RESEARCH ARTICLE | SEPTEMBER 09 2009

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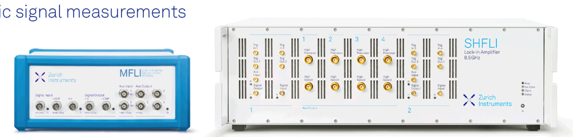
AIP Conf. Proc. 1168, 314–318 (2009)
https://doi.org/10.1063/1.3241458



A New Preconditioner on Gauss-Seidel Method for H-Matrices

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Abstract. In order to accelerate the convergency of Gauss-Seidel method to solve systems of linear equations when the coefficient matrix is an $H-$matrix, a new preconditioner is introduced. The convergency of the new preconditioned method is proved.

Keywords: Gauss-Seidel method; Preconditioning; H-matrix; H-splitting
PACS: 02.60.Dc

INTRODUCTION

Consider the following system:

$$Ax = b,\ x, b \in \mathbb{R}^n, \tag{1}$$

where $A \in \mathbb{R}^{n\times n}$ is a nonsingular matrix. To accelerate convergency of iterative methods to solve (1), the preconditioned methods are often used. Evans et al. [1] presented a preconditioner and improved the convergency rate of AOR iteration method for the original linear system (1) when A is an L-matrix. They showed that, under certain assumptions, some iterative methods which apply on some preconditioned systems are faster than iterative methods when one apply them to the original system (1). Sometimes, their assumptions are too strong in many cases. Hence, Li et al. [2] improved their method and presented a new preconditioner, which overcome the shortcomings in [1]. To solve the linear system (1), many preconditioners have been proposed [3, 4, 5, 6, 7, 8].

For any splitting, $A = M - N$ with $\det(M) \neq 0$, the basic iterative method to solve system (1) is

$$x^{(k+1)} = M^{-1}Nx^{(k)} + M^{-1}b, \qquad k = 0, 1, 2, \ldots$$

where $x^{(0)}$ is an initial vector. Matrix $M^{-1}N$ is called an iteration matrix of the basic iterative method and the method is convergent if $\rho(M^{-1}N) < 1$.
In comparison of two convergent iterative methods, the faster method is that with smaller spectral radius of its iteration matrix. The effective method to decrease the spectral radius is to precondition the linear system (1), namely,

$$PAx = Pb, \tag{2}$$

where P is a nonsingular matrix. The corresponding basic preconditioned iterative method is given in general by

$$x^{(k+1)} = M_P^{-1}N_P x^{(k)} + M_P^{-1}b, \qquad k = 0, 1, 2, \ldots$$

where $PA = M_P - N_P$. Authors in [8] set $P_\alpha = I + S_\alpha$ and authors in [9] set $P_\beta = I + K_\beta$ where S_α and K_β have the following forms

$$S_\alpha = \begin{pmatrix} 0 & -\alpha_1 a_{1,2} & 0 & \cdots & 0 \\ 0 & 0 & -\alpha_2 a_{2,3} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & -\alpha_{n-1}a_{n-1,n} \\ 0 & 0 & \ldots & 0 & 0 \end{pmatrix},\ K_\beta = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ -\beta_1 a_{2,1} & 0 & 0 & \cdots & 0 \\ 0 & & \ddots & \ldots & 0 \\ \vdots & \vdots & & \ddots & 0 \\ 0 & 0 & \cdots & -\beta_{n-1}a_{n,n-1} & 0 \end{pmatrix}.$$

CP1168, Vol. 1, *Numerical Analysis and Applied Mathematics, International Conference 2009*
edited by T. E. Simos, G. Psihoyios, and Ch. Tsitouras

In this paper we introduce a new preconditioner by setting $\widetilde{P} = I + S_\alpha + K_\beta$ with above S_α and K_β. Numerical results show that the spectral radius of iterative matrix in our new preconditioned system is lower than spectral radii of iterative matrices of preconditioned systems in [8] and [9]. First we change system (1) to $\widetilde{P}Ax = \widetilde{P}b$ or

$$A_{\alpha\beta}x = b' \tag{3}$$

where $A_{\alpha\beta} = (I + S_\alpha + K_\beta)A,\ b' = (I + S_\alpha + K_\beta)b$. By splitting $A = I - L - U$ we have

$$A_{\alpha\beta} = I - L - U + S_\alpha + K_\beta - S_\alpha L - S_\alpha U - K_\beta L - K_\beta U.$$

Therefore, the iteration matrix of Gauss-Seidel(GS) method for system (3) is

$$T_{\alpha\beta} = M_{\alpha\beta}^{-1} N_{\alpha\beta} = (I - L + K_\beta - P_{\alpha\beta}L)^{-1}(U - S_\alpha + P_{\alpha\beta}L)$$

where $P_{\alpha\beta} = (S_\alpha + K_\beta)$.
We would like $M_{\alpha\beta}$ be a nonsingular matrix, as $M_{\alpha\beta}$ is a triangular matrix, hence it is enough that its diagonal elements be nonzero. The diagonal elements of $M_{\alpha\beta}$ are $1 - \alpha_i a_{i,i+1}.a_{i+1,i}, \quad i = 1, 2, \ldots, n-1$, therefore, $M_{\alpha\beta}$ is a nonsingular matrix if

$$\alpha_i a_{i,i+1}.a_{i+1,i} \neq 1, \quad i = 1, 2, \ldots, n-1. \tag{4}$$

From now on, by assuming (4), we show that $\rho(M_{\alpha\beta}^{-1} N_{\alpha\beta}) < 1$, that is, the Gauss-Seidel method for (3) converges.

PRELIMINARIES AND NEW PRECONDITIONER

Without loss of generality, we split A in (1) as $A = I - L - U$, where I is the identity matrix, $-L$ and $-U$ are strictly lower and upper triangular matrices of A, respectively. Also we assume that $a_{i,i+1} \neq 0$ and $a_{i,i-1} \neq 0$.

Definition 1. *[10]. The splitting $A = M - N$ is called an H-splitting if $< M > -|N|$ is an M-matrix, where $|N| = (|n_{ij}|)$ and $< M >$ is the comparison matrix of M.*

Lemma 1. *[10]. Let $A = M - N$ be a splitting. If it is an H-splitting, then A and M are H-matrices and $\rho(M^{-1}N) \leq \rho(< M >^{-1} |N|) < 1$.*

Lemma 2. *[11]. Let a real matrix A have non-positive off-diagonal entries. Then matrix A is an M-matrix if and only if there exist some positive vectors $u = (u_1, \cdots, u_n)^T > 0$ such that $Au > 0$.*

Theorem 1. *[8]. Let A be an H-matrix with unit diagonal elements, $A_\alpha = (I + S_\alpha)A = M_\alpha - N_\alpha$, $M_\alpha = I - L - S_\alpha L$ and $N_\alpha = U - S_\alpha + S_\alpha U$. Let $u = (u_1, \ldots, u_n)^T$ be a positive vector such that $< A > u > 0$. Assume that $a_{i,i+1} \neq 0$ for $i = 1, 2, \cdots, n-1$, and*

$$\alpha_i' = \frac{u_i - \sum_{j=1}^{i-1} |a_{i,j}| u_j - \sum_{j=i+2}^{n} |a_{i,j}| u_j + |a_{i,i+1}| u_{i+1}}{|a_{i,i+1}| \sum_{j=1}^{n} |a_{i+1,j}| u_j}$$

then $\alpha_i' > 1$ for $i = 1, 2, \cdots, n-1$ and for $0 \leq \alpha_i < \alpha_i'$, the splitting $A_\alpha = M_\alpha - N_\alpha$ is an H-splitting and $\rho(M_\alpha^{-1} N_\alpha) < 1$ so that the iterative method (2) converges to the solution of system (1).

Theorem 2. *[9]. Let A be an H-matrix with unit diagonal elements, $A_\beta = (I + K_\beta)A = M_\beta - N_\beta$, $M_\beta = I - L + K_\beta - K_\beta L$ and $N_\beta = U + K_\beta U$. Let $u = (u_1, \ldots, u_n)^T$ be a positive vector such that $< A > u > 0$. Assume that $a_{i,i-1} \neq 0$ for $i = 2, \cdots, n$, and*

$$\beta_i' = \frac{u_i - \sum_{k=1}^{i-2} |a_{i,k}| u_k - \sum_{k=i+1}^{n} |a_{i,k}| u_k + |a_{i,i-1}| u_{i-1}}{|a_{i,i-1}| \sum_{k=1}^{n} |a_{i-1,k}| u_k}$$

then $\beta_i' > 1$ for $i = 2, \cdots, n$ and for $0 \leq \beta_i < \beta_i'$, the splitting $A_\beta = M_\beta - N_\beta$ is an H-splitting and $\rho(M_\beta^{-1} N_\beta) < 1$ so that the iterative method (2) converges to the solution of system (1).

Theorem 3. *By assumptions of Theorems 1 and 2, if* $A_{\alpha\beta}=(I+P_{\alpha\beta})A=M_{\alpha\beta}-N_{\alpha\beta}$ *where* $M_{\alpha\beta}=(I-L+K_\beta-P_{\alpha\beta}L)$ *and* $N_{\alpha\beta}=(U-S_\alpha+P_{\alpha\beta}U)$, *suppose that* $u=(u_1,u_2,\ldots,u_n)^T$ *is a positive vector such that* $<A>u>0$, *also* $a_{i,i+1}\neq 0$ *for* $i=1,2,\ldots,n-1$ *and* $a_{j,j-1}\neq 0$ *for* $j=2,3,\ldots,n$, *also* α_i', β_i' *are defined in Theorems 1 and 2, respectively. Then* $\alpha_i'>1$, $\beta_j'>1$ *and for any* $0\le\alpha_i<\alpha_i'$, $0\le\beta_j<\beta_j'$ *every splitting* $A_{\alpha\beta}=M_{\alpha\beta}-N_{\alpha\beta}$ *is an* H*-splitting and* $\rho(M_{\alpha\beta}^{-1}N_{\alpha\beta})<1$.

Proof. By Theorems 1 and 2, we have $\alpha_i'>1$ and $\beta_j'>1$. Thus it is enough to show that $A_{\alpha\beta}=M_{\alpha\beta}-N_{\alpha\beta}$ is an H-splitting and $\rho(M_{\alpha\beta}^{-1}N_{\alpha\beta})<1$.
By Lemma 1 it is enough to show that splitting $A=M_{\alpha\beta}-N_{\alpha\beta}$ is an H-splitting. But by Definition 1 we must show that $<M_{\alpha\beta}>-|N_{\alpha\beta}|$ is an M-matrix. Also, by Lemma 2 if there exists a positive vector u such that $<A>u>0$ then A is an M-matrix. Thus it is enough to show that there exists $u>0$ such that $(<M_{\alpha\beta}>-|N_{\alpha\beta}|)u>0$.
We know that

$$\begin{array}{ll} M_\alpha=I-L-S_\alpha L, & N_\alpha=U-S_\alpha+S_\alpha U,\\ M_\beta=I-L+K_\beta-K_\beta L, & N_\beta=U+K_\beta U,\\ M_{\alpha\beta}=(I-L+K_\beta-S_\alpha L-K_\beta L), & N_{\alpha\beta}=(U-S_\alpha+S_\alpha U+K_\beta U),\end{array}$$

therefore

$$<M_{\alpha\beta}>-|N_{\alpha\beta}|=<M_\alpha+K_\beta(I-L)>-|N_\alpha+K_\beta U|$$

now, we prove the following

$$[(<M_{\alpha\beta}>-|N_{\alpha\beta}|)u]\ge[(<M_\alpha>-|N_\alpha|)u]+[(<M_\beta>-|N_\beta|)u], \tag{5}$$

or we must prove that

$$\begin{aligned}(<M_\alpha+M_\beta-(I-L)>)u-|N_\alpha+N_\beta-U|u\\ \ge(<M_\alpha>+<M_\beta>)u-(|N_\alpha|+|N_\beta|)u\end{aligned} \tag{6}$$

relation (6) holds if we have the followings

$$|N_\alpha+N_\beta-U|u\le(|N_\alpha|+|N_\beta|)u \tag{7}$$

$$(<M_\alpha+M_\beta-(I-L)>)u\ge(<M_\alpha>+<M_\beta>)u. \tag{8}$$

To prove (7) we have

$$[|N_\alpha+N_\beta-U|u]_i$$

$$=|\beta_{i-1}a_{i,i-1}a_{i-1,i}|u_i+\sum_{j=i+1}^{n}|\beta_{i-1}a_{i,i-1}a_{i-1,j}-(a_{i,j}-\alpha_i a_{i,i+1}a_{i+1,j})|u_j$$

and

$$[(|N_\alpha|+|N_\beta|)u]_i$$

$$=|\beta_{i-1}a_{i,i-1}a_{i-1,i}|u_i+\sum_{j=i+1}^{n}(|\beta_{i-1}a_{i,i-1}a_{i-1,j}-a_{i,j}|+|a_{i,j}-\alpha_i a_{i,i+1}a_{i+1,j}|)u_j$$

hence, it is enough we prove the following

$$\sum_{j=i+1}^{n}(|\beta_{i-1}a_{i,i-1}a_{i-1,j}-a_{i,j}|+|a_{i,j}-\alpha_i a_{i,i+1}a_{i+1,j}|)u_j$$

$$\ge\sum_{j=i+1}^{n}|\beta_{i-1}a_{i,i-1}a_{i-1,j}-(a_{i,j}-\alpha_i a_{i,i+1}a_{i+1,j})|u_j$$

the above relation holds if we have the following

$$|\beta_{i-1}a_{i,i-1}a_{i-1,j}|\le|\beta_{i-1}a_{i,i-1}a_{i-1,j}-a_{i,j}|$$

which it holds, because A is an H-matrix.
To prove (8) we have

$$[(< M_\alpha + M_\beta - (I-L) >)u]_i$$

$$= |1-\alpha_i a_{i,i+1}a_{i+1,i}|u_i - \sum_{j=1}^{i-1} |\alpha_i a_{i,i+1}a_{i+1,j} + \beta_{i-1}a_{i,i-1}a_{i-1,j} - a_{i,j}|u_j$$

and

$$[(< M_\alpha > + < M_\beta >)u]_i$$

$$= (|1-\alpha_i a_{i,i+1}a_{i+1,i}|+1)u_i - \sum_{j=1}^{i-1} (|a_{i,j} - \alpha_i a_{i,i+1}a_{i+1,j}| + |a_{i,j} - \beta_{i-1}a_{i,i-1}a_{i-1,j}|)u_j$$

also, it is enough we prove the following

$$-\sum_{j=1}^{i-1} |\alpha_i a_{i,i+1}a_{i+1,j} + \beta_{i-1}a_{i,i-1}a_{i-1,j} - a_{i,j}|u_j$$

$$\geq u_i - \sum_{j=1}^{i-1} (|a_{i,j} - \alpha_i a_{i,i+1}a_{i+1,j}| + |a_{i,j} - \beta_{i-1}a_{i,i-1}a_{i-1,j}|)u_j$$

as there exists j such that $a_{ij} \neq 0$, therefore

$$\begin{aligned} &|\alpha_i a_{i,i+1}a_{i+1,j} - a_{i,j} + \beta_{i-1}a_{i,i-1}a_{i-1,j} - a_{i,j} + a_{i,j}| \\ &< |\alpha_i a_{i,i+1}a_{i+1,j} - a_{i,j} + \beta_{i-1}a_{i,i-1}a_{i-1,j} - a_{i,j}| \\ &\leq |a_{i,j} - \alpha_i a_{i,i+1}a_{i+1,j}| + |a_{i,j} - \beta_{i-1}a_{i,i-1}a_{i-1,j}| \end{aligned}$$

then

$$\begin{aligned} &|\alpha_i a_{i,i+1}a_{i+1,j} - a_{i,j} + \beta_{i-1}a_{i,i-1}a_{i-1,j} - a_{i,j} + a_{i,j}| \\ &< |a_{i,j} - \alpha_i a_{i,i+1}a_{i+1,j}| + |a_{i,j} - \beta_{i-1}a_{i,i-1}a_{i-1,j}|, \quad j = 1, \cdots, i-1 \end{aligned}$$

adding the above relations completes the proof, because we can select u_i small enough. Note that if $< A > u > 0$ then for all positive n, we have $< A > \frac{u}{n} > 0$. □ □

NUMERICAL EXAMPLE

Example 1. *Consider the following Laplace equation*

$$u_{xx} + u_{yy} = 0, \text{ on } R = [0, 0.5]^2$$

with boundary conditions

$$u(0,y) = 0, \quad u(x,0) = 0, \quad u(x,0.5) = 200x, \quad u(0.5,y) = 200y.$$

Applying the finite difference method with the uniform mesh size and n points, we obtain a linear system $Ax = b$. We solved the system by four different methods and different values of n. In all cases the spectral radii of the new method was the least one.

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