Augmented reality manipulatives: New mathematical tools for classrooms

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We are a group of practitioners and researchers dedicated to exploring Augmented Reality (AR) innovations for use in mathematics education. At the ATM Virtual Conference (October 2022), merging the themes of *Use of Manipulatives* and *Around Computational Thinking*, we led a workshop for members to get to grips with AR manipulatives on their own devices. Using GeoGebra 3D Calculator with AR (henceforth GeoGebra 3D/AR), we designed seven AR manipulatives and associated tasks, two of which were interactively 'walked through' during the workshop. In this article, we present an established framework for AR manipulatives, discuss how these manipulatives may be used to support mathematical thinking, consider the affordances and limitations of these new tools, and briefly cover the range of curriculum-aligned AR tasks we have created. These AR manipulatives and tasks are freely available within an online booklet we created for the conference: http://bit.ly/ATMWS. We conclude by looking at what the future might hold for this exciting and innovative technology.

What are AR manipulatives?

Within school mathematics education, manipulatives, such as Dienes' apparatus or multilink cubes, are often used to teach specific mathematical concepts. But what exactly are *AR* manipulatives and what distinguishes them from their physical and virtual predecessors? AR is a 'hybrid' technology that combines real and virtual, is interactive in real time and is registered in 3-D (Azuma, 1997). The term 'AR manipulative' was first introduced by a group of psychology and computing researchers (Bujak et al., 2013) as a new type of manipulative for the mathematics classroom. They presented a three-strand psychological framework to help us understand the potential affordances of AR manipulatives from physical, cognitive, and contextual perspectives. In the following section, we provide a rationale for using GeoGebra 3D/AR to design our AR manipulatives, then present two of these – 'AR Cola Can' and 'AR Straight Line' – through the three strands of Bujak *et al.*'s (2013) lens.

Why GeoGebra 3D/AR?

In recent years, most mobile devices have become 'AR-enabled' as a standard feature, which means that the majority of students should be able to utilise AR features on their smartphones or tablets. GeoGebra 3D/AR is a free application for any such device, which enables users to explore 3D geometry dynamically. After reviewing the free AR mathematics apps hosted by Apple Store and Google Play, our group concluded that GeoGebra 3D/AR is the most versatile (at the time of writing). Furthermore, we assumed ATM members may already be familiar with the GeoGebra interface. GeoGebra describes any web-based GeoGebra task or problem as an 'activity', and any GeoGebra file embedded within an activity is called an 'applet' (i.e. a very small or simple application program). When a learner opens an applet in GeoGebra 3D/AR and activates the AR function (by touching the "AR" button provided by GeoGebra on the screen) we call this an AR manipulative. This is when the digital construction within the applet loses its own background and instead augments the real world (as seen through the device's camera) by being superimposed onto it (see Figure 1).

We now describe how students at Key Stage 3 (11-14 year old) can use AR manipulatives when learning about the mensuration of solids and equations of linear graphs. We hope that the physical, cognitive, and contextual dimensions offered by Bujak et al.'s (2013) framework provide insight into how these manipulatives may transform mathematical thinking.

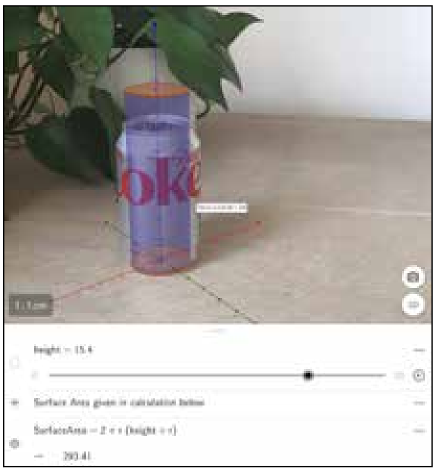

Figure 1: AR Cola Can.

NRICH's Cola Can problem – recommended for students aged 11 to 14 – is a thought-provoking introduction to rates of change (available at https://nrich.maths.org/5888). The problem is in two parts. First, students are asked to calculate the height of a 330ml can of cola if the can's diameter is 6cm, and then asked for the diameter's value if a can (of the same volume) has a height 10cm. For the second part of the problem, students are asked to find which dimensions (diameter and height) would use the least amount of aluminium.

Our group was interested in exploring how AR manipulatives could help explorations of this problem and provide support for different ways to construct mathematical meaning. To do so, we constructed two AR manipulatives using GeoGebra 3D/AR. The first AR manipulative was a dynamic AR cylinder with the numerical value of its volume displayed alongside: AR Cola Can (general). Students are able to change the AR can's height and diameter (the independent variables) and examine how these variables change its volume (the dependent variable). The second AR manipulative was an AR cylinder with a fixed volume of 330ml - AR Cola Can (Fixed Volume) – with this AR manipulative, students are able to vary the height and observe instead how the surface area and radius change to preserve this volume. The GeoGebra formula used to calculate the surface area is displayed adjacent to the camera view containing the AR manipulative. Figure 1 shows the fixed-volume AR manipulative, alongside a short YouTube demonstration (www.bit.ly/ARCanDemo) that shows users how to open these two pre-designed applets. We now consider how these AR manipulatives may support mathematical thinking – specifically through the physical, cognitive and contextual perspectives introduced by Bujak et al. (2013).

Contextual

The physical context for this problem is a can of cola; however, the physical cylinder could have been, for example, a can of beans, a tube of Pringles, a jar of honey – whatever real-world cylinder a student may be drawn to. With AR manipulatives, learners have more agency over the context, which means AR manipulatives can straddle multiple contexts. Think of a class of students all tasked with bringing in a cylindrical object from their respective homes – NRICH's Cola Can problem could be extended to explore a variety of cylinders encompassing a range of volumes. Students could compare their cylindrical dimensions and may feasibly come to discover the general 'critical value' for the ratio of height to the radius that minimises surface area.

Physical

Alongside these multiple contexts, the AR environment allows students to examine AR manipulatives from multiple visual perspectives: students are able to walk around their AR cylinders and view them from different vantage points. Bujak et al. (2013) proposed that leveraging such natural interactions may enable students to build more embodied representations of 3-D shapes, and a greater appreciation of scale.

Cognitive

The hybrid nature of AR enables the dynamic AR cylinders to be 'augmented' alongside a Cola Can, as in Figure 1. AR thus blends together concrete, pictorial, and abstract (CPA) representations of mathematics, potentially allaying concerns (highlighted by Watson and Mason (2019)) around the order in which they should be introduced – which could differ from student to student.

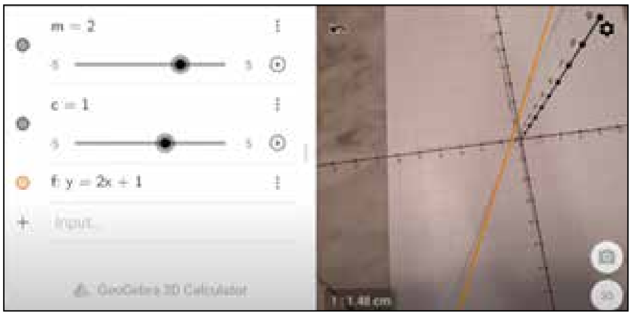

Figure 2: AR Straight Line.

When students are introduced to straight line graphs, they typically spend time substituting values of x into the algebraic equation $y = mx + c$, where m and c are given values, to obtain values for y. These are usually tabulated for equally spaced x-values within a given domain, which enables students to then plot the resulting coordinates (x,y) on a Cartesian grid and determine for themselves that a straight line passes through each plotted coordinate. Various opportunities are then given to perform similar tasks for different values of m and c in the hope that students will notice that, in addition to all the connecting lines they draw being straight, m determines the gradient, and c determines the y-intercept of each graph. This is usually done by comparing different straight line graphs on multiple pages to notice the similarities and differences and relating this to the quantities given in their equations. Ultimately, students should be able to look at any linear graph and determine its equation with minimal measurement or calculation, as well as vice-versa. In the latter case, students should be able to sketch linear graphs and provide the salient information that identifies them as unique to their given equation.

Our group designed an AR manipulative intended to cement the core concepts that enable students to achieve this ultimate aim. So, rather than being a full replacement for the above learning sequence, such an AR manipulative might be well placed within it and especially once an inkling of the effect of varying *m* and *c* arises in the learner's mind, which is most likely after the initial stages of substituting, tabulating, and plotting. Figure 2 shows the AR manipulative as an orange straight line with equation $y = 2x + 1$, which has been overlaid onto a sketch of the same graph, alongside another short YouTube demonstration (www.bit.ly/ARSLDemo). The primary idea of this activity is to gauge how accurately students can sketch (or plot) linear graphs for a given equation – and what information they need to consider and include to achieve this. Once a good match has been obtained between the sketch (or plot) and the AR manipulative, they can also experiment with the sliders for *m* and *c* to understand how they are related to the slope and position of the orange line – and how this compares to the line they drew.

Contextual

Although graphs are fundamentally abstract, by using familiar learning formats such as textbooks or pencil and paper, the subject matter could seem less alien and potentially become more accessible to learners. Many will be used to looking at a Cartesian grid on a page, so a dynamic line on top of it can only add to what they already know. Thus, any new learning about how variables affect features of a straight line graph are put into context. Additionally, students are likely to use smartphones or tablets regularly in a variety of contexts – both in and out of school – providing more familiarity and relevance. This, in turn, could lead to the learner having a more positive attitude towards their mathematics learning.

Physical

AR is not restricted to 3-D solids; even 2-D objects can be brought to life by it. By taking advantage of the time dimension, a straight line graph in a book or on a sheet of paper can appear animated by overlaying a virtual, dynamic line, controlled by on-screen sliders. It also provides a physical idea of where the third dimension (the *z*-axis) would be in relation to the Cartesian plane, to further consolidate the meaning of spatial dimensions.

Cognitive

As before, by overlaying virtual information onto real images, AR blurs the boundaries associated with CPA representations of mathematics. By having variable sliders as part of the virtual information depicting a dynamic straight line graph, a clear cause-effect link between the abstract numbers (or symbols) and pictorial lines can be established in the learner's mind. Although this is true of any dynamic graphing software viewable on a computer screen, by blending it with writing and drawings on paper, AR potentially highlights to the learner how a seemingly static graph they might have drawn has dynamic properties that depend on the equation describing it – and which aspects of it change or stay the same when the sliders are used. They can also use the sliders to match a dynamic graph with a static, pre-drawn one: inspection of the numbers arrived at potentially lends further clarity to the link between the pictorial and the abstract.

It is apparent from both of the above examples that these three strands are not necessarily mutually exclusive; for example, the physical and contextual overlap in terms of both giving the student familiar representations. The entangled nature of these strands is perhaps another attribute of AR's hybrid hallmark.

Other topics

Returning to the definition of AR as a technology that straddles the real and virtual, in 3-D and in real time, which other areas of mathematics may be well suited to this technological environment? And, in turn, which mathematical topics may be enhanced through the use of AR manipulatives? Although curricular strands concerning 3-D space are an obvious starting place for designing AR manipulatives, we have seen that 2-D graph-based topics such as linear equations can also inform the design of AR manipulatives – especially as they can dynamically 'bring to life' graphs in textbooks or sketches. In this sense, 'bridging' the gap between pencil and paper and on-screen activities (as defined by Geraniou and Mavrakis, 2015) is potentially less of a task; combining them using AR perhaps seems more akin to 'fording' the gap. Other 2-D examples include the 'AR Banana' to explore reciprocal functions, and the 'AR Parabola' to experiment with different values of 'g' for a falling object (Figures 5 and 6, respectively). Perhaps less obviously, AR manipulatives could also prove beneficial within the heavily abstract world of post-GCSE mathematics. For example, curve sketching and problem solving in 3-D space become increasingly important skills at these levels. AR activities enable *in situ,* real-time modelling, which makes them strong candidates to align with Overarching Theme 3 (OT3 Modelling) across both A- and AS-Levels within the National Curriculum. Moreover, all the 3-D topics listed as Key Stage 5 in Table 1 are Further Mathematics topics. The AR Bottle (Figure 7) is an example of this.

KS3	KS4	KS5
• Analyse 3D shapes: AR Cola Can (Figure 1) • Explore geometric relationships e.g. similarity: AR Cone (Figure 4.) • 2D linear graphs: AR Straight Line (Figure 2) • Extend and formalise knowledge of ratio and proportion in working with measures and geometry: AR Segment (Figure 3)	• Recognise, sketch and interpret graphs such as the reciprocal function: AR Banana (Figure 5) • Use equation to find line given gradient and/or points: AR Straight Line (Figure 2) • Apply the concepts of congruence and similarity, including the relationships between areas and volumes in similar figures: AR Cone (Figure 4) • Construct and interpret plans and elevations of 3D shapes	• Model motion under gravity in a vertical plane and compare with experimental data: AR Parabola (Figure 6) • Volumes of revolution e.g. AR Bottle (Figure 7) - integrating with hyperbolic tangents. • Differential equations in context: AR Cola Can (Figure 1) • Analyse and model shapes and surfaces in 3D; 3D vectors; 3D matrix transformations; Equations of planes

Table 1: AR activities mapped to National Curriculum objectives (www.bit.ly/ATMWS)

Table 1 summarises the potential topics by key stages of the National Curriculum in England. Figures 1 to 7 illustrate the seven AR manipulatives, each designed by our group to explore one of these diverse strands of mathematics. They all depend on using on-screen sliders to change the dynamic variables being studied, which enable students to understand the effect of quantity variation on spatial objects. Although sliders are a key feature of any dynamic graphic software (including GeoGebra), the cause-effect relationship between learners and their augmented views of the real world due to AR manipulatives may serve to reify mathematical relationships for them. To better understand how this happens for these seven AR manipulatives, the aforementioned contextual, physical, and cognitive dimensions can be similarly applied to each of them.

Issues!

Although these activities highlight the affordances of AR manipulatives, there is an ongoing debate on the extent of their limitations, which we need to understand to inform decisions on whether the use of AR adds value to students' learning experiences in ways that justify the effort needed to design, teach and evaluate AR tasks! The first question a teacher is likely to ask when presented with any new technology they can use in the classroom is: "Is it worth it?" Most teachers are aware of the inevitable glitches and teething problems associated with mainstreaming new technologies. The cognitive load associated with new technologies may result in a possible reduction in learning due to the learner having to think about how to use the technology rather than the subject matter. However, Bujak et al. (2013) argue that, when compared with other technologies, the cognitive load for AR may be lower, by making physical interactions more natural. Moreover, as the technology evolves, AR manipulatives could conceivably become second nature, effortlessly accessible to users – similar to the user interface depicted in the science fiction movie *Minority Report*. For now, however, users still need to master the AR interface, and we still rely on Android or iOS mobile device apps, such as GeoGebra 3D/AR.

Feedback and Discussion with Members

We had two overarching hopes for our workshop: (1). The exemplary AR manipulatives would demonstrate how the 'bridge' between the real world and digital (often abstract) models may no longer be necessary given the visual superimposition – or 'ford' – enabled by AR; and (2). The interest of participating members would be sufficiently piqued for them to want to incorporate AR manipulatives into their own practice. We ran a short, anonymous feedback poll at the end of the session. Of the 13 delegates who responded, 62% thought they would use the AR manipulatives in their respective settings, whereas 38% were unsure. Surprisingly, no one discounted AR manipulatives there and then, as we did not record any 'I will not use these tasks in my classroom' responses! In terms of our ATM session, the majority of attendees found the workshop 'useful' (77%).

The examples presented above also include instructions for teachers and learners to support them to build the manipulatives themselves 'from scratch'. This is aimed at those who are inspired to look beneath the 'AR bonnet' and examine the dynamic mathematical tools that control GeoGebra's 3D/AR engine. Feedback from the workshop participants was that this was less of a preference, and most

Figure 3: AR Segment.

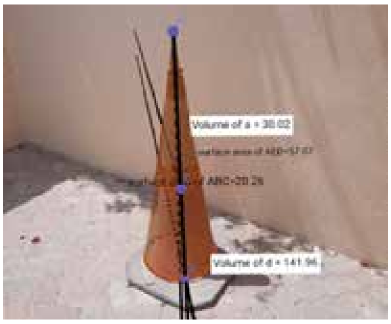

Figure 4: AR Cone.

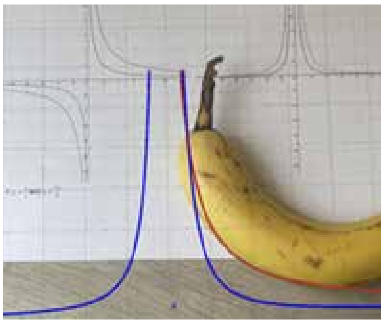

Figure 5: AR Banana.

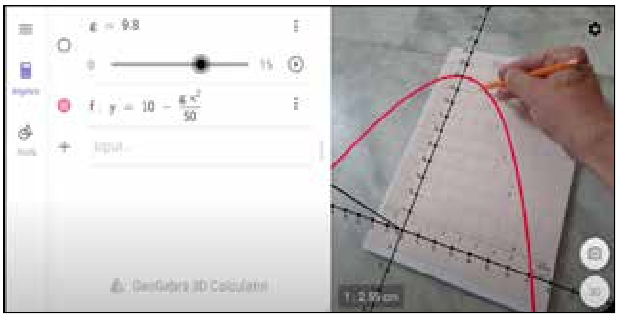

Figure 6: AR Parabola.

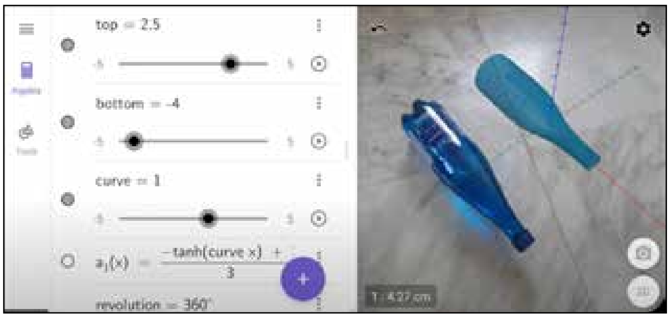

Figure 7: AR Bottle.

were happy to work with prefabricated manipulatives – at least when introducing the tasks. Given that AR manipulatives and resources can take a significant amount of time to create and/or become accustomed to, it is still unclear whether having such scaffolded, technical learning embedded within AR tasks could partially mitigate any resistance to adopt them from time-poor teachers.

The Future

It is clear from the above that more work is needed to help gauge if and how AR manipulatives might either impede or catalyse learning. What type, number and frequency of such tasks might lead to happier, more creative, and more productive learners (and teachers)? How much time should be dedicated to those tasks? Compared with traditional approaches, how might AR technology support a deeper understanding of key mathematical concepts?

One attendee at the workshop raised the highly practical point that many school environments do not allow students to bring their phones into the classroom. Where tablets are not otherwise available, this somewhat restricts the potential for schools to explore AR manipulatives as a matter of course. While there are already alternative innovations available to learners, such as 'AR glasses' and headsets, these are generally more expensive and less accessible. We anticipate a future in which learning with AR will neither be inaccessible nor dominated by mobile handsets. With commercial giants such as Apple and Google both investing in alternative wearables, this future might not be far off. We hope that neither will be the educational merit of any such commercial enterprise! This is perhaps something that can be better assured by earlier adoption of and involvement in the development of this new technology by practising educators themselves.

The authors are a group of academics and postgraduate students affiliated with UCL and East China Normal University.

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