CMC hypersurfaces with isolated singularaties: minimising properties and smooth approximations

by

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A dissertation submitted in partial fulfilment of the requirements for the award of the degree of

Doctor of Philosophy

of

University College London

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September, 2023
Declaration

I, Konstantinos Leskas, confirm that the work presented in this thesis is my own. Where information has been derived from other sources, I confirm that this has been indicated in the thesis.
Abstract

We are interested in the following problem, of local nature. Consider a constant mean curvature (CMC) hypersurface $M$ in an open ball $B$; assume that $M$ is smooth and that taking the closure of $M$ in $B$ only adds a single point. In other words, we are looking at a CMC hypersurface with no boundary in $B$ and with an isolated singularity. The question of interest for us is whether it is possible to approximate $M$ by means of completely smooth CMC hypersurfaces, closed and without boundary in $B$, that have the same mean curvature as $M$.

We look at the problem for non-zero constant mean curvature, since the case of the zero constant (minimal hypersurfaces) was treated in a classical work by R. Hardt and L. Simon. We want to identify hypotheses under which such an approximation is possible. We prove that, if $M$ minimises the prescribed mean curvature functional on one side and if the singularity satisfies a mild assumption (existence of a tangent cone that is smooth away from its vertex), such an approximation is possible in a suitably chosen small ball, in the Hausdorff distance sense; in fact, convergence holds in a stronger (graphical) sense away from the singular point.

In order to prove this result, we use techniques from both Geometric Measure Theory and Elliptic PDE theory. We initially construct an approximation by perturbing the given boundary and solving a suitable minimisation problem, where each CMC hypersurface could a priori have singularities in the open ball $B$. In order to study these potential (unwanted) singularities, we develop a maximum principle for CMC hypersurfaces without boundary in $B$ and with an isolated singularity of the same local structure as that of $M$. Once this maximum principle is established, a contradiction argument coupled with a blow-up procedure excludes the presence of singularities (making use of the aforementioned result by Hardt-Simon).

Once we have established the (abstract) smooth approximation result, we complete the theory by showing the (concrete) existence of CMC hypersurfaces with an isolated singularity that satisfy all the required hypotheses (that is, that can be smoothly approximated in the above sense).
Impact Statement

This thesis is about constant mean curvature (CMC) hypersurfaces with isolated singularities. In the calculus of variations, such objects can be thought as critical points of the prescribed mean curvature functional. CMC hypersurfaces appear in a wide spectrum of mathematics from isoperimetric problems, to the double bubble conjecture, to general relativity. It is known that CMC hypersurfaces can develop singularities and in the field of Geometric Measure Theory and Geometric Analysis in general it is of crucial importance to understand the structure of the singular set.

The scope of this thesis is to shed some light on the nature of isolated singularities of CMC hypersurfaces. We use techniques from Geometric Measure Theory and Elliptic PDEs to identify the necessary assumptions in order to prove a smooth approximation property for CMC hypersurfaces with an isolated singularity. Such a result can be useful in the perturbation of isolated singularities of CMC hypersurfaces. That is to find a smooth solution near a singular solution of the prescribed mean curvature problem.

Furthermore, examples of singular CMC hypersurfaces that minimise the prescribed mean curvature functional are few and far between. We offer a wide range of such examples that also satisfy all the assumptions of the smooth approximation property.
Acknowledgements

First and foremost, I would like to thank my supervisor Prof. Costante Bellettini whose support, advice and guidance was indispensable all these years. I would also like to thank Kobe Marshall-Stevens for many helpful discussions. The feedback of my examiners Prof. Mahir Hadzic and Prof. Giuseppe Tinaglia has helped me to improve the thesis. Finally, I would like to thank my family and friends for their moral support.
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Chapter 1

Introduction

Constant mean curvature (CMC) hypersurfaces have origin in one of the most famous and oldest problem in mathematics, that is the isoperimetric problem. One way to state it in modern mathematical language is the following: Among hypersurfaces of $\mathbb{R}^{n+1}$, with a fixed manifold boundary, that enclose a fixed amount of volume what is the one that has the least area? It turns out that solutions to this minimisation problem are CMC hypersurfaces. An equivalent way to construct CMC hypersurfaces is to minimise the prescribed mean curvature functional. Both approaches lead us to investigate a minimisation problem of a functional in the class of smooth hypersurfaces. However, it is known that such a class is not suitable due to the lack of compactness properties.

Finding the suitable class of a minimisation problem of a functional is usually a hard task and has led to the development of Geometric Measure Theory. It turns out that one way to think of (generalised) CMC hypersurfaces-with-boundary in $\mathbb{R}^{n+1}$ is as integral currents that are critical points to the prescribed mean curvature functional $E_\lambda$ under compactly supported variations that leave the boundary fixed. Alternatively, the associated integral varifold has constant generalised mean curvature. One reason why the class of integral currents is the most suitable is due to the Federer-Fleming compactness theorem, Theorem 2.1.5, which is one of the pillars of Geometric Measure Theory. Based on this compactness it turns out that we can solve certain minimisation problems of the prescribed mean curvature functional as we show in Theorem 3.1.1. However, enlarging the admissible class
of a functional comes at a cost since now solutions can be singular.

One of the main goals of Geometric Measure Theory and Geometric Analysis in general is to understand and analyse the structure of the singular set of CMC hypersurfaces of $\mathbb{R}^{n+1}$. From Allard’s regularity theorem, see Theorem 2.1.2, we can deduce that the singular set of a CMC hypersurface forms a closed and nowhere dense subset. However, this is not enough to give us any control on the size of the singular set. In fact we still do not know whether the singular set can have Hausdorff dimension $n$ or not. If we impose certain restrictions on the CMC hypersurface then a recent result of [BW18], see Theorem 3.3.1, that generalises the work of [SS81] and [WIC14a] gives that the singular set has Hausdorff dimension at most $(n - 7)$.

Another important aspect of the regularity theory developed in [BW18] is that it is optimal. In particular, there are examples of CMC hypersurfaces that satisfy all the necessary assumptions of the regularity theory of [BW18] and are singular. For the minimal case, that is when the mean curvature is zero, this is known from [BDG69] while for the CMC case, with non-zero mean curvature, this is known from the work of [IRV17] that generalises the work of [CHS84]. The simplest possible structure of the singular set is that of an isolated singularity and this is the type of singularities that we want to investigate.

Based on the existence of CMC hypersurfaces with an isolated singularity one could pose the following interesting problem. If a variational problem with respect to a fixed metric in $\mathbb{R}^{n+1}$, or in fact in any other smooth Riemannian manifold, has a singular solution can we perturb the metric and find a smooth solution of the same variational problem close to the singular one. If the variational problem is that of minimising the area functional in the class of integral currents, subject to a boundary condition, an indispensable tool to investigate such a problem is the foliation result developed from Hardt and Simon in [HS85] for area minimising cones in $\mathbb{R}^{n+1}$ with an isolated singularity. This foliation result has been recently generalised in [LIU19] to one-sided area minimising cones of $\mathbb{R}^{n+1}$ with an isolated singularity.

Inspired by the work of [HS85] and based on the extension proved in [LIU19] the scope of this thesis is to develop a smooth approximation result, Theorem 6.2.
for CMC hypersurfaces that minimise $E_\lambda$ on one side. In particular, given a CMC hypersurface with an isolated singularity that minimises $E_\lambda$ on one side we want to construct a sequence of smooth CMC hypersurfaces that minimise $E_\lambda$ on one side and converges to the initial one in the Hausdorff distance sense. Our results rely heavily on the regularity theory developed in [BW18]. Their main theorem essentially says that (weakly) stable CMC hypersurfaces that have no classical singularities and have a small set of touching singularities have a singular set of Hausdorff dimension at most $(n - 7)$.

In order to prove the approximation theorem the strategy is as follows. We perturb the boundary of the initial hypersurface in an appropriately small ball around the singular point and set up on its minimising side a minimisation problem for $E_\lambda$ with this boundary data. Hence, the initial hypersurface will act as an obstacle. This leads us to investigate properties of minimisers with a CMC hypersurface that has an isolated singularity as an obstacle. One of the main tools developed is a multiplicity-1 lemma, Lemma 3.3.2, that will allow us to implement the regularity theory of [BW18] in order to obtain the regularity theorem, Theorem 3.3.2 and Theorem 4.1.2. We also prove an extension of the monotonicity of the mass ratio, Lemma 4.1.3 that generalises the work of [SIM82] for the prescribed mean curvature functional. In this way we construct, in a small ball around the singular point, a sequence of CMC hypersurfaces that minimise $E_\lambda$ on one side and converges to the initial hypersurface. However, the hypersurfaces might be singular.

The second part of our strategy is to develop a singular maximum principle for CMC hypersurfaces with an isolated singularity. Maximum principles for minimal hypersurfaces even in the presence of large singular sets have been investigated in the work of [SIM87], [ILM96] and [WIC14b] to name a few. The main idea in these maximum principles is, roughly speaking, to use an appropriate regularity theory, together with a sheeting theorem, to construct a Jacobi field of the tangent cone at the point where the hypersurfaces meet that does not satisfy the expected behaviour near the origin. For CMC hypersurfaces there is a maximum principle proved in [WHI09] when one of the hypersurfaces is smooth everywhere. We develop a maximum principle for CMC hypersurfaces that both have an isolated singularity. In order to prove it we too investigate properties of Jacobi fields of
stable, minimal cones of $\mathbb{R}^{n+1}$ with an isolated singularity. The simplicity of the singular set of the cone makes our analysis much easier.

The final part of our strategy is to use a blow-up argument along the possible singularities of the constructed sequence. The maximum principle that we have developed makes this argument possible. At this point the work of [LIU19] on the Hardt-Simon foliation of one-sided area minimising cones is indispensable to conclude that the sequence is actually smooth.

Another scope of this thesis is to offer examples of CMC hypersurfaces that satisfy all the necessary assumptions for the approximation theorem to be valid. The main difficulty is to prove that these examples are one-sided minimisers of $E_\lambda$. There are not many known techniques to prove that a critical point is a minimiser of a functional. One way is to show that the hypersurface is actually calibrated as it is done for example in [DP09]. However, the methods from Calibrated Geometry seem unfit to our context. We manage to prove the minimising property, Theorem 7.3, using the growth rate of the Hardt-Simon’s foliation of a strictly stable, strictly area minimising cone at infinity.

Summary of Results

Chapter 3
We define the prescribed mean curvature functional $E_\lambda$ in the admissible class of integral currents of $\mathbb{R}^{n+1}$ supported in a compact set. Enlarging the set of admissible hypersurfaces of a functional allow us to use compactness properties to get existence results on minimisation problems. This is illustrated in Theorem 3.1.1 where we prove existence to certain minimisation problems of $E_\lambda$. This is done using the direct method from the calculus of variations and the main tool is the Federer-Fleming compactness theorem, Theorem 2.1.5.

It turns out that in order to apply the regularity theory of [BW18] to CMC hypersurfaces that minimise $E_\lambda$ on one side we need to rule out classical singularities. We prove the multiplicity-1 lemma, Lemma 3.3.2, where we show that the tangent cone at a singular point of the hypersurface, that is not a touching singularity, has multiplicity-1 almost everywhere, minimises the area functional on one side and the convergence is in the Radon measure sense. This allow us to exclude classical
singularities. Hence we get the regularity theorem, Theorem 3.3.2, for CMC hypersurfaces that minimise $E_\lambda$ on one side, in the absence of touching singularities, where we prove that the singular set has Hausdorff dimension at most $(n - 7)$.

Chapter 4
We introduce the notion of minimisers of $E_\lambda$ with a CMC hypersurface as an obstacle. The properties of the obstacle are crucial to investigate those of the minimiser. In our situation the obstacle is a CMC hypersurface given by an integral current $T$ in the unit ball $B_1$ with an isolated singularity at the origin, that is not a touching singularity. One crucial point is that the minimisers will not necessarily satisfy the variational equations in $B_1$ because of the obstacle. However, since the obstacle is smooth everywhere except at the origin we manage to extend the variational equations on $B_1 \setminus \{0\}$. The real difficulty is to extend them for vector fields that move the origin as well.

It is well known that monotonicity formulas and variational equations are interlinked. We extend the monotonicity formula or rather the monotonicity of the mass ratio, Lemma 4.1.3, for balls centered around the origin. This idea is also present in [SIM82] but we manage to extend it for the prescribed mean curvature functional. Once Lemma 4.1.3 is established a standard capacity argument gives that such minimisers are actually critical points of $E_\lambda$ in $B_1$. Since in this setting we can exclude touching singularities, Lemma 4.1.3 together with the regularity theorem, Theorem 3.3.2 gives the same regularity theory for minimisers of $E_\lambda$ with $T$ as an obstacle.

The focus on the rest of this chapter is to perturb the boundary of $T$ and set up a minimisation problem with that prescribed boundary data and with $T$ as an obstacle, as in Theorem 4.2.1. In this way we manage to construct a sequence of CMC hypersurfaces, possibly singular, in an appropriately small ball around the origin. Using a cut and paste argument along with the minimisation property of $T$ and the regularity theory we have developed we prove that the sequence converges to the initial hypersurface $T$ in that small ball.

Chapter 5
It is common in the field of Geometric Analysis and Geometric Measure Theory
that, at least on a local level, there is a non-linear PDE associated to the objects of interest. In our situation we show in Lemma 5.1.3 that there is a quasilinear operator that we can associate to a CMC hypersurface near its singular point and is related to the tangent cone at that point.

A standard technique in the theory of non-linear PDEs is to analyse them through some other linear PDE. That linear operator is given in Lemma 5.1.4 and Lemma 5.1.5. It turns out that the leading term of the linear PDE is the Jacobi operator $L_C$. We analyse the positive functions on the kernel of $L_C$ for stable minimal cones. At this point the work of [CHS84] is crucial to prove the formula in Lemma 5.2.2. We observe that such solutions explode as we approach the origin at a certain rate.

To prove the maximum principle, Theorem 5.3.1, for singular CMC hypersurfaces with an isolated singularity that admit a tangent cone of multiplicity-1 that is smooth except at the origin we first use Allard’s regularity theorem to get graphical pieces of the hypersurfaces over that cone. Then we rescale the height difference of those pieces so that the limit function $f$ is positive and lies on the kernel of $L_C$. The sequence is chosen so that the limit $f$ does not even explode as we approach the origin something that cannot happen from Lemma 5.2.2.

Chapter 6

We give the necessary conditions that the hypersurface $T$ must satisfy so that the smooth approximation theorem, Theorem 6.2, is valid. Essentially the hypersurface must minimise $E_\lambda$ on one side and has an isolated singularity at the origin with a tangent cone that is smooth except at the origin. The reason for choosing this structural condition at the singularity is, not only to apply the singular maximum principle Theorem 5.3.1 but also to make the Hardt-Simon foliation of [HS85] generalised in [LIU19], Theorem 6.1, applicable. From Theorem 4.2.1 we have a sequence of CMC hypersurfaces that minimise $E_\lambda$ on one side and converges to $T$. To prove that it is smooth we perform a blow-up argument along the possible singularities of the sequence. The singular maximum principle makes the blow-up possible. Then the Hardt-Simon foliation together with Allard’s regularity theorem gives the desired smoothness of the sequence.
Chapter 7
We want to show that certain examples of [IRV17], that generalise the work of [CHS84], satisfy all the assumptions of the approximation theorem, Theorem 6.2. These examples are constructed as normal perturbations of an initial cone $C$ that is strictly stable, minimal and smooth everywhere except at the origin. It turns out that the main difficulty is in proving the one-sided minimising property. From the ideas developed in [HS85] we assume that the initial cone is also strictly area minimising on one side. Such cones not only enjoy a Hardt-Simon foliation but in [HS85] a certain growth rate at infinity is proven, see Theorem 7.2, for the leaves of this foliation. Based on this growth rate we prove the one-sided minimising property. The idea is to assume that there are one-sided minimisers of $E_\lambda$ other than the initial hypersurface and then consider the height function between the graphical pieces of these hypersurfaces. We then construct a barrier between these graphical pieces that, after appropriate rescaling, gives a growth rate at infinity different to the one proven in [HS85] for the leaves of the foliation of the initial cone $C$.

Chapter 8
We briefly discuss how the approximation theorem extends to arbitrary Riemannian manifolds. From the local nature of the theorem we consider that the ambient space is $\mathbb{R}^{n+1}$ with a smooth metric $g$. Most of our arguments from the Euclidean case extend with minor modifications.
Chapter 2

Background material

2.1 Preliminaries from Geometric Measure Theory

We want to investigate minimisation problems for functionals like the prescribed mean curvature functional and for that we need compactness properties. We also want to investigate the singular set of CMC hypersurfaces. In order to do that we need the language and the techniques used in Geometric Measure Theory. In this section we gather definitions and fundamental results that will prove indispensable to our considerations. For a more detailed exposition of the subject one can see [FED69] or [SIM84] which will be our main references.

2.1.1 Integral Varifolds

It is known that varifolds are the suitable objects in order to make sense of the first variation of the area functional for a class weaker than that of smooth manifolds. We gather some definitions and theorems from varifold theory that will be used throughout this thesis.

Definition 2.1.1. (Radon measures) An outer measure $\mu$ in $\mathbb{R}^n$ is a Radon measure if it is Borel regular and finite on compact sets. By Borel regular we mean that all Borel sets are $\mu$-measurable and for any $A \subset \mathbb{R}^n$ there exists a Borel set $B$ with $A \subset B$ and $\mu(A) = \mu(B)$. 
Remark 2.1.1. The above definition can be extended to any outer measure $\mu$ on a topological space $X$ that is Hausdorff, separable and locally compact. In particular, $X$ can be a manifold.

Definition 2.1.2. (Radon measure convergence) Let $\mu_j, \mu$ be Radon measures in $\mathbb{R}^n$. We say that $\mu_j \xrightarrow{j \to \infty} \mu$ as Radon measures if $\mu_j(f) \xrightarrow{j \to \infty} \mu(f)$, for any $f \in C^0_c(\mathbb{R}^n)$.

Remark 2.1.2. We would like to mention here that one reason why Radon measures are useful is that they enjoy compactness properties under the Radon measure convergence. This is a direct application of the Banach-Alaoglu theorem from functional analysis. Another important property is the Radon-Nikodym differentiation theorem. Since we do not want to make this an exposition in measure theory the interested reader can see Theorem 4.4 and Theorem 4.7 in [SIM84] for more details.

We will keep considering for simplicity that we work in $\mathbb{R}^n$ but from Remark 2.1.1 all of our definitions extend to an arbitrary smooth Riemannian manifold. Let $U \subset \mathbb{R}^n$ be an open set and $G_m(U)$ the $m$-Grassmanian bundle over $U$. We denote by $\pi : G_m(U) \to U$ the projection map.

Definition 2.1.3. An $m$-varifold $V$ in $U$ is a Radon measure $||V||$ in $G_m(U)$.

For an $m$-varifold $V$ we define $\mu_V(A) = ||V||(\pi^{-1}(A))$, for $A \subset U$, to be the weight measure of $V$. The weight measure is a Radon measure in $U$. The mass $M(V)$ of the varifold $V$ is given by

$$M(V) = \mu_V(U).$$

We define the support of the varifold $V$, denoted by $\text{spt} V$, to be

$$\text{spt} V = \text{spt} \mu_V.$$

Recall that the support of the measure $\mu_V$ is given by

$$\text{spt} \mu_V = \{x \in U \mid \mu_V(N_x) > 0 \text{ for any open neighborhood } N_x \text{ of } x\}.$$
The most important class of varifolds is that of integral varifolds due to Allard’s regularity theorem, Theorem 2.1.2. We give the following:

Definition 2.1.4. Let $V$ be an $m$-varifold in $U$. We say that $V$ is an integral $m$-varifold if for $\mu_V$-a.e. $x \in U$ there is an $m$-plane $T_x$ and a locally integrable function $\theta : U \to \mathbb{Z}_{>0}$ such that

$$\lim_{\lambda \searrow 0} ||V_{x,\lambda}|| = \theta(x)||T_x||,$$

where the convergence is in the sense of Radon measures in $G_m(U)$ and $V_{x,\lambda}$ is the varifold given by

$$V_{x,\lambda}(A) = \lambda^{-n} \{ (\lambda y + x, T) \mid (y, T) \in G_m(U) \cap A \}, \ A \subset G_m(U).$$

We denote the space of integral $m$-varifolds of $U$ by $IV_m(U)$. We discuss now how we can describe integral $m$-varifolds from their weight measure $\mu_V$.

Recall that an $H^m$-measurable set $M \subset \mathbb{R}^n$ has locally finite measure if for any compact $K \subset \mathbb{R}^n$ we have that $H^m(M \setminus K) < \infty$.

Definition 2.1.5. Let $M \subset \mathbb{R}^n$ be an $H^m$-measurable set with locally finite measure. Then $M$ is a countably $m$-rectifiable set if

$$M \subset M_0 \bigcup (\bigcup_{j=1}^\infty M_j),$$

where $H^m(M_0) = 0$ and $M_j$ are the images of Lipschitz maps

$$F_j : \mathbb{R}^m \to \mathbb{R}^n.$$ 

From the above definition one can easily prove the following:

Lemma 2.1.1. (Lemma 11.1 [SIM84]) Let $M \subset \mathbb{R}^n$ be an $H^m$-measurable set with locally finite measure. Then $M$ is a countably $m$-rectifiable set if and only if

$$M = \bigcup_{j=0}^\infty M_j,$$

with $H^m(M_0) = 0$ and $M_j \subset N_j$ for $N_j$ $C^1$-submanifolds of $\mathbb{R}^n$ for all $j \geq 1$. The
union can be taken to be disjoint.

The most important characterisation of \( m \)-rectifiable sets that validates their significance in the field of Geometric Measure Theory and justifies the reason why sometimes they are called generalised surfaces is the following:

**Lemma 2.1.2.** *(Theorem 11.6 [SIM84])* Let \( M \subset \mathbb{R}^n \) be an \( \mathcal{H}^m \)-measurable set of locally finite measure. Then \( M \) is a countably \( m \)-rectifiable set if and only if for \( \mathcal{H}^m \)-a.e. point \( x \in M \) there exists a positive \( \mathcal{H}^m \)-measurable function \( \theta \) that is locally integrable and an \( m \)-plane \( P_x \) such that

\[
\lim_{\lambda \to 0} \int_{\eta_{x,\lambda}(M)} f(y)\theta(x + \lambda y)d\mathcal{H}^m(y) = \theta(x) \int_{P_x} f(y)d\mathcal{H}^m(y),
\]

for any \( f \in C^0_c(\mathbb{R}^n) \).

The function \( \theta \) is called the multiplicity function and the \( m \)-plane \( P_x \) the approximate tangent space at \( x \). If the function \( \theta \) is integer valued then \( M \) is called an integer \( m \)-rectifiable set.

We can relate the notions of an integer \( m \)-rectifiable set and that of an integral \( m \)-varifold from the first rectifiability theorem, Theorem 38.3 of [SIM84]. In particular, if \( V \) is an integral \( m \)-varifold then the set

\[
M = \left\{ x \in \text{spt} V \mid T_x \text{ and } \theta(x) \text{ both exist} \right\},
\]

is an \( \mathcal{H}^m \)-measurable, countably \( m \)-rectifiable set. Moreover, from Lemma 38.4 of [SIM84] we have that

\[
\int_{G_m(U)} f(x, S)d||V||([x, S]) = \int_M f(x, T_x)d\mu_V,
\]

for any \( f \in C^0_c(G_m(U)) \).

In particular, this allows us to think of integral varifolds in the same way as in Chapter 4 of [SIM84]. That is we can view integral varifolds as the equivalence class \( v(M, \theta) \), where \( M \) is an integer \( m \)-rectifiable set, \( \theta \) a locally \( \mathcal{H}^m \)-integrable function, with positive integer values and equivalence relation \( (M, \theta) \sim (M', \theta') \) if and only if \( \mathcal{H}^m((M \setminus M') \cup (M' \setminus M)) = 0 \) and \( \theta(x) = \theta'(x) \) for \( \mathcal{H}^m \)-a.e \( x \in M \cap M' \).
The associated Radon measure, which is the weight measure of the varifold, is given by

\[ \mu_V(A) = \int_{A \cap M} \theta d\mathcal{H}^m, \]

for any \( \mathcal{H}^m \)-measurable set \( A \subset U \).

We shall adopt both points of view for an integral \( m \)-varifold. It is important to stress out that one has to be careful when it comes to varifold convergence. The varifold topology is the one that comes from the Radon measure convergence on \( G_m(U) \). Hence if we only have convergence on the level of the weight measures \( \mu_V \) this is not equivalent to the varifold convergence.

We now compute the first variation of the area functional. Although this can be done for an arbitrary \( m \)-varifold, see Chapter 39 of [SIM84], we will limit ourselves to integral varifolds. In fact the integrality of the varifold plays no role in the following considerations, only the rectifiability of the varifold.

Let \( U \) be an open subset of \( \mathbb{R}^n \) and \( M \subset U \) an \( m \)-rectifiable set. We need to define the Jacobian of a Lipschitz function over \( M \). Let \( f : U \to \mathbb{R} \) be a Lipschitz function. From Lemma 2.1.1 we get the disjoint decomposition

\[ M = \bigcup_{j=0}^{\infty} M_j, \]

where \( \mathcal{H}^m(M_0) = 0 \) and \( M_j \subset N_j \), for \( N_j \) \( C^1 \)-submanifolds of \( U \) and \( j \geq 1 \). Then combined with Lemma 2.1.2 we have for \( \mathcal{H}^m \)-a.e. \( x \in M \) that \( T_x M = T_x N_j \), for an index \( j \). Thus we define

\[ \nabla^M f(x) = \nabla^{N_j} f(x). \]

The above definition is independent of the decomposition given by \( M_j \), see Remark 12.2 of [SIM84].

Dually we can define for \( \mathcal{H}^m \)-a.e. \( x \in M \) the linear map

\[ d_x^M f : T_x M \to \mathbb{R}, \]

as \( d_x^M f(v) = v \cdot \nabla^M f(x) \), where \( (,) \) denotes the Euclidean inner product. We can naturally extend this to define \( d_x^M f \) for \( f : M \to \mathbb{R}^l \) a Lipschitz function and \( l \geq 1 \).
The Jacobian of a Lipschitz function $f : M \to \mathbb{R}^l$ is defined as

$$J_M f(x) = \sqrt{\det((d^M_M f)^* \circ (d^M_M f))},$$

where $(d^M_M f)^* : \mathbb{R}^l \to T_xM$ denotes the adjoint map of the differential $d^M_M f$.

Finally, for $X \in C^1(U; \mathbb{R}^l)$ let $(E_i)$ be an orthonormal basis of $T_xM$. We define

$$\text{div}_M X = \sum_{i=1}^m E_i \cdot \nabla_{E_i} X,$$

where $(\cdot)$ denotes the Euclidean inner product and $\nabla$ is the Euclidean Levi-Civita connection. The above definition is independent of the chosen orthonormal basis of $T_xM$ as one can readily verify.

To make sense of the first variation of the area functional we need the notion of push-forward for varifolds. Again this can be defined on arbitrary varifolds but we restrict ourselves to integral ones. Let $V = v(M, \theta)$ be an integral $m$-varifold and $f : U \cap \text{spt} V \to \tilde{U}$ a Lipschitz, $1-1$ proper map, where $U, \tilde{U}$ are open subsets of $\mathbb{R}^n$. Recall that the fact that $f$ is proper means that $f^{-1}(K) \cap \text{spt} V$ is compact whenever $K$ is compact in $\tilde{U}$. We define the push-forward varifold to be

$$f_* V = v(f(M), \theta \circ f^{-1}).$$

We briefly justify why this is again an integral $m$-varifold. For that we recall the area formula

$$\int_{f(M) \cap K} \theta \circ f^{-1} d\mathcal{H}^m = \int_{M \cap f^{-1}(K)} J_M f \theta \ d\mathcal{H}^m, \quad (2.1.1)$$

where $K \subset \tilde{U}$ a compact subset. From [2.1.1] and since the map is proper we get that $\theta \circ f^{-1}$ is locally $\mathcal{H}^m$-integrable. The Lipschitz image of a rectifiable set is again a rectifiable set thus $f_* V$ is indeed an integral $m$-varifold.

We can now state the first variation formula for the area functional in the class of integral varifolds. Let $X \in C^1_c(U; \mathbb{R}^n)$ be a vector field and $\theta_t$ be the local flow generated by $X$ in $U$. The vector field $X$ is sometimes called a variation field and it
locally moves the varifold $V$ through its flow $\theta_t$. We have that
\[
\frac{d}{dt} \bigg|_{t=0} M(\theta_t^*, V) = \int_M \text{div}_M X d\mu_V. \tag{2.1.2}
\]

For a proof of the above equation one can see Chapter 4 of \[SIM84\]. An integral $m$-varifold $V$ in $U$ is stationary in $U$ if
\[
\int_M \text{div}_M X d\mu_V = 0, \tag{2.1.3}
\]
for any $X \in C^1_c(U; \mathbb{R}^n)$. In particular the first variation of the area functional vanishes. This is a generalisation of smooth minimal submanifolds to a wider class of geometric objects more suitable for variational problems.

More generally we say that the varifold $V$ has generalised mean curvature $\vec{H}$ in $U$ if
\[
\int_M \text{div}_M X d\mu_V = -\int_M X \cdot \vec{H} d\mu_V, \tag{2.1.4}
\]
for any $X \in C^1_c(U; \mathbb{R}^n)$.

We mention now the most fundamental results of integral varifold theory that will be crucial to almost all of our considerations in the sequel. The first one is the monotonicity formula.

**Theorem 2.1.1.** (Theorem 17.6 \[SIM84\]) Let $V \in IV_m(U)$ with generalised mean curvature $\vec{H}$ and $|\vec{H}| \in L^p_{\text{loc}}(\mu_V)$. Let $x_0 \in U$ and $R > 0$ such that $B_R(x_0) \subset U$ and $|\vec{H}| \leq \lambda$. Then for any $0 < \sigma < \rho < R$ there exist functions $F(\rho) \in [e^{-\lambda \rho}, e^{\lambda \rho}]$ and $G(\sigma, \rho) \in [e^{-\lambda R}, e^{\lambda R}]$ such that
\[
F(\rho) \frac{\mu_V(B_{\rho}(x_0))}{\omega_m \rho^m} - F(\sigma) \frac{\mu_V(B_{\sigma}(x_0))}{\omega_m \sigma^m} = 
\]
\[
= G(\sigma, \rho) \int_{B_{\rho}(x_0) \setminus B_{\sigma}(x_0)} \frac{|(x - x_0)\perp|^2}{|x - x_0|^{m+2}} d\mu_V.
\]

We would like to mention at this point that there is a monotonicity formula, similar to the one above, for $|\vec{H}| \in L^p_{\text{loc}}(\mu_V)$ with $p > n$. One uses the same test vector field as in the proof of Theorem 2.1.1 but now estimates the terms that involve $\vec{H}$ using Hölder’s inequality which leads to a slightly different integrating factor than
the one in the proof of Theorem 2.1.1. For more details one can see Theorem 17.7 of [SIM84].

One of the most important consequences of the monotonicity formula is the monotonicity of the mass ratio. In particular, from Theorem 2.1.1 we obtain that

$$F(\rho) \frac{\mu_V(B_\rho(x_0))}{\omega_m \rho^m},$$

is monotonically non-decreasing. A consequence of this monotonicity is the upper semi-continuity of the density of the Radon measure $\mu_V$. Recall that the density of the measure $\mu_V$ at a point $x_0 \in U$ is defined to be

$$\Theta^m(\mu_V, x_0) = \lim_{\rho \to 0} \frac{\mu_V(B_\rho(x_0))}{\omega_m \rho^m},$$

provided that the limit exists. For more about densities and their importance in measure theory one can see Chapter 1 of [SIM84].

**Corollary 2.1.1. (upper semi-continuity)** Let $V \in IV_m(U)$ with generalised mean curvature $|H| \in L^p_{\text{loc}}(\mu_V)$, $p > n$ and $x_0 \in U$. Then the density $\Theta^m(\mu_V, x_0)$ exists and is a real number. Moreover if $x_j \in U$ with $x_j \xrightarrow{j \to \infty} x_0$ then

$$\limsup_{j \to \infty} \Theta^m(\mu_V, x_j) \leq \Theta^m(\mu_V, x_0).$$

This is a consequence of the monotonicity of the mass ratio, as it can be seen from Corollary 17.8 of [SIM84].

**Remark 2.1.3.** In fact we can prove the following version of upper semi-continuity. Let $V_j, V \in IV_m(U)$ with $\mu_{V_j} \xrightarrow{j \to \infty} \mu_V$, where the latter convergence is in the sense of Definition 2.1.2. If the varifolds have generalised mean curvatures in $L^\infty_{\text{loc}}(\mu_V)$ and $x_j, x_0 \in U$ with $x_j \xrightarrow{j \to \infty} x_0$ then

$$\limsup_{j \to \infty} \Theta^m(\mu_{V_j}, x_j) \leq \Theta^m(\mu_V, x_0).$$
Proof. From Theorem 2.1.1 we have that
\[ \Theta^m(\mu_{V_j}, x_j) \leq F(\rho) \frac{\mu_{V_j}(B_{\rho}(x_j))}{\omega_m \rho^m}, \]
for any \( \rho > 0 \) and since \( x_j \to x_0 \) we get that for any \( \epsilon > 0 \) and \( j \) large enough
\[ \Theta^m(\mu_{V_j}, x_j) \leq F(\rho) \frac{\mu_{V_j}(B_{\rho+\epsilon}(x_0))}{\omega_m \rho^m}. \]
Since \( \mu_{V_j} \to \mu_V \) we get that the right hand side is
\[ \leq F(\rho) \frac{\mu_V(B_{\rho+\epsilon}(x_0))}{\omega_m \rho^m} + \epsilon' \frac{F(\rho)}{\omega_m \rho^m}. \]
We can take \( \epsilon' = \rho^2 \omega_m \) and as \( \epsilon \to 0 \) and \( \rho \to 0 \), since \( F(\rho) \to 1 \), we get the conclusion of Remark 2.1.3. \( \square \)

Remark 2.1.4. Another important consequence of the upper semi-continuity of the density function is the following. Let \( V \in IV_m(U) \) with generalised mean curvature \( \overrightarrow{H} \in L^p_{\text{loc}}(\mu_V) \), for \( p > n \), \( 0 \in \text{spt} V \) and \( B_R \subset U \). We can take the representatives of the equivalence class \( \nu(M, \theta) \) of the varifold to be
\[ M = \{ x \in U \mid \Theta^m(\mu_V, x) > 0 \} \]
and \( \theta(x) = \Theta^m(\mu_V, x) \).

See also 17.9 in [SIM84] for more details.

We now mention Allard’s regularity theorem. Note that by \( \text{gr} u \) we denote the graph of a function \( u : U \to \tilde{U} \) for \( U \subset \mathbb{R}^m \) and \( \tilde{U} \subset \mathbb{R}^l \).

Theorem 2.1.2. (Theorem 23.1 [SIM84]) Let \( V \in IV_m(U) \) with generalised mean curvature \( \overrightarrow{H} \in L^p_{\text{loc}}(\mu_V) \), for \( p > n \), \( 0 \in \text{spt} V \) and \( B_R \subset U \). Let \( \delta > 0 \) such that
\[ R^{1-\frac{p}{n}} \left( \int_{B_R(0)} |\overrightarrow{H}|^p \, d\mu_V \right)^{1/p} \leq \delta \]
and
\[ \frac{\mu_V(B_R(0))}{\omega_m R^m} \leq 1 + \delta. \]
Then there exist $\delta_0, \gamma$ positive constants that depend on $n, m, p$ such that if $\delta \leq \delta_0$ there exists $u \in C^{1,1-\frac{n}{p}}((B^m_{\gamma R}(0)); \mathbb{R}^{n-m})$ with

$$\text{gr } u \cap B_{\gamma R}(0) = \text{spt} V \cap B_{\gamma R}(0).$$

Moreover $u$ satisfies the estimates

$$\frac{1}{R} \sup |u| + \sup |Du| + R^{1-\frac{n}{p}} \sup_{x \neq y} \frac{|Du(x) - Du(y)|}{|x - y|^{1-\frac{n}{p}}} \leq C_{(n,m,p)} \delta^{\frac{1}{2n+2}},$$

where $C_{(n,m,p)}$ a constant that depends on $n, m$ and $p$.

For a detailed exposition on the proof of Allard’s regularity theorem one can see Chapter 5 of [SIM84]. Another excellent treatise can be found in [DEL18]. We give the following:

**Definition 2.1.6.** Let $V \in IV_m(U)$. We say that $x \in \text{spt} V$ is a regular point of $V$ if there exists $\rho > 0$ such that $B_\rho(x) \cap \text{spt} V$ is an $m$-dimensional $C^1$-submanifold of $\mathbb{R}^n$. We denote the set of regular points of $V$ by

$$\text{reg} V = \{ x \in \text{spt} V \mid x \text{ is a regular point of } V \}.$$  

The set

$$\text{sing} V = \text{spt} V \setminus \text{reg} V,$$

is the set of singular points of $V$.

**Corollary 2.1.2.** Let $V \in IV_m(U)$ with generalised mean curvature $|\vec{H}| \in L^p_{\text{loc}}(\mu_V)$ for $p > n$. Then $\text{reg} V$ is a relatively open dense set in $\text{spt} V$ and $\text{sing} V$ is a relatively closed, nowhere dense set in $\text{spt} V$.

At this point we would like to mention that Corollary 2.1.2 does not give any control on the size of the singular set of a varifold $V$ even in case $V$ is just stationary in codimension one, thus for $m = n - 1$. In fact it is still a wide open question whether there exists or not a stationary integral $m$-varifold with singular set of Hausdorff dimension $m$. Given suitable restrictions on the varifold $V$ there are optimal theorems in codimension one, see Theorem 3.3.1 regarding the size of the singular set.
In some occasions such as in the proofs of Theorem 5.3.1 and Theorem 7.3 we will use Allard’s regularity theorem in the following form as well.

**Corollary 2.1.3.** Let $V_j, V \in IV_{n-1}(U)$ with generalised mean curvatures $\overrightarrow{H}_j, \overrightarrow{H}$ respectively, $|\overrightarrow{H}_j| = \lambda_j$, $|\overrightarrow{H}| = \lambda$ for constants $\lambda_j, \lambda$, $\mu_{V_j} \xrightarrow{j \to \infty} \mu_V$ and $x \in \text{reg} V$ such that $\Theta^{n-1}(\mu_V, x) = 1$. Then for all sufficiently large $j$ there exists $\rho > 0$, uniform in $j$, such that $\text{spt} V_j \cap B_\rho(x) \subset \text{reg} V_j$. Moreover there are $C^{1,\alpha}$ - maps

$$\phi_j : \text{reg} V \cap B_\rho(x) \to \text{reg} V_j \cap B_\rho(x),$$

such that

$$||\phi_j - i_{\text{reg} V \cap B_\rho(x)}||_{1,\alpha} \xrightarrow{j \to \infty} 0.$$

**Proof.** Let $\delta_0$ be the universal constant from Theorem 2.1.2. Since the density of $V$ at $x$ is 1 for $\epsilon \leq \frac{\delta_0}{2}$ there exists $R > 0$ such that

$$\frac{\mu_V(B_R(x))}{R^{n-1} \omega_{n-1}} \leq 1 + \epsilon.$$

Since $\mu_{V_j} \to \mu_V$ we have, for all sufficiently large $j$, that

$$\frac{\mu_{V_j}(B_R(x))}{R^{n-1} \omega_{n-1}} \leq \frac{\mu_V(B_R(x))}{R^{n-1} \omega_{n-1}} + \epsilon,$$

Thus in total

$$\frac{\mu_{V_j}(B_R(x))}{R^{n-1} \omega_{n-1}} \leq 1 + 2\epsilon.$$

From Theorem 2.1.2 there exists $\rho > 0$, uniform in $j$ and $u_j, u$ functions such that

$$\text{gr} u_j = \text{spt} V_j \cap B_\rho(x),$$

$$\text{gr} u = \text{spt} V \cap B_\rho(x).$$

From the estimates of Theorem 2.1.2 we see that,

$$||u_j - u||_{1,\alpha} \leq C \epsilon^{\frac{1}{n+2}},$$

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for $\epsilon > 0$ arbitrary and $C$ a constant that depends on $n, p$. Thus we have that

$$\text{reg} V_j \cap B_\rho(x) \rightarrow \text{reg} V \cap B_\rho(x),$$

in $C^{1,\alpha}$-sense, as graphs. The $C^{1,\alpha}$-graph convergence implies the existence of $C^{1,\alpha}$-maps

$$\phi_j : \text{reg} V \cap B_\rho(x) \rightarrow \text{reg} V_j \cap B_\rho(x),$$

with

$$\|\phi_j - i_{\text{reg} V \cap B_\rho(x)}\|_{1,\alpha} \rightarrow 0.$$

\[\square\]

Remark 2.1.5. Using a covering argument we can get the following refinement of Corollary 2.1.3. Given any $W \subset \subset \mathbb{R}^n \setminus \text{sing} V$ then there exists an open set $U \subset \subset \text{reg} V$ with $W \cap \text{reg} V \subset U$ so that the conclusion of Corollary 2.1.3 holds for $\phi_j \in C^2(U; \text{reg} V_j \cap U)$.

2.1.2 Integral Currents

We are interested in integral currents for two main reasons. One is that, unlike varifolds, there is a natural notion of boundary. The other reason is that the direct method from calculus of variations is applicable. These two reasons make the class of integral currents suitable for setting up minimisation problems with prescribed boundary data, which is something that we want to do in the sequel. In what follows, $U$ will denote an open subset of $\mathbb{R}^n$ and $\Omega^m_c(U)$ the space of smooth compactly supported $m$-forms in $U$ endowed with the standard topology.

Definition 2.1.7. A general $m$-current is a continuous linear map $T : \Omega^m_c(U) \rightarrow \mathbb{R}$. We denote the space of general $m$-currents in $U$ by $\mathcal{D}_m(U)$.

Let $T \in \mathcal{D}_m(U)$ then its current boundary is defined to be the $(m - 1)$-current $\partial T$ given by

$$\partial T(\omega) = T(d\omega), \text{ for any } \omega \in \Omega^{m-1}_c(U).$$

Note that if $M$ is a smooth $m$-manifold-with-boundary then $M$ can be thought as
the $m$-current given by integration over $M$ and by Stoke’s theorem its manifold boundary coincides with its current boundary.

The support of a general $m$-current $T$ in $U$ is defined to be

$$\text{spt}T = U \setminus \bigcup W,$$

where the union is taken over all open sets $W \subset U$ where $T(\omega) = 0$, for any $\omega \in \Omega^m_c(U)$ with $\text{spt}\omega \subset W$.

**Definition 2.1.8.** (current convergence) Let $T_j, T \in D_m(U)$. We say that $T_j$ converges to $T$ as $j \to \infty$ and denote it by $T_j \rightharpoonup T$ if and only if $T_j(\omega) \xrightarrow[j \to \infty]{} T(\omega)$, for any $\omega \in \Omega^m_c(U)$.

The boundary operator is continuous with respect to current convergence. Thus if $T_j \rightharpoonup T$ then $\partial T_j \rightharpoonup \partial T$.

The mass $M(T)$ of an $m$-current $T$ is given by

$$M(T) = \sup_{\omega \in \Omega^m_c(U)} \left\{ T(\omega) \mid |\omega| \leq 1 \right\}.$$ 

For any open $W \subset U$ we define

$$M_W(T) = \sup_{\omega \in \Omega^m_c(U)} \left\{ T(\omega) \mid \text{spt}\omega \subset W, |\omega| \leq 1 \right\}.$$ 

The mass is lower semi-continuous with respect to current convergence. This means that if $T_j \rightharpoonup T$ then

$$M_W(T) \leq \lim \inf_{j \to \infty} M_W(T_j).$$ 

**Definition 2.1.9.** Let $T \in D_m(U)$. Then $T$ is an integer rectifiable $m$-current if

$$T(\omega) = \int_M \langle \omega_x, \overrightarrow{T}_x \rangle \theta(x) \, d\mathcal{H}^m, \quad \omega \in \Omega^m_c(U),$$

where $M$ is a countably $m$-rectifiable set, $\theta$ a locally $\mathcal{H}^m$-integrable, positive integer valued function and $\overrightarrow{T}$ an $\mathcal{H}^m$-measurable function such that for $\mathcal{H}^m$-a.e. $x \in M$ it can be expressed as $\tau_1 \land \ldots \land \tau_n$ for $(\tau_i)$ an orthonormal basis of $T_x M$, the
approximate tangent space of $M$ at $x$.

Notice that the above definition makes sense since the approximate tangent space $T_x M$ exists for $\mathcal{H}^m$-a.e. $x \in M$ from Lemma 2.1.2. For an integer rectifiable $m$-current $T$ we shall write $T = (M, \overrightarrow{T}, \theta)$. We call $\theta$ the multiplicity function and $\overrightarrow{T}$ the orientation of $T$.

The mass $M(T)$ of an integer rectifiable $m$-current $T$ is given by

$$M(T) = \int_M \theta \, d\mathcal{H}^m.$$ 

Its associated Radon measure is

$$\mu_T(A) = \int_{A \cap M} \theta \, d\mathcal{H}^m, \text{ for any } \mathcal{H}^m\text{-measurable } A \subset U.$$ 

We will sometimes denote the integer rectifiable $m$-current $T$ by $(M, \overrightarrow{T}, \mu_T)$, where $\mu_T$ is the associated Radon measure. Notice that the support of $T$ is given by $\text{spt} T = \text{spt} \mu_T$.

For an integer rectifiable $m$-current $T$ we can associate an integral $m$-varifold simply by dropping the orientation. So if $T = (M, \overrightarrow{T}, \theta)$ then the associated varifold is $V_T = \mathbf{v}(M, \theta)$. Notice that $\text{spt} V_T = \text{spt} T$, however we would like to stress out that the objects are different, especially when it comes to their respective topologies.

We also fix the following notation. Let $T = (M, \overrightarrow{T}, \mu_T)$ and $f$ a locally $\mathcal{H}^m$-integrable function. We define the current $T \lfloor f$, which is not necessarily an integer rectifiable $m$-current, as

$$(T \lfloor f)(\omega) = \int_M \langle \omega, \overrightarrow{T} \rangle f \, d\mu_T.$$ 

In particular, if $A \subset U$ is an $\mathcal{H}^m$-measurable set then $T \lfloor_A = T \lfloor_{\chi_A}$.

We will make use of the following constancy theorem for top-dimensional currents in $U$.

**Theorem 2.1.3.** (4.1.7 [FED69]) *Let $T \in \mathcal{D}_n(U)$ and $A \subset U$ open and connected. If $\text{spt} T \subset U \setminus A$ then there exists $c \in \mathbb{R}$ such that*

$$\text{spt}(T - cE^n \lfloor_U) \subset U \setminus A,$$
where $E^n$ is the current given by Lebesgue integration.

Similar to the varifold case we define the notion of push-forward for integer rectifiable $m$-currents. Let $T = (M, \overrightarrow{T}, \theta)$ be an integer rectifiable $m$-current and $f : U \cap \text{spt } T \to \tilde{U}$ a Lipschitz, $1 - 1$ proper map, where $U, \tilde{U}$ are open subsets of $\mathbb{R}^n$. We define the push-forward current to be

$$f_* T = (f(M), \theta \circ f^{-1}, N),$$

where $N_x = \frac{\langle \Lambda_m Df_x \rangle(T)}{|\langle \Lambda_m Df_x \rangle(T)|}.$

Similar to the varifold case, from the area formula 2.1.1, we have that $f_* T$ is again an integer rectifiable $m$-current. Moreover, if $V_T$ is the associated integral varifold of $T$ and $f_* V_T$ the push-forward varifold then, from Remark 27.2 (3) of [SIM84]

$$\mu_{f_* V_T} = \mu_{f_* T}.$$

We now define the space of currents that will concern us the most, that is the space of integral currents.

**Definition 2.1.10.** Let $T$ be an integer rectifiable $m$-current in $U$. We call $T$ an integral $m$-current if $M(T) < \infty$ and $M(\partial T) < \infty$. We denote the space of integral $m$-currents in $U$ as $I_m(U)$.

**Remark 2.1.6.** We would like to stress out that if $T \in I_m(U)$ then its current boundary $\partial T \in I_{m-1}(U)$. This is a consequence of the boundary rectifiability theorem, see theorem 30.3 from [SIM84]. Moreover, the push-forward current $f_* T \in I_m(U)$ where $f$ is a Lipschitz, $1 - 1$ proper map and $\partial f_* T = f_* \partial T$.

The following decomposition result for integral currents will be useful.

**Lemma 2.1.3.** (4.5.17 [FED69]) Let $T \in I_{n-1}(\mathbb{R}^n)$ with $\partial T = 0$. Then there exist $\mathcal{L}^n$-measurable sets $M_i$ with $M_i \subset M_{i-1}$ such that

$$T = \sum_{i \in \mathbb{Z}} \partial (E^n[M_i])$$

and
\[ M(T) = \sum_{i \in \mathbb{Z}} M(\partial(E^n | M_i)). \]

In fact \( T = \partial Q \) for \( Q \in I_n(\mathbb{R}^n) \) and \( Q = E^n[f] \), for \( f \) an \( \mathcal{L}^n \)-integrable and integer valued function.

We now give an equivalent characterisation of current convergence in the class of integer rectifiable \( m \)-currents. Let \( W \subset U \) and \( T_1, T_2 \) integer rectifiable \( m \)-currents such that \( M_W(\partial T_1) + M_W(\partial T_2) < \infty \). Define

\[
d_W(T_1, T_2) = \inf_{R, S} \{ M_W(R) + M_W(S) \mid T_1 - T_2 = \partial R + S, \quad R \in \mathcal{D}_{m+1}(U), S \in \mathcal{D}_m(U) \text{ integer rectifiable} \}.
\]

**Theorem 2.1.4.** (Theorem 31.2 [Sim84]) Let \( T_j, T \) be integer rectifiable \( m \)-currents with \( \sup_j (M_W(T_j) + M_W(\partial T_j)) < \infty \), for any \( W \subset U \). Then \( T_j \rightharpoonup T \) if and only if \( d_W(T_j, T) \xrightarrow{j \to \infty} 0 \), for any \( W \subset U \).

One of the most fundamental results of Geometric Measure Theory is the following compactness theorem known as Federer-Fleming compactness.

**Theorem 2.1.5.** (Theorem 27.3 [Sim84]) Let \( T_j \) be a sequence of integer rectifiable \( m \)-currents with \( \sup_j (M_W(T_j) + M_W(\partial T_j)) < \infty \), for any \( W \subset U \). Then there exists a subsequence \( T_j' \) and an integer rectifiable \( m \)-current \( T \) such that \( T_j' \rightharpoonup T \).

**Remark 2.1.7.** From Theorem 2.1.5 we conclude that the space

\[ \left\{ T \in I_m(U) \mid spt T \subset K, M(T) + M(\partial T) \leq M \right\}, \]

for some compact \( K \subset U \) and \( M \) a positive constant is compact when equipped with the current convergence. It is in this form that we will usually make use of Theorem 2.1.5. We also mention that the set \( K \) could be any compact smooth Riemannian manifold.
We now briefly discuss the slicing theory of integer rectifiable $m$-currents.

**Definition 2.1.11.** Let $T$ be an integer rectifiable $m$-current and $u : \mathbb{R}^n \to \mathbb{R}$ a Lipschitz function. We define the slice of $T$ by $u$ to be the current

$$< T, u; r > = \partial(T \{ x \in \mathbb{R}^n \mid u(x) \leq r \}) - (\partial T) \{ x \in \mathbb{R}^n \mid u(x) \leq r \} = (\partial T) \{ x \in \mathbb{R}^n \mid u(x) > r \} - \partial(T \{ x \in \mathbb{R}^n \mid u(x) > r \}).$$

**Lemma 2.1.4.** (4.2.1 [FED69]) Let $T$ be an integer rectifiable $m$-current in $U$ and $u : \mathbb{R}^n \to \mathbb{R}$ a Lipschitz function, with Lipschitz constant $\text{Lip}_u$. Then

$$\int_a^b M_W(< T, u; r >) \, dr \leq (\text{Lip}_u) M_W(T),$$

for any $W \subset \subset U$ and any $-\infty < a < b < \infty$.

**Remark 2.1.8.**

1. From Definition 2.1.11 we can easily see that

$$\partial < T, u; r > = - < \partial T, u; r >.$$

2. One can prove, see Lemma 4.2.15 of [FED69], that the slice of an integer rectifiable $m$-current is an integer rectifiable $(m-1)$-current and the slice of an integral $m$-current is an integral $(m-1)$-current for $L^1$-a.e. $r$.

3. If the function $u(x) = |x - a|$ for $a \in \mathbb{R}^n$ then we denote the slice as $< T; r >$ and notice that if $\partial T = 0$ then $< T; r > = \partial(T \setminus B_r(a))$.

We close this section with a discussion on tangent cones. This will prove to be an indispensable tool in the singularity analysis. We give the following definition that will be suitable to our purposes. For a more general definition see 4.3.16 of [FED69].

**Definition 2.1.12.** (current tangent cone) Let $T$ be an integer rectifiable $m$-current and a point $x \in \text{spt} T \setminus \text{spt} \partial T$. We say that an integer rectifiable $m$-current $C$ is an oriented tangent cone of $T$ at $x$ if $\eta_{0, \lambda} C = C$ for any $\lambda > 0$ and there exists a sequence $r_j \searrow 0$ such that

$$\eta_{x, r_j} T \to C.$$
We give the analogous definition for integral varifolds here in order to compare it with the current analogue.

**Definition 2.1.13.** (varifold tangent cone) Let $V \in IV_m(U)$ and $x \in \text{spt}V$. We say that the integral $m$-varifold $C$ is a tangent cone of $V$ at $x$ if $\eta_{0,\lambda}*C = C$ for any $\lambda > 0$ and there exists a sequence $r_j \downarrow 0$ such that

$$||\eta_{x,r_j}*V|| \rightarrow ||C||.$$  

The push-forward in the above definition is to be understood in the sense of varifolds and the convergence is in the sense of Radon measures of $G_m(U)$.

**Remark 2.1.9.** We see that if $C$ is a current tangent cone of $T$ it does not necessarily imply that the associated varifold $V_C$ is a varifold tangent cone of $V_T$. Conversely, if $V_C$ is a varifold tangent cone of $V_T$ it does not necessarily imply that $C$ is a current tangent cone of $T$.

We will discuss the existence of tangent cones when the current $T$ is associated to a very specific variational problem, see Lemma 3.3.2. We would also like to mention that the uniqueness of tangent cones still remains a wide open problem even for stationary integral varifolds.

### 2.1.3 Sets of Locally Finite Perimeter

We define here what turns out to be a special class of codimension one integer rectifiable currents. For an excellent treatise on this topic, along with applications to certain variational problems, one can see [MAG12].

**Definition 2.1.14.** Let $E$ be an $\mathcal{L}^n$-measurable set in $\mathbb{R}^n$. We say that $E$ is a set of locally finite perimeter in $\mathbb{R}^n$ if for any compact $K \subset \mathbb{R}^n$

$$\sup \left\{ \int_E \text{div}X \, d\mathcal{L}^n \mid X \in C^1(\mathbb{R}^n; \mathbb{R}^n), \text{spt}X \subset K, \sup_{\mathbb{R}^n} |X| \leq 1 \right\} < \infty.$$  

If it is bounded independently of the compact set $K$ then it is called a set of finite perimeter.
Let $E$ be a set of locally finite perimeter. From Riesz’s representation theorem, see Theorem 4.7 of [MAG12], there is a vector valued Radon measure $\mu_E$ such that

$$\int_E \text{div} X \, d\mathcal{L}^n = \int_E X \cdot d\mu_E, \quad (2.1.5)$$

where $(\cdot)$ denotes the Euclidean inner product. Equation 2.1.5 generalises the well-known divergence theorem valid for open sets $U$ in $\mathbb{R}^n$ with $C^1$-boundary to sets of locally finite perimeter.

We define the total variation measure $|\mu_E|$ of $\mu_E$ as

$$|\mu_E|(K) = \sup \left\{ \int_E X \cdot d\mu_E \mid X \in C_c^0(\mathbb{R}^n; \mathbb{R}^n), \text{spt}X \subset K, \sup_{\mathbb{R}^n} |X| \leq 1 \right\},$$

for any $K \subset \mathbb{R}^n$ compact. The measure $|\mu_E|$ is a Radon measure of $\mathbb{R}^n$.

For a set of locally finite perimeter $E$ we define the perimeter $P(E) = |\mu_E|(\mathbb{R}^n)$. Its relative perimeter in $U \subset \mathbb{R}^n$ is defined by $P(E; U) = |\mu_E|(U)$. With this notation, if $E$ is a set of locally finite perimeter $P(E; K) < \infty$ for any compact $K \subset \mathbb{R}^n$ and if $E$ is a set of finite perimeter then $P(E) < \infty$.

We briefly describe now why sets of locally finite perimeter are a special class of integer rectifiable currents. For more details see Section 15 of [MAG12]. Let $E$ be a set of locally finite perimeter then we define its reduced boundary

$$\partial^* E = \left\{ x \in \mathbb{R}^n \mid \lim_{\rho \searrow 0} \frac{\mu_E(B_\rho(x))}{|\mu_E|(B_\rho(x))} \right\} \text{ exists and has length } 1 \mu_E\text{-a.e.}. \right\}.$$

We define the function $\nu_E(x) = \lim_{\rho \searrow 0} \frac{\mu_E(B_\rho(x))}{|\mu_E|(B_\rho(x))}$, for $x \in \partial^* E$. This is called the measure-theoretic (outer) unit normal to $E$.

**Theorem 2.1.6.** (De Giorgi’s structure theorem, Theorem 15.9 [MAG12]) Let $E$ be a set of locally finite perimeter. Then $\partial^* E$ is a countably $(n - 1)$-rectifiable set with multiplicity-1 and $\mu_E = \nu_E \mathcal{H}^{n-1} |\partial^* E|$.

**Remark 2.1.10.** From Theorem 2.1.6 any set of locally finite perimeter $E$ gives rise to an integer rectifiable $(n - 1)$-current $T$ given by the three-tuple $(\partial^* E, \star(\nu_E), 1)$,
where $\ast$ is the Hodge star operator. Thus the current $T$ acts on $\omega \in \Omega_{c}^{n-1}(\mathbb{R}^n)$ by

$$T(\omega) = \int_{\partial^* E} \omega, \ast \nu_E^\top > d\mathcal{H}^{n-1}.$$  

Notice that $T = \partial [E]$, where $[E]$ is the current given by Lebesgue integration over $E$ and $\partial$ is the current boundary operator.

We briefly discuss compactness for sets of locally finite perimeter. We mention that it is much easier to establish compactness properties in that class and we do not need the Federer-Fleming compactness, Theorem 2.1.5.

**Definition 2.1.15.** Let $E_j, E$ be sets of locally finite perimeter. We say that $E_j$ converges to $E$ and write $E_j \xrightarrow{\text{loc}} E$ if $L^n(K \setminus (E_j \Delta E)) \xrightarrow{j \to \infty} 0$, where $E_j \Delta E$ denotes the symmetric difference $(E_j \setminus E) \cup (E \setminus E_j)$ and $K$ is any compact subset of $\mathbb{R}^n$.

Under this notion of convergence and using Remark 2.1.10 we have the following:

**Theorem 2.1.7.** (Corollary 12.27 [MAG12]) Let $E_j$ be a sequence of sets of locally finite perimeter in $\mathbb{R}^n$ with

$$\sup_j P(E_j; B_R) < \infty,$$

for any $R > 0$. Then there exists a subsequence $E_{j'}$ and a set of locally finite perimeter $E$ in $\mathbb{R}^n$ such that

$$E_{j'} \xrightarrow{\text{loc}} E,$$

$$\partial [E_{j'}] \rightharpoonup \partial [E].$$

It will be useful to know if we can modify a set of locally finite perimeter in order to relate it with its topological boundary. This is shown in the following:

**Proposition 2.1.1.** (Proposition 12.19 [MAG12]) Let $E$ be a set of locally finite perimeter in $\mathbb{R}^n$. Then there exists a Borel set $F \subset \mathbb{R}^n$ with $L^n(E \triangle F) = 0$ and

$$\text{spt} \mu_F = \partial F,$$

where $\partial F$ denotes the topological boundary of $F$.

We now give the following lemma, which is just a topological property, but will be useful when combined with Proposition 2.1.1.
Lemma 2.1.5. Let $A \subset X$, where $X$ is a connected topological space and $\partial A$ is connected. Then $\overline{A}$ is connected.

Proof. Suppose $\overline{A}$ is not connected. Then there are $B, C$ non-empty, disjoint open sets in $\overline{A}$, such that

$$\overline{A} = B \cup C.$$ 

Since $B, C$ are open, with respect to the relative topology in $\overline{A}$, there are open sets $U, V$ in $X$ so that

$$B = \overline{A} \cap U, \quad C = \overline{A} \cap V.$$ 

Thus

$$\partial A = (\partial A \cap B) \cup (\partial A \cap C) = (\partial A \cap U) \cup (\partial A \cap V).$$

Since $\partial A$ connected, one of the sets in the union will be empty. We assume that $\partial A \cap C = \emptyset$ thus $\partial A \cap V = \emptyset$. Then

$$C = A^o \cap V,$$

where $A^o$ is the interior in $X$. Thus $C$ is open in $X$. Finally since $C$ is also closed in $\overline{A}$ then $C$ is closed in $X$ and since $C$ is non-empty and not $X$ it contradicts the connectedness of $X$. $\square$

Remark 2.1.11. From Remark 2.1.10 and Proposition 2.1.1, combined with Lemma 2.1.5, we see that if $E$ is a set of locally finite perimeter and $T = \partial[E]$ with $\text{spt} T$ connected then we can assume that $\partial E = \text{spt} T$ and $E$ is a closed and connected subset of $\mathbb{R}^n$.

We close this section with an equivalent characterisation for sets of locally finite perimeter related to functions of bounded variation.

Definition 2.1.16. Let $U \subset \mathbb{R}^n$ open. We say that a function $u \in L^1_{\text{loc}}(U)$ is in $\text{BV}_{\text{loc}}(U)$, the space of functions of locally bounded variation, if for any $W \subset\subset U$

$$\sup \left\{ \int_U u \, \text{div} X \, d\mathcal{L}^n \mid X \in C^1_c(W; \mathbb{R}^n), \sup_U |X| \leq 1 \right\} < \infty.$$ 

If it is bounded independently of the set $W$ then it is in the space of functions of bounded variation denoted by $\text{BV}(U)$. 

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If \( u \in BV_{\text{loc}}(U) \) then from Riesz’s representation theorem, Theorem 4.7 of [MAG12], there is a Radon measure \( \mu \) and a vector field \( \nu \) that is \( \mu \)-measurable and \( |\nu| = 1 \) \( \mu \)-a.e. such that

\[
\int_U u \, \text{div} X \, d\mathcal{L}^n = \int_U X \cdot \nu \, d\mu, \quad X \in C^1_c(U; \mathbb{R}^n).
\]

The measure \( \mu \) is called the distributional derivative of \( u \) and we denote it as \( Du = \nu d\mu \).

In conclusion a Lebesgue measurable set \( E \) in \( U \) is a set of locally finite perimeter in \( U \) if and only if \( \chi_E \in BV_{\text{loc}}(U) \) and a set of finite perimeter if \( \chi_E \in BV(U) \). The perimeter is then given by

\[
P(E) = \int_U |D\chi_E|,
\]

where \( |D\chi_E| \) is the total variation measure of \( D\chi_E \). The relative perimeter is defined analogously. For more details on this equivalent characterisation see Remark 26.28 of [SIM84].

The compactness theorem, Theorem 2.1.7, is essentially equivalent to the following:

**Theorem 2.1.8. (Theorem 6.3 [SIM84])** Let \( u_j \in BV_{\text{loc}}(U) \) with

\[
\sup_j \left( \|u_j\|_{L^1(W)} + \int_W |Du_j| \right) < \infty, \quad \text{for any } W \subset \subset U,
\]

where \( \|\cdot\|_{L^1(W)} \) denotes the \( L^1 \)-norm in \( W \). Then there exists a subsequence \( u_{j'} \) and a function \( u \in BV_{\text{loc}}(U) \) such that

\[
u_j \rightharpoonup u, \quad \int_W |Du| \leq \liminf_{j' \to \infty} \int_W |Du_j'|, \quad \text{for any } W \subset \subset U.
\]
2.2 Preliminaries from PDEs

The objects that we are interested in are associated to a certain non-linear elliptic operator as we show in Lemma 5.1.3. In order to investigate a non-linear operator we need some fundamental results from linear PDE theory. From the local nature of these results, we consider a bounded domain $\Omega \subset \mathbb{R}^n$. Our main reference is [GT77].

In what follows $L$ will denote a second order linear differential operator given by

$$Lu = a^{ij}(x)D_{ij}u(x) + b^i(x)D_iu(x) + c(x)u(x),$$

where $u \in C^2(\Omega)$ and we adopt the Einstein summation convention.

For the coefficients $a_{ij}$ we assume that $a_{ij} = a_{ji}$ and that there exist $\Lambda, \lambda > 0$ such that

$$0 < \lambda(x)|\xi|^2 \leq a^{ij}(x)\xi_i\xi_j \leq \Lambda(x)|\xi|^2,$$

for any $\xi \in \mathbb{R}^n \setminus \{0\}$ and $x \in \Omega$.

Moreover we assume that there exists a uniform $\lambda_0 > 0$ such that $\lambda \geq \lambda_0$ and $\frac{\Lambda}{\lambda}$ is bounded in $\Omega$. Such an operator $L$ is called uniformly elliptic.

One of the main features of uniformly elliptic linear operators is Hopf’s maximum principle. We mention an alternative version of Theorem 3.5 of [GT77] so that we do not have a sign restriction on $c$. The proof is an immediate consequence of Theorem 3.5 of [GT77].

**Theorem 2.2.1.** (Hopf’s maximum principle) Let $L$ be a uniformly elliptic linear operator with bounded coefficients in $\Omega$ and $u, v \in C^2(\Omega)$ such that

$$u \leq v \text{ in } \Omega,$$

$$Lu \geq Lv \text{ in } \Omega.$$

Then $u \equiv v$ or $u < v$.

We also give the following boundary version of Theorem 2.2.1 that again follows from Theorem 3.5 of [GT77].
Theorem 2.2.2. Let $L$ be a uniformly elliptic linear operator with bounded coefficients in $\Omega$, where $\Omega$ has a $C^2$-boundary and $u, v \in C^2(\Omega)$ such that

\[ u \leq v \text{ in } \Omega, \]
\[ Lu \geq Lv \text{ in } \Omega, \]

and furthermore there exists $y \in \partial \Omega$ where $u, v$ are differentiable and

\[ u(y) = v(y), \]
\[ \nabla u(y) = \nabla v(y). \]

Then $u \equiv v$.

Definition 2.2.1. Let $u \in C^2(\Omega)$. The prescribed mean curvature operator of the graph of $u$ is defined as

\[ H_u = \text{div} \left( \frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \right). \]

The graph of $u$ is minimal if $H_u = 0$ and has constant mean curvature if $H_u = \lambda$ for some constant $\lambda$.

Notice that the operator $H_u$ is non-linear. In particular it is a quasilinear operator. This essentially means that the dependence on the second order derivatives of the function is linear. The maximum principle extends for the prescribed mean curvature operator. One can either use the linearisation technique as in Lemma 1.26 of [CM11] or Theorem 10.1 of [GT77].

Theorem 2.2.3. Let $u_1, u_2 \in C^2(\Omega)$ be two solutions of the minimal surface equation with $u_1 \leq u_2$ and $u_1(x_0) = u_2(x_0)$, for some $x_0 \in \Omega$. Then $u_1 \equiv u_2$.

The above theorem extends to minimal hypersurfaces. Recall the following:

Definition 2.2.2. Let $M \subset \mathbb{R}^n$ be a smooth hypersurface. Then $M$ is a minimal hypersurface of $\mathbb{R}^n$ if for any $X \in C^1_c(\mathbb{R}^n; \mathbb{R}^n)$

\[ \int_M \text{div}_M X \, d\mathcal{H}^{n-1} = 0. \]
Notice that the above definition is the smooth analogue of stationary varifolds as defined in 2.1.3.

**Theorem 2.2.4.** Let $M_1, M_2$ be oriented minimal hypersurfaces of $\mathbb{R}^n$ such that $M_1$ lies on one side of $M_2$. Then either $M_1 \cap M_2 = \emptyset$ or $M_1 \equiv M_2$.

**Remark 2.2.1.** Theorem 2.2.4 holds for CMC hypersurfaces with the same scalar mean curvature provided that at the point of touching the mean curvature vector of both hypersurfaces points in the same direction. This is a necessary assumption as the example of two touching spheres that curve oppositely shows.

We now turn our attention to second order differential operators that are of divergence form in a bounded domain $\Omega$ of $\mathbb{R}^n$. In what follows $L$ denotes the operator given by

$$Lu = D_i(a^{ij}(x)D_j u + b^i(x)u) + c^i(x)D_i u + d(x)u.$$  

We assume as before that $a_{ij} = a_{ji}$ and that $L$ is uniformly elliptic with bounded coefficients in $\Omega$.

One of the most fundamental properties of such operators is the Harnack inequality.

**Theorem 2.2.5.** (Corollary 8.21 [GT77]) Let $u \in C^2(\Omega)$ with $Lu = 0$ and $u \geq 0$. Then

$$\sup_K u \leq C_K \inf_K u,$$

where $K$ is any compact subset of $\Omega$ and $C_K$ is a constant that depends on $K$, the dimension $n$ and the operator $L$.

**Remark 2.2.2.** Theorem 2.2.5 is still valid in case $u$ is a weak solution of $Lu = D_i f^i + g$ with $f^i \in L^p$, $g \in L^{p/2}$ for some $p > n$. Then there will be an extra constant on the right hand side that depends on the $L^p$-norm of $f^i$ and the $L^{p/2}$-norm of $g$. For more details the interested reader can see [GT77], Theorem 8.17 and Theorem 8.18.

**Definition 2.2.3.** Let $L$ be a second order linear differential operator in divergent form that is uniformly elliptic, with $a^{ij}, b^i \in C^\alpha(\overline{\Omega})$, for some $0 < \alpha < 1$ and
We say that \( u \in C^{1, \alpha}(\Omega) \) is a weak solution of

\[
Lu = D_i f^i + g,
\]

with \( f^i \in C^\alpha(\Omega), g \in L^\infty(\Omega) \) if

\[
\int_\Omega \left( (a^{ij} D_j u(x) + b^i u) D_i v - (c^i D_i u + d u) v \right) = \int_\Omega (f^i D_i v - g v),
\]

for any \( v \in C^1_c(\Omega) \).

We now mention the interior Schauder estimates which are of fundamental importance in linear elliptic PDE theory.

**Theorem 2.2.6.** *(Theorem 8.32 [GT77])* Let \( u \in C^{1, \alpha}(\Omega) \) be a weak solution of

\[
Lu = D_i f^i + g,
\]

\( f^i \in C^\alpha(\Omega), g \in L^\infty(\Omega) \). Then for any \( \Omega' \subset\subset \Omega \) we have that

\[
||u||_{1,\alpha;\Omega'} \leq C(||u||_{0,\Omega} + ||g||_{0,\Omega} + ||f||_{0,\alpha;\Omega}),
\]

where \( C \) is a constant independent of \( u, f^i \) and \( g \).

**Theorem 2.2.7.** *(Corollary 8.11 [GT77])* Let \( u \in C^{1, \alpha}(\Omega) \) be a weak solution of

\[
Lu = D_i f^i + g.
\]

If the coefficients of \( L \) are in \( C^\infty(\Omega) \) and \( f^i, g \in C^\infty(\Omega) \) then \( u \in C^\infty(\Omega) \) and hence an actual solution of the PDE.

Depending on the regularity of the coefficients and the initial data of the PDE the elliptic regularity theorem, Theorem 2.2.7, can be stated in different ways. However, the above version will be sufficient to our purposes.

**Remark 2.2.3.** We would like to mention that using local coordinates and standard covering arguments all of the above theorems extend to uniformly elliptic linear operators of second order on bounded domains of Riemannian manifolds.
Chapter 3

Generalised CMC hypersurfaces

In this chapter we introduce the prescribed mean curvature functional and the notion of generalised CMC hypersurfaces. We investigate existence properties to certain minimisation problems and define the notion of one-sided minimisers of the prescribed mean curvature functional. Based on [BW18] we also develop a regularity theory for one-sided minimisers.

3.1 The prescribed mean curvature functional

In what follows let $K \subset \mathbb{R}^{n+1}$ be a compact set, with $\mathbb{R}^{n+1} \setminus K$ connected and $\text{vol}(K) = \mathcal{L}^{n+1}(K) > 0$. We denote by $\mathcal{I}_{m,K}(\mathbb{R}^{n+1})$ the set of integral $m$-currents of $\mathbb{R}^{n+1}$ supported in $K$. In order to define the prescribed mean curvature functional we need a notion of enclosed volume.

**Definition 3.1.1.** Let $\lambda \in \mathbb{R}$ and $T, T_0 \in \mathcal{I}_{n,K}(\mathbb{R}^{n+1})$ with $\partial T = \partial T_0$. We define the $\lambda$-volume between $T$ and $T_0$ to be

$$V_\lambda(T, T_0) = Q_{T,T_0}(\omega),$$

where $Q_{T,T_0} \in \mathcal{I}_{n+1,K}(\mathbb{R}^{n+1})$ the unique current of finite mass with $\partial Q_{T,T_0} = T - T_0$ and $\omega = \lambda \, dx^1 \wedge \ldots \wedge dx^{n+1}$.

**Remark 3.1.1.** We need to make sure that the notion of the $\lambda$-volume is well defined. Hence we need to show that there always exists a current $Q_{T,T_0}$ with these...
properties and that it is unique. First notice that for \( T - T_0 \) we can apply Lemma 2.1.3. Thus there exists an integral \((n + 1)\)-current \( Q_{T,T_0} \), with \( Q_{T,T_0} = E^{n+1}[\theta_{T,T_0}\) and \( \theta_{T,T_0} \) is an \( L^{n+1} \)-integrable, integer valued function. So we have that \( Q_{T,T_0} \) is of finite mass and it is unique. It remains to show that \( \text{spt}Q_{T,T_0} \subset K \). From our initial assumptions we can apply Theorem 2.1.3 for \( U = \mathbb{R}^{n+1} \) and \( A = \mathbb{R}^{n+1} \setminus K \). Thus there exists \( c \in \mathbb{R} \) such that

\[
Q_{T,T_0} = cE^{n+1} \text{ in } \mathbb{R}^{n+1} \setminus K,
\]

which contradicts the fact that \( Q_{T,T_0} \) has finite mass unless \( c = 0 \). This shows that \( \text{spt}Q_{T,T_0} \subset K \).

**Remark 3.1.2.** From the uniqueness property of the \( \lambda \)-volume we have the following useful splitting property. Let \( T_1, T_2, T_3, T_4 \in \mathcal{I}_{n,K}(\mathbb{R}^{n+1}) \) with \( \partial T_1 = \partial T_3 \) and \( \partial T_2 = \partial T_4 \) then

\[
V_\lambda(T_1 + T_2, T_3 + T_4) = V_\lambda(T_1, T_3) + V_\lambda(T_2, T_4).
\]

**Proposition 3.1.1.** (isoperimetric inequality) Let \( T, T_0 \in \mathcal{I}_{n,K}(\mathbb{R}^{n+1}) \) with \( \partial T = \partial T_0 \) and \( Q_{T,T_0} = E^{n+1}[\theta_{T,T_0}\) the current that gives the \( \lambda \)-volume between \( T, T_0 \). Then the function \( \theta_{T,T_0} : K \to \mathbb{Z} \) belongs in \( BV(\mathbb{R}^{n+1}) \) and satisfies the inequality

\[
\left( \int_K |\theta_{T,T_0}| d\mathcal{L}^{n+1} \right)^{n/n+1} \leq \left( \int_K |\theta_{T,T_0}|^{1+1/n} d\mathcal{L}^{n+1} \right)^{n/n+1} \leq C_n M(T - T_0),
\]

where \( C_n \) is a constant that depends only on the dimension \( n \).

**Proof.** From the definition of the space \( BV(\mathbb{R}^{n+1}) \), see Definition 2.1.16 it suffices to show that

\[
\sup \left\{ \int_K \theta_{T,T_0} \text{div} X d\mathcal{L}^{n+1} \mid X \in C^1(\mathbb{R}^{n+1}; \mathbb{R}^{n+1}), |X|_\infty \leq 1, \text{spt}X \subset K \right\} < \infty.
\]

Let \( X = (X_1, \ldots X_{n+1}) \) be a vector field supported in \( K \) with \( |X|_\infty \leq 1 \) and define
the form
\[ \eta = (-1)^{i-1} X^i dx^1 \wedge \ldots \wedge \hat{dx}^i \wedge \ldots \wedge dx^{n+1}, \]
where \( \hat{dx}^i \) means that the \( i \)-th term is omitted in the wedge product. Then
\[ d\eta = (\text{div} X) dx^1 \wedge \ldots \wedge dx^{n+1}. \]

Since \( |\eta| \leq 1 \) and \( Q_{T,T_0}(d\eta) = (T - T_0)(\eta) \) from the definition of the mass of a current we have that
\[ \int_K \theta_{T,T_0} \text{div} X d\mathcal{L}^{n+1} = Q_{T,T_0}(d\eta) \leq M(T - T_0) < \infty, \]
proving that \( \theta_{T,T_0} \in BV(\mathbb{R}^{n+1}) \).

Since the function is integer valued and in \( BV(\mathbb{R}^{n+1}) \) inequality 3.1.1 follows directly from the isoperimetric inequality 4.5.9 (31) of [FED69]. \( \square \)

**Definition 3.1.2.** Let \( \lambda \in \mathbb{R} \) and \( T_0 \in \mathcal{J}_{n,K}(\mathbb{R}^{n+1}) \). The prescribed mean curvature functional of \( T \in \mathcal{J}_{n,K}(\mathbb{R}^{n+1}) \) with \( \partial T = \partial T_0 \) is defined to be
\[ E_\lambda(T) = M(T) + V_\lambda(T,T_0). \]

We now give the main existence result, which is similar to the one found in Theorem 2.1 of [DUZ93], where a more general version of \( E_\lambda \) is investigated.

**Theorem 3.1.1.** Let \( \lambda \in \mathbb{R} \) with \( |\lambda| < \frac{1}{C_n \text{vol}(K)^{1/n+1}} \), where \( C_n \) is the constant from inequality 3.1.1 and a fixed \( T_0 \in \mathcal{J}_{n,K}(\mathbb{R}^{n+1}) \). Then there exists \( T \in \mathcal{J}_{n,K}(\mathbb{R}^{n+1}) \) with \( \partial T = \partial T_0 \) such that
\[ E_\lambda(T) = \inf \left\{ E_\lambda(S) \mid S \in \mathcal{J}_{n,K}(\mathbb{R}^{n+1}), \partial S = \partial T_0 \right\}. \]

**Proof.** We first have to prove that \( E_\lambda \) is bounded below in this admissible class and hence we can take the infimum. Fix \( S \in \mathcal{J}_{n,K}(\mathbb{R}^{n+1}) \) with \( \partial S = \partial T_0 \). We compute
\[ V_\lambda(S,T_0) = Q_{S,T_0}(\omega) \geq \int_K -|\lambda| \|\theta_{S,T_0}\| d\mathcal{L}^{n+1}. \]

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From Hölder’s inequality for $\frac{1}{p} = \frac{1}{n+1}$ and $\frac{1}{q} = \frac{n}{n+1}$ we get

$$\geq -\left( \int_K |\lambda|^{n+1} d\mathcal{L}^{n+1} \right)^{1/n+1} \left( \int_K |\theta_{S,T_0}|^{1+1/n} d\mathcal{L}^{n+1} \right)^{n/n+1}.$$

Applying inequality 3.1.1 of Proposition 3.1.1 we get that

$$V_\lambda(S,T_0) \geq -|\lambda|\text{vol}(K)^{1/n+1}C_nM(S-T_0).$$

So in total we get that

$$E_\lambda(S) \geq M(S)(1 - |\lambda|\text{vol}(K)^{1/n+1}C_n) - |\lambda|\text{vol}(K)^{1/n+1}C_nM(T_0). \quad (3.1.2)$$

Combined with the initial assumption on $|\lambda|$ we get

$$E_\lambda(S) \geq -C_{(n,\lambda,K,T_0)},$$

where $C_{(n,\lambda,K,T_0)}$ denotes a constant that depends on $n, \lambda, T_0, K$. Thus indeed $E_\lambda$ is bounded below.

For the existence part, we make use of the direct method from calculus of variations. Let $T_j \in \mathcal{J}_{n,K}(\mathbb{R}^{n+1})$ with $\partial T_j = \partial T_0$ such that

$$E_\lambda(T_j) \to m, \quad (3.1.3)$$

where $m = \inf \left\{ E_\lambda(S) \mid S \in \mathcal{J}_{n,K}(\mathbb{R}^{n+1}), \partial S = \partial T_0 \right\}$. We can also choose the sequence $T_j$ so that

$$E_\lambda(T_j) \leq m + 1. \quad (3.1.4)$$

From 3.1.2 combined with 3.1.3 and 3.1.4 we have

$$M(T_j) \leq \frac{m + 1 + |\lambda|\text{vol}(K)^{1/n+1}C_nM(T_0)}{1 - |\lambda|\text{vol}(K)^{1/n+1}C_n},$$

thus

$$\sup_j M(T_j) < \infty. \quad (3.1.5)$$

From Remark 2.1.7 there exists a subsequence, which we still index as $j$ and
\( T \in \mathcal{I}_{n,K}(\mathbb{R}^{n+1}) \) such that
\[
T_j \to T.
\]
From the continuity of the boundary operator we get \( \partial T_j \to \partial T \) proving that
\[
\partial T = \partial T_0.
\]
Now let \( Q_j \in \mathcal{I}_{n+1,K}(\mathbb{R}^{n+1}) \) such that \( V_\lambda(T_j, T_0) = Q_j(\omega) \), \( Q_j = E^{n+1}[\theta_{T_j, T_0}] \). From inequality 3.1.1 combined with 3.1.5 we have that
\[
\sup_j \|\theta_{T_j, T_0}\|_{L^1(K)} < \infty.
\]
From Proposition 3.1.1 we have that \( \theta_{T_j, T_0} \in BV(\mathbb{R}^{n+1}) \) and since
\[
M(T_j - T_0) = \int_K |D\theta_{T_j, T_0}|,
\]
combined with 3.1.5, we get that
\[
\sup_j \left( \|\theta_{T_j, T_0}\|_{L^1(K)} + \int_K |D\theta_{T_j, T_0}| \right) < \infty.
\]
From Theorem 2.1.8 there exists a subsequence, which we still index as \( j \) and \( \theta \in BV(\mathbb{R}^{n+1}) \) such that \( \text{spt}\theta \subset K \) and
\[
\theta_{T_j, T_0} \xrightarrow{L^1(K)} \theta, \quad (3.1.6)
\]
\[
\int_K |D\theta| \leq \liminf_j \int_K |D\theta_{T_j, T_0}|.
\]
From 3.1.6 we have that the current \( Q = E^{n+1}[\theta] \) is the current limit of \( Q_j \), has current boundary \( T - T_0 \) and
\[
V_\lambda(T_j, T_0) \to V_\lambda(T, T_0), \quad (3.1.7)
\]
From 3.1.7 and the lower semi-continuity of the mass functional we conclude that
\[
E_\lambda(T) \leq \liminf_j E_\lambda(T_j) = m,
\]
proving that $T$ attains the infimum and thus minimises $E_\lambda$. □

**Remark 3.1.3.** Notice that the value of $E_\lambda$ depends on the choice of $T_0$. However, the choice of $T_0$ does not change where the minimum is achieved. To see that, let $T'_0 \in \mathcal{J}_{n,K}(\mathbb{R}^{n+1})$ with $\partial T'_0 = \partial T_0$. From the splitting property, Remark 3.1.2, we have

$$V_\lambda(T; T_0) = V_\lambda(T; T'_0) + V_\lambda(T'_0; T_0),$$

for any $T \in \mathcal{J}_{n,K}(\mathbb{R}^{n+1})$ with $\partial T = \partial T_0$. Thus $E_\lambda$ changes by the constant $V_\lambda(T'_0; T_0)$ and so the minimiser does not depend on the choice of $T_0$.

We now investigate the variational equations of the prescribed mean curvature functional. In what follows we are going to assume for convenience that $K = \overline{B_1}$.

**Definition 3.1.3.** Let $T \in \mathcal{J}_{n,\overline{B_1}}(\mathbb{R}^{n+1})$ with $T = (M, \overrightarrow{T}, \mu_T)$, $\lambda \in \mathbb{R}$ and $T_0 \in \mathcal{J}_{n,\overline{B_1}}(\mathbb{R}^{n+1})$ with $\partial T = \partial T_0$. We say that $T$ is a critical point of $E_\lambda$ in $B_1$ if for any $X \in C^1_c(B_1; \mathbb{R}^{n+1})$ with $\text{spt}X \cap \text{spt}\partial T = \emptyset$

$$\frac{d}{dt} \Big|_{t=0} E_\lambda(\theta_t T) = 0,$$

where $\theta_t$ is the local flow of $X$.

**Proposition 3.1.2.** Let $T \in \mathcal{J}_{n,\overline{B_1}}(\mathbb{R}^{n+1})$ with $T = (M, \overrightarrow{T}, \mu_T)$, $\lambda \in \mathbb{R}$ and $T_0 \in \mathcal{J}_{n,\overline{B_1}}(\mathbb{R}^{n+1})$ with $\partial T_0 = \partial T$. If $T$ is a critical point of $E_\lambda$ in $B_1$ then

$$\int_M \text{div}_M X d\mu_T + \lambda \int_M X \cdot (\ast \overrightarrow{T}) d\mu_T = 0, \quad (3.1.8)$$

for any $X \in C^1_c(B_1; \mathbb{R}^{n+1})$ with $\text{spt}X \cap \text{spt}\partial T = \emptyset$.

*Proof.* Since the derivative exists it suffices to compute only the right derivative thus we consider $t \geq 0$. For a given vector field $X$ with $X \in C^1_c(B_1; \mathbb{R}^{n+1})$ and $\text{spt}X \cap \text{spt}\partial T = \emptyset$ let

$$\theta : [0, t] \times B_1 \to B_1,$$

the local flow of $X$ and $\theta_t(p) = \theta(t, p)$. From the first variation of the area functional, see 2.1.2

$$\frac{d}{dt} \Big|_{t=0} M(\theta_t T) = \int_M \text{div}_M X d\mu_T.$$
For the $\lambda$-volume term let $Q_{T, T_0}$ be the current from Definition 3.1.1. From the homotopy formula, see (26.22) in [SIM84], we have that
\[
\theta_t^* T - \theta_0^* T - \theta_*([0, t] \times \partial T) = \partial(\theta_*([0, t] \times T)).
\]
Notice that since the vector field $X$ does not act on the current boundary of $T$ we have that
\[
\theta_t^* T - T = \partial(\theta_*([0, t] \times T)).
\]
Thus the integral $(n + 1)$-current $Q_t = \theta_*([0, t] \times T)$ has current boundary $\theta_t^* T - T$ and in fact
\[
V_\lambda(\theta_t^* T, T_0) = Q_t(\omega).
\]
From the splitting property, Remark 3.1.2 of the $\lambda$-volume we have
\[
V_\lambda(\theta_t^* T, T_0) = V_\lambda(\theta_t^* T, T) + V_\lambda(T, T_0),
\]
so it suffices to compute
\[
\lim_{t \searrow 0} \frac{V_\lambda(\theta_t^* T, T)}{t}.
\]
From the formula in (26.23) of [SIM84] we have
\[
\theta_*([0, t] \times T)(\omega) = \int_0^t \int_{B_1} <\omega_{\theta_s(x)}, \dot{\theta}_s \wedge \bigwedge_n D\theta_s(\overrightarrow{T}) > ds \, d\mu_T,
\]
where $\dot{\theta}_s$ denotes the time derivative of $\theta_s$ and $<, >$ the coupling of an $(n + 1)$-form with an $(n + 1)$-vector field. Since the integrand is continuous with respect to $s$ from the fundamental theorem of calculus we have that
\[
\frac{1}{t} \int_0^t <\omega_{\theta_s(x)}, \dot{\theta}_s \wedge \bigwedge_n D\theta_s(\overrightarrow{T}) > ds \xrightarrow{t \searrow 0} <\omega_x, X_x \wedge \overrightarrow{T}_x >.
\]
So
\[
\lim_{t \searrow 0} \frac{V_\lambda(\theta_t^* T, T)}{t} = \int_{B_1} <\omega, X \wedge \overrightarrow{T} > d\mu_T,
\]
and since \( \omega = \lambda dx^1 \wedge \ldots \wedge dx^{n+1} \) we get that

\[
\lim_{t \searrow 0} V_\lambda(\theta_t T, T) = \lambda \int_{B_1} X \cdot (\ast T) \, d\mu_T,
\]

and this concludes the proof of Proposition 3.1.2.

**Remark 3.1.4.** We mention at this point that in view of constructing CMC hypersurfaces that minimise \( E_\lambda \) in some compact set, as we want to do later in the proof of Theorem 4.2.1 from Remark 3.1.3 and the variational equations 3.1.8 the choice of the initial current \( T_0 \) does not play any role in that matter. Thus the choice of \( T_0 \) can and will be made accordingly.

Based on the variational equations 3.1.8 we give the following:

**Definition 3.1.4.** Let \( T \in \mathcal{I}_{n,B_1}(\mathbb{R}^{n+1}) \) be a critical point of \( E_\lambda \) in \( B_1 \). Then \( sptT \) is called a generalised CMC hypersurface of \( B_1 \).

**Remark 3.1.5.** If \( T \in \mathcal{I}_{n,B_1}(\mathbb{R}^{n+1}) \) is a critical point of \( E_\lambda \) in \( B_1 \) then from 3.1.8 the associated varifold \( V_T \) satisfies 2.1.4 and thus it has generalised mean curvature \( |H| \in L^\infty(\mu_{V_T}) \). Thus from Corollary 2.1.2 we have that \( \text{reg} V_T \) is a dense open subset of \( \text{spt} V_T \). We have seen that for integral currents \( \text{spt} T = \text{spt} V_T \) hence \( \text{reg} T = \text{reg} V_T \) and thus \( \text{reg} T \) is a dense open subset of \( \text{spt} T \).

### 3.2 One-sided minimisers and stability

In this section we introduce the notion of one-sided minimisers of \( E_\lambda \). We also introduce the notion of stability for \( E_\lambda \) and prove that one-sided minimisers that are also generalised CMC hypersurfaces, thus satisfy the variational equations 3.1.8 are stable.

**Definition 3.2.1.** Let \( A \subset \mathbb{R}^{n+1} \) be a set of locally finite perimeter and \( \lambda \in \mathbb{R} \). Then \( T = (\partial A)[B_1] \) is a one-sided minimiser of \( E_\lambda \) if

\[
E_\lambda(T) \leq E_\lambda(X),
\]

for any \( X \in \mathcal{I}_{n,K}(\mathbb{R}^{n+1}) \) where \( K = A \cap B_1 \) and \( \partial X = \partial T \).
Remark 3.2.1. 1) Let $A$ be a set of locally finite perimeter in $\mathbb{R}^{n+1}$ and $U$ any $\mathcal{L}^{n+1}$-measurable set. We will denote by $\partial[A]|U$ the current $(\partial[A])|U$, unless otherwise stated.

2) Definition [3.2.1] is given on the unit ball $B_1$ because it will be the case that will concern us the most, but it can be given on any open set $U$ of $\mathbb{R}^{n+1}$.

3) The variational equations [3.1.8] are not valid for a one-sided minimiser of $E_\lambda$ in the sense of Definition [3.2.1]. However, following the steps of the proof of Proposition [3.1.2] we have that for any $X \in C^1_c(B_1;\mathbb{R}^{n+1})$ with $\text{spt} X \subset \bar{A}$, $\text{spt} X \cap \text{spt} \partial T = \emptyset$ and $X \cdot \nu_A \geq 0$, where $\nu_A$ the inward pointing unit normal associated to $A$ then

$$\frac{d}{dt} \bigg|_{t=0} E_\lambda(\theta_t T) \geq 0,$$

where $\theta_t$ is the flow of the vector field $X$.

4) From Definition [3.2.1] it is evident that if $T$ is a one-sided minimiser then it minimises only on one of its two sides. Thus for $T$ to be a minimiser of $E_\lambda$ in $B_1$ it has to minimise on both sides. We mention at this point that there is an example of a one-sided minimiser that is not minimising on the other side, when $\lambda = 0$, given by the cone

$$C^{1,5} = \{(x,y) \in \mathbb{R}^2 \times \mathbb{R}^6 \mid 5|x|^2 = |y|^2\}.$$

One can see [DAV04] for more details regarding the cone $C^{1,5}$ and how this property is proved. Based on this example and Theorem [7.3] we see that there are examples of one-sided minimisers of $E_\lambda$, for $\lambda \neq 0$, that will not minimise on the other side.

From 3) of Remark [3.2.1] we shall need to assume that a one-sided minimiser of $E_\lambda$ is also a critical point of $E_\lambda$ in $B_1$. The reason we need to make this assumption is because without it there is no hope of a regularity theorem as the one we prove in Theorem [3.3.2].

To see the latter, we mention the following example when $\lambda = 0$. Consider a convex polygon $P$ in $\mathbb{R}^2$. Then $\mathbb{R}^2 \setminus P$ has a bounded and an unbounded region. In
the unbounded region $P$ minimises the area functional. However it is not stationary in $\mathbb{R}^2$ since at the vertices of $P$ the variational equations will not hold.

**Definition 3.2.2.** Let $T \in \mathcal{J}_n, \mathcal{P}_1(\mathbb{R}^{n+1})$ be a critical point of $E_\lambda$ in $B_1$, for some $\lambda \in \mathbb{R}$. We say that $T$ is a stable point of $E_\lambda$ in $B_1$ if for any $\eta \in C^1_c(\Sigma)$

$$
\int_{\Sigma} |A\Sigma|^2 \eta^2 \, d\mathcal{H}^n \leq \int_{\Sigma} |\nabla^\Sigma \eta|^2 \, d\mathcal{H}^n,
$$

(3.2.1)

where $\Sigma = \text{reg}T$, $A\Sigma$ the second fundamental form of $\Sigma$ in $\mathbb{R}^{n+1}$, $|.|$ the norm induced by the Euclidean metric and $\nabla^\Sigma$ the Levi-Civita connection on $\Sigma$.

**Remark 3.2.2.** We mention here that to each $\eta$ of Definition 3.2.2 we can associate a normal variation of $T$ given by the vector field $X = \eta N$, where $N$ the unit inward-pointing normal of $\Sigma$. Then the stability inequality [3.2.1] comes from the second variation of the prescribed mean curvature functional under such normal variations $X$. When needed we shall adopt this point of view as well. One can see [BD84] for more details on the topic of stability.

We also mention that, using a standard approximation argument, the stability inequality makes sense for $\eta \in C^0_c(\Sigma)$, see also the discussion in Lemma 1 of [SS81].

**Proposition 3.2.1.** Let $T = \partial[A]\lfloor B_1$ be a one-sided minimiser of $E_\lambda$, for some $\lambda \in \mathbb{R}$, that is also a critical point of $E_\lambda$ in $B_1$. Then $T$ is a stable point of $E_\lambda$ in $B_1$.

**Proof.** Let $N$ be the unit inward-pointing normal of $\Sigma = \text{reg}T$. From Remark 3.2.2 for any vector field $X = \eta N$, where $\eta \in C^{0,1}_c(\Sigma)$ with $\eta \geq 0$, the stability inequality [3.2.1] is valid since $T$ is a one-sided minimiser of $E_\lambda$.

For an arbitrary vector field $X = \eta N$, $\eta \in C^{0,1}_c(\Sigma)$, we decompose

$$
X = \eta^+ N - \eta^- N,
$$

where $\eta^+, \eta^- \geq 0$, $\eta^+, \eta^- \in C^{0,1}_c(\Sigma)$, $\eta^+ \eta^- \equiv 0$ and $\nabla \eta^+ \cdot \nabla \eta^- \equiv 0$. Then we compute

$$
\int_{\Sigma} |\nabla^\Sigma \eta|^2 - \int_{\Sigma} |A\Sigma|^2 \eta^2 =
$$

$$
= \int_{\Sigma} \left( |\nabla^\Sigma \eta^+|^2 + |\nabla^\Sigma \eta^-|^2 - |A\Sigma|^2 (\eta^+)^2 - |A\Sigma|^2 (\eta^-)^2 \right),
$$
and since inequality \[3.2.1\] is valid for \(\eta^+\) and \(\eta^-\) we get the inequality for any \(\eta \in C^{0,1}_c(\Sigma)\).

\[\Box\]

Remark 3.2.3. We mention that the assumption on criticality of \(T\) in \(B_1\) made in Proposition \[3.2.1\] was used in order to make sense of the stability inequality in \(B_1\).

### 3.3 Size of the singular set

In this section we investigate the size of the singular set of a one-sided minimiser of \(E_\lambda\) that is also a critical point of \(E_\lambda\) in \(B_1\). We will need the following definitions.

**Definition 3.3.1.** Let \(V \in IV_n(B_1)\). A point \(x \in \text{sing}V\) is called a classical singularity of \(V\) if there exists \(\rho > 0\) such that \(\text{spt}V \cap B_\rho(x)\) is the union of three or more embedded, \(C^{1,\alpha}\) hypersurfaces-with-boundary that meet pairwise along their common \(C^{1,\alpha}\)-boundary \(\Gamma\), with \(x \in \Gamma\) and at least one pair of them meet transversally everywhere along \(\Gamma\).

**Definition 3.3.2.** Let \(V \in IV_n(B_1)\). A point \(x \in \text{sing}V\) is called a touching singularity if it is not a classical singularity and there exists \(\rho > 0\) such that \(\text{spt}V \cap B_\rho(x) = M_1 \cup M_2\) where \(M_i\) is an embedded \(C^{1,\alpha}\)-hypersurface with \((M_i \setminus M_i) \cap B_\rho(x) = \emptyset\), for \(i = 1, 2\). For any \(\sigma > 0\) the coincidence set of \(x\) is defined to be \(M_1 \setminus M_2 \setminus B_\sigma(x)\). The set of touching singularities will be denoted by \(\text{sing}_tV\).

**Remark 3.3.1.** Definitions \[3.3.1\] and \[3.3.2\] make sense for any varifold \(V \in IV_m(U)\), \(U \subset \mathbb{R}^{n+1}\) open and any \(m \leq n + 1\). We have given them for \(m = n\) since this is the only case that will concern us.

We now state the following regularity theorem that can be found in [BW18] or [BCW19] and will be the main tool in proving the regularity theorem for one-sided minimisers of \(E_\lambda\), Theorem \[3.3.2\].

**Theorem 3.3.1.** (Theorem 2.1 [BW18]) Let \(V \in IV_n(B_1)\) that satisfies the following:

1) The first variation of the area functional is in \(L_p^p(\mu_V)\) with \(p > n\), absolutely continuous with respect to \(\mu_V\) and \(\text{reg}V\) is a CMC hypersurface in the classical sense.
2) $V$ has no classical singularities.

3) For any touching singularity the coincidence set has $H^n$-measure zero.

4) The set $\text{gen-reg} V = \text{reg} V \cup \text{sing}_V$ is weakly stable. This means that inequality \[ \text{3.2.1} \] is satisfied for any $\eta \in C^1_c(\text{gen-reg} V)$ with $\int_{\text{gen-reg} V} \eta = 0$.

Then $\text{sing} V \setminus \text{sing}_V$ is empty for $n \leq 6$, discrete for $n = 7$ and has Hausdorff dimension at most $n - 7$ for $n \geq 8$.

We will also need the notion of one-sided minimisers for the area functional as well as a regularity theory for them.

Definition 3.3.3. Let $E$ be a set of locally finite perimeter in $\mathbb{R}^{n+1}$. Then $C = \partial [E]$ is locally a one-sided minimiser of the area functional if for any $x \in E$ there exists $\rho > 0$ such that $M(C|B_\rho(x)) \leq M(X)$, for any $X \in I_n(\mathbb{R}^{n+1})$ with $\partial X = \partial (C|B_\rho(x))$ and $\text{spt} X \subset E \cap \overline{B}_\rho(x)$.

Definition 3.3.4. We will say that $T \in I_n(\mathbb{R}^{n+1})$ is stationary if the associated integral varifold $V_T$ is stationary in the sense of \[ \text{2.1.3} \].

Lemma 3.3.1. If $C$ is locally a one-sided minimiser of the area functional and stationary in $\mathbb{R}^{n+1}$ then $\text{spt} C$ is smoothly embedded everywhere except on a set of Hausdorff dimension at most $(n - 7)$.

The above lemma is proved using the regularity result of \[ \text{[WIC14a]} \]. The main point is to exclude classical singularities for a locally one-sided area minimiser. One can see Sections 2.3 and 9 of \[ \text{[BW18]} \] or Proposition 2.1 found in \[ \text{[LIU19]} \] on how this is done.

One of the main results of this section, that will be used in the sequel, is the following:

Lemma 3.3.2. (multiplicity-1 lemma) Let $T = \partial [A]|B_1$ be a one-sided minimiser of $E_\lambda$ that is also a critical point of $E_\lambda$ in $B_1$. Then for any $x \in \text{spt} T \setminus \text{spt} \partial T$, $x \notin \text{sing}_V T$. 

and any sequence \( r_j \downarrow 0 \) there exists a subsequence \((r'_j)\) and a set of locally finite perimeter \( E \) such that \( C = \partial[E] \) is locally a one-sided minimiser of the area functional, stationary in \( \mathbb{R}^{n+1} \), \( \eta_{0,r_j} C = C \) for any \( r > 0 \) and

\[
\mu_{\eta_{r',r_j}}, T \xrightarrow{j \to \infty} \mu C.
\]

**Proof.** Without loss of generality we may assume that \( x = 0 \). Let \( A_j = \eta_{0,r_j}(A) \) and \( T_j = \eta_{0,r_j}, T \), the integer \((n-1)\)-rectifiable current associated to \( A_j \) as described in Remark 2.1.10. Since \( T \) is a critical point of \( E_\lambda \) in \( B_1 \), \( T_j \) is a critical point of \( E_{\lambda_j} \) in \( B_{1/r_j} \), where \( \lambda_j = \lambda r_j \). From Theorem 2.1.1 and the scale invariance of the mass ratio we get

\[
\sup_j P(A_j, B_R) < \infty,
\]

for any \( R > 0 \).

Thus from Theorem 2.1.7 there exists a subsequence, which we still index as \( j \) and a set of locally finite perimeter \( E \) such that

\[
A_j \xrightarrow{\text{loc}} E,
\]

\[
T_j \to C.
\]  

(3.3.1)

where \( C = \partial[E] \). From Theorem 2.1.4 we have that (3.3.1) is equivalent to

\[
d_W(T_j, C) \xrightarrow{j \to \infty} 0,
\]

for any \( W \subset \subset \mathbb{R}^{n+1} \). So for any \( \epsilon > 0 \) there exists \( j_0 \) such that for any \( j \geq j_0 \)

\[
T_j |B_1 - C|B_1 = \partial Q_j + S_j,
\]  

(3.3.2)

where \( Q_j \) is an integer rectifiable \((n+1)\)-current, \( S_j \) an integer rectifiable \( n \)-current and

\[
M(Q_j) + M(S_j) < \frac{\epsilon}{6}.
\]  

(3.3.3)
We now make the following observation. Define the set
\[ B_j = \left\{ \rho \in (0, 1) \mid M(<Q_j; \rho>) \leq 2M(Q_j) \right\}. \]

Then from Lemma 2.1.4 we can see that for each \( j \) we have \( \mathcal{H}^1(B_j) \geq \frac{1}{2} \). From standard properties of outer measures we know that

\[ \mathcal{H}^1(\limsup_j B_j) \geq \limsup_j \mathcal{H}^1(B_j) \geq \frac{1}{2}. \]

In particular, this means that there is \( \rho \in \limsup_j B_j \) and from the definition of the set theoretic \( \limsup \) we have that \( \rho \in B_j \) for infinitely many \( j \). Thus after passing to a subsequence, which we still index as \( j \), we have that there is \( 0 < \rho < 1 \), uniform in \( j \), such that

\[ M(<Q_j; \rho>) \leq 2M(Q_j). \quad (3.3.4) \]

We will prove now that \( C = \partial[E] \) is locally a one-sided minimiser of the area functional. We only need to do so when \( C \) is not the zero current. Assume that \( C \) is not locally a one sided-minimiser of the area functional. Then there exists an integer rectifiable \( n \)-current \( R \) such that

\[ \partial R = \partial(C|B_\rho), \]
\[ \text{spt} R \subset E \cap B_\rho, \]
\[ M(R) < M(C|B_\rho). \]

Notice that from the lower semi-continuity of the mass functional

\[ \liminf_j M(T_j|B_\rho) \geq M(C|B_\rho), \]

and for any \( \epsilon > 0 \) there exists a large index \( j \) such that

\[ M(T_j|B_\rho) > \liminf_j M(T_j|B_\rho) - \frac{\epsilon}{3}. \quad (3.3.5) \]
Finally, for all sufficiently large $j$ we have 

$$|\lambda_j| < \min \left\{ \frac{\epsilon}{3M(Q_{R,C,B_\rho})}, 1 \right\}, \quad (3.3.6)$$

where $Q_{R,C,B_\rho} \in I_{n+1}(\mathbb{R}^{n+1})$ the unique current of finite mass that bounds $R - C |B_\rho|$, as defined in Definition $3.1.1$.

In conclusion for 

$$\epsilon = \lim \inf_j M(T_j |B_\rho| - M(R),$$

we can choose a large index $j$ such that $3.3.1$, $3.3.2$, $3.3.3$, $3.3.4$, $3.3.5$ and $3.3.6$ are valid. For that index $j$ define 

$$X_j = R + S_j |B_\rho| - \langle Q_j; \rho > .$$

From $3.3.2$ and Remark $2.1.8$ we compute 

$$\partial X_j = \partial (T_j |B_\rho|),$$

and since $T_j$ is a one-sided minimiser of $E_{\lambda_j}$ we get that 

$$E_{\lambda_j}(T_j |B_\rho|) \leq E_{\lambda_j}(X_j). \quad (3.3.7)$$

Fix $T^0_j \in I_n(\mathbb{R}^{n+1})$ with $\partial T^0_j = \partial X_j$ and $\text{spt}T^0_j \subset \overline{A_j} \cap \overline{B_\rho}$. We want to estimate

$$E_{\lambda_j}(X_j) = M(X_j) + V_{\lambda_j}(X_j; T^0_j).$$

We have 

$$M(X_j) \leq M(R) + M(S_j |B_\rho|) + M(< Q_j; \rho >),$$

so from inequalities $3.3.3$ and $3.3.4$ we get 

$$M(X_j) \leq M(R) + \frac{\epsilon}{2}.$$

For the $\lambda_j$-volume using $3.3.2$ inequalities $3.3.4$ and $3.3.6$ and the splitting property
from Remark 3.1.2 we get

\[ V_{\lambda_j}(X_j, T_j^0) = V_{\lambda_j}(R, C|B_\rho) + V_{\lambda_j}(T_j|B_\rho, T_j^0) - V_{\lambda_j}(\partial(Q_j|B_\rho), 0) \leq \]

\[ \leq |\lambda_j|M(Q_{R,C|B_\rho}) + V_{\lambda_j}(T_j|B_\rho, T_j^0) + |\lambda_j|M(Q_j) \leq \]

\[ \leq \frac{\varepsilon}{3} + \frac{\varepsilon}{6} + V_{\lambda_j}(T_j|B_\rho, T_j^0) = \]

\[ = \frac{\varepsilon}{2} + E_{\lambda_j}(T_j|B_\rho) - M(T_j|B_\rho). \]

In conclusion we get that

\[ E_{\lambda_j}(X_j) \leq E_{\lambda_j}(T_j|B_\rho) + M(R) - M(T_j|B_\rho) + \varepsilon. \]

Using inequality 3.3.5 and the definition of \( \varepsilon \) we get

\[ E_{\lambda_j}(X_j) \leq E_{\lambda_j}(T_j|B_\rho) - \frac{2\varepsilon}{3}, \]

which contradicts inequality 3.3.7. Therefore \( C = \partial[E] \) is locally a one-sided minimiser of the area functional.

In order to prove the Radon measure convergence assume that there exists a compact set \( K \) of \( \mathbb{R}^{n+1} \) such that

\[ \liminf_j M_K(T_j) > M_K(C). \]

Let \( \delta > 0 \) be arbitrary and \( K \subset B_R \) and define the function \( \phi : B_R \to \mathbb{R} \) such that \( \phi \equiv 1 \) in a neighborhood of \( K \) inside \( B_R \) and \( \text{spt} \phi \subset \{ x \in B_R | d(x, K) < \delta \} \), where \( 0 < \delta < d(K, \partial B_R) \).

For \( 0 < \mu < 1 \) let \( W_\mu = \{ x \in B_R | \phi(x) > \mu \} \). Then

\[ K \subset W_\mu \subset \{ x \in B_R | d(x, K) < \delta \}. \]

Let \( \epsilon = \frac{\liminf_j M_K(T_j) - M_K(C)}{2} \) and we can work similarly as before to get a large.
index \( j \) such that [3.3.2] is valid in \( W_\mu \), inequality [3.3.3] is valid and

\[
M(<Q_j, \phi; \mu>) \leq 2M(Q_j),
\]

(3.3.8)

for \( 0 < \mu < 1 \) independent of \( j \). We also have that inequality [3.3.5] is valid in the form

\[
M_K(T_j) > \lim \inf_j M_K(T_j) - \frac{\epsilon}{3},
\]

(3.3.9)

and also

\[
|\lambda_j| < 1.
\]

(3.3.10)

Let

\[
X_j = C[W_\mu + S_j[W_\mu + <Q_j, \phi; \mu>].
\]

Notice that now we use the fact that \( <Q_j, \phi; \mu> = \partial Q_j[W_\mu - \partial(Q_j[W_\mu]) \). We can do the exact same computations as before using the choice of \( \epsilon \) and an index \( j \) where [3.3.2] is valid in \( W_\mu \), [3.3.3] [3.3.8] [3.3.9] and [3.3.10] are valid and the fact that \( K \subset W_\mu \) to get that

\[
E_{\lambda_j}(X_j) \leq E_{\lambda_j}(T_j[W_\mu]) + M(C[W_\mu] - \lim \inf_j M_K(T_j) + \epsilon.
\]

We let \( \delta \searrow 0 \) and then \( W_\mu \equiv K \) thus we get

\[
E_{\lambda_j}(X_j) \leq E_{\lambda_j}(T_j[K]) - \epsilon,
\]

contradicting the minimising property of \( T_j \). Therefore, combined with the lower semi-continuity of the mass functional, we have proved that

\[
\lim \inf_j M_K(T_j) = M_K(C),
\]

for any compact set \( K \) in \( \mathbb{R}^{n+1} \). So after passing to a subsequence, which we still index as \( j \), we have

\[
M_K(T_j) \rightarrow M_K(C),
\]

which shows that

\[
\mu_{\eta_0, r_j \cdot T} \rightarrow \mu_C.
\]

(3.3.11)
From [3.3.11] we conclude that $C$ is not the zero current, otherwise the monotonicity formula for $T_j$ would be contradicted.

Furthermore, since $V_T$ has first variation in $L^\infty(\mu_{V_T})$, see Remark 3.1.5 we can apply Theorem 42.7 combined with Theorem 37.4 of [SIM84] to get that every varifold tangent of $V_T$ is a stationary cone. From [3.3.11] we conclude that the associated varifold $V_C$ of $C$ is a varifold tangent, thus $C$ is stationary and $\eta_{0,r}C = C$ for any $r > 0$.

Finally, from Lemma 3.3.1 we have that $C$ is smoothly embedded everywhere except on a set of Hausdorff dimension at most $(n - 7)$. The smoothly embedded part has density 1 almost everywhere and since $C$ is not the zero current we can apply the constancy theorem, Theorem 4.1.31 of [FED69], to conclude that the smoothly embedded part has density 1 everywhere. This concludes the proof of Lemma 3.3.2.

From Theorem 3.3.1 it is evident that in order to prove a regularity theorem for one-sided minimisers of $E_\lambda$ we have to investigate the set of classical and touching singularities. We are going to assume that the one sided-minimiser has no touching singularities. We do that since our goal is to prove the regularity theorem, Theorem 4.1.2, where the presence of touching singularities can be excluded.

**Theorem 3.3.2.** Let $T = \partial[A][B_1$ be a one-sided minimiser of $E_\lambda$ that is also a critical point of $E_\lambda$ in $B_1$ and has no touching singularities. Then $\text{spt}T \setminus \text{spt}\partial T$ is smooth everywhere except on a set of Hausdorff dimension at most $(n - 7)$.

**Proof.** We are going to use Theorem 3.3.1 so we check that conditions 1) – 4) are satisfied. Let $V_T$ be the associated varifold of $T$. Then from Proposition 3.1.2 and Remark 3.1.5 we have that condition 1) is satisfied. Condition 3) is satisfied since from our assumption $\text{sing}_1 V_T = \emptyset$. From Proposition 3.2.1 and since there are no touching singularities we have that condition 4) is also satisfied. It remains to check condition 2). If we have a classical singularity then from Lemma 3.3.2 and Lemma 35.5 of [SIM84] we have that the current limit cone at that point is of the form

$$C = \mathbb{R}^{n-1} \times C_0,$$

where $C_0$ is a multiplicity-1 cone in $\mathbb{R}^2$, that minimises the area functional on one
side and is stationary in $\mathbb{R}^2$. But then, similar to Proposition\textsuperscript{3.2.1} we have that $C_0$ is a stable, minimal cone in $\mathbb{R}^2$ and by [SIM68] we have that $C_0$ is a multiplicity-1 line and thus $C$ is a multiplicity-1 $n$-plane. From Theorem 5.4.6 of [FED69] we have that $T$ is smooth at that point and hence so is $V_T$, thus $V_T$ has no classical singularities and condition 2) is satisfied. We can now apply Theorem \textsuperscript{3.3.1} to get that $V_T$ is smooth everywhere except on a set of Hausdorff dimension at most $(n - 7)$. Since the regular parts of $V_T$ and $T$ are the same this concludes the proof of Theorem \textsuperscript{3.3.2}.

\textbf{Remark 3.3.2.} In order to prove Theorem \textsuperscript{3.3.2} we used Lemma \textsuperscript{3.3.2} to do a tangent cone analysis that allowed us to exclude classical singularities. Another way to exclude classical singularities and get Theorem \textsuperscript{3.3.2} would be to apply the one-sided deformations used in Section 9 part i) of [BW18].

\textbf{Remark 3.3.3.} In the presence of touching singularities then we can still apply Theorem \textsuperscript{3.3.1} to conclude that if $T = \partial[A] \mid B_1$ is a one-sided minimiser that is a critical point of $E_\lambda$ in $B_1$ then $\text{sing} V_T \setminus \text{sing} V_T$ is empty for $n \leq 6$, discrete for $n = 7$ and has Hausdorff dimension at most $(n - 7)$ for $n \geq 8$.

To see the latter, conditions 1) and 2) will be satisfied as explained already in the proof of Theorem \textsuperscript{3.3.2}. Since now we may have $\text{sing} V_T \neq \emptyset$ we need to check that condition 3) is satisfied. From De Giorgi’s structure theorem, Theorem \textsuperscript{2.1.6} we have that $\mathcal{H}^n(\text{spt} V_T \setminus \partial^* E) = 0$ and $\Theta^n(\mu V_T, x) = 1$ for any $x \in \partial^* E$. Thus since at a point $x \in \text{sing} V_T$ we have $\Theta^n(\mu V_T, x) \geq 2$ we conclude that the coincidence set at a touching singularity has $\mathcal{H}^n$-measure zero.

Finally, condition 4) will be satisfied in gen-reg $V_T$ since we can repeat the same proof as in Proposition\textsuperscript{3.2.1} to get stability of the immersion gen-reg $V_T$. Thus the conclusion follows from Theorem \textsuperscript{3.3.1}.

At this point we would like to mention that the set $\text{sing} V_T$ is of Hausdorff dimension at most $(n - 1)$, see the discussion in Remark 2.6 of [BW18].
Chapter 4

The perturbation problem

In what follows $A$ will denote a set of locally finite perimeter and $T = \partial[A]|B_1$ a one-sided minimiser of $E_\lambda$ that is also a critical point of $E_\lambda$ in $B_1$, where $\lambda > 0$. Moreover, we assume that $\text{spt} T$ is connected in $\mathbb{R}^{n+1}$, $0 \in \text{spt} T$, $\text{spt} T \setminus \{0\}$ is a $C^\infty$-hypersurface and $0$ is not a touching singularity of $T$. From Remark 2.1.11 we can assume that $\partial A \cap \overline{B_1} = \text{spt} T$ and that $A$ is a closed and connected set of $\mathbb{R}^{n+1}$. In this chapter we construct a sequence of generalised CMC hypersurfaces that minimise $E_\lambda$ on one side and converges to $T$. In order to set up and prove the main theorem, Theorem 4.2.1, we need to investigate some properties of minimisers with respect to the obstacle $T$.

4.1 Minimisers with respect to an obstacle

We first prove the following result regarding the direction of the mean curvature of $T$.

Lemma 4.1.1. The mean curvature vector of $\text{reg}T$ points inside $A$, the minimising side of $T$.

Proof. Let $\nu_A$ be the inward pointing unit normal associated to the reduced boundary $\partial^* A$ of $A$. We extend the normal $\nu_A$ in a tubular neighborhood of $K$, where $K \subset \text{reg}T$. Consider then $X = f\nu_A$ for $f \in C^1_c(B_1)$ and $f > 0$ as a test field in the
variational equations \[3.1.8\] Since \( \lambda > 0 \) and \( f > 0 \) we get that
\[
\int_{\partial^*A} \text{div}_{\partial^*A} X d\mathcal{H}^n \leq 0,
\]
thus the area decreases as we move inwards and so the mean curvature vector points inwards.

**Definition 4.1.1.** Let \( T' \in J_{n,K}(\mathbb{R}^{n+1}) \) where \( K = A \cap \overline{B}_1 \) and \( T' \) minimises \( E_\lambda \) in \( A \cap \overline{B}_1 \). Then \( T' \) is called a minimiser of \( E_\lambda \) with respect to the obstacle \( T \).

**Remark 4.1.1.** Let \( T' \) be a minimiser of \( E_\lambda \) with respect to the obstacle \( T \) and \( X \in C^1_c(A \cap B_1; \mathbb{R}^{n+1}) \) with \( \text{spt}X \cap \partial T = \emptyset \). Then if \( \text{spt}X \subset \subset A^\circ \cap B_1 \), where \( A^\circ \) denotes the interior of \( A \) in \( \mathbb{R}^{n+1} \), from Proposition \[3.1.2\] we have that the variational equations \[3.1.8\] are valid. On the other hand, if \( \text{spt}X \cap \partial A \cap B_1 \neq \emptyset \) and \( X \cdot \nu_A \geq 0 \), for \( \nu_A \) the inward pointing unit normal associated to \( A \) then from Remark \[3.2.2\] 3) we have
\[
\frac{d}{dt} \bigg|_{t=0} E_\lambda(\theta_t T') \geq 0 \quad (4.1.1)
\]
where \( \theta_t \) is the local flow of \( X \).

The variational inequality \[4.1.1\] is essential since it allows us to use White’s maximum principle, Theorem 7 of [WHI09], in order to get the variational equations \[3.1.8\] in \( B_1 \setminus \{0\} \) for \( T' \). We prove this in the following:

**Lemma 4.1.2.** Let \( T' \in J_{n,K}(\mathbb{R}^{n+1}) \), where \( K = A \cap \overline{B}_1 \), be a minimiser of \( E_\lambda \) with respect to the obstacle \( T \). Then \( T' \) is a critical point of \( E_\lambda \) in \( B_1 \setminus \{0\} \).

**Proof.** Let \( x \in (\text{spt} T' \setminus \partial^* T') \cap B_1 \), then we have the following cases. If \( x \notin \text{spt}T \) then there exists \( r > 0 \) such that
\[
B_r(x) \subset \subset A^\circ \cap B_1,
\]
where \( A^\circ \) denotes the interior of \( A \) in \( \mathbb{R}^{n+1} \). Thus \( T' \) minimises \( E_\lambda \) in \( B_r(x) \) and as discussed in Remark \[4.1.1\] we have that the variational equations \[3.1.8\] are valid around that point.

If \( x \in \text{spt}T \) and \( x \neq 0 \) then around that point we are in the situation where we have a smooth obstacle \( T \), that is a critical point of \( E_\lambda \) for \( \lambda > 0 \), connected, with
mean curvature that points inside $A$ and $T'$ satisfies 4.1.1. Then we can apply the maximum principle, Theorem 7 of \[WHI09\] to conclude that locally around $x$, $T$ and $T'$ agree, thus $T'$ again satisfies the variational equations 3.1.8. In total, we obtain that $T'$ is a critical point of $E_\lambda$ in $B_1 \setminus \{0\}$. \hfill \Box

In order to extend the variational equations of $T'$ around the origin we shall need an extension of the monotonicity formula. We follow \[SIM82\] to prove the following:

**Lemma 4.1.3. (extension of monotonicity formula)** Let $T' = (M, \overrightarrow{T'}, \mu)$ be a minimiser of $E_\lambda$ with respect to the obstacle $T$. Then $F(\tau) \tau^{-n} \mu(B_\tau(0))$ is monotonically non-decreasing, where $F(\tau) \in [e^{-\lambda\tau}, e^{\lambda\tau}]$ and $F(\tau) \to 1$ as $\tau \to 0$.

**Proof.** Fix $\rho_0 < 1$ and let

$$X(x) = \psi(|x|)x,$$

be the position vector field, with $\psi \in C^1_c((0, \rho_0))$. We can plug in $X$ in the variational equations 3.1.8 since from Lemma 4.1.2 we know that $T'$ is a critical point of $E_\lambda$ in $B_1 \setminus \{0\}$ and $X \in C^1(B_1 \setminus \{0\}; \mathbb{R}^{n+1})$. We compute

$$\text{div}_M X = n\psi(|x|) + |x|\psi'(|x|) |\nabla M||x|.$$  

For $0 < \sigma < \rho < \rho_0$ we make the further choices

$$\psi_{\sigma, \rho}(t) = \phi_\sigma(t) \psi\left(\frac{t}{\rho}\right),$$

where

$$\phi_\sigma \equiv 0 \text{ for } t < \sigma/2,$$

$$\phi_\sigma \equiv 1 \text{ for } t > \sigma,$$

$$\phi'_\sigma \geq 0 \text{ for all } t,$$

$$\psi \equiv 0 \text{ for } t \geq 1,$$

$$\psi \equiv 1 \text{ for } t \in [0, 1 - \epsilon],$$

$$\psi' \leq 0 \text{ for all } t,$$
where $\epsilon > 0$ is arbitrary. If we plug in $X(x) = \psi_{\sigma,\rho}(|x|)x$ in the variational equations 3.1.8 we get

$$
\int n\phi_{\sigma}(|x|)\psi\left(\frac{|x|}{\rho}\right)d\mu + \int |x| |\nabla^M |x||^2 \frac{1}{\rho} \psi'\left(\frac{|x|}{\rho}\right)\phi_{\sigma}(|x|)d\mu +
+ \int \psi\left(\frac{|x|}{\rho}\right)\phi_{\sigma}(|x|) x \cdot H \mu \leq 0,
$$

where $H = \lambda \ast \left(\overrightarrow{T\sigma}\right)$. Multiplying by $\rho^{-n-1}$ we get

$$
\rho^{-n-1}n \int \psi\left(\frac{|x|}{\rho}\right) \phi_{\sigma}(|x|)d\mu + E_0(\rho) - \rho^{-n} \frac{d}{d\rho} \int \psi\left(\frac{|x|}{\rho}\right) \phi_{\sigma}(|x|) |\nabla^M |x||^2 d\mu \leq 0,
$$

where $E_0(\rho) = \rho^{-n} \int \psi\left(\frac{|x|}{\rho}\right) \phi_{\sigma}(|x|) \frac{x \cdot H}{\rho} d\mu$.

Using the property that

$$
|\psi'\left(\frac{|x|}{\rho}\right)|\rho^{-n} \geq \frac{(1-\epsilon)^n}{|x|^n} |\psi'\left(\frac{|x|}{\rho}\right)|
$$

we get

$$
\frac{d}{d\rho} \left(\rho^{-n} \int \psi\left(\frac{|x|}{\rho}\right) \phi_{\sigma}(|x|)d\mu\right) - E_0(\rho) - \frac{d}{d\rho} \int \psi\left(\frac{|x|}{\rho}\right) \phi_{\sigma}(|x|)(1 - |\nabla^M |x||^2)|x|^{-n}(1-\epsilon)^n d\mu \geq 0,
$$

We can write

$$
E_0(\rho) = E(\rho) \rho^{-n} \int \psi\left(\frac{|x|}{\rho}\right) \phi_{\sigma}(|x|)d\mu,
$$

with $E(\rho) \in [-\lambda, \lambda]$.

Finally, multiplying by the integrating factor $F(\rho) = e^{-\int_0^\rho E(t)dt}$ and since
\[ F(\rho) \in [e^{-\lambda \rho}, e^{\lambda \rho}] \] we get

\[
\frac{d}{d\rho} \left( F(\rho) \rho^{-n} \int \psi \left( \frac{|x|}{\rho} \right) \phi_\sigma(|x|) d\mu \right) \geq \]

\[
\geq e^{-\lambda} \frac{d}{d\rho} \int \psi \left( \frac{|x|}{\rho} \right) \phi_\sigma(|x|) \left(1 - \frac{\nabla^M |x|^2}{|x|^n (1 - \epsilon)^{-n}} \right) d\mu. \tag{4.1.3}
\]

If we now integrate inequality (4.1.3) from \(\tau\) to \(\rho\), for fixed values \(\tau, \rho\) with \(\sigma < \tau\) and use the properties of \(\psi\) we get

\[
F(\tau) \tau^{-n} \int \psi \left( \frac{|x|}{\tau} \right) \phi_\sigma(|x|) d\mu \leq F(\rho) \rho^{-n} \int \psi \left( \frac{|x|}{\rho} \right) \phi_\sigma(|x|) d\mu, \tag{4.1.4}
\]

Since \(T'\) is an integral current it has finite mass and so \(\mu(B_\tau) < \infty\). Thus we are allowed in inequality (4.1.4) to let \(\sigma, \epsilon \to 0\) to get

\[
F(\tau) \tau^{-n} \mu(B_\tau) \leq F(\rho) \rho^{-n} \mu(B_\rho),
\]

which proves the desired monotonicity formula around the origin. In particular we have the following growth estimate that will be useful in the sequel

\[
\mu(B_\rho) \leq C \rho^n, \tag{4.1.5}
\]

where \(C\) is a constant that depends on \(\lambda\) and the dimension \(n\).

We are now in position to use a standard capacity argument in order to get the desired variational equations in the whole unit ball.

**Theorem 4.1.1.** Let \(T' = (M, \vec{T}', \mu)\) be a minimiser of \(E_\lambda\) with respect to the obstacle \(T\). Then \(T'\) is a critical point of \(E_\lambda\) in \(B_1\).

**Proof.** From Lemma 4.1.2 we know that \(T'\) is a critical point of \(E_\lambda\) in \(B_1 \setminus \{0\}\), hence the variational equations 3.1.8 are valid for any \(X \in C^1_c(B_1 \setminus \{0\}; \mathbb{R}^{n+1})\). For an arbitrary vector field \(X\) that is compactly supported in \(B_1\) we consider the
function $\eta_\epsilon$ with
\[ 0 \leq \eta_\epsilon \leq 1, \]
\[ \eta_\epsilon \equiv 0 \text{ in } B_{\epsilon/2}, \]
\[ \eta_\epsilon \equiv 1 \text{ in } B_1 \setminus B_\epsilon, \]
\[ |\nabla \eta_\epsilon| \leq \frac{C}{\epsilon}. \]

For the vector field $\eta_\epsilon X$ we can apply the variational equations \[3.1.8\] to get
\[ \int_M \eta_\epsilon \text{div}_M X \, d\mu + \int_M \nabla^M \eta_\epsilon \cdot X \, d\mu + \lambda \int_M \eta_\epsilon X \cdot (\nabla) \, d\mu = 0. \]

From the extension of monotonicity formula, Lemma \[4.1.3\], we can use the growth estimate \[4.1.5\] to get
\[ \int_M |\nabla^M \eta_\epsilon| |X| \, d\mu \leq \frac{C}{\epsilon} |X|_\infty \mu(B_\epsilon) \leq C \epsilon^{n-1}. \]

Letting $\epsilon \to 0$ we have that $\eta_\epsilon \to 1$ $\mu$-a.e. and since the gradient term vanishes in the limit we get the variational equations \[3.1.8\] for any vector field $X \in C^1_c(B_1; \mathbb{R}^{n+1})$, proving that $T'$ is a critical point of $E_\lambda$ in $B_1$. \hfill \Box

We now turn our attention to the size of the singular set of a minimiser of $E_\lambda$ with respect to the obstacle $T$. In order to apply Theorem \[3.3.2\] we first need to exclude the presence of touching singularities.

**Lemma 4.1.4.** Let $T' = \partial(B) \setminus B_1$ be a minimiser of $E_\lambda$ with respect to the obstacle $T$, where $B$ a set of locally finite perimeter such that $B \subset A$. Then $T'$ has no touching singularities.

**Proof.** Assume for the contrary that there is a point $x \in (\text{spt} T' \setminus \text{spt} \partial T') \cap B_1$ that is a touching singularity. Then we have the following cases. If $x \notin \text{spt} T$ there exists $r > 0$ such that
\[ B_r(x) \subset A^\circ \cap B_1, \]
where $A^\circ$ denotes the interior of $A$ in $\mathbb{R}^{n+1}$. Thus $T'$ minimises $E_\lambda$ in $B_r(x)$ and we
can use the density estimates of Theorem 21.11 from \cite{MAG12} to get that
\[
\frac{\text{vol} (B \cap B_\rho(x))}{\rho^{n+1} \omega_{n+1}} \leq 1 - \frac{1}{4^{n+1}},
\]
for any \( \rho < r \).

In particular, this estimate gives us that after rescaling \( B \) at \( x \), the volume term must stay strictly below the volume of the ball. On the other hand, since \( x \) is a touching singularity we have two smooth sheets meeting tangentially at \( x \) thus, after rescaling \( B \), the volume term from the density estimates gets closer and closer to the volume of the ball, since in the limit we have two half-balls meeting tangentially along a common tangent plane and so we have a contradiction.

If \( x \in \text{spt} T \) and \( x \neq 0 \) from White’s maximum principle, Theorem 7 of \cite{WHI09}, we have that locally around \( x \), \( T \) and \( T' \) must agree but then \( T \) must have a touching singularity at \( x \) which cannot happen since \( T \) is smooth away from the origin.

If \( x \in \text{spt} T \) and \( x = 0 \) then we are at the situation where Theorem 7 of \cite{WHI09} does not apply. However, If we rescale at \( x \) we get \( B_j \subset A_j \) where \( A_j \) and \( B_j \) are rescalings of \( A \) and \( B \) respectively. In the limit the inclusion will remain the same, see Lemma \[4.2.2\] as well. However \( B_j \) converges to two half-planes meeting tangentially at a common \( n \)-plane while \( A_j \) converges to a set \( A' \) such that \( C = \partial [A'] \) and \( C \) is a cone. So again we get a contradiction and this finishes the proof of lemma.

\begin{theorem}
Let \( T' \) be a minimiser of \( E_\lambda \) with respect to the obstacle \( T \) then \( \text{spt} T' \setminus \text{spt} T' \) is smooth everywhere, except on a set of Hausdorff dimension at most \((n - 7)\).
\end{theorem}

\begin{proof}
We want to apply Theorem \[3.3.2\]. Since \( T' \) is a minimiser of \( E_\lambda \) we can easily see that
\[
M(T') \leq M(T' + X) + Q_X(\omega),
\]
for any \( X \in I_n(\mathbb{R}^{n+1}) \) with \( \text{spt} X \subset A \cap \overline{B}_1 \), \( \partial X = 0 \) and \( Q_X \) the integral \((n+1)\)-current from Definition \[3.1.1\]. We can use Lemma \[2.1.3\] to decompose \( T' \) as
\[
T' = \sum_{i \in \mathbb{Z}} T_i,
\]

\end{proof}

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\[ M(T') = \sum_{i \in \mathbb{Z}} M(T_i), \]

where \( T_i = \partial[U_i] \cap B_1 \) for \( U_i \subset U_{i-1} \) subsets of \( A \cap \overline{B_1} \). Then for a fixed index \( i \) and a fixed \( X \in I_n(\mathbb{R}^{n+1}) \) with \( \text{spt}X \subset A \cap \overline{B_1} \), \( \partial X = 0 \) we have

\[ M(T') \leq M(T' - T_i + T_i + X) + Q_X(\omega) \leq M(T' - T_i) + M(T_i + X) + Q_X(\omega) = M\left(\sum_{i \neq j} T_j\right) + M(T_i + X) + Q_X(\omega) \leq \sum_{i \neq j} M(T_j) + M(T_i + X) + Q_X(\omega) = M(T') - M(T_i) + M(T_i + X) + Q_X(\omega), \]

which shows that each \( T_i \) is a minimiser of \( E_\lambda \) with respect to the obstacle \( T \). Thus from Lemma 4.1.4 each \( T_i \) has no touching singularities. From Theorem 4.1.1 we have that each \( T_i \) is a critical point of \( E_\lambda \) in \( B_1 \). Moreover, each \( T_i \) is a one-sided minimiser of \( E_\lambda \), so we can apply Theorem 3.3.2 to get that there are open sets \( V_i \) such that \( \text{spt}T_i \cap V_i \) are smooth and

\[ \mathcal{H}^{n-7+\delta}(V^c_i \cap B_1) = 0, \]

for any \( \delta > 0 \). Let \( V = B_1 \setminus \bigcup_i (\text{spt}T_i \setminus V_i) \), then

\[ \mathcal{H}^{n-7+\delta}(V^c \cap B_1) = 0. \]

For \( x \in \text{spt}T' \cap V \) we have from inequality 4.1.5 that \( x \in \text{reg} T_i \) for finitely many \( i \)'s. If \( U_i \subset U_j \) then for the tangent spaces at \( x \) we have \( T_y T_i = T_x T_j \).

In conclusion, we can describe \( \text{spt}T_i, \text{spt}T_j \) around \( x \) as graphs of smooth functions \( u_i, u_j \) with \( u_i \geq u_j, u_i(0) = u_j(0), \nabla u_i(0) = \nabla u_j(0) \) and since the mean curva-
ture vectors point in the same direction for both $T_i, T_j$

$$\text{div}\left(\frac{\nabla u_i}{\sqrt{1 + |\nabla u_i|^2}}\right) = \text{div}\left(\frac{\nabla u_j}{\sqrt{1 + |\nabla u_j|^2}}\right) = \lambda.$$  

From the maximum principle for CMC hypersurfaces, see Remark 2.2.1, we have that $u_i = u_j$ and so $\text{spt}T' = \text{spt}T_i$ in that neighborhood of $x$, proving that $x$ is a regular point. Thus $\text{spt}T'$ is smooth everywhere except on a set of Hausdorff dimension at most $(n - 7)$.

**Remark 4.1.2.** We mention here that the one-sided minimising property of $T$ was not needed for Theorem 4.1.1 hence it will remain valid if $T$ is just a critical point of $E_\lambda$ in $B_1$. The same is true for Theorem 4.1.2 provided that the obstacle $T$ admits a varifold tangent cone of multiplicity-1 at the origin, that is non-planar, since in that way we can extend Lemma 4.1.4 as well. This will be our setting in all of our considerations hence Lemma 4.1.4 and Theorem 4.1.2 will still be valid. We will make use of the one-sided minimising property in the next section.

### 4.2 The main theorem

In order to prove the main result of this section we make some further assumptions on the current $T$. We assume that there are $0 < r_1 < r_2 < 1$ such that $\Gamma_0 = \partial(T \cap B_{r_1})$ is smoothly embedded, connected, $(n - 1)$- submanifold of $\partial B_{r_1}$, $A \cap \overline{B}_{r_2}$ has a connected complement and

$$0 < \lambda < \min \left\{ \frac{1}{C_n \text{vol}(A \cap \overline{B}_{r_2})^{1/n+1}}, \frac{n}{r_2} \right\},$$

where $C_n$ the constant from Proposition 3.1.1. We now state two lemmas that will be useful.

**Lemma 4.2.1.** Let $S_j, S \in \mathcal{J}_{n,K}(\mathbb{R}^{n+1})$, $K \subset \mathbb{R}^{n+1}$ compact, such that $S_j$ minimises $E_\lambda$ in $K$ and $S_j \rightharpoonup S$. Then $S$ minimises $E_\lambda$ in $K$ and

$$\mu_{S_j} \xrightarrow{j \to \infty} \mu_S.$$  

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Proof. The proof is an immediate adaptation of the proof of Lemma 3.3.2 exploiting the minimising property on $K$. \hfill \Box

For the next lemma we will need the following:

Definition 4.2.1. Let $X, Y \subset \mathbb{R}^{n+1}$ and for $x \in \mathbb{R}^{n+1}$ $d(x, Y)$ denotes the distance of $x$ from the set $Y$, thus $d(x, Y) = \inf_{y \in Y} |x - y|$. Then the Hausdorff distance between $X$ and $Y$ is given by

$$d_H(X, Y) = \max \left\{ \sup_{x \in X} d(x, Y), \sup_{y \in Y} d(y, X) \right\}.$$

Lemma 4.2.2. Let $S_j, S \in \mathcal{I}_{n, \mathbb{R}}(\mathbb{R}^{n+1})$ be critical points of $E_\lambda$ in $B_1$ with $\mu_{S_j} \to \mu_S$. Then for any compact $K \subset B_1$,

$$\text{spt} S_j \cap K \xrightarrow{d_\infty} \text{spt} S \cap K.$$

Proof. Assume that there exists an $\epsilon > 0$ and a compact set $K \subset B_1$ such that for all $j$

$$d_H(\text{spt} S_j \cap K, \text{spt} S \cap K) > \epsilon.$$

Then from the definition of the Hausdorff distance $d_H$, see Definition 4.2.1, we have two cases.

In the first case there are $x_j \in \text{spt} S_j \cap K$ such that

$$|x_j - x| > \epsilon, \quad (4.2.1)$$

for any $x \in \text{spt} S \cap K$. After passing to a subsequence, which we still index as $j$, $x_j \to x_0 \in K$ and from 4.2.1 we have that $x_0 \notin \text{spt} S$. On the other hand from Remark 2.1.3 we have that

$$\limsup_j \Theta^n(\mu_{S_j}, x_j) \leq \Theta^n(\mu_S, x_0),$$

thus $\Theta^n(\mu_S, x_0) \geq 1$ proving that $x_0 \in \text{spt} S$, contrary to our assumption.
In the second case there are \( x_j \in \text{spt} S \cap K \) with
\[
|x_j - x| > \epsilon, \tag{4.2.2}
\]
for any \( x \in \text{spt} S \cap K \). After passing to a subsequence, which we still index as \( j \), \( x_j \to x_0 \in \text{spt} S \cap K \) and from (4.2.2) we see that \( B_{\epsilon/2}(x_0) \cap \text{spt} S \) has no mass. Thus in the limit we have
\[
\mu_S(B_{\epsilon/2}(x_0)) = 0,
\]
which contradicts the fact that \( x_0 \in \text{spt} S \) and finishes the proof. \( \square \)

We now state and prove the main result of this section which is the first crucial step towards the proof of the smooth approximation theorem, Theorem 6.2.

**Theorem 4.2.1.** Let \( \phi_j \in C^2(\Gamma_0; \partial B_{r_1}) \) with
\[
|\phi_j - i_{\Gamma_0}|_{C^2} \leq 1/j,
\]
where \( i_{\Gamma_0} \) denotes the inclusion map of \( \Gamma_0 \) into \( \partial B_{r_1} \), \( \Gamma_j = \phi_j \ast \Gamma_0 \), \( \Gamma_j \subset A \) and
\[
0 < \lambda < \min \left\{ \frac{1}{C_n \text{vol}(A \cap B_{r_2})^{1/n+1}}, \frac{n}{r_2} \right\}.
\]
Then there exist integral \( n \)-currents \( S_j \) that minimise \( E_\lambda \) in \( A \setminus \overline{B_{r_2}} \) subject to the boundary condition \( \partial S_j = \Gamma_j \) and satisfy the following properties:

1) \( \text{spt} S_j \subset \overline{B_{r_1}} \),

2) there exist sets of locally finite perimeter \( B_j \) with \( \overline{B_j} \subset A \cap \overline{B_{r_1}} \) such that \( S_j = \partial[B_j] \setminus B_{r_1} \),

3) \( \Gamma_j = \text{spt} S_j \cap \partial B_{r_1} \),

4) \( S_j \) is a critical point of \( E_\lambda \) in \( B_{r_1} \). In fact it is a critical point of \( E_\lambda \) in \( B_{r_2} \setminus \Gamma_j \),

5) \( \mu_{S_j} \xrightarrow{j \to \infty} \mu_{T \setminus \overline{B_{r_1}}} \).

**Proof.** We set up for each \( j \) the following minimisation problem
\[
\inf \left\{ E_\lambda(S) \mid S \in I_n(\mathbb{R}^{n+1}), \text{spt} S \subset A \cap \overline{B_{r_2}}, \partial S = \Gamma_j \right\}.
\]
From the initial assumptions we can apply Theorem 3.1.1 so there exist $S_j$ where
the infimum is achieved. From the initial bound on the $C^2$-norm of $\phi_j$ and the area
formula 2.1.1 we estimate

$$M(\Gamma_j) = \int_{\Gamma_0} |J_{\phi_j}| \, dH^{n-1} \leq C,$$

where $C$ is a constant independent of $j$.

In order to bound the mass of $S_j$ uniformly in $j$ we proceed as follows. Since $\Gamma_j \rightharpoonup \Gamma_0$ we have that, for large $j$, $\Gamma_j$ is a graph over a tubular neighborhood $T_0 \subset A \cap \partial B_{r_1}$ of $\Gamma_0$. Thus there are smooth integral $n$-currents $R_j$ in $T_0$ such that

$$\partial R_j = \Gamma_j - \Gamma_0,$$

with $\text{spt} R_j \subset T_0$. Notice that $M(R_j) \leq M(T_0) \leq C$, where $C$ is a constant independent of $j$. We can use the current $R_j + T[B_{r_1}$ as an energy competitor and since the initial fixed current is chosen arbitrarily, see Remark 3.1.3, we conclude that

$$E_\lambda(S_j) \leq M(R_j + T[B_{r_1}) \leq C,$$

where $C$ is another constant independent of $j$.

From the estimate 3.1.2 of Theorem 3.1.1 we have that

$$C \geq E_\lambda(S_j) \geq M(S_j)(1 - \lambda \text{vol}(A \cap B_{r_2})^{1/n+1}C_n) -$$

$$\lambda \text{vol}(A \cap B_{r_2})^{1/n+1}C_n M(R_j + T[B_{r_1}),$$

thus

$$M(S_j) \leq C,$$

for a constant $C$ independent of $j$. In conclusion

$$\sup_j (M(S_j) + M(\Gamma_j)) < \infty.$$

From Theorem 2.1.5 and Remark 2.1.7 we have that there exists a subsequence, which we still index as $j$ and $S \in I_n(\mathbb{R}^{n+1})$ with $\text{spt} S \subset A \cap B_{r_2}$ such
that
\[ S_j \to S. \]

From the continuity of the boundary operator we conclude that \( \partial S = \Gamma_0 \). Moreover from Lemma 4.2.1 we have that
\[ \mu S_j \to \mu S, \]
and \( S \) minimises \( E_\lambda \) in \( A \cap \overline{B_{r_2}} \). Consider the following integral \( n \)-current
\[ T' = T - T | B_{r_1} + S, \]
then \( \partial T' = \partial T \). For a fixed \( T_0 \in I_n(\mathbb{R}^{n+1}) \) with \( \text{spt} T_0 \subset A \cap \overline{B_1}, \partial T_0 = \partial T \) and a fixed \( T_1 \in I_n(\mathbb{R}^{n+1}) \) with \( \text{spt} T_1 \subset A \cap \overline{B_{r_2}}, \partial T_1 = \Gamma_0 \) we compute
\[
E_\lambda(T') = M(T - T | B_{r_1} + S) + V_\lambda(T - T | B_{r_1} + S, T_0) \leq \\
\leq M(T - T | B_{r_1}) + M(S) + V_\lambda(T, T_0) - \\
- V_\lambda(T | B_{r_1}, T_1) + V_\lambda(S, T_1).
\]

From elementary properties of outer measures and the minimising property of \( S \) the right hand side becomes
\[
= M(T) + V_\lambda(T, T_0) + M(S) + V_\lambda(S, T_1) - \\
- M(T | B_{r_1}) - V_\lambda(T | B_{r_1}, T_1) = \\
= E_\lambda(T) + E_\lambda(S) - E_\lambda(T | B_{r_1}) \leq E_\lambda(T).
\]

Since \( T \) is a one-sided minimiser of \( E_\lambda \) we conclude that \( T' \) minimises \( E_\lambda \) with respect to the obstacle \( T \). From Theorem 4.1.1 and Theorem 4.1.2 we have that \( T' \) is a critical point of \( E_\lambda \) in \( B_1 \) and that \( T' \) is smooth everywhere except on a set of Hausdorff dimension at most \( (n - 7) \). Moreover, by definition of \( T' \) we see that there exists a neighborhood \( V \) of the form \( B_1 \setminus B_R \), for some \( R < 1 \), such that
Let now

\[ r_0 = \inf \{ r > 0 \mid T' \text{ agrees with } T \text{ in } B_1 \setminus B_r \}. \]

Notice that the set over which we take the infimum is non-empty because of the existence of the neighborhood \( V \). From the regularity theory, Theorem 4.1.2, applied to \( T' \) there exists a regular point \( x_0 \in \text{spt} T' \setminus \partial B_{r_0} \) and we must have that \( T \) and \( T' \) intersect tangentially at that point since they agree on \( B_1 \setminus B_{r_0} \). We can describe \( T \) and \( T' \) locally around \( x_0 \) as graphs of smooth functions denoted respectively as \( u \), \( v \) over the common tangent plane at \( x_0 \) and since \( T' \) lies on one side of \( T \) we have in total

\[ u(x_0) = v(x_0), \]
\[ \nabla u(x_0) = \nabla v(x_0), \]
\[ v \geq u. \]

From Lemma 4.1.1 we know that the mean curvature vector of \( T \) points inwards and then the same must happen for the mean curvature of \( T' \). To see the latter, if that was not true then from the assumptions on \( u \), \( v \) around \( x_0 \) we would get a contradiction to the Hopf maximum principle, Theorem 2.2.3. Thus \( u \) and \( v \) satisfy the same prescribed mean curvature equation and from the Hopf maximum principle, Theorem 2.2.3, we conclude that \( u \equiv v \). Thus \( \text{spt} T = \text{spt} T' \) and so \( T = T' \), see Remark 4.2.1 below for details. Hence we showed that \( S = T \lfloor B_{r_1} \) and so

\[ \mu_{S_j} \to \mu_{T \lfloor B_{r_1}}. \]

From Theorem 4.1.1 we have that \( S_j \) is a critical point of \( E_\lambda \) in \( B_{r_2} \setminus \Gamma_j \). We now have to ensure that, for \( j \) large enough, \( \text{spt} S_j \subset B_{r_2} \). From the Radon measure convergence and Lemma 4.2.2 we have that for any compact \( K \subset B_{r_2} \) and any \( j \) large enough \( \text{spt} S_j \cap K \subset B_{r_2} \). Thus the only way for \( S_j \) to intersect \( \partial B_{r_2} \) is if it has a disjoint portion, say \( X_j \), that lives in \( \partial B_{r_2} \) and \( \partial X_j = 0 \). Then we have that

\[ S_j = (S_j - X_j) + X_j \text{ with } M(S_j - X_j) + M(X_j) = M(S_j). \]

This contradicts the fact
that $S_j$ minimises $E_\lambda$. To see the latter we compute

$$E_\lambda(S_j) = M(S_j - X_j) + M(X_j) + V_\lambda(S_j - X_j, T_0^j) + V_\lambda(X_j, 0),$$

where $T_0^j$ is an integral $n$-current supported in $A \cap \overline{B_{r_2}}$ with $\partial T_0^j = \partial S_j$. Using the definition of $V_\lambda$, Holder’s inequality and the isoperimetric inequality as in the proof of Theorem 3.1.1 we get that

$$E_\lambda(S_j) \geq E_\lambda(S_j - X_j) + M(X_j)(1 - \lambda C_n \operatorname{vol}(A \cap \overline{B_{r_2}})^{1/n+1}),$$

and from the choice of $\lambda$ we get that $E_\lambda(S_j) > E_\lambda(S_j - X_j)$, which contradicts the minimising property of $S_j$. Thus we showed that, for all sufficiently large $j$, $\text{spt} S_j \subset \subset B_{r_2}$.

Consider now the associated varifold $V_{S_j}$ of $S_j$. Then from Remark 3.1.5 we have that $V_{S_j}$ is a varifold with generalised mean curvature $H$ and $|H| = \lambda < \frac{n}{2r_2}$. We can apply Lemma 19.1 of [SIM84] for the open set $U = B_{r_2} \setminus \overline{B_{r_1}}$ and the varifold $V_{S_j}$ to conclude that $\text{spt} S_j \subset B_{r_1}$.

We now want to prove that $\Gamma_j = \text{spt} S_j \cap \partial B_{r_1}$. Clearly we have that $\Gamma_j \subset \text{spt} S_j \cap \partial B_{r_1}$. Assume now that there is a point $x_j \in \text{spt} S_j \setminus \Gamma_j$ and $x_j \in \partial B_{r_1}$. Since $x_j$ is not a boundary point we have that $S_j$ is a critical point of $E_\lambda$ in a neighborhood of $x_j$ and so we can apply the maximum principle, Theorem 7, of [WHI09] to get a contradiction.

Finally, since $\Gamma_j$ is smoothly embedded and connected we can find a smooth domain $H_j$ in $\partial B_{r_1}$ with $\partial H_j = \Gamma_j$. Then $S_j - [H_j]$ is a cycle in $\overline{B_{r_1}}$ so there exist $R_j$ integral $(n + 1)$-current with $\partial R_j = S_j - [H_j]$. Since $\text{spt} [H_j] \subset \partial B_{r_1}$ we have that

$$S_j = \partial [B_j] \mid B_{r_1},$$

for $B_j$ a set of locally finite perimeter with $\overline{B_j} \subset A \cap \overline{B_{r_1}}$. This finishes the proof of Theorem 4.2.1. \qed

**Remark 4.2.1.** In the proof of Theorem 4.2.1 we claim that $\text{spt} T' = \text{spt} T''$ implies that $T' = T$. To see this we can use the Constancy Theorem, Theorem 41.1 from [SIM84] or more explicitly the Constancy Lemma from Appendix A of [BW18] for
$V = \text{spt}T \setminus \{0\}$ and $M = \text{spt}T \setminus \{0\}$ to conclude that $T = \theta T$, for some constant $\theta \in \mathbb{R}$. Since there is a neighborhood $V$ where $T|_V = T'|_V$ we get that $\theta = 1$. 
Chapter 5

Jacobi fields and the maximum principle

In this chapter we prove a maximum principle for critical points of \( E_\lambda \) in \( B_1 \), for some \( \lambda > 0 \), with an isolated singularity. We will assume that one of them is a one-sided minimiser of \( E_\lambda \). This assumption will simplify the proof since we will be able to apply Lemma 3.3.2. We will also add a structural condition on the isolated singularity. In the case of critical points with respect to the area functional there are many known maximum principles, even in the presence of more complicated singular sets, see for example [SIM87], [ILM96] or [WIC14b].

5.1 CMC hypersurfaces over cones

In this section we gather together some preliminary results from [HS85] regarding regular cones that are minimal in \( \mathbb{R}^{n+1} \) and the mean curvature operator of graphs over such a cone.

Definition 5.1.1. Let \( \Sigma \) be a smooth, compact, \((n-1)\)-dimensional submanifold of \( S^n \), with no manifold boundary. Then the set

\[
C = \{ \lambda y \mid \lambda \geq 0, \ y \in \Sigma \},
\]

is called a regular cone of \( \mathbb{R}^{n+1} \).
Notice that a regular cone is smooth everywhere except at the origin, unless it is a plane. We are interested in regular cones $C$ that are oriented and minimal in $\mathbb{R}^{n+1}$. This means that there is a smooth choice of a unit normal vector field $N$ in the regular part of $C$ and a smooth $n$-vector field $\nu$ such that $N \wedge \nu = e_1 \wedge \ldots \wedge e_{n+1}$ where $e_1, \ldots, e_{n+1}$ is the standard basis of $\mathbb{R}^{n+1}$ and $C \setminus \{0\}$ is minimal in the classical sense. Notice that the latter is equivalent to $\Sigma = C \cap S^n$ being minimal in $S^n$, see Lemma 2.39 of [CM11]. The submanifold $\Sigma$ is called the link of the cone $C$. For such a cone $C$ the link $\Sigma$ is connected. This follows from the following:

**Lemma 5.1.1.** If $\Sigma_1, \Sigma_2$ smooth, minimal, $(n-1)$-submanifolds of $S^n$ then they have a non-empty intersection.

**Proof.** If $\Sigma_1 \cap \Sigma_2 = \emptyset$ then using an orthogonal transformation of $\mathbb{R}^{n+1}$ we can get $\Sigma'_2$ that touches $\Sigma_1$ and lies on one side. Since orthogonal transformations are isometries for $S^n$ we have that $\Sigma'_2$ is minimal as well and thus we get a contradiction to the classical maximum principle, Theorem 2.2.4.

**Remark 5.1.1.** The result in Lemma 5.1.1 is valid in the more general case of ambient manifolds with positive Ricci curvature, see [PW03].

**Remark 5.1.2.** We would like to mention here that a minimal cone is always oriented. Thus our assumption on the orientation of $C$ can be made without any loss of generality. For more details one can see Remark 2.11 of [BW18].

We now prove the following:

**Lemma 5.1.2.** If $C$ is a minimal oriented regular cone then $\mathbb{R}^{n+1} \setminus C$ has two connected components.

**Proof.** Since the link $\Sigma$ of $C$ is minimal from Lemma 5.1.1 we know that $\Sigma$ is connected. From Alexander’s duality theorem, see Corollary 3.45 of [HAT01], we have that

$$\tilde{H}_k(S^n \setminus \Sigma) \cong \tilde{H}^{n-k-1}(\Sigma),$$

for $k \geq 0$, where $\tilde{H}$ denotes the reduced homology and cohomology with coefficients in $\mathbb{Z}$ respectively.
Thus \( \tilde{H}_0(S^n \setminus \Sigma) \cong \tilde{H}^{n-1}(\Sigma) \) and the latter is isomorphic to \( \mathbb{Z} \) since \( \Sigma \) is connected. From the definition of the reduced homology we deduce that \( H_0(S^n \setminus \Sigma) \cong \mathbb{Z} \oplus \mathbb{Z} \). Thus \( S^n \setminus \Sigma \) has two connected components.

Finally, since the radial projection from \( \mathbb{R}^{n+1} \setminus C \to S^n \setminus \Sigma \) is a deformation retraction we have that the same is true for the complement of \( C \) in \( \mathbb{R}^{n+1} \) and this concludes the proof. \( \square \)

From Lemma 5.1.2 let \( E_\pm \) be the two connected components of \( \mathbb{R}^{n+1} \setminus C \), where \( E_+ \) is chosen so that the unit normal vector field \( \vec{N} \) points inside \( E_+ \). Then for the associated current of \( C \), still denoted as \( C \), we have that

\[
C = \partial[E],
\]

where \( E = E_+ \) and \( E \) is a set of locally finite perimeter.

Let \( C \) be a minimal oriented regular cone of \( \mathbb{R}^{n+1} \) with \( \vec{N} \) its unit normal vector field and \( C_1 = C \cap B_1 \). For a function \( u \in C^2(C_1 \setminus \{0\}) \) the graph over \( C_1 \) is

\[
gr_C u = \left\{ x + u(x) \vec{N}(x) \mid x \in C_1 \setminus \{0\} \right\}.
\]

We denote by \( \mathcal{M}_C \) the Euler-Lagrange operator associated to the area functional of \( gr_C u \) and we call \( \mathcal{M}_C \) the prescribed mean curvature operator of \( C \). For \( gr_C u \) to be a CMC hypersurface in \( \mathbb{R}^{n+1} \) we must have

\[
\mathcal{M}_C u = \lambda,
\]

for some constant \( \lambda \). We next analyse the operator \( \mathcal{M}_C \) in case the function \( u \) satisfies the following radial decay, which is the only case that concerns us in this section as well as in the sequel

\[
\frac{|u(x)|}{|x|} + |\nabla u(x)| + |x||\nabla^2 u(x)| \xrightarrow{|x| \to 0} 0.
\]

(5.1.1)

**Lemma 5.1.3.** Let \( u \in C^2(C_1 \setminus \{0\}) \) that satisfies (5.1.1) then

\[
\mathcal{M}_C u = L_C u + \text{div}_C \left( P \left( x, \frac{u}{|x|}, \nabla u \right) \nabla u \right) + \frac{1}{|x|} S \left( x, \frac{u}{|x|}, \nabla u \right),
\]

(5.1.2)
where $L_C u = \Delta_C u + |A_C|^2 u$, for $\Delta_C$ the Laplace-Beltrami operator, $A_C$ the second fundamental form of $C$ in $\mathbb{R}^{n+1}$, $\|\cdot\|$ denotes the norm induced by the Euclidean inner product and

$$P : C_1 \times \mathbb{R} \times TC_1 \to \text{End}(TC_1),$$

$$S : C_1 \times \mathbb{R} \times TC_1 \to \mathbb{R}.$$

Moreover we have the following estimates

$$\left| P\left(x, \frac{u}{|x|}, \nabla u\right) \right| \leq C \Sigma \left( \frac{|u(x)|}{|x|} + |\nabla u(x)| \right),$$

$$\left| S\left(x, \frac{u}{|x|}, \nabla u\right) \right| \leq C \Sigma \left( \frac{|u(x)|}{|x|} + |\nabla u(x)| \right)^2,$$

and if $(x, z, p) \in C_1 \times \mathbb{R} \times TC_1$ with $|z| \leq 1$ and $|p| \leq 1$ then

$$|S_z| + |S_p| + |x| |P_z| \leq C (|z| + |p|),$$

$$|x| |P_{xz}| + |x| |P_{xp}| + |P_z| + |P_p| \leq C,$$

where the subscripts denote partial differentiation and $C, C_\Sigma$ are constants that depend on the dimension $n$ and the link $\Sigma$ of the cone.

**Proof.** The proof is an extension of Lemma 2.26 of [CM11] which is restricted in case $n = 2$. Let $(e_1, \ldots, e_n)$ be a local orthonormal frame of $C \setminus \{0\}$ and extend $N, (e_i)$ and $u$ to a neighborhood of $C \setminus \{0\}$ so that they are constant in the direction of $N$. Let $X : C_1 \to \mathbb{R}^{n+1}$ be the map that sends $x \in C_1$ to $x + u(x)N(x)$ and define the energy functional

$$E(u) = \int_{C_1} \det(g_{ij}),$$

where $g_{ij}$ denotes the pull-back metric of the Euclidean metric via the map $X$. We first compute the metric $g_{ij}$ with respect to the orthonormal frame $(e_i)$ that we have fixed in the beginning. The tangents to $gr_C u$ are given by

$$X_i = e_i + u_i N - u A_{ik} e_k,$$

where $u_i$ denotes $\nabla e_i u$ for $\nabla$ the gradient on $C$ and $A_{ik}$ is the second fundamental
form of the cone. Thus the metric is given by
\[ g_{ij} = \langle X_i, X_j \rangle = \delta_{ij} - 2uA_{ij} + (u_iu_j + u^2A_{ik}A_{kj}). \]

Set \( Q_{ij} = u_iu_j + u^2A_{ik}A_{kj} \) and we have that
\[ |Q_{ij}| \leq C \left( \frac{|u|}{|x|} + |\nabla u| \right)^2. \]

We want to compute
\[ \left. \frac{d}{ds} \right|_{s=0} E(u + sv) = \int_{C_1} \frac{d}{ds} \left|_{s=0} \det(g_{ij}(s)), \right. \]
for \( v \in C^1_0 (C_1 \setminus \{0\}) \). We know that
\[ \left. \frac{d}{ds} \right|_{s=0} \det(g_{ij}(s)) = \det(g_{ij}(0)) \tr(g^{ij}(0)g_{ij}'(0)). \]

The inverse matrix of \( g_{ij} \) is given by
\[ g^{ij} = \delta_{ij} + 2uA_{ij} + \tilde{Q}_{ij}, \]
where \( \tilde{Q}_{ij} \) satisfies the same quadratic inequality as \( Q_{ij} \). We also have that
\[ \det(g_{ij}(0)) = 1 + Q, \]
and we estimate the term \( Q \) as follows
\[ \det(g_{ij}(0)) \leq (\tr(\delta_{ij} - 2uA_{ij} + Q_{ij}))^n n^{-n}, \]
and since the cone is minimal we get \( \tr A_{ij} = 0 \) and therefore
\[ Q \leq \sum_{k=0}^{n-1} \binom{n}{k} Q_{ii}^{n-k}, \]
and using the initial assumption 5.1.1 we get again the same quadratic estimate for \( Q \) as for \( Q_{ij} \).
Finally, for the variation \( x + u(x)N + sv(x)N \) the pull-back metric of the Euclidean metric is given by
\[
g_{ij}(s) = \delta_{ij} + u_i u_j + u^2 A_{ik} A_{jk} - 2sv A_{ij} + 2sv u A_{ik} A_{kj} +
+ svv_j + suv_j + O(s^2).
\]

Thus we compute
\[
g'_{ij}(0) = v_i u_j + v_j u_i - 2v A_{ij} + 2uv A_{ik} A_{kj}.
\]

If we put everything together we get
\[
\frac{d}{ds} \left|_{s=0} \det(g_{ij}(s)) = (1 + Q) \text{tr}((I d + B) D), \right. \tag{5.1.3}
\]
where \( B_{ij} = 2u A_{ij} + \tilde{Q}_{ij} \) and \( D_{ij} = v_i u_j + v_j u_i - 2v A_{ij} + 2uv A_{ik} A_{kj} \). We compute
\[
\text{tr} D = 2v_i u_i - 2v A_{ii} + 2uv A_{ik}^2 = 2 < \nabla u, \nabla v > + 2uv |A_C|^2,
\]
\[
(BD)_{ij} = 2u A_{il} v_l u_j + 2u A_{il} v_j u_l - 4uv A_{il} A_{lj} + 4u^2 v A_{il} A_{lk} A_{kj} +
+ \tilde{Q}_{il} v_l u_j + \tilde{Q}_{il} v_j u_l - 2v \tilde{Q}_{il} A_{ij} + 2uv A_{ik} A_{kj} \tilde{Q}_{il},
\]
and so for the trace of \( BD \) using the symmetry of \( A_{ij} \) and \( \tilde{Q}_{ij} \) we get
\[
4u < A \nabla u, \nabla v > - 4uv |A_C|^2 + < \tilde{Q} \nabla u, \nabla v > +
+(4u^2 A_{il} A_{lk} A_{ki} + 2u A_{ik} A_{ki} \tilde{Q}_{il} - 2 \tilde{Q}_{il} A_{kl}) v,
\]
where \( A \) the matrix of \( A_{ij} \) and \( \tilde{Q} \) the matrix of \( Q_{ij} \). Put together we get that 5.1.3 is equal to
\[
(1 + Q) \left( 2 < \nabla u, \nabla v > - 2u |A|^2 v + < 4u A + \tilde{Q} \nabla u, \nabla v > +
+(4u^2 A_{il} A_{lk} A_{ki} + 2u A_{ik} A_{ki} \tilde{Q}_{il} - 2 \tilde{Q}_{il} A_{kl}) v \right).
\]
If we integrate and use integration by parts we get the desired form for some oper-
ators \( P, S \) that satisfy the desired inequalities. In particular,

\[
P = (1 + Q)(4uA + \tilde{Q}) + QId,
\]

\[
S = Qu|A|^2|x| - |x|(1 + Q)(4u^2A_{id}A_{ki} + 2uA_{ik}A_{ki}\tilde{Q}_d - 2\tilde{Q}_dA_{ii}).
\]

\[\square\]

**Remark 5.1.3.** We will call the operator \( L_C \) from Lemma 5.1.3 the Jacobi field operator of the cone \( C \).

**Remark 5.1.4.** Notice that the assumption 5.1.1 on the radial decay of \( u \) was used to derive the estimates on \( P, S \). Thus the form of the operator \( \mathcal{M}_C \) will be the same as in 5.1.2 for any \( u \), without the radial decay assumption. However the estimates on \( P, S \) will not be the same. We point out that we will have the same estimates for \( P, S \) if instead of 5.1.1 we simply have that

\[
\frac{|u(x)|}{|x|} + |\nabla u(x)| + |x||\nabla^2 u(x)| \leq 1,
\]

for any \( x \in C_1 \setminus \{0\} \).

From Lemma 5.1.3 we have that \( \mathcal{M}_C \) is a quasilinear elliptic operator of second order. Lemma 5.1.3 together with its estimates is crucial for studying a generalised CMC hypersurface near an isolated singularity. The first important consequence of the estimates established in Lemma 5.1.3 is the following linearisation lemma that associates to \( \mathcal{M}_C \) a linear elliptic operator hence we will be able to apply the linear elliptic theory of Section 2.2.

**Lemma 5.1.4.** *(First linearisation lemma)* Let \( u, v \in C^2(C_1 \setminus \{0\}) \) that satisfy 5.1.1 and \( \mathcal{M}_C u = \mathcal{M}_C v = \lambda \), for some constant \( \lambda \). Then the function \( h = v - u \) satisfies the following linear PDE

\[
L_C h = \text{div}_C(\tilde{P}\nabla h) + \frac{1}{|x|}\tilde{R} \cdot \nabla h + \frac{1}{|x|^2}\tilde{S} h,
\]

where

\[
\tilde{P} : C_1 \times \mathbb{R} \times TC_1 \to \text{End}(TC_1),
\]

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\[ \tilde{R} : C_1 \times \mathbb{R} \times TC_1 \to TC_1, \]
\[ \tilde{S} : C_1 \times \mathbb{R} \times TC_1 \to \mathbb{R}. \]

and \( \tilde{P}, \tilde{R}, \tilde{S} \xrightarrow{|x|\to 0} 0 \), where the convergence for \( \tilde{P}, \tilde{R} \) is taken with respect to the induced norms on \( \text{End}(TC_1),TC_1 \) respectively.

**Proof.** We want to compute the linear elliptic operator \( \mathcal{L} \) such that
\[ \mathcal{L} h = M_C v - M_C u. \]
From Lemma 5.1.3 let \( P, S \) be the operators associated to \( M_C \).

Then
\[ M_C v - M_C u = L_C h + \text{div}_C (P_v \nabla v - P_u \nabla u) + \]
\[ + \frac{1}{|x|} (S_v - S_u), \]
where the subscripts on \( P, S \) denote the dependence on \( v, u \) respectively as described in Lemma 5.1.3.

In order to compute the term \( P_v \nabla v - P_u \nabla u \) we rewrite it as
\[ \int_0^1 \frac{d}{dt} P \left( x, \frac{u}{|x|} + t \frac{v - u}{|x|} \right) \nabla u + t (\nabla v - \nabla u) \left( \nabla u + t (\nabla v - \nabla u) \right) dt, \]
and if we differentiate with respect to \( t \) we get the following terms in the difference \( M_C v - M_C u \)
\[ \text{div}_C (P' \nabla h) + \frac{1}{|x|} P'' \cdot \nabla h + \frac{1}{|x|^2} P''' h, \]
with estimates
\[ |P'| \leq C \left( |P_z| \left| \frac{h}{|x|} \right| + |P_p| |\nabla u| + |P_p| |\nabla h| + |P| \right), \]
\[ |P''| \leq C (|P_z| |\nabla u|), \]
\[ |P'''| \leq C (|P_z| |\nabla u| + |P_{xz}| |\nabla u| + |P_{xz}| |\nabla^2 u| + |P_z| |\nabla u|). \]

For the term \( S_v - S_u \) we make a similar computation to get terms of the form
\[ \frac{1}{|x|} S' \cdot \nabla h + \frac{1}{|x|^2} S'' h, \]
with estimates
\[ |S'| \leq C|S_p|, \]
\[ |S''| \leq C|S_z|. \]

Put together we get operators \( \tilde{P}, \tilde{R}, \tilde{S} \) such that the difference is of the desired form. From the estimates in Lemma 5.1.3, assumption 5.1.1 on the functions \( u, v \) and the estimates above we get that \( \tilde{P}, \tilde{R}, \tilde{S} \xrightarrow{|x| \to 0} 0 \). This finishes the proof of the linearisation lemma.

**Remark 5.1.5.** Using the estimates of Lemma 5.1.3 and the form of \( \tilde{P} \) we can also write the right hand side of \( L_C h \) in Lemma 5.1.4 in the form
\[ \tilde{P} \cdot \nabla^2 h + \frac{1}{|x|} \tilde{R}' \cdot \nabla h + \frac{1}{|x|^2} \tilde{S} h, \]
where \( \tilde{P} \cdot \nabla^2 h \) is to be understood as the induced inner product on \( \text{End}(TC_1) \) and again \( \tilde{R}' \xrightarrow{|x| \to 0} 0 \).

We now prove another linearisation lemma that will be useful when proving Theorem 7.3. We essentially compute the linearisation of \( M_C \) at a point \( u \in C^2(C_1 \setminus \{0\}) \) that is close to the zero function using the first linearisation lemma.

**Lemma 5.1.5.** (Second linearisation lemma) Let \( u \in C^2(C_1 \setminus \{0\}) \) such that \( u \) satisfies the radial decay assumption 5.1.1 and \( V \subset C_1 \setminus \{0\} \). Then for any function \( \psi \in C^2(V) \) such that for any \( x \in V, s \in (0, 1) \)
\[ s \frac{|\psi(x)|}{|x|} + s|\nabla \psi(x)| + s|x||\nabla^2 \psi(x)| \leq \Theta, \] (5.1.5)
where \( \Theta \) is a positive constant we have that
\[ \frac{d}{ds} \bigg|_{s=0} M_C(u + s\psi) = L_C \psi + P' \cdot \nabla^2 \psi + \frac{1}{|x|} R' \nabla \psi + \frac{1}{|x|^2} S' \psi, \] (5.1.6)
where \( |P'|, |R'|, |S'| \leq c \left( \frac{|u|}{|x|} + |\nabla u| + |x||\nabla^2 u| \right) \) in \( V \) for \( c \) a positive constant that depends on \( V \).
Proof. For $s$ small enough we want to evaluate

$$\frac{\mathcal{M}_C(u + s\psi) - \mathcal{M}_Cu}{s},$$

and then let $s \to 0$. Let $f_s(x) = s\psi(x)$, then pointwise in $V$ we have that

$$\frac{f_s(x)}{|x|} + |\nabla f_s(x)| + |x||\nabla^2 f_s(x)| \xrightarrow{s \to 0} 0.$$

From the uniform bound $5.1.5$ in $s$ and $x$ in $V$ we conclude that the convergence is uniform in $V$ as $s \to 0$. Thus in particular for $s$ arbitrarily small, we can follow the proof of the first linearisation lemma to get that

$$\frac{\mathcal{M}_C(u + s\psi) - \mathcal{M}_Cu}{s} = L_C\psi + \tilde{P}_s \cdot \nabla^2 \psi + \frac{1}{|x|}\tilde{R}_s \cdot \nabla \psi + \frac{1}{|x|^2}\tilde{S}_s \psi,$$

where the subscript on $s$ denotes the dependence on $s$. From the estimates developed in the proof of Lemma $5.1.4$ we have that

$$|\tilde{P}_s| \leq C\left(|P_{s;f}|s\psi_f + |P_{s;p}|\nabla u| + s|P_{s;p}|\nabla \psi| + |P_s|\right),$$

where $P_s, P_{s;f}, P_{s;p}$ satisfy the estimates of Lemma $5.1.3$. In particular, we have that $|\tilde{P}_s|$ is uniformly bounded in $s$ thus $\tilde{P}_s \xrightarrow{s \to 0} P'$, for some operator $P'$. Using the estimates on $\tilde{P}_s, P_s, P_{s;f}$ and $P_{s;p}$ and letting $s \to 0$ we have that

$$|P'| \leq c\left(\frac{|u|}{|x|} + |\nabla u| + |x||\nabla^2 u|\right).$$

We treat the terms $\tilde{R}_s, \tilde{S}_s$ similarly to conclude the proof of the second linearisation lemma.

5.2 Jacobi fields of stable minimal cones

In this section we gather together some results regarding stable minimal cones and their Jacobi fields found in [CHS84] and [HS85]. In what follows $C$ denotes
an oriented minimal regular cone of $\mathbb{R}^{n+1}$, see Definition 5.1.1, with $\Sigma = C \cap S^n$. We also assume that $C$ is stable for the area functional, thus it satisfies inequality 3.2.1. There is an equivalent characterisation for the stability of the cone found in [CHS84] that we briefly describe here.

We first write the Jacobi operator $L_C$ of $C$ in spherical coordinates. For $x \in C$ write $x = r\omega$ where $r = |x|$ and $\omega = \frac{x}{|x|} \in \Sigma$. Then the metric is given by

$$g_C = dr^2 + r^2 g_\Sigma,$$

where $g_\Sigma$ is the pull-back metric of the round metric of $S^n$ on $\Sigma$.

From the standard formula in Riemannian geometry for the Laplacian we compute

$$\Delta_C f = r^{1-n} \partial_r (r^{n-1} \partial_r f) + r^{-2} \Delta_\Sigma f,$$

where $f$ is any smooth function on $C \setminus \{0\}$. So we get the following formula

$$L_C f = r^{-2} L_\Sigma f + r^{1-n} \partial_r (r^{n-1} \partial_r f), \quad (5.2.1)$$

where $L_\Sigma = \Delta_\Sigma + |A_\Sigma|^2$ and $A_\Sigma$ denotes the second fundamental form of $\Sigma$ in $S^n$. In what follows $\lambda_1$ will denote the first eigenvalue of $-L_\Sigma$. Then we have the following characterisation:

**Lemma 5.2.1.** *(Theorem 4.5 [CHS84])* The cone $C$ is stable if and only if

$$\gamma^+ \geq \gamma^- \geq 0,$$

where

$$\gamma^\pm = \frac{n - 2}{2} \pm \sqrt{\left(\frac{n - 2}{2}\right)^2 + \lambda_1}.$$

If the inequalities are strict then the cone is strictly stable.

We recall here the definition of strict stability.

**Definition 5.2.1.** The cone $C$ is strictly stable if there exists $\theta > 0$ such that

$$\int_C \eta L_C \eta d\mathcal{H}^n \geq \theta,$$
for any \( \eta \in C^2_c(C \setminus \{0\}) \).

**Remark 5.2.1.** If \( \gamma^- = 0 \) in Lemma [5.2.1] then \( C \) is a plane. To see that, if \( \gamma^- = 0 \) then \( \lambda_1 = 0 \). From the variational characterisation of the first eigenvalue of a linear elliptic operator, see [CM11] Lemma 1.34, we have that

\[
0 = \lambda_1 \leq \left( \int_\Sigma \eta^2 \right)^{-1} \int_\Sigma (|\nabla \eta|^2 - |A\eta|^2),
\]

for any \( \eta \in C^1_c(\Sigma) \). Since \( \Sigma \) is compact we can take \( \eta \) to be a constant function to get that \( |A\Sigma| \equiv 0 \) thus \( |AC| \equiv 0 \) and so \( C \) is indeed a plane.

We now investigate positive solutions of \( LCf = 0 \). For such a solution \( f \) the vector field \( X = fN \), where \( N \) is the unit normal vector field of \( C \), is called a Jacobi field of \( C \).

We prove the following:

**Lemma 5.2.2.** Let \((r, \omega)\) be the spherical coordinates in \( C \). Then every positive solution of \( LCf = 0 \) is of the form

\[
f(r\omega) = \left( \frac{c_1}{r^{\gamma^+}} + \frac{c_2}{r^{\gamma^-}} \right) \phi_1(\omega), \quad (5.2.2)
\]

where \( \phi_1 > 0 \) with \( L\Sigma \phi_1 = -\lambda_1 \phi_1 \), \( c_1, c_2 \) are non-negative constants and \( \gamma^\pm \) are defined in Lemma [5.2.1].

**Proof.** Consider the eigenvalues \( \{\lambda_j\}_{j=1}^\infty \) of the operator \(-L\Sigma\) so we have

\[
\lambda_1 < \lambda_2 \leq \lambda_3 \ldots \rightarrow \infty.
\]

Since \( C \) is stable we have, from Lemma [5.2.1], that \( \lambda_1 \geq -\left( \frac{n-2}{2} \right)^2 \). Let \( \phi_j \) be an orthonormal basis of \( L^2(\Sigma) \) such that \( \phi_j \) is an eigenfunction associated to \( \lambda_j \).

Let \( f \in C^2(\Sigma \setminus \{0\}) \) then for any \( r > 0 \) we have that \( f(r,.) \) is a function on \( \Sigma \) hence of the form \( \sum_{j=1}^\infty a_j(r)\phi_j(\omega) \). Thus in order for \( f \) to solve \( LCf = 0 \) we write \( LC \) in spherical coordinates as in [5.2.1] and we get that \( a_j \) must satisfy the ODE

\[
r^2a''_j(r) + (n-1)a'_j(r) - \lambda_j a_j(r) = 0,
\]

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Thus $a_j(r) = c^+_j r^{-\gamma^+_j} + c^-_j r^{-\gamma^-_j}$ where $\gamma^\pm_j = \frac{n-2}{2} \pm \sqrt{\left(\frac{n-2}{2}\right)^2 + \lambda_j}$ and $c^\pm_j$ are constants. Thus $f$ is of the form

$$f(r\omega) = \sum_{j=1}^{\infty} c^\pm_j r^{-\gamma^\pm_j} \phi_j(\omega).$$

Suppose now that $f > 0$ with $L_C f = 0$ and in order to prove that $f$ has the desired form it suffices to show that there can be no finite linear combination of the functions $r^{-\gamma^\pm_j} \phi_j$ such that $f > 0$. We do that only for $j = 1, 2$ since any other finite linear combination can be treated in the same way. Moreover we assume that $\gamma^+ > \gamma^-$ since the case $\gamma^+ = \gamma^-$ is treated in the same manner. Thus let $f > 0$ with $L_C f = 0$ and

$$f = c_1 r^{-\gamma^+} \phi_1 + c_2 r^{-\gamma^-} \phi_1 + c_3 r^{-\gamma^+_2} \phi_2 + c_4 r^{-\gamma^-_2} \phi_2.$$

There is an open set $U \subset \Sigma$ where $\phi_2 < 0$. From our assumption $f > 0$ in the set $rU$ for any $r > 0$. Thus there is $M > 0$ independent of $r$ such that

$$c_1 r^{-\gamma^+} + c_2 r^{-\gamma^-} > M(c_3 r^{-\gamma^+_2} + c_4 r^{-\gamma^-_2}).$$

If we multiply by $r^{-\gamma^+_2}$ and let $r \to 0$ we get that $c_3 \leq 0$. Similarly if we multiply by $r^{-\gamma^-_2}$ and let $r \to \infty$ we get that $c_4 \leq 0$. Assume now that $c_3 < 0$.

There is an open set $V \subset \Sigma$ where $\phi_2 > 0$ and then

$$r^{\gamma^+} f \underset{r \to 0}{\longrightarrow} -\infty,$$

which implies that $f < 0$ on a neighborhood of $rV$ for small $r$ contradicting the fact that $f > 0$ on the whole $C \setminus \{0\}$. Thus $c_3 = 0$. We can treat the case $c_4 < 0$ in a similar fashion since then

$$r^{\gamma^-} f \underset{r \to \infty}{\longrightarrow} -\infty,$$

hence $c_4 = 0$ as well. This shows that $f$ has the desired form and we can easily prove, in a similar fashion as above, that the constants $c_1, c_2$ must be non-negative.

\[\square\]
5.3 A singular maximum principle

In what follows let $T = \partial A |_{B_1}$, for $A$ a set of locally finite perimeter. We make the following assumptions:

a) $T$ is a critical point of $E_\lambda$ in $B_1$,

b) $\text{spt} T$ is connected in $\mathbb{R}^{n+1}$, $0 \in \text{spt} T$ and $\text{spt} T \setminus \{0\}$ is a $C^\infty$-hypersurface.

c) Regarding the structure of the singularity at the origin we assume that the associated varifold $V_T$ admits a varifold tangent cone $C$ of multiplicity-1 that is an oriented regular cone with an isolated singularity at the origin.

Note that after choosing the orientation on $C$ we have showed, using Lemma 5.1.2, that the associated current to $C$, still denoted as $C$, is $C = \partial [E]$ for some set of locally finite perimeter $E$. In particular we have that the cone $C$ is a tangent cone in the current sense as well.

**Remark 5.3.1.** We will need a uniqueness property for the tangent cone of $V_T$ at the origin. In its full generality this is still an open problem. However, in our situation and since we have made assumption c), from Theorem 5 and Corollary of \cite{SIM83}, we do have that $C$ is the unique tangent cone of $V_T$ at the origin. At this point we would like to stress out that the assumption on the multiplicity of the tangent cone is essential since if $T$ is just a critical point of $E_\lambda$ in $B_1$ then the varifold tangent cone of $V_T$ may have higher multiplicity and the uniqueness theorem of \cite{SIM83} is valid only when the multiplicity of the varifold tangent cone is 1.

We will also make use of the maximum principle for stationary varifolds found in Theorem A of \cite{ILM96}.

**Lemma 5.3.1.** Let $E_1, E_2$ be two sets of locally finite perimeter in some open set $U \subset \mathbb{R}^{n+1}$ with $E_1 \subset E_2$. Let $V_1, V_2$ be the associated varifolds to $\partial^* E_1$, $\partial^* E_2$ respectively. If $V_i$ is a stationary varifold in $U$ with $\text{spt} V_i$ connected and $\mathcal{H}^{n-2}(\text{sing} V_i \cap U) = 0$ for $i = 1, 2$ then either

$$\text{spt} V_1 \cap U = \text{spt} V_2 \cap U \text{ or } \text{spt} V_1 \cap \text{spt} V_2 \cap U = \emptyset.$$
Theorem 5.3.1. Let $T = \partial\lfloor B_1$ satisfy assumptions a), b) and c) and $S = \partial\lfloor B_1$, for $B$ a set of locally finite perimeter, with $\overline{B} \subset \overline{A}$ such that $S$ minimises $E_\lambda$ with respect to the obstacle $T$ and $0 \in \text{spt} S$. Then $T = S$ where the equality is in the sense of currents.

Proof. We follow the main ideas from the proof of Corollary 1.20 of [HS85]. From Theorem 4.1.1 and Remark 4.1.2 we have that $S$ is a critical point of $E_\lambda$ in $B_1$. Let $\rho_j \searrow 0$ then from Lemma 4.1.4 we apply Lemma 3.3.2 combined with Lemma 4.2.2 and the uniqueness of the cone $C$ we have that, after passing to a subsequence that we still index as $j$,

$$\mu_{\eta_0, \rho_j} S \to \mu C',$$

where $C' = \partial F$ is a cone, for $F$ a set of locally finite perimeter, with $\overline{F} \subset \overline{E}$, $C'$ minimises area in $\overline{F}$ and is stationary in $\mathbb{R}^{n+1}$.

Due to a result of Almgren-De Giorgi, see Corollary in [BG72], we have that reg$C$, reg$C'$ are connected and since from Corollary 2.1.2 reg$C = \text{spt} C$, reg$C' = \text{spt} C'$ we have that spt$C$, spt$C'$ are connected. Since $C'$ is a one-sided minimiser of the area functional from Lemma 3.3.1 we have that $\mathcal{H}^{n-2}(\text{sing} C') = 0$ so from Lemma 5.3.1 we conclude that spt$C = \text{spt} C'$ and so $C = C'$.

Thus we showed that $C$ is the unique tangent cone of $S$ at the origin and moreover $C$ is a one-sided minimiser of the area functional so from Proposition 3.2.1 it is a stable minimal cone.

We now prove that the origin is an isolated singularity for $S$. Assume, for the contrary, that there is a sequence $y_j \in \text{sing} S$ with $|y_j| > 0$ and $y_j \to 0$ and let $S_j = \eta_{y_j, |y_j|} S$. Then from Lemma 3.3.2 and the uniqueness of the tangent cone at the origin we have that, after passing to a subsequence that we still index as $j$,

$$\mu_{S_j} \to \mu C.$$

Let $\xi_j = \frac{y_j}{|y_j|}$ then, up to a subsequence, $\xi_j \to \xi \in S^n$. From Remark 2.1.3 we have that

$$\limsup_j \Theta^n(\mu_{S_j}, \xi_j) \leq \Theta^n(\mu_C, \xi).$$
Since $\Theta^n(\mu_C, \xi) = 1$ and $\Theta^n(\mu_{S_j}, \xi_j) = \Theta^n(\mu_S, y_j)$ we have, for all sufficiently large $j$, that

$$\Theta^n(\mu_{S_j}, \xi_j) < 1 + \epsilon,$$

where we choose $\epsilon \leq \delta_0$, for $\delta_0$ the dimensional constant of Theorem 2.1.2. This implies that $y_j \in \text{reg} S$ contrary to our initial assumption.

From Corollary 2.1.3 and Remark 2.1.5 combined with the fact that the singularity is isolated for both $S$ and $T$ and the uniqueness of the tangent cone $C$ for all sufficiently large $j$ there exists $\rho > 0$, uniform in $j$ and $u, v \in C^2(C_\rho \setminus \{0\})$ such that

$$\text{gr}_{C_\rho} u = \text{spt} T \cap (B_\rho \setminus \{0\}),$$

$$\text{gr}_{C_\rho} v = \text{spt} S \cap (B_\rho \setminus \{0\}),$$

$$u \leq v, \text{ in } C_\rho \setminus \{0\},$$

$$\frac{|u(x)|}{|x|} + |\nabla u(x)| + |x||\nabla^2 u(x)| \to 0,$$

$$\frac{|v(x)|}{|x|} + |\nabla v(x)| + |x||\nabla^2 v(x)| \to 0.$$

From Lemma 4.1.1 we have that the mean curvature vectors of $\text{reg} S$ and $\text{reg} T$ point in the same direction and from Lemma 5.1.3 we get that

$$\mathcal{M}_C u = \mathcal{M}_C v = \lambda,$$

where $\mathcal{M}_C$ is of the form 5.1.2.

Let $h = v - u \in C^2(C_\rho \setminus \{0\})$ then $h \geq 0$ and from Lemma 5.1.4 $h$ satisfies the linear PDE

$$L_C h = \text{div}_C(\tilde{P} \nabla h) + \frac{1}{|x|} \tilde{R} \cdot \nabla h + \frac{1}{|x|^2} \tilde{S} h,$$

where $\tilde{P}, \tilde{R}, \tilde{S} \to 0$. Thus for any $K \subset C_\rho \setminus \{0\}$ we can apply the Harnack inequality, see Theorem 2.2.5 and Remark 2.2.2 to get that

$$\sup_K h \leq C_K \inf_K h.$$
Hence either \( h > 0 \) or \( h \equiv 0 \). Assume that \( h > 0 \). We define a sequence of rescalings \( h_j(x) = h(r_j x) \) for \( x \in C_{r_j} \setminus \{0\} \), where \( r_j \) is chosen as follows.

Let a sequence \( \rho'_j \searrow 0 \). From the property that \( h \to 0 \) as \( |x| \to 0 \) we can construct a subsequence of \( \rho'_j \), that we still index as \( j \), such that

\[
\sup_{C_{r'_j}} h < \sup_{C_{r_j}} h.
\]

Let \( x_j \) be the points where \( \sup h \) is achieved and set \( r_j = |x_j| \). Then \( r_j \searrow 0 \) and

\[
\sup_{C_{r_j}} h = \sup_{\partial C_{r_j}} h.
\]

Thus we define \( h_j(x) = h(r_j x), \, x \in C_{r_j} \setminus \{0\} \) and we have that

\[
\sup_{C_{r_j}} h_j = \sup_{\partial C_{r_j}} h_j.
\]

Let \( x'_j \in \partial C_{r_j} \) where \( \sup h_j \) is achieved and set

\[
f_j(x) = \frac{h_j(x)}{M_j}, \text{ for } x \in C_{r_j} \setminus \{0\},
\]

where \( M_j = h_j(x'_j) \). From the PDE for \( h \) we have that \( f_j \) satisfies the following PDE

\[
L_{C_j} f_j = \nabla \cdot (P_j \nabla f_j) + R_j \cdot \nabla f_j + S_j f_j,
\]

where

\[
P_j = r_j^2 \tilde{P} \left( r_j x, \frac{h_j}{r_j |x|}, \nabla h_j \right),
\]

\[
R_j = \frac{r_j^2}{|x|} \tilde{R} \left( r_j x, \frac{h_j}{r_j |x|}, \nabla h_j \right),
\]

\[
S_j = \frac{r_j^2}{|x|^2} S \left( r_j x, \frac{h_j}{r_j |x|}, \nabla h_j \right).
\]

Thus, since \( \tilde{P}, \tilde{R}, S \to 0 \) as \( |x| \to 0 \) we have that \( ||P_j||_{0,\alpha:K} \to 0 \) for some \( \alpha > 0 \)
and \( \|R_j\|_{0;K} \rightarrow 0 \), for any \( K \subset C \setminus \{0\} \).

Fix a set \( K \subset C \setminus \{0\} \) and let \( K' \subset K' \subset C \setminus \{0\} \) and \( x_j' \in K' \). We can apply the \( C^{1,\alpha}_1 \)-Schauder estimates on \( K' \) and \( K \), Theorem 2.2.6, to get that

\[
\|f_j\|_{1,\alpha;K} \leq C (\|f_j\|_{0;K'} + \|P_j \nabla f_j\|_{0,\alpha;K'} + \|R_j \cdot \nabla f_j\|_{0;K'} + \|S_j f_j\|_{0;K'}),
\]

(5.3.1)

where \( C \) is a constant independent of \( j \). From the Harnack inequality, Theorem 2.2.5 and Remark 2.2.2, we get that

\[
\sup_{K'} f_j \leq C_{K'} \inf_{K'} f_j,
\]

and we conclude that

\[
\sup_{K'} f_j \leq C_{K'}.
\]

From the uniform convergence of \( P_j \) on compact sets we get a subsequence, that we still index as \( j \), such that

\[
\|P_j\|_{0,\alpha;K'} \leq \frac{1}{2C} \|\nabla f_j\|_{0;K'}.
\]

Thus we can absorb this term on the left hand side of the inequality 5.3.1. We can then estimate the term that involve \( R_j \) and \( \nabla f_j \) in a similar fashion. Putting everything together yields a subsequence, that we still index as \( j \), such that

\[
\|f_j\|_{1,\alpha;K} \leq C',
\]

for any \( K \subset C \setminus \{0\} \), where \( C' \) is a constant independent of \( j \). From Arzela-Ascoli theorem, after passing to a further subsequence, we have that

\[
f_j \rightarrow f \in C^{1,\alpha}_1(C \setminus \{0\}).
\]

Since the right hand side of the PDE for \( f_j \) converges to zero uniformly on compact
sets we have that \( f \) is a weak solution of

\[
L_C f = 0,
\]

in \( C \setminus \{0\} \) and so from elliptic regularity, Theorem 2.2.7, we get that \( f \) is smooth on \( C \setminus \{0\} \). Furthermore, up to a subsequence, we have that \( x_j' \to x_0 \in \partial C_1 \) and so \( f(x_0) = 1 \). Thus from Harnack inequality \( f > 0 \).

In total, we have constructed a positive solution of \( L_C f = 0 \) that is defined on the whole \( C \setminus \{0\} \) for a stable minimal cone \( C \) of \( \mathbb{R}^{n+1} \) and satisfies

\[
\sup_{C_\rho} f = \sup_{\partial C_\rho} f.
\]

From Lemma 5.2.2 we conclude that such a positive function \( f \) cannot exist since \( f \) must be of the form 5.2.2. Thus we get that \( h \equiv 0 \) and hence there exists \( \rho > 0 \) such that \( T|B_\rho = S|B_\rho \).

From the unique continuation valid for CMC hypersurfaces, see for example [MAG12] Theorem 27.4, we conclude that \( \text{spt} T = \text{spt} S \) and since the currents are of multiplicity-1 and oriented in the same way we have that \( T = S \) in the sense of currents. This concludes the proof of Theorem 5.3.1. \( \square \)
Chapter 6

Perturbation of isolated singularities

In what follows let \( A \) be a set of locally finite perimeter and the current \( T = \partial |A| B_1 \).
We make the following assumptions on \( T \).

i) \( T \) is a critical point of \( E_\lambda \) in \( B_1 \) and a one-sided minimiser of \( E_\lambda \), in the sense of Definition 3.2.1

ii) \( \text{spt} T \) is connected in \( \mathbb{R}^{n+1} \), \( 0 \in \text{spt} T \) and \( \text{spt} T \setminus \{0\} \) is a \( C^\infty \)-hypersurface.
From Remark 2.1.11 we may also assume that \( A \) is closed, connected and \( \partial A \cap \overline{B_1} = \text{spt} T \).

iii) The associated varifold \( V_T \) admits a varifold tangent cone at the origin that is an oriented regular cone, in the sense of Definition 5.1.1 and non-planar.

iv) There are \( 0 < r_1 < r_2 < 1 \) such that the boundary \( \Gamma_0 = \partial (T \cap B_{r_1}) \) is smoothly embedded, connected \((n-1)\)-submanifold of \( \partial B_{r_1} \) and \( A \setminus \overline{B_{r_2}} \) has connected complement.

Remark 6.1. We would like to mention at this point that we do not need to assume that the varifold tangent cone has multiplicity-1 since we can conclude this from Lemma 3.3.2.

We know from Lemma 5.1.2 that the current associated to the varifold tangent cone of \( T \) is of the form \( C = \partial |E| \) for \( E \) a set of locally finite perimeter. Furthermore since \( C \) is the tangent cone of \( T \) in the current sense as well we conclude from
Lemma 3.3.2 that \( C \) is a one-sided minimiser of the area functional. For such cones \( C \) we have the following:

**Theorem 6.1. (Hardt-Simon foliation)** Let \( C = \partial[E] \) be a regular cone, stationary in \( \mathbb{R}^{n+1} \) and a one-sided area minimiser. Then there exists a smooth, embedded, connected hypersurface \( S = \partial[F] \) for \( F \) a set of locally finite perimeter with \( \mathcal{T} \subset E \) such that \( d(\text{spt}S, 0) = 1 \) and

1) \( S \) minimises area in \( E \),

2) if \( R = \partial[H] \) is another one-sided area minimiser with \( \mathcal{H} \subset E \), then either \( R = C \) or \( R = \eta_{0,r}S \), for some \( r > 0 \).

The above theorem is a generalisation of Theorem 2.1 of [HS85] proved in [LIU19] for the case of one-sided area minimising cones. This generalisation is based on the regularity theory of stable stationary varifolds of [WIC14a], see Lemma 3.3.1. We now state two versions of the approximation theorem. One is for suitably small scalar mean curvature \( \lambda \) and the other is for arbitrary \( \lambda > 0 \).

**Theorem 6.2. (Approximation theorem for suitably small scalar mean curvature)** Let \( T = \partial[A] \setminus B_1 \) that satisfies properties i) - iv) and \( \phi_j : \Gamma_0 \rightarrow \partial B_1 \) be \( C^2 \)-maps with

\[
|\phi_j - i_{\Gamma_0}|_{C^2} \leq \frac{1}{j},
\]

and for \( \Gamma_j = \phi_j, \Gamma_0 \) we assume that

\[
\Gamma_j \cap A^\circ \neq \emptyset,
\]

where \( A^\circ \) denotes the interior of \( A \) in \( \mathbb{R}^{n+1} \). If

\[
\lambda < \min \left\{ n, \frac{1}{C_n \text{vol}(B_1)^{1/n+1}} \right\},
\]

then there exist integral \( n \)-currents \( S_j \in I_n(\mathbb{R}^{n+1}) \) that minimise \( E_\lambda \) in \( A \cap \mathbb{B}_{r_j} \) subject to the boundary condition \( \partial S_j = \Gamma_j \) that satisfy the following properties:

1) \( \text{spt}S_j \subset \mathbb{B}_{r_j} \),
2) there exist sets of locally finite perimeter $B_j$ with $B_j \subset A \cap \overline{B_{r_1}}$ such that $S_j = \partial \overline{B_j} | B_{r_1}$.

3) $\Gamma_j = \text{spt} S_j \cap \partial B_{r_1}$.

4) $S_j$ is a critical point of $E_\lambda$ in $B_{r_1}$. In fact it is a critical point of $E_\lambda$ in $B_{r_2} \setminus \Gamma_j$.

5) $\mu S_j \rightarrow \mu_T | B_{r_1}$.

6) for all sufficiently large $j$ there exists $\rho < r_1$, uniform in $j$, such that $\text{spt} S_j \cap B_\rho$ is a smooth hypersurface.

Proof. From the assumptions that we have made on $T$ and $\phi_j$ we can apply Theorem 4.2.1 to get integral $n$-currents $S_j$ that satisfy all properties from 1) - 5). In order to prove property 6) we first show that $0 \notin \text{spt} S_j$.

Assume that $0 \in \text{spt} S_j$ then from the singular maximum principle, Theorem 5.3.1 we get that $S_j = T$. This implies that $\Gamma_j = \Gamma_0$ thus $\Gamma_j \subset \partial A$ which contradicts the initial assumption that $\Gamma_j \cap A^o \neq \emptyset$. Hence $0 \notin \text{spt} S_j$.

Let us now assume that $\text{spt} S_j \cap B_\rho$ is not smooth for any $\rho < r_1$. Then we can take $r_j \downarrow 0$ and there exist $y_j \in \text{spt} S_j \cap B_{r_j}$, thus $|y_j| \rightarrow 0$. Consider the rescalings $\tilde{S}_j = \eta_0 |y_j| S_j$. We can proceed as in Lemma 3.3.2 combined with Lemma 4.2.2 and the uniqueness of $C$, to prove that after passing to a subsequence, that we still index as $j$

$$\mu_\tilde{S}_j \rightarrow \mu_S,$$

where $S = \partial [F]$, for $F$ a set of locally finite perimeter with $\overline{F} \subset \overline{E}$, $S$ is stationary in $\mathbb{R}^{n+1}$ and a one-sided area minimiser. Then from Theorem 6.1 we have that $S$ is either smooth or $C$. So in any case

$$\text{spt} S \cap S^n = \emptyset.$$

Let $\xi_j = \frac{y_j}{|y_j|} \in \text{spt} \tilde{S}_j \cap S^n$. Then up to a subsequence

$$\xi_j \rightarrow \xi \in S^n$$

and so

$$\limsup_j \Theta^n(\mu_\tilde{S}_j, \xi_j) \leq \Theta^n(\mu_S, \xi) = 1.$$
Notice that $\xi \in \text{spt}S$ because else we would have a drop of mass and this cannot happen from the above inequality. Thus for large $j$

$$\Theta^n(\mu_{\tilde{S}_j}, \xi_j) < 1 + \epsilon,$$

where $\epsilon \leq \delta_0$ for $\delta_0$ the dimensional constant of Theorem 2.1.2 and since

$$\Theta^n(\mu_{\tilde{S}_j}, \xi_j) = \Theta^n(\mu_{S_j}, y_j),$$

we conclude that $y_j \in \text{reg}S_j$ which gives the desired contradiction and concludes the proof of Theorem 6.2.

Proof. Consider the dilation map $\mu_\rho : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ with $\mu_\rho(x) = \frac{1}{\rho}x$. Then, from the choice of $\rho$, the current $T' = \mu_\rho^*T$ is defined in $B_1$, has scalar mean curvature $\lambda' = \lambda\rho$ and satisfies all the properties needed for Theorem 6.2 to be valid. So we get a sequence of integral $n$-currents $S_j$ that satisfy properties 1) - 6) of Theorem 6.2. Then for the sequence $\mu_{\rho^{-1}}^*S_j$ we get the approximation theorem for $T|B_\rho$ with arbitrary $\lambda$. 

Theorem 6.3. (approximation theorem for arbitrary scalar mean curvature)

Let $T = \partial|A||B_1$ that satisfies properties i)-iii) and $\rho > 0$ such that

$$\lambda\rho < \min\left\{n, \frac{1}{C_n \text{vol}(B_1)^{1/n+1}}\right\}.$$ 

We assume that the current $T|B_\rho$ satisfies assumption iv) for $0 < r_1 < r_2 < \rho$. Let $\phi_j : \Gamma_0 \to \partial B_{r_1}$ be $C^2$-maps with

$$|\phi_j - i\Gamma_0|_{C^2} \leq \frac{1}{j},$$

such that for $\Gamma_j = \phi_j \ast \Gamma_0$ we have

$$\Gamma_j \cap A^0 \neq \emptyset,$$

where $A^0$ denotes the interior of $A$ in $\mathbb{R}^{n+1}$. Then $T|B_\rho$ satisfies conclusions 1) - 6) of Theorem 6.2.
Remark 6.2. In Theorem 6.2 we prove property 6) away from the boundary that we have assigned on $S_j$. This is because we want to approximate the initial current $T$ by smooth CMC hypersurfaces hence the property we are interested in proving is of local nature. However we can show that $S_j$ is smooth in the whole $B_{r_1}$. We can prove this as follows. Let $y_j \in \text{sing} S_j$ then, from Lemma 4.2.2 and up to a subsequence, we have that $y_j \to x_0$ where $x_0 \in \text{spt} \Gamma \cap B_{r_1}$. From boundary regularity, see [ALL75] or [BOU16], for all sufficiently large $j$ there is a neighborhood $V$ of $\Gamma_j$ independent of $j$ such that $S_j$ is a smooth manifold with boundary $\Gamma_j$ in $V$. Thus since $y_j \in \text{sing} S_j$ we have that
\[
d(y_j, \Gamma_j) \geq C,
\]
for $C$ a constant independent of $j$. So in the limit we get that
\[
d(x_0, \Gamma_0) \geq C,
\]
hence $x_0 \in \text{spt} \Gamma \cap B_{r_1} \setminus \Gamma_0$. If $x_0 \neq 0$ then from Remark 2.1.3
\[
\limsup_j \Theta^n(\mu_{S_j}, y_j) \leq \Theta^n(\mu_{T|B_{r_1}}, x_0) = 1,
\]
and so for $j$ large enough
\[
\Theta^n(\mu_{S_j}, y_j) < 1 + \epsilon,
\]
where $\epsilon \leq \delta_0$ for $\delta_0$ the dimensional constant of Theorem 2.1.2 thus $y_j \in \text{reg} S_j$ which is a contradiction. Therefore $y_j \to 0$ and then we can proceed exactly as in the proof of Theorem 6.2 to get that $S_j$ is smooth in the whole ball $B_{r_1}$. 

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Chapter 7

One-sided minimising examples

In this chapter we give examples of CMC hypersurfaces that satisfy all properties needed for Theorem 6.2 to be valid. These examples originate from the work of [CHS84] and that of [IRV17]. We briefly describe these examples.

In what follows let $C$ be a minimal oriented regular cone of $\mathbb{R}^{n+1}$, as defined in 5.1.1, with $N$ its unit normal vector field. We have defined

$$\text{gr}_C u = \left\{ x + u(x)N(x) \mid x \in C_1 \setminus \{0\} \right\},$$

to be the graph of $u \in C^2(C_1 \setminus \{0\})$ over $C_1$. Let $u$ satisfy the radial decay 5.1.1 thus the Euler-Lagrange operator $M_C$ associated to the area functional of $\text{gr}_C u$ is of the form 5.1.2 and satisfies the estimates from Lemma 5.1.3.

Theorem 7.1. (Theorem 4.6 [CHS84], Theorem 1 [IRV17]) Let $C$ be a minimal oriented regular cone that is also strictly stable. Then there exists $\epsilon_0 > 0$ such that for any $0 < \lambda < \epsilon_0$ the PDE $M_C u = \lambda$ has a solution $u_{\lambda}$ that satisfies the decay assumption 5.1.1. Furthermore, $M_{\lambda} = \text{gr}_C u_{\lambda}$ is a CMC hypersurface in $B_1$, satisfies the stability inequality 3.2.1 it has an isolated singularity at the origin and has a smooth, embedded and connected boundary in $S^n$.

Remark 7.1. The case of $\lambda = 0$ is carried out in [CHS84]. The above theorem is a generalisation of Theorem 3.1 and Theorem 4.6 of [CHS84] to $\lambda \neq 0$ and is obtained in [IRV17].
Let $T_\lambda$ be the integral current associated to the hypersurface $M_\lambda$ of Theorem 7.1. From Lemma 5.1.2 we have that $\mathbb{R}^{n+1} \setminus C$ has two connected components thus the same is true for $\mathbb{R}^{n+1} \setminus M_\lambda$. So there are $E$ and $A_\lambda$ sets of locally finite perimeter such that $C = \partial E$ and $T_\lambda = \partial[A_\lambda]|B_1$, where we choose the sets so that the unit normal vector field of $C$ and $M_\lambda$ points inside $E$ and $A_\lambda$ respectively. Notice that, since $T_\lambda$ is a CMC hypersurface in $B_1$, it satisfies the variational equations 3.1.8 hence it is a critical point of $E_\lambda$ in $B_1$.

Let $F : C_1 \rightarrow \mathbb{R}^{n+1}$ be the map $F(x) = x + u_\lambda(x)N(x)$. Then for any $\rho_j \downarrow 0$ we have that

$$||F_j - i_C||_{C^2} \xrightarrow{j \to \infty} 0,$$

where $F_j(x) = \frac{1}{\rho_j} F(\rho_j x)$. This follows from the radial decay assumption 5.1.1 on $u_\lambda$. Thus $T_\lambda$ converges to $C$ in a $C^2$-sense hence $C$ is the unique varifold tangent cone of $V_{T_\lambda}$.

Furthermore, from the construction of $T_\lambda$ we have that $\Gamma = \partial T_\lambda$ is smoothly embedded and connected in $S^n$ and $\text{spt} T_\lambda$ is connected. So we can assume from Remark 2.1.11 that $A_\lambda$ is closed and connected and we can find $0 < r_1 < r_2 < 1$ such that all the assumptions specified at the beginning of Chapter 6 are satisfied.

In conclusion, if we choose

$$\lambda < \min \left\{ \epsilon_0, \frac{1}{C_n \text{vol}(B_1)^{1/n+1}}, n \right\},$$

where $\epsilon_0$ the constant from Theorem 7.1 we have that $T_\lambda$ satisfies all the necessary assumptions of Theorem 5.2 except the one-sided minimising property.

For the remainder of this chapter we want to investigate when $T_\lambda$ is a one-sided minimiser of $E_\lambda$. From Lemma 3.3.2 we have that the tangent cone $C$ has to be a one-sided minimiser of the area functional in $\mathbb{R}^{n+1}$. So this is a necessary condition for $T_\lambda$ to be a one-sided minimiser of $E_\lambda$. However, following the ideas from [HS85] in Sections 3 and 4 we are going to assume that the initial cone $C$ is strictly area minimizing on one side and not just area minimizing. We give the following:

**Definition 7.1.** (strictly area minimising) The cone $C$ is said to be strictly area
minimising on one side if there exists $\theta > 0$ and $\epsilon_0 > 0$ such that for any $\epsilon \in (0, \epsilon_0)$

$$M(C_1) \leq M(S) - \theta \epsilon^n,$$

for any $S$ integral $n$-current with $\text{spt} S \subset \subset E \setminus B_\epsilon(0), \partial S = \partial C_1$.

**Remark 7.2.** If $C$ is strictly area minimising on one side then from Hardt-Simon foliation we have that there is a smooth, embedded and connected hypersurface $S$ that satisfies all properties of Theorem 6.1. Furthermore, from [HS85] and [LIU19] there exists $R := R(C) \geq 1$ and $v \in C^2(C \setminus C_R), v > 0$, such that $S$ is given as the graph of $v$ over $C \setminus C_R$.

For the function $v$ of Remark 7.2 the following growth rate near infinity is proven in Theorem 3.2 of [HS85].

**Theorem 7.2.** Let $C$ be a regular cone that is strictly area minimising on one side, stationary in $\mathbb{R}^{n+1}$ and strictly stable. Then for the function $v$, as defined in Remark 7.2 and for $\gamma^- = \frac{n-2}{2} - \sqrt{\left(\frac{n-2}{2}\right)^2 + \lambda_1}$, where $\lambda_1$ the first eigenvalue of $-L_\Sigma$ as defined in 5.2.1, we have that

$$\lim_{|x| \to \infty} \frac{v(x)}{|x|^{-\gamma^-}} = C > 0. \quad (7.1)$$

We now prove the main result of this chapter. We describe briefly the main idea of the proof, that is based on Theorem 4.1 of [HS85]. We are going to assume that $T_\lambda$ does not minimise $E_\lambda$ on one side on any ball around the origin. Thus we can consider a sequence of one-sided minimisers of $E_\lambda$, say $S_j$, on arbitrarily small balls with the same boundary as that of $T_\lambda$ in that small ball. Then we show that for all large $j$ the currents $S_j$ are given as graphs of functions over an annulus of the initial cone $C$. The annulus of graphicality for $S_j$ is carefully investigated and the Hardt-Simon foliation, Theorem 6.1, is an indispensable tool. The goal then is to construct a barrier for the height function of the graphical pieces of $S_j$ and $T_\lambda$ over that annulus. The barrier is chosen so that, after taking limits, the growth rate of Theorem 7.2 is not valid.
Theorem 7.3. Let $C$ be strictly area minimising on one side, strictly stable and the unique one-sided area minimiser in the class of integral $n$-currents with current boundary $\partial C_l$, $C_l = C \cap B_l$, for any $l \geq 1$. Then there exists $\rho > 0$ such that $T_\lambda | B_\rho$ minimises $E_\lambda$ on one side.

Proof. Assume that there is no $\rho > 0$ such that $T_\lambda | B_\rho$ minimises $E_\lambda$ on one side. Consider a sequence $\rho_j \searrow 0$ and the following minimisation problem

$$\inf \left\{ E_\lambda(S) \mid S \in \mathcal{I}_n(\mathbb{R}^{n+1}), \ \text{spt} \ S \subset A_\lambda \cap \overline{B_{\rho_j}}, \ \partial S = \partial (T_\lambda | B_{\rho_j}) \right\}.$$ 

Then, for all $j$ large enough, we have that

$$\lambda < \frac{1}{C_n \text{vol}(A \cap \overline{B_{\rho_j}})^{1/(n+1)},}$$

thus we can apply Theorem 3.1.1 to get $S_j$ where the infimum is achieved.

From Theorem 4.1.1 and Remark 4.1.2 we have that $S_j$ is a critical point of $E_\lambda$ in $B_{\rho_j}$ for any $j$. Using the same argument as in the proof of Theorem 4.2.1 we can show that

$$S_j = \partial [B_j] | B_{\rho_j},$$

for $B_j \subset A_\lambda \cap \overline{B_{\rho_j}}$, a set of locally finite perimeter. Furthermore, we have that $0 \notin \text{spt} S_j$ for any $j$ since if $0 \in \text{spt} S_j$ then, from Theorem 5.3.1, $S_j = T_\lambda | B_{\rho_j}$, contradicting the initial assumption that $T_\lambda$ does not minimise $E_\lambda$ in $B_\rho$ for any $\rho > 0$.

Thus we can define

$$\sigma_j = d(\text{spt} S_j, 0) > 0,$$

where $d$ denotes the distance of the set $\text{spt} S_j$ from the origin.

Define the following rescalings of $S_j$,

$$S_j^1 = \eta_{0, \rho_j} S_j \ \text{and} \ S_j^2 = \eta_{0, \sigma_j} S_j.$$ 

From the one-sided minimising property of $S_j$ and since $T_\lambda$ is a critical point of $E_\lambda$ in $B_1$, using Theorem 2.1.1 and the scale invariance of the mass ratio, we get that $M(S_j^1) \leq \Lambda$, where $\Lambda$ is a constant independent of $j$. Thus, an in Lemma 3.3.2 we
get that, up to a subsequence
\[ S_j^1 \to C', \]
where \( C' \in I_n(\mathbb{R}^{n+1}) \), \( \text{spt} C' \subset \overline{E} \cap \overline{B}_1 \) and \( C' \) minimises area in \( \overline{E} \cap \overline{B}_1 \). Moreover, since
\[ \partial S_j^1 = \partial (\eta_{0,\rho_j}(T\lambda B_{\rho_j})) \to \partial C_1, \]
we have that \( \partial C' = \partial C_1 \). Thus from the uniqueness assumption on \( C \) we conclude that \( C' = C_1 \).

From Lemma [4.2.1] we have that
\[ \mu_{S_j^1} \to \mu_{C_1}, \tag{7.2} \]
thus from Lemma [4.2.2] we conclude that
\[ \frac{\sigma_j}{\rho_j} \to 0. \tag{7.3} \]

To extract a convergent subsequence for \( S_j^2 \) let \( R > 0 \) and from [7.3] we have that \( R\sigma_j \leq \rho_j \), for \( j \) large enough. Thus, using Theorem [2.1.1] and the scale invariance of the mass ratio we get that \( M(S_j^2 \mid B_R) \leq \Lambda \), where \( \Lambda \) is a constant independent of \( j \). We can now proceed as in the proof of Lemma [3.3.2] to get that, up to a subsequence, there exists \( S' \in I_n(\mathbb{R}^{n+1}) \) such that
\[ \mu_{S_j^2} \to \mu_{S'}, \tag{7.4} \]
\( S' = \partial[F] \) for \( F \subset E \), \( S' \) is stationary in \( \mathbb{R}^{n+1} \), minimises the area functional in \( E \) and from Lemma [4.2.2] we have that \( d(\text{spt} S', 0) = 1 \). Thus from Theorem [6.1] we conclude that \( S' = S \), where \( S \) satisfies all the properties of Theorem [6.1] and Theorem [7.2].

From [7.2] and Allard’s boundary regularity theorem, see [ALL75] or [BOU16], there exists \( v_j^1 \in C^2(C_{\rho_j} \setminus C_{\rho_j/2}) \) such that \( \text{spt} S_j \cap (B_{\rho_j} \setminus B_{\rho_j/2}) = \text{gr}_C v_j^1 \) for all sufficiently large \( j \) and \( v_j^1 \) satisfies the boundary value problem

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\[ M_C v_1^j = \lambda, \]
\[ v_1^j |_{\partial C_{\rho_j}} = u_\lambda |_{\partial C_{\rho_j}}, \]

where \( M_C \) the operator from Lemma 5.1.3.

From 7.4, Corollary 2.1.3, Remark 2.1.5 and since, as discussed in Remark 7.2, \( S_{\sigma_j} \) is a graph over \( C \setminus C_{R(C)\sigma_j} \), where \( S_{\sigma_j} = \eta_{0,\sigma_j} S \), there exist constants \( K' \), \( K \), independent of \( j \) with \( K' > K > R(C) \geq 1 \) and \( v_2^j \in C^2(C_{K'\sigma_j} \setminus C_{K\sigma_j}) \) such that \( \text{spt} S_j \cap (B_{K'\sigma_j} \setminus B_{K\sigma_j}) = \text{gr}_C v_2^j \) and

\[ M_C v_2^j = \lambda, \]

for all sufficiently large \( j \), where \( M_C \) the operator from Lemma 5.1.3.

From 7.3 we have that for all large \( j \) the domains of \( v_1^j \) and \( v_2^j \) are disjoint. Thus we cannot glue them to get a function \( v_j \in C^2(C_{\rho_j} \setminus C_{K\sigma_j}) \).

**Claim 1.** For all sufficiently large \( j \) there exists a function \( v_j \in C^2(C_{\rho_j} \setminus C_{K\sigma_j}) \) such that

\[ v_j |_{C_{\rho_j} \setminus C_{\rho_j}/2} = v_1^j, \]
\[ v_j |_{C_{K'\sigma_j} \setminus C_{K\sigma_j}} = v_2^j, \]
\[ \text{gr}_C v_j = \text{spt} S_j \cap (B_{\rho_j} \setminus B_{K\sigma_j}). \]

For the function \( v_j \) of Claim 1, using 7.4 and Corollary 2.1.3 we have that

\[ \frac{v_j(\sigma_j x)}{\sigma_j} \to v(x), \quad (7.5) \]

uniformly in \( C \setminus C_K \). From 7.5 and since \( v > 0 \) we have that there exist \( m, M > 0 \) such that for all sufficiently large \( j \)

\[ m \sigma_j \leq v_j |_{\partial C_{K\sigma_j}} \leq M \sigma_j. \quad (7.6) \]

We are going to make use of the spherical coordinates \((r, \omega)\) in \( C \). Recall that for \( x \in C \) we have \( r = |x| \) and \( \omega = \frac{x}{|x|} \in \Sigma \), where \( \Sigma \) is the link of the cone \( C \). We
introduce now the function

$$\psi(r\omega) = \phi_1(\omega) r^{-\gamma^+ + \alpha},$$

where \( L_\Sigma \phi_1 = -\lambda_1 \phi_1 \) for \( L_\Sigma \) as defined in 5.2.1, \( \phi_1 > 0 \) eigenfunction, \( \lambda_1 \) the first eigenvalue of \(-L_\Sigma, \gamma^+ = \frac{n-2}{2} + \sqrt{(\frac{n-2}{2})^2 + \lambda_1} \) as defined in Lemma 5.2.1 and \( \alpha > 0 \) to be chosen later. Using the formula 5.2.1 for \( L_C \) we compute that

$$L_C \psi = -r^{-\gamma^+ + \alpha - 2} \phi_1 \left( -\lambda_1 + (-\gamma^+ + \alpha)(-\gamma^+ + \alpha + n - 2) \right) =$$

$$= -r^{-\gamma^+ + \alpha - 2} \phi_1 \left( (\gamma^+)^2 - (n-2)\gamma^+ - \lambda_1 \right) - 2\gamma^+ \alpha + \alpha^2 + (n-2)\alpha$$

$$= -r^{-\gamma^+ + \alpha - 2} \phi_1 \alpha (\alpha + (n-2) - 2\gamma^+),$$

where we used that \((\gamma^+)^2 - (n-2)\gamma^+ - \lambda_1 = 0\). Since \( C \) is strictly stable from Lemma 5.2.1 and Remark 5.2.1 we have that \( \gamma^+ > \gamma^- > 0 \) and if we choose \( \alpha < \gamma^+ - \gamma^- \) we get that

$$L_C \psi \leq -C(n, \Sigma, \alpha) r^{-\gamma^+ + \alpha - 2}, \quad (7.7)$$

where \( C(n, \Sigma, \alpha) > 0 \) is a constant that depends on \( n, \Sigma, \) and \( \alpha \).

Claim 2. There exists a constant \( C \) such that for all sufficiently large \( j \) the following inequality holds

$$v_j - u_\lambda \leq C \delta_j \psi, \quad (7.8)$$

on \( C_{\rho_j} \setminus C_{K \sigma_j} \) where \( \delta_j = \sigma_j^{\gamma^+ - \alpha + 1} \).

From Claim 2 we get the desired contradiction. To see this we rescale the inequality 7.8 by \( \sigma_j \) to get that

$$v_j(\sigma_j x) - u_\lambda(\sigma_j x) \leq \delta_j \psi(\sigma_j x),$$

for any \( x \in C_{\rho_j} \setminus C_K \). From the definition of \( \delta_j \) and \( \psi \) we get that

$$\frac{v_j(\sigma_j x)}{\sigma_j} - \frac{u_\lambda(\sigma_j x)}{\sigma_j} \leq C r^{-\gamma^+ + \alpha},$$

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where $C$ is a constant independent of $j$. Thus if we let $j \to \infty$ from 7.3 and 7.5 and Theorem 7.1 we get that
\begin{equation*}
v \leq Cr^{-\gamma^+ + \alpha},
\end{equation*}
for any $x \in C \setminus C_K$ and $K > R(C)$. Since $0 < \alpha < \gamma^+ - \gamma^-$ from the above inequality we get that
\begin{equation*}
v_{r^\gamma^-} \xrightarrow{r \to \infty} 0,
\end{equation*}
which contradicts Theorem 7.2 and gives the conclusion of Theorem 7.3. It remains now to prove Claims 1 and 2.

**Proof of Claim 1.** To prove Claim 1 it suffices to show that there exists a function $v_j \in C^2(C_{\rho_j/2} \setminus C_{K'} \sigma_j)$ such that $gr^C v_j = \text{spt} S_j \cap (B_{\rho_j/2} \setminus B_{K' \sigma_j})$, for all $j$ large enough.

For the latter it is enough to assume, for the contrary, that there exist $x_j \in C_{\rho_j/2} \setminus C_{K'} \sigma_j$ such that for any $\rho > 0$ and any index $j$ there are no $C^2$-maps
\begin{equation*}
\phi_j : (C_{\rho_j/2} \setminus C_{K'} \sigma_j) \cap B_\rho(x_j) \to \text{reg} S_j \cap B_\rho(x_j),
\end{equation*}
that satisfy the properties of Corollary 2.1.3. Consider the rescalings $S_j^3 = \eta_0 |x_j| \ast S_j$. Then, up to a subsequence, $\frac{\rho_j}{|x_j|} \to l$, where $l \in [2, \infty]$.

If $l < \infty$ then we can bound, as we did for $S_j^1$, $M(S_j^3) \leq \Lambda$ where $\Lambda$ is a constant independent of $j$. Thus in this case, using again the uniqueness of $C$ as a one-sided area minimiser, we have that
\begin{equation*}
\mu_{S_j^3} \to \mu_{C_l},
\end{equation*}
where $l \geq 2$ and $C_l = C \cap B_l$. Furthermore we have that, up to a subsequence, $\frac{x_j}{|x_j|} \to \xi \in S^n$ and in fact from Corollary 2.1.1 we have that $\xi \in C_1$. Thus $\xi$ is a smooth point of the limit of $S_j^3$ and hence from Corollary 2.1.3 for all sufficiently large $j$ there exists $\rho > 0$, independent of $j$ and $C^2$-maps
\begin{equation*}
\phi_j : (C_{|x_j| \setminus \{0\}} \cap B_\rho(x_j)(\xi|x_j|)) \to \text{reg} S_j \cap B_\rho(x_j)(\xi|x_j|),
\end{equation*}
that satisfy all the properties of Corollary 2.1.3. For large $j$ we have that $x_j \in (C_{|x_j| \setminus \{0\}} \cap B_\rho(x_j)(\xi|x_j|))$ and this contradicts our initial assumption on $x_j$.

In case $l = \infty$ we can bound, as we did for $S_j^2$, $M(S_j^3) \leq \Lambda$ where $\Lambda$ is indepen-
dent of $j$. Then we can proceed as in Lemma 3.3.2 to get that, after passing to a subsequence that we still index as $j$

$$\mu_{S_j^3} \to \mu_S,$$

where $S = \partial[F]$ for $F \subset E$ a set of locally finite perimeter, $S$ minimises area in $E$ and is stationary in $\mathbb{R}^{n+1}$. Notice also that since $\frac{\sigma_j}{|x_j|} \leq \frac{1}{K'}$ we get that $d(\text{spt}S, 0) \leq \frac{1}{K'}$.

Thus from Theorem 6.1 we have that the limit $S$ is either $C$ or a leaf of the foliation with distance from the origin at most $\frac{1}{K'}$.

In case it is the cone $C$ we proceed as before to get a contradiction. In case $S$ is a leaf of the foliation of Theorem 6.1 and since, up to a subsequence, $x_j |x_j| \to \xi \in S^n$ we have from Corollary 2.1.3 and Remark 2.1.5 that for all sufficiently large $j$ there exists $\rho > 0$, independent of $j$ and $C^2$-maps

$$\phi_j : S_{|x_j|} \cap B_{\rho|x_j|}(\xi|x_j|) \to \text{reg}S_j \cap B_{\rho|x_j|}(\xi|x_j|),$$

that satisfy all the properties of Corollary 2.1.3, where $S_{|x_j|} = \eta_0|x_j|S$. From the graphicality of $S$ over $C \setminus C \frac{K}{K'}|x_j|$, see Remark 7.2, we have that $S_{|x_j|}$ is graphical over $C \setminus C \frac{K}{K'}|x_j|$ thus there are $C^2$-maps

$$h_j : (C \setminus C \frac{K}{K'}|x_j|) \cap B_{\rho|x_j|}(\xi|x_j|) \to S_{|x_j|} \cap B_{\rho|x_j|}(\xi|x_j|),$$

that satisfy the properties of Corollary 2.1.3 and in conclusion we get $C^2$-maps

$$\tilde{\phi}_j : (C \setminus C \frac{K}{K'}|x_j|) \cap B_{\rho|x_j|}(\xi|x_j|) \to \text{reg}S_j \cap B_{\rho|x_j|}(\xi|x_j|),$$

that satisfy the properties of Corollary 2.1.3 and for large $j$ we have that $x_j \in (C \setminus C \frac{K}{K'}|x_j|) \cap B_{\rho|x_j|}(\xi|x_j|)$ which contradicts our initial assumption on $x_j$ and finishes the proof of Claim 1.

**Proof of Claim 2.** Assume that inequality 7.8 is not valid. Then for any $C > 0$ and any index $j$ there exist $x_j \in C_{\rho_j} \setminus C_{K\sigma_j}$ such that

$$v_j(x_j) - u_\lambda(x_j) > C\delta_j \psi(x_j).$$
Let \( d_j = \sup_{C_{\rho_j} \setminus C_{K\sigma_j}} \frac{v_j - u_\lambda}{\psi} \). Then from the above inequality we have that

\[
\frac{d_j}{\delta_j} \xrightarrow{j \to \infty} \infty.
\] (7.9)

Moreover we have that

\[
v_j - u_\lambda \leq d_j \psi,
\] (7.10)

for any \( x \in C_{\rho_j} \setminus C_{K\sigma_j} \), with equality at some interior point \( \tilde{x}_j \) of \( C_{\rho_j} \setminus C_{K\sigma_j} \).

To see the latter, equality in (7.10) cannot happen on \( \partial C_{\rho_j} \) since \( v_j |_{\partial C_{\rho_j}} = u_\lambda |_{\partial C_{\rho_j}} \) and \( d_j > 0 \). If there exist \( y_j \in \partial C_{K\sigma_j} \) such that \( v_j(y_j) - u_\lambda(y_j) = d_j \psi(y_j) \) we get

\[
v_j(y_j) - u_\lambda(y_j) \geq C \frac{d_j}{\delta_j} \sigma_j,
\]

where \( C \) a positive constant independent of \( j \). From inequality 7.6 and the construction of \( u_\lambda \) we get

\[
\frac{d_j}{\delta_j} \leq \frac{M}{2C},
\]

which contradicts 7.9.

**Claim 3.** For all large \( j \) there exists a neighborhood \( U_j \) of \( \tilde{x}_j \) such that

\[
d_j \frac{\psi(x)}{r} + d_j |\nabla \psi(x)| + d_j r |\nabla^2 \psi(x)| \leq \Theta,
\] (7.11)

for any \( x \in U_j \) and \( \Theta > 0 \) a constant that is independent of \( j \) and \( x \).

Fix a large index \( j \), then from 7.11 we can apply Lemma 5.1.5 in the fixed neighborhood \( U_j \) and since \( M_{C_u}u_\lambda = \lambda \) we get that

\[
M_{C_u}(u_\lambda + d_j \psi) = \lambda + d_j L_{C_u} \psi + d_j \left( \tilde{P}_j \cdot \nabla^2 \psi + \frac{1}{r} \tilde{R}_j \cdot \nabla \psi + \frac{1}{r^2} \tilde{S}_j \psi \right) + \epsilon d_j,
\]

where \( \epsilon > 0 \) can be chosen arbitrarily.

From the estimates in Lemma 5.1.5 and since \( u_\lambda \) satisfies the decay assumption 5.1.1 we have that \( |\tilde{P}_j|, |\tilde{R}_j|, |\tilde{S}_j| \leq c\tilde{M} \), for some constant \( c \) and \( \tilde{M} \) can be made as
small as we want. We estimate using inequality [7.7]

$$M_C(u_\lambda + d_j \psi) \leq \lambda - C(n,\Sigma,\alpha) d_j r^{-\gamma^+ + a^2} + d_j r^{-\gamma^+ + a^2} cM + \epsilon d_j.$$  

We can now take $\tilde{M} < \min \left\{ 1, \frac{C(n,\Sigma,\alpha)}{2c} \right\}$. Thus in $U_j$

$$M_C(u_\lambda + d_j \psi) < \lambda + d_j (\epsilon - M_1),$$

where $M_1 > 0$ constant and we can take $\epsilon < M_1$ to get that

$$M_C(u_\lambda + d_j \psi) < \lambda = M_C(v_j).$$

Then there exists a uniformly elliptic linear operator $L$ such that

$$Lh = M_C(u_\lambda + d_j \psi) - M_C(v_j) < 0,$$  (7.12)

where $h = u_\lambda - v_j + d_j \psi \geq 0$ in $U_j$. Since there exists an interior point of $U_j$ where $h$ vanishes we get from the Hopf maximum principle, Theorem [2.2.1], that $h \equiv 0$ in $U_j$ and this contradicts [7.12] and finishes the proof of Claim 2.

Proof of Claim 3. We have that at $\tilde{x}_j$

$$\frac{d_j |\psi(\tilde{x}_j)|}{|\tilde{x}_j|} = \frac{v_j(\tilde{x}_j)}{|\tilde{x}_j|} - \frac{u_\lambda(\tilde{x}_j)}{|\tilde{x}_j|}.$$

From the construction of $u_\lambda$ we have that $\frac{u_\lambda(\tilde{x}_j)}{|\tilde{x}_j|} < \epsilon$, for all sufficiently large $j$ and $\epsilon > 0$ chosen arbitrarily. Furthermore, we have that

$$\frac{v_j(\tilde{x}_j)}{|\tilde{x}_j|} \leq C_{(K)} \frac{v_j(\sigma_j(\sigma_j^{-1} \tilde{x}_j)))}{\sigma_j},$$

where $C_{(K)}$ denotes a constant that depends on $K$ and from [7.5] we have that for all large $j$

$$\frac{v_j(\sigma_j(\sigma_j^{-1} \tilde{x}_j)))}{\sigma_j} < v(\sigma_j^{-1} \tilde{x}_j) + 1.$$  

From Theorem [7.2] we have that for all $x \in C \setminus C_K$ with $|x|$ large enough $v(x) \leq \frac{C}{|x|^\gamma}$.  

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where \( \gamma^- > 0 \) thus we can bound the right hand side of the above inequality by some constant \( C \). In total, if we choose \( \epsilon < \frac{C}{2} \) we have for all sufficiently large \( j \)

\[
d_j \frac{\psi(\tilde{x}_j)}{|\tilde{x}_j|} \leq C.
\]

From continuity of the function we can find a neighborhood \( U_j \) of \( \tilde{x}_j \) such that for any \( x \in U_j \)

\[
d_j \frac{\psi(x)}{r} \leq \Theta
\]

for some \( \Theta > 0 \), independent of \( j \) and the point \( x \). From the definition of \( \psi \), computing \( |\nabla \psi|, |\nabla^2 \psi| \) and using the above inequality we get that

\[
d_j \frac{\psi}{r} + d_j |\nabla \psi| + d_j r |\nabla^2 \psi| \leq \Theta, \tag{7.13}
\]

for any \( x \in U_j \) thus proving Claim 3. This finishes the proof of Theorem 7.3.

\( \square \)

**Remark 7.3.** Let \( \rho > 0 \) as in Theorem 7.3. Then if we rescale \( T_\lambda \) to get \( T_\lambda' = \eta_{0,\rho} T_\lambda \)

we have that \( T_\lambda' \) is a one sided-minimiser of \( E_{\lambda'} \), where \( \lambda' = \lambda \rho \). Thus if we choose \( \lambda \) such that

\[
\lambda \rho < \min \left\{ \epsilon_0, \frac{1}{C n \text{vol}(B_1)^{1/n + 1}}, n \right\},
\]

we have that Theorem 6.2 is valid for \( T_{\lambda'} \).

**Remark 7.4.** We would like to mention that there is an abundance of examples of strictly area minimising cones. In particular all the cones from Lawson’s list in \([\text{LAW72}]\) are strictly area minimising as shown in Remark 3.3 of \([\text{HS85}]\). Furthermore, to our knowledge there are no known examples of area minimising cones that are not strictly area minimising apart from \( \mathbb{R}^2 \subset \mathbb{R}^3 \).

The latter is a trivial conclusion of the growth estimates of Theorem 3.2 of \([\text{HS85}]\) established for the case \( \gamma^+ = \gamma^- \). For \( \mathbb{R}^2 \) we have \( \gamma^+ = \gamma^- = 0 \) and the function \( v \) is constant thus Theorem 3.2 of \([\text{HS85}]\) immediately implies that \( \mathbb{R}^2 \) cannot be strictly area minimising.
Chapter 8

The Riemannian case

In this chapter we explain briefly how Theorem 6.2 extends when the ambient space is any smooth Riemannian manifold \((N, g)\).

From the local nature of Theorem 6.2 it suffices to assume that the ambient space is \(\mathbb{R}^{n+1}\) with an arbitrary smooth metric \(g\). The definitions of integral varifolds, integral currents and sets of locally finite perimeter extend to the case of an arbitrary smooth metric \(g\) as well as the monotonicity of the mass ratio, Theorem 2.1.1 and Allard’s regularity, Theorem 2.1.2. We shall need the following version of Corollary 2.1.3.

**Lemma 8.1.** Let \(V_j, V \in IV_n(U), U \subset \mathbb{R}^{n+1}\) open set such that \(V_j, V\) have generalised mean curvatures \(|\vec{H}_j| = \lambda_j, |\vec{H}| = \lambda\) respectively, with respect to smooth metrics \(g_j, g\) in \(U\), such that \(g_j \xrightarrow{j \to \infty} g\) smoothly as metrics and \(\mu_{V_j} \xrightarrow{j \to \infty} \mu_V\). Then for any \(x \in \text{reg} V\) with \(\Theta^n(\mu_V, x) = 1\) the conclusion of Corollary 2.1.3 holds in this case as well.

The definition of \(E_\lambda\), see Definition 3.1.2, extends to \(E_{\lambda, g}\) where now the \(\lambda\)-volume is defined with respect to the Riemannian volume form thus \(\omega = \lambda \text{vol}_g\). This is well defined, as explained in Remark 3.1.1, for an arbitrary smooth metric \(g\) since the constancy theorem, Theorem 2.1.3, remains valid.

In order to extend Theorem 3.1.1 for \(E_{\lambda, g}\) we note that the isoperimetric inequality 3.1.1 is valid in \((\mathbb{R}^{n+1}, g)\), where now the isoperimetric constant \(C\) depends on both the dimension and the metric \(g\). Thus we can repeat the exact same steps of the proof of Theorem 3.1.1 to get the following:
Theorem 8.1. Let $\lambda \in \mathbb{R}$ with $|\lambda| < \frac{1}{C \text{vol}_g(K)^{1/n+1}}$, where $C$ is the isoperimetric constant of $(\mathbb{R}^{n+1}, g)$ and a fixed $T_0 \in \mathcal{I}_{n,K}(\mathbb{R}^{n+1})$. Then there exists $T \in \mathcal{I}_{n,K}(\mathbb{R}^{n+1})$ with $\partial T = \partial T_0$ such that

$$E_{\lambda,g}(T) = \inf \left\{ E_{\lambda}(S) \mid S \in \mathcal{I}_{n,K}(\mathbb{R}^{n+1}), \partial S = \partial T_0 \right\}.$$ 

The variational equations for a critical point of $E_{\lambda,g}$ are exactly as in 3.1.8 where now the divergence, the inner product and the $\ast$-operator are taken with respect to the metric $g$. However, the stability inequality 3.2.1 for $\Sigma = \text{reg} T$ becomes

$$\int_{\Sigma} \left( |A_\Sigma|^2 + \text{Ric}_g(N,N) \right) \eta^2 dV_g \leq \int_{\Sigma} |\nabla^\Sigma \eta|^2 dV_g,$$

where $A_\Sigma$ is the second fundamental form of $\Sigma$ with respect to the metric $g$, $\text{Ric}_g$ is the Ricci curvature tensor of $\mathbb{R}^{n+1}$ with respect to the metric $g$ evaluated on the unit normal vector field $N$ of $\Sigma$, $\nabla^\Sigma$ is the Levi-Civita connection of $\Sigma$ induced by $g$ and the norms are induced by the metric $g$. The exact same computations of Proposition 3.2.1 using the same test field in inequality 8.1, give that one-sided minimisers of $E_{\lambda,g}$ that are critical points in $B_1$ are stable in $B_1$.

We have the following version of the multiplicity-1 lemma, Lemma 3.3.2.

Lemma 8.2. Let $T = \partial[A][B_1$ be a one-sided minimiser of $E_{\lambda,g}$ that is also a critical point of $E_{\lambda,g}$ in $B_1$. Then for any $x \in \text{spt} T \setminus \text{spt} \partial T$, $x \notin \text{sing}_1 V_T$ and any sequence $r_j \searrow 0$ there exists a subsequence $(r_j')$ and a set of locally finite perimeter $E$ such that $C = \partial[E]$ is locally a one-sided minimiser of the area functional with respect to the Euclidean metric, stationary in $\mathbb{R}^{n+1}$, $\eta_{0,r} C = C$ for any $r > 0$ and

$$\mu_{\eta_{0,r_j} , T} \xrightarrow{j \to \infty} \mu_C.$$ 

Proof. The only change in the proof will be that when rescaling the current by the sequence $r_j$ we also have to rescale the metric as $g_j(\cdot) = g(r_j \cdot)$ and the rescalings $T_j$ are critical points of $E_{\lambda_j,g_j}$. Thus, using normal coordinates at the point $x$ where we rescale, we get that the limit $C$ lives in $T_x \mathbb{R}^{n+1}$ endowed with the Euclidean metric. Furthermore, Lemma 3.3.1 is valid on any ambient Riemannian manifold as shown in [WIG14]. Now the exact same steps and arguments of the proof of
Lemma 3.3.2 give Lemma 8.2 for $E_{\lambda,g}$.

In [BW20] the regularity theory, Theorem 3.3.1, is generalised to any ambient Riemannian manifold. In fact, in [BW20] regularity and compactness properties of prescribed mean curvature varifolds, that are not necessarily CMC, on ambient Riemannian manifolds are investigated. We can thus follow the exact same steps of the proof of Theorem 3.3.2 where we use Lemma 8.2 instead of Lemma 3.3.2 to get the following:

**Theorem 8.2.** Let $T = \partial[A] \mid B_1$ be a one-sided minimiser of $E_{\lambda,g}$ that is also a critical point of $E_{\lambda,g}$ in $B_1$ and has no touching singularities. Then $\text{spt}T \setminus \text{spt}\partial T$ is smooth everywhere, except on a set of Hausdorff dimension at most $(n-7)$.

If now $T' \in I_n(\mathbb{R}^{n+1})$ is a minimiser of $E_{\lambda,g}$ with respect to the obstacle $T$, as defined in Definition 4.1.1 and $T$ satisfies the exact same properties of Chapter 4 for $E_{\lambda,g}$ instead of $E_\lambda$ we want to get the variational equations, with respect to the metric $g$, in the whole $B_1$ as in Theorem 4.1.1.

**Theorem 8.3.** Let $T' = (M, \overrightarrow{\nabla}, \mu)$ be a minimiser of $E_{\lambda,g}$ with respect to the obstacle $T$. Then $T'$ is a critical point of $E_{\lambda,g}$ in $B_1$.

**Proof.** The proof of Lemma 4.1.2 remains the same for $E_{\lambda,g}$ since Theorem 7 of [WHI09] is valid on any ambient Riemannian manifold. Thus again we have the variational equations in $B_1 \setminus \{0\}$. To include the origin we note that Lemma 4.1.3 remains valid on any codimension. From Nash’s isometric embedding theorem we can embed $(\mathbb{R}^{n+1}, g) \hookrightarrow \mathbb{R}^N$, for some large $N$. Then $T'$ is a generalised CMC hypersurface, away from its boundary, in $\mathbb{R}^N \setminus \{0\}$. Since Lemma 4.1.3 is valid in $\mathbb{R}^N$ and $T'$ can be of any codimension we extend the Euclidean variational equations to include the origin. Thus pulling back the metric we get that $T'$ is a critical point of $E_{\lambda,g}$ in $B_1$. \qed

We can now prove the following:

**Theorem 8.4.** Let $T'$ be a minimiser of $E_{\lambda,g}$ with respect to the obstacle $T$, then $\text{spt}T' \setminus \text{spt}\partial T'$ is smooth everywhere, except on a set of Hausdorff dimension at most $(n-7)$.
Proof. We follow the exact same steps of the proof of Theorem 4.1.2 where we now use Theorem 8.2 and Theorem 8.3. The only difference is that towards the end of the proof the functions \( u_i, u_j \) will satisfy the prescribed mean curvature equation with respect to the metric \( g \), which is still a uniformly elliptic PDE thus the maximum principle Theorem 2.2.1 is again valid.

We need the following versions of Lemma 4.2.1 and Lemma 4.2.2.

Lemma 8.3. Let \( g_j, g \) smooth metrics of \( \mathbb{R}^{n+1} \) such that \( g_j \xrightarrow{j \to \infty} g \) smoothly and \( S_j, S \) integral \( n \)-currents supported in a compact set \( K \subset \mathbb{R}^{n+1} \) such that \( S_j \) minimises \( E_{\lambda, g_j} \) in \( K \) and \( S_j \rightharpoonup S \). Then \( S \) minimises \( E_{\lambda, g} \) in \( K \) and \( \mu_{S_j} \xrightarrow{j \to \infty} \mu_S \).

Lemma 8.4. Let \( g_j, g \) smooth metrics of \( \mathbb{R}^{n+1} \) such that \( g_j \xrightarrow{j \to \infty} g \) smoothly and \( S_j, S \) integral \( n \)-currents supported in \( B_1 \) that are critical points of \( E_{\lambda, g_j} \), \( E_{\lambda, g} \) in \( B_1 \) respectively and \( \mu_{S_j} \xrightarrow{j \to \infty} \mu_S \). Then for any compact \( K \subset B_1 \) we have that \( spt S_j \cap K \xrightarrow{d_{\text{Haus}}} spt S \cap K \) where \( d_{\text{Haus}} \) is the Hausdorff distance with respect to the metric \( g \).

In order to obtain Theorem 4.2.1 we start with the same assumptions on \( T \) and \( \phi_j \) as made in Section 4.1 and Section 4.2, where we have \( E_{\lambda, g} \) in place of \( E_{\lambda} \). We only need to take extra care in the choice of \( \lambda \) and \( r_2 \). We can choose \( r_2 \) small enough so that the second fundamental form of the geodesic sphere \( \partial B_r \) with respect to the metric \( g \) is positive definite for all \( r \leq r_2 \), see [CV81]. Let \( H = \inf \{ H_{\partial B_r} \mid r_1 \leq r \leq r_2 \} \) where \( H_{\partial B_r} \) denotes the mean curvature of \( \partial B_r \) with respect to the metric \( g \). Thus \( H > 0 \) and we take

\[
\lambda < \min \left\{ \frac{1}{C \text{Vol}_g(A \cap B_{r_2})^{1/n+1}}, H \right\},
\]

where \( C \) is the isoperimetric constant with respect to the metric \( g \). We then have the following:

Theorem 8.5. Let \( \phi_j \in C^2(\Gamma_0; \partial B_{r_1}) \) with

\[
|\phi_j - i\Gamma_0|_{C^2} \leq 1/j,
\]
where \(|.|_C^2\) is taken with respect to the metric \(g\), \(\iota_{\Gamma_0}\) denotes the inclusion map of \(\Gamma_0\) into \(\partial B_{r_1}\), \(\Gamma_j = \phi_j \ast \Gamma_0\), \(\Gamma_j \subset A\) and
\[
0 < \lambda < \min \left\{ \frac{1}{C \text{vol}_g(A \cap B_{r_2})^{1/n+1}}, H \right\}.
\]

Then there exist integral \(n\)-currents \(S_j\) that minimise \(E_{\lambda,g}\) in \(A \cap B_{r_2}\) subject to the boundary condition \(\partial S_j = \Gamma_j\) and satisfy the following properties:

1) \(\text{spt} S_j \subset \overline{B_{r_1}}\),

2) there exist sets of locally finite perimeter \(B_j\) with \(\overline{B_j} \subset A \cap \overline{B_{r_1}}\) such that \(S_j = \partial [B_j] \mid B_{r_1}\),

3) \(\Gamma_j = \text{spt} S_j \cap \partial B_{r_1}\),

4) \(S_j\) is a critical point of \(E_{\lambda,g}\) in \(B_{r_1}\). In fact it is a critical point of \(E_{\lambda,g}\) in \(B_{r_2} \setminus \Gamma_j\),

5) \(\mu_{S_j} \xrightarrow{j \to \infty} \mu_{T \mid B_{r_1}}\).

**Proof.** We follow the same steps of the proof of Theorem 4.2.1, using Theorem 8.1 to construct \(S_j\) from the same minimisation problem with \(E_{\lambda,g}\) in place of \(E_{\lambda}\). Using Theorem 8.3, Theorem 8.4 and Lemma 8.3 we get that \(\mu_{S_j} \to \mu_{T \mid B_{r_1}}\).

For the final steps of the proof, in order to show that \(\text{spt} S_j \subset \overline{B_{r_1}}\), we use Lemma 8.4 and the same reasoning as that of Theorem 4.2.1 to get that \(\text{spt} S_j \subset \subset B_{r_2}\). Now instead of using Lemma 19.1 of [SIM84], which we do not know if it extends for an arbitrary metric \(g\), we use the following argument. Let \(r_1 < r < r_2\) and assume that \(\text{spt} S_j\) meets \(\partial B_r\) at some point \(x\). If \(x\) is a regular point of \(\text{spt} S_j\) we get a contradiction from the classical maximum principle since \(\lambda < H\) and \(H < H_{\partial B_r}\), where \(H_{\partial B_r}\) the mean curvature of \(\partial B_r\) with respect to the metric \(g\). If \(x\) is a singular point of \(\text{spt} S_j\) then blowing-up at that point and since the variational equations for \(S_j\) with respect to the metric \(g\) are valid at that point we get in the limit a stationary cone of \(\mathbb{R}^{n+1}\) that lies on a half-space. This cannot happen from Theorem 36.5 of [SIM84] and thus \(\text{spt} S_j \subset \overline{B_{r_1}}\).

Furthermore, since \(\lambda < H\) we can use Theorem 7 of [WHI09], which is valid on any ambient Riemannian manifold, to get that \(\Gamma_j = \text{spt} S_j \cap \partial B_{r_1}\). The final step of
the proof remains the same since it is of topological nature and we are still working in the Euclidean topology. This proves Theorem 8.5.

The remaining ingredient is the singular maximum principle Theorem 5.3.1. We make the same assumptions on the current $T$, thus $T = \partial [A] \lfloor B_1$ is a critical point of $E_{\lambda, g}$ in $B_1$ for $\lambda > 0$, $\text{spt} T$ is connected, $\text{spt} \setminus \{0\}$ is a $C^\infty$-hypersurface and the associated varifold $V_T$ has a varifold tangent cone $C$ of multiplicity-1 that is a regular cone in the sense of Definition 5.1.1 with an isolated singularity at the origin.

**Theorem 8.6.** Let $S = \partial [B] \lfloor B_1$, for $B$ a set of locally finite perimeter, with $\overline{B} \subset \overline{A}$ such that $S$ minimises $E_{\lambda, g}$ with respect to the obstacle $T$ and $\{0\} \in \text{spt} S$. Then $T = S$, where the equality is in the sense of currents.

**Proof.** We follow the same steps as in the proof of Theorem 5.3.1. Using Lemma 8.2, Theorem 8.3, Lemma 8.4 combined with Lemma 5.3.1, which is valid in any ambient Riemannian manifold, we get that the currents have the same, unique, tangent cone $C$ at the origin that is minimal and stable for the area functional with respect to the Euclidean metric.

From Lemma 8.1 we get functions $u, v \in C^2(C \setminus \{0\})$, for some $\rho > 0$ that satisfy the same properties as in the proof of Theorem 5.3.1. The only difference now is that $h = v - u$ satisfies the following PDE

$$L^g_C h = \text{div}_C (P \nabla h) + R \cdot \nabla h + Sh,$$

with $|P|, |R|, |S| \to 0$ as $|x| \to 0$, where the norms are taken with respect to the Euclidean metric and

$$L^g_C h = \Delta^g_C h + |A^g_C|^2 h + \text{Ric}_g(N, N),$$

where in the above equation everything is taken with respect to the metric $g$ and $\text{Ric}_g$ is the Ricci curvature of $(\mathbb{R}^{n+1}, g)$.

We repeat the same rescaling argument for the same sequence $r_j$ as chosen in Theorem 5.3.1 but we also rescale the metric by $g_j(x) = g(r_j x)$. Thus, working in normal coordinates at the origin, we know that the metric goes to the Euclidean one. Since the right hand side of the PDE converges to zero, the limit function $f$
of $f_j = \frac{v(r_j x) - u(r_j x)}{M_j}$, where $M_j$ is chosen exactly as in proof of Theorem 5.3.1 solves $L_C f = 0$ and is not of the form 5.2.2.

We need to change the last argument of the proof of Theorem 5.3.1 because it relies on unique continuation and since the metric $g$ is assumed to be smooth and not analytic we cannot repeat it. After proving that there exists $\rho \leq 1$ such that $T|B_\rho = S|B_\rho$ we can define the following

$$\sup\{r \leq 1 \mid T|B_r = S|B_r\}.$$ 

Clearly the set over which we take the supremum is non-empty from the existence of $\rho$. Let $r_0$ be the supremum and $x_0 \in \partial B_{r_0}$. We can assume, from the regularity theory Theorem 8.4, that $x_0$ is a smooth point and the currents meet tangentially at $x_0$. We can then apply the Hopf maximum principle, Theorem 2.2.2, to get $r > r_0$ where the currents agree. In conclusion, we get that $T|B_1 = S|B_1$. \hfill \Box

In order to state the main theorem of this chapter we make the following assumptions. Let $A$ be a set of locally finite perimeter and the current $T = \partial A|B_1$. We assume that $T$ is a critical point of $E_{\lambda,g}$ in $B_1$ for some $\lambda > 0$ and a one-sided minimiser of $E_{\lambda,g}$. Furthermore we assume that $\text{spt}T$ is connected in $\mathbb{R}^{n+1}$, $0 \in \text{spt}T$ and $\text{spt}T \setminus \{0\}$ is a $C^\infty$-hypersurface. Notice that from Remark 2.1.11 we may assume that $A$ is closed, connected and $\partial A \cap \overline{B_1} = \text{spt}T$. We also assume that the associated varifold $V_T$ admits a varifold tangent cone at the origin that is an oriented regular cone with an isolated singularity at the origin. We do not need to assume that the varifold tangent cone is of multiplicity-1 since now it will follow from Lemma 8.2.

Assume that there are $0 < r_1 < r_2 < 1$ such that the boundary $\Gamma_0 = \partial(T|B_{r_1})$ is smoothly embedded, connected $(n - 1)$-submanifold of $\partial B_{r_1}$ and $A \cap \overline{B_{r_2}}$ has connected complement. Furthermore, we assume that $r_2$ is chosen so that the second fundamental form of $\partial B_r$ with respect to the metric $g$ is positive definite for all $r \leq r_2$.

Finally, let $\phi_j : \Gamma_0 \to \partial B_{r_1}$ be $C^2$-maps with

$$|\phi_j - i_{r_0}|_{C^2} \leq \frac{1}{j},$$
where \( |.|_{C^2} \) is taken with respect to the metric \( g \) and for \( \Gamma_j = \phi_j \ast \Gamma_0 \) we assume that

\[
\Gamma_j \cap A^\circ \neq \emptyset,
\]

where \( A^\circ \) denotes the interior of \( A \) in \( \mathbb{R}^{n+1} \). We then have the following:

**Theorem 8.7.** (approximation theorem on arbitrary ambient manifolds)

Let

\[
\lambda < \min \left\{ \frac{1}{C \text{vol}_g(B_1)^{1/n+1}}, H \right\},
\]

where \( H = \inf \{ H_{\partial B_r} \mid r_1 \leq r \leq r_2 \} \), then there exist integral \( n \)-currents \( S_j \) that minimise \( E_{\lambda,g} \) in \( A \cap \overline{B_{r_1}} \) subject to the boundary condition \( \partial S_j = \Gamma_j \) and satisfy the following properties:

1) \( \text{spt} S_j \subset \overline{B_{r_1}} \),

2) there exist sets of locally finite perimeter \( B_j \) with \( \overline{B_j} \subset A \cap \overline{B_{r_1}} \) such that \( S_j = \partial [B_j] \vert \overline{B_{r_1}} \),

3) \( \Gamma_j = \text{spt} S_j \cap \partial B_{r_1} \),

4) \( S_j \) is a critical point of \( E_{\lambda,g} \) in \( B_{r_2} \). In fact it is a critical point of \( E_{\lambda,g} \) in \( B_{r_2} \setminus \Gamma_j \),

5) \( \mu_{S_j} \xrightarrow{j \to \infty} \mu_{T\vert \overline{B_{r_1}}} \),

6) for all sufficiently large \( j \) there exists \( \rho < r_1 \), uniform in \( j \), such that \( \text{spt} S_j \cap B_\rho \) is a smooth hypersurface.

**Proof.** We repeat the same steps of the proof of Theorem 6.2 where we now use Theorem 8.5 and Theorem 8.6 to get the same contradiction from Theorem 6.1.
## Notation

- $\mathcal{H}^m$: Hausdorff measure in $\mathbb{R}^n$ of dimension $m$
- $\mathcal{L}^n$: Lebesgue measure in $\mathbb{R}^n$
- $\eta_{x,\lambda} : \mathbb{R}^n \rightarrow \mathbb{R}^n$: $\eta_{x,\lambda}(y) = \frac{y - x}{\lambda}$ for $\lambda > 0$ and $x \in \mathbb{R}^n$
- $B_r$: Open ball of radius $r$ centered at the origin
- $B_r(y)$: Open ball of radius $r$ centered at $y$
- $C^0_c(U)$: Space of compactly supported continuous functions $u : U \rightarrow \mathbb{R}$, for $U \subset \mathbb{R}^n$ open
- $\Omega^m_c(U)$: Space of compactly supported smooth $m$-forms for $U \subset \mathbb{R}^n$ open
- $W \subset \subset U$: For a set $U \subset \mathbb{R}^n$ we write $W \subset \subset U$ when $W$ is a precompact set of $U$
- $\nabla$: Euclidean connection
- $C^1(U; V)$: Space of maps $u : U \rightarrow V$ that are $C^1$-differentiable, where $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$ open sets
- $C^1_c(U; V)$: Space of maps that lie in $C^1(U; V)$ and are compactly supported in $U$
- $C^\alpha(U)$: Space of functions $u : U \rightarrow \mathbb{R}$ of Hölder continuous functions, for $U \subset \mathbb{R}^n$ open and $0 < \alpha < 1$
- $C^{k,\alpha}(U)$: Space of functions $u : U \rightarrow \mathbb{R}$ that are $k$-times differentiable and have Hölder continuous $k$-th derivative, for $U \subset \mathbb{R}^n$ open and $k \geq 1$, $0 < \alpha < 1$
\[ L_{\text{loc}}^p(\mu) \] Space of \( \mu \)-measurable functions with \( (\int_K |f|^p d\mu)^{1/p} < \infty \) for any \( K \subset U \) compact, for a Borel measure \( \mu \) in \( U \subset \mathbb{R}^n \) open and \( p \geq 1 \)

\[ x^\perp \] For \( x \in \mathbb{R}^n \), \( x^\perp \) denotes the orthogonal projection onto \((V)^\perp\) for \( V \) an \( m \)-dimensional subspace of \( \mathbb{R}^n \)

\[ \omega_m \] Volume of the \( m \)-ball of radius 1

\[ i_A \] Inclusion map for \( A \subset \mathbb{R}^n \)

\[ |\omega| \] For \( \omega \in \Omega^m_c(U) \) denotes the norm induced by the Euclidean inner product

\[ Df \] For \( f \in C^1(U; V) \) we denote by \( Df \) the differential of \( f \)

\[ \nabla u \] For \( u \in C^1(U) \) we denote by \( \nabla u \) the gradient of \( u \)

\[ \Lambda_m T \] If \( T : V_1 \to V_2 \) is a linear map between two vector spaces of dimension \( n \) and \( m \leq n \) then \( \Lambda_m T \) denotes the induced map from \( \Lambda_m V_1 \to \Lambda_m V_2 \)

\[ * : \Lambda_{n-1} \mathbb{R}^n \to \mathbb{R}^n \] The isomorphism given by the Hodge star operator.

\[ \chi_A \] For \( A \subset \mathbb{R}^n \) we denote by \( \chi_A \) the characteristic function of \( A \)

\[ C^0_c(U) \] Space of Lipschitz functions in \( U \) that are compactly supported
Bibliography


