



Mean-field limit of age and leaky memory dependent Hawkes processes

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Abstract

We propose a mean-field model of interacting point processes where each process has a memory of the time elapsed since its last event (age) and its recent past (leaky memory), generalizing Age-dependent Hawkes processes. The model is motivated by interacting nonlinear Hawkes processes with Markovian self-interaction and networks of spiking neurons with adaptation and short-term synaptic plasticity.

By proving propagation of chaos and using a path integral representation for the law of the limit process, we show that, in the mean-field limit, the empirical measure of the system follows a multidimensional nonlocal transport equation.

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1. Introduction

The dynamics of many interacting particle systems can be approximated, when the size of the system tends to infinity, by a partial differential equation (PDE) [26]. This not only links microscopic and macroscopic scales but also stochastic and deterministic models. For mean-field models, one can prove this type of results by exploiting the propagation of chaos phenomenon, i.e. for *i.i.d.* initial conditions, particles become asymptotically independent in the mean-field limit [31,47].

Propagation of chaos arguments have been applied to the study of interacting point processes [9,15,29,30]. This has been particularly important for the field of theoretical neuroscience as it has provided a rigorous footing to the *population density* formalism, where the

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dynamics of a population of neurons is described by a PDE (see [22, Part III]). An example of population density equation is the refractory density (or age-structured) equation [18,20–22, 36,37,46], which has recently been proved to be exact in the mean-field limit [4,8,42].

The models considered in [4,8,42] all assume that the point processes are ‘renewal’ (in some loose sense), i.e. each process has a memory of its past that is restricted to the time elapsed since its last event. The fact that, in the ‘renewal’ case, the mean-field limit can be characterized by relatively simple deterministic equations has long been recognized in theoretical neuroscience and has led to a large body of work [18,19,46,52]. In contrast, the case where point processes are not ‘renewal’ is much less understood. In particular, even though some heuristic population density equations have been proposed for the ‘non-renewal’ case [33,38,48], their exactness in the mean-field limit has not been proved. The aim of this work is therefore to propose a general framework for relating interacting ‘non-renewal’ point processes with PDEs, in the mean-field limit. This framework relies on the definition of an abstract interacting point process model, which generalizes Age-dependent Hawkes processes [4,43].

1.1. Interacting age and leaky memory dependent Hawkes processes

Consider a system of N interacting point-processes, interacting through a common variable X_t^N . Each point process i is associated with $1 + d$ variables (for d a positive integer): an age variable $A_t^{i,N}$ which represents the time elapsed since the last event of process i and a d -dimensional vector of leaky memory variables $\mathbf{M}_t^{i,N}$ which models the effect of the recent past of process i . The point process i has stochastic intensity $(f(A_{t-}^{i,N}, \mathbf{M}_{t-}^{i,N}, X_{t-}^N))_{t \in \mathbb{R}_+}$ where $f : \mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R} \rightarrow \mathbb{R}_+$ is the intensity function. Intuitively, this means that if we write $(Z_t^{i,N})_{t \in \mathbb{R}_+}$ the counting process associated with the point process i , the instantaneous probability for $Z^{i,N}$ to jump in $]t, t + dt]$, given the past \mathcal{F}_t , is

$$\mathbb{P}(Z_{t+dt}^{i,N} > Z_t^{i,N} | \mathcal{F}_t) = f(A_t^{i,N}, \mathbf{M}_t^{i,N}, X_t^N) dt.$$

Between events (jumps) of process i , the age variable $A_t^{i,N}$ grows linearly with time whereas the leaky memory variables $\mathbf{M}_t^{i,N}$ drift following the vector field $b : \mathbb{R}^d \rightarrow \mathbb{R}^d$.

At each event of process i , its age $A_t^{i,N}$ is reset to 0 and its leaky memory $\mathbf{M}_t^{i,N}$ jumps to $\mathbf{M}_t^{i,N} + \Gamma(\mathbf{M}_t^{i,N})$, where $\Gamma : \mathbb{R}^d \rightarrow \mathbb{R}^d$ is the jump function. The fact that the variables $\mathbf{M}_t^{i,N}$ are not reset to a fixed value at each event allows them to accumulate the effect of successive events.

Finally, the time-dependent effect of an event of point process i on point process j is determined by the interaction function $h : \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}^d \rightarrow \mathbb{R}$ which depends on $A_t^{j,N}$ and $\mathbf{M}_t^{j,N}$. Since the function h is the same for all i and j , the interaction is said to be of mean-field type.

The model can be described by a system of stochastic integral equations: for $i = 1, \dots, N$,

$$A_t^{i,N} = A_0^i + t - \int_0^t A_{s-}^{i,N} dZ_s^{i,N}, \tag{1a}$$

$$\mathbf{M}_t^{i,N} = \mathbf{M}_0^i + \int_0^t b(\mathbf{M}_s^{i,N}) ds + \int_0^t \Gamma(\mathbf{M}_{s-}^{i,N}) dZ_s^{i,N}, \tag{1b}$$

$$Z_t^{i,N} = \int_{[0,t] \times \mathbb{R}_+} \mathbb{1}_{z \leq f(A_{s-}^{i,N}, \mathbf{M}_{s-}^{i,N}, X_{s-}^N)} \pi^i(ds, dz), \tag{1c}$$

with

$$X_t^N = \frac{1}{N} \sum_{j=1}^N H_t^j + \frac{1}{N} \sum_{j=1}^N \int_0^t h(t-s, A_{s-}^{j,N}, \mathbf{M}_{s-}^{j,N}) dZ_s^{j,N}. \tag{1d}$$

The collection $\{\pi^i\}_{i \in \mathbb{N}^*}$ is a sequence of independent Poisson random measures on $\mathbb{R}_+ \times \mathbb{R}_+$ with Lebesgue intensity measure. We work on the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in \mathbb{R}_+}, \mathbb{P})$ where $\{\pi^i\}_{i \in \mathbb{N}^*}$ is independent of \mathcal{F}_0 and $\mathcal{F}_t = \mathcal{F}_0 \cup \sigma(\{\pi^i([0, t], B)\}_{i \in \mathbb{N}^*, B \in \mathcal{B}(\mathbb{R}_+)})$. For all $i \in \mathbb{N}^*$, A_0^i and \mathbf{M}_0^i are \mathcal{F}_0 -measurable random variables taking values in \mathbb{R}_+ and \mathbb{R}^d respectively and $(H_t^i)_{t \in \mathbb{R}_+}$ is a \mathcal{F}_0 -measurable $\mathcal{C}(\mathbb{R}_+)$ random function. The $1/N$ scaling in (1d) will allow us to take the mean-field limit $N \rightarrow \infty$.

If f does not depend on the leaky memory variables \mathbf{M} and h does not depend on the age A nor \mathbf{M} , (1) reduces to a system of interacting Age-dependent Hawkes processes [4,43]. If, in addition, f does not depend on A , the model further reduces to a mean-field system of interacting nonlinear Hawkes processes (with vanishing self-interaction) [9,10]. The model (1) has two motivations: first, it is general enough to encompass several concrete examples from the theory of nonlinear Hawkes processes and neuroscience (see below); second, its mean-field limit can be characterized by a PDE.

1.2. Motivating examples

Hawkes processes [23] provide a flexible and intuitive model for point processes with dependence on the past. They have found applications in finance [1,24], seismology [35], social systems [6], genomics [45] and neuroscience [17,28,41,44,49,50], among other fields. Neuroscience research has mainly focused on *nonlinear* Hawkes processes [2] since they are closely related to well-established neuron models such as the Spike Response Model [19,21,22,25] and the Recursive Linear-Nonlinear Poisson Model [41], both variations of Generalized Linear Models (see [22] Part II and references therein). However, the models differ from the nonlinear Hawkes processes considered in [9,10] in that, even when N is large, self-interaction (the effect of process i on itself) does not vanish. Self-interaction vanishes in [9,10] because it is scaled by $1/N$. Let us now consider the case where self-interaction \mathfrak{h} can be different from hetero-interaction h (the effect of a process on the other processes) and only hetero-interaction is scaled by $1/N$: for $i = 1, \dots, N$,

$$Z_t^{i,N} = \int_{[0,t] \times \mathbb{R}_+} \mathbb{1}_{\{z \leq f(X_{s-}^{i,N})\}} \pi^i(ds, dz), \tag{2a}$$

$$X_t^{i,N}(i) = \mathfrak{H}_t^i + \frac{1}{N} \sum_{j \neq i} H_t^j + \int_0^t \mathfrak{h}(t-s) dZ_s^{i,N} + \frac{1}{N} \sum_{j \neq i} \int_0^t h(t-s) dZ_s^{j,N}, \tag{2b}$$

where $f : \mathbb{R} \rightarrow \mathbb{R}_+$ is monotonically increasing and $\{(\mathfrak{H}_t^i)_{t \in \mathbb{R}_+}\}_{i=1, \dots, N}$ are \mathcal{F}_0 -measurable random $\mathcal{C}(\mathbb{R}_+)$ functions. The model (2) is a mean-field system of interacting nonlinear Hawkes processes with non-vanishing self-interaction. In the context of neuroscience, (2) can be seen as a mean-field network of Generalized Linear Model/Spike Response Model neurons.

Let us now assume that \mathfrak{h} is an Erlang kernel, i.e. there exists $d \in \mathbb{N}^*$ such that

$$\mathfrak{h}(t) = ce^{-\alpha t} \frac{t^{d-1}}{(d-1)!},$$

for some $c \in \mathbb{R}$ and $\alpha > 0$. Self-interaction can then be translated into a ‘Markovian cascade of memory terms’ [10,11]: adjusting the initial conditions, (2) can be equivalently written: for $i = 1, \dots, N$,

$$Z_t^{i,N} = \int_{[0,t] \times \mathbb{R}_+} \mathbb{1}_{\{z \leq f(\mathbf{M}_{s-}^{i,N}(1) + X_{s-}^{i,N})\}} \pi^i(ds, dz), \tag{3a}$$

$$\mathbf{M}_t^{i,N} = \mathbf{M}_0^i + \int_0^t \mathfrak{A} \mathbf{M}_s^{i,N} ds + \mathbf{c} Z_t^{i,N}, \tag{3b}$$

$$X_t^{i,N} = \frac{1}{N} \sum_{j \neq i} H_t^j + \frac{1}{N} \sum_{j \neq i} \int_0^t h(t-s) dZ_s^{j,N}, \tag{3c}$$

where $\mathbf{M}_{s-}^{i,N}(1)$ denotes the first element of the vector $\mathbf{M}_{s-}^{i,N}$. The d -by- d matrix \mathfrak{A} has all diagonal terms equal to $-\alpha$, all superdiagonal terms equal to 1, and all other terms equal to 0; the d dimensional vector \mathbf{c} is defined by $\mathbf{c}(k) = \mathbb{1}_{k=d}c$. This is equivalent to setting $b(\mathbf{m}) = \mathfrak{A}\mathbf{m}$ and $\Gamma(\mathbf{m}) = (0, \dots, 0, c)$ in (1b). The fact that there are N distinct variables $X_t^{i,N}$ instead of a common shared variable X_t^N as in (1) does not affect the mean-field limit since the difference between X_t^N and $X_t^{i,N}$ is of order $1/N$ (see [4]).

The generalization of (3) to the case where \mathfrak{h} is a sum of Erlang kernels is straightforward. Of course, a sum of Erlang kernels can simply be a sum of exponential kernels, which is more common in neuroscience [22,33,34,48]. Notably, taking one exponential kernel with $c < 0$ is enough to model the effects of neuronal refractoriness and spike-frequency adaptation [33].

In the example (3), the leaky memory variables \mathbf{M} influence the intensity function f but does not influence the interaction function h . However, in the general model (1), h can depend on \mathbf{M} . In the context of neuronal modeling, this dependence can be used to account for the effects of short-term synaptic plasticity (STP) [53]. Using the notation of the general model (1), we can describe a network of spiking neurons with refractoriness and ‘Tsodyks–Markram’ STP [51]. The Tsodyks–Markram model [51] captures the interplay between synaptic depression and facilitation and has been used to model working memory [32], chaotic dynamics [5] and learning in hierarchical circuits [39]. Taking $d = 2$, the leaky memory variables (the STP variables of the Tsodyks–Markram model) follow, for initial conditions \mathbf{M}_0^i supported in $[U, 1] \times [0, 1]$ with $U \in]0, 1[$,

$$\mathbf{M}_t^{i,N} = \mathbf{M}_0^i + \int_0^t b(\mathbf{M}_s^{i,N}) ds + \int_0^t \Gamma(\mathbf{M}_{s-}^{i,N}) dZ_s^{i,N}, \tag{4a}$$

with the vector field

$$b(\mathbf{m}(1), \mathbf{m}(2)) = \left(\frac{U - \mathbf{m}(1)}{\tau_F}, \frac{1 - \mathbf{m}(2)}{\tau_D} \right), \tag{4b}$$

where $\tau_F > 0$ and $\tau_D > 0$ are the facilitation and depression timescales respectively, and with the jump function

$$\Gamma(\mathbf{m}(1), \mathbf{m}(2)) = (U(1 - \mathbf{m}(1)), -\mathbf{m}(1)\mathbf{m}(2)). \tag{4c}$$

It is easy to verify that the leaky memory variables $\mathbf{M}_t^{i,N}$ then take values in $[U, 1] \times [0, 1]$. Finally, we take f independent of the leaky memory variables and h of the form $h(t, a, \mathbf{m}) = \mathbf{m}(1)\mathbf{m}(2)\bar{h}(t)$. The model we just described generalizes interacting Age-dependent Hawkes processes [4,43] and is more detailed than the model with purely facilitating synapses and without refractoriness studied in [16].

These two motivating examples are clearly special cases of the general model (1). The fact that in both examples, the variables $\mathbf{M}_t^{i,N}$ relax to some fixed value in the absence of jumps motivates the name ‘leaky memory’. Importantly, both examples satisfy the main assumptions we will use in this work (see Section 2).

1.3. Methods and relation to previous work

To prove propagation of chaos in the mean-field limit, we use the method of coupling *à la Sznitman* [47]: to show the convergence of the time-marginals, we follow Fournier and Löcherbach [14] (see also [16]); to show the convergence of the processes, we use the method of Delattre, Fournier and Hoffmann [9] (later used by Chevallier [4] and Ditlevsen and Löcherbach [10]). Our approach for relating the limit process with the limit PDE differs from previous work [4] for it relies on a path integral representation. This representation turns an earlier heuristic method from Naud and Gerstner [34] into a rigorous argument. Contrarily to [4] where PDE solutions in measure space are considered, our method treats PDE solutions in L^1 space and does not involve the semigroup theory results of [3]. More importantly, the path integral method allows us to derive a representation formula for the solution to the PDE. The limit PDE we obtain is a generalization of the Time Elapsed Neuron Network Model of Pakdaman, Perthame and Salort [36] and of the refractory density equation [18,20,22] to the case of neurons with adaptation and short-term synaptic plasticity.

1.4. Plan of the paper

The main results of this work, namely propagation of chaos (Theorem 1) and the characterization of the mean-field limit by a PDE (Theorem 2), are presented in Section 2, together with the assumptions required. The proof of Theorem 1 is presented in Section 3. In Section 4, we show that under more restrictive assumptions, we can get a propagation of chaos result analogous to that of [4,9,10]. Finally, the proof of Theorem 2 is presented in Section 5.

2. Assumptions and main results

General notations. The uniform and Euclidean norms are denoted by $\|\cdot\|_\infty$ and $\|\cdot\|$ respectively. We write $\| \Gamma \|_\infty := \sup_{\mathbf{m} \in \mathbb{R}^d} \| \Gamma(\mathbf{m}) \|$. We use C, C_T and $C_{T,0}$ to denote positive constants (that can change from line to line) where the subscript T signals the dependence on time and the subscript 0 the dependence on the law of $(A_0^1, \mathbf{M}_0^1, (H_t^1)_{t \in \mathbb{R}_+})$.

In this work, we always assume that the functions f, h, b and Γ satisfy:

Assumption 1.

- (i) The functions f, h and Γ are bounded.
- (ii) There exists a bounded, strictly increasing and continuously differentiable function $\psi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ with $\psi(0) = 0$ and satisfying

$$|\psi'(a) - \psi'(a^*)| \leq \kappa |\psi(a) - \psi(a^*)|, \quad \forall a, a^* \in \mathbb{R}_+,$$

for some $\kappa > 0$, such that, for all $(a, \mathbf{m}, x, a^*, \mathbf{m}^*, x^*) \in (\mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R})^2$, and for all $t \in \mathbb{R}_+$,

$$\begin{aligned} |f(a, \mathbf{m}, x) - f(a^*, \mathbf{m}^*, x^*)| &\leq L_f (|\psi(a) - \psi(a^*)| + \|\mathbf{m} - \mathbf{m}^*\| + |x - x^*|), \\ |h(t, a, \mathbf{m}) - h(t, a^*, \mathbf{m}^*)| &\leq L_h (|\psi(a) - \psi(a^*)| + \|\mathbf{m} - \mathbf{m}^*\|), \end{aligned}$$

for some L_f and $L_h > 0$.

(iii) The vector field b and jump function Γ are Lipschitz continuous.

The fact that f is bounded guarantees the well-posedness of the system (1) and a path-wise unique càdlàg strong solution to (1) can be constructed using a standard thinning procedure. (ii) says that f and h are Lipschitz continuous with respect to a ψ -modified metric on the age variable. An example of possible function is $\psi(a) = 1 - \exp(-ak)$. The ψ -modified Lipschitz continuity of f implies that the effect of a on f saturates for large a . Note that in the STP example (4), since the leaky memory variables take values in the compact $[U, 1] \times [0, 1]$, the jump function (4c) is effectively Lipschitz.

To prove propagation of chaos, we need some assumptions on $\{(A_0(i), \mathbf{M}_0(i), (H_t(i))_{t \in \mathbb{R}_+})\}_{i \in \mathbb{N}}$:

Assumption 2.

- (i) The 3-tuples $\{(A_0(i), \mathbf{M}_0(i), (H_t(i))_{t \in \mathbb{R}_+})\}_{i \in \mathbb{N}}$ are *i.i.d.*
- (ii) The random function $(H_t^1)_{t \in \mathbb{R}_+}$ is such that $(\mathbb{E}[H_t^1])_{t \in \mathbb{R}_+} \in \mathcal{C}(\mathbb{R}_+)$.
- (iii) For all $T > 0$, there exists $C_{T,0} > 0$ such that $\sup_{t \in [0, T]} \text{Var}[H_t^1] \leq C_{T,0}$.

A condition similar to (iii) is used in [4]. Note that in [4], Chevallier considers *i.i.d.* random interaction functions instead of a deterministic function h , common to all the point processes. As he proved that, under some square integrability condition, the randomness in the interaction functions averages out in the mean-field limit, we focus here on the fixed h case.

The first main result of this work is a quantified propagation of chaos theorem:

Theorem 1. *Grant Assumptions 1 and 2. For all $T > 0$, there exists $C_{T,0} > 0$ such that*

$$\sup_{t \in [0, T]} \mathbb{E} \left[|\psi(A_t^{1,N}) - \psi(A_t^1)| + \|\mathbf{M}_t^{1,N} - \mathbf{M}_t^1\| + |X_t^N - x_t| \right] \leq C_{T,0} N^{-1/2}, \tag{5}$$

where $(A_t^1, \mathbf{M}_t^1, x_t)_{t \in \mathbb{R}_+}$ (the limit process) is given by the path-wise unique càdlàg strong solution to

$$A_t^1 = A_0^1 + t - \int_0^t A_{s-}^1 dZ_s^1, \tag{6a}$$

$$\mathbf{M}_t^1 = \mathbf{M}_0^1 + \int_0^t b(\mathbf{M}_s^1) ds + \int_0^t \Gamma(\mathbf{M}_{s-}^1) dZ_s^1, \tag{6b}$$

$$Z_t^1 = \int_{[0, t] \times \mathbb{R}_+} \mathbb{1}_{z \leq f(A_{s-}^1, \mathbf{M}_{s-}^1, x_s)} \pi^1(ds, dz), \tag{6c}$$

$$x_t = \mathbb{E}[H_t^1] + \int_0^t \mathbb{E}[h(t-s, A_s^1, \mathbf{M}_s^1) f(A_s^1, \mathbf{M}_s^1, x_s)] ds. \tag{6d}$$

Furthermore, for all $t \in [0, T]$, writing $\mathcal{L}(\psi(A_t^1), \mathbf{M}_t^1)$ the law of $(\psi(A_t^1), \mathbf{M}_t^1)$, there exists $C'_{T,0} > 0$ such that

$$\sup_{t \in [0, T]} \mathbb{E} \left[W_1 \left(\frac{1}{N} \sum_{i=1}^N \delta_{(\psi(A_t^{i,N}), \mathbf{M}_t^{i,N})}, \mathcal{L}(\psi(A_t^1), \mathbf{M}_t^1) \right) \right] \leq C'_{T,0} N^{-1/2}, \tag{7}$$

where W_1 denotes the 1-Wasserstein distance.

It directly follows from Theorem 1 and the Continuous mapping theorem that for all $t > 0$, the empirical measure of the system (1) at time t converges in probability to the time-marginal

of the law of the limit process:

$$\frac{1}{N} \sum_{i=1}^N \delta_{(A_i^{i,N}, \mathbf{M}_i^{i,N}, X^N)} \xrightarrow[N \rightarrow \infty]{\mathbb{P}} \mathcal{L} \left(A_t^1, \mathbf{M}_t^1, x_t \right). \tag{8}$$

The second main result relates the time-marginals $\mathcal{L} \left(A_t^1, \mathbf{M}_t^1, x_t \right)$ of the law of the limit process with the solution to a nonlocal transport equation. To formulate the transport equation, we define the *jump mapping* $\gamma(\mathbf{m}) = \mathbf{m} + \Gamma(\mathbf{m})$ and we write $\nabla \cdot$ the divergence operator in \mathbb{R}^d . To stay within the standard framework of (mass-conservative) transport equation with solutions in L^1 [40], we need:

Assumption 3.

- (i) The vector field satisfies $b \in \mathcal{C}^1(\mathbb{R}^d, \mathbb{R}^d)$ and $\nabla \cdot b \in \mathcal{C}^1(\mathbb{R}^d, \mathbb{R}^d)$.
- (ii) The jump mapping γ is a proper local \mathcal{C}^1 -diffeomorphism.

Theorem 2. *Grant Assumptions 1 and 3. Further assume that the law of the initial condition (A_0^1, \mathbf{M}_0^1) is the absolutely continuous probability measure $u_0(a, \mathbf{m})dad\mathbf{m}$ and $(\mathbb{E}[H_t])_{t \in \mathbb{R}_+} = (\bar{H}_t^1)_{t \in \mathbb{R}_+} \in \mathcal{C}(\mathbb{R}_+)$. Then, the time-marginals $\rho_t \otimes \delta_{x_t} := \mathcal{L} \left(A_t^1, \mathbf{M}_t^1, x_t \right)$ of the law of the limit process (6) correspond to the unique weak solution to*

$$\partial_t \rho_t(a, \mathbf{m}) + \partial_a \rho_t(a, \mathbf{m}) + \nabla \cdot (b(\mathbf{m})\rho_t(a, \mathbf{m})) = -f(a, \mathbf{m}, x_t)\rho_t(a, \mathbf{m}), \tag{9a}$$

$$\rho_t(0, \cdot) = \gamma_* \left(\int_{\mathbb{R}_+} f(a, \cdot, x_t)\rho_t(a, \cdot)da \right), \tag{9b}$$

$$x_t = \bar{H}_t + \int_0^t \int_{\mathbb{R}^d} \int_{\mathbb{R}_+} h(t-s, a, \mathbf{m})f(a, \mathbf{m}, x_s)\rho_s(a, \mathbf{m})dad\mathbf{m}ds, \tag{9c}$$

$$\rho_0(a, \mathbf{m}) = u_0(a, \mathbf{m}), \tag{9d}$$

(where $\nabla \cdot$ denotes the divergence on the variables \mathbf{m} and $\gamma_*(\dots)$ denotes the pushforward measure by γ in the sense that $(\rho, x) \in \mathcal{C}(\mathbb{R}_+, L^1(\mathbb{R}_+ \times \mathbb{R}^d)) \times \mathcal{C}(\mathbb{R}_+)$ and, for all $G \in \mathcal{C}_c^\infty(\mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}^d)$,

$$0 = \int_{\mathbb{R}^d} \int_{\mathbb{R}_+} G(0, a, \mathbf{m})u_0(a, \mathbf{m})dad\mathbf{m} + \int_{\mathbb{R}_+} \int_{\mathbb{R}^d} \int_{\mathbb{R}_+} \left\{ [\partial_t + \partial_a + b(\mathbf{m}) \cdot \nabla]G(t, a, \mathbf{m}) + (G(t, 0, \gamma(\mathbf{m})) - G(t, a, \mathbf{m}))f(a, \mathbf{m}, x_t) \right\} \rho_t(a, \mathbf{m})dad\mathbf{m}dt, \tag{10}$$

where ∇ denotes the gradient operator on the variables \mathbf{m} .

Assumption 3 (ii) guarantees that for all $\mathbf{m} \in \gamma(\mathbb{R}^d)$, the preimage $\gamma^{-1}(\mathbf{m})$ is a finite set of points and (9b) can be more explicitly written

$$\rho_t(0, \mathbf{m}) = \mathbb{1}_{\gamma(\mathbb{R}^d)}(\mathbf{m}) \sum_{\mathbf{m}' \in \gamma^{-1}(\mathbf{m})} \frac{1}{|\det(\mathbf{J}_\gamma(\mathbf{m}'))|} \int_{\mathbb{R}_+} f(a, \mathbf{m}', x_t)\rho_t(a, \mathbf{m}')da, \tag{11}$$

where $\det(\mathbf{J}_\gamma(\mathbf{m}'))$ denotes the determinant of the Jacobian matrix $\mathbf{J}_\gamma(\mathbf{m}')$.

All the results and proofs can be adapted to the simpler case where the system is not age-dependent, as in the Erlang kernel example (3). For this example, the limit PDE (9) becomes

$$\partial_t \rho_t(\mathbf{m}) + \mathfrak{A} \nabla \cdot (\mathbf{m}\rho_t(\mathbf{m})) = f(\mathbf{m}(1) + x_t)\rho_t(\mathbf{m} - \mathbf{c}) - f(\mathbf{m}(1) + x_t)\rho_t(\mathbf{m}),$$

$$x_t = \bar{H}_t + \int_0^t h(t-s) \int_{\mathbb{R}^d} f(\mathbf{m}(1) + x_s) \rho_s(\mathbf{m}) d\mathbf{m} ds,$$

$$\rho_0(\mathbf{m}) = u_0(\mathbf{m}).$$

For the STP example (4), the limit PDE reads

$$\partial_t \rho_t(a, \mathbf{m}) + \partial_a \rho_t(a, \mathbf{m}) + \nabla \cdot (b(\mathbf{m}) \rho_t(a, \mathbf{m})) = -f(a, x_t) \rho_t(a, \mathbf{m}),$$

$$\rho_t(0, \mathbf{m}) = \mathbb{1}_{\gamma(D)}(\mathbf{m}) \frac{1}{1 - \mathbf{m}(1)} \int_{\mathbb{R}_+} f(a, x_t) \rho_t(a, \gamma^{-1}(\mathbf{m})) da,$$

$$x_t = \bar{H}_t + \int_0^t \bar{h}(t-s) \int_{\mathbb{R}^d} \int_{\mathbb{R}_+} \mathbf{m}(1) \mathbf{m}(2) f(a, x_s) \rho_s(a, \mathbf{m}) da d\mathbf{m} ds,$$

$$\rho_0(a, \mathbf{m}) = u_0(a, \mathbf{m}),$$

where $D =]U, 1[\times]0, 1[$ and $\gamma(\mathbf{m}) = (U + [1 - U]\mathbf{m}(1), [1 - \mathbf{m}(1)]\mathbf{m}(2))$.¹

Theorems 1 and 2 have two important implications for neuronal modeling: first, they provide a rigorous footing to multidimensional population density equations, which could be simulated using mesh-based methods described in [7,27,33]; second, they confirm that, not only in the simple ‘renewal’ cases, the PDE point of view can be used to study the nonlinear dynamics of large networks of spiking neurons [12].

3. Proof of Theorem 1 (Propagation of chaos)

The approach here is standard. We use a fixed-point argument to show that the limit process (6) is well-defined. Then, we use the coupling method [47] to prove that a typical particle converges to the limit process.

3.1. Well-posedness of the limit process

Proposition 1. *Grant Assumption 1 and assume that $(\mathbb{E}[H_t^1])_{t \in \mathbb{R}_+} \in \mathcal{C}(\mathbb{R}_+)$. There exists a path-wise unique càdlàg strong solution $(A_t^1, \mathbf{M}_t^1, x_t)_{t \in \mathbb{R}_+}$ taking values in $\mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}$ to (6). Furthermore, $(x_t)_{t \in \mathbb{R}_+} \in \mathcal{C}(\mathbb{R}_+)$.*

Proof. For all $y \in \mathcal{C}(\mathbb{R}_+)$, let us write $(A_t^y, \mathbf{M}_t^y, x_t^y)_{t \in \mathbb{R}_+}$ the càdlàg strong solution to

$$A_t^y = A_0^1 + t - \int_0^t A_{s-}^y dZ_s^y,$$

$$\mathbf{M}_t^y = \mathbf{M}_0^1 + \int_0^t b(\mathbf{M}_s^y) ds + \int_0^t \Gamma(\mathbf{M}_{s-}^y) dZ_s^y,$$

$$Z_t^y = \int_{[0,t] \times \mathbb{R}_+} \mathbb{1}_{z \leq f(A_{s-}^y, \mathbf{M}_{s-}^y, y_s)} \pi^1(ds, dz).$$

Then, we set

$$x_t^y = \mathbb{E}[H_t^1] + \int_0^t \mathbb{E}[h(t-s, A_s^y, \mathbf{M}_s^y) f(A_s^y, \mathbf{M}_s^y, y_s)] ds.$$

Since f and h are bounded, by dominated convergence, we have that $(x_t^y)_{t \in \mathbb{R}_+} \in \mathcal{C}(\mathbb{R}_+)$. Thus, for all $T > 0$, we can define the operator

$$\Phi_T : \mathcal{C}([0, T]) \rightarrow \mathcal{C}([0, T]), \quad (y_t)_{t \in [0, T]} \mapsto (x_t^y)_{t \in [0, T]}.$$

¹ Using (11), a simple calculation gives $\det(\mathbf{J}_\gamma(\gamma^{-1}(\mathbf{m}))) = 1 - \mathbf{m}(1)$.

The solution $(A_t^y, \mathbf{M}_t^y, x_t^y)_{t \in [0, T]}$ is a solution to (6) on $[0, T]$ if and only if $(y_t)_{t \in [0, T]} = (x_t^y)_{t \in [0, T]}$, or equivalently, if and only if $(y_t)_{t \in [0, T]}$ is a fixed point of Φ_T . We are going to show that for T small enough, Φ_T is a contraction for the uniform norm.

For all y and $y^* \in \mathcal{C}([0, T])$, by triangular inequality and using the Lipschitz continuity and the boundedness of f and h , we have, for all $t \in [0, T]$,

$$\left| \int_0^t \mathbb{E}[h(t-s, A_s^y, \mathbf{M}_s^y) f(A_s^y, \mathbf{M}_s^y, y_s)] ds - \int_0^t \mathbb{E}[h(t-s, A_s^{y^*}, \mathbf{M}_s^{y^*}) f(A_s^{y^*}, \mathbf{M}_s^{y^*}, y_s^*)] ds \right| \leq C \int_0^t \Delta_s ds,$$

where

$$\Delta_s := \mathbb{E} \left[|\psi(A_s^y) - \psi(A_s^{y^*})| + \|\mathbf{M}_s^y - \mathbf{M}_s^{y^*}\| + |y_s - y_s^*| \right].$$

By Itô’s formula for jump processes,

$$\psi(A_t^y) = \psi(A_0^1) + \int_0^t \psi'(A_s^y) ds - \int_{[0, t] \times \mathbb{R}_+} \psi(A_{s-}^y) \mathbb{1}_{z \leq f(A_{s-}^y, \mathbf{M}_{s-}^y, y_s)} \pi^1(ds, dz).$$

Whence,

$$\mathbb{E}[|\psi(A_t^y) - \psi(A_t^{y^*})|] \leq \mathbb{E} \left[\left| \int_0^t \psi'(A_s^y) - \psi'(A_s^{y^*}) ds \right| + \mathbb{E} \left[\left| \int_{[0, t] \times \mathbb{R}_+} \psi(A_{s-}^y) \mathbb{1}_{z \leq f(A_{s-}^y, \mathbf{M}_{s-}^y, y_s)} - \psi(A_{s-}^{y^*}) \mathbb{1}_{z \leq f(A_{s-}^{y^*}, \mathbf{M}_{s-}^{y^*}, y_s^*)} \pi^1(ds, dz) \right| \right] \right].$$

Notice that by Assumption 1, $|\psi'(A_s^y) - \psi'(A_s^{y^*})| \leq \kappa |\psi(A_s^y) - \psi(A_s^{y^*})|$. Then, by triangular inequality and using the Lipschitz continuity and the boundedness of f and ψ , we easily get

$$\mathbb{E}[|\psi(A_t^y) - \psi(A_t^{y^*})|] \leq C \int_0^t \Delta_s ds;$$

similarly, using the Lipschitz continuity of b , Γ and f and the boundedness of Γ and f , we get $\mathbb{E}[\|\mathbf{M}_t^y - \mathbf{M}_t^{y^*}\|] \leq C \int_0^t \Delta_s ds$. Thus, for all $t \in [0, T]$,

$$\Delta_t \leq C \int_0^t \Delta_s ds + \|y - y^*\|_\infty,$$

and by Grönwall’s lemma, $\Delta_t \leq \|y - y^*\|_\infty \exp(Ct)$. Whence,

$$\|\Phi_T(y) - \Phi_T(y^*)\|_\infty \leq C'T \exp(CT) \|y - y^*\|_\infty.$$

For T small enough, Φ_T is a contraction and has a unique fixed point by Banach’s fixed-point theorem. The fixed point gives the unique solution to (6) on $[0, T]$. Since the constants C and C' do not depend on T nor on the law of $(A_0^1, \mathbf{M}_0^1, (H_t^1)_{t \in \mathbb{R}_+})$, we can iterate the argument above on successive time intervals of length T to obtain the solution to (6) on \mathbb{R}_+ . \square

3.2. Convergence

Proof. The proof of the convergence (5) follows the same general strategy as in [9, Theorem 7].

For all $i = 1, \dots, N$, we define the *coupled* limit process $(A_t^i, \mathbf{M}_t^i, Z_t^i)_{t \in \mathbb{R}_+}$ as the path-wise unique càdlàg strong solution to

$$\begin{aligned} A_t^i &= A_0^i + t - \int_0^t A_{s-}^i dZ_s^i, \\ \mathbf{M}_t^i &= \mathbf{M}_0^i + \int_0^t b(\mathbf{M}_s^i) ds + \int_0^t \Gamma(\mathbf{M}_{s-}^i) dZ_s^i, \\ Z_t^i &= \int_{[0,t] \times \mathbb{R}_+} \mathbb{1}_{z \leq f(A_{s-}^i, \mathbf{M}_{s-}^i, x_s)} \pi^i(ds, dz), \\ x_t &= \mathbb{E}[H_t^1] + \int_0^t \mathbb{E}[h(t-s, A_s^1, \mathbf{M}_s^1) f(A_s^1, \mathbf{M}_s^1, x_s)] ds. \end{aligned}$$

The process $(A_t^i, \mathbf{M}_t^i, Z_t^i)_{t \in \mathbb{R}_+}$ is coupled to $(A_t^{i,N}, \mathbf{M}_t^{i,N}, Z_t^{i,N})_{t \in \mathbb{R}_+}$ in the sense that it shares the same $(A_0^i, \mathbf{M}_0^i, (H_t^i)_{t \in \mathbb{R}_+})$ and the same Poisson random measure π^i . The variable x_t has no index i as it is the same for all i ; it can be interpreted as the deterministic time-varying ‘mean field’ which acts uniformly on all the individual processes. Importantly, the limit processes $\{(A_t^i, \mathbf{M}_t^i, Z_t^i)_{t \in \mathbb{R}_+}\}_{i=1}^N$ are *i.i.d.* For all $t \geq 0$, let us define

$$\Delta_t^{1,N} := \mathbb{E} \left[|\psi(A_t^{1,N}) - \psi(A_t^1)| + \|\mathbf{M}_t^{1,N} - \mathbf{M}_t^1\| + |X_t^N - x_t| \right].$$

Arguing as in the proof of Proposition 1, we get

$$\mathbb{E} \left[|\psi(A_t^{1,N}) - \psi(A_t^1)| + \|\mathbf{M}_t^{1,N} - \mathbf{M}_t^1\| \right] \leq C \int_0^t \Delta_s^{1,N} ds.$$

It remains to control the term $\mathbb{E} [|X_t^N - x_t|]$:

Fix $T > 0$. For all $t \in [0, T]$, by triangular inequality,

$$\begin{aligned} \mathbb{E} [|X_t^N - x_t|] &\leq \mathbb{E} \left[\left| \frac{1}{N} \sum_{i=1}^N H_t^i - \mathbb{E}[H_t^1] \right| \right] \\ &+ \mathbb{E} \left[\left| \frac{1}{N} \sum_{i=1}^N \int_{[0,t] \times \mathbb{R}_+} h(t-s, A_{s-}^{i,N}, \mathbf{M}_{s-}^{i,N}) \mathbb{1}_{z \leq f(A_{s-}^{i,N}, \mathbf{M}_{s-}^{i,N}, x_{s-}^N)} \pi^i(ds, dz) \right. \right. \\ &\quad \left. \left. - \frac{1}{N} \sum_{i=1}^N \int_{[0,t] \times \mathbb{R}_+} h(t-s, A_{s-}^i, \mathbf{M}_{s-}^i) \right. \right. \\ &\quad \left. \left. \times \mathbb{1}_{z \leq f(A_{s-}^i, \mathbf{M}_{s-}^i, x_s)} \pi^i(ds, dz) \right| \right] \\ &+ \mathbb{E} \left[\left| \frac{1}{N} \sum_{i=1}^N \int_{[0,t] \times \mathbb{R}_+} h(t-s, A_{s-}^i, \mathbf{M}_{s-}^i) \mathbb{1}_{z \leq f(A_{s-}^i, \mathbf{M}_{s-}^i, x_s)} \pi^i(ds, dz) \right. \right. \\ &\quad \left. \left. - \int_0^t \mathbb{E}[h(t-s, A_s^1, \mathbf{M}_s^1) \right. \right. \\ &\quad \left. \left. \times f(A_s^1, \mathbf{M}_s^1, x_s)] ds \right| \right] \\ &=: Q_t^N + R_t^N + S_t^N. \end{aligned} \tag{12}$$

By Cauchy–Schwarz inequality and Assumption 2,

$$Q_i^N \leq \left(\frac{\text{Var}[H_t^1]}{N} \right)^{1/2} \leq C_{T,0} N^{-1/2}.$$

By exchangeability, triangular inequality and by the Lipschitz continuity and boundedness of f and h ,

$$R_t^N \leq C \int_0^t \Delta_s^{1,N} ds.$$

By Cauchy–Schwarz inequality,

$$\begin{aligned} S_t^N &\leq \mathbb{E} \left[\left(\frac{1}{N} \sum_{i=1}^N \int_{[0,t] \times \mathbb{R}_+} h(t-s, A_{s-}^i, \mathbf{M}_{s-}^i) \mathbb{1}_{z \leq f(A_{s-}^i, \mathbf{M}_{s-}^i, x_s)} \pi^i(ds, dz) \right. \right. \\ &\quad \left. \left. - \int_0^t \mathbb{E}[h(t-s, A_s^1, \mathbf{M}_s^1) \times f(A_s^1, \mathbf{M}_s^1, x_s)] ds \right)^2 \right]^{1/2} \\ &= \text{Var} \left[\int_{[0,t] \times \mathbb{R}_+} h(t-s, A_{s-}^1, \mathbf{M}_{s-}^1) \mathbb{1}_{z \leq f(A_{s-}^1, \mathbf{M}_{s-}^1, x_s)} \pi^1(ds, dz) \right]^{1/2} N^{-1/2}. \end{aligned}$$

However, writing $\tilde{\pi}^1(ds, dz) := \pi^1(ds, dz) - dsdz$ the compensated Poisson random measure, we have, by Itô isometry for compensated jump processes,

$$\begin{aligned} &\text{Var} \left[\int_{[0,t] \times \mathbb{R}_+} h(t-s, A_{s-}^1, \mathbf{M}_{s-}^1) \mathbb{1}_{z \leq f(A_{s-}^1, \mathbf{M}_{s-}^1, x_s)} \pi^1(ds, dz) \right] \\ &= \mathbb{E} \left[\left(\int_{[0,t] \times \mathbb{R}_+} h(t-s, A_{s-}^1, \mathbf{M}_{s-}^1) \mathbb{1}_{z \leq f(A_{s-}^1, \mathbf{M}_{s-}^1, x_s)} \tilde{\pi}^1(ds, dz) \right)^2 \right] \leq T \|h\|_\infty^2 \|f\|_\infty. \end{aligned}$$

Hence, $S_t^N \leq C_T N^{-1/2}$. Gathering the bounds, we get

$$\mathbb{E} \left[|X_t^N - x_t| \right] \leq C \int_0^t \Delta_s^{1,N} ds + C_{T,0} N^{-1/2}.$$

Finally,

$$\Delta_t^{1,N} \leq C \int_0^t \Delta_s^{1,N} ds + C_{T,0} N^{-1/2}, \quad \forall t \in [0, T],$$

and by Grönwall’s lemma,

$$\Delta_t^{1,N} \leq C_{T,0} \exp(C_T T) N^{-1/2}, \quad \forall t \in [0, T], \tag{13}$$

which concludes the proof of (5).

By exchangeability,

$$\begin{aligned} &\mathbb{E} \left[W_1 \left(\frac{1}{N} \sum_{i=1}^N \delta_{(\psi(A_t^{i,N}), \mathbf{M}_t^{i,N})}, \mathcal{L}(\psi(A_t^1), \mathbf{M}_t^1) \right) \right] \\ &\leq \mathbb{E} \left[|\psi(A_t^{1,N}) - \psi(A_t^1)| + \|\mathbf{M}_t^{1,N} - \mathbf{M}_t^1\| \right] \end{aligned}$$

$$+ \mathbb{E} \left[W_1 \left(\frac{1}{N} \sum_{i=1}^N \delta_{(\psi(A_t^i), \mathbf{M}_t^i)}, \mathcal{L} \left(\psi(A_t^1), \mathbf{M}_t^1 \right) \right) \right].$$

Then, we simply use (13) and a result on the convergence of the empirical measures in Wasserstein distance [13, Theorem 1] to get (7). \square

4. Alternative propagation of chaos result

Theorem 1 guarantees the convergence of the time-marginals (see (8)), which is sufficient for relating the empirical measure of the system (1) with the PDE (9). However, under more restrictive assumptions on the vector field b and the jump mapping γ , it is possible to get the convergence of the processes, as in [4,9,10].

Assumption 4.

- (i) Writing $(B_t)_{t \in \mathbb{R}_+}$ the flow associated with the vector field b , for all $t \geq 0$, B_t is 1-Lipschitz for the Euclidean distance.
- (ii) The jump mapping γ is 1-Lipschitz for the Euclidean distance.

Theorem 3. Grant Assumptions 1, 2 and 4. For all $T > 0$, there exists $C_{T,0} > 0$ such that

$$\mathbb{E} \left[\sup_{t \in [0, T]} |\psi(A_t^{1,N}) - \psi(A_t^1)| + \|\mathbf{M}_t^{1,N} - \mathbf{M}_t^1\| \right] \leq C_{T,0} N^{-1/2}, \tag{14}$$

where $(A_t^1, \mathbf{M}_t^1)_{t \in \mathbb{R}_+}$ is given by the path-wise unique strong solution to (6).

Proof. The well-posedness of the limit process $(A_t^1, \mathbf{M}_t^1, x_t)_{t \in \mathbb{R}_+}$ has already been proved in Proposition 1. For the convergence, we follow the same strategy as in [9, Theorem 8] (see also [4, Theorem IV.1] and [10, Theorem 1]).

Let $\{(A_t^i, \mathbf{M}_t^i, Z_t^i)_{t \in \mathbb{R}_+}\}_{i=1}^N$ be the same coupled limit process as in Section 3.2. The integral $\int_0^t |d(Z_s^{1,N} - Z_s^1)|$ counts the number of times one counting process jumps whereas the other does not, on the time interval $[0, t]$. We define

$$\delta_t^N := \mathbb{E} \left[\int_0^t |d(Z_s^{1,N} - Z_s^1)| \right] = \int_0^t \mathbb{E}[|f(A_s^{1,N}, \mathbf{M}_s^{1,N}, X_s^N) - f(A_s^1, \mathbf{M}_s^1, x_s)|] ds.$$

The key observation is that Assumption 4 guarantees

$$\mathbb{E} \left[\sup_{s \in [0, t]} |\psi(A_s^{1,N}) - \psi(A_s^1)| + \|\mathbf{M}_s^{1,N} - \mathbf{M}_s^1\| \right] \leq C \delta_t^N : \tag{15}$$

Clearly, $\sup_{s \in [0, t]} |\psi(A_s^{1,N}) - \psi(A_s^1)| \leq \|\psi\|_\infty \cdot \mathbb{1}_{\int_0^t |d(Z_s^{1,N} - Z_s^1)| > 0} \leq \|\psi\|_\infty \int_0^t |d(Z_s^{1,N} - Z_s^1)|$, which implies that $\mathbb{E}[\sup_{s \in [0, t]} |\psi(A_s^{1,N}) - \psi(A_s^1)|] \leq \|\psi\|_\infty \delta_t^N$. On the other hand, by Assumption 4(i), in a time interval with no jumps in $Z^{1,N}$ nor Z^1 , $\|\mathbf{M}_t^{1,N} - \mathbf{M}_t^1\|$ cannot increase. If both $Z^{1,N}$ and Z^1 jump at time t , by Assumption 4(ii), $\|\mathbf{M}_t^{1,N} - \mathbf{M}_t^1\| \leq \|\mathbf{M}_{t-}^{1,N} - \mathbf{M}_{t-}^1\|$. Hence, the only way to have $\|\mathbf{M}_t^{1,N} - \mathbf{M}_t^1\| > \|\mathbf{M}_{t-}^{1,N} - \mathbf{M}_{t-}^1\|$ is if $Z^{1,N}$ jumps at time t but not Z^1 or vice versa. However, in these cases, the increase is bounded by $\| \Gamma \|_\infty$. In summary, we have that $\sup_{s \in [0, t]} \|\mathbf{M}_s^{1,N} - \mathbf{M}_s^1\| \leq \| \Gamma \|_\infty \int_0^t |d(Z_s^{1,N} - Z_s^1)|$, which concludes the verification of (15).

We now have to control δ_t^N . Using (15), we have that

$$\begin{aligned} \delta_t^N &\leq L_f \int_0^t \mathbb{E} \left[|\psi(A_s^{1,N}) - \psi(A_s^1)| + \|\mathbf{M}_s^{1,N} - \mathbf{M}_s^1\| + |X_s^N - x_s| \right] ds \\ &\leq L_f \int_0^t C \delta_s^N + \mathbb{E}[|X_s^N - x_s|] ds. \end{aligned}$$

Fix $T > 0$, for all $s \in [0, T]$, we can bound $\mathbb{E}[|X_s^N - x_s|]$ as in the proof of Theorem 1 (see (12)):

$$\mathbb{E}[|X_s^N - x_s|] \leq Q_s^N + R_s^N + S_s^N.$$

with the same variance bounds $Q_s^N + S_s^N \leq C_{T,0} N^{-1/2}$. By exchangeability, triangular inequality and using (15), we have

$$R_s^N \leq \|h\|_\infty \delta_s^N + \|f\|_\infty L_h \int_0^s \mathbb{E}[|\psi(A_u^{1,N}) - \psi(A_u^1)| - \|\mathbf{M}_u^{1,N} - \mathbf{M}_u^1\|] du \leq C_T \delta_s^N.$$

Gathering the bounds, we get

$$\delta_t^N \leq C_T \int_0^t \delta_s^N ds + C_{T,0} N^{-1/2}, \quad \forall t \in [0, T],$$

and we conclude using Grönwall’s lemma. \square

By exchangeability, (14) implies that for all $T > 0$, there exists $C'_{T,0} > 0$ such that

$$\mathbb{E} \left[\sup_{t \in [0, T]} |\psi(A_t^{1,N}) - \psi(A_t^1)| + \|\mathbf{M}_t^{1,N} - \mathbf{M}_t^1\| + |\psi(A_t^{2,N}) - \psi(A_t^2)| + \|\mathbf{M}_t^{2,N} - \mathbf{M}_t^2\| \right] \leq C'_{T,0} N^{-1/2}.$$

By standard arguments on the Skorokhod metric and the Continuous mapping theorem, we have the weak convergence

$$\left(A_t^{1,N}, \mathbf{M}_t^{1,N}, A_t^{2,N}, \mathbf{M}_t^{2,N} \right)_{t \in \mathbb{R}_+} \xrightarrow[N \rightarrow \infty]{w} \left(A_t^1, \mathbf{M}_t^1, A_t^2, \mathbf{M}_t^2 \right)_{t \in \mathbb{R}_+}.$$

Since $(A_t^1, \mathbf{M}_t^1)_{t \in \mathbb{R}_+}$ and $(A_t^2, \mathbf{M}_t^2)_{t \in \mathbb{R}_+}$ have the same law, by [47, Proposition 2.2], we have the convergence in probability of the empirical measure of the system (1) to the law of the limit process:

$$\frac{1}{N} \sum_{i=1}^N \delta_{(A_t^{i,N}, \mathbf{M}_t^{i,N})_{t \in \mathbb{R}_+}} \xrightarrow[N \rightarrow \infty]{\mathbb{P}} \mathcal{L} \left((A_t^1, \mathbf{M}_t^1)_{t \in \mathbb{R}_+} \right) \quad \text{in } \mathcal{P}(\mathcal{D}(\mathbb{R}_+, \mathbb{R}_+ \times \mathbb{R}^d)), \quad (16)$$

where $\mathcal{P}(\mathcal{D}(\mathbb{R}_+, \mathbb{R}_+ \times \mathbb{R}^d))$ denotes the space of probability measures on the Skorokhod space $\mathcal{D}(\mathbb{R}_+, \mathbb{R}_+ \times \mathbb{R}^d)$ of càdlàg functions $\mathbb{R}_+ \rightarrow \mathbb{R}_+ \times \mathbb{R}^d$ and $\mathcal{L} \left((A_t^1, \mathbf{M}_t^1)_{t \in \mathbb{R}_+} \right)$ denotes the law of the process $(A_t^1, \mathbf{M}_t^1)_{t \in \mathbb{R}_+}$.

The convergence (16) is clearly stronger than the convergence of the time-marginals (7) but it requires the additional Assumption 4, which is somewhat restrictive.

5. Proof of Theorem 2 (Transport equation for the empirical measure)

Here, our aim is to show that if we write $(\rho_t \otimes \delta_{x_t})_{t \in \mathbb{R}_+}$ the time-marginals of the law of the process (6), then $(\rho_t, x_t)_{t \in \mathbb{R}_+}$ is a weak solution to (9). To show this, we use the limit

process to construct a representation formula for ρ_t . The representation formula is obtained by making rigorous the heuristic ‘path integral’ method in [34]. We then show that the path integral representation gives a weak solution to (9). Finally, we prove that weak solution to (9) is unique.

5.1. Path integral representation for the time-marginals of the law of the limit process

To formulate the path integral representation, we first need to introduce some notations and definitions.

Let $(x_t)_{t \in \mathbb{R}_+}$ be given by the limit process (6) and let us write $(A_t^*, \mathbf{M}_t^*)_{t \in \mathbb{R}_+}$ the càdlàg process following the dynamics (6a) and (6b) given $(x_t)_{t \in \mathbb{R}_+}$ and the initial condition $(a_0, \mathbf{m}_0) \in \mathbb{R}_+ \times \mathbb{R}^d$. For all $t > 0$, (A_t^*, \mathbf{M}_t^*) is deterministic given the initial condition and the jump times in $[0, t]$. Hence, for all $k \in \mathbb{N}$ (the number of jumps in $[0, t]$) and for all $0 < t_1 < \dots < t_k \leq t$ (the jump times in $[0, t]$), we can define, recursively, the mappings $\theta_t^k(t_1, \dots, t_k) : \mathbb{R}^d \rightarrow \mathbb{R}^d$, giving \mathbf{M}_t^* as a function of the initial condition \mathbf{m}_0 :

$$\theta_t^0 := B_t, \tag{17a}$$

$$\forall k \geq 1, \quad \theta_t^k(t_1, \dots, t_k) := B_{t-t_k} \circ \gamma \circ \theta_{t_k}^{k-1}(t_1, \dots, t_{k-1}), \tag{17b}$$

where $(B_t)_{t \in \mathbb{R}_+}$ is the flow associated with the vector field b .

For all $k \geq 1$, we can now define the mapping

$$\phi_t^k \left(\begin{pmatrix} t_1 \\ \vdots \\ t_{k-1} \\ t_k \\ \mathbf{m}_0 \end{pmatrix} \right) = \begin{pmatrix} t_1 \\ \vdots \\ t_{k-1} \\ t - t_k \\ \theta_t^k(t_1, \dots, t_{k-1}, t_k)(\mathbf{m}_0) \end{pmatrix}.$$

If there are k jumps in the time interval $[0, t]$ and if these jumps occur at times t_1, \dots, t_k , then $(A_t^*, \mathbf{M}_t^*) = (t - t_k, \theta_t^k(t_1, \dots, t_k)(\mathbf{m}_0))$. For $k = 0$, we simply have

$$\phi_t^0 \left(\begin{pmatrix} a_0 \\ \mathbf{m}_0 \end{pmatrix} \right) = \begin{pmatrix} a_0 + t \\ \theta_t^0(\mathbf{m}_0) \end{pmatrix}.$$

If f is bounded (Assumption 1), we can write $\eta^k(t_1, \dots, t_k; a_0, \mathbf{m}_0) dt_1 \dots dt_k$ the probability density over the k -first jump times of the process $(A_t^*, \mathbf{M}_t^*)_{t \in \mathbb{R}_+}$ having initial condition (a_0, \mathbf{m}_0) . We further define the sub-probability density $v_t^k(t_1, \dots, t_k; a_0, \mathbf{m}_0) dt_1 \dots dt_k$:

$$v_t^k(t_1, \dots, t_k; a_0, \mathbf{m}_0) := \mathbb{1}_{t_k \leq t} \int_t^\infty \eta^{k+1}(t_1, \dots, t_k, t_{k+1}; a_0, \mathbf{m}_0) dt_{k+1}.$$

Note that the mass of v_t^k is the probability of having exactly k jumps in the time interval $[0, t]$. Hence, $v_t^k / \int v_t^k$ can be interpreted as the probability density over the k jump times knowing that there are exactly k jumps in the time interval $[0, t]$.

Lastly, for all $k \geq 1$, we denote by $\Pi_{t_1, \dots, t_k, \mathbf{m}_0}^k$ and $\Pi_{a, \mathbf{m}}^k$ the projections

$$\begin{aligned} \Pi_{t_1, \dots, t_k, \mathbf{m}_0}^k &: (t_1, \dots, t_k, a_0, \mathbf{m}_0) \mapsto (t_1, \dots, t_k, \mathbf{m}_0), \\ \Pi_{a, \mathbf{m}}^k &: (t_1, \dots, t_{k-1}, a, \mathbf{m}) \mapsto (a, \mathbf{m}). \end{aligned}$$

By convention, for $k = 0$, these projections are the identity.

We have the path integral representation:

Lemma 1. Grant *Assumption 1*. Let $(A_t^1, \mathbf{M}_t^1, x_t)_{t \in \mathbb{R}_+}$ denote the limit process (6) for the initial condition $(A_0^1, \mathbf{M}_0^1) \sim u_0$ and $(\mathbb{E}[H_t])_{t \in \mathbb{R}_+} \in \mathcal{C}(\mathbb{R}_+)$. Then, for all $t > 0$, the time-marginal $\rho_t := \mathcal{L}(A_t^1, \mathbf{M}_t^1)$ is given by the representation formula

$$\rho_t = \sum_{k=0}^{\infty} (\Pi_{a, \mathbf{m}}^k \circ \phi_t^k \circ \Pi_{t_1, \dots, t_k, \mathbf{m}_0}^k)_*(v_t^k u_0). \tag{18}$$

If we further grant *Assumption 3* and if u_0 is absolutely continuous, then ρ_t is also absolutely continuous.

Proof. Let τ_k be the time of the k th jump of $(A_t^1, \mathbf{M}_t^1, x_t)_{t \in \mathbb{R}_+}$. Since (A_t^1, \mathbf{M}_t^1) is a function of the initial conditions (A_0^1, \mathbf{M}_0^1) and the jump times $\{\tau_k\}_{k \in \mathbb{N}^*}$, for any continuous and bounded test function F on $\mathbb{R}_+ \times \mathbb{R}^d$, we can write $\mathbb{E}[F(A_t, \mathbf{M}_t)]$ as a ‘path integral’:

$$\begin{aligned} \mathbb{E}[F(A_t^1, \mathbf{M}_t^1)] &= \mathbb{E}\left[F(A_t^1, \mathbf{M}_t^1) \mathbb{1}_{\{t < \tau_1\}}\right] + \sum_{k=1}^{\infty} \mathbb{E}\left[F(A_t^1, \mathbf{M}_t^1) \mathbb{1}_{\{\tau_k \leq t < \tau_{k+1}\}}\right] \\ &= \mathbb{E}\left[F(\phi_t^0(A_0^1, \mathbf{M}_0^1)) \mathbb{1}_{\{t < \tau_1\}}\right] + \sum_{k=1}^{\infty} \mathbb{E}\left[F(\Pi_{a, \mathbf{m}}^k \circ \phi_t^k(\tau_1, \dots, \tau_k, \mathbf{M}_0^1)) \mathbb{1}_{\{\tau_k \leq t < \tau_{k+1}\}}\right] \\ &= \int_{\mathbb{R}^d} \int_{\mathbb{R}_+} F(\phi_t^0(a_0, \mathbf{m}_0)) v_t^0(a_0, \mathbf{m}_0) u_0(da_0, d\mathbf{m}_0) \\ &\quad + \sum_{k=1}^{\infty} \int_{\mathbb{R}^d} \underbrace{\int_0^t \dots \int_0^t}_{k \text{ times}} F(\Pi_{a, \mathbf{m}}^k \circ \phi_t^k(t_1, \dots, t_k, \mathbf{m}_0)) \\ &\quad \times \underbrace{v_t^k(t_1, \dots, t_k; a_0, \mathbf{m}_0) u_0(da_0, d\mathbf{m}_0) dt_1 \dots dt_k}_{(\Pi_{t_1, \dots, t_k, \mathbf{m}_0}^k)_*(v_t^k u_0)} \\ &= \int_{\mathbb{R}^d} \int_{\mathbb{R}_+} F(a, \mathbf{m}) \left(\sum_{k=0}^{\infty} (\Pi_{a, \mathbf{m}}^k \circ \phi_t^k \circ \Pi_{t_1, \dots, t_k, \mathbf{m}_0}^k)_*(v_t^k u_0) \right) (da, d\mathbf{m}), \end{aligned}$$

whence the representation formula (18).

If u_0 is absolutely continuous, then $v_t^k u_0$ is absolutely continuous for all $k \geq 0$. If, in addition, *Assumption 3* is granted, then ϕ_t^k is a proper local diffeomorphism and $(\Pi_{a, \mathbf{m}}^k \circ \phi_t^k \circ \Pi_{t_1, \dots, t_k, \mathbf{m}_0}^k)_*(v_t^k u_0)$ is absolutely continuous for all $k \geq 0$. The probability measure ρ_t is therefore absolutely continuous. \square

5.2. From the limit process to weak solutions

Proposition 2. Grant *Assumptions 1* and *3*. Further assume that the law of the initial condition (A_0^1, \mathbf{M}_0^1) is the absolutely continuous probability measure $u_0(a, \mathbf{m}) da d\mathbf{m}$ and $(\mathbb{E}[H_t])_{t \in \mathbb{R}_+} = (\bar{H}_t^1)_{t \in \mathbb{R}_+} \in \mathcal{C}(\mathbb{R}_+)$. Then, the time-marginals $(\rho_t)_{t \in \mathbb{R}_+}$ of the law of the process (given by the path integral representation (18)) and $(x_t)_{t \in \mathbb{R}_+}$ is a weak solution to (9).

Proof. First, we use the path integral representation (18) to prove that $\rho \in \mathcal{C}(\mathbb{R}_+, L^1(\mathbb{R}_+ \times \mathbb{R}^d))$. We have to show that for all $T > 0$, $\rho \in \mathcal{C}([0, T], L^1(\mathbb{R}_+ \times \mathbb{R}^d))$. Let us take Z_t^1 from (6c), which counts the number of events in the time interval $[0, t]$. For any $t \in [0, T]$ and any $l \in \mathbb{N}$,

$$\sum_{k=l}^{\infty} \left\| v_t^k u_0 \right\|_{L^1} = \mathbb{P}(Z_t^1 \geq l) \leq \mathbb{P}(Z_T^1 \geq l).$$

Hence, for all $t', t \in [0, T]$,

$$\begin{aligned} & \left\| \rho_{t'} - \rho_t \right\|_{L^1} \\ & \leq \sum_{k=0}^{\infty} \left\| \left(\Pi_{a,m}^k \circ \phi_{t'}^k \circ \Pi_{t_1, \dots, t_k, \mathbf{m}_0}^k \right)_* (v_{t'}^k u_0) - \left(\Pi_{a,m}^k \circ \phi_t^k \circ \Pi_{t_1, \dots, t_k, \mathbf{m}_0}^k \right)_* (v_t^k u_0) \right\|_{L^1} \\ & \leq \sum_{k=0}^l \left\| \left(\Pi_{a,m}^k \circ \phi_{t'}^k \circ \Pi_{t_1, \dots, t_k, \mathbf{m}_0}^k \right)_* (v_{t'}^k u_0) - \left(\Pi_{a,m}^k \circ \phi_t^k \circ \Pi_{t_1, \dots, t_k, \mathbf{m}_0}^k \right)_* (v_t^k u_0) \right\|_{L^1} \\ & \quad + 2\mathbb{P}(Z_T > l). \end{aligned}$$

Since $\mathbb{P}(Z_T > l) \rightarrow 0$ as $l \rightarrow \infty$, to show that $\rho \in \mathcal{C}([0, T], L^1(\mathbb{R}_+ \times \mathbb{R}^d))$, it suffices to show that for all $k \in \mathbb{N}$,

$$\left(\left(\Pi_{a,m}^k \circ \phi_t^k \circ \Pi_{t_1, \dots, t_k, \mathbf{m}_0}^k \right)_* (v_t^k u_0) \right)_{t \in [0, T]} \in \mathcal{C}([0, T], L^1(\mathbb{R}_+ \times \mathbb{R}^d)).$$

By the density of $\mathcal{C}_c(\mathbb{R}_+ \times \mathbb{R}^d)$ in $L^1(\mathbb{R}_+ \times \mathbb{R}^d)$, for any $\epsilon > 0$, there exists $\tilde{u}_0 \in \mathcal{C}_c(\mathbb{R}_+ \times \mathbb{R}^d)$ such that $\|\tilde{u}_0 - u_0\|_{L^1} < \frac{\epsilon}{3\|f\|_{\infty}^k}$. For all $t \in [0, T]$,

$$\begin{aligned} & \left\| \left(\Pi_{a,m}^k \circ \phi_t^k \circ \Pi_{t_1, \dots, t_k, \mathbf{m}_0}^k \right)_* (v_t^k \tilde{u}_0) - \left(\Pi_{a,m}^k \circ \phi_t^k \circ \Pi_{t_1, \dots, t_k, \mathbf{m}_0}^k \right)_* (v_t^k u_0) \right\|_{L^1} \\ & = \left\| v_t^k (\tilde{u}_0 - u_0) \right\|_{L^1} \\ & \leq \|f\|_{\infty}^k \|\tilde{u}_0 - u_0\|_{L^1} \leq \frac{\epsilon}{3}. \end{aligned}$$

Hence, by triangular inequality, it only remains to show that for all $\tilde{u}_0 \in \mathcal{C}_c(\mathbb{R}_+ \times \mathbb{R}^d)$,

$$\left(\left(\Pi_{a,m}^k \circ \phi_t^k \circ \Pi_{t_1, \dots, t_k, \mathbf{m}_0}^k \right)_* (v_t^k \tilde{u}_0) \right)_{t \in [0, T]} \in \mathcal{C}([0, T], L^1(\mathbb{R}_+ \times \mathbb{R}^d)). \tag{*}$$

Since \tilde{u}_0 is compactly supported, there exists $C > 0$ such that $\text{Supp}(\tilde{u}_0) \subset [0, C] \times [-C, C]^d$. For all $t \in [0, T]$,

$$\left(\Pi_{a,m}^k \circ \phi_t^k \circ \Pi_{t_1, \dots, t_k, \mathbf{m}_0}^k \right)_* (v_t^k \tilde{u}_0) \leq \mathbb{1}_{[0, C+T] \times [-C-k\|f\|_{\infty}, C+k\|f\|_{\infty}]^d} \|\tilde{u}_0\|_{\infty} \in L^1(\mathbb{R}_+ \times \mathbb{R}^d).$$

Therefore, (*) is verified by dominated convergence. This achieves the proof that $\rho \in \mathcal{C}(\mathbb{R}_+, L^1(\mathbb{R}_+ \times \mathbb{R}^d))$.

Now, we verify that ρ satisfies (10) for all test functions. For any $G \in \mathcal{C}_c^{\infty}(\mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}^d)$ and any $T > 0$, by Itô’s formula for jump processes,

$$\begin{aligned} G(T, A_T^1, \mathbf{M}_T^1) &= G(0, A_0^1, \mathbf{M}_0^1) + \int_0^T [\partial_t + \partial_a + b(\mathbf{M}_t^1) \cdot \nabla] G(t, A_t^1, \mathbf{M}_t^1) dt \\ & \quad + \int_{[0, T] \times \mathbb{R}_+} \left(G(t, 0, \gamma(\mathbf{M}_{t-}^1)) - G(t, A_{t-}^1, \mathbf{M}_{t-}^1) \right) \mathbb{1}_{z \leq f(A_{t-}^1, \mathbf{M}_{t-}^1, x_t)} \pi^1(dt, dz). \end{aligned}$$

Taking the expectation,

$$\begin{aligned} \mathbb{E}[G(T, A_T^1, \mathbf{M}_T^1)] &= \mathbb{E}[G(0, A_0^1, \mathbf{M}_0^1)] + \int_0^T \mathbb{E} \left[[\partial_t + \partial_a + b(\mathbf{M}_t^1) \cdot \nabla] G(t, A_t^1, \mathbf{M}_t^1) \right] dt \\ &+ \int_0^T \mathbb{E} \left[\left(G(t, 0, \gamma(\mathbf{M}_t^1)) - G(t, A_t^1, \mathbf{M}_t^1) \right) f(A_t^1, \mathbf{M}_t^1, x_t) \right] dt, \end{aligned}$$

which is equivalent to

$$\begin{aligned} &\int_{\mathbb{R}_+} \int_{\mathbb{R}^d} G(T, a, \mathbf{m}) \rho_t(a, \mathbf{m}) da d\mathbf{m} \\ &= \int_{\mathbb{R}_+} \int_{\mathbb{R}^d} G(0, a, \mathbf{m}) u_0(a, \mathbf{m}) da d\mathbf{m} + \int_0^T \int_{\mathbb{R}_+} \int_{\mathbb{R}^d} \left\{ [\partial_t + \partial_a + b(\mathbf{m}) \cdot \nabla] G(t, a, \mathbf{m}) \right. \\ &\quad \left. + (G(t, 0, \gamma(\mathbf{m})) - G(t, a, \mathbf{m})) f(a, \mathbf{m}, x_t) \right\} \rho_t(a, \mathbf{m}) da d\mathbf{m} dt. \end{aligned} \tag{19}$$

Since G is compactly supported, the $T \rightarrow \infty$ limit of (19) is (10). This concludes the proof. \square

5.3. Uniqueness of weak solutions

Proposition 3. Grant Assumptions 1 and 3. For any $(u_0, \tilde{H}) \in L^1(\mathbb{R}_+ \times \mathbb{R}^d, \mathbb{R}_+) \times \mathcal{C}(\mathbb{R}_+)$, the solution to (9) is unique.

Proof. Let (ρ, x) be a weak solution for some $(u_0, \tilde{H}) \in L^1(\mathbb{R}_+ \times \mathbb{R}^d, \mathbb{R}_+) \times \mathcal{C}(\mathbb{R}_+)$. By Assumption 3, the border condition (9b) can be written like (11) and the function

$$(t, \mathbf{m}) \mapsto \mathbb{1}_{\gamma(\mathbb{R}^d)}(\mathbf{m}) \sum_{\mathbf{m}' \in \gamma^{-1}(\mathbf{m})} \frac{1}{|\det(\mathbf{J}_\gamma(\mathbf{m}'))|} \int_{\mathbb{R}_+} f(a, \mathbf{m}', x_t) \rho_t(a, \mathbf{m}') da =: p_t(\mathbf{m}) \tag{20}$$

is in $\mathcal{C}(\mathbb{R}_+, L^1(\mathbb{R}^d))$ since f is bounded and Lipschitz with respect to the third variable.

By the standard theory of transport equations with initial datum in L^1 (see [40]) and treating (20) as a source term, ρ solves

$$\begin{aligned} &\rho_t(a, \mathbf{m}) \\ &= \begin{cases} u_0(a - t, B_t^{-1}(\mathbf{m})) \exp \left(\int_0^t (\nabla \cdot b)(B_{t-s}^{-1}(\mathbf{m})) \right. \\ \quad \left. - f(a - t + s, B_{t-s}^{-1}(\mathbf{m}), x_s) ds \right) & \text{if } a \geq t, \\ p_{t-a}(B_a^{-1}(\mathbf{m})) \exp \left(\int_{t-a}^t (\nabla \cdot b)(B_{t-s}^{-1}(\mathbf{m})) \right. \\ \quad \left. - f(a - t + s, B_{t-s}^{-1}(\mathbf{m}), x_s) ds \right) & \text{if } 0 < a < t. \end{cases} \end{aligned} \tag{21}$$

Using (20) and (21), we have the rough bound $\|\rho_t\|_{L^1} \leq \|u_0\|_{L^1} \exp(t\|f\|_\infty)$:

$$\begin{aligned} \|\rho_t\|_{L^1} &\leq \|u_0\|_{L^1} + \int_0^t \int_{\mathbb{R}^d} p_{t-a}(\mathbf{m}) d\mathbf{m} da = \|u_0\|_{L^1} + \int_0^t \int_{\mathbb{R}^d} p_s(\mathbf{m}) d\mathbf{m} ds \\ &\leq \|u_0\|_{L^1} + \|f\|_\infty \int_0^t \int_{\mathbb{R}^d} \int_{\mathbb{R}_+} \rho_s(a, \mathbf{m}) da d\mathbf{m} ds \end{aligned}$$

and the bound is obtained using Grönwall’s lemma.

Let (ρ^*, x^*) be another weak solution to (9) for the same (u_0, \bar{H}) . In the following, we derive bounds on the distance $\|\rho_t - \rho_t^*\|_{L^1} + |x_t - x_t^*|$ and apply Grönwall’s lemma. This is relatively straightforward since f and h are bounded and f is Lipschitz with respect to the third variable. For all finite time $T > 0$ and for all $t \in [0, T]$,

$$\begin{aligned} & \|\rho_t - \rho_t^*\|_{L^1} \\ & \leq \int_{\mathbb{R}^d} \int_t^\infty \left| u_0(a-t, B_t^{-1}(\mathbf{m})) \right. \\ & \quad \times \exp\left(\int_0^t (\nabla \cdot b)(B_{t-s}^{-1}(\mathbf{m})) - f(a-t+s, B_{t-s}^{-1}(\mathbf{m}), x_s) ds\right) \\ & \quad \left. - u_0(a-t, B_t^{-1}(\mathbf{m})) \right. \\ & \quad \times \exp\left(\int_0^t (\nabla \cdot b)(B_{t-s}^{-1}(\mathbf{m})) - f(a-t+s, B_{t-s}^{-1}(\mathbf{m}), x_s^*) ds\right) \Big| dad\mathbf{m} \\ & + \int_{\mathbb{R}^d} \int_0^t \left| p_{t-a}(B_a^{-1}(\mathbf{m})) \right. \\ & \quad \times \exp\left(\int_{t-a}^t (\nabla \cdot b)(B_{t-s}^{-1}(\mathbf{m})) - f(a-t+s, B_{t-s}^{-1}(\mathbf{m}), x_s) ds\right) \\ & \quad \left. - p_{t-a}^*(B_a^{-1}(\mathbf{m})) \right. \\ & \quad \times \exp\left(\int_{t-a}^t (\nabla \cdot b)(B_{t-s}^{-1}(\mathbf{m})) - f(a-t+s, B_{t-s}^{-1}(\mathbf{m}), x_s^*) ds\right) \Big| dad\mathbf{m} \\ & =: Q_1 + Q_2. \end{aligned}$$

But $Q_1 \leq \|u_0\|_{L_f} \int_0^t |x_s - x_s^*| ds$, and by triangular inequality (using the shorthand $f(x_t) := f(a, \mathbf{m}, x_t)$),

$$\begin{aligned} Q_2 & \leq \left(\int_0^t \int_{\mathbb{R}^d} \rho_s(0, \mathbf{m}) d\mathbf{m} ds \right) L_f \int_0^t |x_s - x_s^*| ds + \int_0^t \int_{\mathbb{R}^d} |p_s(\mathbf{m}) - p_s^*(\mathbf{m})| d\mathbf{m} ds \\ & = \left(\int_0^t \int_{\mathbb{R}^d} \int_{\mathbb{R}_+} f(x_t) \rho_s dad\mathbf{m} ds \right) L_f \int_0^t |x_s - x_s^*| ds \\ & \quad + \int_0^t \int_{\mathbb{R}^d} \int_{\mathbb{R}_+} |f(x_t) \rho_s - f(x_t^*) \rho_s^*| dad\mathbf{m} ds \\ & \leq \|f\|_\infty \int_0^t \|\rho_s\|_{L^1} ds L_f \int_0^t |x_s - x_s^*| ds + \sup_{s \in [0, t]} \|\rho_s\|_{L^1} L_f \int_0^t |x_s - x_s^*| ds \\ & \quad + \|f\|_\infty \int_0^t \|\rho_s - \rho_s^*\| ds. \end{aligned}$$

By the rough bound on $\|\rho_t\|_{L^1}$ established above, we get

$$\|\rho_t - \rho_t^*\|_{L^1} \leq C_{T,0} \int_0^t \|\rho_s - \rho_s^*\|_{L^1} + |x_s - x_s^*| ds.$$

On the other hand,

$$|x_t - x_t^*| \leq \|h\|_\infty \int_0^t \int_{\mathbb{R}^d} \int_{\mathbb{R}_+} |f(x_s) \rho_s - f(x_s^*) \rho_s^*| dad\mathbf{m} ds,$$

which can be bounded as shown above. Whence,

$$\|\rho_t - \rho_t^*\|_{L^1} + |x_t - x_t^*| \leq C_{T,0} \int_0^t \|\rho_s - \rho_s^*\|_{L^1} + |x_s - x_s^*| ds.$$

By Grönwall's lemma, $\|\rho_t - \rho_t^*\|_{L^1} + |x_t - x_t^*| = 0$ for all $t \in [0, T]$. Since this is true for all $T > 0$, $(\rho, x) = (\rho^*, x^*)$, which concludes the proof. \square

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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