How do computational thinking and mathematical thinking interact (in terms of knowledge, ways of thinking and competencies)?

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Introduction

Over the past decade, there has been an upsurge of interest in the research on computational thinking (CT). Many countries are in the process of introducing computational thinking into their school curricula, either as a new dedicated subject, a cross-curricular theme, or integrated within an existing subject, such as computing (e.g., Kafai & Proctor, 2022) or mathematics (e.g., Bocconi et al., 2018; Royal Society, 2018; Nordby, Bjerke & Mifsud, 2022). The relationship between computational thinking and mathematical thinking has been of particular interest (e.g., Perez, 2018, Kallia et al., 2021, Geraniou & Hodgen, 2022a; 2022b) and there is variation in how computational thinking is perceived by different stakeholders in the mathematics education discipline, including researchers, teacher practitioners, and teacher educators.

In this interactive research panel, the panellists discussed characterizations of computational thinking and how it relates to mathematical thinking in terms of knowledge, ways of thinking, and competency. There were two sessions. The first session started off with an introductory task for the audience, seeking participants' past experiences with CT in mathematics, their views on CT's importance for different stakeholders from the mathematics education broader community and its relevance to today's society, all of which were recorded using a padlet. This activity was intended to set the scene for participants and supported them in reflecting on what followed in this panel session. Paul addressed the importance of computational thinking in mathematics education in the 21st century and its potential links to mathematical thinking from an historical, research-based perspective. Elise and Eirini reacted by sharing their own positionalities and personal interests, what they found interesting, and what future work in this field should entail. This first session ended with the panellists responding to some ideas shared in the padlet. The second session focused on the presentation of two engaging, yet contrasting, vignettes regarding the impact of CT in mathematics education, envisioning what is to happen in the year 2040 in a mathematics classroom and in the mathematics education scene in general from a societal, educational and political perspective. The participants were asked to use another padlet to respond to vignette-specific reflective questions, but also some general questions: 1. Does the scenario in this vignette excite you? Concern you; 2. Are constructs like "computational thinking" or "mathematical thinking" evident in this vignette? If so, what are the similarities or differences between them? The panellists were then invited to react to the points raised in the padlet. The panel session ended with each panellist sharing their one-line take-home message that captured their key reaction.

In the next pages, we present our gathered reflections from the various parts of this panel session, highlighting questions and issues related to computational thinking that, in our view, are of great interest to the mathematics education research community.

How do Computational Thinking and Mathematical Thinking interact?

Why are we interested in Computational Thinking? As Paul argued, our private and professional lives are full of interactions with digital technology that are based on computations. As (future) citizens and professionals, we need to be able to deal with this and ensure we have a good insight into the computational processes involved. But one could wonder whether CT is really a "new" concept. Papert mentioned CT in his Mindstorms book and argued that "...visions of how to integrate computational thinking into everyday life was insufficiently developed" (Papert, 1980, p.182). Most of us have seen the definitions of CT offered by various authors, For example, Wing (2010) defined CT as "the thought processes involved in formulating problems and their solutions so that the solutions are represented in a form that can be effectively carried out by an information-processing agent" (Cuny, Snyder & Wing, 2010). Lodi (2020) characterised CT as the "*disciplinary way of thinking*" of Informatics and argued that computational problem solving involves the formulation and the solution of the problem expressed in a way that allows an "external" processing agent (a human or a machine) to carry it out.

Some research also indicated the various processes involved with CT. Angeli et al. (2016) argued that CT involves five processes: Abstraction, Generalization, Decomposition, Algorithms, and Debugging. Some of these processes have been mentioned by Grover and Pea (2013) and Selby and Woollard (2015), who added a few more: Problem-solving, Abstraction, Generalization, Decomposition, Algorithmic thinking and Modularization. But how do computational thinking and mathematical thinking relate? Kallia et al. (2021) claimed that CT in mathematics education is seen as "a structured problem-solving approach in which one is able to solve and/or transfer the solution of a mathematical problem to other people or a machine by employing thinking processes that include abstraction, decomposition, pattern recognition, algorithmic thinking, modelling, logical and analytical thinking, generalisation and evaluation of solutions and strategies" (Kallia et al. 2021, pp.20-21). Considering such arguments, should we all agree that Mathematical Thinking (MT) is a subset of CT (Fig 1a)? Or do CT and MT have an intersection (Fig 1b)? Or is CT a subset of MT (Fig 1c)?

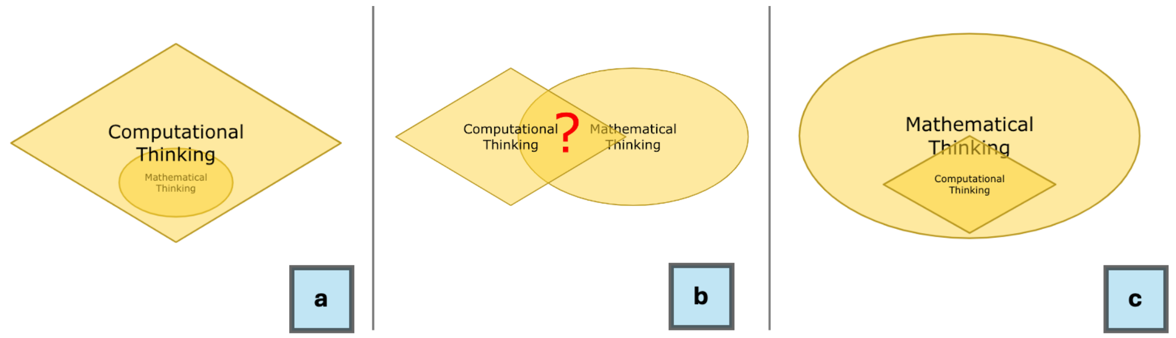

Figure 1. How do CT and MT relate?

So, which of these three reflects the best view? And what about the potential links between CT and programming and coding, or even artificial intelligence? As Paul concluded, the landscape of CT in education is still quite scattered, and even though CT and MT share common ground, it is still unclear how to cultivate that common ground.

Eirini reacted to Paul's ideas by posing a question that has been in her mind for the past few years: "Why is CT '***important***' in mathematics education and mathematics education research?" and in fact "why does it seem to be appearing in discussions within the mathematics education community and beyond?". Her own research interests regarding computational thinking focus on: (a) Investigating mathematics educators and mathematics teachers' views and perceptions on CT and its impact on mathematical learning; and on (b) identifying strategies for promoting CT and its integration in mathematics teaching and learning. In order to look at these 2 research foci, Eirini looked into what computational thinking is, considering some different definitions to those presented by Paul. It has been characterized as an essential competency for a digital society (Inprasitha, 2021) or the "new digital age competency" (e.g., Grover & Pea, 2013). It has been defined as "the thought processes involved in formulating problems and their solutions so that the solutions are represented in a form that can be effectively carried out by an information-processing agent." (Cuny, Snyder and Wing, 2010). It involves practices that are also required in mathematics, such as "decomposition, abstraction, algorithm design, debugging, iteration, and generalization" (Li et al., 2020, p.156). Perez's (2018) review of CT in mathematics education highlights a range of practices, including elements, such as "developing algorithms and automations" as well as composite practices, such as "efficient and effective combinations of resources, testing and debugging" (p. 428). As one notices, there are some highlighted words, such as "competency" (in fact a "new digital age competency"), "thought processes", "practices", "a range of practices". Reflecting on these definitions, Eirini drew attention to the idea of whether CT is indeed a competency and how it relates to MT. Paul looked at these two being separate, but also having an intersection and sharing common ground. In Sands et al.'s (2018) research, CT was synonymous to doing mathematics using digital tools. So, are CT and MT only linked when doing mathematics using digital technologies? Recent work by Geraniou and Hodgen (2023) indicated that some mathematics educators struggle to articulate what the relationship between CT and Mathematics is and, as a result, view CT as closely tied to computers and other digital tools. Carrying out this research work with mathematics educators led them to think about the relationship between mathematical knowledge, thinking, competency and where CT fits in. Having in mind that CT was perceived in the literature as a "competency" as mentioned earlier, Eirini reflected on definitions regarding mathematical competence to identify potential links. In fact, she wondered about whether there is a link between CT and mathematical digital competency, which is a construct developed recently by Geraniou and Jankvist (2019). They argued that when solving a mathematical problem using a digital tool, having mathematical competency and mathematical knowledge and digital competency and knowledge of the digital tool equips you to solve the problem. Mathematical Digital Competency was defined as the interplay of mathematical competency and digital competency and the MDC framework was developed (for more information, please read Geraniou & Jankvist, 2019). Going back to the work of Geraniou and Hodgen (2023) on mathematics educators' views on CT, Eirini shared some reflections regarding a recent study involving interviews with mathematics educators seeking to find out more about their view on what CT is and how it relates to MT. In those interviews, mathematics educators were asked to bring a mathematical task involving the use of any digital tool that they are familiar with and the interview was based on their chosen tasks. After presenting one of these case studies, Eirini argued that it became clear to them that possessing computational thinking is a competency, in fact an MDC, and the MDC framework was adapted (Geraniou & Hodgen, 2023). Eirini therefore concluded with the argument that CT may indeed be a Mathematical Digital Competency.

Elise shared her own research path regarding mathematical and computational thinking, before reacting to Paul's contribution. Her research focuses on investigating undergraduate students' reasoning about combinatorics (counting) problems, which is a specific mathematical topic and most of her work did not consider computing at all. Indeed, Elise noted that she has little programming or computational experience. However, her motivation to engage with computing is driven by her mathematics education research, as a central finding of her work in combinatorics emphasizes the value of helping students focus on sets of outcomes. She elaborated on this by presenting and discussing a task: "How many three-digit sequences can be made from the numbers 1-6 that have no repeated digits?" Figure 2 demonstrates an example of how a small bit of Python code offers an insightful representation to consider a solution to this counting task. The four loops in the code represent the multiplicative process in permutations, and the conditional statement ensure that elements are not repeated.

$$\underline{6} \cdot \underline{5} \cdot \underline{4} = 120 \qquad P(6,3) = \frac{6!}{3!}$$

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Numbers = [1, 2, 3, 4, 5, 6]

Counter = 0

for i in Numbers:
    for j in Numbers:
        if j != i:
            for k in Numbers:
                if k != i and k != j:
                    print(i, j, k)
                    Counter = Counter + 1
print(Counter)
```

123	213	...	512	612
124	214	...	513	613
125	215	...	514	614
126	216	...	516	615
132	231	...	521	621
134	234	...	523	623
135	235	...	524	624
136	236	...	526	625
...	...	...	...	...
164	264	...	563	653
165	265	...	564	654

Figure 2. "How many three-digit sequences can be made from the numbers 1-6 that have no repeated digits?"

Elise used this task as an example of how CT and MT can intersect, and in her work she has argued that the two can be mutually beneficial; here, student reasoning about the computational representation of the code (an instance of CT) may support students' understanding of combinatorial processes (an instance of MT). In this way, Elise finds herself aligning with Paul's characterization in Figure 1b above, and her work seeks to focus on that intersection. She concluded by sharing her future research goals, which involve (a) exploring ways in which CT builds upon what we know about mathematics education research and (b) finding out if even modest engagement with CT and activity may support students' MT and activity (and vice versa). She argued that she benefitted from engagement with computationally-minded researchers in other disciplines, and hopes for a future with meaningful interdisciplinary collaborations. She finally put forward a question for all to reflect upon: is there a future where computational and mathematical thinking reflexively develop and grow?

The importance and value of Computational Thinking in Mathematics Education

The panellists wanted to inspire participants in considering the potential impact of CT in mathematics education and therefore carefully chose two engaging, yet contrasting, vignettes, envisioning what is to happen in the year 2040 in a mathematics classroom and in the mathematics education scene in general from a societal, educational and political perspective.

The first scenario was titled "The Adventure Travel List problem" and it stated "It's year 2040 and the maths teacher sets the following homework task to their Year 9 class: "Harry needs to visit 5 different places in the world. He starts his adventure travel in London (UK) and then he needs to go to all the following 5 places: Kathmandu (Nepal), Nairobi (Kenya), Machu Picchu (Peru), Cairo (Egypt), Sydney (Australia). What is the shortest route that Harry can take to visit all 5 places and return to London?", which is an adaptation of the well-known "travelling salesman problem". The scenario presented the dialogue between two students, Mark and Sofia, and Mark was seeking help from Sofia to solve the problem. There was an element of surprise at the end as Sofia proved to be an Artificial Intelligent Tutor, programmed to support Mark, find a solution and reach an answer. Participants were asked to reflect on questions like: Do Mark and Sofia have Mathematical Digital Competency? What about CT? Do they both have it? How would you feel about this task? Why would this be an activity worthwhile to teach in 2040? Would we have AI tutors supporting students to develop CT? Is there evidence of the interplay between CT and MDC in this vignette?

The second scenario involved a fictitious scene taking place in 2040, during which politicians recognised that having a computationally fluent workforce and citizenry was increasingly necessary for global competition, and were motivated to create governmental action plans to expedite the development of computing education. Twenty years later, in 2060, efforts to integrate computing into schools have broadly been implemented in many countries around the world. A graduation requirement from secondary school involves a computational data fluency certificate in any subject a student may choose, while at university level, most institutions for any major offer a comprehensive common first-year slate of courses that integrate computing with a student's discipline of choice. The education scene in 2060 looks very different. Computing and computational thinking has become the hub that connects all disciplines and academic journals favor interdisciplinary collaboration to represent excellence in scholarly educational research. At school, all students attend mathematics lessons for 20h/week, as it is widely recognized that in this era of computational thinking, mathematical skills are the only ones that matter for problem solving and for dialogue with intelligent computing tools. Classes are set up in traditional ways in old-fashioned classrooms, and they focus on ancient geometry, number theory, logic, and rhetorics, as to prepare students for the 22nd-century future. In some of these countries, this approach is only applied to the top-5% of high-achieving students, as they are identified in a problem-solving test for 4-year-old children. They are trained to become the engineers and scientific leaders for the future; being selected for this elitarian group is considered an honor. Participants were asked to reflect on questions like: From an instructional perspective, what might be gained or what would be lost in these two approaches? From a research perspective, what would be gained or lost in these two approaches?

The participants were also asked to respond to some general questions for each scenario: 1. Does the scenario in this vignette excite you? Concern you?; 2. Are constructs like "computational thinking" or "mathematical thinking" evident in this vignette? If so, what are the similarities or differences between them?

Both scenarios brought forward arguments about how realistic these cases are and participants raised some concerns about AI taking over the teacher's role for example (first scenario) or whether we would need to teach our students how to code if AI can do this for us (second scenario).

We, the panellists, see the value of both mathematical thinking and computational thinking, but are concerned about whether and how the two should be integrated. For example, one can go for the "pure and isolated" option and focus on MT, or for the embedded, real-life option of CT in applied situations, and perhaps competency (and potentially mathematical digital competency) should also be explored. We, as mathematics educators, should find ways, through our work and research, to influence any decisions taken by different stakeholders (e.g. government, policy-makers, schools and other institutions) regarding the integration of CT and MT. We know that digital technologies are part of our lives, AI is evolving, possibly even faster than anticipated, and any research we conduct will shape the mathematics education field in the coming decades. So, the key question to ask ourselves is: What should the mathematics education scene look like in the year 2050?

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