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Nonlinear optimal feedback control of open two-level non-markovian stochastic quantum system

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Abstract In this paper, a nonlinear optimal feedback tracking control based on the nonlinear state estimator for a two-level bilinear open non-Markovian stochastic quantum system is proposed. The proposed nonlinear state estimator is designed by using state-dependent differential Riccati equation and constructed to optimally estimate the state based on from the measurement output of the controlled quantum system. The estimated state is continuously updated by the output data of continuous weak measurement and used to design the nonlinear state feedback controller to track the output of the reference model. The output state of reference model is chosen as the desired performance. The numerical simulation results verify the achievability of the proposed state feedback control method, and the capability to steer the state of system from any arbitrary initial state to the final reference target state with high state transfer success rate $\geq 90\%$.

Keywords Open stochastic quantum system, Nonlinear optimal control · state-dependent Riccati equation · Continuous measurement, Nonlinear state estimation

1 Introduction

Quantum information technologies consist of the quantum communication, quantum computation, and quantum control, among them quantum control is one of very important areas and plays a key role in all application areas of quantum systems [1, 2, 3]. When quantum systems interact with the environment, there will appear the quantum disturbance, decoherence or/and white noise. The measurement is also a type of interaction with the quantum system, which produces the back-action to the system

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and makes the system become an open stochastic quantum system [4,5]. The open quantum system in which the environment memory effect is ignored can be described by Lindblad-type Markovian master equation under Born or Markovian approximation [33,34,35]. This model is widely used in many fields of quantum optics [36]. However, in some other cases, such as the initial state of correlation and entanglement and the quantum system interaction with a nanostructure environment, there exists a longer environment memory effect which makes the Markovian approximation invalid and the systems show non-Markovianain characteristics [37,38,39,40] existing in spin echo [41], quantum spot [42] and fluorescence systems [43]. Due to the existence of memory effect, non-Markovian quantum systems show more complex characteristics and the state manipulations become more difficult.

An important task of quantum control is to design an efficient control law for a given quantum system by means of appropriate control theory to manipulate the initial states to the desired target states. Many researchers have designed control laws for the quantum systems [6,7], however, what they designed was the open loop control strategy, or the states of quantum system used for computational control law were not estimated by the estimator in real time, but determined from the system dynamical equation. To overcome this issue, an estimator is required to estimate the state of the system by using the output values of the quantum system, and then the estimated state is used to design the state feedback control laws [8]. Some work have been presented for the feedback control based on estimate of linear stochastic quantum system [9,10]. When the quantum system has some nonlinearities, such as the bilinearities, to design a state feedback controller for the bilinear stochastic quantum systems is a challenging work. There are several techniques to design the nonlinear optimal feedback control for the nonlinear system, among them the optimal control with nonlinearty is based on the state-dependent Riccati equation (SDRE).

Feedback control and state estimation using continuous weak measurement (CWM) signal for the linear quantum system were studied in [11,12,13]. The measurement output continuously updates the state of the nonlinear state estimator (NSE), which goes asymptotically towards the state of system. Using CWM, one can measure a quantum ensemble and obtain the system measurement records. The measurement records are used to construct a serial of optimization problems. Then reconstruct the state of the quantum state by means of an appropriate optimal algorithm.

This paper focuses on the study of state transfer control of a two-level bilinear open non-Markovian stochastic quantum system (ONMSQS) beneath spontaneous emission of spin-$\frac{1}{2}$ particle, i.e., the decoherence factor of quantum system. It is known that in quantum computation the state transfer of ONMSQS with high state transfer success rate is critical in the presence of environmental noise and decoherence effect. Therefore, the objective is to move the state of ONMSQS to the reference target state from any of this paper arbitrary initial state with high state transfer success rate $\geq 90\%$.

The main contribution and motivation in this paper is that we propose a nonlinear optimal feedback tracking control (NOFTC) based on nonlinear state estimator (NSE) for the two-level bilinear ONMSQS. The output data is used to estimate the state continuously by NSE, and based on these estimated state. The NOFTC is designed based on state-dependent differential Riccati equation (SDRE) to capture the nonlinearities of the system. The SDRE depends only on the current state. The computation is carried out online, and the control goal is to minimize the quadrature distance between the dynamical variable of the quantum system and that of the reference system model. The proposed NSE and NOFTC law are designed separately but both parts can be combined to perform the desired control task. As far as we know, such NOFTC has not been applied to the bilinear ONMSQS, and the work done in this paper has the potential interest.

This paper is arranged as follows: Section 2 gives the description of the bilinear two-level ONMSQS and NSE. NOFTC law designed is described in Section 3. Closed-loop dynamics of the quantum estimated state feedback control system is designed in Section 4. Section 5 is the numerical simulations and results discussion. Finally, Section 6 is the conclusion of the proposed work.

The dynamics of open quantum system can be described by the stochastic quantum master equation, which interacts with surrounding environment. The general form of the total Hamiltonian is

$$H_{total} = H_s + H_c(t) + H_b + H_{int} \tag{1}$$

where H_{total}, H_s, $H_c(t)$, H_b are the total, free, control and bath Hamiltonian, respectively. H_{int} represents the interaction Hamiltonian between the environment and system controlled. The bath Hamiltonian H_b is consisted of Harmonic oscillators masses m_i and frequencies ω_i as

$$H_b = \sum_{i=1}^{N}\left(\frac{p_i^2}{2m_i} + \frac{m_i}{2}x_i^2\omega_i^2\right) \tag{2}$$

where m_i and ω_i are the Harmonic oscillators masses and frequencies, respectively. $(x_1,\ x_2,\ ..\ .\ ,\ x_N,\ p_1,\ p_2,\ .\ ..\ p_N)$ are the coordinates and their conjugate moments. Assuming that the quantum system initial state is defined by density matrix $\rho(0) = \rho_0$, for simplicity $= 1$. The Hamiltonian H_{int} of interaction between the H_s and H_b is assumed to be bilinear and can be defined as

$$\begin{aligned} H_{int}(t) &= e^{i(H_s+H_b)t} H_{int} e^{-i(H_s+H_b)t} \\ &= \alpha \sum_n A_n(t) \otimes B_n(t) \end{aligned} \tag{3}$$

where $\otimes$ denotes the tensor between two systems $A_n(t)$ and $B_n(t)$, α is the coupling constant, and there are

$$A_n(t) = e^{iH_st} A_n e^{-iH_st} B_n(t) = e^{iH_bt} B_n e^{-iH_bt}$$

The effect of the environment on the quantum system can be seen as interplay between the dissipation and fluctuation phenomena, which makes the quantum system lose the coherence, also called decoherence of the system.

According to the quantum dissipation theory, the dynamical equation of a stochastic quantum process can be described as a reduced density matrix ρ_t of the quantum system as

$$d\rho_t = -i[H_s,\rho]\,dt + \sum_{k=1}^{2} u_k(-i[H_{ck},\rho_t])\,dt + L_1(t)[\rho_t]\,dt + L_2[\rho_t] \tag{4}$$

where ρ_t indicates state density matrix, $u_x(t)$ and $u_y(t)$ are the control fields, $L_1[\rho_t]dt$ and $L_2(t)$ depicts the Lindblad terms, ω_0 indicates the transition frequency. σ_x, σ_y and σ_z are the Pauli matrices. H_{ck} is the control Hamiltonian, and u_k are the control fields. The term $\sum_{k=1}^{2} u_k H_{ck} = u_x(t)\sigma_x + u_y(t)\sigma_y$, and $H_s = H_0 = \sigma_z$, where ω_0 is the transition frequency. σ_x, σ_y and σ_z are the Pauli matrices defined as

$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},\ \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},\ \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

The Lindblad terms $L_1[\rho_t]dt$ and $L_2(t)$ can be described as

$$\begin{aligned} L_1(t)[\rho_t]dt = [\Delta(t)(t) + \gamma(t)]\mathscr{D}\left[\sigma^-\right]\rho_t dt + [\Delta(t) + \gamma(t)]\ \mathscr{D}\left[\sigma^+\right]\rho_t dt \\ + M\mathscr{D}[\sigma_z]\rho_t dt \end{aligned} \tag{5}$$

where the parameters $\Delta(t)$ and $\gamma(t)$ represents the diffusion and dissipation coefficients of the quantum system, σ^- and σ^+ are the lowering and raising operators, respectively. M is the interaction strength between the system and the measurement, η is the detection efficiency and W_t is the Weiner process noise. and

$$L_2[\rho_t] = \sqrt{M\eta}\mathscr{H}[\sigma_z]\rho_t dW_t \tag{6}$$

and dW_t can be described

$$dW_t = dY_t - \sqrt{M\eta}tr(\sigma_z\rho_t)dt \tag{7}$$

where dY_t is the output of the observation process.

Then dynamics of system (4) can be written as

$$\begin{aligned} d\rho_t = -i\omega_0[\sigma_z,\ \rho_t]dt + -iu_x(t)[\sigma_x,\ \rho_t]dt - iu_y(t)[\sigma_y,\ \rho_t]dt + [\Delta(t) + \gamma(t)]\mathscr{D}\left[\sigma^-\right]\rho_t dt \\ + [\Delta(t) + \gamma(t)]\ \mathscr{D}\left[\sigma^+\right]\rho_t dt + M\mathscr{D}[\sigma_z]\rho_t dt\ + \sqrt{M\eta}\mathscr{H}[\sigma_z]\rho_t dW_t \end{aligned} \tag{8}$$

where $\sigma^- = 1/2(\sigma_x - i\sigma_y)$ and $\sigma^+ = 1/2(\sigma_x + i\sigma_y)$ are the lowering and raising operators, $M \geq 0$ is the interaction strength between the system and the measurement, $0 < \eta < 1$ is the detection efficiency, and dW_t is the Weiner process which satisfies the quantum Itô's rules, $E[dW_t] = 0$, $[dW_t]^2 = dt$. $\mathscr{D}$ and $\mathscr{H}$ denote the superoperators.

The superoperators $\mathscr{D}$ and $\mathscr{H}$ are

$$\begin{aligned} \mathscr{D}[\sigma_z]\rho_t &= \sigma_z\rho_t\sigma_z^\dagger - \frac{1}{2}\sigma_z^\dagger\sigma_z\rho_t - \frac{1}{2}\rho_t\sigma_z^\dagger\sigma_z \\ \mathscr{H}[\sigma_z]\rho_t &= \sigma_z\rho_t + \rho_t\sigma_z - [tr(\sigma_z\rho_t + \rho_t\sigma_z)]\rho_t \end{aligned}$$

The parameters $\Delta(t)$ and $\gamma(t)$ in Eq. (8) are the diffusion and dissipation coefficient of the quantum system. The parameters $\Delta(t)$ and $\gamma(t)$ are given by [19].

$$\Delta(t) = \int_0^t d\tau k(\tau)\cos(\omega_0\tau) \tag{9}$$

$$\gamma(t) = \int_0^t d\tau \mu(\tau) \cos(\omega_0 \tau) \tag{10}$$

with

$$k(\tau) = 2\int_0^\infty d\omega J(\omega) \coth[\omega/2k_B T] \cos(\omega\tau) \tag{11}$$

$$\mu(\tau) = 2\int_0^\infty d\omega J(\omega) \sin(\omega\tau) \tag{12}$$

where $k_B T$ is the environmental temperature. $J(\omega)$ is the spectral density of the environment which has the form

$$J(\omega) = \frac{2}{\pi}\omega \frac{\omega_c^2}{\omega_c^2 + \omega^2} \tag{13}$$

where ω is the bath frequency, and ω_c is the high-frequency cutoff.

Then the analytical expression in Eq. (10) can be expressed as

$$\gamma(t) = \frac{\alpha^2 \omega_0 r^2}{1+r^2}\left[1 - e^{-r\omega_0 t}\cos(\omega_0 t) - rsin(\omega_0 t)\right] \tag{14}$$

where r is the ratio of ω_c and ω_0.

Analytical expression of Eq. (9) can be expressed as

$$\begin{aligned}
\Delta(t) = {} & \alpha^2 \omega_0 \frac{r^2}{1+r^2}\Bigg(\coth(\pi r_0) - \cot(\pi r_c) e^{-\omega_c t}\left[rcos(\omega_0 t) - \sin(\omega_0 t)\right] \\
& + \frac{1}{\pi r_0}\cos(\omega_0 t)\left[\bar{F}(-r_c,t) + \bar{F}(r_c,t) - \bar{F}(ir_c,t) - \bar{F}(-ir_c,t)\right] \\
& - \frac{1}{\pi r_0}\sin(\omega_0 t)\Bigg\{\frac{e^{-\nu_1 t}}{2r_0\left(1+r_0^2\right)}\left[(r-i)\ \bar{G}(-r_0,t) + (r+i)\ \bar{G}(r_0,t)\right] \\
& + \frac{1}{2r}\left[\bar{F}(-r_c,t) - \bar{F}(r_c,t)\right]\Bigg\}\Bigg)
\end{aligned} \tag{15}$$

where $r = {}^{\omega_c}/_{\omega_0}$ and $r_0 = {}^{\omega_0}/_{2\pi k_B T}/$, in which $k_B T$ is the environmental temperature, and

$$\bar{F}(x,t) \equiv {}_2F_1\left(x,\ 1,\ 1,\ 1+x, e^{-\nu_1 t}\right),\ \ \bar{G}(x,t) \equiv {}_2F_1\left(2,\ 1+x, 2+x,\ e^{-\nu_1 t}\right)$$

where ${}_2F_1(a,b,c,z)$ is the Guass hyoergeometric function and defined as

$$\begin{aligned}
{}_2F_1(a,\ b,\ c,\ z) &= 1 + \frac{ab}{1!c}z + \frac{a(a+1)b(b+1)}{2!c(c+1)z^2}z^2 + \dots \\
&= \sum_{n=0}^{\infty}\frac{(a)_n (b)_n z^n}{(c)_n}\frac{z^n}{n!}
\end{aligned}$$

where$(a)_n$ is a pochhammer symbol. Under high temperature limit, the diffusion coefficient $\Delta(t)$ in Eq. (15) has the form,

$$\Delta(t) = 2\alpha^2 k_B T \frac{r^2}{1+r^2}\left\{1 - e^{-r\omega_0 t}\left[\cos(\omega_0 t) - \frac{1}{r}\sin(\omega_0 t)\right]\right\} \tag{16}$$

The parameters $\Delta(t)$ and $\gamma(t)$ contain very important Markovian and non-Markovian features of open quantum systems. The indispensable difference between Markovian systems and non-Markovian systems is the presence of environmental memory effect. The environment behaves as a sink for the quantum system information. The system loses quantum information into the environment, due to the system reservoir interaction. If the environment has a non-trivial structure, the apparently lost information can go back to the system in future time, which leads to the non-Markovian dynamics with memory [20,21]. Define the decay rate as $\beta_{\pm}(t) = \frac{\Delta(t)\pm\gamma(t)}{2}$, $\beta \geq 0$, the system behaves in a Markovian dynamics; when $\beta < 0$, the system presents the non-Markovian dynamical characteristics. From Eqs. (14) and (16) one can observe that for the high temperature, $\mid \Delta(t) \gg \gamma(t) \mid$ and $\gamma(t) \approx 0$, which means that the diffusion coefficient $\Delta(t)$ plays a dominant role in affecting the dynamical behavior of system. In high temperature dynamics, $\beta_{+}(t) = \beta_{-}(t) = \frac{\Delta(t)}{2}$, as $\gamma(t) \approx 0$. In order to carefully investigate the transfer of quantum state, the effect of the dissipation coefficient $\gamma(t)$ can no longer be negligible, although it is small under high temperature, thus $\beta(t)$ depends on both $\gamma(t)$ and $\Delta(t)$.

The density matrix ρ_t of a two-level quantum system can be defined by $(x_t,\, y_t,\, z_t)$ in the Cartesian coordinate system as

$$\rho_t = \frac{1}{2}\left(I + x_t\sigma_x + y_t\sigma_y + z_t\sigma_z\right) = \frac{1}{2}\begin{pmatrix} 1+z_t & x_t - iy_t \\ x_t + iy_t & 1-z_t \end{pmatrix} \tag{17}$$

where x_t , y_t, and z_t represent the Cartesian coordinate system, and they are real numbers. $tr(\rho_t) = 1$, and $tr(\rho_t^2) \leq 1$.

$$\begin{aligned} dx_t &= -\left(\Delta(t) + \frac{M}{2}\right)x_t dt - \omega_0 y_t + u_y(t) z_t dt + \sqrt{M\eta}x_t z_t dW_t \\ dy_t &= -\left(\Delta(t) + \frac{M}{2}\right)y_t dt + \omega_0 x_t - u_x(t) z_t dt + \sqrt{M\eta}y_t z_t dW_t \\ dz_t &= -2\gamma(t)dt - 2\Delta(t) z_t dt - u_y x_t dt + u_x(t) y_t dt + \sqrt{M\eta}\left(z_t^2 - 1\right)dW_t \end{aligned} \tag{18}$$

In the weak measurement, a laser probe (along the z-axis) interacts with the atomic ensemble, and then the output of the interacted system system is detected using a photodetector which is called a homodyne detector. This provides a continuous measurement of the spin component along the propagation direction, the measured observable is σ_z in the z-direction. The quantum Weiner noise after interaction with the system is also in the output of the quantum system. From Eq. (7) the stochastic differential equation of the output of quantum system can also be described as

$$dY_t = -\sqrt{M\eta}z_t dt + dW_t \tag{19}$$

where M is the interaction strength between the system and the measurement, η is the detection efficiency and W_t is the Weiner process noise. dY_t shows the output of the observation process, and Y_t is the measurement process output of homodyne detector [14,22].

Equation (19) shows that the output of the system not only affects the Wiener noise but also contains some information of the system state, and one can extract the information about quantum states by making measurement output of the system. Therefore, the output of the system can be processed to design an estimator and an estimated state feedback control of quantum system [14].

Consider the equivalent dynamical system of SME (18) and output measurement process (19),

$$dX_t = A_0(t)dt + A(t)X_t dt + B(X_t)U(t)dt + G(X_t)dW_t \tag{20}$$

$$dY_t = CX_t dt + dW_t \tag{21}$$

where X_t is the actual state of the system, and $X_t = [x_t\ y_t\ z_t]^T \in \{X_t \in \mathbb{R}^n \,||X_t| \le |1|\}$ are values of ρ_t; x_t, y_t and z_t are the qubit Bloch sphere coordinates corresponding to the density matrix ρ_t, $A_o \in R^{(n\times p)}$, $A \in R^{(n\times n)}$, $B(X_t) \in R^{(n\times m)}$, $C \in R^{(p\times n)}$, $G(X_t) \in R^{(n\times p)}$ and $W_t \in R^p$ are real polynomial functions of X_t and $Y_t \in R^{p\times p}$. n, m and p are positive integers, respectively,

$$A_0 = \begin{bmatrix} 0 \\ 0 \\ -2\gamma(t) \end{bmatrix},\ A = \begin{bmatrix} -\left(\Delta(t) + \frac{M}{2}\right) & -\omega_0 & 0 \\ \omega_) & -\left(\Delta(t) + \frac{M}{2}\right) & 0 \\ 0 & 0 & -2\Delta(t) \end{bmatrix},\ B(X_t) = \begin{bmatrix} z_t & 0 \\ 0 & -z_t \\ -x_t & y_t \end{bmatrix}$$

$$U(t) = \begin{bmatrix} u_y(t) \\ u_x(t) \end{bmatrix},\ G(X_t) = \begin{bmatrix} \sqrt{M\eta}x_t z_t \\ \sqrt{M\eta}y_t z_t \\ \sqrt{M\eta}\left(z_t^2 - 1\right) \\ \end{bmatrix},\ C = \begin{bmatrix} 0\ 0\ -\sqrt{M\eta} \end{bmatrix}$$

The vector X_t is a collection of $2^2 - 1$ Gell-mann matrices, generating $SU(2)$, which should satisfy the following commutation relationship [23,24],

$$\left[X_t, X_t^T\right] = X_t X_t^T - \left(X_t X_t^T\right)^T = 2i\Theta(X_t)$$

where $\Theta(X_t)$ is the linear mapping.

The system Eq. (18) and Eq. (19) is realizable if there exists Hamiltonian operator $\mathsf{H} = \alpha X_t$ with $\alpha \in \mathbb{R}^3$ and coupling operator $L = \Omega X_t$ with $\Omega \in \mathbb{C}^3$, such that, $A_0 = 2i\Theta(\Omega)\Omega^\dagger, A = -2i\Theta(\alpha) + \Omega^\dagger\Omega + \Omega^T\Omega^* - 2\Omega^\dagger\Omega I, B = 2i\Theta[-\Omega^\dagger\ \ \Omega^T)]\Gamma$, where Γ is the complex matrix, $G = \Theta(i(\Omega^* - \Omega))$ and $C = \Omega + \Omega^*$.

The quantum system coherence can also be described with purity. For an arbitrary state ρ_t, the purity p_t can be expressed as

$$p_t = tr(\rho_t^2) \tag{22}$$

with $p_t = 1$ denotes the pure state, and $p_t < 1$ the mixed state. The coherence of quantum state is an essential feature of quantum systems and is usually represented by the non-diagonal elements of the density matrix ρ_t in Block spheres as:

$$\Lambda_{\mathrm{t}} = \frac{\sqrt{x_t^2 + y_t^2}}{2} \tag{23}$$

and the population

$$\rho_{t_{11}} = \frac{1+z_t}{2},\ \rho_{t_{22}} = \frac{1-z_t}{2} \tag{24}$$

where parameters Λ_t and p_t relate with the coherence and purity of the real system. $\rho_{t_{11}}$ and $\rho_{t_{22}}$, are elements of the density matrix

Figure 1 is the schematic diagram of dynamical stochastic quantum system (20) and measurement output process (21).

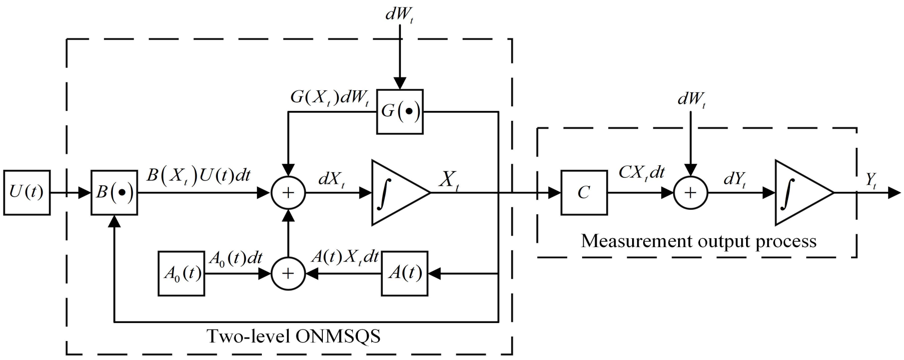

Fig. 1 Schematic diagram of two-level ONMSQS with output measurement process

2 Design of nonlinear optimal feedback tracking control of ONMSQS

In this section, nonlinear state estimator (NSE) and nonlinear optimal feedback tracking control (NOFTC) based on NSE are proposed for the ONMSQS. The NOFTC includes the estimated behavior of the quantum system for reconstructing the dynamical variables of the ONMSQS. Continuous weak measurement (CWM) is usually used for the quantum feedback control system. The state of system can be estimated based on the output of CWM process, to design a state feedback control law. However, the state ρ_t cannot be completely estimated with only one measurement record Y_t, it should be estimated by the output data obtained from CWM, because the measurement operator related with Y_t contains the partial information of the system state. Therefore, it is an optimal estimation problem, to do this the output Y_t are applied to drive the state estimator to estimate the system state. Assuming that all the parameters of the system are known except the state, and the state ρ_t can be estimated precisely by state-dependent differential Riccati equation (SDRE) estimator. Then the NOFTC law is designed based on estimated state of NSE to perform desired control task.

2.1 Nonlinear state estimator design of ONMSQS

The estimator of ONMSQS is designed by constructing a NSE with the structure of a linear observer, whose matrices and the gain can be depend on the state estimate $\hat{X}_t$.

The proposed NSE is based on the dual of the SDDRE nonlinear control by allowing the coefficient matrices of state-dependent [25,26,27,28].

Consider the system (20) and measurement output (21) the objective is to design the NSE to be able to estimate $\hat{X}_t$ of X_t that minimizes the error at time t by means of measurement data $[Y_s : t_0 \leqslant s \leqslant t]$: the conditional expectation value of X_t. We establish the estimator with the same construction of the dynamic system 20, in which by adding the correction term $K_e(\hat{X}_t,t)(dY_t - d\hat{Y}_t)$ into (20) to ensure the realization of state estimation. Where $K_e(\hat{X}_t,t)$ is the NSE gain matrix, $dY_t - d\hat{Y}_t$ is the innovated error between the system output and that of NSE, which describes the information gain from output Y_t, dY_t is the actual measurement output of the system , and $d\hat{Y}_t$ is the estimated output of the NSE. Thus the proposed NSE in this paper has the form

$$d\hat{X}_t = A_0(t)dt + A(t)\hat{X}_t dt + B(\hat{X}_t)U(t)dt + K_e(\hat{X}_t,t)(dY_t - C\hat{X}_t dt) \tag{25}$$

$$d\hat{Y}_t = C\hat{X}_t dt + dW_t \tag{26}$$

The Eq. (25) can also be written as

$$\begin{aligned} d\hat{X}_t = & \left(A(t) - K_e\left(\hat{X}_t,t\right)C\right)\hat{X}_t dt + B\left(\hat{X}_t\right)U(t)dt + K_e\left(\hat{X}_t,t\right)CX_t dt \\ & + K_e\left(\hat{X}_t,t\right)dW_t + A_0(t)dt \end{aligned} \tag{27}$$

where $\hat{X}_t$ is the estimated state, and $\hat{X}_t = [\hat{x}_t\ \hat{y}_t\ \hat{z}_t]^T \in \left\{\hat{X}_t \in \mathbb{R}^n \left|\left|\hat{X}_t\right| \leq |1|\right.\right\}$ are the estimated variables of $\hat{\rho}_t$ in Eq.(18), where $\hat{x}_t$, $\hat{y}_t$ and $\hat{z}_t$ are the qubit Bloch sphere coordinates corresponding to the density matrix $\hat{\rho}_t$. $K_e(\hat{X}_t,t)$ is the estimator gain matrix.

Considering density matrix must satisfy the conditions: $tr(\hat{\rho}_t) = 1$ and $tr(\hat{\rho}_t^2) \leqslant 1$, with $\hat{X}_t$=$\hat{X}_0$ and the commutation relationship

$$\hat{X}_t\hat{X}_t^T - (\hat{X}_t\hat{X}_t^T)^T = 0$$

Notice that there is the partial information of ρ_t in the output Eq. (18), and in such a case, if there is no noise, we can estimate the two-level state's variables in Eq. (17), but the noise makes the estimated state remains with error, we need to design an optimal algorithm to minimize this error. The error between the X_t and $\hat{X}_t$, can be defined as

$$e_t = X_t - \hat{X}_t$$

where rate of error is

$$de_t = dX_t - d\hat{X}_t$$

The design purpose of the NSE is to find the estimator gain $K_e(\hat{X}_t,t)$ that minimizes the estimation error covariance matrix $P_e(\hat{X}_t,t) = E[e_t \cdot e_t^T]$, so the rate of error dynamics of the systems (20) and (25), we have

$$\begin{aligned} de_t = & \left[A(t) - K_e\left(\hat{X}_t,t\right)C\right]e_t dt - B(X_t)U(t)dt + B\left(\hat{X}_t\right)U(t)dt \\ & + \left(G(X_t) - K_e\left(\hat{X}_t,t\right)\right)dW_t \end{aligned} \tag{28}$$

If the gain matrix $K_e(\hat{X}_t,t)$ is properly designed, then the error will approach to zero with arbitrary decay. The fulfillment of above requirement means that the system is to be asymptotically stable. Now the rate of error covariance has the form

$$dP_e(\hat{X}_t,t) = E\left[de_t \cdot e_t^T + e_t \cdot de_t^T\right] \tag{29}$$

where $E[\cdot]$ is the expectation function.

Substituting Eq. (28) in the above Eq. (29), and by using then by straight forward computation we have

$$\begin{aligned}\dot{P}_e(\hat{X}_t,t) =& [A - K_e(\hat{X}_t,t)C]P_t(\hat{X}_t,t) + P_e(\hat{X}_t,t)[A - K_e(\hat{X}_t,t)C]^T \\ &+ G(X_t)F_W G^T(X_t) + K_e(\hat{X}_t,t)F_W K_e^T(\hat{X}_t,t)\end{aligned} \tag{30}$$

where F_W is noise intensity, and $F_W dt = E[dW_t dW_t^T]$.

The objective is to minimize the error covariance, i.e., $\min\limits_{K_e(\hat{X}_t,t)} tr\ \dot{P}_e(\hat{X}_t,t)$ w.r.t., observer gain $K_e(\hat{X}_t)$, we have

$$\frac{\partial\left(tr\ \dot{P}_e(\hat{X}_t,t)\right)}{\partial K_e(\hat{X}_t,t)} = -P_e(\hat{X}_t,t)C^T - P_e(\hat{X}_t,t)C^T + 2K_e(\hat{X}_t,t)F_W = 0$$

where tr denotes the trace of matrix.

Rearranging the above equation to find the observer gain $K_e(\hat{X}_t,t)$, we get

$$K_e(\hat{X}_t,t) = P_e(\hat{X}_t,t)C^T F_W^{-1} \tag{31}$$

By using Eqs. (31) and (30), the state dependent differential Riccati (SDRE) equation can be expressed as

$$\dot{P}_e(\hat{X}_t,t) = AP_e(\hat{X}_t,t) + P_e(\hat{X}_t,t)A + Q_W(\hat{X}_t) - P_e(\hat{X}_t,t)C^T F_W^{-1} CP_e(\hat{X}_t,t) \tag{32}$$

where $Q_W(\hat{X}_t)dt = G(\hat{X}_t)F_W G^T(\hat{X}_t)dt$.

From Eq. (32) one can see that the SDRE is the function of state variables and can be integrated with some initial condition of P_{e_0} to compute $P_e(\hat{X}_t,t)$. Therefore, its computation needs to be carried out on-line at each update step to obtain estimation of the system states continuously. The error between the NSE and the ONMSQS tends to zero if the gain matrix $K_e(\hat{X}_t,t)$ is properly designed. Then the error will approach to zero asymptotically and it will tracks the systems states trajectory. This means that the estimator is asymptotically stable and detectable.

Figure 2 shows the schematic diagram of system model (25) and output (26), where EMO stands for estimated measurement output $\hat{Y}_t$ and Table 1 depicts the entire procedure of estimation of the system

Table 1 depicts the entire procedure of estimation of the system.

2.2 Design of nonlinear optimal feedback tracking control law of ONMSQS

In this subsection, NOFTC law is designed by using the state dependent Riccati equation (SDRE) for control of ONMSQS. The whole closed-loop control is performed online due to state-dependent equation of both controller and NSE, which allows the NOFTC to achieve the control results even the quantum system is under the environmental noises.

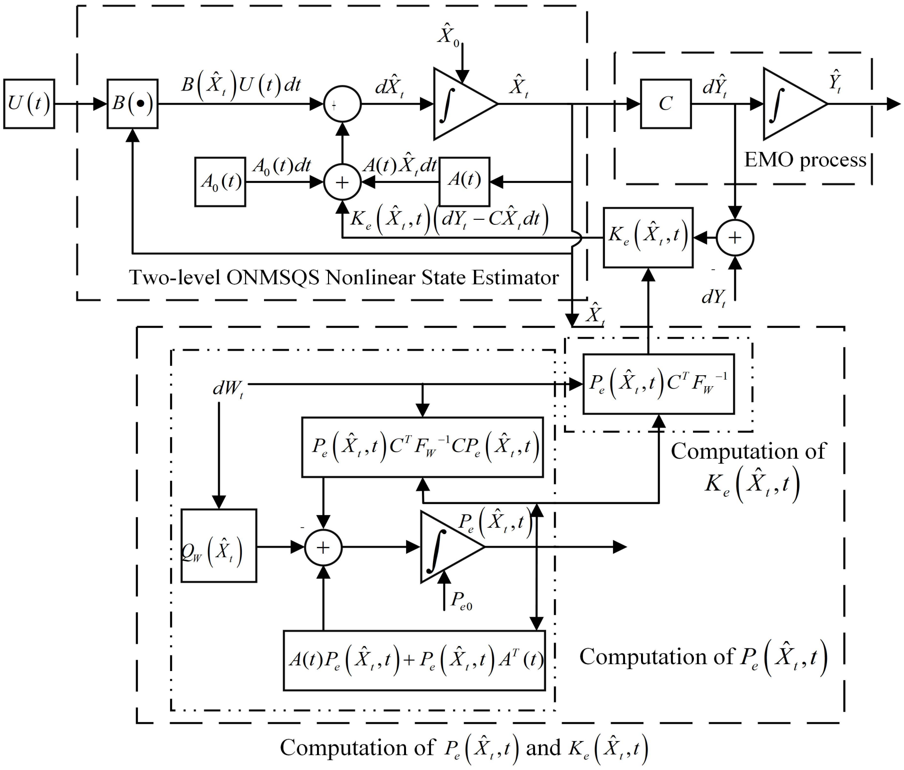

Fig. 2 Schematic diagram of NSE of ONMSQS

Table 1 Procedure of the ONMSQS state estimation

Consider the system 20 and output 21

$$dX_t = A_0 dt + AX_t dt + B(X_t)U(t)dt + G(X_t)dW_t$$
$$dY_t = CX_t dt + dW_t$$

Initialize

$$P_e(0) = P_{e_0},\ \hat{X}(0) = \hat{X}_0$$

Update $P_e(\hat{X}_t, t)$

$$\dot{P}_e(\hat{X}_t, t) = AP_e(\hat{X}_t, t) + P_e(\hat{X}_t, t)A^T + Q_W(\hat{X}_t) - P_e(\hat{X}_t, t)C^T F_w^{-1} CP(\hat{X}_t)$$

Estimator gain $K_e(\hat{X}_t, t)$

$$K_e(\hat{X}_t, t) = P_e(\hat{X}_t, t)C^T F_W^{-1}$$

Estimate update

$$d\hat{X}_t = A_0 dt + A\hat{X}_t dt + B(\hat{X}_t)U(t)dt + K_e(\hat{X}_t, t)\left(dY_t - C\hat{X}_t dt\right)$$

2.2.1 Reference system model

A reference system model should have the required dynamics, the output of reference system is selected as a desired quantum state as

$$\hat{X}_t^{\,r} = A_r \hat{X}_t^r \tag{33}$$

with

$$A_r = \begin{bmatrix} \Pi_1 & 0 & 0 \\ 0 & \Pi_2 & 0 \\ 0 & 0 & \Pi_3 \end{bmatrix}$$

where $-1 \leqslant \Pi_i \leqslant 1$, $i = 1,\ 2,\ 3$ are the constant of the reference model. The dynamical variable vector $\hat{X}_t^r$ of this reference is denoted as $\hat{X}_t^r = [\hat{x}_t^r\ \hat{y}_t^r\ \hat{z}_t^r]^T$, where $\hat{X}x_t^r\ \hat{y}_t^r$ and $\hat{z}_t^r$ are the desired Cartesian coordinates of the reference model state in the Bloch sphere. When the dynamical ONMSQS closely tracks these reference system state variables, then the desired quantum control goal can be achieved.

2.2.2 Control problem formulation

Considering the non-Markovian SME (20), measurement output (21) and the estimator (25), and the reference system model (33). The NOFTC problem is to design the control field $U(t)$ based on estimated state $\hat{X}_t$ and desired state $\hat{X}_t^r$ using SDRE strategy to minimize the following cost functional of error

$$J\left(X_f, \hat{X}_t,\ U(t)\right) = E\left[\frac{1}{2}\hat{e}_{t_f}^T S \hat{e}_{t_f} + \frac{1}{2}\int_{t_0}^{t_f} \left(\hat{e}_t^T Q\left(\hat{X}_t\right)\hat{e}_t + U^T\left(\hat{X}_t, t\right) R\left(\hat{X}_t\right) U\left(\hat{X}_t, t\right)\right) dt\right] \tag{34}$$

where $\hat{e}_t = \hat{X}_t - \hat{X}_t^r$, $\hat{X}_t^r$ is the desired final state, $Q(\hat{X}_t)$, and S are the symmetric positive semi-definite matrices, and $R(\hat{X}_t)$ is a symmetric definite matrix. The $\hat{e}_t^T Q(\hat{X}_t)\hat{e}_t$ and $U^T(\hat{X}_t, t) R(\hat{X}_t) U(\hat{X}_t, t)$ represents the control accuracy and measure of control effort.

The utmost advantage of SDRE control is that there always exists physical intuition, and designers can directly control performance by adjusting the weight matrix $Q(\hat{X}_t)$ and $R(\hat{X}_t)$. Furthermore, $Q(\hat{X}_t)$ and $R(\hat{X}_t)$ not only are constants, but also can be varying functions of state. In this way, different behavior patterns can be applied in different regions of the system's state space [29,30].

2.2.3 NOFTC law design

The optimal form of the SDRE control can be obtained by using Hamilton-Jacobi equation (HJB) theory. The corresponding stochastic HJB expression for the cost functional Eq. (34) and estimated state equation Eq. (25) can be derived as

$$\begin{aligned} -\partial_t V\left(\hat{X}_t, t\right) = \min_{u_t: t_f} \Bigg(& \frac{1}{2} U^T\left(\hat{X}_t, t\right) R\left(\hat{X}_t\right) U\left(\hat{X}_t, t\right) + \frac{1}{2}\hat{e}_t^T Q\left(\hat{X}_t\right)\hat{e}_t \\ & + \left(\nabla_{\hat{X}_t} V\left(\hat{X}_t\right)\right)^T \left(A_0(t) + A(t)\hat{X}_t + B\left(\hat{X}_t\right) U\left(\hat{X}_t, t\right)\right) \\ & + \frac{1}{2} G\left(\hat{X}_t\right) \nabla_{\hat{X}_t \hat{X}_t} V\left(\hat{X}_t, t\right) G^T\left(\hat{X}_t\right) \Bigg) \end{aligned} \tag{35}$$

where $V(\hat{X}_t, t)$ is the optimal value function, and $\nabla_{\hat{X}_t}$, $\nabla_{\hat{X}_t \hat{X}_t}$ depict the partial derivatives of $V(\hat{X}_t, t)$ w.r.t., $\hat{X}_t$ and twice w.r.t., $\hat{X}_t$, respectively. In order to solve for the optimal value function we will guess its parametric form which should satisfy the HJB Eq. (35). Due to quadratic nature of Eq. (35), we consider the value function of the form as

$$V\left(\hat{X}_t,t\right)=\frac{1}{2}\hat{X}_t^T P_c\left(\hat{X}_t,t\right)\hat{X}_t-\hat{X}_t^T s\left(\hat{X}_t,t\right)+q\left(\hat{X}_t,t\right) \tag{36}$$

where $V(\hat{X}_t,t)$ is symmetric with boundary condition $V(\hat{X}_{t_f},t_f)=\hat{X}_{t_f}^T S\hat{X}_{t_f}$. Now from Eq. (36) we can compute the required derivatives for HJB Eq. (35), i.e.,

$$\partial_t V\left(\hat{X}_t,t\right)=\frac{1}{2}\hat{X}_t^T\dot{P}_c\left(\hat{X}_t,t\right)\hat{X}_t-\hat{X}_t^T\dot{s}\left(X_t,t\right)+\dot{q}\left(\hat{X}_t,t\right)$$
$$\nabla_{\hat{X}_t}V\left(\hat{X}_t,t\right)=P_c\left(\hat{X}_t,t\right)\hat{X}_t+s\left(\hat{X}_t,t\right)$$
$$\nabla_{\hat{X}_t\hat{X}_t}V\left(\hat{X}_t,t\right)=P_c\left(\hat{X}_t,t\right)$$

The approximations are done for the NOFTC law designed in skipping terms, i.e., $\frac{\partial P_c(\hat{X}_t,t)}{\partial\hat{X}_t}$, $\frac{\partial s(\hat{X}_t,t)}{\partial\hat{X}_t}$ and $\frac{\partial q(\hat{X}_t,t)}{\partial\hat{X}_t}$ are skipped in state-dependent differential equations and in the control equation for the sake of computational simplicity and avoiding mathematical complexity [18].

Substituting the above derivatives $\partial_t V(\hat{X}_t,t)$, $\nabla_{X_t}V(\hat{X}_t,t)$ and $\nabla_{\hat{X}_t\hat{X}_t}V(\hat{X}_t,t)$ into Eq. (35), one can get

$$\begin{aligned}
-\frac{1}{2}\hat{X}_t^T\dot{P}_c\left(\hat{X}_t,t\right)\hat{X}_t+\hat{X}_t^T\dot{s}\left(\hat{X}_t,t\right)-\frac{1}{2}\dot{q}\left(\hat{X}_t,t\right)&\\
=\min_{u_t:t_f}\Bigg(\frac{1}{2}U^T\left(\hat{X}_t,t\right)R\left(\hat{X}_t\right)\left(\hat{X}_t\right)U\left(\hat{X}_t,t\right)+\frac{1}{2}\hat{e}_t^T Q\left(\hat{X}_t\right)\hat{e}_t&\\
+\left(P_c\left(\hat{X}_t,t\right)\hat{X}_t+s\left(\hat{X}_t,t\right)\right)^T\left(A_0\left(t\right)+A\left(t\right)\hat{X}_t+B\left(\hat{X}_t\right)U\left(\hat{X}_t,t\right)\right)&\\
+\frac{1}{2}tr\left(G\left(\hat{X}_t\right)G^T\left(\hat{X}_t\right)P_c\left(\hat{X}_t,t\right)\right)\Bigg)&
\end{aligned} \tag{37}$$

Taking derivative of Eq. (37) w.r.t., $U\left(\hat{X}_t,t\right)$ and set to zero, we can obtain the corresponding nonlinear optimal control $U^*\left(\hat{X}_t,t\right)$ as

$$U^*\left(\hat{X}_t,t\right)=-R^{-1}\left(\hat{X}_t\right)B\left(\hat{X}_t\right)^T\left(P_c\left(\hat{X}_t,t\right)\hat{X}_t-s\left(X_t,t\right)\right) \tag{38}$$

Substituting the value of $U^*\left(\hat{X}_t,t\right)$ in Eq. (37), the term HJB can be re-written as

$$\begin{aligned}
&-\frac{1}{2}\hat{X}_t^T\dot{P}\left(\hat{X}_t,t\right)\hat{X}_t+\hat{X}_t^T\dot{s}\left(\hat{X}_t,t\right)-\frac{1}{2}\dot{q}\left(\hat{X}_t,t\right)\\
&=-\frac{1}{2}\hat{X}_t^T\Big[A(t)^T P_c\left(\hat{X}_t,t\right)+P_c\left(\hat{X}_t,t\right)A\left(t\right)-P_c\left(\hat{X}_t,t\right)B\left(\hat{X}_t\right)R^{-1}\left(\hat{X}_t\right)B^T\left(\hat{X}_t\right)P_c\left(\hat{X}_t,t\right)\\
&+Q\left(\hat{X}_t\right)\Big]\hat{X}_t-\hat{X}_t^T\Big[\left(A\left(t\right)-B\left(\hat{X}_t\right)R^{-1}\left(\hat{X}_t\right)B^T\left(\hat{X}_t\right)P_c\left(\hat{X}_t,t\right)\right)^T s\left(\hat{X}_t,t\right)+Q\left(\hat{X}_t\right)\hat{X}_f^r\\
&-P_c\left(\hat{X}_t,t\right)A_0\left(t\right)\Big]-s\left(\hat{X}_t,t\right)^T A_0\left(t\right)-\frac{1}{2}s\left(\hat{X}_t,t\right)^T B\left(\hat{X}_t\right)R^{-1}\left(\hat{X}_t\right)B^T\left(\hat{X}_t\right)s\left(\hat{X}_t,t\right)\\
&+\frac{1}{2}G\left(\hat{X}_t\right)P_c\left(\hat{X}_t,t\right)G^T\left(\hat{X}_t\right)
\end{aligned} \tag{39}$$

From Eq. (39) one can get,

$$\begin{aligned}-\dot{P}_c\left(\hat{X}_t,t\right) =& A(t)^T P_c\left(\hat{X}_t,t\right) + P_c\left(\hat{X}_t,t\right) A\left(t\right) + Q\left(\hat{X}_t\right)\\ &- P_c\left(\hat{X}_t,t\right) B\left(\hat{X}_t\right) B\left(\hat{X}_t\right)^T P_c\left(\hat{X}_t,t\right)\end{aligned} \tag{40}$$

$$\begin{aligned}\dot{s}\left(\hat{X}_t,t\right) =& -\left(A\left(t\right) - B\left(\hat{X}_t\right) R^{-1}\left(\hat{X}_t\right) B^T\left(\hat{X}_t\right) P_c\left(\hat{X}_t,t\right)\right)^T s\left(\hat{X}_t,t\right)\\ &- Q\left(\hat{X}_t\right)\hat{X}_t^r + P_c\left(\hat{X}_t,t\right) A_0\end{aligned} \tag{41}$$

$$\begin{aligned}\dot{q}\left(\hat{X}_t,t\right) =& s\left(\hat{X}_t,t\right)^T B\left(\hat{X}_t\right) R^{-1}\left(\hat{X}_t\right) B^T\left(\hat{X}_t\right) s\left(\hat{X}_t,t\right) + 2s\left(\hat{X}_t,t\right)^T A_0\left(t\right)\\ &- G\left(\hat{X}_t\right) P_c\left(\hat{X}_t,t\right) G^T\left(\hat{X}_t\right)\end{aligned} \tag{42}$$

where $P_c(\hat{X}_t,t)$ is symmetric, and it is the suboptimal solution of SDRE with $P_c(\hat{X}_{t_f},t_f) = S$. The terms $s(\hat{X}_t,t)$ is the solution of the nonhomogeneous equation with $s\left(\hat{X}_{t_f},t_f\right) = \hat{X}_{t_f}^r$ and $q\left(\hat{X}_{t_f},t_f\right) = 0$. Obviously, the characterization advantage of resulting NOFTC Eq. (38) by solving the corresponding Eqs. (40) and (41) is that it provides a possibility to deal with the nonlinearities of the systems. Moreover, because the SDRE depends only on the current state, the computational calculations can be performed online, in which case the Eqs. (40) and (41) are then solved at each point $\hat{X}_t$ to obtain $U^*(\hat{X}_t,t)$ in NOFTC .

2.2.4 Approximate Lyapunov-based solution

There still is a problem in the solution of Eq. (38) by using Eqs. (40) and (41) with the final conditions $P_c(\hat{X}_{t_f},t_f) = S$ and $s\left(\hat{X}_{t_f},t_f\right) = \hat{X}_{t_f}^r$: the ststes used in the future are not known ahead of time. One cannot calculate SDCs of Eqs. (40) and (41) backward in time from t_f to t_0. To solve this problem, the original nonlinear SDRE is converted to the differential Lyapunov equation (DLE) using an approximate analytical method, which can be solved in closed form at each time step [20, 22, 23, 36]. Furthermore, the states' values in the Riccati equation are frozen their current values from each time step of the current time t to the final time t_f.

Now in order to get the closed-form solution of SDRE (40) and non-homogeneous Eq. (41) we have following steps:

(a) Closed-form solution for Riccati equation

In order to get the solution of $P_c(\hat{X}_t,t)$ in Eq. (40), solve the algebraic Riccati equation (ARE) to calculate the steady state value $P_{ss}(\hat{X}_t,t)$

$$\begin{aligned}&A(t)^T P_{ss}\left(\hat{X}_t,t\right) + P_{ss}\left(\hat{X}_t,t\right) A\left(t\right) - P_{ss}\left(\hat{X}_t,t\right) B\left(\hat{X}_t\right) B\left(\hat{X}_t\right)^T P_{ss}\left(\hat{X}_t,t\right)\\ &\quad + Q\left(\hat{X}_t\right) = 0\end{aligned} \tag{43}$$

Subtract Eq. (43) from Eq. (40)

$$\begin{aligned}-\dot{P}_c\left(\hat{X}_t,t\right) =& A(t)^T\left(P_c\left(\hat{X}_t,t\right) - P_{ss}\left(\hat{X}_t,t\right)\right) + \left(P_c\left(\hat{X}_t,t\right) - P_{ss}\left(\hat{X}_t,t\right)\right) A\left(t\right)\\ &- P_{ss}\left(\hat{X}_t,t\right) B\left(\hat{X}_t\right) B\left(\hat{X}_t\right)^T P_{ss}\left(\hat{X}_t,t\right) + P_{ss}\left(\hat{X}_t,t\right) B\left(\hat{X}_t\right) B\left(\hat{X}_t\right)^T P_{ss}\left(\hat{X}_t,t\right)\end{aligned} \tag{44}$$

Use the change of variables in Eq. (44) and let

$$K_P\left(\hat{X}_t,t\right)=\left[P_c\left(\hat{X}_t,t\right)-P_{ss}\left(\hat{X}_t,t\right)\right]^{-1} \tag{45}$$

The final condition of Eq. (45) is

$$K_P\left(\hat{X}_t,t_f\right)=\left[S-P_{ss}\left(\hat{X}_t,t\right)\right]^{-1} \tag{46}$$

Calculate the closed-loop value $A_{cl}(\hat{X}_t)$

$$A_{cl}\left(\hat{X}_t\right)=A\left(t\right)-B\left(\hat{X}_t\right)R^{-1}\left(\hat{X}_t\right)B\left(\hat{X}_t\right)^T P_{ss}\left(\hat{X}_t,t\right) \tag{47}$$

Solve the DLE

$$\dot{K}_P\left(\hat{X}_t,t\right)=K_P\left(\hat{X}_t,t\right)A_{cl}^T\left(\hat{X}_t\right)+A_{cl}\left(\hat{X}_t\right)K_P\left(\hat{X}_t,t\right)-B\left(\hat{X}_t\right)R^{-1}\left(\hat{X}_t\right)B\left(\hat{X}_t\right)^T \tag{48}$$

The solution $K_P(\hat{X}_t,t)$ of equation (48) is [16]

$$K_P\left(\hat{X}_t,t\right)=e^{A_{cl}\left(\hat{X}_t\right)\left(t-t_f\right)}\left[K_P\left(\hat{X}_t,t_f\right)-\mathfrak{D}\right]e^{A_{cl}^T\left(\hat{X}_t\right)\left(t-t_f\right)}+\mathfrak{D} \tag{49}$$

where $\mathfrak{D}$ is the solution of the following algebraic Lyapunov function [32,31]

$$A_{cl}\left(\hat{X}_t\right)\mathfrak{D}+\mathfrak{D}A_{cl}^T\left(\hat{X}_t\right)-B\left(\hat{X}_t\right)R^{-1}\left(\hat{X}_t\right)B^T\left(\hat{X}_t\right)=0 \tag{50}$$

Then calculate the value of $P_c(\hat{X}_t,t)$

$$P_c\left(\hat{X}_t,t\right)=K_P^{-1}\left(\hat{X}_t,t\right)+P_{ss}\left(\hat{X}_t,t\right) \tag{51}$$

(b) Closed-Form Solution for Non-homogeneous Equation

Calculate the steady state value of $s_{ss}(\hat{X}_t,t)$ according to Eq. (41)

$$s_{ss}\left(\hat{X}_t,t\right)=\left(A\left(t\right)-B\left(\hat{X}_t\right)R^{-1}\left(\hat{X}_t\right)B^T\left(\hat{X}_t\right)P_{ss}\left(X_t,t\right)\right)^{T^{-1}}\left(Q\left(\hat{X}_t\right)\hat{X}_t^r-P_{ss}\left(X_t,t\right)A_0\left(t\right)\right) \tag{52}$$

Use the change of variables and define

$$K_s\left(\hat{X}_t,t\right)=\left[s\left(\hat{X}_t,t\right)-s_{ss}\left(\hat{X}_t,t\right)\right] \tag{53}$$

The solution $K_s(\hat{X}_t,t)$ is given as

$$K_s\left(\hat{X}_t,t\right)=e^{-\left(A(t)-B\left(\hat{X}_t\right)R^{-1}\left(\hat{X}_t\right)B^T\left(\hat{X}_t\right)P_{ss}(X_t,t)\right)^T\left(t-t_f\right)}\left[s\left(\hat{X}_t,t_f\right)-s_{ss}\left(\hat{X}_t,t\right)\right] \tag{54}$$

Then calculate the value of $s(\hat{X}_t,t)$ from the equation below as

$$s\left(\hat{X}_t,t\right)=K_s\left(\hat{X}_t,t\right)+s_{ss}\left(\hat{X}_t,t\right) \tag{55}$$

Finally calculate the NOFTC law $U^*(\hat{X}_t,t)$ by using Eqs. (51) and (55)

$$U^*\left(\hat{X}_t,t\right)=-R^{-1}\left(\hat{X}_t\right)B\left(\hat{X}_t\right)^T\left(P_c\left(X_t,t\right)\hat{X}_t-s\left(X_t,t\right)\right) \tag{56}$$

In summary, the online implementation of NOFTC in Finite-horizon SDRE strategy is shown in the following Algorithm.

Algorithm: Steps for online implementation of NOFTC SDRE strategy

---

At each time-step repeat:

{

1: Estimate the system state X_t using NSE (25) and calculate $A_0(t)$, $A(t)$ and $B(\hat{X}_t)$.
2: Solve SDARE (43) for $P_{ss}(\hat{X}_t, t)$.
3: Evaluate $K_P(\hat{X}_t, t_f)$ using Eq. (46).
4: Calculate $A_{cl}(\hat{X}_t)$ in Eq. (47).
5: Solve algebraic Lyapunov function (50) to find $\mathfrak{D}$.
6: Find $K_P(\hat{X}_t, t)$ using Eq. (49).
7: Calculate $P_c(\hat{X}_t, t)$ using Eq. (51).
8: Using Eq. (52) calculate the steady state value of $s_{ss}(\hat{X}_t, t)$.
9: Solve Eq. (54) to calculate $K_s(\hat{X}_t, t_f)$.
10: Find $s(\hat{X}_t, t)$ using Eq. (55).
11: Calculate the NOFTC law using Eq. (56).
12: Apply NOFTC law on the ONMSQS and NSE.

}

Figure 3 shows the flow chat of the NSE based ONFTC process for ONMSQS.

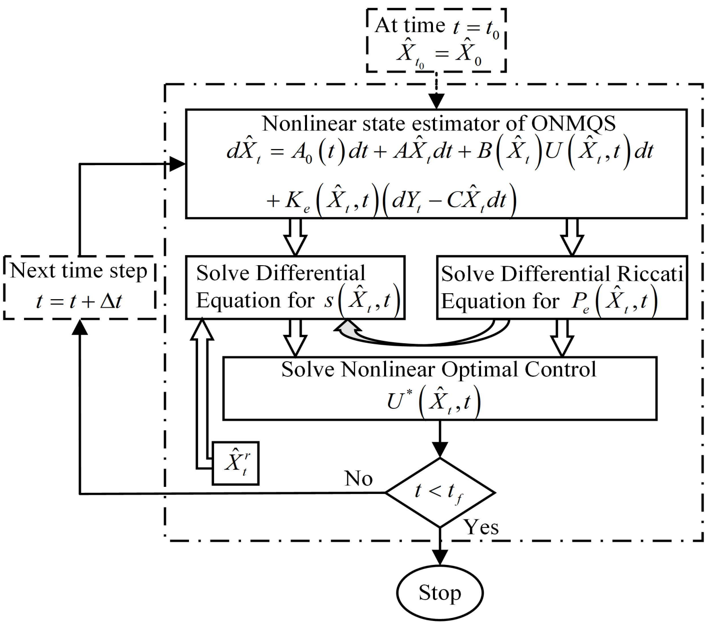

Fig. 3 Flowchart of the NSE based NOFTC process for ONMSQS

3 Closed-loop system dynamics

By applying the NSE-based NOFTC to the ONMSQS, we can get the closed-loop dynamics of the system which includes the NOFTC, principle ONMSQS, the NSE and reference system model. Denoting the closed-loop dynamical variables as $\check{X}_t = \left[X_t \; \hat{X}_t\right]^T$ and $\check{X}_t = \left[Y_t \; \hat{Y}_t\right]^T$, from the system (20) and measurement process output (21), NSE (25) , estimated output (26), NOFTC (38), and the reference system model (33), the dynamical equation of the closed-loop system is described as

$$d\check{X}_t = (\check{A}_0 + \check{A}\check{X}_t + \check{B}(\check{X}_t))dt + \check{G}(\check{X}_t)dW_t \tag{57}$$

$$d\begin{bmatrix} X_t \\ \hat{X}_t^r \\ \hat{X}_t \end{bmatrix} = \left(\check{A}_0(t) + \check{A}(t) \begin{bmatrix} X_t \\ \hat{X}_t^r \\ \hat{X}_t \end{bmatrix} + \check{B}\left(\check{X}_t\right) \begin{bmatrix} X_t \\ \hat{X}_t^r \\ \hat{X}_t \end{bmatrix} \right) dt + \check{G}\left(\check{X}_t\right) dW_t$$

$$d\check{Y}_t = \check{C}\check{X} + \check{D}\begin{bmatrix} dW_t \\ 0 \end{bmatrix} \tag{58}$$

$$d\begin{bmatrix} Y_t \\ \hat{Y}_t \end{bmatrix} = \check{C}\begin{bmatrix} X_t \\ \hat{X}_t \end{bmatrix} + \check{G}\begin{bmatrix} dW_t \\ 0 \end{bmatrix}$$

where

$$\check{A}_0(t) = \begin{bmatrix} A_0(t) \\ 0_{3\times 1} \\ A_0(t) \end{bmatrix}, \; \check{A}\,(\mathrm{t}) = \begin{bmatrix} A(t) & 0_{3\times 3} & 0_{3\times 3} \\ 0_{3\times 3} & A_r & 0_{3\times 3} \\ K_e\left(\hat{X}_t, t\right)C & 0_{3\times 3} & A(t) - K_e\left(\hat{X}_t, t\right)C \end{bmatrix}$$

,

$$\check{B}(\check{X}_t) \begin{bmatrix} 0_{3\times 3} & B(X_t)\,\Xi_2 & -B(X_t)\,\Xi_1 \\ 0_{3\times 3} & 0_{3\times 3} & 0_{3\times 3} \\ 0_{3\times 3} & B\left(\hat{X}_t\right)\Xi_2 & -B\left(\hat{X}_t\right)\Xi_1 \end{bmatrix}, \; \check{G}(\check{X}_t) = \begin{bmatrix} G(X_t) \\ 0_{3\times 1} \\ K_e\left(\hat{X}_t, t\right) \end{bmatrix}$$

,

$$\check{C} = \begin{bmatrix} C & 0_{1\times 3} & 0_{1\times 3} \\ 0_{1\times 3} & 0_{1\times 3} & 0_{1\times 3} \\ 0_{1\times 3} & 0_{1\times 3} & C \end{bmatrix}, \; \check{D} = \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix}$$

where $\Xi_1 = -R^{-1}\left(\hat{X}_t\right)B\left(\hat{X}_t\right)^T P_c\left(\hat{X}_t, t\right)$ and $\Xi_2 = R^{-1}\left(\hat{X}_t\right)B\left(\hat{X}_t\right)^T s\left(\hat{X}_t\right)$.

The Eqs. (57) and (58) are the closed-loop dynamical model of NSE based NOFTC for the ONMSQS. The closed-loop NSE-based NOFTC structure for the ONMSQS is shown in Figure 4.

4 Numerical simulation and results analyses

In this section, we apply NOFTC on a two-level ONMSQS to demonstrate the advantages of NSE-based NOFTC system for the state transfer of ONMSQS. The control fields transfer the system from arbitrary initially state ρ_0 to the target eigenstate ρ_f. For the two-level ONMSQS (20) there are all two eigenstates: $|0\rangle$ and $|1\rangle$. Experimental performance of the state transfer success rate $\geq 90\%$ is selected for the quantum system state transfer from arbitrary state to the desired state.

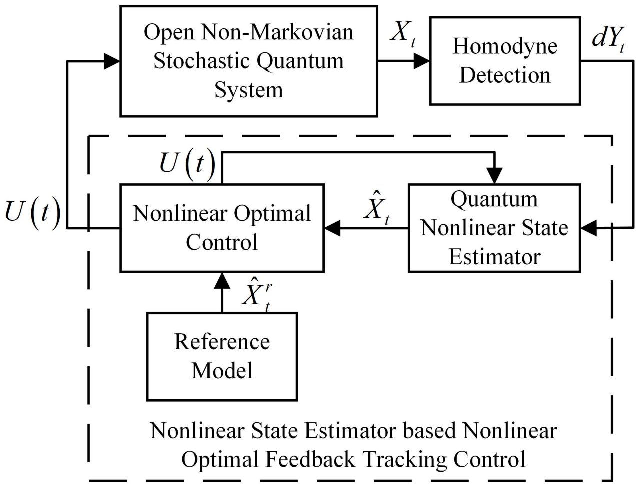

Fig. 4 Schematic diagram of the closed-loop NOFTC of the system

We first do the experiment of the free evolution of the system without external control fields, i.e., $u_x(t) = u_y(t) = 0$. The initial state is chosen as: $X_0 = [x_0\ y_0\ z_0]^T = [0.1\ -0.2\ 0.85]^T$, and other parameters are selected as: $\alpha = 0.1$, $\omega_0 = 10$, $K_B T = 30$, $r = 0.2$, $M = 2$ and $\eta = 1$. Figure 5 shows the free evolution of the system (20) in the Bloch sphere, in which m_0 and m_f are the initial and final state of the ONMSQS , from which one can see that the system states eventually terminate on the equilibrium state: the maximum mixed state, which is the stable point of the system. Due to the characteristics of the non-Markovian, the trajectory of the state does not directly but spirally attenuate to the center of ball.

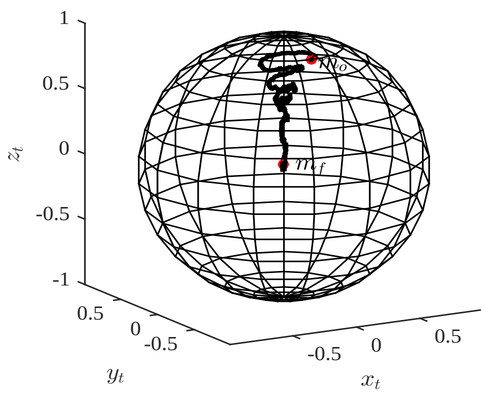

Fig. 5 Free time evolution trajectory of the ONMSQS with $X_0 = [x_0\ y_0\ z_0]^T = [0.1\ -0.2\ 0.85]^T$

It should be noted that moving the system to the desired state with high transfer rate depends not only on the control fields but also the joint relationship of system parameters, i.e., the relations of α, ω_0, K_BT, r, M, and η and $[u_x(t);\ u_y(t)]$. we do the experiments to verify the validity of the NOFTC, and the comparisons between the estimated and real states, in which the initial values of the ONMSQS controlled has been selected randomly and the NSE is initialized at $\hat{X}_0 = [-0.2\ 0.15\ 0.6]^T$, at $t = 0$, and $F_w = 1$ and $P_{e_0} = I_{3\times3}$. The respective state and input weighting matrices are selected as $Q = 3\times diag[2\ 12\ 5]$, $R = 10\times diag[0.01\ 0.09]$, $S = diag[111]$, and the reference targeted final state is the eigenstate $|1\rangle$, i.e., $\hat{X}_t^r = [0\ 0\ -1]^T$. $\alpha = 0.1$, $\omega_0 = 10$, $K_BT = 30$, $r = 0.1$, $M = 3$, and $\eta = 1$. $t_f = 4$ seconds. The variance of dW_t is 0.1 and $C = [0\ 0\ \sqrt{M\eta}]$. Figure 6 shows the state transfer of real and estimated ONMSQS with NSE-based NOFTC, from which one can see that the estimated state can track the system state and achieve error within $\pm2\%$ about 1 sec. time, and NOFTC designed can drive the system in the direction of the given desired trajectory with high state transfer success rate, i.e., $\geq 90\%$.

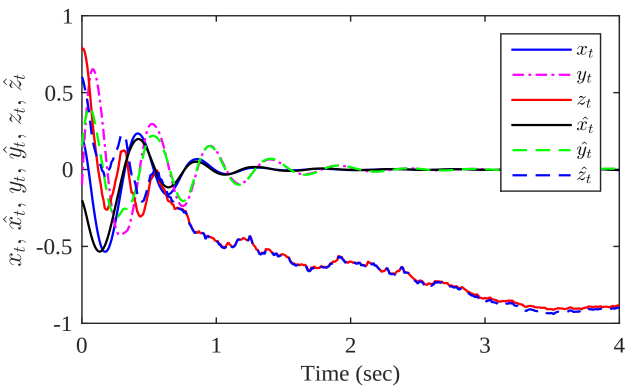

Fig. 6 State transfer of the actual and the estimated ONMSQS with $F_w = 1$ and $P_{e_0} = I_{3\times3}$

Figure 7 shows the control fields of NOFTC. Figures 8, 9 and 10 are the time

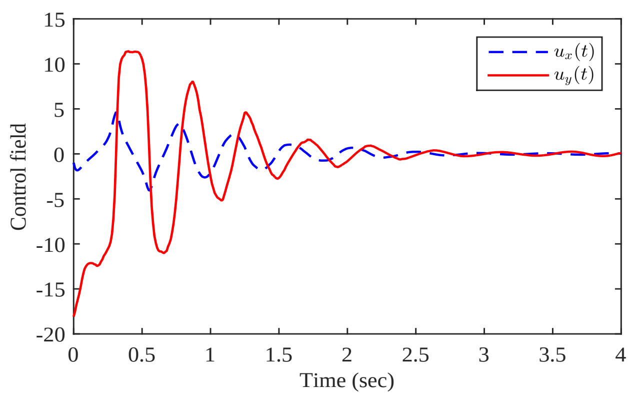

Fig. 7 State transfer control fields with $Q = 3\times diag[2\ 12\ 5]$, $R = 10\times diag[0.01\ 0.09]$ and $S = diag[111]$

evolution of coherence (23), purity (22) and occupation probability of the real and the estimated system (24), respectively. Figure 11 shows the errors between the

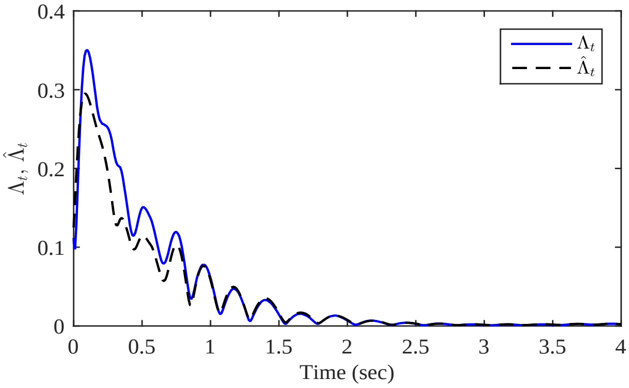

Fig. 8 Evolution of coherence function Λ_t

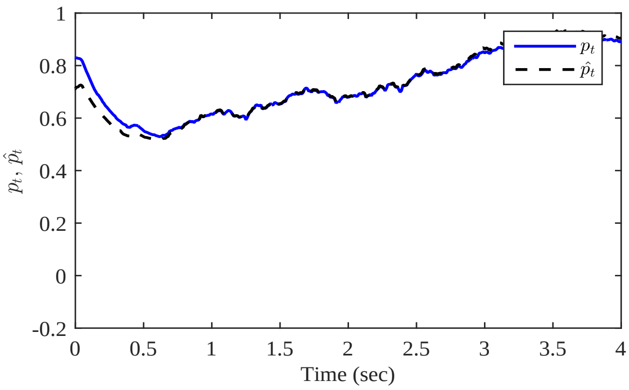

Fig. 9 Evolution of the purity of the system

Cartesian coordinates of the real and estimated ONMSQS, i.e., $e_1 = x_t - \hat{x}_t$, $e_1 = y_t - \hat{y}$ and $e_3 = z_t - \hat{z}$, from which one can see that the error between the estimated and real system is small enough for each individual value about 1 sec. time.

Non-Markov quantum system has many parameters, and some of them are changing with the time. In order to demonstrate the robustness of the performances of the system for the different parameters, in the experiments, for the total 6 different parameters, we first do one experiment for a given group parameters, then we let one of them are equal to different two values and at the same time let the other parameters un-changed, the total number of the experiments of desired and designed states of the system for these 6 parameters are 2+12=14. i.e., $\alpha = 0.1$, $\omega_0 = 10$, $K_BT = 30$, $r = 0.1$, $M = 3$, $\eta = 1$, $\alpha = 0.05$, $\omega_0 = 10$, $K_BT = 30$, $r = 0.1$, $M = 3$, $\eta = 1$, $\alpha = 0.1$

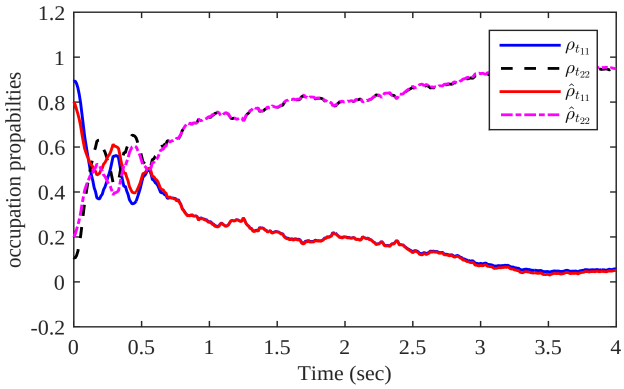

Fig. 10 Evolution of occupation probabilities of the system

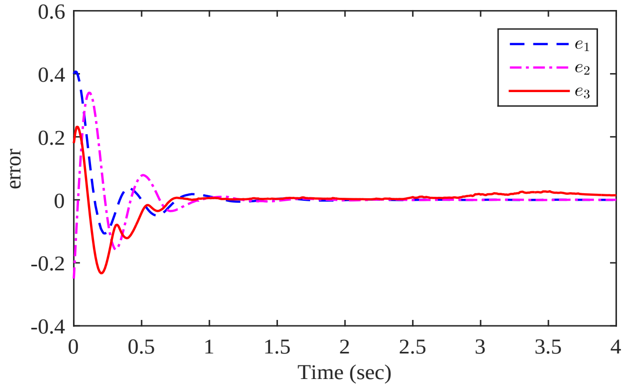

Fig. 11 Error between real and estimated system with $e_1 = x_t - \hat{x}_t$, $e_1 = y_t - \hat{y}$ and $e_3 = z_t - \hat{z}$

, $\omega_0 = 15$, $K_BT = 30$, $r = 0.1$, $M = 3$, $\eta = 1$, $\alpha = 0.1$, $\omega_0 = 10$, $K_BT = 50$, $r = 0.1$, $M = 3$, $\eta = 1$, $\alpha = 0.1$, $\omega_0 = 10$, $K_BT = 30$, $r = 0.2$, $M = 3$, $\eta = 1$, $\alpha = 0.1$, $\omega_0 = 10$, $K_BT = 30$, $r = 0.1$, $M = 5$, $\eta = 1$ and $\alpha = 0.1$, $\omega_0 = 10$, $K_BT = 30$, $r = 0.1$, $M = 3$, $\eta = 0.8$. Figure 12 shows the states transfers of estimated and real ONMSQS with different system parameters, from which one can see that with higher M and the more noise introduced in the system, when the effect of K_BT and r increase the coherence becomes shorter, and with smaller r the system can achieve high success state transfer rate. Under the all different parameters, the characteristics of the quantum system have little changed, and the designed control quantum system can well track the desired state and have good performances in any cases. The experiments demonstrate that the NOFTC designed has ability to track the states of the control quantum system to the desired trajectory.

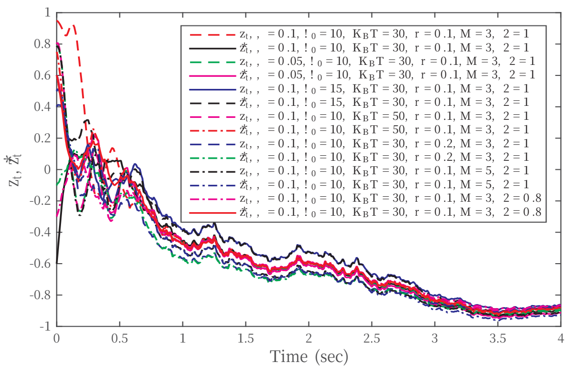

Fig. 12 Evolution of state transfer of the real and estimated system with different system parameters

5 Conclusion

In this paper, a closed-loop NSE-based NOFTC was proposed and designed for the state transfer of two-level ONMSQS. The proposed, which included a NSE to continuously estimate the system state by the measurement data, and then the NOFTC law was designed via SDRE. The closed-loop NSE based NOFTC for ONMSQS was implemented online at each time step to get the high control performance for the tracking of reference model. Numerical simulations results demonstrate that the NSE was able to track the real ONMSQS with the error $\pm 2\%$, and the NOFTC proposed had the ability to steer the state of ONMSQS from any initial arbitrary state to the final target state.

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