for any $\rho \subseteq \{i \mid i < N\}$, let

$$B^l(\rho) = (a^l)^{\uparrow} \cup \bigcup_{i \in \rho} (b_i^l)^{\uparrow} \cup \bigcup_{i<N,\ i \notin \rho} (c_i^l)^{\uparrow},$$
$$B^r(\rho) = (a^r)^{\uparrow} \cup \bigcup_{i \in \rho} (b_i^r)^{\uparrow} \cup \bigcup_{i<N,\ i \notin \rho} (c_i^r)^{\uparrow},$$

and observe that these sets are prime.

The set of nodes N of $\mathcal{N}$ is defined to be

$$\begin{aligned} &\{x, y, z, u, \times\} \\ \cup &\{w_{l,r} \mid l \in L,\ r \in R,\ l \bullet r = p'\} \\ \cup &\{v_{\rho,\rho'} \mid \rho, \rho' \subseteq \{i \mid i < N\}\} \end{aligned}$$

Edge labels between nodes are as shown in Figure 4.8, all edges not shown have empty edge labels. Let the index of x be 1, the indices of $v_{\rho,\rho'}$ and $w_{l,r}$ be 2 and the indices of y, z be 3 ($u, \times$ have no indices). For the node labelling of $w \in \{x, v_{\rho,\rho'}, w_{l,r} \mid \rho, \rho' \subseteq \{i \mid i < N\},\ r \in R,\ l \in L\}$ with index $i = 1, 2, 3$, let $\bot(w) = \{\{\}\} \cup \{S \in A_n \mid \delta(S) \neq i\}$, for the remaining nodes $w \in \{u, \times\}$ let $\bot(w) = \{\{\}\}$.

It can be checked exhaustively that for every pair but $\{p', d_1\} \neq \{p, p', d_1\}$, there exists a $x, y \in N$ where $\top(x, y)$ includes one but not the other of the pair, for example, if $B, C, B', C' \subseteq \{i \mid i < N\}$, $(B, C) \neq (B', C')$, so $\{d_1\} \cup \{b_i^l \mid i \in B\} \cup \{c_i^l \mid i \in C\} \neq \{d_1\} \cup \{b_i^l \mid i \in B'\} \cup \{c_i^l \mid i \in C'\}$, this pair is discriminated on $(x, v_{\rho,\rho'})$ provided $\rho \subseteq B,\ \rho \not\subseteq B'$, or if $\rho' \cup C = \{i \mid i < N\},\ \rho' \cup C' \neq \{i \mid i < N\}$, or the other way round. It can also be checked that the network is saturated. □

Before we prove the next lemma, we define prime and indexed networks. An $\mathcal{A}_n$-network $\mathcal{N}$ is *prime* if for all $x, y \in \mathcal{N},\ \top(x,y)$ is a prime subset of A_n. A network $\mathcal{N}$ is *indexed* if there exists a node $\times \in N$ with no outgoing edges and each node $x \in N \setminus \{\times\}$ has an index $\iota(x) \in \{1,2,3\}$ such that for all $x, y \in \mathcal{N},\ (S \in \top(x,y) \Rightarrow \delta(S) = d_{\iota(x)})$ and $\top(x, \times) = d_{\iota(x)}^{\uparrow}$.

Lemma 122. *Let $\mathcal{N}$ be a closed, prime, consistent, indexed network. Then $\exists$ has a winning strategy in $\Gamma_n(\mathcal{A}_n, \mathcal{N}, x_0, y_0, s_\perp)$.*

Proof. Assume $\mathcal{N}$ satisfies the conditions. Extend the labelling of $\mathcal{N}$ so that

$$\perp(x) = \{S \in A_n : \delta(S) \neq d_{\iota(x)}\}$$

The network remains prime, closed, indexed and consistent, by Lemma 118, and this ensures that all join moves are trivial. Note that by the closure of $\mathcal{N}$, all composition moves are trivial, and since edge labels are prime sets, choice moves are also trivial.

That leaves witness moves. If $\forall$ plays a witness move (x, y, α, β) where $\alpha;\beta \in \top(x,y)$, by the additive definition of ; and since $\top(x,y)$ is prime, we can find join irreducibles $\alpha_0 \subseteq \alpha,\ \beta_0 \subseteq \beta$ where $\alpha_0;\beta_0 \in \top(x,y)$.

Let $\delta(\beta_0) = d_i$ where $i \in \{2,3\}$. She takes a new node z with index i, and extends the network to z by $\top(x,z) = \alpha_0^{\uparrow},\ \top(z,y) = \beta_0^{\uparrow},\ \top(z,\times) = d_i^{\uparrow}$, and $\perp(z) = \{S \in A_n \mid \delta(S) \neq d_i\}$. Since $\alpha_0;\beta_0 \in \top(x,y)$ it follows that $\alpha_0 \subseteq \{d_1\} \cup L$ and $\beta \subseteq \{d_2\} \cup R$. Hence the resulting network will be closed. If α_0, β_0 are primes she can play a prime network and from there she can win the game of length $n-1$ by the induction hypothesis. So we may assume either α_0 or β_0 is irreducible but not prime, and since $\alpha_0;\beta_0 \in \top(x,y)$, either $\alpha_0 = \{a^l, d_1\}$ or $\beta_0 = \{a^r, d_2\}$, without loss assume the former $\alpha_0 = \{a^l, d_1\}$, and $\top(x,y) = \{p\}^{\uparrow}$. If β_0 is prime, then choose some prime set π, such that $\pi;\beta_0^{\uparrow} = p^{\uparrow}$ and let $\top(x,z) = \pi$.

Finally, suppose neither α_0 nor β_0 is prime, so $\alpha_0 = \{a^l, d_1\},\ \beta_0 = \{a^r, d_2\}$. We must show that $\exists$ can survive $n-1$ choice moves, the remainder of the game. Inductively, she maintains a contiguous set $\Delta \subseteq \{i \mid i < N\}$ whose complement in $\{i \mid i < N\}$ has size more than 2^k where k is the number of rounds remaining, and a partition of Δ into two successor-free sets $\Delta_b \uplus \Delta_c = \Delta$, for parity. Initially $\Delta = \emptyset$. She ensures that $S \in \top(x,z) \to S \subseteq \{d_1, a^l\} \cup \{b_i^l \mid i \in \Delta_b\} \cup \{c_i^l \mid i \in \Delta_c\}$, and the similar for $T \in \top(z,y)$. This ensures $S;T \in p^{\uparrow}$ for $S \in \top(x,z),\ T \in \top(z,y)$ which proves closure of the network. To maintain this induction hypothesis, when $\forall$ plays a choice move $\{d_1, b_i^l\} \sqcup \{d_1, c_i^l\} \in \top(x,y)$ she extends Δ to include i, most economically, and this ensures that the complement in $\{i \mid i < N\}$ will be at least half its previous size, which maintains the induction hypothesis.

Note that at no point in this strategy does $\exists$ amend any $\top$-labels to the original $\mathcal{N}$ network. This also ensures that $s_\perp$ is never added to $\top(x_0, y_0)$. □

Lemma 123. *$\exists$ has a winning strategy in $\Gamma_n(\mathcal{A}_n)$.*

Proof. In the initial round let $\forall$ play $a \neq b \in A_n$. There is a prime $\pi \in \mathcal{S}_n$ such that $\pi \subseteq a,\ \pi \not\subseteq b$ or $\pi \not\subseteq a,\ \pi \subseteq b$, without loss assume the former. $\exists$ plays the prime network $\mathcal{N}_0$ with nodes x_0, y_0 and lets $\top(x_0, y_0) = \pi^{\uparrow}$, otherwise $\top(x', y')$ is empty. She defines a suitable node index function ι from π, e.g. if $\pi = \{d_1, b_i^l\}$ she lets $\iota(x_0) = 1,\ \iota(y_0) = 2$. Then $\mathcal{N}_0$ is closed, prime, consistent and has a suitable index function. By Lemma 122 she can survive another n-rounds. □

By Lemmas 116, 123, we claim

Theorem 124. $\mathsf{R}(\sqcup, ;)$ *is not finitely axiomatisable.*

4.4 Demonic Meet and Demonic Lattice Semigroups

In this section, we look at the class of representable semigroups with the demonic meet $\mathsf{R}(\sqcap, ;)$ and show that the representation problem is undecidable for finite $(\sqcup, \sqcap, ;)$-structures. We define $\sqcap$ as the greatest lower bound with respect to $\sqsubseteq$. We have seen in the previous sections that not every pair of binary relations has a common refinement, but this could be fixed by adjoining an extra point $\times$ to the base of the relations and replacing each relation R by $R \cup \{(x, \times) \mid (x, x) \in \mathrm{D}(R)\}$. This is to model the 'magic' state $\perp$ where all conditions are true, as discussed in Section 4.1.

Our proof of the failure of the finite representation property for $(\sqcap, ;)$ uses a structure, similar to a $(\cdot, ;)$-subreduct of the point algebra $\mathcal{P}$. However, as $\cdot$ and $\sqcap$ are not the same operations, the structure in the proof below cannot be considered a true subreduct of the point algebra.

Theorem 125. *The finite representation property fails for finite structures in* $\mathsf{R}(\sqcap, ;)$.

Proof. We explicitly define a finite algebra $\mathcal{A} = (A, \sqcap, ;)$. The carrier set of the algebra is $A = \{z, e, g\}$ and the two binary operations $\sqcap, ;$ are defined below.

$\sqcap$	z	e	g
z	z	z	z
e	z	e	z
g	z	z	g

$;$	z	e	g
z	z	z	z
e	z	e	g
g	z	g	g

See how this algebra is representable over the base $\mathbb{Q} \cup \{\times\}$ by representation h where

$$h(z) = \{(q, \times) \mid q \in \mathbb{Q}\} \cup \{(\times, \times)\},$$
$$h(e) = \{(q, q) \mid q \in \mathbb{Q}\} \cup h(z),$$
$$h(g) = \{(q, r) \mid q < r \in \mathbb{Q}\} \cup h(z).$$

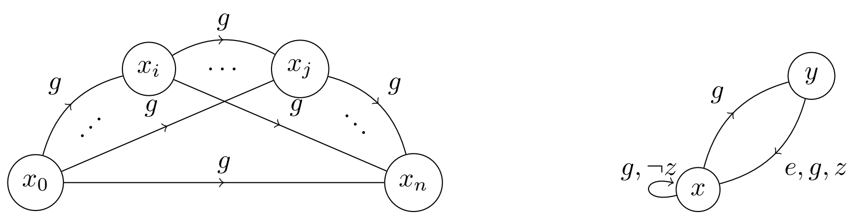

Figure 4.9: The proof of Theorem 125, visualised.

Now let us show that any representation h of $\mathcal{A}$ must be over an infinite base X. The argument is heavily based on that of Theorem 70, however, due to the nature of $\sqcap$, some additional claims need to be shown.

While we are allowed to assume that because $0 \leq g$ in the point algebra, there will exist a pair of points in the image of g but not 0, we cannot do the same about $z \sqsubseteq g$ in A due to the domain condition in the demonic refinement. This is why we need to prove the fact that such a pair of points exists.

We can assume though that because $z \neq g$, there must either exist a pair of points $(x_0, x_n) \in h(g)$ but not in $h(z)$ or $(x_0, x_n) \in h(z)$ but not in $h(g)$. In the latter case, because $z \sqcap g = z$ this means that $(x_0, x_0) \notin \mathrm{D}(g)$. This is contradicted by the fact that $(x_0, x_n) \in h(z)$ and $z = g;z$. Thus there exists a pair of points $(x_0, x_n) \in h(g)$ but not in $h(z)$.

From here on, we reiterate the argument for Theorem 70 by assuming there exists a representation h over X such that $|X| = n$. To witness the composition $g = g;g = \cdots = \underbrace{g;\ldots;g}_{n\text{-times}}$, we need $n-1$ points $x_1, \ldots, x_{n-1}$ where n is the size of the finite representation h, see Figure 4.9 left. By the pigeonhole principle and the fact that z is preserved under composition, there will exist a point $x \in \{x_i \mid 0 \leq i \leq n\}$ such that $(x, x) \in h(g)$ but not $(x, x) \in h(z)$. As $g = g;e$, there must exist such a y to witness this composition, see Figure 4.9 right. Through a series of compositions, we get that $(y, x) \in h(g)$ as well, and thus in $h(g) \sqcap h(e) = h(g \sqcap e) = h(z)$. As z is preserved under composition, this yields a contradiction. □

Finally, we look at the signature of demonic join-meet semigroups $(\sqcup, \sqcap, ;)$. Similar to its 'angelic' version $(+, \cdot, ;)$, this signature is quite badly behaved. Below we show that the $(\sqcup, \sqcap, ;)$-representation problem is undecidable.

Lemma 126. *Let $\mathcal{A} = (A, \sqcup, \sqcap, ;)$, with least element $\mathbf{0}$ such that $a;\mathbf{0} = \mathbf{0}$ for all $a \in A$. Define $\mathcal{A}' = (A, +, \cdot, ;)$ with the same carrier set and $+, \cdot, ;$ defined as $\sqcup, \sqcap, ;$ for $\mathcal{A}$. Then $\mathcal{A}' \in \mathsf{R}(+, \cdot, ;)$ if and only if $\mathcal{A} \in \mathsf{R}(\sqcup, \sqcap, ;)$. Furthermore, $\mathcal{A}$ is representable over a finite base if and only if $\mathcal{A}'$ is.*

Proof. Let h be a representation of $\mathcal{A}'$ over base X. Define $h' : A \to \wp((X \cup \{\times\}) \times (X \cup \{\times\}))$ by $h'(a) = h(a) \cup \{(x, \times) \mid x \in X \cup \{\times\}\}$ where $\times \notin X$, so h' is a $(+, \cdot, ;)$-representation, representing each $a \in \mathcal{A}$ as a left-total binary relation over $X \cup \{\times\}$. Since demonic and angelic operators agree over left-total relations, it follows that ϕ is a $(\sqcup, \sqcap, ;)$-representation of $\mathcal{A}$.

Conversely, suppose h is a representation of $\mathcal{A}$ over X. Since $\mathbf{0} \leq a$ in $\mathcal{A}$ we have $h(\mathbf{0}) \sqsubseteq h(a)$ so $\mathrm{D}(h(\mathbf{0})) \supseteq \mathrm{D}(h(a))$. Since $\mathbf{0} = a;\mathbf{0}$ we get $\mathrm{D}(h(\mathbf{0})) \subseteq \mathrm{D}(h(a))$, hence we get $\mathrm{D}(h(a)) = \mathrm{D}(h(\mathbf{0}))$ for all $a \in A$. Since $\mathrm{D}(h(a))$ is constant, demonic and angelic operators agree, so h is also a $(\cup, \cap, ;)$-representation of $\mathcal{A}'$.

Observe that in both implications, the base set of the representation for the signature in the conclusion is at most one element larger than the base set of the representation in the assumption. Thus the finiteness of the representation is also preserved. □

By using the undecidability result for the [finite] representation problem of $(+, \cdot, ;)$-structures [HHJ21], where structures with such a least element are used, we conclude the theorem below.

Theorem 127. *The [finite] representation problem is undecidable for finite* $(\sqcup, \sqcap, ;)$*-structures.*

4.5 Conclusion

In this chapter, we have examined the finite representation property and related properties in signatures containing demonic lattice operations, together with ordinary composition. We have shown that the finite representation property holds in $\mathsf{R}(\sqsubseteq, ;)$ but the class is not finitely axiomatisable. We then showed the failure of finite axiomatisability of $\mathsf{R}(\sqcup, ;)$ and the failure of the finite representation property of $\mathsf{R}(\sqcup, ;)$. We concluded with an undecidability result for $\mathsf{R}(\sqcup, \sqcap, ;)$, implying both properties fail in that class. These results are independent of each other and although they agree with their angelic counterparts (with the exception of NFA for $\mathsf{R}(\sqcup, ;)$), they do not trivially follow from them. The results regarding FRP agree with the wider statement of the main conjecture. Further, the results in the section conclude the proof that the finite representation property and finite axiomatisability are independent for relation algebra reductions. In the chapters that follow, we continue to examine FRP results in other non classical relation algebraic settings, namely domain-range algebras of binary relations and weakening relation algebras.

Chapter 5

Domain-Range Semigroups

We have now examined some cases of Conjecture 27 in its classical relation algebraic as well as its wider setting. We continue the thesis by looking at finite representability with yet more structures motivated by program correctness – domain-range semigroups of binary relations. More specifically, we show that the finite representation property holds for a large number of conjectured signatures, containing domain and range, an extension of the result in [HE13], presented in Chapter 2. Further, we show some nonfinite axiomatisability results, which, although related to the conjecture, do not contradict it. We speculate on possible routes of showing FRP for signatures with domain and range that are conjectured to have it.

A key element in program behaviour modelling is representing conditions. As discussed in the previous chapter, more specifically Section 4.1, when we take the relational approach towards it, they will be modelled as reflexive relations. In the abstract case, those can be presented in a number of ways. For example, in the Kleene algebra expansions, they are abstracted as tests and guarded equations. However, in relation algebra, the domain and range operations have been used to model conditions and enabledness, see [DJS09].

This chapter examines the behaviour of signatures containing domain and range. We do this separately from the remainder of the relation algebra reductions because of the strange nature of the domain and range operations. Recall that $\mathrm{D}(R) = 1' \cdot R;\breve{R}$ and $\mathrm{R}(R) = 1' \cdot \breve{R};R$. Strictly speaking, these operations require meet — and by extension negation — to be defined. In Chapter 3, we, therefore, state that these signatures are not to be considered by the finite representation property conjecture.

Again, the semigroup operation in a signature containing domain and range can be concretely interpreted both 'angelically' as relational composition ; or demonically as the demonic composition $*$. It turns out that the latter may also be used to model total as opposed to partial correctness. The approach towards that differs from that in Section 4.1, see [HMS20] for more detail.

We conjecture that we can extend the positive side of the FRP conjecture to signatures containing domain and range. In fact, we show that all signatures containing domain, converse, and composition, but not negation, join, or meet, will have the finite representation property. This extends the results for ordered domain algebras from [HE13]. We conjecture that the result can be extended further.

This is because, inevitably, some relations in any proper domain-range semigroup ('angelic' or demonic) will necessarily be partial functions. An easy example of this is the domain elements. Observe how for a domain element, any point in the base will either be in a pair with exactly one point (in this case itself) or none at all, thus making it a partial function. In addition to this simple example, other elements will always end up being represented as partial functions and the characterisation of those can get more elaborate. Furthermore, this concept may enable us to define a generalised explicit finite representation for a large number of signatures.

As discussed in Section 2.3, a number of results regarding (the failure) of the finite axiomatisability of the representation class exist. For any $(\mathrm{D}, ;) \subseteq \tau \subseteq (\mathrm{D}, \mathrm{R}, 0, 1', ;)$, the class $\mathsf{R}(\tau)$ is not finitely axiomatisable [HM11]. However, the signature $(\leq, \smile, \mathrm{D}, \mathrm{R}, 1', 0, ;)$ has its representation class finitely axiomatisable with its finite members admitting finite representations [HM13, HE13]. Furthermore, the varieties generated by $\mathsf{R}(\mathrm{D}, \mathrm{R}, ;)$ and $\mathsf{R}(\mathrm{D}, \mathrm{R}, *)$ — that is the closures of the two classes under homomorphisms — are finitely axiomatisable [JM19]. In [JM19], the authors leave the finite axiomatisability of $\mathsf{R}(\mathrm{D}, \mathrm{R}, *)$ open.

In the following sections, we answer that question by showing that $\mathsf{R}(\mathrm{D}, \mathrm{R}, *)$ is in fact not finitely axiomatisable. We then prove that the finite representation property for all signatures $(D, \smile, ;) \subseteq \tau \subseteq (\tau_{\mathsf{RA}} \setminus (-, +)) \cup (\mathrm{D}, \mathrm{R}, \leq)$. For the remaining (converse-free) signatures, we conjecture that a possible route of proving the finite representation property would be by defining Zareckiĭ-style representations with special considerations for the 'functional elements', that is, all the elements of the algebra that will necessarily be represented as partial functions.

5.1 Demonic Domain-Range Semigroups

In this section, we show that the class $R(\mathrm{D}, \mathrm{R}, *)$ is not finitely axiomatisable, by games. The definition of the network and the game are going to be, again, slightly different from those in Sections 2.2 and 4.3. We will again rely on the reader to check some lemmas, however, those will be easily translatable from the proofs in previous sections.

Define a network $\mathcal{N} = (N, \bot, \top)$ where $\bot, \top : N \times N \to \wp(A)$ are both labelling functions for node pairs and $\mathcal{A} = (A, \mathrm{D}, \mathrm{R}, *)$ a structure in the signature of domain-range demonic semigroups. We say the network $\mathcal{N}$ is consistent if and only if

$$\forall x, y \in N : \top(x, y) \cap \bot(x, y) = \emptyset,$$
$$\forall x \neq y \in N, \forall s \in A : ((\exists t : R(t) = s \vee D(t) = s) \Rightarrow s \notin \top(x, y)).$$

Again, note that unlike with the networks from Section 2.2, this network has two labelling functions and unlike with those from Section 4.3, both of these are edge-labellings. We add the additional consistency condition for the domain and the range in order to ensure that domain-range elements are not put in the 'true' labels of some (x, y) where $x \neq y$.

We can now define a game for a $(D, R, *)$-structure $\mathcal{A}$. Again, It is played by two players $\forall, \exists$. The game $\Gamma_n(\mathcal{A})$ starts with the initialisation move and then continues for n moves. At each move, $\forall$ challenges $\exists$ to amend the initially empty network $\mathcal{N}$. After n moves, it is determined that $\exists$ is the winner if and only if the network is consistent.

Initialisation On initialisation, $\forall$ picks a pair $a \neq b \in A$. $\exists$ returns a network $\mathcal{N}_0 = (N_0, \top_0, \bot_0)$ such that there exist some $x, y \in N_0$ with either $a \in \top_0(x, y), b \in \bot_0(x, y)$ or $b \in \top_0(x, y), a \in \bot_0(x, y)$.

Witness Move $\forall$ picks a pair of nodes $x, z \in N_i$ in the network $\mathcal{N}_i = (N_i, \top_i, \bot_i)$ and a pair of elements $a, b \in A$ such that $a * b \in \top_i(x, z)$. $\exists$ returns a network $\mathcal{N}_{i+1} = (N_{i+1}, \top_{i+1}, \bot_{i+1}) \supseteq \mathcal{N}_i$ such that there exists a $y \in N_{i+1}$ with $a \in \top_{i+1}(x, y), b \in \top_{i+1}(y, z)$.

Composition Move $\forall$ picks some $x, y, z \in N_i$ in the current network $\mathcal{N}_i = (N_i, \top_i, \bot_i)$ along with a, b such that $a \in \top_i(x, y)$ and $b \in \top_i(y, z)$. $\exists$ returns a network $\mathcal{N}_{i+1} = (N_{i+1}, \top_{i+1}, \bot_{i+1}) \supseteq \mathcal{N}_i$ with either $a * b \in \top_{i+1}(x, z)$ or some $w \in N_{i+1}$ with $a \in \top_{i+1}(x, w), \mathrm{D}(b) \in \bot_{i+1}(w, w)$.

Composition Path Move $\forall$ picks, some $x, y \in N_i$ in the current network $\mathcal{N}_i = (N_i, \top_i, \bot_i)$ with $a \in \top_i(x, y)$ and $\mathrm{D}(a * b) \in \top_i(x, x)$ and $\exists$ must return $\mathcal{N}_{i+1} \supseteq \mathcal{N}_i$ with $\mathrm{D}(b) \in \top_{i+1}(y, y)$.

Domain-Range Move $\forall$ picks $x, y \in N_i$ in the current network $\mathcal{N}_i = (N_i, \top_i, \bot_i)$ such that $a \in \top_i(x, y)$ and $\exists$ must return $\mathcal{N}_{i+1} \supseteq \mathcal{N}_i$ with $\mathrm{D}(a) \in \top_{i+1}(x, x), \mathrm{R}(a) \in \top_{i+1}(y, y)$.

Domain Path Move $\forall$ picks a node $x \in N_i$ in the current network $\mathcal{N}_i = (N_i, \top_i, \bot_i)$ such that $\mathrm{D}(a) \in \top_i(x, x)$ and $\exists$ must return a network $\mathcal{N}_{i+1} \supseteq \mathcal{N}_i$ with some $y \in N_{i+1}$ such that $a \in \top_{i+1}(x, y)$.

Range Path Move $\forall$ picks a node $x \in N_i$ in the current network $\mathcal{N}_i = (N_i, \top_i, \bot_i)$ such that $\mathrm{R}(a) \in \top_i(x, x)$ and $\exists$ must return a network $\mathcal{N}_{i+1} \supseteq \mathcal{N}_i$ with some $y \in N_{i+1}$ such that $a \in \top_{i+1}(y, x)$.

Again, using arguments, analogous to those in Section 2.2, we can prove the following lemmas.

Lemma 128. *A structure* $\mathcal{A} = (A, \mathrm{D}, \mathrm{R}, *)$ *is representable (a member of* $\mathsf{R}(\mathrm{D}, \mathrm{R}, *)$*) if and only if for all* $n < \omega$*,* $\exists$ *has a winning strategy for* $\Gamma_n(\mathcal{A})$*.*

Lemma 129. *There exists a countable first-order theory* $\Sigma = \{\sigma_1, \sigma_2, ...\}$ *such that*

(1) $\exists$ *has a winning strategy for* $\Gamma_i(\mathcal{A})$ *if and only if* $\mathcal{A} \models \sigma_i$*,*

(2) a structure $\mathcal{A} = (A, \mathrm{D}, \mathrm{R}, *)$ *is in* $\mathsf{R}(\mathrm{D}, \mathrm{R}, *)$ *if and only if* $\mathcal{A} \models \Sigma$*.*

Lemma 130. *To show that* $\mathsf{R}(\mathrm{D}, \mathrm{R}, *)$ *is not finitely axiomatisable, it suffices to show that there exists, for every* $i < \omega$ *a structure* $\mathcal{A} = (A, \mathrm{D}, \mathrm{R}, *) \notin \mathsf{R}(\mathrm{D}, \mathrm{R}, *)$ *for which* $\exists$ *has a winning strategy in* $\Gamma_i(\mathcal{A})$*.*

Before we define these structures in order to prove nonfinite axiomatisability, we recursively define an infinitary predicate $\preceq$ using infinitary $(\mathrm{D}, \mathrm{R}, *)$-formulas such that for every structure $\mathcal{A}$ with a representation h we will have $\forall s, t \in A : s \preceq t \Rightarrow h(s) \sqsubseteq h(t)$. Recall that $\sqsubseteq$ is defined as the set of pairs of binary relations (R, S) such that $\mathrm{D}(S) \leq \mathrm{D}(R)$, however, when restricting to the domain of S we have $\mathrm{D}(S);R \leq S$.

The base case of the following definition is quite involved. Do not worry, the proof of Lemma 131 will make the intuitions behind it very clear. For algebras $\mathcal{A} = (A, \mathrm{D}, \mathrm{R}, *)$ with associative $*$ we define

$$s \preceq_1 t \iff \exists u, v : \mathrm{R}(u * v) = u * v \wedge s = \mathrm{R}(u * \mathrm{D}(v)) \wedge t = s * v * u,$$

$$s \preceq_{n+1} t \iff \begin{pmatrix} & \exists s', t', u, v : s' \preceq_n t' \wedge s = u * s' * v \wedge t = u * t' * v \\ \vee & \exists v : s \preceq_n v \wedge v \preceq_n t \end{pmatrix},$$

and

$$\preceq = \bigcup_{n<\omega} \preceq_n .$$

Lemma 131. *For any* $s, t \in A$ *where* $\mathcal{A} = (A, \mathrm{D}, \mathrm{R}, *) \in \mathsf{R}(\mathrm{D}, \mathrm{R}, *)$*, we have that if* $s \preceq t$*, it is true that for any* $(\mathrm{D}, \mathrm{R}, *)$*-representation* h *we have* $h(s) \sqsubseteq h(t)$*.*

Proof. We show this by induction over n.

Base case In the base case, there exist such u, v that $u * v = \mathrm{D}(u * v)$ and $s = \mathrm{R}(u * \mathrm{D}(v))$ and $t = s * v * u$. First see how if $(x, x) \in \mathrm{D}(h(t))$, there must exist a witness for $s * v * u$ and since s is a domain-range element, it must hold that $(x, x) \in h(s)$. Since $\mathrm{D}(h(s)) = h(s)$, we have $\mathrm{D}(h(t)) \subseteq \mathrm{D}(h(s))$. Furthermore, assume that $(x, x) \in \mathrm{D}(h(t))$ and $(x, x) \in h(s)$. See how there must exist a y such that $(y, x) \in h(u * \mathrm{D}(v))$. There must also exist a z such that $(x, z) \in h(v)$. Since $(y, y) \in h(\mathrm{D}(u * \mathrm{D}(v)))$, we can see that $(y, z) \in h(u * v)$ and since $u * v$ is a domain element, $y = z$. And because $(x, x) \in h(\mathrm{D}(t))$ and because $(x, z) \in h(s * v)$ and $(z, x) \in h(u)$, we conclude $(x, x) \in h(s * v * u) = h(t)$.

Induction case The induction case follows from the fact that $\sqsubseteq$ is transitive as well as left and right monotone for $*$, as discussed in the introduction of Section 4.2. □

Defining a refinement predicate in order to prove non-finite axiomatisability is a similar approach to the one described in [HM11] where the predicate $\lhd$ is defined as the monotone, transitive closure of $\mathrm{D}(s);\mathrm{D}(t)\lhd\mathrm{D}(t)$ to signify ordinary inclusion ($\leq$) for the 'angelic' signature. However, for the demonic signature, $\lhd$ can be simply described as $D(s) * t \lhd t$ as the following axiom is sound for $\mathsf{R}(\mathrm{D}, \mathrm{R}, *)$

$$\forall s, t : \mathrm{D}(s * \mathrm{D}(t)) * s = s * \mathrm{D}(t).$$

Thus, $\lhd$ does not show useful when trying to show $\mathsf{R}(\mathrm{D}, \mathrm{R}, *)$ is not finitely axiomatisable. This is because the definition of $\lhd$ (where we replace ; with $*$) can equivalently be defined via a single formula $s \lhd t \Leftrightarrow (\exists u, v : s = u \wedge t = \mathrm{D}(v) * u)$. For example, $a * \mathrm{D}(b) * c \lhd a * c$ but $a * \mathrm{D}(b) * c = \mathrm{D}(a * \mathrm{D}(b)) * a * c$. Therefore, the nontrivial $\lhd$-cycles in the structures used in [HM11] to show nonfinite axiomatisability for $\mathsf{R}(\mathrm{D}, \mathrm{R}, ;)$ violate a single axiom.

The reader can check that when the $\preceq$ formulas are stated as $(\mathrm{D}, \mathrm{R}, ;)$-formulas (by replacing $*$ with ;), we get that $s \preceq t$ if and only if for all representations h we have $h(s) \leq h(t)$ in the 'angelic' interpretation. This implies that the predicate $\lhd$ is not a complete characterisation of $\leq$ for $\mathsf{R}(\mathrm{D}, \mathrm{R}, ;)$.

Now we define for every $n < \omega$ an algebra $\mathcal{A}_n = (A_n, \mathrm{D}, \mathrm{R}, *)$. It contains, for all $i < n$, the domain-range elements $d, r, 0, m_i, \varepsilon_i$ with all pairwise compositions of two different domain-range elements evaluating to 0. Then we define, for all $i < n$, the elements $a_i, b_i, c_i, d_i, ac_i, acd_i, cdb_i, db_i, ab_i$ with d being the domain of a_i, ac_i, acd_i, ab_i, m_i the domain of c_i, b_i, cd_i, cdb_i and the range of a_i, d_i, cd_i, acd_i, ε_i the domain of d_i and the range of c_i, ac_i and r the range of ab_i, cdb_i, db_i, b_i. The reader may find it useful to refer to Figure 5.1 when determining these domain-range elements.

We define the demonic composition $*$ for the following pairs as

$$\begin{array}{cccc}
a_i * b_i = ab_i, & a_i * c_i = ac_i, & a_i * cd_i = acd_i, & a_i * cdb_i = ab_{i+1}, \\
c_i * d_i = cd_i, & c_i * db_i = cdb_i, & d_i * c_i = \varepsilon_i & d_i * b_i = db_i \\
ac_i * d_i = acd_i & ac_i * db_i = ab_{i+1} & acd_i * c_i = ac_i & acd_i * cdb_i = ab_{i+1} \\
acd_i * b_i = ab_{i+1} & cd_i * b_i = cdb_i & cd_i * c_i = c_i & cd_i * cd_i = cd_i
\end{array}$$

where $+$ denotes addition modulo n. All other compositions are either the mandatory domain-range compositions or they evaluate to 0. Again, Figure 5.1 may show useful for the reader.

Lemma 132. *The algebra $\mathcal{A}_n$ is not in $\mathsf{R}(\mathrm{D}, \mathrm{R}, *)$, for any $0 < n < \omega$.*

Proof. Observe how $m_i \preceq c_i * d_i$ and thus $ab_i = a_i * m_i * b_i \preceq a_i * c_i * d_i * b_i = ab_{i+1}$ for all $i < n$ with $+$ being the addition modulo n. This means by transitivity of $\preceq$ that for all $i, j < n$ we have $ab_i \preceq ab_j$. Now assume that there existed a representation h. By Lemma 131 would have $h(ab_i) \sqsubseteq h(ab_j), h(ab_j) \sqsubseteq h(ab_i)$ even where $i \neq j$. Since $\sqsubseteq$ is antisymmetric for binary relations, we would have $h(ab_i) = h(ab_j)$ for $i \neq j$. Therefore, no such mapping h can exist. □

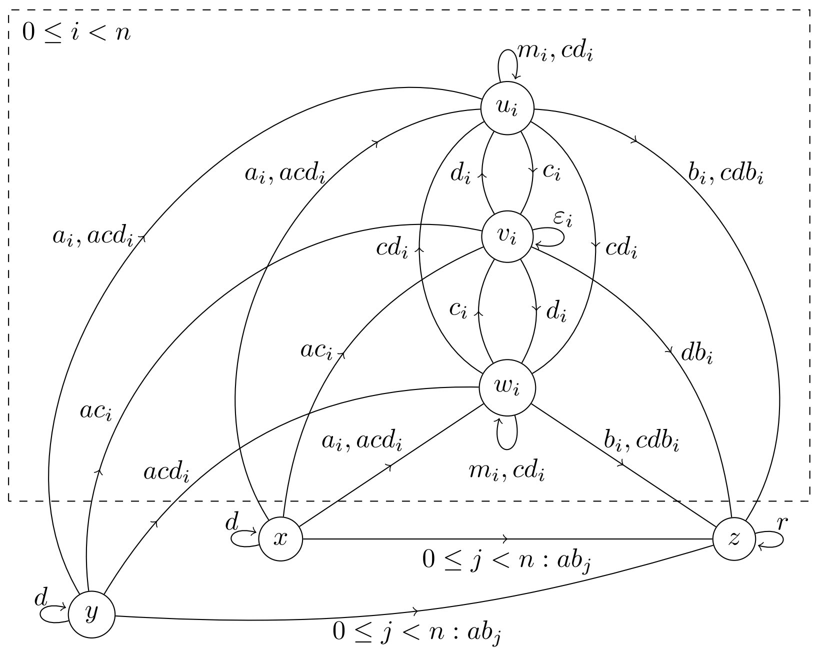

Figure 5.1: A homomorphism $h : A_n \to \wp(X \times X)$ such that $h(s) \neq h(t)$ for all $s \neq t \in A_n$, except ab_i, ab_j or b_i, cdb_i for $i, j < n$.

Lemma 133. *For $\mathcal{A}_n$, $\exists$ will have a winning strategy for a game $\Gamma_k(\mathcal{A}_n), k < \lfloor n/2 \rfloor$.*

Proof. First, let us show that $\exists$ can win a game if the initialisation pair is not $ab_i \neq ab_j$ or $b_i \neq cdb_i$. She can, on such an initialisation move, fix $X = \{x, y, z\} \cup \{u_i, v_i, w_i \mid 0 \leq i < n\}$ and return a network $\mathcal{N}$, corresponding to a homomorphism $h : \mathcal{A} \to \wp(X \times X)$ as presented in Figure 5.1. The reader can check that $\mathcal{N}$ is consistent and that no matter what move $\forall$ calls in the subsequent rounds, $\exists$ can keep returning the same network.

A similar homomorphism with two nodes with r in the reflexive label and one with d can be constructed to have all pairs but ab_i, ab_j or a_i, acd_i faithfully discriminated. Thus, the only way that $\exists$ can surely lose the game is if $\forall$ requests an initialisation with $ab_i \neq ab_j$. Note that either $i - j \geq \lfloor n/2 \rfloor$ or $j - i \geq \lfloor n/2 \rfloor$ where $-$ is subtraction modulo n. Thus we can assume, without loss, that the initialisation pair is ab_0, ab_k where $k \geq \lfloor n/2 \rfloor$.

$\exists$'s strategy will be as follows. Let x_0, y_0 be the nodes that discriminate the initialisation pairs. Her strategy will maintain a network for which $\forall$ can only call a witness move over (x_0, y_0), the label $\top(x_0, y_0)$ remains at ith move to be $ab_j, j \leq i$, the only non-empty $\bot$ label will be that of x_0, y_0 containing ab_k only, and there will be no non-reflexive incoming edges into x_0 and none outgoing from y_0. We check the strategy by induction. In both the base case and the

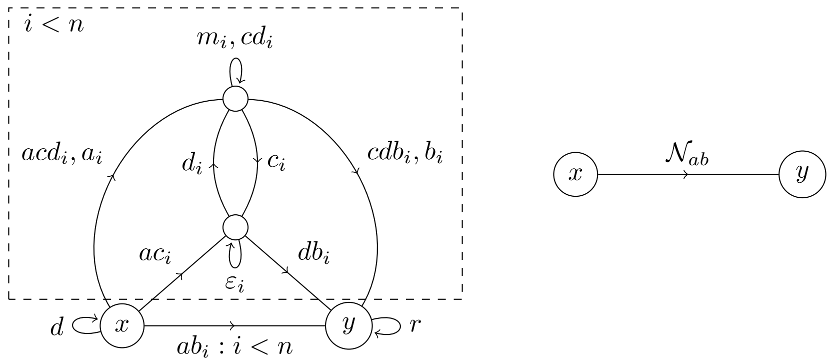

Figure 5.2: The definition of the building block network $\mathcal{N}_{ab}$ used in the proof of ∃'s winning strategy for $\Gamma_k(\mathcal{A}_n)$ for $k < \lfloor n/2 \rfloor$ when $ab_0 \neq ab_k$ is played at initialisation (left) and the notation used in further proofs (right).

induction case we will use a network $\mathcal{N}_{ab}$ as a building block, we define it in Figure 5.2 left. The right part of the figure shows the notation we will use for it when we define bigger networks that contain $\mathcal{N}ab$ later in the proof.

Base Case In the base case, ∃ returns the network from Figure 5.3 top left. The reader can check that the network is closed under all moves, but witnesses over x_0, y_0 and that it satisfies the remaining conditions.

Induction Case In the induction case at the ith move, ∀ may play a witness move $a_j * b_j$ where $0 \leq j < n$. ∃ may respond by adding the non x_0, y_0 nodes and labels from the top right Figure 5.3 where the node denoted • is the witness. This will at most add ab_i to $\top(x_0, y_0)$ and preserve the induction hypothesis. In all other cases, that is $a_j * cd_j, ac_j * db_j, acd_j * b_j$, and $acd_j * cdb_j$ where $0 \leq j < i - 1$ or $j = n - 1$, she may respond with the $*$-closure (with no $\perp$ labels) of the Figure 5.3 bottom. In this case nothing needs to be added to $\top(x_0, y_0)$ and the induction hypothesis also remains preserved. The node denoted • is a witness for $a_j * cdb_j$, □ for $ac_j * db_j$, ■ for $acd_j * b_j$, and •, ■ are both also witnesses for $acd_j * cdb_j$.

□

Directly from Lemmas 130, 132, and 133, we conclude the below theorem.

Theorem 134. $\mathsf{R}(\mathrm{D}, \mathrm{R}, *)$ *is not finitely axiomatisable.*

5.2 FRP for Convoluted Domain Semigroups

In this section, we extend the results from [HE13, HM13] and show that any signature $(\smile, \mathrm{D}, ;) \subseteq \tau \subseteq (\leq, 0, 1, 1', \mathrm{D}, \mathrm{R}, \smile, ;)$ has the finite representation

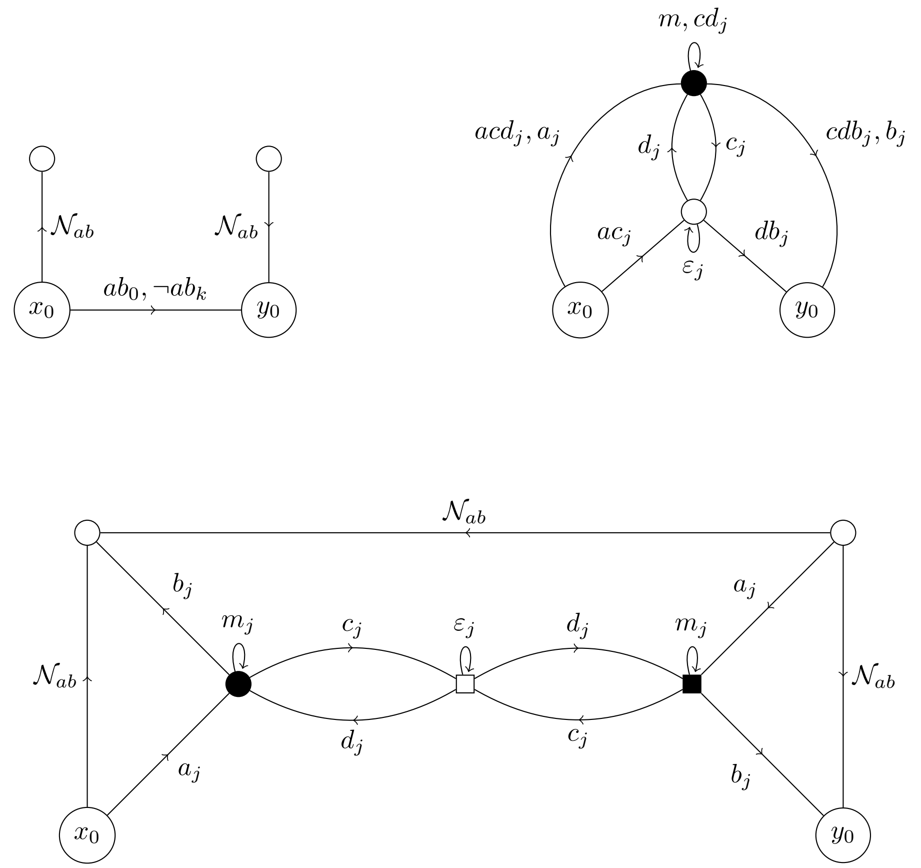

Figure 5.3: The visualisation of a winning strategy for $\exists$ in $\Gamma_k(\mathcal{A}_n)$ for $k < \lfloor n/2 \rfloor$ when $ab_0 \neq ab_k$ is played at initialisation (top left), the $a_j * b_j$ witness (top right) and all other possible witness moves (bottom).

property for finite members of $\mathsf{R}(\tau)$. This similarity class of signatures likely contains further examples of signatures that do not have their representation class finitely axiomatisable but have the finite representation property for finite members.

Furthermore, for all join-free signatures τ, conjectured to have the finite representation property in Conjecture 27, an interesting consequence emerges. A finite τ-structure is finitely representable if and only if it embeds into a finite representable $(\tau \cup (\mathrm{D}, \smile, ;))$-structure. This presents another possible route of showing a large portion of the still open part of Conjecture 27.

Lemma 135. *Take a signature $(\smile, \mathrm{D}) \subseteq \tau \subseteq (\leq, 0, 1, 1', \mathrm{D}, \mathrm{R}, \smile, ;)$ and a finite structure $\mathcal{A} = (A, \tau) \in \mathsf{R}(\tau)$. There exists a finite representable $(\tau \cup (\leq, \mathrm{R}))$-structure $\mathcal{A}'$ such that $\mathcal{A}$ is its τ-reduct, that is, $\mathcal{A}$ and $\mathcal{A}'$ have the same carrier set A and all operations in $\mathcal{A}$ are interpreted in the same way as in $\mathcal{A}'$.*

Proof. The range operation R is term-definable as $\mathrm{R}(s) = \mathrm{D}(\breve{s})$ and the ordering can be defined semantically as all pairs $s, t \in A$ such that in any representation h of $\mathcal{A}$ we have $h(s) \leq h(t)$. Observe that if $\leq \in \tau$, this is the same as the interpretation of $\leq$, else, $\leq$ is always correctly represented in any h, by definition. □

Lemma 136. *Take a signature $(\leq, \smile, \mathrm{D}, \mathrm{R}, ;) \subseteq \tau \subseteq (\leq, 0, 1, 1', \mathrm{D}, \mathrm{R}, \smile, ;)$ and a structure $\mathcal{A} = (A, \tau) \in \mathsf{R}(\tau)$. If $\mathcal{A}$ is representable, then there exists a finite representable structure $\mathcal{A}' = (A', ((0, 1') \cup \tau))$ such that $\mathcal{A}$ is a subreduct of $\mathcal{A}'$.*

Proof. Take a representation h over X of $\mathcal{A}$. If any of the elements is already represented as either $0, 1'$, set the interpretation of the symbol to that element. Else, we examine separately how we can amend the set with two new elements $0, 1'$ such that $h' : A' \to \wp(X \times X)$ where $h'(s) = h(s)$ for all $s \in A$, $h(1') = \{(x, x) \mid x \in X\}$, and $h(0) = \emptyset$, is a representation for $\mathcal{A}'$.

The element 1' is defined to have $\mathrm{D}(1') = \mathrm{R}(1') = \breve{1'} = 1'$, $1' \leq 1'$, $1' \leq s$ for all $s \in A$ such that $\{(x, x) \mid x \in X\} \leq h(s)$, and $s \leq 1'$ for all $s \in A$ such that $h(s) \leq \{(x, x) \mid x \in X\}$. Finally, 1' is defined to be the left-right identity for the composition operation (;). Clearly, amending h to get h' that represents 1' as the diagonal over X will represent all these operations and predicates correctly. Additionally, since no other element is represented as the diagonal, h' remains faithful.

The element 0 is defined to have $\mathrm{D}(0) = \mathrm{R}(0) = 0$, $0 \leq s$, and $0;s = s;0 = 0$ for all $s \in A'$. Similarly, amending h to get h' that represents 0 as the empty set will represent all these operations and predicates correctly. Additionally, since no other element is represented as the empty set, h' remains faithful. □

The above two lemmas and the results from [HE13, HM13] enable us to prove the below theorem.

Theorem 137. *For any signature $(\smile, \mathrm{D}, ;) \subseteq \tau \subseteq (\leq, 0, 1, 1', \mathrm{D}, \mathrm{R}, \smile, ;)$, finite members of $\mathsf{R}(\tau)$ will have a finite representation.*

Proof. Take a finite representable τ-structure $\mathcal{A}$. By Lemmas 135, 136 that there will exist a finite representable τ' structure $\mathcal{A}'$ and $(\leq, 0, 1', \mathrm{D}, \mathrm{R}, \smile, ;) \subseteq \tau' \subseteq (\leq, 0, 1, 1', \mathrm{D}, \mathrm{R}, \smile, ;)$ and $\mathcal{A}$ is a subreduct of $\mathcal{A}'$. As finite representability is trivially preserved by subreducts, it suffices to show that $\mathcal{A}'$ is finitely representable.

The $(\leq, 0, 1', \mathrm{D}, \mathrm{R}, \smile, ;)$-reduct of $\mathcal{A}'$ is finitely representable by the constructions shown in [HE13, HM13] where the set of closed sets $X \subseteq \wp(A')$ is defined as the set of all upwardly closed sets not containing 0 that satisfy some additional domain-range conditions. Importantly, a mapping $h : A' \to \wp(X \times X)$ is defined for all $s \in A'$ as

$$h(s) = \{(T, U) \in X \times X \mid T;s \subseteq U,\ U;\breve{s} \subseteq T\}$$

where a for a set $S \subseteq A'$ and an element $t \in A'$, we define $S;t$ as

$$S;t = \{s;t \mid s \in S\}.$$

Finally, the fact that 1 is correctly represented follows from $\leq$ being correctly represented. $\square$

Two interesting novel consequences of this theorem are stated below.

Corollary 138. *The class* $\mathsf{R}(\leq, 0, 1, 1', \mathrm{D}, \mathrm{R}, \smile, ;)$ *is axiomatised by the axioms for* $\mathsf{R}(\leq, 0, 1', \mathrm{D}, \mathrm{R}, \smile, ;)$ *from [HM13], as well as*

(1) $s \leq 1$,

(2) $\breve{1} = 1$.

Corollary 139. *For signatures* $\tau \subseteq (\tau_{\mathsf{RA}} \setminus (-, +)) \cup (\mathrm{D}, \mathrm{R})$*, to show that a finite* τ*-structure is finitely representable, it suffices to show that it embeds into a finite* $(\tau \cup (\smile, \mathrm{D}))$*-structure in* $\mathsf{R}(\tau \cup (\smile, \mathrm{D}))$.

Although this type of an embedding presents a possible route to show the finite representation property, there exist structures where this embedding would not be trivial, i.e. the structure $\mathcal{A}$ would have to be a proper subreduct of the structure $\mathcal{A}'$. We examine this in the signature of domain-range semigroups (without the converse) $R(\mathrm{D}, \mathrm{R}, ;)$.

If we have a structure $\mathcal{A} \in R(\mathrm{D}, \mathrm{R}, ;)$, it may be the case that some pairs of elements will indeed always be represented as converses. For example, domain-range elements will always be represented as self-converse, however, there also exist a number of non-domain elements with a converse defined. Similarly to the definition of $\preceq_1$ in the previous section, if we take some $s;t$ with $\mathrm{D}(s;t) = s;t$, one can show that $s;\mathrm{D}(t)$ and $\mathrm{R}(s);t$ will be a proper converse pair, more details in the next section.

However, take an element s with its converse $\breve{s}$ in the structure and say $s = t; u$ and $\mathrm{R}(t) = \mathrm{D}(u)$. The converses of t, u may not be defined. Both t and u have a *semiconverse*. That is, for every representation h, $\overbrace{(h(t))} \leq h(u;\breve{s})$ and $\overbrace{(h(u))} \leq h(\breve{s};t)$, but the $\geq$ inclusions do not necessarily hold, see Figure 5.4.

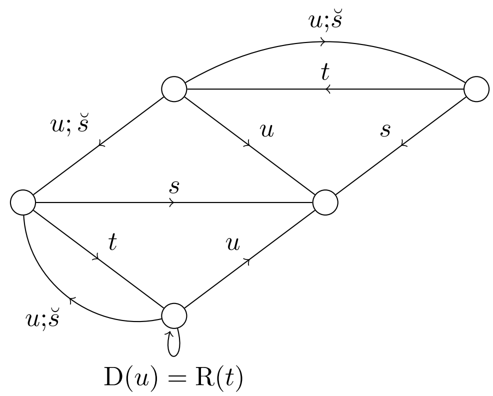

Figure 5.4: Partial converse of t, i.e. $\breve{t} \leq u;\breve{s}$, but $u;\breve{s} \not\leq \breve{t}$, where s, t, u are elements of a representable domain-range semigroup

5.3 Partial functions in Domain-Range Semigroups

In this section, we explore another possible route via which we could potentially prove the finite representation property for converse-free signatures containing domain and range operations. When we defined $\preceq$ in Section 5.1, we were taking advantage of the fact that some elements will always be represented as partial functions. This can lead to simpler recursive axiomatisations and perhaps even the finite representation property.

The simplest example of partial functions in some $\mathcal{A} \in \mathsf{R}(\mathrm{D}, \mathrm{R}, ;)$ are the domain-range elements. However, they are not the only such examples. We hypothesise that the set of these function-like elements $F \subseteq A$ is as follows.

$$f \in F \Leftrightarrow \exists s, t : s;t = \mathrm{D}(s;t) \wedge \mathrm{R}(s);t = f$$

From the above equivalence, we can define a function $\rho : F \to A$ where

$$\rho(f) = s;D(t) \qquad\qquad \text{where } f = R(s);t, \quad s;t = D(s;t).$$

In the 'angelic' signature $\rho(f)$ will be the converse of f. If we replace ; with $*$ in the above definitions, we observe that the notion of a converse does not really have a demonic counterpart, so we say that $\rho(f)$ will be a converse-like image of f. In either case, it will be true that $h(\mathrm{D}(f;\rho(f))) \leq h(f;\rho(f))$, or equivalently $h(\mathrm{D}(f * \rho(f))) \sqsubseteq h(f * \rho(f))$, for representable structures in $\mathsf{R}(\mathrm{D}, \mathrm{R}, ;)$ and $\mathsf{R}(\mathrm{D}, \mathrm{R}, *)$ respectively.

This enables us to define an ordering as a monotone, transitive closure of $\mathrm{D}(f;\rho(f)) \leq f;\rho(f)$, the angelic version of the $\preceq$ predicate from Section 5.1. However, at the moment we do not know if this is a complete characterisation of refinement, 'angelic' or demonic. Furthermore, some axioms are only valid for functions, for example for all $f, g, h \in F$ we have

$$f;g = f;h \Rightarrow \mathrm{R}(f);g = \mathrm{R}(f);h.$$

A complete recursive simplified axiomatisation of either the angelic or the demonic class can now account for these as well. However, most importantly, the existence of partial functions enables us to hypothesise about expanding on the idea of Cayley–Zareckiĭ representation for semigroups by defining separate rules for functions and relations. We conclude this chapter with a conjecture.

Conjecture 140. *The extensions of the Cayley–Zareckiĭ representation of groups where partial functions will be given a different set of rules than non-functional relations to hold on a pair of points will help us show finite representation property for signatures including* $(\leq, 1', ;), (\mathrm{D}, \mathrm{R}, ;),$ *and* $(\mathrm{D}, \mathrm{R}, *)$*, bringing us a step closer towards proving the positive implication of Conjecture 27.*

5.4 Conclusion

This chapter is the penultimate results chapter in the thesis and in it we have shown that many signatures containing domain and range have the finite representation property. These are examples for the main conjecture addressed in this work. Further, we have examined some related properties, namely we have shown the nonfinite axiomatisability of $\mathsf{R}(\mathrm{D}, \mathrm{R}, *)$, which builds on results in [HM11], but does not follow trivially from them by any means. We then speculate how to show the remaining cases of the main conjecture for signatures with domain and range, namely by taking advantage of the necessity to represent some elements of such algebras as partial functions. We now move on to the final chapter of the thesis, where we examine finite representability and related properties for weakening relation algebras — yet another nonclassical relation algebra variation.

Chapter 6

Weakening Relation Algebras

The results presented in this chapter were shown in collaboration with Peter Jipsen. The author thanks him for agreeing to present these results in this thesis.

This is the final results chapter of the thesis. It examines the finite representability and related topics in yet another non-classical setting — the area of weakening relation algebras. Although these algebras have been of interest in the literature, the existing body of research regarding these structures isn't as vast as was the case in the previous section. This is why we begin the section by showing some properties in which the class of representable weakening relation algebras agrees with that of representable relation algebras, but also prove some major differences — namely the failure of closure under homomorphisms. Further, we examine (finite) representability of small weakening relation algebras on a case by case basis. Only then do we revisit the main conjecture and show some cases of it hold in the weakening setting as well. The weakening representation classes are bigger than classical representation classes, however this does not mean that the finite weakening representation property trivially implies its classical counterpart. The cases we show to have the finite representation property in the weakening setting remain open in the classical setting.

We have seen that demonic operations, although not included in the Tarskian signature, have been studied extensively due to their usefulness in computer science. Similarly, the class of weakening relation algebras, motivated by the weakening rule in formal systems have been recently becoming of interest [GJ20a, GJ20b, Jip17, Ste12, Ste15].

The main idea in weakening relation algebra is to represent the constant 1' as an arbitrary partial order, rather than the diagonal, though we still require 1' to be a two-sided identity for the ; operation. This is based on the various weakening rules in formal systems, for example, the rules of consequence in Hoare Logic [Hoa69]. In addition to their definition being motivated by formal systems, weakening relation algebras have applications in sequent calculi [GJ17], quasi-proximity lattices/spaces [Smy92], order-enriched categories [KV16], mathematical morphology [Ste15], and program semantics, for example, via separation logic [Rey02].

As we will see later on, all operations in the signature of relation algebra, except complementation and converse are closed under weakening. Interestingly, however, the combined complement-converse operation is. This new operation as well as all the other operations in the signature of relation algebras closed under weakening are included in the signature of weakening relation algebras. Additionally, these are sometimes studied with a variation of the Heyting implication (different to that in Section 3.3), however, here we study the fragment without it.

In this chapter, we present a number of novel results in the area, resulting from our studies. We begin the chapter by defining the class of [diagonally] representable weakening relation algebras and show that for both classes representation problem is undecidable. We then present our adaptation of the representation games to weakening relation algebras. We continue with the frame presentation of weakening relation algebras, analogous to atom structures. This enables us to define and work towards the abstract classes of weakening relation algebras akin to RA_n for $2 < n \leq \omega$. We show that, unlike the class of representable weakening relation algebras, the class of diagonally representable weakening relation algebras forms a variety. Finally, we study the [finite] representability of some small algebras and show that when the identity (and by extension the domain and range) does not have to be contained within the diagonal, the finite axiomatisability of the representation class and the finite representation property can be shown for some signatures where the property either fails or remains open in the classical relation-algebraic setting.

6.1 Weakening Relation Algebras

In this section, we define the concept of weakening relations, the class of representable weakening relation algebras, as well as diagonally representable weakening algebras. We then show that membership in both of those classes is undecidable for finite structures.

Definition 141. *Let $\mathcal{X} = (X, 1')$ be a poset, that is, $1'$ is a partial order over the set X. A relation $R \subseteq X \times X$ is a* weakening relation *for $\mathcal{X}$ if $1';R = R;1' = R$.*

We can quickly see that $0, 1, +, ;$ preserve weakening. However, that is not necessarily the case with $-$ or $\smile$. Take A poset $\mathcal{X} = (X, 1')$ where $X = \{x, y\}, 1' = \{(x, x), (y, y), (x, y)\}$ and a binary relation $R = \{(x, y)\}$, see Figure 6.1. Observe that $(x, x) \in -R$, however, because $(x, y) \in 1'$, that would mean that (x, y) would also have to be in $-R$. Similarly because $(y, x) \in \breve{R}$, we'd require that $(y, y) \in \breve{R}$ if $\smile$ preserved weakening.

While the closure under complement and converse fails individually, if we take the term-definable unary operation $\sim$ — we'll call it the complement-

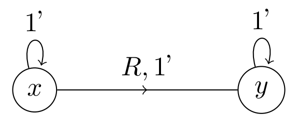

Figure 6.1: Weakening relations are not closed under complementation and converse.

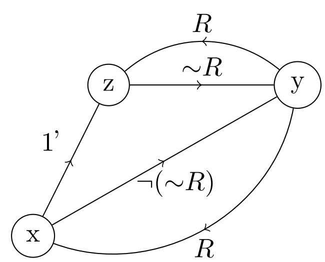

Figure 6.2: The proof of Proposition 143, visualised.

converse — we see that the weakening relations are closed under it. We define the operation in terms of τ_{RA} as

$$\sim R = -(\breve{R}) = \overbrace{(-R)},$$

or equivalently,

$$\sim R = \{(x, y) \in \top \mid (y, x) \notin R\}.$$

Remark 142. *In the context of weakening relation algebras,* $\top$ *will be an arbitrary equivalence relation. The closure under* 1', $\sim$, ; *is sufficient for reflexivity, symmetry, and transitivity, respectively.*

Proposition 143. *The complement-converse operation* $\sim$ *preserves weakening.*

Proof. For contradiction, suppose that the operation was not closed under weakening. That would mean that for some weakening relation R there exists a pair $(x, y) \notin \sim R$ where, without loss, (x, y) witnesses a 1';($\sim R$) composition, via some z, see Figure 6.2. As $(x, y) \notin \sim R$, we know $(y, x) \in R$ and thus, by R being a weakening relation and $(x, z) \in$ 1', we have that $(y, z) \in R$. This contradicts the fact that $(z, y) \in \sim R$. □

Now we have everything to define the class of proper weakening relation algebras below.

Definition 144. *The class of* proper weakening relation algebras PwkRA *is the class of all structures* $\mathcal{A} = (A, 0, 1, +, 1', \sim, ;)$ *where there exists an equivalence relation* $\top \subseteq X \times X$ *such that*

(1) $A \subseteq \wp(\top)$,

(2) 0 *is interpreted as* $\emptyset$ *and* 1 *as* $\top$,

(3) $1'$ *is a partial order over* X,

(4) $+, \sim, ;$ *are interpreted as join, complement-converse, and composition.*

The class of representable weakening relation algebras RwkRA *is the closure of* PwkRA *under isomorphic copies. Again, an isomorphism* h *mapping a representable weakening relation algebra to a proper one is called a* representation.

Remark 145. *The reader can check that meet* $(\cdot)$ *is term-definable in* RwkRA *as* $R \cdot S = \sim(\sim R + \sim S)$.

Remark 146. *In the weakening setting, the diversity element will be* $0' = \sim 1'$.

Remark 147. *We will refer to the signature of weakening relation algebras* $(0, 1, +, 1', \sim, ;)$ *as* τ_{wkRA}.

Remark 148. *Again, we may require* $\top$ *to be equal to* $X \times X$. *If an algebra is representable in such a way, it is* square-representable.

Observe how we no longer require the identity to be equal to the *diagonal*, that is $\{(x, x) \mid x \in X\}$. When we require this to be the case, we say that we are talking about diagonally representable algebras.

Definition 149. *The class of all proper weakening relation algebras where* $1' = \{(x, x) \mid x \in X\}$ *is called the class of* proper diagonal weakening relation algebras PDwkRA. *Its isomorphism closure is called the class of* diagonally representable weakening relation algebras DRwkRA.

Remark 150. $\mathsf{PDwkRA} = \mathsf{P}(\tau_{\mathsf{wkRA}})$ *and* $\mathsf{DRwkRA} = \mathsf{R}(\tau_{\mathsf{wkRA}})$.

We now prove the following Lemma that will enable us to show the undecidability result.

Lemma 151. *For every finite* $\mathcal{A} \in \mathsf{RA}$, *the following statements are equivalent:*

(1) $\mathcal{A} \in \mathsf{RRA}$,

(2) The τ_{wkRA}*-reduct of* $\mathcal{A}$ *is in* DRwkRA,

(3) The τ_{wkRA}*-reduct of* $\mathcal{A}$ *is in* RwkRA.

Proof. The implication (1) $\Rightarrow$ (2) is trivial. For the converse, observe that because $\mathcal{A}$ is finite, it will have atoms. Let h be τ_{wkRA}-representation of $\mathcal{A}$. Take any two atoms $a, b \in A$ and observe that $h(a) \cdot h(b) = \emptyset$ as meet is term-definable from τ_{wkRA} and thus every pair of points (x, y) will be in the image of at most one atom. Furthermore, for all atoms, any two points must be in either $h(a)$ or $h(-a)$ as $a + (-a) = 1$ and $+ \in \tau_{\mathsf{wkRA}}$. If there existed a pair of points where no atom a existed such that $(x, y) \in h(a)$, that would mean that it would be in the meet of $h(-a)$ for all atoms a, which (because $\mathcal{A}$ is a finite structure) is defined to be 0. Thus no such pair exists and for every pair (x, y) there exists a unique atom a such that $(x, y) \in h(a)$. Thus $-$ is represented by h and so is $\sim$ since $\smile$ is term-definable from $-, \sim$.

The implication (2) $\Rightarrow$ (3) is, again, trivial. For the converse, observe that if there existed a pair $(x, y) \in h(1')$ where h is a weakening representation and $x \neq y$, that would imply that $(y, x) \notin h(1')$ and thus $(x, y) \in h(\sim 1')$. As meet is correctly represented, this would imply that $1' \cdot \sim 1'$ is nonzero, which is not the case in τ_{wkRA}-reducts of relation algebras. □

From the above lemma and the fact that the representation problem is undecidable for relation algebras [HH01] we get the following claims.

Theorem 152. *The representation problem is undecidable for finite [diagonally] representable weakening relation algebras.*

Corollary 153. *The classes* $\mathsf{RwkRA}, \mathsf{DRwkRA}$ *are not finitely axiomatisable. Furthermore, the finite representation property for finite structures fails in both classes.*

6.2 Weakening Relation Algebra by Games

In this section, we adapt the relation algebra representation games [HH02] as presented in Section 2.2 to work for the context of weakening relation algebras. We begin by defining the basic axioms for weakening relation algebras. These will be analogous to the axioms for Boolean involutive unital magmas. In this section, we state the axioms, for justification see Section 6.3.

Definition 154. *A* bounded cyclic involutive unital distributive lattice-ordered magma $\mathcal{A} = (A, 0, 1, +, 1', \sim, ;)$ *is an algebra such that for all* $s, t, u, v \in A$

(1) $(A, 0, 1, \cdot, +)$ *is a bounded distributive lattice,*

(2) $(s + t); (u + v) = s; u + s; v + t; u + t; v,$

(3) $s;0 = 0 = 0;s,$

(4) $s;1' = s = 1';s,$

(5) $\sim(\sim s) = s,$

(6) $\sim(s \cdot t) = \sim s + \sim t.$

Note that $s \leq t$ if and only if $s + t = t$, or equivalently $\sim s \cdot \sim t = \sim t$ which can be rewritten as $\sim t \leq \sim s$, hence $\sim$ is order-reversing. The adjective "cyclic" is included in the name to contrast it to the non-cyclic general case the are two unary operations $\sim, \neg$ in the language that satisfy $\sim\neg s = s = \neg\sim s$, see [GFJL23]. In the cyclic case $\sim, \neg$ have the same interpretation. Distributive lattice-ordered magmas are abbreviated as $d\ell$-magmas. Additionally, we define the *diversity* as 0' = $\sim$1'.

The networks we define in this section will — much like those in Section 2.2 — only have one labelling function. However, the key difference arises when we talk about consistency as we do not have negation in the signature. Additionally, we must require that 1 is in every label. This will help us show faithfulness is assured by the initialisation move.

Definition 155. *A* network (for a bounded cyclic involutive unital $d\ell$-magma $\mathcal{A}$) *is a tuple* $\mathcal{N} = (N, \lambda)$ *where* N *is a set of* nodes *and* $\lambda : N^2 \to \wp(A)$ *is a* labelling function *such that for all* $x, y \in N$, $1' \in \lambda(x, x)$ *and* $1 \in \lambda(x, y)$. *Such a network is* consistent *if and only if for all* $x, y \in N$ *we have that*

$$\lambda(x, y) \cap \{\sim a \mid a \in \lambda(y, x)\} = \emptyset.$$

A network $\mathcal{N} = (N, \lambda)$ *is a* prenetwork *of* $\mathcal{N}' = (N', \lambda')$ *— denoted* $\mathcal{N} \subseteq \mathcal{N}'$ *— if and only if* $N \subseteq N'$ *and for all* $x, y \in N$ *we have* $\lambda(x, y) \subseteq \lambda'(x, y)$.

The definition of the game largely follows the definition of the game for relation algebras from Section 2.2. The key difference is the initialisation move and the moves pertinent to the complement-converse operation.

Definition 156. *An* n-round representation game*, denoted* $\Gamma_n(\mathcal{A})$*, for some* $n \leq \omega$ *is a two player game played between the challenger* $\forall$ *(Abelard) and the responder* $\exists$ *(Héloïse) over* n *moves, where* $\exists$ *returns a network* $\mathcal{N}_i$ *after the* i*th move for* $0 \leq i < n$ *such that if* $0 \leq i \leq j < n$, $\mathcal{N}_i \subseteq \mathcal{N}_j$. *The game is won by* $\forall$ *if* $\exists$ *returns an inconsistent network. Otherwise* $\exists$ *wins.*

initialisation move $\forall$ *picks a pair of elements* $s \not\leq t \in A$ *and* $\exists$ *must return a network* $\mathcal{N}_0$ *with some* $(x, y) \in N_0^2$ *such that* $s \in \lambda(x, y)$ *and* $\sim t \in \lambda(y, x)$.

On the i*th move for* $0 < i < n$, $\forall$ *may challenge* $\exists$ *with any of the following four moves.*

join move $\forall$ *picks* $x, y \in N_{i-1}$, *some* $s \in \lambda_{i-1}(x, y)$, *and some* $t, u \in A$ *such that* $s \leq t + u$. $\exists$ *must return a* $\mathcal{N}_i$ *with* $t \in \lambda_i(x, y)$ *or* $u \in \lambda_i(x, y)$.

involution move $\forall$ *picks* $x, y \in N_{i-1}$ *and some* $s, t \in A$ *such that* $t = \sim s$. $\exists$ *must return a* $\mathcal{N}_i$ *with* $s \in \lambda_i(x, y)$ *or* $t \in \lambda_i(y, x)$.

composition move $\forall$ *picks* $x, y, z \in N_{i-1}$ *and* $s \in \lambda_{i-1}(x, y), t \in \lambda_{i-1}(y, z)$. $\exists$ *must return a* $\mathcal{N}_i$ *with* $s;t \in \lambda_i(x, z)$.

witness move $\forall$ *picks* $x, y \in N_{i-1}$, $s;t \in \lambda_{i-1}(x,y)$. $\exists$ *must return a* $\mathcal{N}_i$ *with some* $z \in N_i$ *such that* $s \in \lambda_i(x,z), t \in \lambda_i(z,y)$.

We carefully check the proof of the below proposition as it is significantly different to the argument presented in Theorem 57. Namely, we check that the initialisation move does indeed ensure faithfulness and propose a different equivalence relation by which we quotient the limit networks to build a representation.

Proposition 157. *A countable bounded cyclic unital dℓ magma $\mathcal{A}$ is representable if and only if $\exists$ has a winning strategy for $\Gamma_\omega(\mathcal{A})$.*

Proof. If $\mathcal{A}$ is representable, then $\exists$ can take some representation h over X. Let $s \not\leq t$ be the pair played on initialisation move. There will exist some maximal $X' \subseteq X$ such that $\exists x, y \in X' : (x,y) \in h(s), (x,y) \notin h(t)$ and $\forall z, w \in X' : (z,w) \in h(1)$. On initialisation move, $\exists$ can return the network $\mathcal{N} = (X', \lambda)$ where $\lambda(x,y) = \{u \in A \mid (x,y) \in h(u)\}$. Because h preserves all the operations in the language, all moves $\forall$ may call are trivially responded to by returning the same network after every move.

Conversely, because $\mathcal{A}$ is countable then $\forall$ can schedule his moves in a way that every move will be called eventually. Let $\mathcal{N}_0^{s,t}, \mathcal{N}_1^{s,t}, \mathcal{N}_2^{s,t}, \ldots$ be the networks during an $\exists$-winning play of $\Gamma_\omega(\mathcal{A})$ where $\forall$ scheduled his moves in such a way and the initialisation move was called for the pair $s \not\leq t$. Define $N_\omega^{s,t}$ as $\{x \mid \exists i < \omega : j \geq i \Rightarrow x \in N_i^{s,t}\}$, $\lambda_\omega^{s,t}(x,y)$ as $\{u \mid \exists i < \omega : j \geq i \Rightarrow (x, y \in N_j^{s,t} \wedge u \in \lambda_j^{s,t}(x,y))\}$, and a relation $\equiv$ as $\{(x,y) \in (N_\omega^{s,t})^2 \mid 1\text{'} \in \lambda_\omega^{s,t}(x,y), 1\text{'} \in \lambda_\omega^{s,t}(y,x)\}$. It is symmetric by definition, reflexive because networks are defined as having $1\text{'} \in \lambda_\omega^{s,t}(x,x)$ and transitive because all composition moves were called eventually and $1\text{'};1\text{'} = 1\text{'}$. Therefore, we can define $h^{s,t} : A \to \big((N_\omega^{s,t}/{\equiv})^2\big)$ where for all $u \in A$ we have $h^{s,t}(u) = \{([x]_\equiv, [y]_\equiv) \mid u \in \lambda_\omega^{s,t}(x,y)\}$.

The reader can check that $h^{s,t}$ is a homomorphism (because all moves were called eventually) for $\mathcal{A}$ discriminating $s \not\leq t$ (because of initialisation). Thus let $h(u)$ for all $u \in A$ be the disjoint union $\biguplus_{s \not\leq t \in A} h^{s,t}(u)$. Because h is a homomorphism that discriminates all $s \not\leq t$ pairs, it is a representation. $\square$

The remainder of the arguments to show the below theorems can be reconstructed from Section 2.2 with relative ease.

Theorem 158. *A bounded cyclic unital dℓ magama $\mathcal{A}$ is representable if and only if $\exists$ has a winning strategy for $\Gamma_\omega(\mathcal{A})$.*

Theorem 159. *The class* RwkRA *has a recursively enumerable axiomatisation.*

6.3 Frames and Frame Games

In this section, we present finite weakening relation algebras as frames, similar to Routley-Meyer frames or relevance frames for relevance logic [BDM09]. These are the analogous notion to atom structures of atomic relation algebras. We then define a modified version of the representation game that utilises frames, similar to $\mathcal{G}$ from Section 2.2.

Then, we define an n-pebble version of the frame game. Analogous to the abstract classes of relation algebras $\mathsf{RA}_\omega \subseteq \ldots \subseteq \mathsf{RA}_4 = \mathsf{RA} \subseteq \mathsf{RA}_3$, this gives rise to classes of weakening relation algebras $\mathsf{wkRA}_\omega \subseteq \ldots \subseteq \mathsf{wkRA}_3 \subseteq \mathsf{wkRA}_2$. Unlike with relation algebras, we also define wkRA_2, see Remark 178 for justification. Clearly $\mathsf{RA}_\omega, \mathsf{wkRA}_\omega$ are the classes of representable relation algebras and weakening algebras, respectively. Furthermore, similarly to RA_4, we say that wkRA_4 is the class of weakening relation algebras.

First, observe that the language of RwkRA does not include negation and hence the lattice need not be Boolean. As we will see in Section 6.6, the smallest representable non-Boolean algebra is a 4-element chain $\mathcal{S}_4$. Thus we cannot present finite weakening relation algebras using atoms. Instead, we make use of join-irreducibles.

Definition 160. *A non-*0 *element a of a representable weakening relation algebra is* join-irreducible *if and only if for all s, t if $a = s + t$ then $a = s$ or $a = t$. It is* join-prime *if and only if for all s, t if $a \leq s + t$ then $a \leq s$ or $a \leq t$.*

Because $\cdot$ distributes over $+$ we have that an element is join-irreducible if and only if it is join-prime. In the finite case, every algebra will have join-irreducibles and every element is a join of join-irreducibles. In general, this is only true for *perfect* algebras. In fact, by definition, a distributive lattice is *join-perfect* if every element is a join of completely join-irreducible elements. This generalises the concept of *atomic* for Boolean algebras.

The element a in the result below is called the *join-irreducible label* of (x, y).

Proposition 161. *In a representation h of a finite representable weakening relation algebra $\mathcal{A}$, for any pair (x, y) there exists a join-irreducible $a \in \mathcal{A}$ such that*

$$a^{\uparrow} = \{s \in \mathcal{A} \mid (x, y) \in h(s)\}$$

where $\uparrow$ denotes the upward closure of a with respect to $\leq$.

Proof. A representation h maps joins to unions, hence the set $\{s \mid (x, y) \in h(s)\}$ is upward closed and if $(x, y) \in h(a+b)$ then it is also in $h(a)$ or $h(b)$. Hence the base set of the representation is itself a union of upward closures of join-primes. Now if it is above $a^{\uparrow}$ and $b^{\uparrow}$ then it must be the case that (y, x) is in neither $h(\sim a)$ nor $h(\sim b)$ and thus $(x, y) \in h(\sim(\sim a + \sim b)) = h(a \cdot b)$. Thus the meet of all such join-irreducibles must also be a non-0 element that is join-prime and below all elements in the set. ☐

Although the converse operation is not defined in our language, we can use the following trick to define a useful unary operation on the join-irreducibles.

Definition 162. *For every join-irreducible a in a finite algebra, define $\hat{a} = \sim\sum_{a \not\leq s} s$ where $\sum$ is with respect to join $(+)$.*

The join $\sum_{s\not\leq t} t$, defined for all s in a finite algebra $\mathcal{A}$, is usually denoted $\kappa(s)$. If we take $s \leq s' \in A$ we have $s \not\leq t \Rightarrow s' \not\leq t$ and thus $\kappa(s) \leq \kappa(s')$, hence κ is order-preserving. Because $\sim$ is order-reversing and κ is order-preserving we have that $\hat{}$ is order-reversing.

Proposition 163. *In any finite bounded distributive involutive additive algebra $\mathcal{A}$, if a is a join-irreducible, so is $\hat{a}$.*

Proof. It is well known that $\kappa(a)$ of a join-irreducible a in a lattice is meet irreducible and because $\sim$ is order reversing, that means that $\hat{a} = {\sim}\kappa(a)$ is a join-irreducible. □

Proposition 164. *If a pair (x, y) in a representation has the join-irreducible label a, then (y, x) has label $\hat{a}$. Moreover, $\hat{\hat{a}} = a$.*

Proof. ${\sim}s \in h(y, x)$ if and only if $s \notin h(x, y)$, i.e. $a \not\leq s$. Thus, by the argument from Proposition 161 the join-irreducible label of (y, x) can be written as $\prod_{a\not\leq s} {\sim}s$ where $\prod$ is with respect to meet $(\cdot)$ and this is equivalent to ${\sim}\sum_{a\not\leq s} s$ by the De Morgan equivalence. □

Finally to characterise composition, we need to define a ternary predicate, similar to the set of allowed triangles in relation algebras.

Definition 165. *Let $\mathcal{A}$ be a finite bounded cyclic involutive dℓ-magma and define a ternary relation* R *on the set of join-irreducibles of $\mathcal{A}$ by*

$$\mathrm{R}(a, b, c) \Longleftarrow a \leq b;c.$$

Recall that for relation algebras with atoms a, b, c the Peircian triangle law says that

$$a \leq b; c \iff \hat{a} \leq \hat{c}; \hat{b} \iff b \leq a; \hat{c} \iff \hat{b} \leq c; \hat{a} \iff c \leq \hat{b}; a \iff \hat{c} \leq \hat{a}; b.$$

As we will see in the next section, this law does not hold for the class of representable weakening relation algebra frames. However, atom structures for relation algebras generalise to the weakening setting as follows.

Definition 166. *A* relevance frame $\mathcal{F} = (F, \mathrm{I}, \leq, \mathrm{R}, \hat{})$ *is a structure with a carrier set F, a unary predicate* I*, a partial order predicate $\leq$, a ternary predicate* R*, and an order-reversing involution operation $\hat{}$ where for all a, b, c, d in F*

(F1) $a \leq b \Leftrightarrow \exists e : \mathrm{I}(e) \wedge \mathrm{R}(a, e, b)$,

(F2) $a \leq b \Leftrightarrow \exists e : \mathrm{I}(e) \wedge \mathrm{R}(a, b, e)$,

(F3) $a \leq b \wedge \mathrm{R}(b, c, d) \Rightarrow \mathrm{R}(a, c, d)$,

(F4) $b \leq c \wedge \mathrm{R}(a, b, d) \Rightarrow \mathrm{R}(a, c, d)$,

(F5) $c \leq d \wedge \mathrm{R}(a, b, c) \Rightarrow \mathrm{R}(a, b, d)$.

Proposition 167. *A relevance frame $\mathcal{F} = (F, \mathrm{I}, \leq, \mathrm{R}, \hat{\ })$ defines a bounded involutive unital $d\ell$-magma $\mathcal{A} = (A, 0, 1, +, 1', \sim, ;)$ by taking the joins of downsets in $(F, \leq)$ as the elements of A with their partial order and for all $s, t \in A$*

$$1' = \sum_{\mathrm{I}(a)} a, \qquad \sim s = \sum_{\hat{a} \nleq s} a, \qquad s;t = \sum_{b \leq s, c \leq t, \mathrm{R}(a,b,c)} a,$$

where $a, b, c \in F$.

Proof. The construction from the converse part of Birkhoff's representation theorem [Bir35] states that a distributive lattice may be recovered from a partial order as the set of its downsets. Clearly $\emptyset, F$ are both downward closed so the lattice is bounded. To show that the magma is unital, we can see that no term of the join defining the composition with the identity is above the identity by (F1), (F2) and because $\leq$ is reflexive, there will exist, for every join-irreducible a term in the composition with the identity (on either side) equal to that join-irreducible. Thus 1' is precisely the identity. Composition is additive by definition. $\sim$ is an involution because a join-irreducible $a \leq \sim(\sim s)$ if and only if $\hat{a} \nleq \sim s$ which is true if and only if $a = \hat{\hat{a}} \leq s$. For the De Morgan equivalence, $a \leq \sim(\sim s + \sim t)$ if and only if $\hat{a} \nleq \sim s + \sim t$, or equivalently $\hat{a} \nleq \sim s \wedge \hat{a} \nleq \sim t$ which by definition is true if and only if $a = \hat{\hat{a}} \leq s$ and $a = \hat{\hat{a}} \leq t$, or simply $a \leq s \cdot t$. □

Proposition 168. *Every finite bounded cyclic involutive unital $d\ell$-magma has a unique equivalent relevance frame.*

Proof. Finite distributive lattices are determined by their poset of join-irreducibles, and from Proposition 163 they have a unique $\hat{\ }$ defined on the join-irreducibles. The mapping to R is unique as (F3), (F4), (F5) ensure that R is downward closed in the first argument and upward closed in the other arguments. □

Proposition 169. *For finite algebras and finite frames, the mappings described in the previous two lemmas are inverses of each other.*

Proof. Finite distributive lattices correspond uniquely to their posets of join-irreducibles. The preservation of identity and the composition follow trivially from the definition. For $\sim, \hat{\ }$ observe that $\sim\kappa(a) = \hat{a} \nleq s$ if and only if $\sim s \nleq \kappa(a) = \sum_{a \nleq} t$, or equivalently $a \leq \sim s$. For the converse note that $\sim\kappa(a) = \sum_{\hat{b} \nleq \kappa(a)} b = \sum_{a \leq \hat{b}} b = \sum_{b \leq \hat{a}} b = \hat{a}$. □

Although we have defined these frames for finite algebras, we can say that a possibly infinite algebra is *frame-definable* if it can be defined by a relevance frame. In the context of relation algebras, this corresponds to complete and atomic relation algebras. Similarly to that class, we state that every non-frame definable algebra embeds into a frame-definable algebra — its *cannonical extension* — with equivalent representability. The proof of this claim is out of scope of this thesis, but follows along the similar lines as Monk's proof, as reiterated in [HH02, Theorem 3.36] where prime filters are used instead of ultrafilters. For a frame-definable algebra, we say that $\mathcal{F}(\mathcal{A})$ is the frame $\mathcal{F}$ for which $\mathcal{A}$ is the complex algebra of $\mathcal{F}$. This generalises to non-frame-definable structures via canonical extensions.

We now continue by defining frame networks, analogous to atomic networks from Section 2.2.

Definition 170. *A* frame network $\mathcal{N} = (N, \lambda)$ *is defined for a frame* $\mathcal{F} = (F, \mathrm{I}, \leq, \mathrm{R}, \hat{\ })$ *with* N *being the set of nodes and* $\lambda : N^2 \to F$ *is the labelling function. The network is said to be* consistent *if and only if for all* $x, y, z \in N$ *we have* $\lambda(x, y) = \widehat{\lambda(y, x)}$, $\mathrm{R}(\lambda(x, y), \lambda(x, z), \lambda(z, y))$, *and* $\mathrm{I}(\lambda(x, x))$.

We say for two frame networks $\mathcal{N} = (N, \lambda)$, $\mathcal{N}' = (N', \lambda')$ *that* $\mathcal{N} \subseteq \mathcal{N}'$ *if and only if* $N \subseteq N'$ *and* $\lambda = \lambda' \restriction_{N^2}$ *where* $\restriction$ *denotes the restriction of the function to the domain in the subscript.*

Definition 171. *An infinite length* frame game $\mathcal{G}(\mathcal{F})$ *where* $\mathcal{F} = (F, \mathrm{I}, \leq, \mathrm{R}, \hat{\ })$ *is a relevance frame is defined for two players* $\forall$ *and* $\exists$.

The game starts with $\forall$ *picking a join-irreducible* a *and* $\exists$ *must return a frame network* $\mathcal{N}_0 = (N_0, \lambda_0)$ *such that there exists* $x, y \in N_0$ *such that* $\lambda_0(x, y) = a$.

At the ith move for $0 < i < \omega$ $\forall$ *picks a pair* $x, y \in N_{i-1}$ *and a pair of join-irreducibles* a, b *such that* $\mathrm{R}(\lambda(x, y), a, b)$ *and for all* $a' \leq a, b' \leq b \in N_{i-1}$ *if* $R(\lambda(x, y), a', b')$ *then* $a = a'$ *and* $b = b'$. $\exists$ *must return a network* $\mathcal{N}_i = (N_i, \lambda_i)$ *such that* $\mathcal{N}_{i-1} \subseteq \mathcal{N}_i$ *and* $\exists z \in N_i$ *such that* $\lambda(x, z) = a, \lambda(z, y) = b$.

$\forall$ *wins if and only if* $\exists$ *returns an inconsistent network at any point in the game.*

Proposition 172. *Let* $\mathcal{F}(\mathcal{A})$ *be the frame of a cyclic involutive unital bounded* $d\ell$*-magma (or its canonical extension).* $\exists$ *has a winning strategy for* $\mathcal{G}(\mathcal{F}(\mathcal{A}))$ *if and only if she has a winning strategy for* $\Gamma_\omega(\mathcal{A})$.

Proof. If $\exists$ has a winning strategy for $\Gamma_\omega(\mathcal{A})$, then $\mathcal{A}$ and its cannonical extension $\mathcal{A}'$ are both representable. For two points x, y in a representation of $\mathcal{A}'$ we know that there exists a join irreducible a for which $(x, y) \in h(s)$ if and only if $a \leq s$, for all $s \in A'$. Thus if a is the initial join-irreducible $\exists$ can map all her moves from the representation, starting with a pair of points in $h(a)$ but not $\kappa(a)$.

For the converse, assume she has a winning strategy for $\mathcal{G}(\mathcal{F}(\mathcal{A}))$. To respond to the initialisation move with $s \not\leq t$ there will exist a join-irreducible a such that $a \leq s$ but $a \not\leq t$ or rather $t \leq \kappa(a)$ so returning the initial network for a will ensure that $a \leq s$ and $\hat{a} \leq \tilde{t}$. Any witness move called can be responded to by minimal join-irreducible pairs, which makes any other witness moves called by $\forall$ redundant. □

We now define for every $2 \leq n \leq \omega$ the n-pebble equivalent version of the frame game as follows. We choose the lower bound for n to be 2 (unlike 3 for RA) due to some interesting behaviour by the identity element in weakening relation Algebras, as we will see in the following section.

Definition 173. *The n-pebble infinite move game $\mathcal{G}^n(\mathcal{F})$ for a frame $\mathcal{F}$ is defined exactly as $\mathcal{G}(\mathcal{F})$, except before $\forall$ calls a witness move, he takes $\mathcal{N}' \subseteq \mathcal{N}_{i-1}$ such that $|N'| < n$ and then proceeds to call the witness move.*

In particular, the frame game $\mathcal{G}$ is equivalent to $\mathcal{G}^\omega$. Next we define wkRA_n and wkRA analogous to RA_n, the variety of all n-dimensional relation algebras, and RA, the variety of all (4-dimensional) relation algebras [Mad83, HH02].

Definition 174. *The class wkRA_n is the class of all bounded cyclic involutive unital $d\ell$-magmas $\mathcal{A}$ for which $\exists$ has a winning strategy for $\mathcal{G}^n(\mathcal{F}(\mathcal{A}))$. The class of weakening relation algebras wkRA is defined as wkRA_4.*

It follows that wkRA_ω is equivalent to RwkRA and $\mathsf{wkRA}_\omega \subseteq \ldots \subseteq \mathsf{wkRA}_4 \subseteq \mathsf{wkRA}_3 \subseteq \mathsf{wkRA}_2$.

6.4 The Abstract Class

In this section, we provide finite axiomatisations for wkRA_2 and wkRA_3. We leave open the problem of whether — similarly to RA_4 — the axiomatisation for wkRA_4 consists of axioms of wkRA_3 and associativity of ;. We begin by axiomatising wkRA_2. This will be done using the axiomatisation of bounded cyclic involutive unital $d\ell$-magmas together with the theory Φ_2, defined below.

Definition 175. *Let Φ_2 be the first order theory given by the following quasiequations:*

(2A1) $s \cdot {\sim}s \leq 0'$,

(2A2) $s \leq t \Leftrightarrow s;{\sim}t \cdot 1' \leq 0'$,

(2A3) $s \leq t;u \wedge s;t \leq {\sim}u \Rightarrow s \cdot 1' \leq 0'$,

(2A4) $s \leq t;u \wedge u;s \leq {\sim}t \Rightarrow s \cdot 1' \leq 0'$,

(2A5) $s \leq t;u \wedge (s \cdot 1' \cdot t;(u \cdot v)) + (1' \cdot s \cdot (t \cdot {\sim}v);u) \leq 0' \Rightarrow s \cdot 1' \leq 0'$.

Before we prove the soundness and completeness, we introduce a ternary predicate for the language of frames $\mathrm{R}^{\min}$ from the equivalence below.

$$\mathrm{R}^{\min}(a,b,c) \Leftrightarrow \mathrm{R}(a,b,c) \wedge \forall b', c' : (\mathrm{R}(a,b',c') \wedge b' \leq b \wedge c' \leq c \Rightarrow b' = b \wedge c' = c)$$

Note that since the union of a chain of prime filters is again a prime filter, frames of the form $\mathcal{F}(\mathcal{A})$ have the property that $\mathrm{R}(a,b,c)$ can be refined to $\mathrm{R}^{\min}(a,b',c')$ for some prime filters $b' \leq b$ and $c' \leq c$.

Lemma 176. *Let a be a join irreducible in a finite bounded cyclic involutive unital $d\ell$-magma $\mathcal{A}$ such that $\mathcal{A} \models \Phi_2$. We have*

(1) $a \not\leq \kappa(a)$,

(2) $a \leq 0' \Leftrightarrow \hat{a} \not\leq 1'$,

(3) $(a \leq 1\text{'} \wedge a \not\leq 0\text{'}) \Leftrightarrow (a = \hat{a} \wedge a \leq 1\text{'})$.

Proof. For (1), see that because the structure is finite, the join of all s such that $a \not\leq s$ is precisely the join of all join irreducibles b such that $a \not\leq b$. Thus $a \not\leq \kappa$. For (2), see that $a \leq 0\text{'}$ if and only if $0\text{'} \not\leq \kappa(a)$ and thus $\hat{a} = \sim\kappa \not\leq \sim 0\text{'} = 1\text{'}$. For (3), see that if $a \not\leq 1\text{'}$, the equivalence holds trivially. Otherwise, we examine two cases. Suppose $a = \hat{a} \wedge a \leq 0\text{'}$. By (2) we have that $a = \hat{a} \not\leq 1\text{'}$ and we have a contradiction. Otherwise, assume you had $a \not\leq 0\text{'}$ and $\hat{a} \neq a$. Note that by (2) we have $\hat{a} \leq 1\text{'}, \hat{a} \not\leq 0\text{'}$. Because $a \neq \hat{a}$ we either have $a \leq \kappa(\hat{a})$ or $\hat{a} \leq \kappa(a)$. Without loss, take $a \leq \kappa(\hat{a})$ which means that $a = \hat{\hat{a}} \leq \sim a$, which implies $a = a \cdot \sim a \leq 0\text{'}$, yielding a contradiction. □

Lemma 177. *Let $\mathcal{A}$ be a bounded cyclic involutive unital $d\ell$-magma. $\mathcal{A} \models \Phi_2$ if and only if $\mathcal{F}(\mathcal{A})$ satisfies*

(2F1) $\forall a \exists b : \mathrm{I}(b) \wedge \hat{b} = b \wedge \mathrm{R}(b, a, \hat{a})$,

(2F2) $\forall a, b : \mathrm{I}(a) \wedge \hat{a} = a \wedge \mathrm{R}(a, b, \hat{b}) \Rightarrow R(b, a, b)$,

(2F3) $\forall a, b : \mathrm{I}(a) \wedge \hat{a} = a \wedge \mathrm{R}(a, b, \hat{b}) \Rightarrow R(\hat{b}, \hat{b}, a)$,

(2F4) $\forall a, b, c : \mathrm{I}(a) \wedge \hat{a} = a \wedge \mathrm{R}^{\min}(a, b, c) \Rightarrow b = \hat{c}$.

Proof. For the left to right implication, observe that for any join-irreducible a, we know that $a \not\leq \kappa(a)$ so $(a; \sim\kappa(a) \cdot 1\text{'}) \not\leq 0\text{'}$ by (2A2). Thus there must exist a join-irreducible $b \leq 1\text{'}, b \not\leq 0\text{'}, b \leq a;\hat{a}$. From Lemma 176(3) we have $\hat{b} = b$ and we have proven that (2F1) follows from Φ_2. For (2F2) assume we have $a \leq 1\text{'}, \hat{a} = a$ then $a \not\leq \sim 1\text{'} = 0\text{'}$. Thus $a = a \cdot 1\text{'} \not\leq 0\text{'}$ and $a \leq b;\sim\kappa(b)$ implies $a;b \not\leq \sim\sim\kappa(b)$ by (2A3) or simply $b \leq a;b$. By a similar argument, we get (2F3) from (2A4). Finally, assume the left hand side of (2F4). We know that $a \leq b;c$ and $a = a \cdot 1\text{'} \not\leq 0\text{'}$ so by (2A3) we have that $a;b \not\leq \sim c$ and because $a \leq 1\text{'}$ and monotonicity $a;b \leq b$ so $b \not\leq \sim c$. This is the same as $\sim c \leq \kappa(b)$ and thus $\hat{b} \leq c$. Before we move on, see how if a is a join irreducible such that $\hat{a} = a, \mathrm{I}(a)$, and for some s, t we have $a \cdot 1\text{'} \cdot s;t \not\leq 0\text{'}$ there must exist some $a' \leq a$ such that $a' \leq s;t$ and $a \not\leq 0\text{'}$. Because $a' \leq a \leq 1\text{'}$ and $a \not\leq 0\text{'}$, we have $\hat{a'} = a'$ so from $a' \leq a$ we also get $a = \hat{a} \leq \hat{a'} = a'$. Therefore, we have $a \leq s;t$. Now, we know that $a \leq b;c$ and $a \cdot 1\text{'} \not\leq 0\text{'}$ so by (2A5) we must have either $a \leq b;(\hat{b} \cdot c)$ or $a \leq (b \cdot \kappa(b));c$. As $b;c$ is a minimal witness for a, that means that either $c \leq \hat{b}$ (and together with $\hat{b} \leq c$ this makes $\hat{b} = c$) or $b \leq \kappa(b)$ which is a contradiction.

For the right to left implication, we start with proving (2A1). Suppose $s \cdot \sim s \not\leq 0\text{'}$. Then there exists some join irreducible $a \leq s \cdot \sim s$ not below 0'. Thus $0\text{'} \leq \kappa(a)$ and $\hat{a} \leq 1\text{'}$ and from (2F1) we know there exists a join-irreducible $b = \hat{b}, b \leq a; \hat{a} \leq (s \cdot \sim s);1\text{'} = s \cdot \sim s$ and that contradicts $b = \hat{b}$. For (2A2)($\Leftarrow$) assume $s \not\leq t$. That is true if there exists a join-irreducible a such that $a \leq s, \hat{a} \leq \sim t$. Thus by (2F1) there exists a join-irreducible $b = b \cdot 1\text{'} \not\leq 0\text{'}$ below $1\text{'} \cdot a; \hat{a} \leq 1\text{'} \cdot s;\sim t$ and we conclude $1\text{'} \cdot s;\sim t \not\leq 0\text{'}$. For $\Rightarrow$, if $1\text{'} \cdot s;\sim t \not\leq 0\text{'}$ then there exist join-irreducibles a, b, c such that $\mathrm{I}(a), a = \hat{a}, b \leq s, c \leq \sim t$ and b, c

also being minimal and hence $\hat{b} = c$ by (2F4). Therefore $\hat{b} \leq {\sim}t$ or simply $s \not\leq t$. For (2A3) assume $s \cdot 1' \not\leq 0'$ and $s \leq t;u$. Thus there exists $\mathrm{I}(a), \hat{a} = a \leq s$ and b, c below t, u respectively such that $\mathrm{R}^{\min}(a, b, c)$. By (2F4) we have $c = \hat{b}$ so we have by (2F2) $\mathrm{R}(b, a, b)$ implying $b \leq a;b$, or equivalently $a;b \not\leq \kappa(b)$. Because $\hat{b} \leq u$ we have ${\sim}u \leq \kappa(b)$ and so $a;b \not\leq {\sim}u$ concluding $s;t \not\leq {\sim}u$. Similarly we get (2A4) using (2F3). Finally, $s \cdot 1' \not\leq 0'$ and $s \leq t;u$ iff there exist some a, b, c in the corresponding frame such that $a \leq s \cdot 1', a \not\leq 0', b \leq t, c \leq u, \mathrm{I}(a), \hat{a} = a, \mathrm{R}^{\min}(a, b, c)$ and thus $\hat{b} = c$ by (2F4). Observe that for every v either $c \leq v$ or $\hat{c} = b \leq {\sim}v$ and thus $a \leq t; (v \cdot u)$ or $a \leq (t \cdot {\sim}v);u$ and the join of the two terms is not below 0'. □

Remark 178. *Unlike in the axioms (T6) atom structures, the domain self-hat join-irreducible for (2F1) does not have to be unique, which leads to some interesting behaviour, even for two-pebble games. This is why we set the lower bound for n with wkRA_n at 2.*

Theorem 179. wkRA_2 *is axiomatised by the basic axioms for bounded cyclic involutive unital $d\ell$-magmas and Φ_2.*

Proof. By Lemma 177 this axiomatisation is equivalent to the frame conditions, enumerated (2F1)–(2F4). First, we show these are sound for the two pebble game. If there existed a join-irreducible a with no $b, \mathrm{I}(b), \hat{b} = b$ with $R(b, a, \hat{a})$, then $\forall$ would win on initialisation with a because if $\lambda(x, y) = a$, no consistent b would exist for $\lambda(x, x)$. We show (2F4) next. If this didn't hold for some a, b, c then $\forall$ could start by asking a on the initial move. By order reversing of $\hat{}$ and the identity, a is the only join-irreducible to be set as $\lambda(x, x)$ where $\lambda(x, y) = a$. For the second move, $\forall$ calls the witness $b;c$ on (x, x) and we get an inconsistency because $b \neq \hat{c}$. For (2F2) and (2F3) see that if $\mathrm{R}(a, b, \hat{b})$ we have $\mathrm{R}^{\min}(a, b, \hat{b})$ by order reversing properties of $\hat{}$ and (2F4). Thus, if $\forall$ again starts by forcing $\lambda(x, x) = a$ then calling the witness $b;\hat{b}$, then both $\mathrm{R}(b, a, b), \mathrm{R}(\hat{b}, \hat{b}, a)$ must hold to keep the network consistent.

To show completeness, it suffices to say that $\exists$ can respond to any initialisation with a by returning a network with two nodes x, y with $\lambda(x, y) = a, \lambda(y, x) = \hat{a}$ and by (2F1) there exists a b for a and b' for $\hat{a}$ to be set as $\lambda(x, x)$ and $\lambda(y, y)$ respectively and by (2F2)(2F3) all other triangles are also consistent. A witness move can only be called on a reflexive node (x, x) and that means that by (2F2)(2F3)(2F4) any witness will be consistent and by the same reasoning as with initialisation, $\exists$ can put a label on $\lambda(y, y)$ and keep the network consistent. □

In order to axiomatise wkRA_3, we need to add two well-known axioms as well as a set of quasiequations. The first axiom is called *rotation* for involutive semirings [GFJL23] and the second one was found by Maddux in [Mad22] as an axiom that holds for binary relations, but not for relevance logic frames.

Definition 180. *Let Φ_3 be the first order theory containing all the formulas in Φ_2 as well as*

(3A1) $s{;}t \leq {\sim}u \Rightarrow t{;}u \leq {\sim}s$,

(3A2) $s \cdot t{;}u \leq ((s{;}v) \cdot t){;}u + t{;}(u \cdot {\sim}v)$,

(3A3) $1\text{'} \cdot {\sim}s'{;}s \cdot t{;}{\sim}t' \leq 0\text{'} \Rightarrow s{;}t \leq (s \cdot s'){;}t + s{;}(t \cdot t')$,

(3A4) $1\text{'} \cdot s \cdot 0\text{'} = 0 \Rightarrow (s \cdot 1\text{'}){;}(t{;}u) \leq ((s \cdot 1\text{'}){;}t){;}u$,

(3A5) $1\text{'} \cdot u \cdot 0\text{'} = 0 \Rightarrow (s{;}t){;}(u \cdot 1\text{'}) \leq s{;}(t{;}(u \cdot 1\text{'}))$.

Lemma 181. *Let $\mathcal{A}$ be a bounded cyclic involutive unital $d\ell$-magma. $\mathcal{A} \models \Phi_3$ if and only if for $\mathcal{F}(\mathcal{A})$ all the formulas from Lemma 177 hold as well as*

(3F1) $\forall a, b, c : \mathrm{R}^{\min}(a, b, c) \Rightarrow \mathrm{R}(b, a, \hat{c})$,

(3F2) $\forall a, b, c : \mathrm{R}(a, \hat{b}, \hat{c}) \Rightarrow \mathrm{R}(b, \hat{c}, \hat{a})$,

(3F3) $\forall a, b, c : \mathrm{R}^{\min}(a, b, c) \Rightarrow \exists d : d = \hat{d} \wedge \mathrm{I}(d) \wedge \mathrm{R}(d, \hat{b}, b) \wedge \mathrm{R}(d, c, \hat{c})$,

(3F4) $\forall a, b, c, d : d = \hat{d} \wedge \mathrm{I}(d) \wedge \mathrm{R}(a, d, a) \wedge \mathrm{R}^{\min}(a, b, c) \Rightarrow \mathrm{R}(b, d, b)$,

(3F5) $\forall a, b, c, d : d = \hat{d} \wedge \mathrm{I}(d) \wedge \mathrm{R}(a, a, d) \wedge \mathrm{R}^{\min}(a, b, c) \Rightarrow \mathrm{R}(c, c, d)$.

Proof. For the left to right implication of (3F1) if a, b, c are join-irreducibles with $a \leq b{;}c$ as well as the minimality condition for b, c then see that $a = a \cdot b{;}c \leq (a{;}\hat{c} \cdot b){;}c + b{;}(c \cdot \kappa(c))$. $c \cdot \kappa(c)$ is strictly below c and due to minimality of b, c for this composition $a \not\leq b{;}(c \cdot \kappa(c))$. Thus $a \leq (a{;}\hat{c} \cdot b){;}c$ and again by minimality $a{;}\hat{c} \cdot b = b$ or simply $\mathrm{R}(b, a, \hat{c})$. For (3F2) observe that $a \not\leq \hat{b}{;}\hat{c}$ is the same as ${\sim}\kappa(b){;}{\sim}\kappa(c) \leq \kappa(a)$ and by rotate we get ${\sim}\kappa(c){;}{\sim}\kappa(a) \leq \kappa(b)$ and ${\sim}\kappa(a){;}{\sim}\kappa(b) \leq \kappa(c)$ so $\mathrm{R}(a, \hat{b}, \hat{c}), \mathrm{R}(b, \hat{c}, \hat{a}), \mathrm{R}(c, \hat{a}, \hat{b})$ are equivalent. For (3F3) if $\mathrm{R}^{\min}(a, b, c)$ then we know $b{;}c \not\leq (b \cdot \kappa(b)){;}c + b{;}(c \cdot \kappa(c))$ and thus by (3A3), $1 \cdot (\hat{b}{;}b) \cdot (c{;}\hat{c}) \not\leq 0\text{'}$ and we can find a d satisfying $\mathrm{I}(d), \hat{d} = d, \mathrm{R}(d, \hat{b}, b), \mathrm{R}(d, c, \hat{c})$. For (3F4), see that $1\text{'} \cdot d \cdot 0\text{'} = 0$ and thus $a \leq d{;}a \leq d{;}(b{;}c) \leq (d{;}b){;}c$. By minimality $b = d{;}b$. By a similar argument, we get (3F5).

For the right to left implication, if $s{;}t \leq {\sim}u$ observe that for all join-irreducibles a, b, c such that $a \leq s, b \leq t, c \leq u$ we have $a{;}b \leq \kappa(\hat{c})$ and thus $\neg\mathrm{R}(\hat{c}, a, b)$ and by (3F2) we have $\neg\mathrm{R}(\hat{a}, b, c)$ and thus $b{;}c \leq \kappa(\hat{a}) = {\sim}a$. If for all join-irreducibles a, b, c below s, t, u respectively that holds then $t{;}u \leq {\sim}s$. To show $s \cdot t{;}u \leq ((s{;}v) \cdot t){;}u + t{;}(u \cdot {\sim}v)$ take any $a \leq s \cdot t{;}u$ and some minimal b, c witnessing the $t{;}u$ composition. Then all v will either have $c \leq {\sim}v$ or $\hat{c} \leq v$, in either case, the term is above a by monotonicity. Finally if $s{;}t \not\leq (s \cdot s'){;}t \cdot s{;}(t \cdot t')$ it means that $s{;}t$ is non-empty and as such there exists some $a \leq s{;}t$ and some $\mathrm{R}^{\min}(a, b, c)$ and as such $b \not\leq s'$ and $c \not\leq t'$ and thus $\hat{b}{;}b \leq {\sim}s'{;}s$ and $c{;}\hat{c} \leq t{;}{\sim}t'$ and there exists a $d \not\leq 0\text{'}$ such that $d \leq 1\text{'} \cdot {\sim}s'{;}s \cdot t{;}{\sim}t'$ and therefore the term cannot be below 0'. Take any join-irreducible $a \leq (s \cdot 1\text{'}){;}t{;}u$. There will exist a self-ˆ join-irreducible $d \leq s \cdot 1$ such that $d \leq d{;}a$ and a minimal b, c below t, u such that $a \leq b{;}c$ and so we have by (3F4) $b \leq d{;}b$ and thus $a \leq b{;}c \leq (d{;}b){;}c \leq ((s \cdot 1\text{'}){;}t){;}u$. The dual is shown similarly from (3F5). $\square$

Theorem 182. wkRA_3 *is axiomatised by the basic axioms for bounded cyclic involutive unital $d\ell$-magmas and Φ_3.*

Proof. First, we show that all the formulas from Lemma 181 are sound. If we have a, b, c such that $\mathrm{R}^{\min}(a, b, c)$ then $\forall$ calls a on initialisation and calls the witness $\mathrm{R}^{\min}(a, b, c)$ on the $\lambda(x, y) = a$ and $\exists$ must return such a network where $\lambda(x, z) = a, \lambda(y, z) = \hat{c}$ so $R(b, a, \hat{c})$ must hold for consistency and we have (3F1). For (3F2), assume, without loss, that we have $\mathrm{R}(a, \hat{b}, \hat{c})$ so there must be some minimal $\hat{b'} \leq \hat{b}, \hat{c'} \leq \hat{c}$ to call the witness on the initial pair a. Observe that for consistency $b \leq b' \leq \hat{c'}; \hat{a} \leq \hat{c}; \hat{a}$, by monotonicity. For (3F3), if $\forall$ initialises with a and calls the b, c witness, $\exists$ needs a join-irreducible d to put on the reflexive edge of the added node.

From Lemma 177, Theorem 179 we have that $\exists$ can survive the initial move and we only need to examine the two possible witness moves, that on a non-reflexive edge in a two-node network and that on a reflexive edge. If a witness move $\mathrm{R}^{min}(a, b, c)$ is called on a non-reflexive edge (x, y), check that all Peircian transformations of this triangle hold. By (3F1), we have $\mathrm{R}(b, a, \hat{c})$ and through (3F2) we get $\mathrm{R}(\hat{b}, c, \hat{a}), \mathrm{R}(\hat{c}, \hat{a}, b)$ from $\mathrm{R}(a, b, c)$ and $\mathrm{R}(\hat{a}, \hat{c}, \hat{b}), \mathrm{R}(c, \hat{b}, a)$ from $\mathrm{R}(b, a, \hat{c})$. For the reflexive edge on (z, z) you can see that $\exists$ can add $\lambda(z, z) = d$ from (3F3) and by similar reasoning to Theorem 179 all triangles including (z, z) are consistent. Finally let $\lambda(x, x) = d$. By (3F4) $\mathrm{R}(b, d, b)$ and by (3F2) $\mathrm{R}(\hat{d} = d, b, \hat{b})$. The consistency of other triangles follows from formulas in Lemma 177. Similarly, we get consistency for $\lambda(y, y)$. For the reflexive witness $\mathrm{R}^{min}(d, a, \hat{a})$ on (x, x) observe due to order reversing of $\hat{}$, $\exists$ can either find a join-irreducible c such that $\mathrm{R}^{\min}(\lambda(x, y), a, c)$ or $\mathrm{R}^{\min}(\lambda(y, x), c, \hat{a})$ and $\exists$ can use the same strategy as for the non-reflexive witness move. □

To axiomatise the class $\mathsf{wkRA} = \mathsf{wkRA}_4$, we would at least need to add associativity for composition. For RA, it is precisely the axioms for RA_3 and composition that axiomatise RA_4, however, whether this also holds for wkRA remains open.

6.5 The Diagonally Representable Class

In this section, we examine representable diagonal weakening relation algebras, that is, those relation algebras where 1' can be represented as an antichain. Therefore, when in this section we talk about the concrete binary relation 1', we mean the diagonal on X. The algebras with this property are the members of RwkRA that satisfy the identity $1' \cdot 0' = 0$. We show that the simple representable diagonal relation algebras have a discriminator term. A neat consequence is that, unlike representable weakening relation algebras, representable diagonal weakening relation algebras can be defined by an equational theory.

In universal algebra, a *discriminator term* is a term d in terms of variables

a, b, c for which the below property holds.

$$d(a,b,c) = \begin{cases} c & \text{if } a = b \\ a & \text{otherwise} \end{cases}$$

Lemma 183. *For all* $R \subseteq X^2$ *we have* $1' \cdot (R;\sim R) = 0$.

Proof. Suppose there exists $(x, x') \in 1' \cdot (R;(R \cdot \sim R))$. Because $(x, x') \in 1'$ we have $x = x'$. Thus there must exist a y to witness the composition by having $(x, y) \in R, (y, x) \in \sim R$. This means that $(x, y) \in R$ and $(x, y) \notin R$ and we have reached a contradiction. □

Let us now define term d_1 for binary relations $R, S \subseteq X \times X$ as

$$d_1(R, S) = 1' \cdot \big(R;(\sim S)\big)$$

Lemma 184. *If* $R \cdot (-S) \neq 0$ *for* $R, S \subseteq X^2$ *then* $d_1(R, S) \neq 0$.

Proof. Take any $(x, y) \in R \cdot (-S)$, we know that $(x, x) \in 1'$ and $(y, x) \in \sim S$ because $(x, y) \notin S$. Therefore, $(x, x) \in 1' \cdot (R;\sim S)$ and $d_1(R, S)$ is nonempty. □

Theorem 185. *Simple diagonally representable weakening relation algebras have a discriminator term.*

Proof. It is easy to see that in simple weakening relation algebras $1;s;1 = 1$ if $s \neq 0$ and $1;s;1 = 0$ otherwise. By the lemmas above, we have for representable algebras that $a = b$ if and only $d_1(a, b) + d_1(b, a) = 0$. Thus $d(a, b, c) = 1; (d_1(a, b) + d_1(b, a)); 1 \cdot a + \sim(1; (d_1(a, b) + d_1(b, a)); 1) \cdot c$ will equal to c if $a = b$ and a otherwise. □

Corollary 186. *Diagonally representable weakening relation algebras form a discriminator variety.*

Proof. The representation game defined for weakening relation algebras only needs an additional move where $\exists$ is requested to add $1'$ to $\lambda(y, x)$ if $1' \in \lambda(x, y)$ and this game gives rise to a similar style of a recursive axiomatisation as presented in Lemma 56. If all variables are given unique names, the universal quantifiers can also be moved to the beginning of all these formulas. Observe that although these formulas apply to all algebras, the game is played on the homomorphic image of the algebra where 1 maps to $1;a;1$ where $a \not\leq b$ is the initialisation pair of σ_n. Thus we can construct a term from any universally quantified first-order formula that is equal to $1;a;1$ if and only if the formula is true and 0 otherwise. For equations $t = t'$ we take $1;a;1 \cdot \sim d(t, t', 1;a;1)$. If a term t corresponds to a formula, then $\sim t \cdot 1;a;1$ corresponds to its negation and for disjunctions, we can take the join of the corresponding terms. Thus every formula σ_n has an equivalent equation. □

6.6 Finite Representability of Small Algebras

In this section, we study the examples of weakening relation algebras up to size six. Although relation algebras require at least eight elements in order to only admit infinite representations, it only takes four elements with weakening relation algebras.

We begin by looking at Sugihara monoids. Sugihara monoids are commutative distributive idempotent involutive residuated lattices. This variety is semilinear, i.e., generated by linearly ordered algebras, and the structure of these algebras is well known. In particular, the Sugihara monoid $\mathcal{S}_n = (S_n, \tau_{\mathsf{wkRA}})$ is a chain with n elements $\{a_{-k}, a_{-k+1}, \ldots, a_{-1}, a_0, a_1, \ldots, a_{k-1}, a_k\}$ if $n = 2k+1$ is odd, and otherwise for even n, $S_n = S_{n+1} \setminus \{a_0\}$. The involution operation is given by $\sim a_i = a_{-i}$ and the multiplication $a_i;a_j$ is defined to be the rightmost of a_i, a_j in the ordering $a_0, a_1, a_{-1}, a_2, a_{-2}, \ldots$ where a_0 is omitted when n is even. It follows that in the odd case the identity element is $1' = a_0$ and in the even case it is $1' = a_1$.

Note that $\mathcal{S}_2$ is the 2-element Boolean algebra and that for even n, there is a surjective homomorphism from S_n to S_{n-1} that identifies a_1 and a_{-1}. It is proved in [Mad10] that the even Sugihara chains can be represented by algebras of weakening relations. For $\mathcal{S}_2$ this is clear since $\mathcal{S}_2 \cong \mathrm{Rel}(1)$. For $\mathcal{S}_4$ an infinite base set is needed with a dense order. E.g., we can take $(\mathbb{Q}, \leq)$ be the poset of rational numbers with the standard order and check that $S_4 \cong \{\emptyset, <, \leq, \mathbb{Q}^2\}$ is a representation.

It follows from the consistency of networks that no nontrivial member of wkRA_2 has an element that satisfies $a = \sim a$. Hence any finite member of wkRA has an even number of elements. In particular, the odd Sugihara chains do not have a representation by weakening relations. However, they are in the variety generated by all algebras of weakening relations since they are homomorphic images of even Sugihara chains. This shows that RwkRA is not closed under homomorphic images, so it is a proper quasivariety.

Let $\mathbf{2} = \{0, 1\}$ be the two element chain with $0' < 1'$. The algebra $\mathcal{W}(\mathbf{2})$ is shown in Figure 6.4, and it has the following six elements: $\emptyset$, $\{(0,1)\}$, $\{(0,0),(0,1)\}$, $\{(0,1),(1,1)\}$, $\leq$, $\mathbf{2} \times \mathbf{2}$. The τ_{wkRA}-reduct of the point algebra $\mathcal{P}$ is shown in Figure 6.3. It has two weakening proper subreducts, $\mathcal{S}_4 = (\{\emptyset, <, \leq, \top\}, \tau_{\mathsf{wkRA}})$ and $\mathcal{W}_{6,1} = (\{\emptyset, =, <, \leq, <\cup>, \top\}, \tau_{\mathsf{wkRA}})$. Like the point algebra, both of these algebras can only be represented on an infinite set. Note that $\mathcal{W}_{6,1}$ is diagonally representable, while $\mathcal{S}_4$ is not.

Since wkRA_3 is finitely axiomatised, one can use a model finder such as Mace4 [McC] to compute all members of cardinality n for small values of n. Up to isomorphism there are 14 algebras with 6 elements or fewer in wkRA_3 such that ; is associative, shown in Figure 6.4. We now briefly describe their representations by weakening relations. Note that when we describe their representations as infinite, this is because the algebra does not admit finite representations.

The first 5 are symmetric representable relation algebras, hence they are diagonally representable weakening relation algebras. As mentioned above, the Sugihara algebra $\mathcal{S}_4$ and the algebra $\mathcal{W}_{6,1}$ are representable as subalgebras of

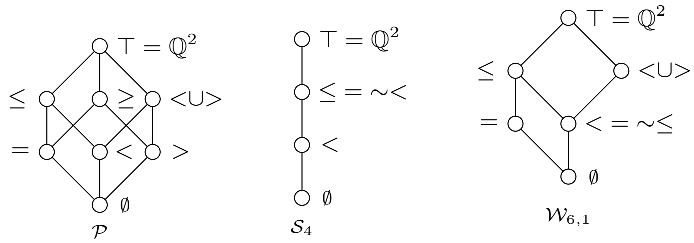

Figure 6.3: The representations over the set $\mathbb{Q}$ of the point algebra $\mathcal{P}$, the weakening subalgebra $\mathcal{S}_4$ and the diagonally representable weakening subalgebra $\mathcal{W}_{6,1}$.

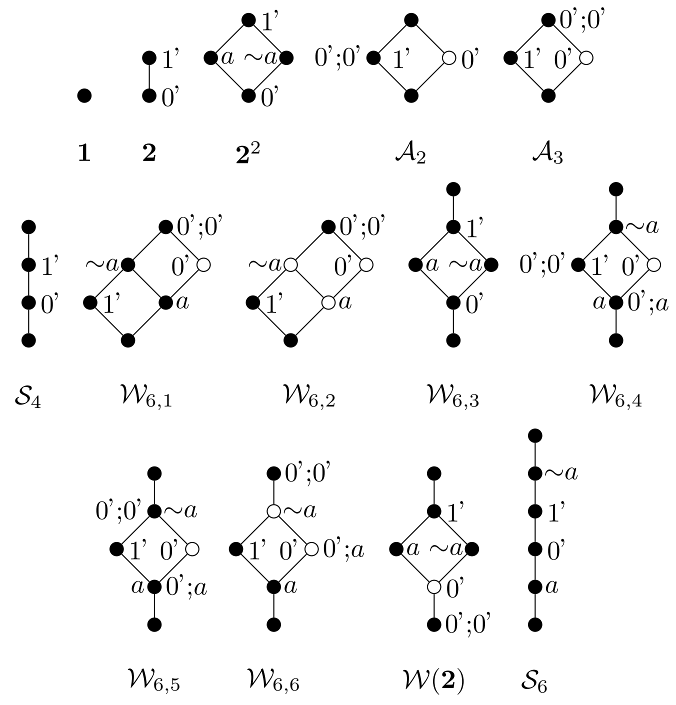

Figure 6.4: All algebras in wkRA_4 up to 6 elements. Black nodes denote idempotent elements ($s;s = s$).

the τ_{wkRA}-reduct of the point algebra (Figure 6.3) while the $\mathcal{S}_6$ as well as all other even length Sugihara monoids is representable [Mad10].

The reader can check that the remaining representations are given as follows.

(1) $\mathcal{W}_{6,2}$ is representable as τ_{wkRA}-subreduct of the complex algebra of $\mathbb{Z}_7$, where the element $a = \{1, 2, 4\}$ and $1' = \{0\}$,

(2) $\mathcal{W}_{6,3}$ is subdirectly embedded in a direct product of two copies of $\mathcal{S}_4$, hence it is representable over the union of two disjoint copies of $\mathbb{Q}$,

(3) $\mathcal{W}_{6,4}$ is represented over $X = \{0, 1\} \times \mathbb{Q}$ with a partial order $(i, p) \leq (j, q) \iff p < q$ or $p = q, i = j$; the identity 1' maps to $\leq$ and the element a maps to the relation $\{((i, p), (i, q)) \mid i = 0, 1,\ p < q\}$,

(4) $\mathcal{W}_{6,5}$ requires three copies of $\mathbb{Q}$, that is, $\{1, 2, 3\} \times \mathbb{Q}$; the partial order $\leq$ is defined in the same way and a is mapped to the relation $\{((i, p), (i, q)) \mid i = 0, 1, 2,\ \ p < q\}$,

(5) $\mathcal{W}_{6,6}$ is represented over $X = \{0, 1\} \times \mathbb{Q}$ with order $(i, p) \leq (j, q) \iff i = j$ and $p \leq q$; the identity 1' maps to $\leq$ and the element a maps to the relation $\{((i, p), (i, q)) \mid i = 0, 1,\ \ p < q\}$.

6.7 FRP and FA with Weakening

In this section, we define the weakening representation classes for reductions of τ_{wkRA}, analogous to $\mathsf{R}(\tau)$ for τ_{RA}. We then examine the finite axiomatisability and the finite representation property for two signatures. Note that the weakening representation classes are not the same as weakly representable classes, like those studied in [And94].

Definition 187. *For a signature $\tau \subseteq \tau_{\mathsf{wkRA}}$, we define the* weakening representation class $\mathsf{wkR}(\tau)$ *as the class of all structures $\mathcal{A} = (A, \tau)$ for which there exists an injective mapping $h : A \to \wp(X \times X)$ where $\mathcal{X} = (X, \leq)$ such that all operations and predicates in τ are preserved by h, with $1'$ mapping to $\leq$.*

The signature τ has the finite weakening representation property *if and only if all finite members of* $\mathsf{wkR}(\tau)$ *admit finite representations.*

Note that for any 1'-free signature, we have $\mathsf{wkR}(\tau) = \mathsf{R}(\tau)$. Furthermore, the domain and range can be defined without the need for converse as $\mathrm{D}(s) = 1' \cdot s;1, \mathrm{R}(s) = 1' \cdot 1;s$ and as such still preserve weakening.

We now examine the first signature $\tau = (\leq, 1', ;)$. We know that $\mathsf{R}(\leq, 1', ;)$ is not finitely axiomatisable [HM13], with the finite representation property remaining an open question for the signature. This is in stark contrast to $\mathsf{wkR}(\leq, 1', ;)$ for which we get the following result with relatively minor modifications of the Zareckiĭ representation [Zar59].

Theorem 188. *The class $\mathsf{wkR}(\leq, 1', ;)$ is axiomatised by associativity, partial order, identity, and monotonicity. Its finite members admit finite representations.*

Proof. The axioms are clearly sound. Take a structure $\mathcal{A} = (A, \leq, 1', ;)$ with associativity, partial order, identity, and monotonicity. Now take the mapping $h : A \to \wp(A \times A)$ such that

$$h(s) = \{(t, u) \mid t;s \geq u\}.$$

From [Zar59], this mapping clearly represents $\leq, ;$ correctly. To show it is injective, take $s \not\leq t$ and observe that $(1', s) \in h(s)$, but not in $h(t)$. To show that $h(1')$ is a partial order, observe that the representation of 1' is precisely $\geq$, which is a partial order.

Finally, observe that if A is finite, the structure $\mathcal{A}$ will be representable over the finite base (A itself). □

Finally, we look at another signature we have examined in this document quite closely — $(\mathrm{D}, \mathrm{R}, ;)$. Again, $\mathsf{R}(\mathrm{D}, \mathrm{R}, ;)$ is not finitely axiomatisable [HM11] with the finite representation property remaining open. The proof of nonfinite axiomatisability of $\mathsf{R}(\mathrm{D}, \mathrm{R}, ;)$ neatly works for $\mathsf{wkR}(\mathrm{D}, \mathrm{R}, ;)$ as well. That is, by defining a predicate $\lhd$ by infinitary first-order predicate where $\lhd^{n+1}$ for $n < \omega$ is the transitive, monotone closure of $\lhd^n$ and $\mathrm{D}(s);\mathrm{D}(t) \lhd^0 \mathrm{D}(t)$, we can prove the below two theorems.

Theorem 189. *The class* $\mathsf{wkR}(\mathrm{D}, \mathrm{R}, ;)$ *is axiomatised by the following axioms:*

(1) the operation ; *is associative,*

(2) ; *acts as a meet operation for the domain-range elements,*

(3) $\mathrm{D}(s) \lhd \mathrm{D}(t) \Rightarrow \mathrm{D}(s);\mathrm{D}(t) = \mathrm{D}(t)$,

(4) $s \lhd t \Rightarrow \mathrm{D}(s) \lhd \mathrm{D}(t)$,

(5) $\mathrm{D}(\mathrm{D}(s)) = \mathrm{R}(\mathrm{D}(s)) = \mathrm{D}(s)$ *and* $\mathrm{D}(\mathrm{R}(s)) = \mathrm{R}(\mathrm{R}(s)) = \mathrm{R}(s)$,

(6) $\mathrm{D}(s);s = s;\mathrm{R}(s) = s$,

(7) $\lhd$ *is antisymmetric,*

for all $s \in A$*. Furthermore, finite members of* $\mathsf{wkR}(\mathrm{D}, \mathrm{R}, ;)$ *admit finite representations.*

Proof. The axioms are, again, trivially sound. Now take a structure $\mathcal{A} = (A, \mathrm{D}, \mathrm{R}, ;)$ that is a model for the above theory and define a mapping $h : A \to \wp(A \times A)$ such that

$$h(s) = \{(t, u) \mid t;\mathrm{D}(s;\mathrm{R}(u)) = t,\ \ u \lhd t;s\}.$$

For faithfulness, see that for $s \not\lhd t$, we have $(\mathrm{D}(s), t) \in h(s)$, but not $h(t)$. Observe that $h(s);h(t) \subseteq h(s;t)$ by monotonicity as well as the domain condition. For $h(u);h(v) \supseteq h(u;v)$, take $(s, t) \in h(u;v)$ and observe that $s;u;\mathrm{D}(v;\mathrm{R}(t))$ witnesses the composition.

If $(s,t) \in h(u)$, then by the domain condition of h, we have that $(s,s) \in \mathrm{D}(u)$. Furthermore, $(t,t) \in \mathrm{R}(u)$. Finally, observe that $\lhd$ is a partial order as it is transitive and reflexive by definition as well as antisymmetric. Furthermore, the proper binary relation $\lhd$ over the base A is the partial order used for weakening within which all domain-range elements are contained.

Again, observe that if A is finite, the algebra $\mathcal{A}$ will be representable over the finite base (A itself). □

This implies that unlike with $\mathsf{R}(\mathrm{D},\mathrm{R},;)$, $\lhd$ is a complete characterisation of the partial order in $\mathsf{wkR}(\mathrm{D},\mathrm{R},;)$. Although we get the finite representation property for finite structures, we can still prove nonfinite axiomatisability, using the same argument as [HM11].

Theorem 190. *The class* $\mathsf{wkR}(\mathrm{D},\mathrm{R},;)$ *is not finitely axiomatisable.*

Proof. This argument is an outline. If we for $n < \omega$ define a structure $\mathcal{A}_n = (A_n, \mathrm{D}, \mathrm{R}, ;)$ where for all $i < n$ we have the domain elements $0, d, m_i, \varepsilon_i, r$ as well as $a_i, b_i, am_i, mb_i, ab_i$ where d is the domain for a_i, am_i, ab_i, the element m_i is the range for am_i and the domain for mb_i, the element ε_i is the range for a_i and the domain for b_i, and r is the range for ab_i, b_i, mb_i, ab_i. The compositions for domain elements are defined according to 0 being the bottom domain element with respect to $\lhd^0$ as well as $m_i \lhd^0 \varepsilon_i$. The mandatory domain-range compositions are also defined as well as $a_i;m_i = am_i$, $m_i;b_i = mb_i$, $a_{i+1};mb_{i+1} = am_{i+1};b_{i+1} = a_i;b_i = ab_i$ where $+$ denotes addition modulo n. All other compositions result in 0.

See that the structure obeys the basic axioms for domain-range semigroups. However, it is not representable. That is because it has a $\lhd$-cycle. This arises from $ab_i = a_{i+1};m_{i+1};b_{i+1} \lhd a_{i+1};b_{i+1} = ab_{i+1}$ and by the transitive closure $ab_i \lhd ab_j$ for all $i,j < n$. This contradicts $\lhd$ being antisymmetric, an axiom for $\mathsf{wkR}(\mathrm{D},\mathrm{R},;)$. However, $ab_i \neq ab_j$ for all $i \neq j$ breaking the antisymmetry of $\lhd$ is the only axiom that is broken by these structures. In order to reach the full cycle though, a number of transitive steps increasing with n had to be taken. Thus the smallest k for which $\lhd_k$ is antisymmetric increases with n. Via a number of arguments (for example, using the compactness of first order logic) we get that $\mathsf{wkRA}(\mathrm{D},\mathrm{R},;)$ is not finitely axiomatisable. □

6.8 Conclusion

This chapter concludes the novel results presented in this thesis. In it, we have shown a number of properties for weakening relation algebras. These were previously unknown, but were necessary to examine finite representability in the context of weakening relation algebras. In this chapter, we have revisited the main conjecture examined in this text and showed some examples of it that hold in the weakening setting, but remain open in the classical setting. Further, all the results in this chapter are related to but independent from the analogous results for relation algebras (in the non-weakening setting). Now that we have examined a number of results in the area of finite representability (and related properties), especially in connection to Conjecture 27, the only thing that remains to be addressed is some open problems and the potential direction of future work. This is examined in the next chapter.

Chapter 7

Problems

In this chapter, we present the open problems in the area of finite representability of algebras as binary relations and related topics. We begin with Chapter 3, where we make significant progress in proving the below conjecture.

Conjecture 27. $\mathsf{R}(\tau)$ *will have the finite representation property if and only if* $(-,;) \not\subseteq \tau \not\supseteq (\cdot,;)$.

Our results, together with those from [Mad16, Neu17], reduce the proof of this conjecture to the below problem. While we leave it open, some possible proof ideas are sketched in Section 3.4.

Problem 191. *For signatures* $\tau \subseteq \tau_{\mathsf{RA}} \setminus (-)$*, show that any finite structure* $\mathcal{A} \in \mathsf{R}(\tau)$ *embeds into a finite member of* RA_4.

Furthermore, our case-by-case analysis of the finite representation property for individual signatures reveals three types of problems when representing them finitely — or equivalently, embedding them into finite members of RA. We identify three signatures that we believe should have their finite representation property studied individually. This is because we speculate that the techniques developed in proving FRP for these could easily generalise to a larger number of signatures.

Problem 192. *For the following signatures* τ*, do finite members of* $\mathsf{R}(\tau)$ *admit finite representations?*

(1) $(+,;)$,

(2) $(\smile,;)$,

(3) $(\leq, 1', ;)$.

We have discussed how the decidability of the representation problem can be shown via finite axiomatisability and finite representation property. Furthermore, we have shown that $\mathsf{R}(\sqsubseteq,;)$ is an example of a representation class that is not finitely axiomatisable, but with all finite members admitting finite representations. Together with [Mad16, BS78], this shows that neither of the two properties is stronger than the other. However, the below problem remains open.

Problem 193. *Does there exist a signature* $\tau \subseteq \tau_{\mathsf{RA}}$ *with membership in* $\mathsf{R}(\tau)$ *decidable for finite structures, but there exists no finite axiomatisation of* $\mathsf{R}(\tau)$ *and not all finite members of* $\mathsf{R}(\tau)$ *have the finite representation property?*

We show that $\mathsf{R}(-, ;)$ is not finitely axiomatisable nor can all its finite members be finitely represented. In a forthcoming paper, Jackson shows that if we require square representability, the membership in the representation class is undecidable for finite structures. However, the below problem remains open.

Problem 194. *Is membership in* $\mathsf{R}(-, ;)$ *decidable for finite structures?*

Finally, in Chapter 3, we discuss the operation $\rightarrow$, closely related to negation. Residuals for binary relations $\backslash, /$ are another example of such operations. Recently, it has been shown that $(\leq, \backslash, /, ;)$ does not have the finite representation property [Mik22]. The following problems remain open.

Problem 195. *Is the inclusion of* $(/, ;)$ *or* $(\backslash, ;)$ *in the signature* τ *a sufficient condition for the finite representation property to fail?*

In Chapter 4, we study the finite representation property for algebras with some demonic operations. We define the demonic meet — a previously unstudied operation in the context of binary relations — but demonic negation and converse remain undefined.

Problem 196. *Define the demonic version of converse and negation for binary relations, motivated by Dijkstra's demonic model of computation [DS90]. This defines the class of representable demonic relation algebras. Is the class axiomatised by the same axioms as the class* RRA*? Which reduction signatures have the finite representation property?*

We prove in Chapter 5 that the inclusion of $(\mathrm{D}, \smile, ;)$ in the signature is a sufficient condition for FRP when $-, +, \cdot \notin \tau$. The below problem remains open.

Problem 197. *Do converse-, join-, negation-, and meet-free signatures with domain and range have the finite representation property?*

In Chapter 6, we axiomatise the abstract class wkRA_3, but fail to do the same for $\mathsf{wkRA}_4 = \mathsf{wkRA}$. We study the finite representation property and finite axiomatisability of $\mathsf{wkR}(\mathrm{D}, \mathrm{R}, ;)$ and $\mathsf{wkR}(\leq, 1', ;)$, but there remain a number of signatures to be studied.

Problem 198. *Is the abstract class of weakening relation algebras* wkRA *finitely axiomatisable?*

Problem 199. *For signatures* τ *where* $\mathsf{R}(\tau) \neq \mathsf{wkR}(\tau)$*, determine whether the class* $\mathsf{wkR}(\tau)$ *is finitely axiomatisable and whether its finite members admit finite representations.*

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