Some Analytic Systems of Rules

Timo Lang(✉)

University College London, London, UK
timo.lang@ucl.ac.uk

Abstract. We define two simple systems of rules, i.e. calculi with a global condition on the order of rule instances in a proof, for the modal logics of shift-reflexive and Euclidean frames respectively. Cut-elimination, and therefore the subformula property, can be derived directly from the cut-elimination property of adjacent logics. We compare our system to the calculus of grafted hypersequents, which has previously been used to capture both logics.

We then discuss an attempt to obtain similar 'modular' cut-elimination proofs in other systems of rules. This general attempt is carried out for two more logics, namely the modal logic of serial frames and the intermediate logic axiomatised by the law of the weak excluded middle.

1 Introduction

Among the various proof frameworks used in the investigation of nonclassical logics, *systems of rules* as introduced by Negri [16] remain relatively little studied. Broadly speaking, a system of rules is a sequent-type calculus with a global correctness condition on the order in which rules may be applied; they form an instance of *higher-level rules* [20]. In [16], for example, it is shown that extending the sequent calculus for intuitionistic logic with the system of rules

$$\begin{array}{ccc} \dfrac{A, B, \Gamma_1 \Rightarrow \Pi_1}{A, \Gamma_1 \Rightarrow \Pi_1}\,(A,B)_L & & \dfrac{A, B, \Gamma_2 \Rightarrow \Pi_2}{B, \Gamma_2 \Rightarrow \Pi_2}\,(A,B)_R \\ \vdots & & \vdots \\ \Gamma \Rightarrow \Pi & & \Gamma \Rightarrow \Pi \\ \hline & \Gamma \Rightarrow \Pi & \end{array}\;(Lin)$$

yields a calculus for *Gödel Logic*, i.e. the extension of intuitionistic logic by the linearity axiom $(A \to B) \vee (B \to A)$. The schematic representation of the system above is understood as follows: Both rules $(A, B)_L$ and $(A, B)_R$ can be used in branches of the proof tree as long as those branches meet below in an instance of (Lin). By using such global conditions it is possible to capture analytically various logics that do not have a cutfree sequent calculus. For example, [16] develops systems of rules based on the labelled sequent calculus for all normal modal logics axiomatised by (generalised) Sahlqvist formulas. In [9] it is shown that proofs in the hypersequent calculus can be rewritten as particular systems of sequent rules, called *2-systems* (and vice versa). A different use of global conditions is shown in [1]: By replacing the (local) eigenvariable condition in

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first-order **LK**-proofs by a global condition, one obtains sound but potentially much shorter proofs.

The study of cut-elimination in systems of rules is in a rather unsatisfying stage. In [9] the analyticity of the systems of rules is obtained, but only indirectly via cut-elimination in the hypersequent calculus. [16] argues that a standard cut reduction argument goes through in the system of rules and illustrates one reduction step. As already remarked in [9], the argument seems to apply only to rules handling atomic formulas. This restriction is possible in the labelled sequent calculus but is too strong in an unlabelled system.

In the first part of this article we develop *grounded proofs*, a simple system of rules for the modal logics $\mathbf{KT}^{\Box}$ and **K5** of shift-reflexive and Euclidean frames respectively. These logics are of interest because their proof theory is less straightforward than that of other modal logics. In particular, neither shift-reflexivity nor Euclideanness is a *simple frame property* [13] which would guarantee the existence of a cutfree hypersequent calculus. The most elementary proof system for $\mathbf{KT}^{\Box}$ and **K5** seems to be the *grafted hypersequent calculus* of Lellmann and Kuznets [12]. Nested [7], prefixed tableaux [14] and labelled sequent calculi [15] are also available.

Our systems can be succinctly described as follows. For $\mathbf{KT}^{\Box}$, grounded proofs can make use of all rules of a sequent calculus for **KT**, with the proviso that every unsound modal rule has an instance of the rule (K) below it. For **K5**, grounded proofs can make use of all rules of a hypersequent calculus for **S5**, with the proviso that every unsound modal rule has an instance of the rule (MM) below it:

$$\frac{\Gamma \Rightarrow A}{\Box\Gamma \Rightarrow \Box A}\ (K) \qquad \frac{\Gamma_1 \Rightarrow A_1 \mid \ldots \mid \Gamma_n \Rightarrow A_n}{\Box\Gamma_1, \ldots, \Box\Gamma_n \Rightarrow \Box A_1, \ldots, \Box A_n}\ (MM)$$

It is a remarkable feature of both systems that their cutfree completeness can be proved *directly*, using only the deduction theorem and the cutfreeness of the (hyper)sequent calculi for **K**, **KT** and **S5**. With these ingredients the proof is almost trivial for $\mathbf{KT}^{\Box}$; for **K5** we additionally have to prove a combinatorial lemma about hypersequent derivations. In retrospect, grounded proofs can be seen as proofs in the grafted hypersequent calculus that satisfy a normal form. We make this observation precise by defining a translation from our system into the grafted hypersequent calculus, thereby obtaining a new (and arguably much simpler) proof of cut-elimination for the latter calculus.

In the second part of this article we explore the theme of *strongly modular proofs of cut-elimination*, i.e.: Proofs of cut-elimination that build on the cut-elimination property of adjacent logics (**K**, **KT** and **S5** in our example) but do not require knowledge about *how* cut-elimination for these systems was obtained. In other words, a proof of cut-elimination is strongly modular if it uses other cut-elimination theorems as 'blackboxes'. What is the scope of strongly modular proofs? We show that for many logics, strongly modular proofs of cut-elimination are possible in a simple sequent system with a global correctness condition called *revivability*. This condition however is defined only abstractly, and so the usefulness of said result depends on finding a simpler equivalent characterisation of revivability. We conclude by showing two examples where such a simple characterisation is possible: The modal logic **KD** of serial frames and the intermediate logic **LQ** axiomatised by the law of the weak excluded middle.

2 Preliminaries

Modal Logics. By a *modal logic* we mean any set of formulas in the language $\{\bot, \neg, \wedge, \vee, \rightarrow, \Box\}$ that contains all propositional tautologies, the normality axiom $\Box(p \rightarrow q) \rightarrow (\Box p \rightarrow \Box q)$, and is closed under uniform substitution, *Modus Ponens* (from A and $A \rightarrow B$ infer B) and *Necessitation* (from A infer $\Box A$).

The smallest modal logic (with respect to $\subseteq$) is $\mathbf{K}$. For any modal logic $\mathbf{L}$ and formula C, $\mathbf{L}+C$ denotes the smallest extension of $\mathbf{L}$ to a modal logic containing all instances of C. The table below lists some modal logics relevant to this paper, together with their corresponding frame condition (for proofs, see e.g. [5]).

modal logic	frame condition	first-order formula
$\mathbf{KT} := \mathbf{K} + \Box p \rightarrow p$	reflexive	$\forall x\, xRx$
$\mathbf{KT}^{\Box} := \mathbf{K} + \Box(\Box p \rightarrow p)$	shift-reflexive	$\forall x \forall y.\, xRy \rightarrow yRy$
$\mathbf{K5} := \mathbf{K} + \neg\Box p \rightarrow \Box\neg\Box p$	Euclidean	$\forall x \forall y \forall z.\, xRy \wedge xRz \rightarrow yRz$
$\mathbf{S5} := \mathbf{K5} + \Box p \rightarrow p$	totally connected	$\forall x \forall y.\, xRy$

The deduction theorem has to be slightly adapted for modal logics. We define $\Box^k A := \Box \dots \Box A$ (k boxes) for $k > 0$ and $\Box^0 A := A$. A *modalized instance of C* is any formula of the form $\Box^k C_0$ where C_0 is an instance of C and $k \geq 0$. Then:

Theorem 1 (essentially [10, Theorem 2]). *$A \in \mathbf{K} + C$ iff $(\bigwedge \Omega) \rightarrow A \in \mathbf{K}$ for some finite set Ω of modalized instances of C.*

Sequent Calculi. A *sequent* is a pair of finite multisets of formulas written $\Gamma \Rightarrow \Delta$. Its *formula interpretation* is $\bigwedge\Gamma \rightarrow \bigvee\Delta$ where $\bigwedge\emptyset := \neg\bot$ and $\bigvee\emptyset := \bot$. We say that a sequent is valid in a logic if its formula interpretation is.

The propositional rules in Fig. 1 constitute a calculus **LK** for classical propositional logic.[1] We obtain sequent calculi

- $\mathcal{C}_{\mathbf{K}}$ by adding the modal rule (K);
- $\mathcal{C}_{\mathbf{KT}}$ by adding both modal rules (K) and (T).

[1] The metavariables in Fig. 1 are chosen such that by enforcing $|\Pi| = 0$ and $|\Delta| \leq 1$ one obtains a calculus for intuitionistic logic. This will be used in Sect. 4.3.

$$\frac{}{p \Rightarrow p}\,(id) \quad \frac{\Gamma \Rightarrow \Pi}{\Sigma, \Gamma \Rightarrow \Pi, \Delta}\,(w) \quad \frac{\Sigma, \Sigma, \Gamma \Rightarrow \Delta, \Pi, \Pi}{\Sigma, \Gamma \Rightarrow \Delta, \Pi}\,(c) \quad \frac{\Gamma \Rightarrow A, \Pi \qquad \Gamma, A \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \Pi}\,(cut)$$

$$\frac{}{\Gamma, \bot \Rightarrow \Delta}\,(\bot_L) \quad \frac{\Gamma \Rightarrow A, \Pi}{\Gamma, \neg A \Rightarrow \Pi}\,(\neg_L) \quad \frac{\Gamma, A \Rightarrow \Pi}{\Gamma \Rightarrow \neg A, \Pi}\,(\neg_R) \quad \frac{\Gamma, A, B \Rightarrow \Delta}{\Gamma, A \wedge B \Rightarrow \Delta}\,(\wedge_L)$$

$$\frac{\Gamma \Rightarrow A, \Pi \qquad \Gamma \Rightarrow B, \Pi}{\Gamma \Rightarrow A \wedge B, \Pi}\,(\wedge_R) \quad \frac{\Gamma, A \Rightarrow \Delta \qquad \Gamma, B \Rightarrow \Delta}{\Gamma, A \vee B \Rightarrow \Delta}\,(\vee_L) \quad \frac{\Gamma \Rightarrow A_i, \Pi}{\Gamma \Rightarrow A_1 \wedge A_2, \Pi}\,(\vee_R)$$

$$\frac{\Gamma, B \Rightarrow \Delta \qquad \Gamma \Rightarrow A, \Pi}{\Gamma, A \rightarrow B \Rightarrow \Delta, \Pi}\,(\rightarrow_L) \quad \frac{\Gamma, A \Rightarrow B, \Delta}{\Gamma \Rightarrow A \rightarrow B, \Delta}\,(\rightarrow_R)$$

$$\frac{\Gamma \Rightarrow A}{\Box\Gamma \Rightarrow \Box A}\,(K) \quad \frac{\Gamma, A \Rightarrow \Delta}{\Gamma, \Box A \Rightarrow \Delta}\,(T)$$

$$\frac{\mathcal{H} \mid \Gamma, A \Rightarrow \Delta}{\mathcal{H} \mid \Box A \Rightarrow \mid \Gamma \Rightarrow \Delta}\,(\Box^5_L) \quad \frac{\mathcal{H} \mid \Rightarrow A}{\mathcal{H} \mid \Rightarrow \Box A}\,(\Box^5_R) \quad \frac{\Gamma_1 \Rightarrow A_1 \mid \ldots \mid \Gamma_n \Rightarrow A_n}{\Box\Gamma_1, \ldots, \Box\Gamma_n \Rightarrow \Box A_1, \ldots, \Box A_n}\,(MM)$$

$$\frac{\mathcal{H}}{\mathcal{H} \mid \Gamma \Rightarrow \Delta}\,(ew) \quad \frac{\mathcal{H} \mid \Gamma \Rightarrow \Delta \mid \Gamma \Rightarrow \Delta}{\mathcal{H} \mid \Gamma \Rightarrow \Delta}\,(ec)$$

Fig. 1. Propositional, modal and structural hypersequent rules.

Derivations in sequent calculi will be denoted by letters α, β. The formula A is said to be derivable in a sequent calculus if the sequent $\Rightarrow A$ is. A sequent calculus is called *adequate for a logic* if the formulas it derives are exactly the theorems of the logic. Finally, a proof in a sequent calculus is *cutfree* if it does not use the rule (*cut*), and a sequent calculus *admits cut-elimination* if every sequent provable in it has a cutfree proof. The following is folklore:

Theorem 2. *The calculi $\mathcal{C}_{\mathbf{K}}$ and $\mathcal{C}_{\mathbf{KT}}$ are adequate for the modal logics* **K** *and* **KT** *respectively and admit cut-elimination.*

3 Two Systems of Rules

The similarity of the modal logics $\mathbf{KT}^{\Box}$ and **K5** lies in the fact that they are both 'one step away' from their companion logics **KT** and **S5** respectively. That is, in any shift-reflexive (Euclidean) frame the subframe induced by all worlds reachable from some fixed world is reflexive (totally connected), and therefore adequate for **KT** (**S5**). We formalize this observation for later reference.

Theorem 3. *Let M be a Kripke model containing a world w, and let M_w be obtained from M by restricting M's frame to worlds that are reachable from w (using one or more steps) via the accessibility relation. Then:*

1. *$M, v \models \varphi \iff M_w, v \models \varphi$ for all worlds v in M_w and modal formulas φ;*
2. *If M is shift-reflexive, then M_w is reflexive;*
3. *If M is Euclidean, then M_w is totally connected.*

From this one can easily deduce the following known equivalences:

Theorem 4. $\Box A \in \mathbf{KT}^{\Box} \iff A \in \mathbf{KT}$ *and* $\Box A \in \mathbf{K5} \iff A \in \mathbf{S5}$.

Theorem 4 implies that we can use the sequent calculus $\mathcal{C}_{\mathbf{KT}}$ and the hypersequent calculus **HS5** (see Sect. 3.2) to derive formulas in the boxed fragment of $\mathbf{KT}^{\Box}$ and **K5**. But it is not immediate what Theorem 4 tells us about the proofs of theorems in $\mathbf{KT}^{\Box}$ and **K5** that are not prefixed with $\Box$, e.g. $\neg\Box p \to \Box\neg\Box p \in \mathbf{K5}$ or $\Box\Box p \to \Box p \in \mathbf{KT}^{\Box}$.

3.1 $\mathbf{KT}^{\Box}$

We start by describing a simple system of rules for $\mathbf{KT}^{\Box}$, which is obtained by imposing a global constraint on $\mathcal{C}_{\mathbf{KT}}$-proofs. The crucial notion is the following:

Definition 1 (grounded $\mathcal{C}_{\mathbf{KT}}$-proof). *A proof in $\mathcal{C}_{\mathbf{KT}}$ is* grounded *if any lowermost modal inference in it is (K).*

In other words, only those instances of (T) are admitted in a grounded $\mathcal{C}_{\mathbf{KT}}$-proof that have an instance of (K) below. No exact pairing is required, i.e. the same instance of (K) can 'ground' multiple instances of (T) above it. Figure 2 (left and middle) shows two grounded $\mathcal{C}_{\mathbf{KT}}$-proofs with the modal rules highlighted.

$$\dfrac{\dfrac{\dfrac{p \Rightarrow p}{\Box p \Rightarrow p}\,(T)}{\Rightarrow \Box p \to p}}{\Rightarrow \Box(\Box p \to p)}\,(K) \qquad \dfrac{\dfrac{\dfrac{\dfrac{p \Rightarrow p}{\Box p \Rightarrow p}\,(T) \quad \dfrac{p \Rightarrow p}{\Box p \Rightarrow p}\,(T)}{\Box p \lor \Box p \Rightarrow p}}{\Box(\Box p \lor \Box p) \Rightarrow \Box p}\,(K)}{\Rightarrow \Box(\Box p \lor \Box p) \to \Box p} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{p \Rightarrow p}{\Rightarrow p \mid \Box p \Rightarrow}\,(\Box^5_L)}{\Rightarrow p \mid \Rightarrow \neg\Box p}}{\Rightarrow \Box p, \Box\neg\Box p}\,(MM)}{\neg\Box p \Rightarrow \Box\neg\Box p}}{\Rightarrow \neg\Box p \to \Box\neg\Box p}$$

Fig. 2. Grounded proofs in **KT** (left and middle) and in **HS5** (right)

Theorem 5 (Soundness of grounded $\mathcal{C}_{\mathbf{KT}}$-proofs). *If there is a grounded $\mathcal{C}_{\mathbf{KT}}$-proof of $\Gamma \Rightarrow \Delta$, then $\Gamma \Rightarrow \Delta$ is valid in $\mathbf{KT}^{\Box}$.*

Proof. It suffices to show that the conclusion of an instance of (K) in a $\mathcal{C}_{\mathbf{KT}}$-proof is valid in $\mathbf{KT}^{\Box}$. Indeed, as the endsequent of a grounded $\mathcal{C}_{\mathbf{KT}}$-proof is derivable from the conclusions of its lowermost instances of (K) using only propositional rules, it then follows that the endsequent is valid in $\mathbf{KT}^{\Box}$ as well.[2] So let

$$\dfrac{\Gamma \Rightarrow A}{\Box\Gamma \Rightarrow \Box A}\,(K)$$

[2] Note that if a grounded proof has no instances of (K) at all, then it is essentially a propositional proof, and so the statement is trivial.

be such an instance. As its premise $\Gamma \Rightarrow A$ is valid in **KT**, we can use the deduction theorem (Theorem 1) to obtain a finite set Ω of modalized instances of the reflexivity axiom $\Box p \to p$ such that the sequent $\Omega, \Gamma \Rightarrow A$ is valid in **K**. Then, by (K), also $\Box\Omega, \Box\Gamma \Rightarrow \Box A$ is valid in **K**. As all formulas in $\Box\Omega$ are modalized instances of the axiom of shift-reflexivity and therefore valid in $\mathbf{KT}^{\Box}$, it follows that the reduced sequent $\Box\Gamma \Rightarrow \Box A$ is valid in $\mathbf{KT}^{\Box}$. □

Theorem 6 (Cutfree completeness of grounded $\mathcal{C}_{\mathbf{KT}}$-proofs). *If $\Gamma \Rightarrow \Delta$ is valid in $\mathbf{KT}^{\Box}$, then there is a grounded cutfree $\mathcal{C}_{\mathbf{KT}}$-proof of it.*

Proof. Let $\Gamma \Rightarrow \Delta$ be valid in $\mathbf{KT}^{\Box}$. By the deduction theorem there is a finite set Ω of modalized instances of $\Box(\Box p \to p)$ such that $\Omega, \Gamma \Rightarrow \Delta$ is valid in **K**. We may write Ω as $\Box\Omega'$, where Ω' is now a set of modalized instances of $\Box p \to p$.

Consider a lowermost instance of (K) in a cutfree $\mathcal{C}_{\mathbf{K}}$-proof α of $\Omega, \Gamma \Rightarrow \Delta$:

$$\frac{\Omega', \Sigma \Rightarrow A}{\Box\Omega', \Box\Sigma \Rightarrow \Box A}\,(K)$$

Here we assume harmlessly that $\Box\Omega'$ in the conclusion of (K) contains exactly the antecessors of $\Omega = \Box\Omega'$ in the endsequent, i.e. no contraction or weakening has been applied to a formula in $\Box\Omega'$ between this instance of (K) and the endsequent. We now construct a cutfree grounded proof as follows. In α, replace the proof of the premise (for all lowermost (K) simultaneously) with a cutfree $\mathcal{C}_{\mathbf{KT}}$-proof of $\Sigma \Rightarrow A$; this is possible as every formula in Ω' is valid in **KT**, and moreover **KT** admits cut-elimination. Apply (K) to obtain the sequent $\Box\Sigma \Rightarrow \Box A$, and now follow the original proof downwards while removing antecessors of $\Box\Omega'$ to eventually obtain $\Gamma \Rightarrow \Delta$. □

3.2 K5

The system of rules for **K5** will involve a hypersequent calculus for **S5**, so we first introduce some notation. A *hypersequent* is a multiset of sequents written $\Gamma_1 \Rightarrow \Delta_1 \mid \ldots \mid \Gamma_n \Rightarrow \Delta_n$ and its (modal) formula interpretation is $\Box(\wedge\Gamma_1 \to \vee\Delta_1) \vee \ldots \vee \Box(\wedge\Gamma_1 \to \vee\Delta_1)$. We say that a hypersequent is valid in a logic if its formula interpretation is.

There are now two ways of assigning a formula to $\Gamma \Rightarrow \Delta$, namely $\Box(\wedge\Gamma \to \vee\Delta)$ "boxed" or $\wedge\Gamma \to \vee\Delta$ "flat", depending on whether we treat $\Gamma \Rightarrow \Delta$ as a one-component hypersequent or as a sequent. To avoid any ambiguity, we will explicitly say in this section that $\Gamma \Rightarrow \Delta$ is *flat-valid in a logic* **L** if $\wedge\Gamma \to \vee\Delta \in \mathbf{L}$. Otherwise, by validity of a hypersequent (possibly with only one component) we always mean the boxed interpretation above. In any modal logic $\mathbf{L} \supseteq \mathbf{KT}$ (so in particular, **S5**) we have the equivalence $A \in \mathbf{L} \iff \Box A \in \mathbf{L}$ and so the notions of valid and flat-valid coincide on sequents. However, we will work in **K5** where such an equivalence does not apply.

Definition 2. *The rules of the hypersequent calculus* **HS5** *are as follows:*

– *Any rule of* **LK***, applied componentwise in a hypersequent;*
– *Additionally, we have rules* (ew) *and* (ec)*, the modal rules* $(\Box^5_L), (\Box^5_R)$ *(see Fig. 1) and the* modal merging rule (MM)*:*

$$\frac{\Gamma_1 \Rightarrow A_1 \mid \ldots \mid \Gamma_n \Rightarrow A_n}{\Box\Gamma_1, \ldots, \Box\Gamma_n \Rightarrow \Box A_1, \ldots, \Box A_n}\,(MM)$$

There are a number of slightly different hypersequent calculi for **S5** (see the survey [3]) and any of these would be suitable for the system of rules we define below. We use a variant due to Restall [18] as this calculus underlies the grafted hypersequent calculus in [12] to which we later relate.

The only change from [18] is that we include the rule (MM). While being redundant—(MM) is derivable from $(\Box^5_L)$ and $(\Box^5_R)$—it will be useful to formulate the system of rules. Note that (MM) has no hypersequent context and so its conclusion is always a sequent. For $n = 1$ the rule coincides with (K).

Theorem 7 ([18]). **HS5** *is adequate for* **S5** *and admits cut-elimination.*

Definition 3. *A proof in* **HS5** *is* grounded *if every lowermost modal rule in it is* (MM)*.*

Figure 2 (right) shows a grounded **HS5**-proof of the characteristic **K5**-axiom. While it is formally possible due to (ew) and (ec) that hypersequents with more than one component appear in the lower part of a grounded **HS5**-proof, it is easy to see that this is never necessary. We will therefore tacitly assume that Definition 3 is extended by the clause: *. . . and every hypersequent that is not above an instance of* (MM) *has exactly one component.* The following Lemma will give us the soundness of grounded **HS5**-proofs.

Lemma 1. *If the premise of an instance of* (MM) *is valid in* **S5***, then its conclusion is flat-valid in* **K5***.*

Proof. Assume contrapositively the conclusion $\Box\Gamma_1, \ldots, \Box\Gamma_n \Rightarrow \Box A_1, \ldots, \Box A_n$ is not flat-valid in **K5**. Then $(\wedge_{i\leq n}\Box\Gamma_i) \rightarrow (\vee_{i\leq n}\Box A_i)$ fails at a world w of an Euclidean model M. In particular, there are worlds $v_1, \ldots, v_n$ accessible from w such that v_i satisfies every formula in Γ_i but falsifies A_i. Now we use Theorem 3. Pick an arbitrary world v in M_w (say, v_1). As M_w is totally connected, every world $v_1, \ldots, v_n$ is accessible from v. Hence $\Box(\wedge\Gamma_i \rightarrow A_i)$ fails at v for every $i \leq n$, and consequently so does $\vee_{i\leq n}\Box(\wedge\Gamma_i \rightarrow A_i)$, which is the (boxed) interpretation of the premise $\Gamma_1 \Rightarrow A_1 \mid \ldots \mid \Gamma_n \Rightarrow A_n$ of (MM). Since M_w is totally connected, it follows that this hypersequent is not valid in **S5**. □

Theorem 8 (soundness of grounded HS5-proofs). *If there is a grounded* **HS5***-proof of* $\Gamma \Rightarrow \Delta$*, then* $\Gamma \Rightarrow \Delta$ *is flat-valid in* **K5***.*

Proof. Similar to the proof of Theorem 5. The endsequent $\Gamma \Rightarrow \Delta$ of a grounded proof is derivable from the conclusions of instances of (MM) using only propositional inferences. As these conclusions are flat-valid in **S5** by Lemma 1, the same follows[3] for $\Gamma \Rightarrow \Delta$. □

[3] Note that propositional rules preserve both validity and flat-validity.

We now turn to the cutfree completeness of grounded **HS5**-proofs. This will again be derived from the deduction theorem and cut-elimination for $\mathcal{C}_{\mathbf{K}}$ and **HS5**. The situation in **K5** is more complicated than in $\mathbf{KT}^{\Box}$ for the following reason: The outermost connective of the axiom $\Box(\Box p \to p)$ is a $\Box$, and thus the first (read bottom-up) rule that will be applied to it when used as an assumption in a $\mathcal{C}_{\mathbf{K}}$-proof is (K), i.e. the very rule that separates the top from the bottom part in our system of rules. In contrast, the outermost connective of $\neg\Box p \to \Box\neg\Box p$ is $\to$. So if we follow an occurrence of the axiom upwards in the proof, it will first be split into two different parts $\Box p$ and $\Box\neg\Box p$ via $(\to_L)$ and $(\neg_R)$ that only later encounter a modal rule. Thus at the part of the proof where we want to introduce the rule (MM) to obtain a system of rules, the constituent formulas of the axiom instances have been scattered among the branches of the $\mathcal{C}_{\mathbf{K}}$-proof. In a first step, we use the hypersequent structure to bring these scattered axiom parts back together.

Lemma 2. *The following rule is admissible in* **S5***:*

$$\frac{\mathcal{H} \mid C, \Gamma_1 \Rightarrow \Delta_1 \qquad \mathcal{H} \mid \neg\Box C, \Gamma_2 \Rightarrow \Delta_2}{\mathcal{H} \mid \Gamma_1 \Rightarrow \Delta_1 \mid \Gamma_2 \Rightarrow \Delta_2}$$

Proof. The rule can easily shown to be sound using the Kripke semantics of **S5**. It can also be derived from the generalised rule for cuts on boxed formulas that Avron uses in his proof [2] of cut-elimination for **S5**. ☐

$$\dfrac{\dfrac{\dfrac{C, \Gamma_1 \Rightarrow A_1}{\Box C, \Box\Gamma_1 \Rightarrow \Box A_1}(K)}{\vdots\,\alpha_1 \atop \Box C, \Gamma \Rightarrow \Delta} \qquad \dfrac{\dfrac{\neg\Box C, \Gamma_2 \Rightarrow A_2}{\Box\neg\Box C, \Box\Gamma_2 \Rightarrow \Box A_2}(K)}{\vdots\,\alpha_2 \atop \Box\neg\Box C, \Gamma \Rightarrow \Delta}}{\neg\Box C \to \Box\neg\Box C, \Gamma \Rightarrow \Delta}(\to_L), (\neg_R) \quad \rightsquigarrow \quad \dfrac{\begin{array}{c} \text{Lemma 2} \\ \dfrac{\Gamma_1 \Rightarrow A_1 \mid \Gamma_2 \Rightarrow A_2}{\Box\Gamma_1, \Box\Gamma_2 \Rightarrow \Box A_1, \Box A_2}(MM) \\ \vdots\,\alpha_1 \\ \Gamma, \Box\Gamma_2 \Rightarrow \Delta, \Box A_2 \\ \vdots\,\alpha_2 \\ \Gamma, \Gamma \Rightarrow \Delta, \Delta \end{array}}{\Gamma \Rightarrow \Delta}$$

Fig. 3. Constructing a grounded **HS5**-proof

At this point we can already illustrate how the grounded **HS5**-proof will be constructed in a very simple case—see Fig. 3. Here we start from a cutfree $\mathcal{C}_{\mathbf{K}}$-proof using only a single non-modalized axiom instance $\neg\Box C \to \Box\neg\Box C$. After breaking up the axiom into two parts $\Box C$ and $\Box\neg\Box C$ using invertible rules, both parts are traced upwards in their respective branch α_1 and α_2 until they are principal in an inference of (K). Then both premises of (K) are rejoined using Lemma 2 into a single hypersequent, thereby eliminating the axiom parts. Below this hypersequent we can simulate both proofs α_1, α_2 (this time omitting the axiom parts) to arrive at the desired $\Gamma \Rightarrow \Delta$.

To deal with the general case, we need to extend Lemma 2. For this we introduce some notation: Given an index set $I = \{1, \dots, n\}$ we write $\Gamma, \{C_i\}_{i \in I} \Rightarrow \Delta$ for the sequent $\Gamma, C_1, \dots, C_n \Rightarrow \Delta$, and $\mathcal{H} \mid [\Gamma_i \Rightarrow \Delta_i]_{i \in I}$ for the hypersequent $\mathcal{H} \mid \Gamma_1 \Rightarrow \Delta_1 \mid \dots \mid \Gamma_n \Rightarrow \Delta_n$.

Lemma 3. *Let $\{C_i \mid i \in I\}$ be a set of formulas. If the hypersequent*

$$\mathcal{H} \mid \{C_j\}_{j \in J}, \{\neg\Box C_k\}_{k \in I \setminus J}, \Gamma_J \Rightarrow \Delta_J$$

is valid in **S5** *for all $J \subseteq I$, then so is $\mathcal{H} \mid [\Gamma_J \Rightarrow \Delta_J]_{J \subseteq I}$.*

Proof. By induction on $|I|$. For $I = \emptyset$ the statement is trivial. Thus let $i_0 \in I$. For $J \subseteq I$ we call S_J the hypersequent

$$\mathcal{H} \mid \{C_j\}_{j \in J}, \{\neg\Box C_k\}_{k \in I \setminus J}, \Gamma_J \Rightarrow \Delta_J.$$

For any $J \subseteq I$ with $i_0 \in J$ and $L \subseteq (I \setminus \{i_0\})$ we apply Lemma 2 (with $C := C_{i_0}$) to S_J and S_L obtaining

$$\mathcal{H} \mid \{C_j\}_{j \in J \setminus \{i_0\}}, \{\neg\Box C_k\}_{k \in I \setminus J}, \Gamma_J \Rightarrow \Delta_J \mid \{C_l\}_{l \in L}, \{\neg\Box C_m\}_{m \in (I \setminus \{i_0\}) \setminus L}, \Gamma_L \Rightarrow \Delta_L$$

Call S^*_J the component with right hand side Δ_J. Keeping J fixed while letting $L \subseteq (I \setminus \{i_0\})$ vary, we can use the induction hypothesis to obtain the hypersequent

$$\mathcal{H} \mid S^*_J \mid [\Gamma_L \Rightarrow \Delta_L]_{L \subseteq I \setminus \{i_0\}}.$$

By another application of the induction hypothesis, now letting J vary across subsets of I containing i_0 (in other words: letting J' vary across subsets of $I \setminus \{i_0\}$ and setting $J := J' \cup \{i_0\}$), we obtain

$$\mathcal{H} \mid [\Gamma_J \Rightarrow \Delta_J]_{J \subseteq I, i_0 \in J} \mid [\Gamma_L \Rightarrow \Delta_L]_{L \subseteq I \setminus \{i_0\}}$$

i.e. $\mathcal{H} \mid [\Gamma_J \Rightarrow \Delta_J]_{J \subseteq I}$. □

Note that Lemma 2 is the instance of Lemma 3 where $|I| = 1$. We can now prove the completeness theorem.

Theorem 9 (Cutfree completeness of grounded HS5-proofs). *If $\Gamma \Rightarrow \Delta$ is flat-valid in* **K5***, then there is a cutfree grounded* **HS5***-proof of it.*

Proof. Let $\Gamma \Rightarrow \Delta$ by flat-valid in **K5**. By the deduction theorem, there is a set Ω of modalized instances of $\neg\Box p \to \Box\neg\Box p$ such that $\Omega, \Gamma \Rightarrow \Delta$ is flat-valid in **K**, and therefore has a cutfree $\mathcal{C}_{\mathbf{K}}$-proof α. We can write Ω as $\Box\Omega_1 \cup \{\Box C_i \to \Box\neg\Box C_i\}_{i \in I}$ where $\Box\Omega_1$ contains modalized instances of the axiom with at least one box. By standard invertibility results in $\mathcal{C}_{\mathbf{K}}$, we may assume that the lowermost inferences in α are $(\to_L)$ and $(\neg_R)$ applied to all axioms $\neg\Box C_i \to \Box\neg\Box C_i$. In this way, we obtain $2^{|I|}$-many premises, which can succinctly be described as follows: For every $J \subseteq I$, we have a premise T_J containing the (negated) antecedents of all axioms with index $j \in J$ and the consequents of all other axioms, i.e.

$$T_J := \Box\Omega_1, \{\Box C_j\}_{j \in J}, \{\Box\neg\Box C_k\}_{k \in I \setminus J}, \Gamma \Rightarrow \Delta.$$

We now fix cutfree $\mathcal{C}_{\mathbf{K}}$-proofs α_J of T_J for every $J \subseteq I$. Letting P_J denote the number of lowermost inferences of (K) in α_J, we enumerate them as

$$\frac{\Omega_1, \{C_j\}_{j\in J}, \{\neg\Box C_k\}_{k\in I\setminus J}, \Gamma_J^p \Rightarrow A_J^p}{\Box\Omega_1, \{\Box C_j\}_{j\in J}, \{\Box\neg\Box C_k\}_{k\in I\setminus J}, \Box\Gamma_J^p \Rightarrow \Box A_J^p}\ (K)_J^p$$

where $0 < p \leq P_J$. Once again we assume harmlessly that the modalized axiom instances and their parts in the antecedent have not been subject to contraction or weakening. Let us assume moreover that $P_J \neq 0$ for all $J \subseteq I$, i.e. there is at least one instance of (K) in every α_J, as the other case is very simple.[4]

As the premise of $(K)_J^p$ is flat-valid in $\mathbf{K}$ and every formula in Ω_1 is valid in $\mathbf{S5}$, it follows that the sequent

$$S_J^p := \{C_j\}_{j\in J}, \{\neg\Box C_k\}_{k\in I\setminus J}, \Gamma_J^p \Rightarrow A_J^p$$

is flat-valid, and therefore also valid, in $\mathbf{S5}$. Define $\mathcal{F} := \{f : \mathcal{P}(I) \to \mathbb{N} \mid 0 < f(J) \leq P_J\}$ and fix one $f \in \mathcal{F}$. We think of f as choosing one specific lowermost instances $(K)_J^{f(J)}$ in every α_J. The family $\{S_J^{f(J)}\}_{J\subseteq I}$ is such that Lemma 3 is applicable to it, and therefore the following hypersequent is valid in $\mathbf{S5}$:

$$\mathcal{H}^f := [\Gamma_J^{f(J)} \Rightarrow A_J^{f(J)}]_{J\subseteq I}$$

We now construct the grounded $\mathbf{HS5}$-proof. Fix cutfree $\mathbf{HS5}$-proofs β^f of $\mathcal{H}^f$ for every $f \in \mathcal{F}$. Below each β^f apply (MM) to obtain the sequent

$$\{\Box\Gamma_J^{f(J)}\}_{J\subseteq I} \Rightarrow \{\Box A_J^{f(J)}\}_{J\subseteq I}.$$

Letting $J_1, J_2, \ldots$ be an enumeration of $\mathcal{P}(I)$, we focus on the subfamily of sequents

$$\{\Box\Gamma_J^{f(J)}\}_{J\subseteq I, J\neq J_1}, \Box\Gamma_{J_1}^p \Rightarrow \Box A_{J_1}^p, \{\Box A_J^{f(J)}\}_{J\subseteq I, J\neq J_1}$$

for fixed $f \in \mathcal{F}$ and varying $0 < p \leq P_{J_1}$. In other words, we consider all possible values of f on J_1 while keeping the other values fixed. Now observe that these P_{J_1}-many sequents look similar to the conclusions of the instances $(K)_{J_1}^p$ where $0 < p \leq P_{J_1}$, only that the axiom parts have been replaced. We can therefore simulate[5] the proof α_{J_1} below these sequents obtaining

$$\{\Box\Gamma_J^{f(J)}\}_{J\subseteq I, J\neq J_1}, \Gamma \Rightarrow \Delta, \{\Box A_J^{f(J)}\}_{J\subseteq I, J\neq J_1}$$

instead of the original endsequent T_J of α_{J_1}. Starting from this new family of sequents (for all $f \in \mathcal{F}$), we can repeat the above steps, simulating the proofs α_{J_2}, α_{J_3}, $\alpha_{J_4} \ldots$ until we eventually arrive at the sequent $\Gamma, \ldots, \Gamma \Rightarrow \Delta, \ldots, \Delta$ from which we then obtain $\Gamma \Rightarrow \Delta$ by contraction. $\Box$

[4] Assume (K) is never applied in α_J. Then no modal formula is ever principal in α_J (note here that modal formulas do not appear in initial sequents, which we require to be atomic). It is then easy to see that the modal formulas in the conclusion of α_J can simply be removed to obtain a (still cutfree) $\mathcal{C}_{\mathbf{K}}$-proof of $\Gamma \Rightarrow \Delta$. This proves the theorem, as a cutfree $\mathcal{C}_{\mathbf{K}}$-proof is also a cutfree grounded $\mathbf{HS5}$-proof.

[5] Note that α_{J_1} has only propositional inferences below $(K)_{J_1}^p$, so we do not have to worry about the changed contexts breaking some instance of (K).

3.3 Grounded Proofs and Grafted Hypersequents

In [12] calculi for the logics $\mathbf{KT}^{\Box}$ and **K5** are defined. These build on the notion of a *grafted hypersequent* $\Gamma \Rightarrow \Delta \mathrel{||} \Sigma_1 \Rightarrow \Delta_1 \mid \ldots \mid \Sigma_n \Rightarrow \Delta_n$ consisting of a sequent $\Gamma \Rightarrow \Delta$ called the *trunk* and a hypersequent $\Sigma_1 \Rightarrow \Delta_1 \mid \ldots \mid \Sigma_n \Rightarrow \Delta_n$ called the *crown*. If the crown is empty, we write $\Gamma \Rightarrow \Delta$ instead of $\Gamma \Rightarrow \Delta \mathrel{||}$. A grafted hypersequent corresponds to the modal formula $(\bigwedge\Gamma \rightarrow \bigvee\Delta) \vee \bigvee_{i=1}^{n}\Box(\bigwedge\Sigma_i \rightarrow \bigvee\Delta_i)$, i.e. one combines the flat interpretation of the trunk with the boxed interpretation of the crown. As pointed out in [12], grafted hypersequents are a restricted form of *nested sequents*.

We can now compare our systems of grounded proofs with the calculi in [12]. Let us first consider the grafted hypersequent calculus $\mathcal{R}_{\mathbf{K5}}$ for **K5**. We refer to [12, Figs. 1 and 2] for a complete list of the rules. The following presentation should suffice for our purposes:

- The *trunk rules* are the rules of **LK** applied to the trunk, the crown remaining unchanged;
- The *crown rules* are the rules of $\mathbf{HS5} \setminus \{(MM)\}$ applied to the crown, where it is required that the trunk is the empty sequent $\Rightarrow$;
- Two *transfer rules* mediate between the trunk and the crown:

$$\frac{\Gamma \Rightarrow \Delta \mathrel{||} \mathcal{H} \mid \Rightarrow A}{\Gamma \Rightarrow \Delta, \Box A \mathrel{||} \mathcal{H}}\,(\Box_R) \qquad \frac{\Gamma \Rightarrow \Delta \mathrel{||} \mathcal{H} \mid \Sigma, A \Rightarrow \Pi}{\Gamma, \Box A \Rightarrow \Delta \mathrel{||} \mathcal{H} \mid \Sigma \Rightarrow \Pi}\,(\Box_L)$$

A grounded **HS5**-proof can be translated into a proof in $\mathcal{R}_{\mathbf{K5}}$ as follows:

1. Replace every non-lowermost (MM) by its derivation via $(\Box_L^5)$ and $(\Box_R^5)$.
2. Replace every hypersequent $\mathcal{H}$ above some instance of (MM) by $\Rightarrow \mathrel{||} \mathcal{H}$.
3. Replace every lowermost (MM)-inference by transfer rules as shown below:

$$\frac{\Gamma_1 \Rightarrow A_1 \mid \ldots \mid \Gamma_n \Rightarrow A_n}{\Box\Gamma_1, \ldots, \Box\Gamma_n \Rightarrow \Box A_1, \ldots, \Box A_n} \rightsquigarrow \frac{\dfrac{\Rightarrow \mathrel{||} \Gamma_1 \Rightarrow A_1 \mid \ldots \mid \Gamma_n \Rightarrow A_n}{\Box\Gamma_1, \ldots, \Box\Gamma_n \Rightarrow \mathrel{||} \Rightarrow A_1 \mid \ldots \mid \Rightarrow A_n}\text{ some }(\Box_L)\text{'s}}{\Box\Gamma_1, \ldots, \Box\Gamma_n \Rightarrow \Box A_1, \ldots, \Box A_n}\text{ some }(\Box_R)\text{'s}$$

The grafted hypersequent calculus $\mathcal{R}_{\mathbf{KT}^{\Box}}$ for the logic of shift-reflexive frames is defined similarly; here it is only componentwise applications of $\mathcal{C}_{\mathbf{KT}}$-rules that are admitted in the crown (it follows that one only needs crowns with one component). An analogous translation from grounded $\mathcal{C}_{\mathbf{KT}}$-proofs to $\mathcal{R}_{\mathbf{KT}^{\Box}}$ can be defined. The translated proofs satisfy a normal form that already appears in [12, see Def. 4.3].

As the translation described above does not introduce cuts, and as there are cutfree grounded proofs for all theorems of $\mathbf{KT}^{\Box}$ (Theorem 6) and **K5** (Theorem 8), we immediately obtain a new proof of the following (first established in [12] via a syntactic reduction procedure):

Theorem 10. *$\mathcal{R}_{\mathbf{K5}}$ and $\mathcal{R}_{\mathbf{KT}^{\Box}}$ admit cut elimination.*

4 Strongly Modular Proofs of Cut-Elimination

The method of the previous section can be summarized as follows: Aiming to show $\Gamma \Rightarrow \Delta$ in an extended system ($\mathbf{KT}^{\Box}$ or $\mathbf{S5}$), we start from a cutfree $\mathcal{C}_{\mathbf{K}}$-proof α of $\Omega, \Gamma \Rightarrow \Delta$ for some (modularized) axiom instances Ω of the extended logic. Then we inspect α and replace some parts of it with cutfree proofs in $\mathcal{C}_{\mathbf{KT}}$ or **HS5**, this way getting rid of the axiom instance in Ω and thereby obtaining a cutfree 'grounded' proof of $\Gamma \Rightarrow \Delta$.

We emphasize the following: *At no point in the argument one needed to understand how cut-elimination for* $\mathcal{C}_{\mathbf{K}}, \mathcal{C}_{\mathbf{KT}}$ *and* **HS5** *is established.* In other words, these cut-elimination results are used as 'blackboxes' in the proof. Let us introduce the following informal terminology: A proof of cut-elimination is

- *weakly modular* if it is obtained by modifying or extending the cut-elimination proof of some other logic;
- *strongly modular* if it is obtained by using the cut-elimination property of some other logic, irrespective of how this property was obtained.

Our proofs of Theorem 6 and Theorem 9 are strongly modular in this sense. We are not aware of other such proofs in the literature.[6] On the other hand, weakly modular proofs are numerous: One might for example argue for cut-elimination in $\mathcal{C}_{\mathbf{KT}}$ by describing how the reduction steps in the cut-elimination algorithm for $\mathcal{C}_{\mathbf{K}}$ have to be extended to accommodate the additional rule (T).[7] The disadvantage of this approach is of course that the reader has to know the algorithm for $\mathcal{C}_{\mathbf{K}}$. If such a proof were to be formalised, one would have to copy and extend the complete formalisation of the proof for $\mathcal{C}_{\mathbf{K}}$, instead of using $\mathcal{C}_{\mathbf{K}}$'s already established cut-elimination as a lemma in the formalised proof for $\mathcal{C}_{\mathbf{KT}}$. The most successful attempts at modularity in cut-elimination have been proofs that are parametrized over a specific class of axioms or rules (e.g. [4,8,13,17]).

We believe strongly modular proofs of cut-elimination are interesting and deserve further study. They have the potential of being both shorter[8] and more reliable through the reuse of already established theorems. Moreover, given the general significance of cut-elimination, any method for obtaining it is important.

Of course, with only two[9] examples at hand there is the possibility that we have encountered a 'happy coincidence' rather than a general idea. Indeed the situation of $\mathbf{KT}^{\Box}$ and $\mathbf{K5}$ is quite special in that they are sandwiched between logics with cutfree calculi, i.e. $\mathbf{K} \subseteq \mathbf{KT}^{\Box} \subseteq \mathbf{KT}$ and $\mathbf{K} \subseteq \mathbf{K5} \subseteq \mathbf{S5}$, and the gap to the 'upper logic' **KT** or **S5** is very small in a precise sense (Theorem 4).

In the remainder of this article we sketch an idea that could be useful for obtaining strongly modular proofs of cut-elimination for other logics. We conduct

[6] We do not count proofs using cutfreeness of another calculus for the *same* logic, or a conservative extension thereof.

[7] Also, a *weakly modular* proof of cut-elimination for grounded **KT**-proofs is obtained by observing that all reduction steps in $\mathcal{C}_{\mathbf{KT}}$'s cut-elimination preserve groundedness.

[8] E.g., compare our proof for **K5** with the one in the grafted hypersequent calculus [12].

[9] Side remark: The result for $\mathbf{KT}^{\Box}$ also applies to all modal logics $\mathbf{K} + \Box C$ where $\mathbf{K} + C$ has a cutfree calculus.

the discussion in a semi-formal style. While there will not be enough evidence for a ‘general method’, we do present two further examples where a strongly modular proof is possible: The modal logic **KD** (using cut-elimination in **K**) and the intermediate logic **LQ** (using cut-elimination in intuitionistic logic).

4.1 Calculi with Ghost Rules

We start from the general situation that $\mathbf{L} \subseteq \mathbf{M}$ where $\mathbf{L}$ is some logic with a cutfree sequent calculus $\mathcal{C}_{\mathbf{L}}$. We seek a calculus for $\mathbf{M}$ that admits a strongly modular proof of cut-elimination, relative to cut-elimination in $\mathcal{C}_{\mathbf{L}}$. We additionally assume that a deduction theorem holds between $\mathbf{L}$ and $\mathbf{M}$. That is, a sequent $\Gamma \Rightarrow \Delta$ is valid in $\mathbf{M}$ iff $\Omega, \Gamma \Rightarrow \Delta$ is valid (and therefore cutfree provable) in $\mathbf{L}$ for a suitable set of formulas Ω.

Our proofs of the completeness theorems (Theorems 6 and 9) suggest that we should attempt to construct a cutfree $\mathbf{M}$-proof of $\Gamma \Rightarrow \Delta$ by somehow transforming a cutfree $\mathcal{C}_{\mathbf{L}}$-proof α of $\Omega, \Gamma \Rightarrow \Delta$. Now one naive transformation might immediately spring to mind: Can we simply take α and remove all occurrences of Ω and its ancestors in α to obtain a cutfree proof $\alpha^\dagger$ of $\Gamma \Rightarrow \Delta$?

The first question then is, in what system does $\alpha^\dagger$ qualify as a proof? Clearly removing formulas from inferences in $\mathcal{C}_{\mathbf{L}}$ creates unsound rules. In a first step, we therefore extend $\mathcal{C}_{\mathbf{L}}$ with *‘ghost rules’*: These are rules in which the principal formula in the conclusion and its ancestors in the premises have been removed. For examples, the ghost rules corresponding to $(\wedge_R)$ and (K) are

$$\frac{\Gamma \Rightarrow \Delta \qquad \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\,(\wedge_R)^\dagger \quad \text{and} \quad \frac{\Gamma \Rightarrow}{\Box\Gamma \Rightarrow}\,(K)^\dagger.$$

Different rules can have the same ghost rules, e.g. $(\wedge_R)^\dagger = (\vee_L)^\dagger$. Some ghost rules, e.g. $(\wedge_L)^\dagger$, are ‘dummy inferences’ $\Gamma \Rightarrow \Delta / \Gamma \Rightarrow \Delta$ that we do not add to the system. If $\mathcal{C}_{\mathbf{L}}$ has initial sequents $p \Rightarrow p$ then one or both occurrences of p can be ancestors of Ω, and thus we need different ghost initial sequents:

$$\frac{}{\Rightarrow p}\,(* \Rightarrow)^\dagger \qquad \frac{}{p \Rightarrow}\,(\Rightarrow *)^\dagger \qquad \frac{}{\Rightarrow}\,(* \Rightarrow *)^\dagger$$

Letting $\mathcal{C}_{\mathbf{L}}^\dagger$ denote the calculus extended by such ghost inferences we see that $\alpha^\dagger$ is (up to dummy inferences) a cutfree $\mathcal{C}_{\mathbf{L}}^\dagger$-proof of $\Gamma \Rightarrow \Delta$. More generally we infer from the deduction theorem that every sequent valid in $\mathbf{M}$ has a cutfree proof in $\mathcal{C}_{\mathbf{L}}^\dagger$. But of course, $\mathcal{C}_{\mathbf{L}}^\dagger$ also has many derivations which do not correspond to proofs in $\mathbf{M}$.

Definition 4. *A class* $\mathbb{P}$ *of* $\mathcal{C}_{\mathbf{L}}^\dagger$*-proofs is* cutfree-adequate for $\mathbf{M}$ *if the endsequent of every* $\mathbb{P}$*-proof is valid in* $\mathbf{M}$ *(‘soundness’) and there is a cutfree* $\mathbb{P}$*-proof of every* $\mathbf{M}$*-valid sequent (‘completeness’).*

Let us informally call $\mathbf{M}$-*revivable* a $\mathcal{C}_{\mathbf{L}}^\dagger$-proof of $\Gamma \Rightarrow \Delta$ if we can insert formulas and inferences into it to obtain a $\mathcal{C}_{\mathbf{L}}$-proof of $\Omega, \Gamma \Rightarrow \Delta$, where Ω is a set of $\mathbf{M}$-valid formulas. The proof $\alpha^\dagger$ from the above discussion is the typical example of an $\mathbf{M}$-revivable proof.

By the deduction theorem and cut-elimination in $\mathcal{C}_{\mathbf{L}}$ it follows that the **M**-revivable proofs in $\mathcal{C}_{\mathbf{L}}^{\dagger}$ form a cutfree-adequate class for **M**.[10] So what we have obtained is indeed a strongly modular proof of cut-elimination for the system of **M**-revivable $\mathcal{C}_{\mathbf{L}}^{\dagger}$-proofs. The property of being **M**-revivable can be seen as a global correcteness condition on $\mathcal{C}_{\mathbf{L}}^{\dagger}$-proofs, and therefore constitutes—in its broadest interpretation—a system of rules for $\mathcal{C}_{\mathbf{L}}^{\dagger}$. But of course this observation is rather[11] useless in practice unless we can express the property of being revivable in simpler terms, say via a condition on the order of rules being applied.

To conclude this article, we now discuss two logics—**KD** and **LQ**—where this is the case. Their similarity lies in the fact that they admit a very strong version of the deduction theorem, and this will allow us to express their notions of 'revivability' in fairly simple terms. In doing so, we obtain both a system of rules and a strongly modular proof of cut-elimination.

4.2 $\mathbf{K} \subseteq \mathbf{KD}$

The modal logic **KD** is the extension of **K** by the seriality axiom $\neg\Box\bot$; in terms of the Kripke semantics, $\neg\Box\bot$ enforces that every world has at least one successor. It is well-known (see, e.g., [13]) that extending $\mathcal{C}_{\mathbf{K}}$ with the rule

$$\frac{\Gamma \Rightarrow}{\Box\Gamma \Rightarrow}\ (D)$$

yields a sequent calculus $\mathcal{C}_{\mathbf{KD}}$ for **KD** admitting cut-elimination. We now present a new proof of cut-elimination for **KD** that is strongly modular.

As the seriality axiom has no variables, the modalized instances of it are exactly the formulas $\Box^k\neg\Box\bot$ for $k \geq 0$. Following the methodology sketched in the previous section, we now extend $\mathcal{C}_{\mathbf{K}}$ to a calculus $\mathcal{C}_{\mathbf{K}}^{\dagger}$ with ghost rules. Crucially, the ghost rule $(K)^{\dagger}$ coincides with the rule (D) above.

Theorem 11. *Those proofs in* $\mathcal{C}_{\mathbf{K}}^{\dagger}$ *whose only ghost rule is* $(K)^{\dagger}$ *form a cutfree-adequate class for* **KD**.

Proof. Let us first deal with completeness. If $\Gamma \Rightarrow \Delta$ is valid in **KD**, then there is a set of modalized instances of $\neg\Box\bot$ such that $\Omega, \Gamma \Rightarrow \Delta$ has a $\mathcal{C}_{\mathbf{K}}$-proof α. Using cut-elimination in $\mathcal{C}_{\mathbf{K}}$, we may assume that α is cutfree. As there is no right rule for $\bot$, the $\mathcal{C}_{\mathbf{K}}$-rules that can be applied in α to an ancestor of a modalized instance of $\neg\Box\bot$ in Ω are only $(\neg_L)$ and (K). Now obtain $\alpha^{\dagger}$ by removing Ω and all its ancestors from the proof. As $(\neg_L)^{\dagger}$ is a dummy rule, the only ghost rule we need to create is $(K)^{\dagger}$. Thus $\alpha^{\dagger}$ is as desired.

[10] The idea of systematically replacing systems of rules with axiom instances in order to prove *soundness* already appears in [16].

[11] One could maybe make the following remark: When looking for a simple cut-free sequent calculus that endowed with *some* global correctness criterion captures the logic **M**, one does not have to look further than $\mathcal{C}_{\mathbf{L}}^{\dagger}$.

We now turn to soundness. For this we have to 'revive' a $\mathcal{C}^\dagger_{\mathbf{K}}$-proof β of $\Gamma \Rightarrow \Delta$ whose only ghost rule is $(K)^\dagger$. This is done as follows:

$$\frac{\Gamma \Rightarrow}{\Box\Gamma \Rightarrow}\,(K)^\dagger \quad\rightsquigarrow\quad \cfrac{\cfrac{\cfrac{\Gamma \Rightarrow}{\Gamma \Rightarrow \bot}\,(w)}{\Box\Gamma \Rightarrow \Box\bot}\,(K)}{\Box\Gamma, \neg\Box\bot \Rightarrow}\,(\neg_L)$$

Now propagate the newly added $\neg\Box\bot$ downwards in the proof. We will have to add $\Box$'s in front of it whenever we encounter the rule (K). Doing so for all instances of $(K)^\dagger$ we eventually obtain a $\mathcal{C}_{\mathbf{K}}$-proof of $\Omega, \Gamma \Rightarrow \Delta$ where Ω contains modalized instances of $\neg\Box\bot$. Thus $\Gamma \Rightarrow \Delta$ is valid in **KD**. □

As restricting the ghost inferences in $\mathcal{C}^\dagger_{\mathbf{K}}$ to $(K)^\dagger$ yields exactly $\mathcal{C}_{\mathbf{KD}}$, we have obtained a new (and strongly modular) proof of cut-elimination for $\mathcal{C}_{\mathbf{KD}}$.

4.3 IL $\subseteq$ LQ

For our final example, we leave the realm of modal logics and consider an intermediate logic instead. **LQ** extends **IL** by the law of *weak excluded middle* $\neg p \vee \neg\neg p$; it is known [11] that the following deduction theorem holds: $A \in \mathbf{LQ} \iff (\wedge_{i \leq n} \neg p_i \vee \neg\neg p_i) \rightarrow A \in \mathbf{IL}$ where $p_1, \ldots, p_n$ are the variables occurring in A. Let $\mathcal{C}_{\mathbf{IL}}$ be the single-conclusion calculus obtained from the first group of rules in Fig. 1 by stipulating that $|\Pi| = 0$ and $|\Delta| \leq 1$. $\mathcal{C}_{\mathbf{IL}}$ is adequate for **IL** and admits cut-elimination.

Definition 5. *A proof in $\mathcal{C}^\dagger_{\mathbf{IL}}$ is* **LQ***-grounded if the following holds:*

1. *The only ghost rules in it are $(\vee_L)^\dagger$ and ghost initial sequents $\Rightarrow p$, $p \Rightarrow$, $\Rightarrow$.*
2. *Letting $(\vee_L)^\dagger_1, \ldots, (\vee_L)^\dagger_n$ denote all instance of $(\vee_L)^\dagger$ in the proof, there are sets $L_1, R_1, \ldots, L_n, R_n$ of ghost initial sequent occurrences such that*
   - *every ghost initial sequent $p \Rightarrow$ (resp. $\Rightarrow p$, resp. $\Rightarrow$) appears in exactly one L_i (resp. exactly one R_i, resp. exactly one R_i and exactly one L_j);*
   - *No two distinct variables appear in connected components, where being connected is the reflexive, transitive and symmetric closure of the relation $L_i \sim R_j \iff i = j \vee L_i \cap R_j \neq \emptyset$*
   - *Every branch of the proof containing a sequent in L_i (R_i) goes through the left (right) premise of $(\vee_L)^\dagger_i$. If it goes through the right premise, it contains a sequent with empty right hand side above $(\vee_L)^\dagger_i$.*

Figure 4 (middle) shows a simple **LQ**-grounded proof where $n = 1$.

Theorem 12. *The class of* **LQ***-grounded $\mathcal{C}^\dagger_{\mathbf{IL}}$-proofs is cutfree-adequate for* **LQ**.

Proof. (Sketch). Completeness is similar to Theorem 11; **LQ**'s special deduction theorem restricts the necessary ghost inferences to initial sequents and $(\vee_L)^\dagger$.

We now show soundness by 'reviving' an **LQ**-grounded proof of $\Gamma \Rightarrow \Delta$. Start by adding variables and $(\neg_R)$-inferences to the ghost initial sequents as follows:

$$(p \Rightarrow)_{\in L_i} \rightsquigarrow (\frac{p \Rightarrow p^{L_i}}{p, \neg p^{L_i} \Rightarrow}) \quad (\Rightarrow p)_{\in R_i} \rightsquigarrow (p^{R_i} \Rightarrow p) \quad (\Rightarrow)_{\in L_i \cap R_j} \rightsquigarrow (\frac{p^{R_j} \Rightarrow p^{L_i}}{p^{R_j}, \neg p^{L_i} \Rightarrow})$$

The superscripts act only as markers, i.e. p, p^{R_i}, p^{L_i} denote the same variable. In replacing $(\Rightarrow) \in L_i \cap R_j$ we add the variable p from a component connected to L_i or R_j (unique if it exists) and an arbitrary variable otherwise; in the other cases the choice of the added variable is forced by the preexisting p. The $\neg p^{L_i}$'s are then propagated downwards until the left premise of $(\vee_L)_i^\dagger$. The p^{R_i}'s are propagated downwards until we encounter the first sequent $\Sigma \Rightarrow$ with empty right hand side, at which point we introduce double negations:

$$(\Sigma \Rightarrow) \rightsquigarrow \left(\frac{\dfrac{\Sigma, p^{R_i} \Rightarrow}{\Sigma \Rightarrow \neg p^{R_i}}}{\Sigma, \neg\neg p^{R_i} \Rightarrow} \right)$$

Propagate the $\neg\neg p^{R_i}$'s down to the right premise of $(\vee_L)_i^\dagger$ and rewrite as follows:

$$\frac{\Sigma \Rightarrow \Pi \qquad \Sigma \Rightarrow \Pi}{\Sigma \Rightarrow \Pi} (\vee_L)_i^\dagger \quad \rightsquigarrow \quad \frac{\Sigma, \neg p^{L_i} \Rightarrow \Pi \qquad \Sigma, \neg\neg p^{R_i} \Rightarrow \Pi}{\Sigma, \neg p \vee \neg\neg p \Rightarrow \Pi} (\vee_L)$$

Propagate the new formula $\neg p \vee \neg\neg p$ to the endsequent. Doing so for all $i \leq n$, we obtain a $\mathcal{C}_{\mathbf{IL}}$-proof of $\Omega, \Gamma \Rightarrow \Delta$ where Ω contains instances of the weak excluded middle axiom. Thus $\Gamma \Rightarrow \Delta$ is valid in **LQ**. □

It is instructive to compare **LQ**-grounded proofs to other calculi in the literature. For example, a hypersequent calculus for **LQ** [8] is obtained by adding the rule (lq) (below left) to a hypersequent calculus for intuitionistic logic.[12] The corresponding 2-system of rules [9] is pictured on the right:

$$\frac{\Sigma, \Sigma' \Rightarrow}{\Sigma \Rightarrow \mid \Sigma' \Rightarrow} (lq) \qquad \frac{\dfrac{}{\Sigma \Rightarrow} \;\; \vdots \;\; \Gamma \Rightarrow \Delta \qquad \dfrac{\Sigma, \Sigma' \Rightarrow}{\Sigma' \Rightarrow} \;\; \vdots \;\; \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta} (bot)$$

Figure 4 hints at the translation of **LQ**-grounded proofs into both calculi.

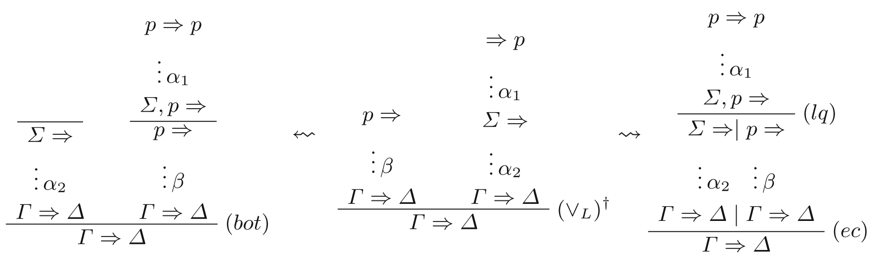

Fig. 4. From **LQ**-grounded proofs to 2-systems (left) and hypersequents (right)

[12] An interesting sequent calculus for **LQ** is presented in [6].

5 Conclusion and Future Work

We have defined *grounded proofs*, a system of rules for $\mathbf{KT}^{\Box}$ and **K5**, and proved the cut-elimination theorem. We showed how grounded proofs relate to grafted hypersequents, thereby recovering and simplifying the cut-elimination theorem for the latter calculus. We then elaborated on *strongly modular proofs of cut-elimination*, providing two more examples through the logics **KD** and **LQ**.

Future work. Strongly modular proofs do not directly yield an algorithm for eliminating cuts. We would like to know whether the arguments given here can be used to write an algorithm that, e.g., eliminates cuts in grounded **K5**-proofs by calling the cut-elimination algorithms for **K** and **S5** as subroutines.

The method of obtaining strongly modular proofs through calculi with ghost rules is in a very early stage and so much remains to be explored. As a first step, one could try to extend the argument for **LQ** to all intermediate logics with a similar deduction theorem, i.e. logics with the *simple substitution property* [19].

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