# Generative Datalog with Continuous Distributions 

Martin Grohe ${ }^{1}$, Benjamin Lucien Kaminski ${ }^{2}$, Joost-Pieter Katoen ${ }^{1}$, and Peter Lindner ${ }^{1}$

${ }^{1}$ \{grohe,katoen,lindner\}@informatik.rwth-aachen.de<br>${ }^{2}$ kaminski@cs.uni-saarland.de<br>${ }^{1}$ RWTH Aachen University, Aachen, Germany<br>${ }^{2}$ Saarland University, Saarland Informatics Campus, Saarbrücken, Germany, and University College London, London, United Kingdom

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#### Abstract

Arguing for the need to combine declarative and probabilistic programming, Bárány et al. (TODS 2017) recently introduced a probabilistic extension of Datalog as a "purely declarative probabilistic programming language." We revisit this language and propose a more principled approach towards defining its semantics based on stochastic kernels and Markov processes - standard notions from probability theory. This allows us to extend the semantics to continuous probability distributions, thereby settling an open problem posed by Bárány et al. We show that our semantics is fairly robust, allowing both parallel execution and arbitrary chase orders when evaluating a program. We cast our semantics in the framework of infinite probabilistic databases (Grohe and Lindner, ICDT 2020), and show that the semantics remains meaningful even when the input of a probabilistic Datalog program is an arbitrary probabilistic database.


## 1. Introduction

Augmenting programming languages with stochastic behavior such as probabilistic choices or random sampling has a long tradition in computer science [59,47]. In recent years, a lot of effort went into the development of dedicated probabilistic programming languages (such as, for example, Anglican [64], Church [31], Figaro [57], Pyro [6], R2 [55], and Stan [13]) that allow the specification and "execution", via probabilistic inference, of sophisticated probabilistic models.

Such languages are nowadays important tools in a large variety of applications in different fields like artificial intelligence, computer vision, and cryptography to name a few [30, 32, 65].

From a database perspective, it is desirable to have a declarative probabilistic programming language that operates on a standard relational data model. Such a language, Probabilistic Programming Datalog (PPDI ${ }^{1}$ ) has recently been introduced by Bárány, ten Cate, Kimelfeld, Olteanu, and Vagena [5]. This language is relational and declarative while still employing main features of probabilistic programming languages such as random sampling and conditioning on observations. A PPDL program has a generative and a constraint part. The generative part is a Datalog program augmented by special tuple-generating rules that involve random sampling. The constraint part conditions the resulting probability space on, say, typical database constraints.

In this work, we focus on the generative part of such a program, which is referred to as Generative Datalog. In a nutshell, a Generative Datalog program is a standard Datalog program with the addition that it may contain rules of the shape

$$
R(\vec{x}) \leftarrow S_{1}\left(\vec{x}_{1}\right), \ldots, S_{m}\left(\vec{x}_{n}\right)
$$

where, as usual, $R, S_{1}, \ldots, S_{m}$ are relation symbols and the $S_{i}\left(\vec{x}_{i}\right)$ are atoms. The difference is that we allow the tuple $\vec{x}$ to contain not only variables and constants, but also references to parameterized distributions as in

$$
R(x, y, \operatorname{Gaussian}\langle x, y\rangle) \leftarrow S_{1}(x), S_{2}(y) .
$$

The standard point of view for generative programs is that they are run in an iterative fashion where in each step an random fact is added to the current database instance, until no rule is applicable anymore (which is made precise later). Intuitively, the head $R(x, y, \operatorname{Gaussian}\langle x, y\rangle)$ of the above rule can be understood as a sampling instruction: If the body $S_{1}(x), S_{2}(y)$ of the rule is satisfied for valuations of $a=\mu$ and $b=\sigma^{2}$ of $x$ and $y$, then we add a fact $R(a, b, X)$ where $X \sim$ Gaussian $\langle a, b\rangle$.

The language of Bárány et al. comes with the restriction that it only allows sampling from discrete probability distributions. Continuous probability distributions are explicitly mentioned as a relevant extension to the language in [5], starting with the definition of its semantics. Yet, acknowledging that this requires even a new, and possibly more challenging definition of the probability spaces in question, their introduction was left open in [5]. We provide such an extension in this work. The main technical questions concern 1. a rigorous definition of its semantics and the well-definedness of outputs, 2. the effect of the order of rule execution on the outcome, and 3 . algorithmic properties. In this paper, we lay the foundations of the extended language by focusing on (1) and (2).

Probabilistic databases (PDBs) are a formal model for describing uncertainty in data [2] 63, 66]. Traditionally, they were limited to probability spaces that consist of a finite number of possible alternative database instances (called possible worlds). Continuous probability distributions, however, arise naturally in many application scenarios that involve uncertain data like noisy sensor measurements [16, 15]. Moreover, for example, a lot of real world statistical

[^0]phenomena, especially those that concern aspects of human behavior, follow normal or lognormal distributions [51]. In computational security, continuous distributions like the Laplacedistribution are common [23].

Unfortunately, generalizing from discrete to continuous distributions usually comes with substantial mathematical overhead. While several systems [3] 41 60] handle continuous probability distributions, only recently [36, 37], Grohe and Lindner proposed a general framework for rigorously dealing with probabilistic databases over continuous domains. Moreover, they establish basic properties such as the measurability of relational calculus and Datalog queries, which in turn allows for formally specifying the semantics of queries over continuous probabilistic databases. This framework and some of the basic results, specifically the measurability of relational calculus queries, is also the foundation for this work. We emphasize that even for infinite probabilistic databases as in [36 37], the definition of database instances remains unchanged: every database instance is still a finite set (or, depending on the context, bag) of facts. Instead, the probability space may be infinite, meaning that there may be infinitely many possible worlds. We note that even though some computations of Generative Datalog programs may run forever, we are always only interested in the finite outcomes, and treat infinite computations as errors where no output is produced.

### 1.1. Contributions

Our main contribution is the introduction of a formal semantics for the probabilistic Datalog language of Bárány et al. [5] that allows sampling from continuous probability distributions. We focus on the generative component of this language, called Generative Datalog (GDatalog). We define how a GDatalog program, on input a single database instance produces an output (sub-)probability distribution over database instances. With our approach (and further extending [5]), we also allow probabilistic inputs.

Bárány et al. [5] introduce the probabilistic semantics of GDatalog as follows.

- To every GDatalog program, they associate an existential Datalog program [10] 12] from which they construct a chase tree. This can be seen as the introduction of a nondeterministic semantics in order to model the possible worlds generated by a GDatalog program.
- This nondeterminism is resolved by weighting the paths of the chase tree according to the distributions that symbolically appear in the GDatalog program.

In a nutshell, Bárány et al. construct a discrete time, discrete space Markov process. We adapt this approach to the continuous setting, and construct a discrete time, but continuous space Markov process that is rooted in the framework of standard PDBs [37]. In order to ensure that the so-obtained probability space is well-defined, we show that the probabilistic transitions described by a GDatalog program satisfy certain technical conditions (that is, that they are stochastic kernels).

The main technical result is that the semantics is independent of the choice of the chase tree and that it is equivalent to the semantics obtained from parallel execution of all applicable rules at any execution step of a GDatalog program.

In addition to the extensions sketched before, we propose slight changes to the semantics introduced by Bárány et al. [5] in order to avoid the following two peculiarities exhibited by the original semantics. Note that with our changes to the semantics however, we lose the feature of FO-rewritability. This is addressed in detail in Section 6.2 (see Remark 6.4).

Example 1.1 (Semantic Continuity). Consider the two GDatalog programs shown in Figure 1 and an input instance $D_{\text {in }}=\{R(0)\}$. A GDatalog program has access to a set $\Psi$ of parameterized distributions. In this case, $\Psi=\{$ Flip $\}$, where Flip $\langle p\rangle$ denotes the Bernoulli distribution with parameter $p$, i. e. the flip of a coin that is biased according to $p$.

| $\mathcal{G}_{0}:$ | $S\left(\operatorname{Flip}\left\langle\frac{1}{2}\right\rangle\right) \leftarrow R(0)$ |
| :--- | :--- |
|  | $S\left(\operatorname{Flip}\left\langle\frac{1}{2}\right\rangle\right) \leftarrow R(0)$ |

$$
\begin{array}{ll}
\mathcal{G}_{\varepsilon}: & S\left(\mathrm{Flip}\left\langle\frac{1}{2}\right\rangle\right) \leftarrow R(0) \\
& S\left(\mathrm{Flip}\left\langle\frac{1}{2}+\varepsilon\right\rangle\right) \leftarrow R(0)
\end{array}
$$

Figure 1.: Two GDatalog programs $\mathcal{G}_{0}$ and $\mathcal{G}_{\varepsilon}$ (where $0<\varepsilon<\frac{1}{2}$ ).

The fact $R(0)$ in $D_{\text {in }}$ is just a dummy fact that makes all the rules applicable. Under the original semantics ${ }^{2}$, the program $\mathcal{G}_{0}$, on input $D_{\mathrm{in}}$, generates the following probabilistic database:

Table 1.: The probabilistic database generated by $\mathcal{G}_{0}$.

| possible world | $\{R(0), S(0)\}$ | $\{R(0), S(1)\}$ |
| :--- | :---: | :---: |
| probability | $\frac{1}{2}$ | $\frac{1}{2}$ |

Yet, for any $\varepsilon \in\left(0, \frac{1}{2}\right)$, the program $\mathcal{G}_{\varepsilon}$ generates three possible worlds under the original semantics:

Table 2.: The probabilistic database generated by $\mathcal{G}_{\varepsilon}$.

| possible world | $\{R(0), S(0)\}$ | $\{R(0), S(1)\}$ | $\{R(0), S(0), S(1)\}$ |
| :--- | :---: | :---: | :---: |
| probability | $\frac{1}{4}-\frac{\varepsilon}{2}$ | $\frac{1}{4}+\frac{\varepsilon}{2}$ | $\frac{1}{2}$ |

The probabilities intuitively arise because the rules "fire" independently. In [5], nondeterminism is introduced by allocating new relations to store the outcomes of the random samplings associated with $\psi$-terms for $\psi \in \Psi$. These relations are called Result ${ }_{n}^{\psi}$ where $\psi$ is the involved distribution (and $n$ is an additional technical parameter related to the arity of the new relation). The relations Result $_{n}^{\psi}$ however do not distinguish which rule of the original program lead to their introduction. Therefore they are, for example, oblivious of multiple occurrences of the same probabilistic rule. Thus, in the probabilistic semantics, both rules of $\mathcal{G}_{0}$ collapse and we arrive at the probabilities shown in Table 1 Yet, for every $\varepsilon>0$, this distinction happens in the semantics of $\mathcal{G}_{\varepsilon}$, yielding the probabilities from Table 2

[^1]Contrary to this, we argue that it is more reasonable to have the probabilistic database produced by $\mathcal{G}_{\varepsilon}$ converging to the one produced by $\mathcal{G}_{0}$ as $\varepsilon \rightarrow 0$. Under the semantics we propose later, both $\mathcal{G}_{0}$ and $\mathcal{G}_{\varepsilon}$ generate the three possible worlds $\{R(0), S(0)\},\{R(0), S(1)\}$, and $\{R(0), S(0), S(1)\}$. The probabilities of these worlds obtained from $\mathcal{G}$ are $\frac{1}{4}, \frac{1}{4}$, and $\frac{1}{2}$, respectively. This coincides (pointwise) with the limits of the probabilities obtained from $\mathcal{G}_{\varepsilon}$ as $\varepsilon \rightarrow 0$ (see Table 2).

Example 1.2 (Independence from Symbolic Names). As another example, consider the following program $\mathcal{G}_{0}^{\prime}$. The difference over $\mathcal{G}_{0}$ is that in the second rule, Flip has been replaced with Flip' which is mathematically the same distribution, but with a different symbolic name.

$$
\begin{array}{ll}
\hline \mathcal{G}_{0}^{\prime}: & S\left(\text { Flip }\left\langle\frac{1}{2}\right\rangle\right) \leftarrow R(0) \\
& S\left(\text { Flip }^{\prime}\left\langle\frac{1}{2}\right\rangle\right) \leftarrow R(0) \\
\hline
\end{array}
$$

Figure 2.: A GDatalog program with two identical, yet differently named distributions.

Under the semantics of [5] (and with removing auxiliary relations), $\mathcal{G}_{0}^{\prime}$ generates three possible worlds, $\{R(0), S(0)\},\{R(0), S(1)\}$, and $\{R(0), S(0), S(1)\}$, with probabilities $\frac{1}{4}, \frac{1}{4}$, and $\frac{1}{2}$. Recall that this differs from the result of $\mathcal{G}_{0}$. The reason is that in [5], the syntactic name of the parameterized distribution is hard-coded into the relations Result ${ }_{n}^{\psi}$. Under the semantics we propose later, the results of $\mathcal{G}_{0}$ and $\mathcal{G}_{0}^{\prime}$ coincide (after removing auxiliary relations). $\triangleleft$

The properties of the original semantics sketched above are not a fundamental problem for the definition of the semantics, and do not raise any questions about the correctness of [5]. Instead, we argue that these are two examples, where the original semantics feels unnatural. Consequently, we adapt our semantics to resolve such effects.

### 1.2. Paper Outline

After concluding the introduction with a short survey of related work, we present the most central mathematical definitions and background results from measure theory and probabilistic databases in Section 2 In Section [3 we introduce the syntax of GDatalog programs together with the backbone of its semantics-the translation into an existential Datalog program. In Section 4 we present our version of a probabilistic chase, generalizing the ideas of [5]. We show that this notion defines a Markov process over database instances. Section5is devoted to establishing similar results when a parallel chase procedure is used (which is novel over [5]). In Section 6 we discuss various properties of the semantics. First, we show that no matter which kind of chase procedure is used, the probabilistic database that is described by its semantics turns out to be the same. We argue that (regarding finite outcomes) our semantics can simulate the original semantics of [5] and discuss the termination behavior of GDatalog programs. We briefly revisit the full Probabilistic Programming Datalog (PPDL) language in Section 7 We conclude the work and indicate topics for future research in Section 8

To ease the accessibility, the longer technical sections of the paper, Sections3to 5 open with a separate in-depth outline. For obtaining a better understanding before reading the full paper
in detail, we advise the reader to check out these introductions, along with the discussions of Section 6.2

### 1.3. Related Work

Both the probabilistic programming and the probabilistic database community have developed a variety of models and systems that allow to specify continuous probability distributions over data. We briefly mention some of these models and compare them to the scope of this paper. As the original introduction of PPDL already features a broad survey of related work [5, Section 7] that we leave as is, we just comment on related work regarding the added support of continuous variables.

Basically all probabilistic programming languages support continuous distributions, for instance Church [31], Anglican [64], and Figaro [57]. Such languages are contrasted by the database-centric nature of PPDL. Conceptually closer to PPDL are languages studied in statistical relational artificial intelligence (StarAI) [20]. A prominent example of such a language is ProbLog [19, 24], a probabilistic variant of Prolog. In ProbLog, standard Prolog rules can be annotated with a probability value ("rule-based" uncertainty). The extension Hybrid ProbLog [38] allows continuous attribute-level uncertainty in rule-heads.

Markov Logic Networks (MLNs) [58] describe joint distributions of variables based on weighted ("soft") first-order constraints. Hybrid MLNs [68] introduce numeric terms and properties to MLNs, although it is not easy to tell the relationship between the resulting system and "pure" continuous attribute-level uncertainty (see the discussion in [38]). Infinite MLNs [61] allow for countably infinitely many variables with countable domains. Similarly, Probabilistic Soft Logic (PSL) [44] is a formalism for specifying joint distributions with weighted rules, but also "soft" truth values. PSL rules are restricted to conjunctive bodies, as encountered in plain Datalog. As MLNs build upon Markov networks, Bayesian Logic (BLOG) [54 53] builds upon Bayesian networks. BLOG is a programming language supporting continuous distributions and a random (say Poisson-distributed) number of objects. Its continuous semantics is formally treated in [69]. In a nutshell, MLNs, PSL, and BLOG provide first-order templates for specifying graphical models [45].

While all formalisms mentioned so far (and, additionally, those discussed in [5]) share individual features with PPDL, conjoining Datalog with classical probabilistic programming was novel to [5]. Also, its introduction of attribute-level uncertainty in rule heads differs from previous probabilistic versions of Datalog that adhere to the rule-based uncertainty and possibly prior uncertainty for given ground facts (like the probabilistic Datalog languages of [26, 27] and [22]). A version of Datalog ${ }^{ \pm}$[33, 34] that is used for specifying ontologies consists of an MLN, and a Datalog program with (among others) tuple-generating dependencies that may be annotated with events in the probability space. On the contrary, the event annotations of fudgeD [67] use distinguished variables solely used to introduce dependencies among the rules. The connection between probabilistic Datalog and database-valued Markov processes is already noted in [22,5]. The Monte Carlo Database System (MCDB) [41] allows for the specification and querying of probabilistic databases in an SQL-like syntax. Its successor SimSQL [11],
in particular, is a framework that allows the definition of Markov processes over database instances.

## 2. Preliminaries

### 2.1. Foundations from Measure Theory

Here we cover the background from measure theory needed for this paper. More details can be found in standard textbooks on measure theory and Borel sets [42, 62]. Some well-known key results we use can moreover be found in Appendix A.

### 2.1.1. Measure Spaces

A family $\mathfrak{X}$ of subsets of a set $\mathbb{X}$ is called a $\sigma$-algebra on $\mathbb{X}$ if it contains $\mathbb{X}$ and is closed under complements and countable unions. A pair ( $\mathbb{X}, \mathfrak{X}$ ) with $\mathfrak{X}$ being a $\sigma$-algebra on $\mathbb{X}$ is called a measurable space. The elements of $\mathfrak{X}$ are called measurable sets or events. A function $\mu: \mathfrak{X} \rightarrow \mathbb{R}_{\geq 0} \cup\{\infty\}$ is a measure on a measurable space $(\mathbb{X}, \mathfrak{X})$ if $\mu(\emptyset)=0$ and $\mu\left(\cup_{i \in \mathbb{N}} X_{i}\right)=$ $\sum_{i \in \mathbb{N}} \mu\left(X_{i}\right)$ for any sequence of pairwise disjoint events $X_{i} \in \mathfrak{X}(i \in \mathbb{N})$. The value $\mu(\mathbb{X})$ is called the mass of $\mu$. Measures with $\mu(\mathbb{X})=1$ are called probability measures, measures with $\mu(\mathbb{X}) \leq 1$ are called sub-probability measures.

A triple $(\mathbb{X}, \mathfrak{X}, \mu)$ is called a measure space if $(\mathbb{X}, \mathfrak{X})$ is a measurable space and $\mu$ is a measure on $(\mathbb{X}, \mathfrak{X})$. If $\mu$ is a (sub-)probability measure, then $(\mathbb{X}, \mathfrak{X}, \mu)$ is called a (sub-)probability space. The measure space $(\mathbb{X}, \mathfrak{X}, \mu)$ (or simply $\mu$ ) is called $\sigma$-finite, if there exists a partition of $\mathbb{X}$ into countably many measurable sets of finite measure. All measures that appear in this paper are $\sigma$-finite.

In this paper, we need four standard constructions of measurable spaces.

1. Generated $\sigma$-algebra. If $\mathbb{X}$ is a non-empty set and $\mathfrak{G} \subseteq \mathcal{P}(\mathbb{X})$, then $\sigma(\mathfrak{G})$ denotes the (unique) inclusionwise smallest $\sigma$-algebra on $\mathbb{X}$ containing $\mathfrak{G}$. We say $\sigma(\mathfrak{G})$ is generated by $\mathfrak{G}$.
2. Trace $\sigma$-algebra. If $(\mathbb{X}, \mathfrak{X})$ is a measurable space and $X \subseteq \mathbb{X}$, then $\mathfrak{X} \upharpoonright_{X}:=\left\{X^{\prime} \cap X: X^{\prime} \in\right.$ $\mathfrak{X}\}$ is a $\sigma$-algebra on $\mathbb{X} \cap X$, called the trace $\sigma$-algebra of $X$. If $X \in \mathfrak{X}$, then $\mathfrak{X} \upharpoonright_{X} \subseteq \mathfrak{X}$.
3. Disjoint union $\sigma$-algebra. Let $(\mathbb{X}, \mathfrak{X})$ and $(\mathbb{Y}, \mathfrak{Y})$ be measurable spaces with $\mathbb{X} \cap \mathbb{Y}=\emptyset$. Then the family $\mathfrak{X} \oplus \mathfrak{Y}:=\{Z \subseteq \mathbb{X} \cup \mathbb{Y}: Z \cap \mathbb{X} \in \mathfrak{X}$ and $Z \cap \mathbb{Y} \in \mathfrak{Y}\}$ is a $\sigma$-algebra on $\mathbb{X} \uplus \mathbf{Y}$. This easily generalizes to $\bigoplus_{i \in I} \mathfrak{X}_{i}$ for any finite index set $I$.
4. Let $\left(\mathbb{X}_{i}, \mathfrak{X}_{i}\right)$, with $i \in I$ for some index set $I$, be a collection of measurable spaces and let $\mathbb{X}:=\prod_{i \in I} \mathbb{X}_{i}$. The product $\sigma$-algebra $\bigotimes_{i \in I} \mathfrak{X}_{i}$ is the coarsest $\sigma$-algebra on $\mathbb{X}$ that makes all canonical projections $\pi_{i}: \mathbb{X} \rightarrow \mathbb{X}_{i}:\left(x_{i}\right)_{i \in I} \mapsto x_{i}$ measurable. If $I$ is countable, then $\bigotimes_{i \in I} \mathfrak{X}_{i}$ is generated by the family of measurable rectangles $\prod_{i \in I} X_{i}$ with $X_{i} \in \mathfrak{X}_{i}$. If $I=\{1, \ldots, n\}$ we write $\bigotimes_{i=1}^{n} \mathfrak{X}_{i}$ or $\mathfrak{X}_{1} \otimes \cdots \otimes \mathfrak{X}_{n}$ for the product $\sigma$-algebra. If all $\left(\mathbb{X}_{i}, \mathfrak{X}_{i}\right)$ are equal, we write $\mathfrak{X}^{\otimes n}$. If $I=\mathbb{N}$, and all $\left(\mathbb{X}_{i}, \mathfrak{X}_{i}\right)$ are equal, we write $\mathfrak{X}^{\otimes \omega}$ for $\bigotimes_{i=0}^{\infty} \mathfrak{X}_{i}$.

Measure theory is closely tied to notions from topology. A topological space is a pair $(\mathbb{X}, \mathfrak{T})$ where $\mathbb{X}$ is a set and $\mathfrak{I}$ is a family of subsets of $\mathbb{X}$, called the open sets, such that $\mathfrak{I}$ contains both $\mathbb{X}$ and $\emptyset$ and is closed under finite intersections and arbitrary unions. The $\sigma$-algebra on a topological space $(\mathbb{X}, \mathfrak{I})$ that is generated by the open sets is called the Borel $\sigma$-algebra on $(\mathbb{X}, \mathfrak{I})$ (resp. on $\mathbb{X}$ if $\mathfrak{I}$ is understood from context). We denote the Borel $\sigma$-algebra on $\mathbb{X}$ by $\mathfrak{B o r}(\mathbb{X})$. Typical examples are $\mathfrak{B o r}(\mathbb{R})$ and $\mathfrak{B o r}[0,1]:=\mathfrak{B o r}([0,1])$.

In probability theory, one often works with the Borel $\sigma$-algebras generated from Polish topological spaces, i.e. from completely metrizable spaces containing a countable dense set. The resulting measurable spaces are called standard Borel spaces. We do not delve into the details here, as all measurable spaces appearing in this paper are standard Borel. For further information, especially in the context of probabilistic databases, see [37].

### 2.1.2. Measurable Functions and Kernels

Let $(\mathbb{X}, \mathfrak{X})$ and $(\mathbb{Y}, \mathfrak{Y})$ be measurable spaces. A function $f: \mathbb{X} \rightarrow \mathbf{Y}$ is called $(\mathfrak{X}, \mathfrak{Y})$-measurable (or simply measurable, if clear from context) if for all $Y \in \mathfrak{Y}$ ) it holds that $f^{-1}(Y):=\{x \in$ $\mathbb{X}: f(x) \in Y\} \in \mathfrak{X}$. The function $f$ is called bimeasurable if additionally $f(X) \in \mathfrak{Y}$ for all $\boldsymbol{X} \in \mathfrak{X}$.

If $f$ is $(\mathfrak{X}, \mathfrak{Y})$-measurable and $\mu$ is a measure on $(\mathbb{X}, \mathfrak{X})$, then $\mu \circ f^{-1}$ is the so-called pushforward measure of $\mu$ along $f$ on ( $\mathbb{Y}, \mathfrak{Y})$. If $\mu$ is a (sub-)probability measure, so is $\mu \circ f^{-1}$.

A function $\kappa: \mathbb{X} \times \mathfrak{Y} \rightarrow[0,1]$ is called a (sub-)stochastic kernel from $(\mathbb{X}, \mathfrak{X})$ to $(\mathbb{Y}, \mathfrak{Y})$ if

- for all $X \in \mathbb{X}$, the function $\kappa(X, \cdot): \mathfrak{Y} \rightarrow[0,1]$ is a (sub-)probability measure on $(\mathbb{Y}, \mathfrak{Y})$, and
- for all $Y \in \mathfrak{Y}, \kappa(\cdot, Y): \mathbb{X} \rightarrow[0,1]$ is $(\mathfrak{X}, \mathfrak{B o r}[0,1])$-measurable.

For every measurable space $(\mathbb{X}, \mathfrak{X})$, the function $\iota: \mathbb{X} \times \mathfrak{X} \rightarrow[0,1]$ with $\iota(x, \boldsymbol{X})=1$ if $x \in X$ and $\iota(x, X)=0$ if $x \notin X$ is a stochastic kernel from $(\mathbb{X}, \mathfrak{X})$ to itself, called the identity kernel on ( $\mathbb{X}, \mathfrak{X}$ ).

### 2.1.3. Graphs, Sections and Product Measures

First we note that any countable product of standard Borel spaces (with the product $\sigma$-algebra) is standard Borel again [62, Proposition 3.1.23]. If $f: \mathbb{X} \rightarrow \mathbf{Y}$ is measurable with $(\mathbb{X}, \mathfrak{X})$ and $(\mathbf{Y}, \mathfrak{Y})$ standard Borel, then the graph of $f$, defined by

$$
\operatorname{graph}(f):=\{(x, f(x)): x \in \mathbb{X}\}
$$

is measurable in $\mathfrak{X} \otimes \mathfrak{Y}$ [62, Proposition 3.1.21 and 2.1.9].
If $Z \subseteq \mathbb{X} \times \mathbb{Y}$ and $x \in \mathbb{X}$, then $Z_{x}:=\{y \in \mathbb{Y}:(x, y) \in Z\}$ is called the $x$-section of $Z$. If $Z \in \mathfrak{X} \otimes \mathfrak{Y}$, then $Z_{x} \in \mathfrak{Y}$. Symmetrically, for any $y$-section $Z_{y}$ of $Z$, we have $Z_{y} \in \mathfrak{X}$.

If $(\mathbb{X}, \mathfrak{X}, \mu)$ and $(\mathbb{Y}, \mathfrak{Y}, v)$ are measure spaces with $\mu$ and $v \sigma$-finite, then there exists a unique product measure $\mu \otimes v$ of $\mu$ and $v$ on $(\mathbb{X} \times \mathbb{Y}, \mathfrak{X} \otimes \mathfrak{Y})$ with the property that $(\mu \otimes v)(X \times Y)=$ $\mu(X) \cdot v(Y)$ for all $X \in \mathfrak{X}$ and $Y \in \mathfrak{Y}$. This can be extended to any finite (nonempty) product
of measures [42] cf. Theorem 1.27 and p. 15]. We use the notation $\bigotimes_{i=1}^{n} \mu_{i}$ and $\mu^{\otimes n}$ analogous to the one for product $\sigma$-algebras.

Fubini's Theorem intuitively states that integration in a product space can be carried out in an arbitrary order.

Fact 2.1 (Fubini's Theorem, cf. [42, Theorem 1.27]). Let $(\mathbb{X}, \mathfrak{X})$ and $(\mathbb{Y}, \mathfrak{Y})$ be measurable spaces and $\mu$ be a $\sigma$-finite measure on $(\mathbb{X}, \mathfrak{X})$. Then for all measurable $f: \mathbb{X} \times \mathbb{Y} \rightarrow \mathbb{R}_{\geq 0}$, it holds that

$$
\int_{\mathbf{X} \times \mathbf{Y}} f d(\mu \otimes v)=\int_{\mathbf{X}}\left(\int_{\mathbf{Y}} f d v\right) d \mu=\int_{\mathbf{Y}}\left(\int_{\mathbf{X}} f d \mu\right) d v
$$

### 2.1.4. Multifunctions and Selections

Let $(\mathbb{X}, \mathfrak{X})$ and $(\mathbb{Y}, \mathfrak{Y})$ be measurable spaces where $(\mathbb{Y}, \mathfrak{Y})$ is standard Borel (with fixed Polish topology $\mathfrak{I}_{\mathrm{Y}}$ ). A function $M: \mathbb{X} \rightarrow \mathcal{P}(\mathrm{Y}) \backslash \emptyset$ is called a multifunction, and is denoted $M: \mathbb{X} \rightrightarrows$ Y . A multifunction $M: \mathbb{X} \rightrightarrows \mathbf{Y}$ is called

- closed-valued, if for every $x \in \mathbb{X}, M(x) \subseteq \mathbb{Y}$ is closed w.r.t. $\mathfrak{I}_{Y}$, and
- $\mathfrak{X}$-measurable, if $M^{-1}(Y):=\{x \in \mathbb{X}: M(x) \cap Y \neq \emptyset\} \in \mathfrak{X}$ for every open set $Y \in \mathfrak{I}_{Y}$.

Similarly to the corresponding statement for measurable functions, if $M: \mathbb{X} \rightrightarrows Y$ is a closedvalued measurable multifunction, then

$$
\operatorname{graph}(M):=\{(x, y): y \in M(x)\}
$$

is a measurable set in $\mathfrak{X} \otimes \mathfrak{Y}$.
A selection of a multifunction $M$ is a function $s: \mathbb{X} \rightarrow \mathbb{Y}$ with $s(x) \in M(x)$ for all $x \in \mathbb{X}$. A well-known result from Kuratowski and Ryll-Nardzewski (Fact A.5, see [49] and 62, Theorem 5.2.1]) states that for ( $\mathbf{Y}, \mathfrak{Y}$ ) standard Borel, every measurable, closed-valued multifunction $M: \mathbb{X} \rightrightarrows \mathbb{Y}$ has a $(\mathfrak{X}, \mathfrak{Y})$-measurable selection.

### 2.1.5. (Discrete-Time) Stochastic Processes

A stochastic process in discrete time is a sequence of random variables in some state space $(\mathbb{X}, \mathfrak{X})$. Intuitively, a (discrete-time) Markov process is a stochastic process where the distribution in the $(i+1)$ th step only depends on the distribution of the previous step $i$. By a theorem of Kolmogorov (Fact A.8), Markov processes in discrete time are guaranteed to exist for any initial distribution and any sequence of stochastic kernels $\kappa_{i}: \mathbb{X} \times \mathfrak{X} \rightarrow[0,1]$, describing the probabilistic transition on the state space from the $i$ th to the $(i+1)$ th step. If $(\mathbb{X}, \mathfrak{X})$ is the state space of the process, then $\left(\mathbb{X}^{\omega}, \mathfrak{X}^{\otimes \omega}\right)$ is its path space.

### 2.2. Parameterized Distributions

Let $\Pi$ be a non-empty set (of parameters) and let $(\mathbb{W}, \mathfrak{W}, \mu)$ be a measure space.

Definition 2.2. A parameterized distribution with parameter space $\Pi$ and underlying space ( $\mathbb{W}, \mathfrak{W}, \mu$ ) is a function $\psi: \Pi \times \mathbb{W} \rightarrow \mathbb{R}_{\geq 0}$ such that for all $p \in \Pi$ it holds that $\psi(p, \cdot)$ is $\left(\mathfrak{B}, \mathfrak{B o r}\left(\mathbb{R}_{\geq 0}\right)\right)$-measurable and that

$$
\begin{equation*}
\int_{\mathbb{W}} \psi(p, \cdot) d \mu=1 \tag{2.1}
\end{equation*}
$$

If $\psi$ is a parameterized distribution, we use $\Pi_{\psi}$ to refer to its parameter space and $\left(\mathbb{W}_{\psi}, \mathfrak{W}_{\psi}, \mu_{\psi}\right)$ to refer to its underlying measure space. Moreover, we usually make the parameter in the argument of $\psi$ explicit by writing $\psi\langle p\rangle(w)$ instead of $\psi(p, w)$, and $\psi\langle p\rangle$ for the function $\psi(p, \cdot)$.

The requirement from (2.1) demands that for a parameterized distribution $\psi$ and any fixed parameter $p$, the function

$$
P_{\psi\langle p\rangle}: \mathfrak{W}_{\psi} \rightarrow[0,1]: W \mapsto \int_{W} \psi\langle p\rangle d \mu_{\psi}
$$

is a probability measure. We will always assume that $(\mathbb{W}, \mathfrak{W}, \mu)$ is either

- a discrete measure space, with $\mathfrak{W}=\mathcal{P}(\mathbb{W})$ being the powerset $\sigma$-algebra and $\mu$ being the counting measure on $(\mathbb{W}, \mathfrak{W})$; or
- the Euclidean space $\mathbb{R}^{n}$ for some $n \in \mathbb{N}_{>0}$, equipped with its Lebesgue-measurable sets $\mathfrak{W}$ and the $n$-dimensional Lebesgue measure $\mu$.

Note that in the first case, the integral from (2.1) collapses to the sum $\sum_{w \in \mathbb{W}} \psi(p, w)$ and $\psi\langle p\rangle$ plays the role of a probability mass function. Accordingly, in the second case, $\psi\langle p\rangle$ is a probability density function. If $\Pi_{\psi}$ is a space of $m$-tuples, then $m=\operatorname{pardim}(\psi)$ is called the parameter dimension of $\psi$. Typically, if pardim $(\psi)=m>1$, then $\Pi_{\psi}$ is the full Cartesian product of $m$ spaces. We refer to parameterized distributions by symbolic names such as Binomial, Poisson or Gaussian if they describe the corresponding well-known distributions. Such examples are shown in Table 3

For our work, we need to discuss situations where the parameters themselves are random variables. Thus, the following result on measurability with respect to parametrizations is central for our work. It is a special case of [28] Theorem 3.2], tailored to our definition of parameterized distributions. It states that, under suitable technical conditions, the probability of a fixed event under a parameterized distribution is a measurable function of the parameters.

Fact 2.3 (Gaudard \& Hadwin [28, Theorem 3.2]). Let $\psi$ be a parameterized distribution such that $\Pi_{\psi}$ is a Borel subset of a Polish space and the following hold.

1. For all $w \in \mathbb{W}_{\psi}$, the function $\Pi_{\psi} \rightarrow[0,1]: \vec{p} \mapsto \psi\langle\vec{p}\rangle(w)$ is continuous.
2. Every $\vec{p}_{0} \in \Pi_{\psi}$ has a neighborhood $N\left(\vec{p}_{0}\right)$ with

$$
\int_{\mathbb{W}_{\psi}}\left(\sup _{\vec{p} \in N\left(\vec{p}_{0}\right)} \psi\langle\vec{p}\rangle\right) d \mu_{\psi}<\infty
$$

Table 3.: Prominent examples of parameterized distributions.

| Parameterized <br> Distribution $\psi$ | Parameter <br> Space $\Pi_{\psi}$ | Underlying <br> Space $\mathbb{W}_{\psi}$ | pmf $/ \mathbf{p d f} \psi\langle\vec{p}\rangle(w)$ |
| :---: | :---: | :---: | :---: |
| Flip | $[0,1]$ | $\{0,1\}$ | Flip $\langle p\rangle(w)= \begin{cases}p & \text { for } w=1, \\ 1-p & \text { for } w=0\end{cases}$ |
| Binomial | $\mathbb{N}_{>0} \times[0,1]$ | $\mathbb{N}_{\geq 0}$ | Binomial $\langle n, p\rangle(k)=\binom{n}{k} p^{k}(1-p)^{n-k}$ |
| Poisson | $\mathbb{R}_{>0}$ | $\mathbb{N}_{\geq 0}$ | Poisson $\langle\lambda\rangle(k)=\frac{\lambda^{k}}{k!} e^{-\lambda}$ |
| Gaussian | $\mathbb{R} \times \mathbb{R}_{>0}$ | $\mathbb{R}$ | Gaussian $\left\langle\mu, \sigma^{2}\right\rangle(x)=\frac{1}{\sqrt{2 \pi \sigma^{2}}} e^{-\frac{(x-\mu)^{2}}{\sigma^{2}}}$ |

3. If $\vec{p}, \vec{q} \in \Pi_{\psi}$ with $\vec{p} \neq \vec{q}$, then $P_{\psi\langle\vec{p}\rangle}$ and $P_{\psi\langle\vec{q}\rangle}$ are different probability measures.

Then for every $\boldsymbol{W} \in \mathfrak{W}_{\psi}$ and $\boldsymbol{B} \in \mathfrak{B o r}[0,1]$, it holds that

$$
\left\{\vec{p} \in \Pi_{\psi}: P_{\psi\langle\vec{p}\rangle}(\boldsymbol{W}) \in \boldsymbol{B}\right\} \in \mathfrak{B o r}\left(\Pi_{\psi}\right)
$$

Let us first describe the three conditions in words. Condition 1 states that for every fixed argument $w$ the density $\psi\langle\vec{p}\rangle(w)$ in $w$ is continuous with respect to $\vec{p}$. Condition 2 means that for every parameter $\vec{p}$, the supremum of densities parameterized from within a neighborhood of $\vec{p}$ is integrable. Finally, condition 3 states that distinct parameters produce different distributions through $\psi$. Together, the conditions enforce that the parameterized distribution is well-behaved with respect to certain topological properties in the parameter space.

We emphasize the crucial role of Fact 2.3 for the results of this paper. We only allow such parameterized distributions in generative Datalog programs, that adhere to the technical preconditions of this theorem. The reason for this is that we need its conclusion at a central point in our constructions.

If the underlying space of a parameterized distribution is countable at most (and hence, $\mu_{\psi}$ is the counting measure), then condition 2 is trivially fulfilled. If additionally, the parameter space is discrete, then condition 1 is always satisfied as well. For uncountable parameter spaces, this need not be the case. However, one may easily verify that, for example, the Binomial and the Poisson distribution meet condition 1 (and 2 and 3, for that matter). The Gaussian distribution is a continuous distribution for which Fact 2.3 is applicable [28, p. 173]. Thus, all the distributions from Table 3 are suitable for our later application. Moreover, according to [28, p. 173], the conditions generally apply "to the most common [parameterized] families". Therefore, they should not be considered as too harsh a restriction for our purposes. We want to point out a particular caveat though. Notably, the theorem is not applicable for the Dirac distribution. This is, because the Dirac distribution is not even a parametrized distribution in the sense of Definition 2.2 to begin with. Related to this, we note that it has recently been pointed out [4], that the class of "allowed" parameterized distributions should also be treated with care in the setting of [5] (where the sample space is always discrete, but the parameter space may be uncountable).

### 2.3. Relational Databases

We fix a countably infinite set Rel, and a non-empty set $\mathbb{U}$. The elements of Rel are called relation symbols and $\mathbb{U}$ is called the universe. We also fix a function ar: Rel $\rightarrow \mathbb{N}$ and call $\operatorname{ar}(R)$ the arity of $R$. The attributes of $R$ are then the numbers $1, \ldots, \operatorname{ar}(R)$. Additionally, we fix a function dom that maps every pair $(R, i)$ with $R \in \operatorname{Rel}$ and $1 \leq i \leq \operatorname{ar}(R)$ to a non-empty subset of $\mathbb{U}$, and we write $\operatorname{dom}_{i}(R)$ instead of $\operatorname{dom}(R, i)$. Then $\operatorname{dom}_{i}(R)$ is called the domain of the $i$ th attribute in $R$. We define the domain of $R$ as

$$
\operatorname{dom}(R):=\prod_{i=1}^{\operatorname{ar}(R)} \operatorname{dom}_{i}(R) \subseteq \mathbb{U}^{\operatorname{ar}(R)}
$$

Throughout the rest of the paper, we assume that Rel, $\mathbb{U}$, ar and dom are fixed.
A database schema $\mathcal{S}$ is a non-empty, finite subset of Rel. A fact (or, $\mathcal{S}$-fact) is an expression of the shape $R\left(u_{1}, \ldots, u_{n}\right)$ where $u_{i} \in \operatorname{dom}_{i}(R)$ for all $i=1, \ldots, \operatorname{ar}(R)$. The set of facts with relation symbol $R$ (or, $R$-facts) is denoted by $\mathbb{F}_{R}$. The set of all $\mathcal{S}$-facts is denoted by $\mathbb{F}_{\mathcal{S}}$. A database instance over $\mathcal{S}$ and $\mathbb{U}$ (or, $\mathcal{S}$-instance) is a finite bag (multiset) of facts from $\mathbb{F}_{\mathcal{S}}$.

The following example is loosely based upon an example from [41] and serves as a running example throughout the paper.
Example 2.4 (Corporate Data). We consider a database that stores data of various companies, with database schema $\mathcal{S}=\{$ PartnerOf, Employee, PayScale $\}$. The tuples of the relations capture the following information:

- PartnerOf $\left(c_{1}, c_{2}\right)$ means that the companies $c_{1}$ and $c_{2}$ are contract partners.
- Employee ( $s, c, d$ ) means that the social security number (SSN) $s$ is associated with an employee at the department $d$ of company $c$.
- PayScale $(c, d, \mu)$ means that employees of department $d$ at company $c$ achieve an average annual income of $\mu$ dollars.

The database is shown in Figure 3below.

Employee

| SSN | Company | Department |
| :---: | :---: | :---: |
| $962-00-3472$ | F-Corp | HR |
| $981-00-8876$ | E-Corp | IT |

PartnerOf

| Company_1 | Company_2 |
| :---: | :---: |
| A-Corp | F-Corp |
| A-Corp | D-Corp |

PayScale

| Company | Department | Average_Salary |
| :---: | :---: | :---: |
| A-Corp | IT | $\$ 55000$ |
| E-Corp | IT | $\$ 63000$ |
| F-Corp | HR | $\$ 56000$ |

Figure 3.: A database with three relations, containing corporate data.

For example, we might assume that dom $_{1}$ (PayScale) is the set of strings over some alphabet, whereas typically $\operatorname{dom}_{3}($ PayScale $)=\mathbb{N}$. For technical reasons, it is convenient to formally let $\operatorname{dom}_{3}($ PayScale $)=\mathbb{R}$ though, in order for it to coincide with the respective parameter domain of the parameterized Gaussian distribution.

### 2.4. Probabilistic Databases

In a nutshell, a probabilistic database (PDB) is a collection of database instances (in the sense of Section 2.3) that is equipped with a probability measure. Throughout this paper, we use the framework of standard probabilistic databases [37] (to be consulted for details). Recall that for a database schema $\mathcal{S}$, we let $\mathbb{F}_{\mathcal{S}}$ denote the set of facts that can be built from $\mathcal{S}$. A basic assumption for standard PDBs is that all attribute domains are standard Borel. Then $\mathbb{F}_{\mathcal{S}}$ is standard Borel as well and we denote its (Borel) $\sigma$-algebra by $\mathfrak{F} s$. The sample space $\mathbb{D}$ of a standard PDB is the set of all database instances over $\mathcal{S}$, that is, finite bags of facts from $\mathbb{F}_{\mathcal{S}}$. We drop the subscript $\mathcal{S}$, if the schema is clear. By a generic construction, $\mathbb{D}$ is equipped with a $\sigma$-algebra $\mathfrak{D}$, turning it into a measurable space. The $\sigma$-algebra $\mathfrak{D}$ is generated by the family of counting events $C(F, n)$ consisting of those instances that contain exactly $n$ facts from $F$, where $F$ is a measurable set of facts.

Definition 2.5. A standard probabilistic database is a probability space $\Delta=(\mathbb{D}, \mathfrak{D}, P)$ where $(\mathbb{D}, \mathfrak{D})$ is the measure space from the construction above.

As we only work with standard PDBs, we omit the term "standard" henceforth.
Fact 2.6 (Measurability of Queries [37]). Relational algebra and aggregate queries are measurable functions on PDBs.

The construction of PDBs sketched before inherently uses bag semantics, meaning that the sample space contains instances with duplicates. For the purpose of this paper, we only want to consider set semantics though. This can either be achieved on the side of measures, i.e. PDBs with almost surely set-valued instances; or by restricting the sample space to the set $\mathbb{D}^{\text {set }}$ of duplicate-free instances from $\mathbb{D}$. Note that $\mathbb{D}^{\text {set }}$ is a measurable subset of $\mathbb{D}$ and, consequentially, $\mathfrak{D}^{\text {set }}:=\mathfrak{D} \upharpoonright_{\mathbb{D}^{\text {set }}}$ a sub- $\sigma$-algebra of $\mathfrak{D}$, that is, $\mathfrak{D}^{\text {set }} \subseteq \mathfrak{D}$. Moreover, $\mathfrak{D}$ is generated by the family of all set-valued counting events $\boldsymbol{C}^{\text {set }}(\boldsymbol{F}, n):=\boldsymbol{C}(\boldsymbol{F}, n) \cap \mathbb{D}^{\text {set }}$ (cf. [62, p. 83]).

Proposition 2.7. For every standard $\operatorname{PDB}(\mathbb{D}, \mathfrak{D}, P)$, the measurable space $(\mathbb{D}, \mathfrak{D})$ is standard Borel, as is its restriction to set instances.

Proof. This is an instantiation of a known result from point process theory and the theory of random measures. We use the notation from [18]. For any standard Borel space $(\mathbb{X}, \mathfrak{X})$, the set $\mathcal{N}_{\mathbf{X}}^{\#}$ of $\mathbb{N} \cup\{\infty\}$-valued measures $\mu$ on $(\mathbb{X}, \mathfrak{X})$ with the property that $\mu(X)<\infty$ for all bounded $\boldsymbol{X} \in \mathfrak{X}$ is a Polish space and its Borel $\sigma$-algebra is generated by the evaluation maps

$$
\operatorname{eval}_{X}: \mathcal{N}_{\mathbf{X}}^{\#} \rightarrow \mathbb{R}: \mu \mapsto \mu(\boldsymbol{X})
$$

where $X \in \mathfrak{X}$ [18, Proposition 9.1.IV]. The subspace $\mathcal{N}_{\mathbf{X}}=\operatorname{eval}_{\mathbf{X}}^{-1}(\mathbb{N})$ of measures of $\mathcal{N}_{\mathbf{X}}^{\#}$ of finite total mass is a measurable subset of $\mathcal{N}_{\mathbf{X}}^{\#}$ and thus, a standard Borel space when equipped
with the corresponding trace $\sigma$-algebra [25] 424G]. It is easy to see that there is a Borel isomorphism between our space $(\mathbb{D}, \mathfrak{D})$ and the space $\left(\mathcal{N}_{\mathbf{X}}, \mathfrak{B o r}\left(\mathcal{N}_{\mathbf{X}}\right)\right)$. Since $\mathbb{D}^{\text {set }}$ is a measurable subset of $\mathbb{D},\left(\mathbb{D}^{\text {set }}, \mathfrak{D} \upharpoonright_{\mathbb{D}^{\text {set }}}\right)$ is standard Borel as well.

With the single exception of the proposition above, all facts we use about standard PDBs in this paper are shown in [37].

Throughout this paper we will exclusively use set instances and set semantics. To simplify notation, we write $(\mathbb{D}, \mathfrak{D})$ instead of $\left(\mathbb{D}^{\text {set }}, \mathfrak{D}^{\text {set }}\right)$ for the measurable space of set instances.

Definition 2.8 (Sub-Probabilistic Databases). If $(\mathbb{D}, \mathfrak{D})$ is the measurable space of a standard PDB, and $P$ is a sub-probability measure on $(\mathbb{D}, \mathfrak{D})$, then $\mathcal{D}=(\mathbb{D}, \mathfrak{D}, P)$ is called a subprobabilistic database.

Let $\mathbb{D}_{\perp}:=\mathbb{D} \uplus\{\perp\}$ and equip this space with the $\sigma$-algebra $\mathfrak{D}_{\perp}:=\mathfrak{D} \cup\{\boldsymbol{D} \cup\{\perp\}: D \in$ $\mathfrak{D}\}$. There is a natural one-to-one correspondence between probability measures on $\left(\mathbb{D}_{\perp}, \mathfrak{D}_{\perp}\right)$ and sub-probabilistic databases on the instance space $(\mathbb{D}, \mathfrak{D})$. A natural interpretation of the "missing" probability mass of a sub-probabilistic database is that it describes the probability of an error event (or the outcome of a draw from the PDB to be undefined). The space $\mathbb{D}_{\perp}$ makes this error event (" $\perp$ ") explicit. Note that with this transformation the results of [37] concerning query measurability directly also apply to sub-probabilistic databases.

### 2.5. Logic and Datalog

We briefly introduce the background from logic that we need throughout the paper. For details, we refer, for example, to [1,50]. Let $\mathcal{S}$ be a relational schema and $\mathbb{U}$ the universe as before. Let $\operatorname{Var} \neq \emptyset$ be a fixed, countably infinite set of variables. As is common in database theory, we do not distinguish between constant symbols and constants from $\mathbb{U}$. An atom is an expression of the shape $R(\vec{u})$ where $R$ is a relation symbol from $\mathcal{S}$ and $\vec{u} \in(\operatorname{Var} \cup \mathbb{U})^{k}$ where $k$ is the arity of $R$. First-order formulas are built from atoms using $\neg, \wedge, \vee, \forall$ and $\exists$. The free variables of $\varphi$ are the variables appearing among its atoms that are not bound by a quantifier. A formula without free variables is called a sentence, or Boolean. We write $\varphi(\vec{x})$ to indicate that $\varphi$ has free variables exactly $\vec{x}$.

Suppose $\vec{x}=\left(x_{1}, \ldots, x_{n}\right)$ is a tuple of variables. A valuation of $\vec{x}$ is a function $\alpha$ mapping every variable $x_{i}$ in $\vec{x}$ to a constant $\alpha\left(x_{i}\right)=a_{i} \in \mathbb{U}$. If $\vec{u}$ is a tuple of variables and constants, and $\alpha$ a valuation of the variables in $\vec{u}$, then $\alpha(\vec{u})$ denotes the tuple obtained by replacing every variable $x$ in $\vec{u}$ with $\alpha(x)$. It is often convenient to identify a valuation $\alpha$ of variables $x_{1}, \ldots, x_{n}$ with the tuple $\vec{a}=\left(a_{1}, \ldots, a_{n}\right)$. If the free variables of $\varphi$ are all contained in the tuple $\vec{x}$, we write $\varphi(\vec{a})$ for the formula that emerges from $\varphi$ by replacing every occurrence of $x_{i}$ with $a_{i}$.
Remark 2.9. Formally, we consider sorted first-order languages. The set of valuations of a single variable $x$ is then given as the intersection of all the attribute domains of the positions where $x$ occurs. For simplicity, we assume that all positions where a variable $x$ occurs are typed equally.

The semantics $\vDash$ of first-order logic are defined in the standard way.

For the introduction of Datalog, we follow [1. Chapter 12], to which we refer the reader for further details. A Datalog rule is a logical expression of the shape

$$
\begin{equation*}
R(\vec{x}) \leftarrow S_{1}\left(\vec{x}_{1}\right), \ldots, S_{m}\left(\vec{x}_{m}\right) \tag{2.2}
\end{equation*}
$$

where $R$ and $S_{i}$ are relation symbols and $\vec{x}, \vec{x}_{i}$ are tuples of variables or constants of the appropriate lengths such that every variable in the tuple $\vec{x}$ appears among the variables of some tuple $\vec{x}_{i}$. The head of the rule (2.2) is $R(\vec{x})$, and the body is $S_{1}\left(\vec{x}_{1}\right), \ldots, S_{m}\left(\vec{x}_{m}\right)$. A Datalog program is a finite set of Datalog rules.

The relation symbols that only occur in the rule bodies of a program are called extensional. The remaining ones (those appearing at least once in a rule head) are called intensional. The extensional (or intensional) relation symbols form the extensional (intensional, resp.) schema of the program. The combined schema consists of both the extensional and intensional relation symbols.

Under the model-theoretic view, a Datalog program $\mathcal{P}$ is a conjunction $\varphi$ of first-order sentences, where all variables in every rule are universally quantified. A model of $\mathcal{P}$ is a database instance over the combined schema satisfying $\varphi$. The input to a Datalog program is a database instance $D$ over the extensional schema. The outcome of $\mathcal{P}$ on $D$ is the minimal model of $\mathcal{P}$ that contains $D$. Such a minimal model always exists, and it contains no constants beyond those present in $D$ and $\mathcal{P}$. It is a superset of $D$ that only contains additional facts from the intensional schema.

Foreshadowing a "generative" point of view, we highlight the equivalent approach to Datalog semantics through fixpoints. Let again $\mathcal{P}$ be a Datalog program and $D$ be database instance over the combined schema. A fact $f$ is an immediate consequence of $D$ subject to $\mathcal{P}$, if there exists a valuation of the variables of a rule such that its body is satisfied in $D$, and such that $f$ is the valuation of the head of the rule. The map that sends any database instance to the set of its immediate consequences is a monotone operator on database instances (with respect to $\subseteq$ ). Then the outcome of $\mathcal{P}$ on $D$ is the minimal fixpoint of the operator. This is unique, and equivalent to the model-theoretic notion of outcome described above. This second definition can be interpreted algorithmically: Given an input instance $D$, at every step in time, we add all immediate consequences to the current database instance, until no further changes occur. This produces exactly the outcome of $\mathcal{P}$ on $D$.

## 3. General Generative Datalog

In this section, we introduce the Generative Datalog language in a version that allows the use of continuous distributions. After specifying the syntax, the main goal of this section is to provide the groundwork for a well-defined semantics of Generative Datalog programs.

## Structure of this section

First of all, in Section 3.1, we establish the syntax of Generative Datalog. For this, we build upon the original syntax of [5] but implement some slight alterations that on the one hand are tailored to the technical developments later on, and on the other hand allow us to overcome
the issues discussed in Section 1.1. In Section 3.2 we set the scene for the later introduction of the semantics, by describing the desired workings of a Generative Datalog program in an abstract, but informal way. The rest of the section prepares key ingredients for the semantics: the translation of Generative Datalog programs into existential Datalog programs (Section3.3), the notion of rule applicability (Section 3.4), the definition and properties of the possible new instances emerging after letting a rule fire (Section 3.5), and a brief discussion of functional dependencies that are induced by the rules of the translated existential Datalog program (Section 3.6). We remark that all of these developments mirror equivalent developments in the original introduction of Generative Datalog in [5], but come with the additional need to discuss various measurability properties due to considering continuous distributions. Based on this, the concrete semantics will be introduced and treated in the sections thereafter.

### 3.1. Syntax of Generative Datalog

We start by introducing the syntax of Generative Datalog programs. We note that, already here, there are slight differences to the version of Bárány et al. that stem from the updated semantics we are going to introduce. At a later point (Section6.2), we will come back to the differences to [5].

We fix two disjoint database schemas $\mathcal{E}$, and $\mathcal{I}$. The schema $\mathcal{E}$ is called the extensional database schema, and $\mathcal{I}$ is called the intensional database schema.

Additionally, we fix a set $\Psi$ of (symbolic names of) parameterized distributions. In order to not worry about which combinations of parameters are "legal", we require that if $\psi$ is a parameterized distribution, then the parameter space $\Pi_{\psi}$ is a full Cartesian product of $m>0$ spaces where $m$ is the number of parameters in $\psi$. All parameterized distributions we would typically want to support (including, in particular, those of Table 3) satisfy this requirement anyway.

The terms of the GDatalog-language (over $\mathcal{E}, \mathcal{I}$ and $\Psi$ ) are defined as follows:

1. All variables and all constants from $\mathcal{E} \cup \mathcal{I}$ are terms. Such terms are called deterministic terms.
2. Let $\psi \in \Psi$ be a parameterized distribution with $\operatorname{pardim}(\psi)=m$, say with parameter space $\Pi_{\psi}=\Pi_{\psi, 1} \times \cdots \times \Pi_{\psi, m}$. Then

$$
\begin{equation*}
\psi\left\langle p_{1}, \ldots, p_{m}\right\rangle \tag{3.1}
\end{equation*}
$$

is a term where $p_{i}$ is either a variable, or a constant in $\Pi_{\psi, i}$. Terms of the shape (3.1) are called probabilistic terms or $\Psi$-terms.

An atom is an expression of the form $R\left(t_{1}, \ldots, t_{n}\right)$ where $n$ is the arity of $R \in \mathcal{E} \cup \mathcal{I}$ and $t_{1}, \ldots, t_{n}$ are terms subject to the following restrictions for all $i=1, \ldots, n$ :

- If $t_{i}=c$ is a constant, then $c \in \operatorname{dom}_{i}(R)$.
- If $t_{i}=\psi\langle\vec{p}\rangle$, then $R \in \mathcal{I}$ and $\mathbb{W}_{\psi} \subseteq \operatorname{dom}_{i}(R)$.

If some term $t_{i}$ is probabilistic, $R\left(t_{1}, \ldots, t_{n}\right)$ is called a probabilistic atom, and a deterministic atom otherwise. If $R \in \mathcal{E}$, then $R\left(t_{1}, \ldots, t_{n}\right)$ is called an $\mathcal{E}$-atom and otherwise, if $R \in \mathcal{I}$, then $R$ is called an $\mathcal{I}$-atom. We emphasize that probabilistic terms are only allowed to occur in $I$-atoms.

Definition 3.1. A GDatalog $[\mathcal{E}, \mathcal{I}, \Psi]$ rule $\varphi$ is an expression

$$
\begin{equation*}
R\left(t_{1}, \ldots, t_{n}\right) \leftarrow S_{1}\left(t_{11}, \ldots, t_{1 n_{1}}\right), \ldots, S_{k}\left(t_{k 1}, \ldots, t_{k n_{k}}\right) \tag{3.2}
\end{equation*}
$$

such that

- $R$ is intensional with $n=\operatorname{ar}(R)$ and $R\left(t_{1}, \ldots, t_{n}\right)$ is an $I$-atom, possibly with $\Psi$-terms;
- $S_{1}, \ldots, S_{k}$ are relation symbols (extensional or intensional) with $n_{i}=\operatorname{ar}\left(S_{i}\right)$ and for all $i=1, \ldots, k, S_{i}\left(t_{i 1}, \ldots, t_{i n_{i}}\right)$ is a deterministic atom; and,
- all variables appearing among $t_{1}, \ldots, t_{n}$ appear in $\left\{t_{i j_{i}}: 1 \leq i \leq k\right.$ and $\left.1 \leq j_{i} \leq n_{i}\right\}$.

Moreover, we require that if $t_{i}=\psi\left\langle p_{1}, \ldots, p_{m}\right\rangle$, and $p_{j}$ is a variable for $j=1, \ldots, m$, then all attribute positions where the variable $p_{j}$ reappears on the right-hand side of (3.2) have the same attribute domain, coinciding with $\Pi_{\psi, j}$.

The last requirement in Definition 3.1 ensures that parameterized distributions cannot be used with malformed parameters. We denote the rule $\varphi$ from (3.2) as

$$
\varphi_{h}\left(\vec{x}_{h}\right) \leftarrow \varphi_{b}(\vec{x})
$$

so that $\varphi_{h}$ is the formula (atom) $R\left(t_{1}, \ldots, t_{n}\right)$ and $\vec{x}_{h}$ is the tuple of variables appearing therein (i. e. $\varphi_{h}$ 's free variables), and, similarly, $\varphi_{b}$ is the conjunction of atoms on the right-hand side of (3.2), with free variables $\vec{x}$. By definition, all variables of $\vec{x}_{h}$ reappear in $\vec{x}$. As in standard Datalog terminology, the formula $\varphi_{h}$ is called the head, and $\varphi_{b}$ is called the body of the rule. A rule $\varphi$ is called deterministic if its head is a deterministic atom, and probabilistic otherwise.

Definition 3.2 (GDatalog Programs). A GDatalog[ $\mathcal{E}, \mathcal{I}, \Psi]$-program is a finite bag $\left\{\left\{\varphi_{1}, \ldots, \varphi_{k}\right\}\right\}$ of GDatalog[ $\left.\mathcal{E}, \mathcal{I}, \Psi\right]$ rules.

The significance of letting a program be a bag rather than a set of rules is that it is our mechanism of sampling multiple times for the same parameters. Every copy of a rule is interpreted as a separate sampling instruction. We expand on this in Sections 3.2 and 6.2
Example 2.4 (continued). We use the database instance shown in Figure3(see Example 2.4) as an input instance for a GDatalog program. Apart from the given extensional schema $\mathcal{E}:=\mathcal{S}=$ \{PartnerOf, Employee, PayScale\}, we let $\mathcal{I}:=\{$ AffilEmployee, Res $\}$. The intended purpose of Res is to store the query result we are interested in, whereas AffilEmployee is just an auxiliary relation. Let $\Psi:=\{$ Gaussian $\}$. Then, for example,

$$
\begin{equation*}
\operatorname{Res}(s, c, \operatorname{Gaussian}\langle\mu, 10000\rangle) \leftarrow \operatorname{Employee}(s, c, d), \operatorname{PayScale}(c, d, \mu) \tag{3.3}
\end{equation*}
$$

is a GDatalog $[\mathcal{E}, \mathcal{I}, \Psi]$ rule. Intuitively, (3.3) is an instruction that defines circumstances under which we should generate new $I$-facts. In this case, we want to sample an income amount for
employees, based on the average income at their workplace. The $\Psi$-term Gaussian $\langle\mu, 10000\rangle$ indicates that the income amount is Normal distributed, parameterized with the average value $\mu$ from the PayScale table and a constant variance 10000.

```
\(\mathcal{G}_{\text {sal }}: \quad \operatorname{AffilEmployee}\left(s, c_{0}, d\right) \leftarrow \operatorname{Employee}\left(s, c_{0}, d\right)\)
    AffilEmployee \((s, c, d) \leftarrow \operatorname{Employee}(s, c, d)\), AffilEmployee \(\left(s^{\prime}, c^{\prime}, d^{\prime}\right)\), PartnerOf \(\left(c, c^{\prime}\right)\)
    AffilEmployee \((s, c, d) \leftarrow\) Employee \((s, c, d)\), AffilEmployee \(\left(s^{\prime}, c^{\prime}, d^{\prime}\right)\), \(\operatorname{PartnerOf}\left(c^{\prime}, c\right)\)
    \(\operatorname{Res}(s, c, \operatorname{Gaussian}\langle\mu, 10000\rangle) \leftarrow \operatorname{AffilEmployee}(s, c, d), \operatorname{PayScale}(c, d, \mu)\)
```

Figure 4.: A GDatalog program for our running example.

Figure 4 shows the GDatalog $[\mathcal{E}, \mathcal{I}, \Psi]$ program $\mathcal{G}_{\text {sal }}$ with three deterministic rules and one probabilistic rule. This program computes tuples ( $s, c, i$ ), such that $s$ is an employee at company $c$ with an annual income of $i$ dollars, where $i$ is sampled from a Gaussian distribution like described before.

Example 3.3. Figure [5 shows the running example ${ }^{3}$ of [5] in our syntax (cf. [5] Figure 3, p. 22:8]).

| $\mathcal{G}_{\text {burglary }}:$ | Earthquake $(c$, Flip $\langle 0.1\rangle) \leftarrow \operatorname{City}(c, r)$ |
| ---: | :--- |
|  | $\operatorname{Unit}(h, c) \leftarrow \operatorname{House}(h, c)$ |
|  | $\operatorname{Unit}(b, c) \leftarrow \operatorname{Business}(b, c)$ |
|  | $\operatorname{Burglary}(x, c, \operatorname{Flip}\langle r\rangle) \leftarrow \operatorname{Unit}(x, c), \operatorname{City}(c, r)$ |
|  | $\operatorname{Trig}(x, \operatorname{Flip}\langle 0.6\rangle) \leftarrow \operatorname{Unit}(x, c), \operatorname{Earthquake}(c, 1)$ |
|  | $\operatorname{Trig}(x, \operatorname{Flip}\langle 0.9\rangle) \leftarrow \operatorname{Burglary}(x, c, 1)$ |
|  | $\operatorname{Alarm}(x) \leftarrow \operatorname{Trig}(x, 1)$ |

Figure 5.: The GDatalog program from the running example of [5] in our syntax.

There, $\mathcal{E}=\{$ City, House, Business $\}$ and $\mathcal{I}=\{$ Earthquake, Unit, Burglary, Trig, Alarm $\}$ with $\Psi=\{$ Flip $\}$. Initially, we have a database instance containing assignments of cities to regions (in City), and of houses and businesses to cities (in House and Business). The program does not further distinguish between houses and businesses, and collects them together as units. With the first rule, we flip a coin, whether city $c$ is struck by an earthquake. Similarly, with the fourth rule, we flip a coin determining whether unit $x$ in city $c$ is burglarized. We assume that every unit is equipped with an alarm system that triggers when someone is trespassing (but may fail to do so). This is captured by rule number six. Yet, an earthquake may also trigger the security system, but with a lower probability, as modeled by rule number five. Finally, when the system is triggered, it sounds the alarm.

From now on, to simplify notation, we assume that our GDatalog programs contain at most one parameterized distribution per rule. Furthermore, we assume that the parameterized distribution (if existing) is invoked in the last attribute of the relation in the rule head. That is, we assume rule heads of probabilistic rules to be of the form $R(\vec{u}, \psi\langle\vec{p}\rangle)$.

[^2]Our proofs generalize to the unrestricted setting. The measurability discussions that follow are not affected by permutations of the attribute positions, and, moreover, for the simultaneous usage of two or more parameterized distributions, the resulting tuples are distributed with their respective product distribution (cf. Fact 2.1).

### 3.2. An Informal Semantics

Before delving into the intricacies of a formal semantics for GDatalog programs, let us explain an informal operational semantics for GDatalog rules and programs. We have already given the intuition of applying a rule in Example 2.4, considering the rule (3.3):

$$
\begin{equation*}
\operatorname{Res}(s, c, \operatorname{Gaussian}\langle\mu, 10000\rangle) \leftarrow \operatorname{Employee}(s, c, d), \operatorname{PayScale}(c, d, \mu) \tag{3.3}
\end{equation*}
$$

Let $\alpha$ be a valuation of $s, c, d, \mu$ such that Employee $(s, c, d)$ and PayScale $(c, d, \mu)$ exist in the input database. This makes the rule applicable for the valuation $\alpha$. Just as in plain Datalog, we generate a fact to add to the current database instance. For this, we sample a random variable $X \sim \operatorname{Gaussian}\langle\alpha(\mu), 10000\rangle$, and generate $\operatorname{Res}(\alpha(s), \alpha(c), X)$.

We run a GDatalog program $\mathcal{G}$ on a database instance over the extensional schema similarly to a normal Datalog program. All intensional relations are initialized to be empty. By repeatedly applying the rules as described above, the program generates (random) facts. All rule applications are stochastically independent. We stipulate that each rule of the program (or more precisely, each occurrence of each rule-remember a program is a bag of rules where a rule may occur several times) can only be applied once for every instantiation of the variables appearing in the head of the rule. The computation terminates if no rule is applicable anymore, and the output consists of the original input, extended by the set of facts generated in the execution, that is, a database instance over the combined (extensional plus intensional) schema ${ }^{4}$ Because of the sampling of values in the rule applications, the output is probabilistic. We interpret it as a probabilistic database. Thus, given a database instance over the extensional schema, a GDatalog program generates a probabilistic database over the combined schema.

However, our informal description of the semantics raises several crucial questions:

1. Does the program always terminate?
2. How can we be sure that the output is indeed a well-defined probabilistic database?
3. In which order do we apply the rules, and does this make a difference?

The answer to Question (1) is simply 'no' (in general), as the following easy example demonstrates.

Example 3.4. Suppose a GDatalog program $\mathcal{G}$ contains the rule

$$
R(\operatorname{Gaussian}\langle\mu, 1\rangle) \leftarrow R(\mu),
$$

[^3]where $R \in I$. Intuitively the program stops, when a value is sampled that we have already seen. For this concrete example, this will happen with probability 0 though. That is, the program almost surely diverges. A similar behavior can already be enforced with deterministic distributions alone, for example with the rule
$$
R(\operatorname{Incr}\langle i\rangle) \leftarrow R(i),
$$
where $\operatorname{Incr}\langle i\rangle$, for parameter $i \in \mathbb{N}$, is a probability mass function on $\mathbb{N}$ with probability 1 on the outcome $i+1$. This program always diverges (provided that the rule fires at all).

The first program of Example 3.4 points at another complicating issue. It may well happen that a program terminates for certain random choices, but does not for others. We resolve this issue by conditioning the output probability distribution on termination, or by saying that a GDatalog program only defines a sub-probabilistic database, where the probability mass of the whole space may be smaller than 1 . The "missing" probability mass is then the probability of divergence.

It is an open research question to understand termination criteria for GDatalog programs. The other two questions are main guiding questions of our paper. The answer to (2) is given by a thorough investigation of the stochastic process sketched above, and regarding (3), we will indeed see that we don't have to worry too much about the order of rule applications in the end.

### 3.3. Translation to Existential Datalog Programs

As in [5], we first introduce a nondeterministic semantics for our programs by translating a given GDatalog program into an existential Datalog program (Datalog ${ }^{\exists}$ program). This is basically the same procedure as in [5] Section 3.2] modulo slight changes that are motivated by the discussions of the previous section.

Intuitively, the probabilistic rules of a GDatalog program introduce attribute values that are the result of some random sampling. In contrast, the rules of a Datalog ${ }^{\exists}$ program may introduce attribute values that are subject to nondeterminism.

A Datalog ${ }^{\exists}$ program is a Datalog program that additionally allows rules of the shape

$$
\exists y: \varphi_{h}\left(\vec{x}_{h}, y\right) \leftarrow \varphi_{b}(\vec{x})
$$

where $\vec{x}$ again contains all variables of $\vec{x}_{h}$. Such rules are called existential rules.
Construction (Associated Datalog ${ }^{\exists}$ Program). Let $\mathcal{G}=\left\{\left\{\varphi_{1}, \ldots, \varphi_{k}\right\}\right\}$ be a GDatalog $[\mathcal{E}, \mathcal{I}, \Psi]$ program. We construct the Datalog ${ }^{\exists}$ program $\mathcal{G}^{\exists}$ as follows. For all $i=1, \ldots, k$ do the following:

1. If $\varphi_{i}$ is a deterministic rule of $\mathcal{G}$, then $\varphi_{i}$ is a rule of $\mathcal{G}^{\exists}$.
2. If $\varphi_{i}$ is a probabilistic rule of $\mathcal{G}$, say

$$
\varphi_{i}(\vec{x})=\left(R(\vec{u}, \psi\langle\vec{p}\rangle) \leftarrow \varphi_{i, b}(\vec{x})\right)
$$

with $\vec{u}=\left(u_{1}, \ldots, u_{\mathrm{ar}(R)-1}\right)$ and $\vec{p}=\left(p_{1}, \ldots, p_{\text {pardim }(\psi)}\right)$ being tuples of variables or constants (such that the variables therein all appear in $\vec{x}$ ), then we add the following two rules to $\mathcal{G}^{\exists}$, where $R_{i}$ is a new, distinguished relation symbol of arity $\operatorname{ar}(R)+\operatorname{pardim}(\psi)$ :

$$
\begin{align*}
& \exists z: R_{i}(\vec{u}, \vec{p}, z) \leftarrow \varphi_{i, b}(\vec{x}) \\
& R(\vec{u}, z) \leftarrow R_{i}(\vec{u}, \vec{p}, z) \tag{3.4}
\end{align*}
$$

We call $\mathcal{G}^{\exists}$ the existential, or Datalog ${ }^{\exists}$ version of $\mathcal{G}$. It inherits the extensional schema $\mathcal{E}^{\exists}:=\mathcal{E}$ from $\mathcal{G}$. The intensional schema $I^{\exists}$ of $\mathcal{G}^{\exists}$ is obtained from $\mathcal{I}$ by adding the new relations $R_{i}$. $\quad \checkmark$

Intuitively, probabilistic rules in the original program $\mathcal{G}$ introduce two new rules (an existential, and a standard one) in $\mathcal{G}^{\exists}$ in order to "decouple" sampling values using parameterized distributions from adding facts to the database. The first rule of (3.4) carries the information which valuation, resp. parametrization, is used for the sampling, and introduces a variable ( $z$ ) storing the sample outcome. With facts produced by the first rule, the second rule specifies how to assemble the tuple with the given "sample outcome". In particular, note the parametrization $\vec{p}$ being projected away. This enables us to sample more than once with different parametrizations $\vec{p}$, without altering the rule applicability in the case where $\vec{p}$ is not contained in $\vec{u}$.
Example 2.4 (continued). Reconsider our running example, and the GDatalog program $\mathcal{G}_{\text {sal }}$ from Figure 4 Its Datalog ${ }^{\exists}$ version $\mathcal{G}_{\text {sal }}^{\exists}$ is shown in Figure 6 below.

```
\(\mathcal{G}_{\text {sal }}^{\exists}: \quad\) AffilEmployee \(\left(s, c_{0}, d\right) \leftarrow\) Employee \(\left(s, c_{0}, d\right)\)
    AffilEmployee \((s, c, d) \leftarrow\) Employee \((s, c, d)\), AffilEmployee \(\left(s^{\prime}, c^{\prime}, d^{\prime}\right)\), PartnerOf \(\left(c, c^{\prime}\right)\)
    AffilEmployee \((s, c, d) \leftarrow\) Employee \((s, c, d)\), AffilEmployee \(\left(s^{\prime}, c^{\prime}, d^{\prime}\right)\), PartnerOf \(\left(c^{\prime}, c\right)\)
    \(\exists z: \operatorname{Res}^{\prime}(s, c, \mu, 10000, z) \leftarrow\) AffilEmployee \((s, c, d)\), PayScale \((c, d, \mu)\)
    \(\operatorname{Res}(s, c, z) \leftarrow \operatorname{Res}^{\prime}(s, c, \mu, 10000, z)\)
```

Figure 6.: The Datalog ${ }^{\exists}$ program associated with the GDatalog program $\mathcal{G}_{\text {sal }}$.
The rules that are typeset in a lighter shade remain unchanged over the original GDatalog program $\mathcal{G}_{\text {sal }}$. The fourth and fifth rules are the new rules introduced by our construction. Therein, Res' is the new relation symbol introduced in the translation of the fourth rule of the original program.

In the following, we prepare the introduction of a chase procedure similar to [5]. With the presence of parameterized distributions with uncountable measure spaces, deriving the probabilistic semantics from $\mathcal{G}^{\exists}$ is more involved than in [5]. In fact, most of their approach towards the construction of a probability space immediately breaks down. There, the authors construct the probability space based on defining the probability of cylinder sets (to be thought of as an initial sequence of generated facts) by multiplying their probabilities. This is only possible, since [5] is restricted to discrete distributions. In particular, after every finite number of steps, their programs have only countably many possible program configurations. In our setting, this might be a continuum, even after just a single step. Therefore, we already need to proceed with additional care regarding the applicability of rules, and the probability distribution induced by
a single program step. In particular, we need more advanced tools from measure theory in order to ensure well-definedness of these concepts. Nevertheless, the whole approach can be thought of as a generalization of ideas already present in [5].

In the next section, we start with a rigorous treatment of the applicability of rules, and the set of result instances after firing a rule.
Remark 3.5. In the remainder of this paper, we predominantly need the existential version $\mathcal{G}^{\exists}$ of our GDatalog program $\mathcal{G}$. Unless explicitly mentioned otherwise, $\varphi$ will denote a rule of $\mathcal{G}^{\exists}$ in the following. Note that unlike the original GDatalog program which is a bag of rules, we can always assume that the constructed program $\mathcal{G}^{\exists}$ is a set of rules $\left\{\varphi_{1}, \ldots, \varphi_{k}\right\}$. The existential rules we created are pairwise different anyway, and for every deterministic rule, we just retain a single copy (for the semantics of existential Datalog, multiple copies have no additional effect).

For simplicity, we always let $\mathcal{G}^{\exists}=\left\{\varphi_{1}, \ldots, \varphi_{k}\right\}$ from now on. In particular, $k$ is to be understood as the number of rules in $\mathcal{G}^{\exists}$.

### 3.4. Rule Applicability

In this subsection, we formalize the notion of rules being enabled to fire in the execution of the program. So far, we have associated an existential Datalog program $\mathcal{G}^{\exists}$ with our original program $\mathcal{G}$. Existential Datalog is already well-established, and has a well-defined semantics [12]. So why do we need to worry about matters of rule applicability in the first place? There are two issues we need to pay attention to:

1. It is not immediately clear, how rule applications adhere to the measurable structure of our underlying spaces of database instances. Quite naturally, multiple rules might be applicable at once, leaving us with the burden to come up with a policy for rule execution. In order to transform the probability measure in a well-defined way, this policy has to be measurable (in a sense that will become clear below).
2. Usually, the choice of policy is not unique. Even if the outcomes of the existential program do not depend on the chosen policy, we still need to argue that different policies do not produce different probability distributions.

We start by formalizing the relevant notions to tackle the first problem, and defer the discussion of the second problem to Section 6

If $\varphi=\varphi(\vec{x})$ is a formula with free variables $\vec{x}$, then we let $\mathbb{V}_{\varphi} \subseteq \mathbb{U}^{|\vec{x}|}$ denote the domain of the valuations of $\vec{x}$, which is the Cartesian product of the attribute domains. The space $\mathbb{V}_{\varphi}$ is naturally equipped with the corresponding product $\sigma$-algebra $\mathfrak{B}_{\varphi}$ obtained from the attribute spaces.

Definition 3.6. Let $\varphi=\varphi(\vec{x}) \in \mathcal{G}^{\exists}$ and let $\vec{u} \in \mathbb{V}_{\varphi}$. Let $D \in \mathbb{D}$ be a database instance. Then $\varphi$ is applicable for valuation $\vec{u}$ if $D \not \vDash \varphi_{h}(\vec{u})$ and $D \vDash \varphi_{b}(\vec{u})$. The rule $\varphi$ is called applicable if such $\vec{u}$ exists.

Suppose $\varphi$ is a rule of $\mathcal{G}^{\exists}$ with $\varphi_{h}=\varphi_{h}\left(\vec{x}_{h}\right)$ and $\varphi_{b}=\varphi_{b}(\vec{x})$. Recall that $\mathbb{V}_{\varphi}$ is the space of valuations of the free variables $\vec{x}$ in the rule. The ground space of $\varphi$ (or, space of head groundings of $\varphi$ ) is defined as follows.

- If $\varphi$ is existential, say $\varphi_{h}\left(\vec{x}_{h}\right)=\exists z: R\left(t_{1}, \ldots, t_{m-1}, z\right)$ for an $m$-ary relation symbol $R$, then $\mathbb{V}_{\varphi}^{(h)}:=\operatorname{dom}_{1}(R) \times \cdots \times \operatorname{dom}_{m-1}(R)$.
- Otherwise, say if $\varphi_{h}\left(\vec{x}_{h}\right)=R\left(t_{1}, \ldots, t_{m}\right)$, then $\mathbb{V}_{\varphi}^{(h)}:=\operatorname{dom}_{1}(R) \times \cdots \times \operatorname{dom}_{m}(R)$.

Again, $\mathbb{V}_{\varphi}^{(h)}$ is equipped with the corresponding product $\sigma$-algebra $\mathfrak{B}_{\varphi}^{(h)}$, which makes $\left(\mathbb{V}_{\varphi}^{(h)}, \mathfrak{B}_{\varphi}^{(h)}\right)$ standard Borel. We consider the function $\pi_{\varphi}: \mathbb{V}_{\varphi} \rightarrow \mathbb{V}_{\varphi}^{(h)}$ that maps valuations of $\vec{x}$ to valuations of the full tuple in the head atom. For example, for the rule $\varphi(x, y)=$ $(\exists z: R(x, x, 1, z) \leftarrow S(x, y), T(y, x))$, we have

$$
\begin{aligned}
\mathbb{V}_{\varphi} & =\operatorname{dom}_{1}(S) \times \operatorname{dom}_{2}(S)=\operatorname{dom}_{2}(T) \times \operatorname{dom}_{1}(T), \\
\mathbb{V}_{\varphi}^{(h)} & =\left\{\left(u_{1}, u_{2}, u_{3}\right): u_{i} \in \operatorname{dom}_{i}(R)\right\} \text { and } \\
\pi_{\varphi}(x, y) & =(x, x, 1) \text { for all }(x, y) \in \mathbb{V}_{\varphi} .
\end{aligned}
$$

Lemma 3.7. For all rules $\varphi$ it holds that $\pi_{\varphi}$ is $\left(\mathfrak{B}_{\varphi}, \mathfrak{B}_{\varphi}^{(h)}\right)$-measurable.
Proof. For all $\varphi, \pi_{\varphi}$ is a composition of the following kinds of functions: projections $(x, y) \mapsto x$, transpositions $(x, y) \mapsto(y, x)$, repetitions $x \mapsto(x, x)$, and appending constants $x \mapsto(x, c)$. All of these are measurable with respect to the corresponding standard Borel product $\sigma$-algebras, and so are their compositions.

Definition 3.8. The set of applicable pairs for an instance $D \in \mathbb{D}$ is given by

$$
\operatorname{App}(D)=\left\{(\varphi, \vec{a}): \varphi \text { applicable for some } \vec{u} \text { with } \pi_{\varphi}(\vec{u})=\vec{a}\right\} .
$$

For any database instance $D$, the set $\operatorname{App}(D)$ tells us which rules are applicable, and under which valuations of their free variables. Later, when we talk about executions of a program, reaching an instance $D$ with $\operatorname{App}(D)=\emptyset$, intuitively means that the program terminates. We are going to analyze the properties of App in order to show that we can select one particular pair from each $\operatorname{App}(D)$ in a measurable way.

Example 2.4 (continued). We consider two artificial examples of database instances $D$ of schema $\left(\mathcal{E}^{\exists} \cup \mathcal{I}^{\exists}\right)$. Suppose that the rules of $\mathcal{G}_{\text {sal }}^{\exists}$ (see Figure 6) are enumerated as $\varphi_{1}, \ldots, \varphi_{5}$, from top to bottom. Thus,

$$
\begin{aligned}
\varphi_{4} & =\varphi_{4}(s, c, d, \mu) \\
& =\left(\exists z: \operatorname{Res}^{\prime}(s, c, \mu, 10000, z) \leftarrow \operatorname{AffilEmployee}(s, c, d), \operatorname{PayScale}(c, d, \mu)\right)
\end{aligned}
$$

and

$$
\begin{aligned}
\varphi_{5} & =\varphi_{5}(s, c, \mu, z) \\
& =\left(\operatorname{Res}(s, c, z) \leftarrow \operatorname{Res}^{\prime}(s, c, \mu, 10000, z)\right)
\end{aligned}
$$

1. Suppose that $D$ contains exactly the facts $\sqrt[5]{ }$

AffilEmployee(981-00-8876, E-Corp, IT),
AffilEmployee(935-00-3912, E-Corp, IT),
PayScale(E-Corp, IT, \$ 63000 ).
Then $\operatorname{App}(D)$ contains exactly the two tuples

$$
\begin{aligned}
& \left(\varphi_{4}, 981-00-8876, \text { E-Corp, } \$ 63000, \$^{2} 10000\right) \text { and } \\
& \left(\varphi_{4}, 935-00-3912, \text { E-Corp, } \$ 63000, \$^{2} 10000\right)
\end{aligned}
$$

Note that the department (for both tuples taking the value "IT") is already projected away, and that the constant " $\left[\$^{2}\right] 10000$ " has been added).
2. Now suppose that $D$ additionally contains the fact

$$
\text { Res }^{\prime}\left(981-00-8876, \text { E-Corp, } \$ 63000, \$^{2} 10000, \$ 62,271\right)
$$

With the presence of the new tuple, $\varphi_{4}$ is no longer applicable for any valuation with

$$
(s, c, \mu) \mapsto(981-00-8876, \text { E-Corp, \$ } 63000)
$$

Yet, with the new tuple, $\varphi_{5}$ is now applicable. In this situation $\operatorname{App}(D)$ contains exactly tuples

$$
\begin{aligned}
& \left(\varphi_{4}, 935-00-3912, \text { E-Corp, } \$ 63000, \$^{2} 10000\right) \text { and } \\
& \left(\varphi_{5}, 981-00-8876, \text { E-Corp, } \$ 62,271\right)
\end{aligned}
$$

We let $(\mathbb{D}, \mathfrak{D})$ be the measurable space of instances associated with the schema $\mathcal{E}^{\exists} \cup \mathcal{I}^{\exists}$ (see Section 2.4). Note that the set $\mathbb{D}_{\text {App }}:=\{D \in \mathbb{D}: \operatorname{App}(D) \neq \emptyset\}$ is measurable in $(\mathbb{D}, \mathfrak{D})$ using Fact[2.6, because the condition $\operatorname{App}(D) \neq \emptyset$ is expressible as a Boolean relational calculus query. We let $\mathfrak{D}_{\text {App }} \subseteq \mathfrak{D}$ denote the trace $\sigma$-algebra of $\mathbb{D}_{\text {App }}$. Formally, App is a multifunction App: $\mathbb{D}_{\text {App }} \rightrightarrows \mathbb{A}=\bigcup_{\varphi \in \mathcal{G}^{\exists}}\left(\{\varphi\} \times \mathbb{V}_{\varphi}^{(h)}\right)$. Therein, $\mathbb{A}$ is naturally equipped with the $\sigma$-algebra $\mathfrak{A}:=\bigoplus_{\varphi \in \mathcal{G}^{\exists}}\left(\{\varphi\} \otimes \mathfrak{B}_{\varphi}^{(h)}\right)$.

Our goal is to show that App is a measurable multifunction, and apply the Theorem of Kuratowski and Ryll-Nardzewski (see Fact A.5) to obtain a measurable selection. Such a measurable selection is the kind of measurable "policy" we sought to obtain.

Lemma 3.9. Let $\boldsymbol{A} \in \mathfrak{A}$. Then $\operatorname{App}^{-1}(\boldsymbol{A})=\left\{D \in \mathbb{D}_{\mathrm{App}}: \operatorname{App}(D) \cap \boldsymbol{A} \neq \emptyset\right\} \in \mathfrak{D}_{\mathrm{App}} \subseteq \mathfrak{D}$. $\triangleleft$
That is, for every measurable set $A$ of potentially applicable pairs, the set of instances where a pair from $A$ really is applicable, is measurable.

[^4]Proof. The function $D \mapsto \operatorname{App}(D)$ can be expressed as a relational algebra view $V$ as follows. For every rule $\varphi(\vec{x})$ there exists a relational algebra query $Q_{\varphi}$ that on input $D$ returns all tuples $\vec{u}$ where $\vec{u}$ is a valuation of the free variables of $\varphi$ making $\varphi$ applicable. Then $\pi_{\varphi} \circ Q_{\varphi}$ (with $\pi_{\varphi}$ being applied pointwise) is a measurable query as well. Our view $V$ finally is the deduplication of $\bigcup_{\varphi \in \mathcal{G}^{\exists}}\{\varphi\} \times\left(\pi_{\varphi} \circ Q_{\varphi}\right)(D)$. Then for all $A \in \mathfrak{A}$ it holds that

$$
\operatorname{App}^{-1}(A)=V^{-1}\left(C(A, 0)^{\mathrm{c}}\right)
$$

where $C(A, 0)^{\text {c }}$ is the set of instances in the associated instance measurable space that contain at least one fact from $A$. Since $V$ is measurable by Fact 2.6, the claim follows.

It easily follows that App is a measurable multifunction on $\mathbb{D}_{\text {App }}$.
Corollary 3.10. There exists a measurable function app: $\mathbb{D}_{\mathrm{App}} \rightarrow \mathbb{A}$ such that for all $D \in \mathbb{D}_{\mathrm{App}}$ it holds that $\operatorname{app}(D) \in \operatorname{App}(D)$.

Subsequently, we use a measurable selection app to resolve the case of multiple rules being applicable in a deterministic way. If multiple rules are applicable (i.e. multiple tuples could be produced, possibly via sampling), the function app selects the rule that is allowed to fire together with the relevant part of the valuation.
Remark 3.11. With the introduction of $\pi_{\varphi}$ into App, we rectify an inaccuracy of the conference version [35]. There, for the sake of simplicity, we did not distinguish valuations $\vec{u}$ that make a rule $\varphi$ applicable, and the resulting tuple $\pi_{\varphi}(\vec{u})$. With the above, we have made this distinction explicit. Technically, the simplification raises no problems due to the measurability of $\pi_{\varphi}$. Yet, possible projections have to be accounted for in the parallel chase procedure. We elaborate on this in the section on the parallel chase (Section 5).

### 3.5. Follow-Up Instances

For upgrading the semantics of Datalog ${ }^{\exists}$ to a probabilistic one (according to the original GDatalog program $\mathcal{G}$ ), we need a measurable correspondence between "intermediate instances" that occur during the execution of the program and all the "follow-up instances" or "extensions" that emerge from such instances by a single rule application. Intuitively, whenever a rule is applicable (that is, its body is satisfied but its head is not), it may fire. If the rule is deterministic, then the ground fact from the head of the rule gets added to the current database instance. If the rule is probabilistic, then the ground fact from the head of the rule gets added with some valuation of the existentially quantified variable and we get a distribution over the follow-up instances according to the parameterized distribution from the original rule. The present section is devoted to formalizing this set-up.

Let $\varphi$ be a rule of $\mathcal{G}^{\exists}$ with ground space $\left(\mathbb{V}_{\varphi}^{(h)}, \mathfrak{B}_{\varphi}^{(h)}\right)$. If $\varphi$ is existential, then its corresponding original rule in $\mathcal{G}$ contains a $\Psi$-term, say using the parameterized distribution $\psi$. We let $\left(\mathbb{W}_{\varphi}, \mathfrak{W}_{\varphi}, \mu_{\varphi}\right):=\left(\mathbb{W}_{\psi}, \mathfrak{W}_{\psi}, \mu_{\psi}\right)$ be the underlying space of $\psi$. The elements of $\mathbb{W}_{\varphi}$ are called the sample outcomes of $\varphi$ and $\left(\mathbb{W}_{\varphi}, \mathfrak{W}_{\varphi}\right)$ the sample space of $\varphi$.
Remark 3.12. For every deterministic rule $\varphi$, we introduce a dummy measure space $\left(\mathbb{W}_{\varphi}, \mathfrak{W}_{\varphi}, \mu_{\varphi}\right)$ with $\mathbb{W}_{\varphi}=\{*\}$ some fixed singleton set, $\mathfrak{B}_{\varphi}$ its powerset, and $\mu_{\varphi}$ the function with $\{*\} \mapsto 1$.

We let $\left(\mathbb{F}_{\varphi}, \mathfrak{F}_{\varphi}\right)=\left(\mathbb{F}_{R}, \mathfrak{F}_{R}\right)$ if $R$ is the relation symbol in the head of $\varphi$. For all existential rules $\varphi$, all $\vec{a} \in \mathbb{V}_{\varphi}^{(h)}$, and all $b \in \mathbb{W}_{\varphi}$, we let $f_{\varphi}(\vec{a}, b) \in \mathbb{F}_{\varphi}$ denote the fact that is obtained by substituting $\vec{a}$ and $b$ into the atom in the head of $\varphi$. We define $f_{\varphi}(\vec{a}, b)$ similarly for deterministic rules, but in this case, the value $b=*$ is just discarded. In particular, for all $\varphi, f_{\varphi}$ is a function $f_{\varphi}: \mathbb{V}_{\varphi}^{(h)} \times \mathbb{W}_{\varphi} \rightarrow \mathbb{F}_{\varphi}$.
Example 2.4 (continued). Consider rule $\varphi_{4}$ of $\mathcal{G}_{\text {sal }}^{\exists}$ (see Figure (6). Let

$$
\vec{a}=\left(981-00-8876, \text { E-Corp, } \$ 63000, \$^{2} 10000\right) \in \mathbb{V}_{\varphi_{4}}^{(h)}
$$

and let $b=\$ 62,271 \in \mathbb{W}_{\varphi_{4}}$ be a sample outcome. Then

$$
f_{\varphi_{4}}(\vec{a}, b)=\operatorname{Res}^{\prime}(981-00-8876, \text { E-Corp, } \$ 63000, \$ 10000, \$ 62,271) .
$$

When $(\varphi, \vec{a})$ is an applicable pair in an instance $D$, then $\varphi$ is applicable for some $\vec{u} \in \pi_{\varphi}^{-1}(\vec{a})$. In that situation, $\varphi$ may fire, which amounts to sampling a value $b \in \mathbb{W}_{\varphi}$, and adding the fact $f_{\varphi}(\vec{a}, b)$ to $D$. We first describe this process of adding facts formally, and regardless of rule applicability. Note that clearly, $f_{\varphi}$ is $\left(\mathfrak{B}_{\varphi}^{(h)} \otimes \mathfrak{B}_{\varphi}, \mathfrak{Y}_{\varphi}\right)$-measurable, as it just prepends the right relation symbol to its argument.

We consider two kinds of extension functions, the sequential extension function and the parallel extension function. The sequential extension function captures the effect of firing a single rule in a database instance. In essence, if $(\varphi, \vec{a})$ is applicable in $D$ and $b$ a possible sample outcome for the parameterized distribution in $\varphi$, the sequential extension function maps $D$ to the instance that is obtained by adding $f_{\varphi}(\vec{a}, b)$ to $D$. The parallel extension function does the same thing, but for the case where multiple rules may fire at once. We will need that both functions obey certain measurability properties. The remainder of this section formally introduces these functions, and establishes various kinds of measurability results that are needed later.

The sequential extension function is defined as follows. Recall that $\mathcal{G}^{\exists}=\left\{\varphi_{1}, \ldots, \varphi_{k}\right\}$. First, let

$$
\mathbb{G}:=\mathbb{D} \times \bigcup_{i=1}^{k}\left(\left\{\varphi_{i}\right\} \times \mathbb{V}_{\varphi}^{(h)} \times \mathbb{W}_{\varphi}\right)=\left\{(D, \varphi, \vec{a}, b): D \in \mathbb{D}, \varphi \in \mathcal{G}^{\exists}, \vec{a} \in \mathbb{V}_{\varphi}^{(h)}, \text { and } b \in \mathbb{W}_{\varphi}\right\}
$$

and let $\mathfrak{G}$ denote the $\sigma$-algebra on $\mathbb{G}$ constructed in the straight-forward way using the disjoint union and product constructions. Then the function ext: $\mathbb{G} \rightarrow \mathbb{D}$ with

$$
\operatorname{ext}(D, \varphi, \vec{a}, b)=D \cup\left\{f_{\varphi}(\vec{a}, b)\right\}
$$

is the sequential extension function where $f_{\varphi}$ is defined as indicated above.
For every tuple $\vec{\ell}:=\left(\ell_{1}, \ldots, \ell_{k}\right) \in \mathbb{N}^{k}$, the parallel extension function with firing configuration $\vec{\ell}$ is defined as follows. Let

$$
\begin{aligned}
\mathbb{G}_{\vec{\imath}} & :=\mathbb{D} \times \prod_{i=1}^{k}\left(\left\{\varphi_{i}\right\} \times\left(\mathbb{V}_{\varphi_{i}}^{(h)} \times \mathbb{W}_{\varphi_{i}}\right)^{\ell_{i}}\right) \\
& \left.=\left\{\left(D,\left(\varphi_{i}, \vec{a}_{i_{j}}, b_{i j}\right)_{i_{j}}\right): D \in \mathbb{D}, \varphi_{i} \in \mathcal{G}^{\exists}, \vec{a}_{i j} \in \mathbb{V}_{\varphi_{i}}^{(h)}, \text { and } b_{i j} \in \mathbb{W}_{\varphi_{i}}\right)\right\} .
\end{aligned}
$$

As above, we equip $\mathbb{G}_{\vec{\ell}}$ with its canonical $\sigma$-algebra $\mathscr{G}_{\vec{\ell}}$. The function $\operatorname{Ext}_{\vec{\ell}}: \mathbb{G}_{\vec{\ell}} \rightarrow \mathbb{D}$ with

$$
\operatorname{Ext}_{\vec{\ell}}\left(D,\left(\varphi_{i}, \vec{a}_{i j_{i}}, b_{i j_{i}}\right)_{i, j_{i}}\right)=D \cup \bigcup_{i=1}^{k} \bigcup_{j_{i}=1}^{\ell_{i}}\left\{f_{\varphi_{i}}\left(\vec{a}_{i j_{i}}, b_{i j_{i}}\right)\right\}
$$

is called a parallel extension function.
Definition 3.13 (Follow-Up Instances). Let $D \in \mathbb{D}$.

1. If $(\varphi, \vec{a})$ is applicable in $D$, then every instance $\operatorname{ext}(D, \varphi, \vec{a}, b)$ with $b \in \mathbb{W}_{\varphi}$ is called a follow-up instance of $D$ with respect to ( $\varphi, \vec{a}$ ) under sequential rule execution.
2. If $\operatorname{App}(D) \neq \emptyset$, then every instance $\operatorname{Ext}_{\vec{\ell}}\left(D,\left(\varphi_{i}, \vec{a}_{i j_{i}}, b_{i j_{i}}\right)\right)$ is called a follow-up instance of $D$ with respect to $\operatorname{App}(D)$ under parallel rule execution, with firing configuration $\vec{\ell}=\left(\ell_{1}, \ldots, \ell_{k}\right), \operatorname{App}(D)=\left\{\left(\varphi_{i}, \vec{a}_{i j_{i}}\right): 1 \leq i \leq k, 1 \leq j_{i} \leq \ell_{i}\right\}$, and $b_{i j_{i}} \in \mathbb{W}_{\varphi_{i}}$.

Note that when discussing follow-up instances $\operatorname{Ext}_{\vec{\ell}}\left(D,\left(\varphi_{i}, \vec{a}_{i j_{i}}, b_{i j_{i}}\right)\right)$, an entry $\ell_{i}=0$ in the configuration $\vec{\ell}$ corresponds to the rule $\varphi_{i}$ not being able to fire in $D$.

For a given instance and fixed pairs $(\varphi, \vec{a})$, the various sets of follow-up instances are measurable.

Lemma 3.14. Let $D \in \mathbb{D}$.

1. For all $\varphi \in \mathcal{G}^{\exists}$ and all $\vec{a} \in \mathbb{V}_{\varphi}^{(h)}$ it holds that

$$
\bigcup_{b \in \mathbb{W}_{\varphi}} \operatorname{ext}(D, \varphi, \vec{a}, b)=: \operatorname{ext}\left(D, \varphi, \vec{a}, \mathbb{W}_{\varphi}\right) \in \mathfrak{D}
$$

2. For all $\vec{\ell}=\left(\ell_{1}, \ldots, \ell_{k}\right) \in \mathbb{N}^{k}$ and all $\varphi_{i} \in \mathcal{G}^{\exists}$ and $\vec{a}_{i j_{i}} \in \mathbb{V}_{\varphi_{i}}^{(h)}$ where $i=1, \ldots, k$ and $j_{i}=1, \ldots, \ell_{i}$ for all $i$, it holds that

$$
\bigcup_{\substack{b_{i j_{i}} \in \mathbb{W}_{\varphi_{i}} \\ \text { for all } i, j_{i}}} \operatorname{Ext}_{\vec{\ell}}\left(D,\left(\varphi_{i}, \vec{a}_{i j_{i}}, b_{i j_{i}}\right)_{i, j_{i}}\right)=: \operatorname{Ext}_{\vec{\ell}}\left(D,\left(\varphi_{i}, \vec{a}_{i j_{i}}, \mathbb{W}_{\varphi_{i}}\right)_{i, j_{i}}\right) \in \mathfrak{D} .
$$

Proof. Consider the first part of the lemma and fix $D \in \mathbb{D}$ and $(\varphi, \vec{a}) \in \operatorname{App}(D)$. Then $f_{\varphi}\left(\vec{a}, \mathbb{W}_{\varphi}\right)=\left\{f_{\varphi}(\vec{a}, b): b \in \mathbb{W}_{\varphi}\right\} \in \mathfrak{F}_{\varphi}$, and it holds that

$$
\operatorname{ext}\left(D, \varphi, \vec{a}, \mathbb{W}_{\varphi}\right)=\left(\bigcap_{f \in D} C(f, 1)\right) \cap C\left(f_{\varphi}\left(\vec{a}, \mathbb{W}_{\varphi}\right), 1\right) \cap C\left(\left(D \cup f_{\varphi}\left(\vec{a}, \mathbb{W}_{\varphi}\right)\right)^{\mathrm{c}}, 0\right)
$$

This is a finite intersection of counting events, so the claim follows.
The second part of the lemma can be shown analogously.
In the remainder of this section, we show that the sequential and parallel extension functions are measurable. Recall that $\mathfrak{F}$ is the $\sigma$-algebra on the space $\mathbb{F}$ of all facts.

Lemma 3.15. It holds that $\{(D, f) \in \mathbb{D} \times \mathbb{F}: f \in D\} \in \mathfrak{D} \otimes \mathfrak{F}$.
Proof. Recall that $\mathbb{F}$ is a Polish space. We fix a compatible Polish metric on $\mathbb{F}$, as well as a countable dense set $\mathbb{F}_{0} \subseteq \mathbb{F}$. Then for all $D \in \mathbb{D}$ and $f \in \mathbb{F}$ it holds that

$$
f \in D \Longleftrightarrow \forall \varepsilon>0 \exists f_{\varepsilon} \in \mathbb{F}_{0}:(D, f) \in C\left(B_{\varepsilon}\left(f_{\varepsilon}\right),>0\right) \times B_{\varepsilon}\left(f_{\varepsilon}\right)
$$

where $B_{\varepsilon}\left(f_{\varepsilon}\right)$ denotes the open ball of radius $\varepsilon$ around $f_{\varepsilon}$. The above equivalence easily translates to a countable combination of products of counting events and open balls. Thus, it follows that $\{(D, f): f \in D\} \in \mathfrak{D} \otimes \mathfrak{F}$.

It follows from Lemma 3.15 and the measurability of $f_{\varphi}$ that

$$
\begin{equation*}
\left\{(D, \varphi, \vec{a}, b) \in \mathbb{G}: f_{\varphi}(\vec{a}, b) \in D\right\} \in \mathfrak{G} \tag{3.5}
\end{equation*}
$$

as it is the intersection of the set from Lemma 3.15 with

$$
\bigcup_{i=1}^{k}\left(\mathbb{D} \times f_{\varphi_{i}}^{-1}\left(\mathbb{F}_{\varphi_{i}}\right)\right)
$$

Proposition 3.16 (Measurability of the Extension Functions).

1. The function ext: $\mathbb{G} \rightarrow \mathbb{D}$ is $(\mathfrak{G}, \mathfrak{D})$-measurable.
2. For all $\vec{\ell} \in \mathbb{N}^{k}$, the function $\operatorname{Ext}_{\vec{\ell}}: \mathbb{G}_{\vec{\ell}} \rightarrow \mathbb{D}$ is $\left(\mathfrak{G}_{\vec{\ell}}, \mathfrak{D}\right)$-measurable.

Proof. Note that (1) is a consequence of (2), since

$$
\operatorname{ext}^{-1}(\mathcal{D})=\operatorname{Ext}_{\vec{e}_{1}}^{-1}(\mathcal{D}) \cup \cdots \cup \operatorname{Ext}_{\vec{e}_{k}}^{-1}(\mathcal{D})
$$

where $\vec{e}_{1}, \ldots, \vec{e}_{k}$ are the $k$ unit vectors. We will only show (2) for the special case of $k=1$ and $\vec{\ell}=(2)$ and indicate in the end how the proof can be generalized to arbitrary $k$ and $\vec{\ell}$.

Let $F \in \mathscr{F}$ and $n \in \mathbb{N}$. It holds that $\left(D, \varphi, \vec{a}, b, \varphi, \vec{a}^{\prime}, b^{\prime}\right) \in \operatorname{Ext}_{(2)}^{-1}(C(F, n))$ if and only if one of the following holds (with $f:=f_{\varphi}(\vec{a}, b)$ and $f^{\prime}:=f_{\varphi}\left(\vec{a}^{\prime}, b^{\prime}\right)$ ):
(i) $D \in C(F, n-2)$ and $f, f^{\prime} \in F$ with $f \neq f^{\prime}$ and $f, f^{\prime} \notin D$.
(ii) $D \in C(F, n-1)$ and

- $f \in \boldsymbol{F}$ but $f^{\prime} \notin \boldsymbol{F}$ and $f \notin D$, or
- $f=f^{\prime} \in F$ and $f=f^{\prime} \notin D$.
(iii) $D \in C(F, n)$ and
- $f, f^{\prime} \notin F$, or
- $f \in F$ but $f^{\prime} \notin F$ and $f \in D$, or
- $f, f^{\prime} \in F$ and $f, f^{\prime} \in D$.

Thus, $\operatorname{Ext}_{(2)}^{-1}(C(F, n))$ is the union of the sets described in the items above. The individual sets are measurable using the measurability of $f_{\varphi}$, by the measurability of (3.5), and by the measurability of the diagonal $\left\{(f, f): f \in \mathbb{F}_{\varphi}\right\}$ in $\mathbb{F}_{\varphi} \times \mathbb{F}_{\varphi}$.

With the same ideas, the proof can be generalized to any firing configuration $\vec{\ell}$ and any number of rules. All that is needed is a similar case distinction for facts $f_{1}, \ldots, f_{m}$ (with $m=$ $\ell_{1}+\cdots+\ell_{k}$ ) over the number of facts of $\boldsymbol{F}$ that are contained in $D$, over the number of facts among $f_{1}, \ldots, f_{m}$ that belong to $F$, and over whether some of the $f_{1}, \ldots, f_{m}$ are equal.

We let $\xi$ and $\Xi_{\vec{\ell}}$ denote the characteristic functions of the graphs of ext and $\mathrm{Ext}_{\vec{\ell}}$, respectively. That is, $\xi: \mathbb{G} \rightarrow\{0,1\}$ is defined by

$$
\xi\left(D, \varphi, \vec{a}, b, D^{\prime}\right)= \begin{cases}1 & \text { if } \operatorname{ext}(D, \varphi, \vec{a}, b)=D^{\prime} \text { and }  \tag{3.6}\\ 0 & \text { otherwise }\end{cases}
$$

and $\Xi_{\vec{\ell}}$ is defined similarly.
Corollary 3.17.

1. The function $\xi$ is $(\mathfrak{5} \otimes \mathfrak{D}, \mathfrak{B o r}(\mathbb{R}))$-measurable.
2. The function $\Xi_{\vec{\ell}}$ is $(\mathfrak{G} \otimes \mathfrak{D}, \mathfrak{B o r}(\mathbb{R}))$-measurable for all $\vec{\ell} \in \mathbb{N}^{k}$.

Proof. By Proposition 3.16 the functions ext and $\mathrm{Ext}_{\vec{f}}$ are measurable. Thus, their graphs are measurable sets in the corresponding product space. Since characteristic functions of measurable sets are measurable, the claim follows.

### 3.6. Induced Functional Dependencies

Following [5], with every existential rule $\varphi$ of $\mathcal{G}^{\exists}$, we associate a functional dependency $\operatorname{FD}(\varphi)$ in the following way. Recall that we assumed that all existential rules have the atom in their head in the format $R(\vec{u}, \vec{p}, z)$ where $z$ is the existentially quantified variable. Suppose $R$ is the relation symbol in the head of $\varphi$ with attributes $A_{1}, \ldots, A_{k}$. Then $\operatorname{FD}(\varphi)$ is the functional dependency $R: A_{1}, \ldots, A_{k-1} \rightarrow A_{k}$. Then this functional dependency intuitively expresses that there is at most one value of the random (resp. existential) attribute when all other attribute values are fixed, cf. [5] p. 22:8].

Recall that $(\mathbb{D}, \mathfrak{D})$ is the measurable space of database instances of schema $\mathcal{E}^{\exists} \cup I^{\exists}$. Input instances are restricted to the schema $\mathcal{E}^{\exists}$. We denote by ( $\mathbb{D}_{\text {in }}, \mathfrak{D}_{\text {in }}$ ) the measurable space of instances over $\mathcal{E}^{\exists}$. Note that $\mathbb{D}_{\text {in }} \subseteq \mathbb{D}$ and $\mathfrak{D}_{\text {in }} \subseteq \mathfrak{D}$.

The following is easy to check using the definitions of App, $f_{\varphi}$, ext and Ext.
Lemma 3.18 (cf. [5. Proposition 4.2]). Let $\varphi$ be an existential rule of $\mathcal{G}^{\exists}$. Then the following holds:

1. Every database instance $D \in \mathbb{D}_{\text {in }}$ satisfies $\mathrm{FD}(\varphi)$.
2. If $D \in \mathbb{D}$ and $\operatorname{App}(D) \neq \emptyset$, then $D$ satisfies $\operatorname{FD}(\varphi)$. Moreover, for all $(\varphi, \vec{a}) \in \operatorname{App}(D)$, and all $b \in \mathbb{W}_{\varphi}$, the follow-up instance $\operatorname{ext}(D, \varphi, \vec{a}, b)$ satisfies $\mathrm{FD}(\varphi)$ as well. Likewise, it holds that $\operatorname{Ext}_{\vec{\ell}}\left(D,\left(\varphi_{i}, \vec{a}_{i j_{i}}, b_{i j_{i}}\right)_{i, j_{i}}\right)$ satisfies $\operatorname{FD}(\varphi)$ where $\operatorname{App}(D)=\left\{\left(\varphi_{i}, \vec{a}_{i j_{i}}\right): 1 \leq i \leq k, 1 \leq\right.$ $\left.j_{i} \leq \ell_{i}\right\}$ and $b_{i j_{i}} \in \mathbb{W}_{\varphi_{i}}$ for all $i, j_{i}$.

This result intuitively means that in the execution of our programs, we will only ever have a single sample outcome for a given instantiation of head variables. This becomes crucial at a later point, as it allows us to show that the computation steps needed to obtain an intermediate instance $D$ from an input instance $D_{\text {in }}$ to the program are unique.

## 4. Sequential Probabilistic Chase

The chase of a GDatalog program $\mathcal{G}$ corresponds to chasing its Datalog ${ }^{\exists}$ version $\mathcal{G}^{\exists}$. We will construct a chase tree for the Datalog ${ }^{\exists} \operatorname{program} \mathcal{G}^{\exists}$, the nodes of which are labeled with database instances, and the edges of which capture applications of rules. Thus, existential rules lead to nodes with multiple (in our case possibly uncountably many!) children. In the countable case, one can label the edges to these children with probabilities according to the probabilistic rule that the existential Datalog ${ }^{\exists}$ rule was constructed from (cf. Section 3.3). This is the approach [5] took.

We follow the general spirit of this approach. However, the edge labeling outlined above is only sufficient for domains that are countably infinite at most. Instead of labeling edges, we label nodes with the probability distribution over their children. Yet, making the distribution explicit in the chase tree is not necessary, as it is implicit from the current instance $D$ and the applicable pair we use. In this section, we formalize this procedure and demonstrate how such chase trees induce a stochastic process on database instances.

Remark 4.1. In the construction of said stochastic process, implicit independence assumptions are made. Intuitively, we want that if multiple samplings occur along a path, then they are stochastically independent, as long as there is no logical dependence between them. This means that random samplings should only depend on the current state of the database where the corresponding rule and instantiation of variables get applicable, and ultimately comes down to the stochastic process being Markov (cf. Section2.1.5).

Note that from a measure-theoretic point of view, there is no need to associate the execution of a Datalog program to a tree as we are going to do. We believe though, that doing so is beneficial for exposing the intuition behind the underlying stochastic process and for emphasizing the connections to the original approach in [5].

## Structure of this section

In Section 4.1 we introduce the central notion of chase steps, capturing the effect of a single rule application from the existential Datalog program, and, in the case of existential rules providing it with a probabilistic structure based upon the parameterized distribution that induced the existential rule. This can be thought of as the continuous generalization of the notion of chase steps from [5]. Chase steps naturally compose into a chase tree, that, in turn, captures the stepwise execution of the whole program. Along the way, we already make some technical
observations concerning these notions, before focusing, in Section 4.2 on paths in the chase tree. These paths correspond to individual runs of the program, including fixed sampling outcomes for the existential rules. Every chase path either leads to a finite output instance (where no rules are applicable anymore), or is infinite. In Section 4.3 we formalize this in terms of a function that maps terminating paths to their output instance, and non-terminating paths to some error event. Our main concern lies in proving that the probability distribution over chase paths can be described in terms of stochastic kernels comprising the individual chase steps. We show this in Section4.4 This establishes that the chase paths are the paths of a Markov process over the space of database instances. Combining this with the tools from Section 4.3 allows us to associate a well-defined output (sub-)probability distribution for our programs.

### 4.1. Chase Steps and Chase Trees

A chase step captures the semantics of applying a single (applicable) rule to an input instance.
Definition 4.2 (Chase Step). A (sequential) chase step for $\mathcal{G}$ is a tuple ( $D, \varphi, \vec{a}, \boldsymbol{E}, \mu$ ), where

- $D$ is a database instance in $\mathbb{D}_{\text {App }}$ (i. e. $\left.\operatorname{App}(D) \neq \emptyset\right)$
- $(\varphi, \vec{a}) \in \operatorname{App}(D)$ is an applicable pair
- $E=\operatorname{ext}\left(D, \varphi, \vec{a}, \mathbb{W}_{\varphi}\right)$ is the set of follow-up instances of $D$ w.r.t. ( $\varphi, \vec{a}$ ) under sequential rule execution, and
- $\mu$ is the probability measure on $\mathfrak{D} \upharpoonright_{E}$ that is defined as follows:
- If $\varphi$ is an existential rule of $\mathcal{G}^{\exists}$, then for all measurable $D \subseteq E$,

$$
\begin{equation*}
\mu(\boldsymbol{D})=\int_{\mathbb{W}_{\varphi}} \xi(D, \varphi, \vec{a}, \cdot, \boldsymbol{D}) \cdot \psi_{\varphi}\langle\vec{a}\rangle(\cdot) d \mu_{\varphi}, \tag{4.1}
\end{equation*}
$$

where $\psi_{\varphi}$ is the parameterized distribution in the rule of $\mathcal{G}$ that $\varphi$ originated from 6

- Otherwise,

$$
\mu(\boldsymbol{D})= \begin{cases}1 & \text { if } \boldsymbol{D}=\{\operatorname{ext}(D, \varphi, \vec{a}, *)\}, \text { and }  \tag{4.2}\\ 0 & \text { if } \boldsymbol{D}=\emptyset\end{cases}
$$

Recall that $\xi(D, \varphi, \vec{a}, b, D)$ is an indicator telling us whether in $D$ an application of the rule $\varphi$ with head grounding $\vec{a}$ and resulting sample $b$ leads to an instance in $D$. Thus, the integral in (4.1) is the total probability of all samples $b$ that lead from $D$ to an instance in $D$ when rule $\varphi$ is fired with head grounding $\vec{a}$.

[^5]

Figure 7.: Illustration of a (sequential) chase step.

We denote a chase step $(D, \varphi, \vec{a}, \boldsymbol{E}, \mu)$ as

$$
D \xrightarrow{(\varphi, \vec{a})}(E, \mu)
$$

and say that the chase step starts in $D$, uses $(\varphi, \vec{a})$, and goes into $E$ with distribution $\mu$. Note that in such a chase step, $E$ and $\mu$ are determined by $D, \varphi$ and $\vec{a}$.

Figure 7 illustrates a sequential chase step $D \xrightarrow{(\varphi, \vec{a})}(E, \mu)$ as a directed tree of depth 1 with root $D$. The children of $D$ are the follow-up instances of $D$ using $(\varphi, \vec{a})$, and the edge from $D$ to a follow-up instance $D^{\prime}$ corresponds to a sample outcome $b \in \mathbb{W}_{\varphi}$. The illustration also insinuates that the transition from $D$ to $D^{\prime}$ is probabilistic. In particular, note that $D$ has uncountably many children if the random variable that is sampled has an uncountable support.

Remark 4.3. Recall from Remark 3.12 that for deterministic rules $\varphi$, we defined $\mathbb{W}_{\varphi}=\{*\}$ for some dummy singleton $\{*\}$. Letting $\psi_{\varphi}\langle\vec{a}\rangle(*)=\mu_{\varphi}(*)=1$ in this case, we can regard (4.2) as a special case of (4.1). This allows us without loss of generality to uniformly treat all the chase step measures that appear later as if they were of shape (4.1).

In Definition 4.2 we covertly claimed that $\mu$ as in (4.1) and (4.2) is a well-defined probability measure. We repay the debt by showing that this is indeed the case.

Lemma 4.4. The function $\mu$ from Definition 4.2 is well-defined, and a probability measure on $\mathfrak{D r}_{E}$.

Proof. Let $D \xrightarrow{(\varphi, \vec{a})}(E, \mu)$ be a sequential chase step. Then by Lemma 3.14 it holds that $E=$ $\operatorname{ext}\left(D, \varphi, \vec{a}, \mathbb{W}_{\varphi}\right) \in \mathfrak{D}$. For fixed, measurable $\boldsymbol{D} \subseteq E$, consider the function

$$
\xi(D, \varphi, \vec{a}, \cdot, D): \mathbb{W}_{\varphi} \rightarrow\{0,1\}: b \mapsto \begin{cases}1 & \text { if } D \cup\left\{f_{\varphi}(\vec{a}, b)\right\} \in D  \tag{4.3}\\ 0 & \text { otherwise }\end{cases}
$$

Then $\xi(D, \varphi, \vec{a}, \cdot, D)$ maps $b$ to 1 if and only if $b$ is in the $(D, \varphi, \vec{a})$-section of $\operatorname{ext}^{-1}(\boldsymbol{D})$. In particular, $\xi(D, \varphi, \vec{a}, \cdot, D)$ is $\left(\mathfrak{W}_{\varphi}, \mathfrak{B o r}\left(\mathbb{R}_{\geq 0}\right)\right)$-measurable, and it follows that

$$
\xi(D, \varphi, \vec{a}, \cdot, D) \cdot \psi_{\varphi}\langle\vec{a}\rangle
$$

(as a product of real-valued, measurable functions) is $\left(\mathfrak{W}_{\varphi}, \mathfrak{B o r}\left(\mathbb{R}_{\geq 0}\right)\right)$-measurable as well. Thus, the function $\mu$, as defined in (4.1) is a well-defined measure. Note that for $\boldsymbol{D}=\boldsymbol{E}$, the function from (4.3) is the constant 1-function. Since $\psi_{\varphi}$ is a parameterized distribution, it follows that $\mu$ is a probability measure.

Given a database instance $D$, we can now argue about sequences of follow-up instances using sequences of chase steps.

$$
\begin{equation*}
\ldots \quad D \xrightarrow{(\varphi, \vec{a})}(\boldsymbol{E}, \mu) \quad D^{\prime} \xrightarrow{\left(\varphi^{\prime}, \vec{a}^{\prime}\right)}\left(\boldsymbol{E}^{\prime}, \mu^{\prime}\right) \quad \ldots \tag{4.4}
\end{equation*}
$$

Here sequences can branch, when the rules that are applied are existential rules of the Datalog ${ }^{\exists}$ version of $\mathcal{G}$ (cf. Figure 7). What we just described is formalized in the notion of chase trees.

Recall that $\mathbb{D}_{\text {App }}$ denotes the set of instances $D$ with $\operatorname{App}(D) \neq \emptyset$, and that $(\mathbb{A}, \mathfrak{H})$ is the space of pairs $(\varphi, \vec{a})$ with $\varphi \in \mathcal{G}^{\exists}$ and $\vec{a} \in \mathbb{V}_{\varphi}^{(h)}$.

Definition 4.5. A measurable chase policy is a measurable function app: $\{D \in \mathbb{D}: \operatorname{App}(D) \neq$ $\emptyset\} \rightarrow \mathbb{A}$ with the property that $\operatorname{app}(D) \in \operatorname{App}(D)$.

By Corollary 3.10, a measurable chase policy always exists (as long as any rule is applicable in some instance at all). Intuitively, if multiple pairs $(\varphi, \vec{a})$ are applicable in an instance, app $(D)$ stipulates which of these is used for the next chase step.

Definition 4.6 (Chase Tree). Let $D_{\text {in }} \in \mathbb{D}_{\text {in }}$ and let app: $\mathbb{D}_{\text {App }} \rightarrow \mathbb{A}$ be a measurable chase policy. The (sequential) chase tree $T_{\text {app }, D_{\text {in }}}$ for input instance $D_{\text {in }}$ with respect to the GDatalog program $\mathcal{G}$ and app is a labeled countable-depth tree $T_{\text {app }, D_{\text {in }}}=(V, E, \Lambda)$ with labeling function $\Lambda$, root node $r \in V$, and the following properties.

1. The root node $r$ is labeled with $D_{\text {in }}$.
2. If $v \in V$ is a leaf node, then it is labeled with an instance $D_{v}$ such that $\operatorname{App}\left(D_{v}\right)=\emptyset$.
3. If $v \in V$ is an inner node, then it is labeled with an instance $D_{v}$ such that $\operatorname{App}\left(D_{v}\right) \neq \emptyset$ and
a) $D_{v} \xrightarrow{\operatorname{app}\left(D_{v}\right)}\left(\boldsymbol{E}_{v}, \mu_{v}\right)$ is a chase step with $\left(\boldsymbol{E}_{v}, \boldsymbol{\mu}_{v}\right)$ as in Definition4.2
b) the function $v^{\prime} \mapsto D_{v^{\prime}}$ is a bijection between the children of $v$ and $\boldsymbol{E}_{v}$.

In essence, a chase tree captures all possible computations of a GDatalog program according to a (measurable) chase policy. See Figure 8 for an illustration. Observe that for all $D_{\text {in }} \in \mathbb{D}_{\text {in }}$, the chase tree $T_{\text {app }, D_{\text {in }}}$ is uniquely determined by app. The tree may contain paths of (countably) infinite length, and it may contain nodes with uncountably many children.

Lemma 4.7. Let $D_{\text {in }} \in \mathbb{D}_{\text {in }}$ be an input instance and let app be a measurable chase policy. Then $v \neq w$ implies $D_{v} \neq D_{w}$ for all nodes $v \neq w$ in the corresponding chase tree $T_{\mathrm{app}, D_{\mathrm{in}}}$.


Figure 8.: Illustration of a sequential chase tree, cf. Figure 7

This is shown using the functional dependencies introduced by the Datalog ${ }^{\exists}$ program. The short proof below is directly transferred from [5] to our setting.

Proof. Let $D_{\text {in }}$ and app be fixed and suppose there exist $v, w \in V\left(T_{\text {app }, D_{\text {in }}}\right)$ with $v \neq w$ such that $D_{v}=D_{w}$. Let $u$ be the least common ancestor of $v$ and $w$ in $T_{\text {app, } D_{\text {in }}}$. In particular, the node $u$ has multiple child nodes. Thus, the rule $\varphi_{u}$ from $\operatorname{app}\left(D_{u}\right)=\left(\varphi_{u}, \vec{a}_{u}\right)$ is existential. Let $v^{\prime}$ and $w^{\prime}$ be the children of $u$ on the path to $v$ respectively $w$. Then $D_{v^{\prime}}=D_{u} \cup\left\{f_{v}\right\}$ and $D_{w^{\prime}}=D_{u} \cup\left\{f_{w}\right\}$ for some $f_{v}, f_{w} \in \mathbb{F}_{\varphi_{u}}$ with $f_{v} \neq f_{w}$. By the setup of $T_{\text {app, } D_{\text {in }}}, D_{u} \subseteq D_{v}=D_{w}$, so $f_{v}, f_{w} \in D_{v}=D_{w}$. This, however, contradicts Lemma 3.18

Note that if the GDatalog program $\mathcal{G}$ only contains discrete distributions, then any chase policy app is trivially measurable. In this case, and modulo our changes to the existential version $\mathcal{G}^{\exists}$, we obtain the same chase trees as [5] (omitting any kind of probability labels).

### 4.2. Chase Paths

For the remainder of the section, we fix an arbitrary input instance $D_{\text {in }} \in \mathbb{D}_{\text {in }}$, and an arbitrary measurable chase sequence app. Our goal is to construct a Markov process on the space of database instances $(\mathbb{D}, \mathfrak{D})$ by embedding the sequential chase tree $T_{\text {app }, D_{\text {in }}}=: T=(V, E, \Lambda)$ into the path space $\left(\mathbb{D}^{\omega}, \mathfrak{D}^{\otimes \omega}\right)$.

We define a binary relation $\vdash_{\text {app }} \subseteq \mathbb{D} \times \mathbb{D}$ (denoted in infix notation) on $\mathbb{D}$ as follows. Let $D \in \mathbb{D}$.

- If $\operatorname{App}(D)=\emptyset$, then $D \vdash_{\text {app }} D^{\prime}$ if and only if $D^{\prime}=D$.
- If $\operatorname{App}(D) \neq \emptyset$, then there exists a unique chase step $D \xrightarrow{\operatorname{app}(D)}(\boldsymbol{E}, \mu)$. Then $D \vdash_{\text {app }} D^{\prime}$ if and only if $D^{\prime} \in E$.

In this section, we drop the subscript and just write $\vdash$ instead of $\vdash_{\text {app }}$.
Lemma 4.8. The relation $\vdash \subseteq \mathbb{D} \times \mathbb{D}$ is measurable, i.e. $\vdash \in \mathfrak{D} \otimes \mathfrak{D}$.
Proof. We decompose $\vdash$ depending on the rule prescribed by the chase policy. First observe

$$
\left\{\left(D, D^{\prime}\right) \in \mathbb{D}^{2}: \operatorname{App}(D)=\emptyset \text { and } D \vdash D^{\prime}\right\}=\operatorname{diag}\left(\mathbb{D}^{2}\right) \cap\left(\mathbb{D} \backslash\left(\mathbb{D}_{\mathrm{App}}\right) \times \mathbb{D}\right) \in \mathfrak{D} \otimes \mathfrak{D}
$$

Now fix a rule $\varphi \in \mathcal{G}^{\exists}$. We conclude the proof by showing

$$
\begin{equation*}
\left\{\left(D, D^{\prime}\right) \in \mathbb{D}^{2}: \operatorname{App}(D) \ni(\varphi, \vec{a})=\operatorname{app}(D) \text { for some } \vec{a} \in \mathbb{V}_{\varphi}^{(h)} \text { and } D \vdash D^{\prime}\right\} \in \mathfrak{D} \otimes \mathfrak{D} \tag{4.5}
\end{equation*}
$$

Recall that $\left(\mathbb{V}_{\varphi}^{(h)}, \mathfrak{B}_{\varphi}^{(h)}\right)$, the space of head groundings of $\varphi$, is standard Borel. Moreover, recall that the space of all facts over $\mathcal{E} \cup \mathcal{I},(\mathbb{F}, \mathfrak{F})$ is standard Borel.

For each of these spaces, we fix compatible Polish metrics, and countable dense sets $\mathbb{V}_{0} \in$ $\mathfrak{B}_{\varphi}^{(h)}$, respectively $\mathbb{F}_{0} \in \mathfrak{F}$. We let $B_{\varepsilon}(x)$ denote the open ball of radius $<\varepsilon$ around an element $x$ in either space.

Then a pair of instances $\left(D, D^{\prime}\right)$ is contained in the set from Equation (4.5) if and only if $D$ has the shape $\left\{f_{1}, \ldots, f_{n}\right\}$ and it holds that $\operatorname{app}(D)=(\varphi, \vec{a})$ for some $\vec{a}$ such that $D^{\prime}=$ $\left\{f_{1}, \ldots, f_{n}\right\} \cup\left\{f_{\varphi}(\vec{a}, b)\right\}$ for some $b \in \mathbb{W}_{\varphi}$. This is the case if and only if there exists some $\varepsilon_{0}>0$ such that for all $\varepsilon \in\left(0, \varepsilon_{0}\right)$ there exists some $\vec{a}_{\varepsilon} \in \mathbb{V}_{0}$, and facts $f_{1, \varepsilon} \ldots, f_{n, \varepsilon} \in \mathbb{F}_{0}$ of distance at least $\varepsilon_{0} / 3$ to each other such that

1. $\operatorname{app}(D)=(\varphi, \vec{a})$ for some $\vec{a} \in B_{\varepsilon}\left(\vec{a}_{\varepsilon}\right)$, i. e. $D \in \operatorname{app}^{-1}\left(\varphi, B_{\varepsilon}\left(\vec{a}_{\varepsilon}\right)\right)$,
2. for all $i=1, \ldots, n$, both $D$ and $D^{\prime}$ contain exactly one fact from $B_{\varepsilon}\left(f_{i, \varepsilon}\right)$,
3. $D$ contains no fact outside of $\bigcup_{i=1}^{n} B_{\varepsilon}\left(f_{i, \varepsilon}\right)$, and
4. $D^{\prime}$ contains no fact outside of $\bigcup_{i=1}^{n} B_{\varepsilon}\left(f_{i, \varepsilon}\right) \cup f_{\varphi}\left(B_{\varepsilon}\left(\vec{a}_{\varepsilon}\right), \mathbb{W}_{\varphi}\right)$ and contains exactly one fact in $f_{\varphi}\left(B_{\varepsilon}\left(\vec{a}_{\varepsilon}\right), \mathbb{W}_{\varphi}\right)$.

Note that in the condition above, we have that $\bigcap_{\varepsilon \in\left(0, \varepsilon_{0}\right)} f_{\varphi}\left(B_{\varepsilon}\left(\vec{a}_{\varepsilon}\right), \mathbb{W}_{\varphi}\right)=f_{\varphi}\left(\vec{a}, \mathbb{W}_{\varphi}\right)$, and that $f_{\varphi}\left(B_{\varepsilon}\left(\vec{a}_{\varepsilon}\right), \mathbb{W}_{\varphi}\right) \in \mathfrak{F}_{\varphi} \subseteq \mathfrak{F}$.

Our condition then describes a measurable set in $\mathfrak{D} \otimes \mathfrak{D}$, as it can be written as a countable intersection over $\varepsilon=1 / k$, small enough, of counting events and preimages of app.

The relation $\vdash$ is our vehicle for embedding the chase tree $T$ into $\left(\mathbb{D}^{\omega}, \mathfrak{D}^{\otimes \omega}\right)$. Note that every edge $(v, w) \in E$ corresponds to $D_{v} \vdash D_{w}$. The relation $\vdash$, however contains "loops" $D_{v} \vdash D_{v}$ for every leaf $v$ of $T$. Moreover, as the chase tree starts with the fixed root label $D_{\mathrm{in}}$, there may be pairs of instances $\left(D, D^{\prime}\right)$ with $D \vdash D^{\prime}$ that do not appear as instance labels in the tree $T$ at all. We only achieve a direct correspondence between $\vdash$ and the edge relation $E$ of $T$, once we restrict the first component of $\vdash$ to the set of inner node labels, i. e. to $\left\{D_{v}: v \in V: \operatorname{App}\left(D_{v}\right) \neq\right.$ $\emptyset\}$. Let

$$
\begin{aligned}
\text { paths }(\mathrm{app}) & :=\left\{\left(D_{0}, D_{1}, \ldots\right) \in \mathbb{D}^{\omega}: D_{i} \vdash D_{i+1} \text { for all } i \in \mathbb{N}\right\} \text { and } \\
\text { paths }\left(\operatorname{app}, D_{\text {in }}\right) & :=\left\{\left(D_{0}, D_{1}, \ldots\right) \in \text { paths }(\operatorname{app}): D_{0}=D_{\text {in }}\right\} .
\end{aligned}
$$

The elements of paths (app) are called the paths of app and the elements of paths(app, $D_{\text {in }}$ ) are called the paths of app starting in $D_{\text {in }}$. Note that paths(app) is the set of paths in $T$ where all finite maximal paths have been extended to infinite sequences by repeating the label of the leaf node infinitely often.

While the full path space $\left(\mathbb{D}^{\omega}, \mathfrak{D}^{\otimes \omega}\right)$ contains paths completely unrelated to app (and $D_{\text {in }}$ ), the sets paths $(\mathrm{app})$ and paths $\left(\mathrm{app}, D_{\mathrm{in}}\right)$ only contain relevant paths for the given GDatalog program and chase policy.
Remark 4.9. We introduce two separate notions, because we consider two scenarios: the single input instance scenario, where we evaluate a GDatalog program for a given input instance $D_{\text {in }} \in \mathbb{D}_{\text {in }}$; and the PDB input scenario, where the input is already a probability distribution over database instances. Technically, the former can be cast as a special case of the latter. The single input instance scenario however, is the one originally described by Bárány et al. [5], and we thus prefer to give it an explicit treatment.

Using a pairwise intersection of $卜$-pairs in $\mathbb{D}^{\omega}$, we immediately obtain the following from Lemma 4.8

Corollary 4.10. The sets paths(app) and paths $\left(\operatorname{app}, D_{\text {in }}\right)$ are measurable in $\left(\mathbb{D}^{\omega}, \mathfrak{D}^{\otimes \omega}\right)$. $\triangleleft$
This concludes the embedding of chase trees into the path space ( $\mathbb{D}^{\omega}, \mathfrak{D}^{\otimes \omega}$ ). We are not interested in the paths themselves though. Intuitively, finite paths in the chase trees need to be mapped back to database instances.

### 4.3. Limit Instances

Recall Figure 8 By now, we have described how a GDatalog program spans a "computation tree" of database instances. Yet (as discussed in Example 3.4 some of these computations (i. e. paths in the tree) are of infinite length.

The definitions of Bárány et al. [5] also cover infinite paths with their (per path) probability. We choose a different approach and ignore infinite paths altogether, which we justify by the following two reasons.

Infinite paths correspond to infinite results of the computation and such "infinite instances" are not captured by our framework of standard PDBs.

For practical means, arguably any interest lies on finite results. In fact, Bárány et al. put a strong focus on a restriction to their programs that guarantees all computation paths to be finite. Yet, we include in our discussion programs that terminate with a positive probability $\leq 1$. In probabilistic programming, it is common practice to consider programs with infinite computation paths and analyze the termination behavior of such [9 14]. For example, a GDatalog program with infinite paths is as good as a completely finite one if the probability of all infinite computation is 0 .

With the above, we motivate how we wrap up our semantics of GDatalog with continuous distributions. Further discussion on semantic properties and a glimpse into termination behavior is given in Section 6


Figure 9.: Terminating and non-terminating paths in the sequential chase tree.

A path $\vec{D}=\left(D_{0}, D_{1}, D_{2}, \ldots\right) \in \mathbb{D}^{\omega}$ is called terminating after $i$ steps if $D_{i}=D_{j}$ for all $j>i$, but $D_{i} \neq D_{i-1}$ (in the case $i>0$ ). We call a path $\vec{D}$ terminating if such an $i$ exists. The sets of such paths are easily seen to be measurable in $\left(\mathbb{D}^{\omega}, \mathfrak{D}^{\otimes \omega}\right)$. Figure 9 shows our intended mapping from paths to instances. A special error event $\perp$ is used as a sink for non-terminating paths.

We let $\mathbb{D}_{\perp}:=\mathbb{D} \cup\{\perp\}$ denote the augmented instance space with the additional error event, and $\mathfrak{D}_{\perp}=\mathfrak{D} \oplus\{\emptyset,\{\perp\}\}$ its $\sigma$-algebra. Our mapping between $\mathbb{D}^{\omega}$ and $\mathbb{D}_{\perp}$ is defined as follows

$$
\lim _{\text {app }}(\vec{D}):= \begin{cases}D_{i} & \text { if } \vec{D} \in \text { paths }(\text { app }) \text { and } \vec{D} \text { is terminating at position } i,  \tag{4.6}\\ \perp & \text { otherwise. }\end{cases}
$$

If $\lim _{\text {app }}(\vec{D}) \neq \perp$, then $\lim _{\text {app }}(\vec{D})$ is called the limit instance of $\vec{D}$. Just $\operatorname{like}^{\lim } \mathrm{app}_{\text {ap }}$, we define $\lim _{\text {app }, D_{\text {in }}}$ by using paths (app, $D_{\text {in }}$ ) instead of paths (app) in (4.6). If $\lim _{\text {app }, D_{\text {in }}}(\vec{D}) \neq \perp$, then $\lim _{\text {app }, D_{\text {in }}}(\vec{D})$ is called the limit instance of $\vec{D}$ in $T_{\text {app }, D_{\text {in }}}$. We emphasize that the effect of $\lim _{\text {app }}$ is indeed that terminating chase paths are mapped to the instance they terminate in, whereas all infinite chase paths are collectively mapped onto the error event $\perp$.

Lemma 4.11. Both $\lim _{\text {app }}$ and $\lim _{\text {app }, D_{\text {in }}}$ are bimeasurable.
Proof. We only show the assertions for $\lim _{\text {app }}$. The corresponding results for $\lim _{\text {app }, D_{\text {in }}}$ are obtained the same way.

First note that for all $\vec{D} \in \mathbb{D}^{\omega}$ it holds that $\lim _{\text {app }}(\vec{D})=\perp$ if and only if either $\vec{D} \notin$ paths(app), or $\vec{D}$ is not terminating. Thus, $\lim _{\text {app }}^{-1}(\{\perp\}) \in \mathfrak{D}^{\otimes \omega}$.

For $D \in \mathfrak{D}$, it holds that $\vec{D} \in \lim _{\text {app }}^{-1}(D)$ if and only if there exists $i \in \mathbb{N}$ such that $\vec{D}$ is a path of app terminating at position $i$ with the property that $\vec{D} \in \pi_{i}^{-1}(D)$ (i. e. $\pi_{i}(\vec{D}) \in D$ ), where
$\pi_{i}$ is the (measurable) projection to the $i$ th coordinate. Thus, $\lim _{\text {app }}^{-1}(D) \in \mathfrak{D}^{\otimes \omega}$, and together with the above, $\lim _{\text {app }}$ is $\left(\mathfrak{D}^{\otimes \omega}, \mathfrak{D}_{\perp}\right)$-measurable.

It remains to prove that $\lim _{\mathrm{app}}$ maps measurable sets to measurable sets. Lemma 4.7 implies that $\lim _{\text {app }}$ is injective on $\lim _{\text {app }}^{-1}(\mathbb{D})$. The lemma only applies to the chase tree for a fixed input instance $D_{\mathrm{in}}$, but as the input instance remains part of any subsequent instance in the evaluation of a Datalog program, it directly extends to all input instances. As $\lim _{\text {app }}$ is measurable, so is its restriction to $\lim _{\text {app }}^{-1}(\mathbb{D}) \in \mathfrak{D}^{\otimes \omega}$ (with respect to $\mathfrak{D}^{\otimes \omega} \Gamma_{\text {limapp }}^{-1}(\mathbb{D})$. Since $(\mathbb{D}, \mathfrak{D})$ and $\left(\mathbb{D}^{\omega}, \mathfrak{D}^{\otimes \omega}\right)$ are standard Borel (Proposition 2.7), the assertion follows from Fact A.4

### 4.4. Chase Trees as Markov Processes

In this subsection, we establish a correspondence between a chase tree for a given GDatalog program and a discrete-time Markov process whose state space is the (in general not countable) space of database instances. We have seen in the previous subsection how paths in a chase tree naturally correspond to a set of paths of such a process (cf. Section 2.1.5) in the countably infinite product space $\left(\mathbb{D}^{\omega}, \mathfrak{D}^{\otimes \omega}\right)$.

To obtain the correspondence to a Markov process, we need to show that the probabilistic transitions that are encoded within the nodes of any level of the chase tree, or, to be more precise, by its measurable chase policy, describe a stochastic kernel from ( $\mathbb{D}, \mathfrak{D}$ ) to itself.

The interpretation of the GDatalog semantics as a database-valued Markov process (which, by itself, was already recognized in [5, p. 22:14]) makes also apparent that the natural generalization of the GDatalog language is to allow the input to be a (sub-)probabilistic database rather than a single instance. A GDatalog program then induces a mapping from a (sub-)probabilistic database to a sub-probabilistic database ("losing" the mass of "non-terminating" paths). We will come back to this at the end of the subsection.

We extend $\vdash_{\text {app }}$ to a function $\kappa_{\vdash_{\text {app }}}: \mathbb{D} \times \mathfrak{D} \rightarrow[0,1]$ where again, if the reference is clear, we just write $\kappa_{\vdash}$. For $D \in \mathbb{D}$ and $D \in \mathfrak{D}$ we distinguish two cases.

- If $\operatorname{App}(D) \neq \emptyset$ with $\operatorname{app}(D)=(\varphi, \vec{a})$ such that $D \xrightarrow{(\varphi, \vec{a})}(\boldsymbol{E}, \mu)$ is the corresponding chase step, then $\kappa_{\vdash}(D, \boldsymbol{D}):=\mu(\boldsymbol{D} \cap \boldsymbol{E})=\int \xi(D, \varphi, \vec{a}, \cdot, \boldsymbol{D} \cap \boldsymbol{E}) \cdot \psi_{\varphi}\langle\vec{a}\rangle d \mu_{\varphi}$.
- If $\operatorname{App}(D)=\emptyset$, we let $\kappa_{\vdash}(D, \boldsymbol{D})=\iota(D, \boldsymbol{D})$ where $\iota$ is the identity kernel.

Intuitively, $\kappa_{\vdash}(D, D)=\operatorname{Pr}(D \vdash D)$ with the latter referring to the probability space for the chase step starting in $D$. The following proposition resolves the main technical obstacle for turning measurable chase policies and sequential chase trees into Markov processes.

Proposition 4.12. $\kappa_{\vdash}$ is a stochastic kernel.
Proof. Clearly, $\kappa_{\vdash}(D, \cdot)$ is a probability measure for all $D \in \mathbb{D}$. The complicated part of the proof is establishing that $\kappa_{\vdash}(\cdot, \boldsymbol{D})$ is $\left(\mathfrak{D}_{\varphi}, \mathfrak{B o r}[0,1]\right)$-measurable for all $D \in \mathfrak{D}$.

Let $D \in \mathfrak{D}$ be fixed. It suffices to show for all rules $\varphi$ that $\kappa_{\vdash}(\cdot, D)$ is $\left(\mathfrak{D}_{\varphi}, \mathfrak{B o r}[0,1]\right)$ measurable where $\left(\mathbb{D}_{\varphi}, \mathfrak{D}_{\varphi}\right)$ is the restriction of $(\mathbb{D}, \mathfrak{D})$ to the instances $D$ with $\operatorname{App}(D) \neq \emptyset$ and $\operatorname{app}(D)=(\varphi, \vec{a})$ for some $\vec{a} \in \mathbb{V}_{\varphi}^{(h)}$. Note that the corresponding statement for the restriction to the set of instances $D$ with $\operatorname{App}(D)=\emptyset$ clearly holds, as $\kappa_{\vdash}$ is the identity kernel in this case.

Thus, let $\varphi$ be a fixed rule of $\mathcal{G}^{\exists}$. Without restriction (see the discussion below Definition 4.2), we always treat $\varphi$ as an existential rule. By Fact 2.3 , the function $\kappa_{1}: \mathbb{V}_{\varphi}^{(h)} \times \mathfrak{B}_{\varphi} \rightarrow[0,1]$ with

$$
\kappa_{1}(\vec{a}, \boldsymbol{B}):=\int_{B} \psi_{\varphi}\langle\vec{a}\rangle d \mu_{\varphi}
$$

is a stochastic kernel from $\mathbb{V}_{\varphi}^{(h)}$ to $\mathbb{W}_{\varphi}$. Thus, the function $\kappa_{2}:\left(\mathbb{D}_{\varphi} \times \mathbb{V}_{\varphi}^{(h)}\right) \times \mathfrak{W}_{\varphi} \rightarrow[0,1]$ with

$$
\kappa_{2}(D, \vec{a}, B):=\kappa_{1}(\vec{a}, \boldsymbol{B})
$$

is a stochastic kernel from $\mathbb{D}_{\varphi} \times \mathbb{V}_{\varphi}^{(h)}$ to $\mathbb{W}_{\varphi}$. The function $\xi(\cdot, \varphi, \cdot, \cdot, D): \mathbb{D}_{\varphi} \times \mathbb{V}_{\varphi}^{(h)} \times \mathbb{W}_{\varphi} \rightarrow$ $\{0,1\}$ with

$$
\xi(\cdot, \varphi, \cdot, \cdot, D):(D, \vec{a}, b) \mapsto \xi(D, \varphi, \vec{a}, b, D)
$$

is $\left(\mathfrak{D}_{\varphi} \otimes \mathfrak{B}_{\varphi}^{(h)} \otimes \mathfrak{W}_{\varphi}, \mathfrak{B} \mathfrak{p r}[0,1]\right)$-measurable, as it is the characteristic function of the $\varphi$-section of $\operatorname{ext}^{-1}(\boldsymbol{D})$, and $\operatorname{ext}^{-1}(\boldsymbol{D}) \in \mathfrak{D}_{\varphi} \otimes \mathfrak{B}_{\varphi}^{(h)} \otimes \mathfrak{B}_{\varphi}$ by Proposition 3.16. Using Fact A.2 for $\xi(\cdot, \varphi, \cdot, \cdot, D)$ and $\kappa_{2}$, the function

$$
g: \mathbb{D}_{\varphi} \times \mathbb{V}_{\varphi}^{(h)} \rightarrow[0,1]:(D, \vec{a}) \mapsto \int_{\mathbb{W}_{\varphi}} \xi(D, \varphi, \vec{a}, \cdot, \boldsymbol{D}) \cdot \psi_{\varphi}\langle\vec{a}\rangle d \mu_{\varphi}
$$

is $\left(\mathfrak{D}_{\varphi} \otimes \mathfrak{B}_{\varphi}^{(h)}, \mathfrak{B} \mathfrak{p r}[0,1]\right)$-measurable. Observe that the function $h: \mathbb{D}_{\varphi} \rightarrow \mathbb{D}_{\varphi} \times \mathbb{V}_{\varphi}^{(h)}$ with

$$
h(D):=(D, \vec{a})
$$

is $\left(\mathfrak{D}_{\varphi}, \mathfrak{D}_{\varphi} \otimes \mathfrak{B}_{\varphi}^{(h)}\right)$-measurable by the measurability of app, and using Fact A.1 Thus, for all $x \in[0,1]$, it follows that

$$
\kappa_{\vdash}(\cdot, D)^{-1}[0, x)=\left\{D \in \mathbb{D}: \kappa_{\vdash}(D, D)<x\right\}=\left(h^{-1} \circ g^{-1}\right)[0, x) \in \mathfrak{D}
$$

entailing the claim.
Proposition 4.12 directly implies the following by Kolmogorov's existence theorem (Fact A.8).

Corollary 4.13. Let app be a measurable chase policy and let $(\mathbb{D}, \mathfrak{D}, P)$ be a sub-probabilistic database. Then there exists a Markov process with state space $(\mathbb{D}, \mathfrak{D})$, with initial distribution $P$ and with transition kernels $\kappa_{\vdash}$.

Note that every sub-probability measure $\vec{P}$ on the path space $\left(\mathbb{D}^{\omega}, \mathfrak{D}^{\otimes \omega}\right)$ defines a pushforward sub-probability measure $\vec{P} \circ \lim _{\text {app }}^{-1}\left(\right.$ or $\vec{P} \circ \lim _{\text {app }, D_{\text {in }}}^{-1}$, respectively) on $\left(\mathbb{D}_{\perp}, \mathfrak{D}_{\perp}\right)$. Thus, every Markov process like in Corollary 4.13 defines an output sub-probabilistic database. The semantics of our GDatalog program $\mathcal{G}$, finally, is said output.

Theorem 4.14. Let app be a measurable chase policy.

1. For all $D_{\mathrm{in}} \in \mathbb{D}_{\mathrm{in}}$, the program $\mathcal{G}$ on input $D_{\mathrm{in}}$ defines a sub-probabilistic database $\mathcal{G}_{\text {app }}\left(D_{\text {in }}\right)$ with respect to app.
2. For all sub-probabilistic databases $\mathcal{D}=\left(\mathbb{D}_{\mathrm{in}}, \mathfrak{D}_{\mathrm{in}}, P\right)$, the program $\mathcal{G}$ on input $\mathcal{D}$ defines $a$ sub-probabilistic database $\mathcal{G}_{\text {app }}(\mathcal{D})$ with respect to app.

For the first part of the above theorem, we let the initial distribution of Corollary 4.13 be the Dirac one on the instance $D_{0} \square^{7}$ For the second part, the initial distribution is the subprobability distribution of the input sub-probabilistic database. Note that even if the input $\mathcal{D}$ is a probabilistic database (i.e. $P$ has total mass 1 ), the output $D_{\text {app }}$ may be a proper subprobabilistic database.
Remark 4.15. In the end, we might want to get rid of the auxiliary relations that were created in the translation to the Datalog ${ }^{\exists}$ program. This can be done in a measurable way by a relational algebra view (cf. Fact 2.6), yielding again a sub-probabilistic database.

## 5. Parallel Probabilistic Chase

We obtain another variant of the chase procedure if we allow all applicable rules to fire simultaneously. This notion of parallel chase is of interest as it is not depending on having a measurable chase policy at hand.
Remark 4.1 (continued). Recall our discussion of independence assumptions and the Markov property from the beginning of Section 4 . As there, the parallel chase has independent sampling in the absence of logical dependencies. This is, in particular, the case for multiple rules firing in parallel in a single parallel step.

In this section, we again fix a GDatalog program $\mathcal{G}$, with its Datalog ${ }^{\exists}$ version $\mathcal{G}^{\exists}$, and assume that $\mathcal{G}^{\exists}=\left\{\varphi_{1}, \ldots, \varphi_{k}\right\}$.

## Structure of this section

In the following, we introduce the parallel chase for the GDatalog language (Section5.1) and construct a Markov process (Section 5.2) as in the sequential case. Most of the definitions and results are modest extensions of their counterparts in Section 4 thus allowing a briefer presentation.

### 5.1. Parallel Chase Steps and the Parallel Chase Tree

If $D$ is a database instance, the firing configuration of $D$ is the tuple $\vec{\ell}(D)=\left(\ell_{1}, \ldots, \ell_{k}\right) \in \mathbb{N}^{k}$ where $\ell_{i}=\left|\left\{\vec{a}:\left(\hat{\varphi}_{i}, \vec{a}\right) \in \operatorname{App}(D)\right\}\right|$ for all $1 \leq i \leq k$. Note that the set $\mathbb{D}_{\vec{\ell}}$ of database instances having a fixed firing configuration $\vec{\ell}$ is measurable in $(\mathbb{D}, \mathfrak{D})$, since we know that App corresponds to an Relational Calculus view and the cardinalities in question can be obtained by a counting aggregation. This yields a measurable mapping by Fact 2.6

Definition 5.1 (Parallel Chase Step). A parallel chase step for $\mathcal{G}$ is a tuple $(D, \operatorname{App}(D), \boldsymbol{E}, \mu)$ where

[^6]- $D$ is a database instance in $\mathbb{D}_{\text {App }}$, say with firing configuration $\vec{\ell}=\left(\ell_{1}, \ldots, \ell_{k}\right)$ such that

$$
\operatorname{App}(D)=\left\{\left(\varphi_{i}, \vec{a}_{i_{i}}\right): 1 \leq i \leq k \text { and } 1 \leq j_{i} \leq \ell_{i}\right\} .
$$

- $\boldsymbol{E}$ is the event

$$
E=\operatorname{Ext}_{\vec{l}}\left(D, X_{11}, \ldots, X_{1 \ell_{1}}, \ldots, X_{k 1}, \ldots, X_{k \ell_{k}}\right)
$$

with $X_{i j_{i}}=\left\{\left(\varphi_{i}, \vec{a}_{i j_{i}}, b\right): b \in \mathbb{W}_{\varphi_{i}}\right\}$, and

- $\mu$ is the probability measure on $\mathfrak{D} \upharpoonright_{E}$ that is defined by

$$
\begin{equation*}
\mu(\boldsymbol{D})=\int_{\mathbb{W}} \Xi_{\vec{\ell}}\left(D, \varphi_{1}, \vec{a}_{11}, b_{11}, \ldots, \varphi_{k}, \vec{a}_{k \ell_{k}}, b_{k \ell_{k}}, \boldsymbol{D}\right) \cdot \prod_{i=1}^{k} \prod_{j_{i}=1}^{\ell_{i}} \psi_{i}\left\langle\vec{a}_{i j_{i}}\right\rangle\left(b_{i j_{i}}\right) d \mu^{\otimes} \tag{5.1}
\end{equation*}
$$

for all measurable $\boldsymbol{D} \subseteq E$, where $\psi_{i}$ is the parameterized distribution in the rule of $\mathcal{G}$ that $\varphi_{i}$ originated from, $\mathbb{W}=\prod_{i=1}^{k} \mathbb{W}_{\varphi_{i}}^{\ell_{i}}$ and $\mu^{\otimes}$ is the product measure $\mu^{\otimes}=\bigotimes_{i=1}^{k} \mu_{\psi_{i}}^{\otimes \ell_{i}}$ on $\mathbb{W}$ with $\mu_{\psi_{i}}$ being the measure underlying the parameterized distribution $\psi_{i}$ (cf. Section [2.2).

Note that as before (see Section 4.1), we use a dummy integration for non-existential rules to enable the unified expression of (5.1). Then, we again concentrate on the existential rules and interpret the deterministic ones as special cases (cf. Remark 4.3).

Remark 5.2. Note that by definition of $E x t_{\vec{\ell}}$, the case where multiple deterministic rules with the same left-hand sides are applicable, is nicely resolved, and the corresponding fact is only added once in the extension, i.e., the results from applications of deterministic rules may collapse.

Recall that it may well happen that $\operatorname{App}(D)$ contains multiple pairs with first component an existential rule $\varphi$, for example $(\varphi, \vec{a})$ and $\left(\varphi, \vec{a}^{\prime}\right)$. However, for any related sample outcomes $b$ and $b^{\prime}$ it holds that $f_{\varphi}(\vec{a}, b) \neq f_{\varphi}\left(\vec{a}^{\prime}, b^{\prime}\right)$ because $\vec{a} \neq \vec{a}^{\prime}$. That is, if $\varphi_{i}$ is existential, any followup instance will contain $\ell_{i}$ new facts with the relation symbol from the head of $\varphi$. In particular, the results from applications of existential rules do not collapse.

Lemma 5.3. The function $\mu$ from Definition5.1 is well-defined, and a probability measure on $\mathfrak{D} \upharpoonright_{E}$.

Proof. This can be shown analogously to Lemma4.4 That the integral is well-defined follows from the measurability of $\Xi_{\vec{\imath}}$ and the measurability of the $\psi_{i}$. For the integration, we use a product density of our parameterized distributions. Thus, in the case $E=D$, (5.1) collapses to $\int \prod_{i} \prod_{j_{i}} \psi_{i}\left\langle\vec{a}_{i j_{j}}\right\rangle d \mu^{\otimes}=1$ and it follows that $\mu$ is a probability measure.

The use of the product density in (5.1) stipulates the independence assumption we discussed before-all probabilistic rules that fire together in a parallel chase step do so independently. Note that by Fact 2.1 and the definition of $\Xi_{\vec{\ell}}$, the concrete order of the tuples ( $\varphi_{i}, \vec{a}_{i j}, b_{i j}$ ) has no impact on $\mu$ whatsoever. Also note that for firing configurations $\vec{\ell}$ with $\ell_{i}=0$ for all but one $i_{0}$ and $\ell_{i_{0}}=1$, (5.1) coincides with (4.1). Figure 10 contains an illustration of a parallel chase step.


Figure 10.: Illustration of a parallel chase step, continuous case.
We denote a parallel chase step $(D, \operatorname{App}(D), E, \mu)$ as

$$
D \xlongequal{\operatorname{App}(D)}(E, \mu) .
$$

Note that a parallel chase step is already determined by $D$ (and App) alone. That is, if there exists a parallel chase step starting in $D$, it is unique. This is in contrast to the sequential chase step that additionally depends on a measurable chase policy app of App. In the following, we adapt the definitions from Sections 4.1 to 4.3 to the parallel setting. For the definition of a chase tree, all we have to do is take the definition of the sequential chase tree, and replace the sequential chase steps with parallel ones.

Definition 5.4 (Parallel Chase Tree). Let $D_{\text {in }} \in \mathbb{D}_{\text {in }}$. The parallel chase tree $T_{\text {App }, D_{\text {in }}}$ for input instance $D_{\text {in }}$ with respect to the GDatalog program $\mathcal{G}$ is a labeled countable-depth tree $T_{\text {App }, D_{\text {in }}}=(V, E, \Lambda)$ with labeling function $\Lambda$, root node $r \in V$, and the following properties.

1. The root node $r$ is labeled with $D_{\text {in }}$.
2. If $v \in V$ is a leaf node, then it is labeled with an instance $D_{v}$ such that $\operatorname{App}\left(D_{v}\right)=\emptyset$.
3. If $v \in V$ is an inner node, then it is labeled with an instance $D_{v}$ such that $\operatorname{App}\left(D_{v}\right) \neq \emptyset$ and
a) $D_{v} \xrightarrow{\operatorname{App}\left(D_{v}\right)}\left(E_{v}, \mu_{v}\right)$ is a chase step with $\left(E_{v}, \mu_{v}\right)$ as in Definition 5.1 and
b) the function $v^{\prime} \rightarrow D_{v^{\prime}}$ is a bijection between the children of $v$ and $\boldsymbol{E}_{v}$.

For all $D_{\text {in }} \in \mathbb{D}_{\text {in }}$, the parallel chase tree $T_{\text {App }, D_{\mathrm{in}}}$ is unique. As for the sequential chase tree, $T_{\text {App }, D_{\text {in }}}$ has paths of at most countable length but may contain nodes with uncountably many children. Again, the tree is labeled injectively. This can be shown just like Lemma 4.7

Lemma 5.5. Let $D_{\text {in }} \in \mathbb{D}_{\text {in }}$. Then $v \neq w$ implies $D_{v} \neq D_{w}$ for all $v \neq w$ in $T_{\text {App, } D_{\mathrm{in}}}$.

We proceed with the introduction of parallel versions of the various relations and functions encountered in Sections 4.2 and 4.3 First, we define a parallel version of the relation $\vdash$ from Section 4.2 Our new relation $\Vdash \subseteq \mathbb{D}^{2}$, again denoted in infix notation, is defined just like $\vdash$, except that the sequential chase step is once more replaced by a parallel one:

- If $\operatorname{App}(D)=\emptyset$, then $D \Vdash D^{\prime}$ if and only if $D=D^{\prime}$.
- If $\operatorname{App}(D) \neq \emptyset$, there exists a unique parallel chase step $D \xlongequal{\operatorname{App}(D)}(\boldsymbol{E}, \mu)$ starting in $D$. In this case, $D \Vdash D^{\prime}$ if and only if $D^{\prime} \in E$.

We let

$$
\begin{aligned}
\text { paths }(\mathrm{App}) & :=\left\{\left(D_{0}, D_{1}, \ldots\right) \in \mathbb{D}^{\omega}: D_{i} \Vdash D_{i+1} \text { for all } i \in \mathbb{N}\right\} \text { and } \\
\text { paths }\left(\operatorname{App}, D_{\text {in }}\right) & :=\left\{\left(D_{0}, D_{1}, \ldots\right) \in \operatorname{paths}(\mathrm{App}): D_{0}=\mathcal{D}_{\text {in }}\right\} .
\end{aligned}
$$

Again, paths (App, $D_{\text {in }}$ ) corresponds to the paths in $T_{\mathrm{App}, D_{0}}$ where finite paths are continued by repeating the last instance label infinitely often. We keep using the notions of terminating paths and paths terminating at position $i$ from Section 4.2

To map paths to instances in the parallel setting, we define $\lim _{\mathrm{App}}: \mathbb{D}^{\omega} \rightarrow \mathbb{D}_{\perp}$ by

$$
\vec{D} \mapsto \begin{cases}D_{i} & \text { if } \vec{D} \in \text { paths(App) and } \vec{D} \text { is terminating at position } i  \tag{5.2}\\ \perp & \text { otherwise }\end{cases}
$$

Again, we define a version $\lim _{\text {App, } D_{\text {in }}}$ of $\lim _{\text {App }}$ where using paths(App, $D_{\text {in }}$ ) instead of paths(App) in (5.2).

The relations, sets and functions we just defined enjoy properties analogous to their sequential counterparts (cf. Lemmas 4.8 and 4.11 and Corollary 4.10).

## Lemma 5.6

1. The relation $\Vdash \subseteq \mathbb{D}^{2}$ is measurable in $\mathfrak{D} \times \mathfrak{D}$.
2. The sets paths $(\mathrm{App})$ and paths $\left(\mathrm{App}, D_{\mathrm{in}}\right)$ are measurable in $\left(\mathbb{D}^{\omega}, \mathfrak{D}^{\otimes \omega}\right)$.
3. Both $\lim _{\mathrm{App}}$ and $\lim _{\mathrm{App}, D_{\mathrm{in}}}$ are bimeasurable.

Proof. The proofs are easy extensions of the proofs of Lemmas 4.8 and 4.11 and Corollary 4.10

1. Let $\vec{l}$ be a fixed firing configuration. For the set of instances having firing configuration $\vec{\ell}$, we can decompose App into a sequence of $\sum_{i} \ell_{i}$ measurable selections as follows. We start with $\operatorname{App}_{0}:=\operatorname{App}(D)$ and note that $\left|\operatorname{App}_{0}(D)\right|=\sum_{i} \ell_{i}$. As App is a measurable multifunction, it has a measurable selection $\operatorname{app}_{1}$. We set $\operatorname{App}_{1}(D):=\operatorname{App}(D) \backslash\left\{\operatorname{app}_{1}(D)\right\}$. One can easily show that $\left\{D \in \mathbb{D}: \operatorname{App}_{1}(D) \neq \emptyset\right\}$ is measurable, and that $\operatorname{App}_{1}$ is a measurable multifunction on this set (as long as it is non-empty). Repeating this process for $j=1, \ldots, \sum_{i} \ell_{i}$, we obtain functions app $, \ldots, \operatorname{app}_{\sum_{i} \ell_{i}}$. If $\left(\varphi^{(i)}, \vec{a}^{(i)}\right)=\operatorname{app}_{i}(D)$, then for all $b^{(i)} \in \mathbb{W}_{\varphi^{(i)}}$ it holds that

$$
\begin{equation*}
\operatorname{Ext}_{\vec{\ell}}\left(D,\left(\varphi^{(i)}, \vec{a}^{(i)}, b^{(i)}\right)\right)=\operatorname{ext}\left(\ldots\left(\operatorname{ext}\left(D, \varphi^{(1)}, \vec{a}^{(1)}, b^{(1)}\right) \ldots\right), \varphi^{\left(\sum_{i} \ell_{i}\right)}, \vec{a}^{\left(\sum_{i} \ell_{i}\right)}\right) \tag{5.3}
\end{equation*}
$$

To see this note that on any database instance $D$ with $\left|\operatorname{App}_{0}(D)\right|=\sum_{i} \ell_{i}$, our sequence of app-functions yields a sequence of pairs $(\varphi, \vec{a})$. The only situation in which several of these pairs yield the same resulting tuple $f_{\varphi}(\vec{a}, b)=f_{\varphi^{\prime}}\left(\vec{a}^{\prime}, b^{\prime}\right)$ is when $\varphi$ and $\varphi^{\prime}$ are two different deterministic rules (with the same head), with $\vec{a}=\vec{a}^{\prime}$ and $\vec{b}=\vec{b}^{\prime}=*$. Thus, there are two pairs $(\varphi, \vec{a})$ and $\left(\varphi^{\prime}, \vec{a}\right)$ among the sequence $\left(\operatorname{app}_{i}(D)\right)_{i}$. Even though the one of these two that appears "later" is not an applicable pair anymore in the intermediate instance constructed in (5.3), this is no problem, for if $f_{\varphi}(\vec{a}, *)$ is already contained in an intermediate instance $D^{\prime}$, then by definition $\operatorname{ext}\left(D^{\prime}, \varphi^{\prime}, \vec{a}, *\right)=D^{\prime}$.
Now with our sequence of app-functions (resp. of pairs $\left(\varphi^{(i)}, \vec{a}^{(i)}\right)$ ), we can proceed as in Lemma 4.8 where we showed the measurability of the set (4.5). There, we described the membership of a pair ( $D, D^{\prime}$ ) in the relation $\vdash$ by countably approximating $D$ and $D^{\prime}=D \cup\left\{f_{\varphi}(\vec{a}, b)\right\}$ (for some $b$ ). The only thing that changes is that now $D^{\prime}=D \cup$ $\left\{f_{\varphi_{1}}\left(\vec{a}^{(1)}, b^{(1)}\right), \ldots, f_{\varphi_{k}}\left(\vec{a}^{\left(\sum_{i} f_{i}\right)}, b^{\left(\sum_{i} \ell_{i}\right)}\right)\right\}$ (for some $\left.b^{(1)}, \ldots, b^{\left(\sum_{i} f_{i}\right)}\right)$.
2. This follows immediately from (1).
3. This can be shown just like Lemma 4.11 with the only change being the use of App instead of app, and using Lemma 5.5instead of Lemma 4.7

### 5.2. The Markov Process for Parallel Chasing

In analogy to Section 4.4 we show in this section how the parallel chase defines a Markov process of database instances.

We define $\kappa_{\vdash}: \mathbb{D} \times \mathfrak{D} \rightarrow[0,1]$ as follows. Let $D \in \mathbb{D}$ and $\boldsymbol{D} \in \mathfrak{D}$.

- If $\operatorname{App}(D) \neq \emptyset$ and $D \xlongequal{\operatorname{App}(D)}(E, \mu)$ is the corresponding parallel chase step, then $\kappa_{\text {ト }}(D, \boldsymbol{D})=\mu(\boldsymbol{D} \cap \boldsymbol{E})$ with $\mu$ as in Definition 5.1.
- If $\operatorname{App}(D)=\emptyset$, then $\kappa_{\Perp}(D, \boldsymbol{D})=\iota(D, \boldsymbol{D})$ where $\iota$ is the identity kernel.

Intuitively, $\kappa_{\Perp-}(D, D)=\operatorname{Pr}(D \Vdash D)$ with the latter referring to the probability space defined for the chase step starting in $D$.

Proposition 5.7. $\kappa_{\Vdash}$ is a stochastic kernel.
The proof is similar to that of Proposition 4.12
Proof. By Lemma5.3 and the definition of $\kappa_{\kappa_{\Perp}}$, the function $\kappa_{\Vdash}(D, \cdot)$ is a probability measure for all $D \in \mathbb{D}$. So again, the harder part is to show that $\kappa_{\mathbb{l}}(\cdot, \boldsymbol{D})$ is $(\mathfrak{D}, \mathfrak{B o r}[0,1])$-measurable. For this, it suffices to show that the restriction of $\kappa_{\Perp}(\cdot, \boldsymbol{D})^{-1}[0, \alpha)$ to any fixed firing configuration $\vec{\ell}$ is in $\mathfrak{D}$ for all $\alpha \in[0,1]$. The claim then follows since there are only countably many $\vec{\ell}$, since the set $\mathbb{D}_{\vec{\ell}}=\{D \in \mathbb{D}: \vec{\ell}(D)=\vec{\ell}\}$ is measurable, and since the sets $[0, \alpha)$ generate $\mathfrak{B o r}[0,1]$.

We use the trick of the proof of Lemma 5.6(1) and decompose App into multiple measurable selections. With these selections we construct a measurable function $D \mapsto$
( $D, \varphi_{1}, \vec{a}_{11}, \ldots, \varphi_{k}, \vec{a}_{k \ell_{k}}$ ). With a repeated application of Fubini's theorem to the definition of $\mu$ (see Definition5.1), we obtain that for all $D$ it holds that

$$
\begin{aligned}
& \mu(\boldsymbol{D})= \sum_{\vec{l}} \int_{\mathbb{W}} \Xi_{\vec{l}}\left(D, \varphi_{1}, \vec{a}_{11}, b_{11}, \ldots, \varphi_{k}, \vec{a}_{k \ell_{k}}, b_{k \ell_{k}}, D \cap \mathbb{D}_{\vec{l}}\right) \\
& \quad \cdot \prod_{i=1}^{k} \prod_{j_{i}=1}^{\ell_{i}} \psi_{i}\left\langle\vec{a}_{j_{j}}\right\rangle\left(b_{i j_{i}}\right) d\left(\mu_{\psi_{1}}^{\otimes \ell_{1}} \otimes \cdots \otimes \mu_{\psi_{k}}^{\otimes \ell_{k}}\right) \\
&=\sum_{\vec{l}} \int_{\mathbb{W}_{1}} \psi_{1}\left\langle\vec{a}_{11}\right\rangle\left(b_{11}\right) \cdots \int_{\mathbb{W}_{k}} \psi_{k}\left\langle\vec{a}_{k \ell_{k}}\right\rangle\left(b_{k \ell_{k}}\right) \\
& \quad \cdot \Xi_{\vec{l}}\left(\varphi_{1}, \vec{a}_{11}, b_{11}, \ldots, \varphi_{k}, \vec{a}_{k_{k},}, b_{k \ell_{k}}, D \cap \mathbb{D}_{\vec{\ell}}\right) d \mu_{\psi_{k}} \ldots d \mu_{\psi_{1}}
\end{aligned}
$$

(Note that there are $\ell_{1}+\cdots+\ell_{k}$ integrals in every summand of the last expression above.) In this situation, proceeding exactly like in the proof of Proposition 4.12 one can show that the innermost integral is

$$
\begin{equation*}
\left(\mathfrak{D} \otimes \mathfrak{U}_{1}^{\otimes \ell_{1}} \otimes \cdots \otimes \mathfrak{U}_{k-1}^{\otimes \ell_{k}-1} \otimes \mathcal{P}\left(\left\{\varphi_{k}\right\}\right) \otimes \mathfrak{B}_{\varphi_{k}}^{(h)}, \mathfrak{B o r}[0,1]\right) \text {-measurable } \tag{5.4}
\end{equation*}
$$

where $\mathfrak{U}_{i}$ is the $\sigma$-algebra of $\left\{\varphi_{i}\right\} \times \mathbb{V}_{\varphi_{i}}^{(h)} \times \mathbb{W}_{\varphi_{i}}$. Using the sequence of measurable functions we introduced before, we can get rid of the trailing $\mathcal{P}\left(\left\{\varphi_{k}\right\}\right) \otimes \mathfrak{B}_{\varphi_{k}}^{(h)}$ in (5.4) just like in the proof of Proposition 4.12 Propagating the above procedure outwards yields that (the mentioned restriction of) $\kappa_{\Downarrow}$ is $(\mathfrak{D}, \mathfrak{B o r}[0,1])$-measurable.

By the merit of Proposition 5.7 we can construct a Markov process analogously to Section 4.4

Corollary 5.8. Let $(\mathbb{D}, \mathfrak{D}, P)$ be a sub-probabilistic database. Then there exists a Markov process with state space $(\mathbb{D}, \mathfrak{D})$, with initial distribution $P$ and with transition kernels $\kappa_{1}$.

Mapping the paths of the process back to instances using $\lim _{\text {App }}$, the parallel chase also generates an output sub-probabilistic database.

## Theorem 5.9.

1. For all $D_{\mathrm{in}} \in \mathbb{D}_{\mathrm{in}}$, the program $\mathcal{G}$ on input $D_{\mathrm{in}}$ defines a sub-probabilistic database $\mathcal{G}_{\text {App }}\left(D_{\text {in }}\right)$.
2. For all sub-probabilistic databases $\mathcal{D}=\left(\mathbb{D}_{\mathrm{in}}, \mathfrak{D}_{\mathrm{in}}, P\right)$, the program $\mathcal{G}$ on input $\mathcal{D}$ defines $a$ sub-probabilistic database $\mathcal{G}_{\text {App }}(\mathcal{D})$.

## 6. Semantic Properties of Generative Datalog

### 6.1. Chase Independence

Let $\mathcal{G}$ be a GDatalog program. We know from Theorems 4.14 and 5.9 that for every input sub-probabilistic database $\mathcal{D}, \mathcal{G}$ defines outputs $\mathcal{G}_{\text {App }}(\mathcal{D})$ for parallel steps and $\mathcal{G}_{\text {app }}(\mathcal{D})$ for
sequential steps where app is a measurable chase policy. In this section we show, that the output is independent of the chase procedure and, in particular, independent of the choice of policy in the sequential chase.

Theorem 6.1. For all input instances $D_{\mathrm{in}} \in \mathbb{D}_{\mathrm{in}}$ and all measurable chase policies app we have $\mathcal{G}_{\text {app }}\left(D_{\text {in }}\right)=\mathcal{G}_{\text {App }}\left(D_{\text {in }}\right)$.

Remark 6.2. In the light of the above result, it is natural to ask why we introduced the sequential version (Section (4) in the first place, when the semantics could just have been introduced using the parallel chase. Apart from the sequential version being the more typical approach to the semantics, having both the sequential, and the parallel version at hand, together with the fact that they yield the same result, renders the semantics of GDatalog programs quite robust. The original motivation for introducing the parallel chase was, in fact, just to establish that the sequential chase does not depend on the choice of chase policy. The paper [5] features a similar statement but does not introduce a parallel version of the chase. Thus, the sequential chase also serves the purpose of connecting our work to [5].

We outline the proof of Theorem 6.1 and ask the reader to already have a peek at Figure 11 We will fix an event $D$ in the output and show that it has the same measure in both processes. For this, it suffices to come up with an easier to handle countable partition of $D$ into measurable sets and show that the measures coincide on each set of the partition. This partition roughly brings paths in the existential and the parallel chase trees into a one-to-one correspondence with respect to the effect of rule applications.

Proof of Theorem 6.1. Let $\mu_{\mathrm{app}, D_{\text {in }}}^{\otimes}$ denote the probability measure on the path space $\left(\mathbb{D}^{\omega}, \mathfrak{D}^{\otimes \omega}\right)$ of the Markov process associated with the sequential chase using app. Likewise, we let $\mu_{\text {App }, D_{\text {in }}}^{\otimes}$ denote the probability measure on $\left(\mathbb{D}^{\omega}, \mathfrak{D}^{\otimes \omega}\right)$ from the parallel chase. That is, $\mu_{\mathrm{app}, D_{\mathrm{in}}}^{\otimes} \circ \lim _{\mathrm{app}, D_{\mathrm{in}}}^{-1}$ is the sub-probability measure of $\mathcal{G}_{\mathrm{app}}\left(D_{\mathrm{in}}\right)$, and $\mu_{\mathrm{App}, D_{\text {in }}}^{\otimes} \circ \lim _{\mathrm{App}, D_{\text {in }}}^{-1}$ is the sub-probability measure of $\mathcal{G}_{\text {App }}\left(D_{\text {in }}\right)$. Let $\varphi_{1}, \ldots, \varphi_{k}$ be the rules of $\mathcal{G}^{\exists}$.

First note that if $D$ is the label of a leaf in $T_{\text {app }, D_{\text {in }}}$, then it is also the label of a leaf in $T_{\text {App }, D_{\text {in }}}$, and vice versa $8^{8}$ We call the (unique) path from $D_{\text {in }}$ to $D$ (in either tree) the ( $D_{\text {in }}, D$ )-path. If $D$ is a leaf label, then the set of facts that are produced on the $\left(D_{\mathrm{in}}, D\right)$-path in $T_{\text {app }, D_{\text {in }}}$ coincides with the set of facts that are produced on $\left(D_{\mathrm{in}}, D\right)$-path in $T_{\text {App }, D_{\mathrm{in}}}$. We say a rule fires on a path if it appears in one of the chase steps along the path. If $D$ is a leaf label, then for every existential rule $\varphi$ of $\mathcal{G}^{\exists}$, the number of times it fires on the $\left(D_{\mathrm{in}}, D\right)$-path in $T_{\mathrm{app}, D_{\mathrm{in}}}$ is equal to the number of times it fires on the $\left(D_{\mathrm{in}}, D\right)$-path in $T_{\mathrm{App}, D_{\mathrm{in}}}$.

Consider two paths, one in $T_{\mathrm{app}, D_{\text {in }}}$ and one in $T_{\mathrm{App}, D_{\text {in }}}$, that end in a leaf with the same label. For these two paths, we say that the application of $\varphi$ in the sequential chase produces the same fact as the application of $\varphi^{\prime}$ in the parallel chase, if there exist intermediate instances $D$ (in the sequential chase) and $D^{\prime}$ (in the parallel chase) such that all of the following holds.

- For all $b$ it holds that $f_{\varphi}(\vec{a}, b)=f_{\varphi^{\prime}}\left(\vec{a}^{\prime}, b\right)=: f_{b}$ where $(\varphi, \vec{a})=\operatorname{app}(D)$ and $\left(\varphi^{\prime}, \vec{a}^{\prime}\right) \in$ App (D).
- The instance $D \cup\left\{f_{b}\right\}$ is the successor instance of $D$ on the path in $T_{\text {app }, D_{\text {in }}}$.

[^7]- The instance $D^{\prime} \cup\left\{f_{b}\right\}$ is contained in the successor instance of $D^{\prime}$ on the path in $T_{\mathrm{App}, D_{\text {in }}}$.

We say that an instance $D$, the rule application of $\varphi$ in the sequential chase produces the same fact as a rule application of $\varphi^{\prime}$ in the parallel chase, if $f_{\varphi}(\vec{a}, b)=f_{\varphi^{\prime}}\left(\vec{a}^{\prime}, b\right)$

We now fix an arbitrary measurable set $D \in \mathfrak{D}$ and show that

$$
\mu_{\mathrm{app}, D_{\mathrm{in}}}^{\otimes}\left(\lim _{\mathrm{app}, D_{\mathrm{in}}}^{-1}(\boldsymbol{D})\right)=\mu_{\mathrm{App}, D_{\mathrm{in}}}^{\otimes}\left(\lim _{\mathrm{App}, D_{\mathrm{in}}}^{-1}(\boldsymbol{D})\right) .
$$

Let $m, n \in \mathbb{N}$. For $i=1, \ldots, m$ let $C^{(i)}$ be a $\{0,1\}$-valued $n \times k$-matrix

$$
C^{(i)}=\left(\begin{array}{ccc}
c_{11}^{(i)} & \ldots & c_{1 k}^{(i)} \\
\vdots & \ddots & \vdots \\
c_{n 1}^{(i)} & \ldots & c_{n k}^{(i)}
\end{array}\right)
$$

We call such a matrix a correspondence matrix. We use such matrices to fix correspondences in paths of our both chase trees (cf. Figure 11). The gist of our approach is to partition the discussed event $D$ according to correspondence matrices for the paths in our chase trees.

Let $\varphi^{(1)}, \ldots, \varphi^{(m)}$ be a sequence of $m$ rules and let $\vec{\ell}_{1}, \ldots, \vec{\ell}_{n}$ be a sequence of $n$ firing configurations. Then we define a set

$$
D_{\vec{\varphi}, \vec{L}, \vec{C}}=D_{\varphi^{(1)}, \ldots, \varphi^{(m)}, \vec{\ell}_{1}, \ldots, \vec{\ell}_{n}, C^{(1)}, \ldots, C^{(m)}}
$$

by letting $D \in D_{\vec{\varphi}, \vec{L}, \vec{C}}$ if $D \in D$ and

1. $D$ is a leaf label on level $m$ in $T_{\mathrm{app}, D_{\mathrm{in}}}$, such that the sequence of rules firing on the $\left(D_{\mathrm{in}}, D\right)$ path in $T_{\mathrm{app}, D_{\mathrm{in}}}$ is exactly $\varphi^{(1)}, \ldots, \varphi^{(m)}$; and
2. $D$ is a leaf label on level $n$ in $T_{\text {app }, D_{\text {in }}}$, such that the sequence of firing configurations on the $\left(D_{\text {in }}, D\right)$-path in $T_{\text {App }, D_{\text {in }}}$ is exactly $\vec{\ell}_{1}, \ldots, \vec{\ell}_{n}$ (where $\vec{\ell}_{j}=\left(\ell_{j 1}, \ldots, \ell_{j r}\right)$ ); and
3. it holds that $c_{j r}^{(i)}=1$ if and only if the application of $\varphi^{(i)}$ in the $i$ th step in the sequential chase tree $T_{\mathrm{app}, D_{\mathrm{in}}}$ produces the same fact as one of the $\ell_{j r}$ applications of rule $\varphi_{r}$ in the $j$ th step in the parallel chase tree $T_{\mathrm{App}, D_{\text {in }}}$.

In essence, the matrix $C^{(i)}$ tells us how the application of $\varphi^{(i)}$ in the sequential chase corresponds to rule applications in the parallel chase, subject the firing sequence we fixed. For an illustration of the setup of $D_{\vec{\varphi}, \vec{L}, \vec{C}}$, see Figure 11

Note that if $\varphi^{(i)}$ is an existential rule, then $C^{(i)}$ contains exactly one 1-entry (this is because all existential rules come with different head relation symbols and since the same fact cannot be produced by the same existential rule under different instantiations).

If $\varphi^{(i)}$ is deterministic, then all but one row of $C^{(i)}$ are all zeroes (if this were not the case, then the fact produced by $\varphi^{(i)}$ in the sequential chase would get produced in two different rounds of the parallel chase, which is impossible by the definition of rule applicability). It can happen though that $c_{j r}^{(i)}=c_{j r^{\prime}}^{(i)}=1$ for some $r \neq r^{\prime}$. This happens when multiple (deterministic) rules produce the same fact as the application of $\varphi^{(i)}$ in the $i$ th step of $T_{\mathrm{app}, D_{\mathrm{in}}}$.


Figure 11:: Paths obtained from $D_{\vec{\varphi}, \vec{L}, \vec{C}}$ in the sequential (left) and the parallel (right) chase tree.

Every instance $D \in D$ lies in exactly one set $D_{\vec{\varphi}, \vec{L}, \vec{C}}$. That is, the sets $D_{\vec{\varphi}, \vec{L}, \vec{C}}$ partition $D$ into countably many sets. (Moreover, $D_{\vec{\varphi}, \vec{L}, \vec{C}}$ are measurable. They can be expressed using app, sequences of measurable selections app ij $_{i}$ (which are introduced below), the measurable functions $f_{\varphi}$ and the diagonals in the fact spaces.)

Now let $\vec{D}=\vec{D}(\vec{\varphi}, \vec{L}, \vec{C}):=\lim _{\text {App }, D_{\text {in }}}^{-1}\left(\boldsymbol{D}_{\vec{\varphi}, \vec{L}, \vec{C}}\right)$. Then, by definition 9 ,

$$
\begin{align*}
\mu_{\mathrm{App}, D_{\mathrm{in}}}^{\otimes}(\overrightarrow{\boldsymbol{D}})=\iint \cdots \int \mathbb{1}_{\vec{D}}\left(D_{\mathrm{in}}, D_{1}, \ldots, D_{n-1}, D_{n}, D_{n}, \ldots\right)  \tag{6.1}\\
\kappa_{\Vdash}\left(D_{n-1}, d D_{n}\right) \cdots \kappa_{\Vdash}\left(D_{1}, d D_{2}\right) \kappa_{\Vdash}\left(D_{\mathrm{in}}, d D_{1}\right)
\end{align*}
$$

where $\kappa_{\Downarrow-}\left(D_{i}, d D_{i+1}\right)$ is shorthand for $d\left(\kappa_{\Vdash}\left(D_{i}, \cdot\right)\right)$.
Recall that for any firing configuration $\vec{\ell}^{\prime}=\left(\ell_{1}^{\prime}, \ldots, \ell_{k}^{\prime}\right)$ for parallel chasing, we can obtain a sequence of $\sum_{i} \ell_{i}^{\prime}$ measurable selections app $p_{1}, \ldots$, app $_{\sum_{i} \ell_{i}^{\prime}}$ such that for all $D$ with firing configuration $\vec{\ell}^{\prime}$ it holds that

$$
\operatorname{App}(D)=\left\{\operatorname{app}_{j}(D): 1 \leq j \leq \sum_{i} \ell_{i}^{\prime}\right\}
$$

(We have already used this in the proof of Lemma 5.6(1), details can be found ibid.) For all the firing configurations $\vec{\ell}_{1}, \ldots, \vec{\ell}_{n}$ we fixed before, we chose such a sequence of measurable selections app $j_{i j}$ such that $\mathrm{app}_{i j}$ is the $j_{i}$ th measurable selection belonging to $\vec{\ell}_{i}=\left(\ell_{i 1}, \ldots, \ell_{i k}\right)$. In the following we let

$$
\left(\varphi_{i j_{i}}, \vec{a}_{i j_{i}}\right):=\operatorname{app}_{i j_{i}}\left(D_{i}\right)
$$

where $1 \leq j_{i} \leq \ell_{i 1}+\cdots+\ell_{i k}=: s_{i}$ for all $i=1, \ldots, n$. Moreover, we let $\psi_{i j_{i}}$ denote the parameterized distribution from rule $\varphi_{i j_{i}}$ and $\mu_{i j_{i}}$ the associated underlying measure (cf. Section 2.2).

[^8]Note that since we fixed $\vec{\ell}_{1}, \ldots, \vec{\ell}_{n}$, the collections $\left(\varphi_{i j_{i}}\right)_{j_{i}},\left(\psi_{i j_{i}}\right)_{j_{i}}$ and $\left(\mu_{i j_{i}}\right)_{j_{i}}$ do not depend on $D_{i}$.

Now for every measurable function $g$ it holds that

$$
\begin{array}{r}
\int_{\mathbb{D}} g\left(D_{i+1}\right) \kappa_{\Vdash}\left(D_{i}, d D_{i+1}\right)=\int_{\mathbb{W}} g\left(\operatorname{Ext}_{\vec{\imath}_{i}}\left(D_{i}, \varphi_{i 1}, \vec{a}_{i 1}, b_{i 1}, \ldots, \varphi_{i s_{i}}, \vec{a}_{i s_{i}}, b_{i s_{i}}\right)\right)  \tag{6.2}\\
\cdot \psi_{i 1}\left\langle\vec{a}_{i 1}\right\rangle\left(b_{i 1}\right) \cdot \ldots \cdot \psi_{i s_{i}}\left\langle\vec{a}_{i s_{i}}\right\rangle\left(b_{i s_{i}}\right) d\left(\mu_{i 1} \otimes \cdots \otimes \mu_{i s_{i}}\right)
\end{array}
$$

where $\mathbb{W}=\prod_{j=1}^{s_{i}} \mathbb{W}_{\varphi_{i j}}$ and $i \in\{0, \ldots, n-1\}$ with $D_{0}=D_{\text {in }}$. The equality in (6.2) is obtained as follows. We apply the substitution rule (see Fact A.7) for the measurable function Ext ${\overrightarrow{\boldsymbol{l}_{i}}}$ to transform the domain of integration. With applications of the chain rule (see Fact A. 6 we extract $\psi_{i j_{i}}$ from the integration measure.

Now by Fubini's theorem we can rewrite the right hand side of (6.2) as an iterated integration with $s_{i}$ integrals. We do so from the inside to the outside for all of the integrals appearing in (6.1). We obtain that $\mu_{\mathrm{App}, D_{\text {in }}}^{\otimes}(\vec{D})$ is equal to

$$
\begin{array}{r}
\Gamma_{\vec{D}}=\int \mathbb{1}_{D_{\vec{p}, \vec{L}, \vec{C}}}\left(\operatorname{Ext}_{\vec{l}_{n}}\left(\operatorname{Ext}_{\vec{l}_{n-1}}\left(\cdots \operatorname{Ext}_{\vec{l}_{1}}\left(D_{\mathrm{in}},\left(\varphi_{1 j_{1}}, \vec{a}_{1 j_{1}}, b_{1 j_{1}}\right)_{j_{1}}\right) \cdots\right),\left(\varphi_{n j_{n}}, \vec{a}_{n j_{n}}, b_{n j_{n}}\right)_{j_{n}}\right)\right) \\
\cdot \prod_{i=1}^{n} \prod_{j_{i}=1}^{s_{i}} \psi_{i j_{i}}\left\langle\vec{a}_{i j_{i}}\right\rangle\left(b_{i j_{i}}\right) d\left(\mu_{11} \otimes \cdots \otimes \mu_{n s_{n}}\right) . \tag{6.3}
\end{array}
$$

With $\overrightarrow{\boldsymbol{E}}=\overrightarrow{\boldsymbol{E}}(\vec{\varphi}, \vec{L}, \vec{C}):=\lim _{\mathrm{app}, D_{\text {in }}}^{-1}\left(\boldsymbol{D}_{\vec{\varphi}, \vec{L}, \vec{C}}\right)$, applying the same procedure to the corresponding expression for $\mu_{\mathrm{app}, D_{\mathrm{in}}}^{\otimes}(\overrightarrow{\boldsymbol{E}})$ yields

$$
\begin{align*}
& \gamma_{\vec{E}}=\int \mathbb{1}_{D_{\vec{\varphi}, \vec{L}, \vec{C}}}\left(\operatorname{ext}\left(\operatorname{ext}\left(\cdots \operatorname{ext}\left(D_{\mathrm{in}}, \varphi^{(1)}, \vec{a}^{(1)}, b^{(1)}\right), \ldots\right), \varphi^{(m)}, \vec{a}^{(m)}, b^{(m)}\right)\right) \\
& \cdot \prod_{i=1}^{m} \psi^{(i)}\left\langle\vec{a}^{(i)}\right\rangle d\left(\mu^{(1)} \otimes \cdots \otimes \mu^{(m)}\right) . \tag{6.4}
\end{align*}
$$

We have already argued that the rules, parameterized distributions, and underlying measures are fixed in these expressions. Yet, we note that the parametrizations $\vec{a}^{(i)}$ and $\vec{a}_{i j_{i}}$ in (6.4) and (6.3) depend on the outcome of the previous sample via app and the measurable selections app ${ }_{i j_{i}}$ we constructed before. Note that the integrands of (6.3) and (6.4) are functions in $b_{11}, \ldots, b_{n s_{n}}$, respectively $b^{(1)}, \ldots, b^{(m)}$. Every (suitable) tuple ( $b_{1 s_{1}}, \ldots, b_{n s_{n}}$ ) gives rise to a random path

$$
D_{\text {in }} \Vdash D_{1} \Vdash D_{2} \Vdash \ldots \Vdash D_{n}
$$

in $T_{\mathrm{App}, D_{\mathrm{in}}}$, that is almost surely contained in $\vec{D}=\lim _{\mathrm{App}, D_{\mathrm{in}}}^{-1}\left(D_{\vec{\varphi}, \vec{L}, \vec{C}}\right)$. Likewise, every (suitable) tuple $b^{(1)}, \ldots, b^{(m)}$ in (6.4) describes a random path

$$
D_{\text {in }} \vdash E_{1} \vdash E_{2} \vdash \ldots \vdash E_{m}
$$

in $T_{\text {app }, D_{\text {in }}}$ that is a.s. contained in $\vec{E}=\lim _{\text {app }, D_{\text {in }}}^{-1}\left(D_{\vec{\varphi}, \vec{L}, \vec{C}}\right)$. By the definition of $D_{\vec{\varphi}, \vec{L}, \vec{C}}$, we have $D_{n}=E_{m} \in D$ where no rule is applicable anymore. Recall that in $D_{\vec{\varphi}, \vec{L}, \vec{C}}$, the application of rule
$\varphi^{(i)}$ in $T_{\text {app }, D_{\text {in }}}$ has the same resulting fact effect as $c_{j r}^{(i)}$ of the applications of $\varphi_{r}$ in the $j$ th step in $T_{\text {App }, D_{\mathrm{in}}}$. The confinement to $D_{\vec{\phi}, \vec{L}, \vec{C}}$ establishes a one-to-one correspondence between $\vec{D}$ and $\overrightarrow{\boldsymbol{E}}$, resp. between tuples ( $b_{11}, \ldots, b_{n s_{n}}$ ) and ( $b^{(1)}, \ldots, b^{(m)}$ ) (where the first of these sequences may contain redundant *'s as leftovers from collapsing deterministic rules). In this case, we call the tuples equivalent. It remains to show that if $\left(b_{11}, \ldots, b_{n s_{n}}\right)$ and $\left(b^{(1)}, \ldots, b^{(m)}\right)$ are equivalent, then

$$
\begin{equation*}
\prod_{i=1}^{n} \prod_{j_{i}=1}^{s_{i}} \psi_{i j_{i}}\left\langle\vec{a}_{i_{i}}\right\rangle\left(b_{i_{j}}\right)=\prod_{i=1}^{m} \psi^{(i)}\left\langle\vec{a}^{(i)}\right\rangle\left(b^{(i)}\right) . \tag{6.5}
\end{equation*}
$$

This is the case because of the correspondence fixed by $\vec{C}$, which can be seen as follows. First of all, all factors that belong to non-existential rules can be canceled from (6.5) (even though there might be a different number of them on both sides) as they evaluate to 1 (cf. Remark 4.3). The remaining numbers of factors on both sides coincide (see our discussion of $c_{j r}^{(i)}$ before) and, again, are in one-to-one correspondence with each other via $\vec{C}$. In particular, if $\vec{a}_{i_{j} i}, b_{i j_{i}}$ and $\vec{a}^{(i)}, b^{(i)}$ produce the same fact, then $\vec{a}_{i j_{i}}=\vec{a}^{(i)}$ and $b_{i j_{i}}=b^{(i)}$.

A similar argument applies to the product measures (of the $\mu^{(i)}$, resp. the $\mu_{i_{j}}$ ) used for the integration in (6.3) and (6.4)-recalling that the measures themselves are already fixed by fixing $\vec{\varphi}$ and $\vec{L}$ and that the measure spaces from the non-existential rules are trivial (Remarks 4.3 and (3.12).

We have shown that the expressions (6.3) and (6.4) have the same value (for fixed $D$ ), i. e. for all suitable $\vec{\varphi}, \vec{L}$, and $\vec{C}$ it holds that

$$
\begin{equation*}
\mu_{\mathrm{app}, D_{\mathrm{in}}}^{\otimes}(\vec{E}(\vec{\varphi}, \vec{L}, \vec{C}))=\gamma_{\vec{E}(\vec{p}, \vec{L}, \vec{C})}=\Gamma_{\vec{D}(\vec{\varphi}, \vec{L}, \vec{C})}=\mu_{\mathrm{App}, D_{\mathrm{in}}}^{\otimes}(\vec{D}(\vec{\varphi}, \vec{L}, \vec{C})) . \tag{6.6}
\end{equation*}
$$

As $D_{\vec{\varphi}, \vec{L}, \vec{C}}$ is a countable, measurable partition of $D$, we obtain

$$
\mu_{\mathrm{app}, D_{\text {in }}}^{\otimes}\left(\lim _{\mathrm{app}, D_{\text {in }}}^{-1}(D)\right)=\sum_{\vec{\varphi}, \vec{L}, \vec{C}} \gamma_{\vec{E}}(\vec{\varphi}, \vec{L}, \vec{C}) \stackrel{\sqrt{6.6})}{=} \sum_{\vec{\varphi}, \vec{L}, \vec{C}} \Gamma_{\vec{D}(\vec{\varphi}, \vec{L}, \vec{C})}=\mu_{\text {App }, D_{\text {in }}}^{\otimes}\left(\lim _{A \mathrm{App}, D_{\text {in }}}^{-1}(D)\right) .
$$

When fed an initial distribution instead of a single instance $D_{\text {in }}$, we directly obtain the following corollary.

Corollary 6.3. For all sub-probabilistic databases $\mathcal{D}=\left(\mathbb{D}_{\mathrm{in}}, \mathfrak{D}_{\mathrm{in}}, P\right)$, and all measurable chase policies app it holds that $\mathcal{G}_{\text {app }}(\mathcal{D})=\mathcal{G}_{\text {App }}(\mathcal{D})$.

Because of the results in this subsection, we just write $\mathcal{G}(\mathcal{D})$ for the sub-probabilistic database that is obtained by running $\mathcal{G}$ on $\mathcal{D}$. Recall that we can also eliminate the auxiliary tuples if we want to (see Remark 4.15).

### 6.2. Comparison to the Original Semantics

We continue from the discussions surrounding Examples 1.1 and 1.2 The key difference between our semantics and Bárány et al.'s semantics is in the mechanism that prevents generating infinitely many facts that only differ in a sampled value. We tie this to the rules: each probabilistic rule is only permitted to fire once for each setting of the parameters. Bárány et al.
tie it to the distributions: each distribution is only permitted to produce one sample for each parameter setting. This difference in the semantics (in particular, the decoupling of sampling from distribution names) resolves the issue sketched in Example 1.2

Both we and Bárány et al. have mechanisms to relax the requirement of only sampling once: we allow it to repeat rules; Bárány et al. introduce symbolic parameters that can be added to a distribution to allow several applications. Multiple copies of the same rule have a different behavior under our semantics as they would exhibit in the original one. We treat multiple copies as separate instructions, while they collapse to the effect of just a single such rule in the original semantics. We achieve this by associating different existential rules to the individual copies during the translation to the existential program, leading to the semantic difference pointed out in Example 1.1

With our version of the semantics, we can, however, simulate the semantics of Bárány et al., as far as only the distribution over finite outcomes is concerned. This means that for a program $\mathcal{G}$ using the original semantics, we can construct a new program $\mathcal{H}$ that, under our semantics, has the property that the sup-probabilistic database it produces on (finite) outcome instances coincides with the distribution over the finite outcomes of $\mathcal{G}$. As database instances are finite per definition, this is, after all, the interesting part of the distribution. The semantics of Bárány et al. extends towards a distribution also over the infinite chase paths, that is, over infinite outcomes. Recall that in our semantics, all infinite chase paths are merged into a single error event. Therefore, we necessarily lose the information about the infinite program executions that are present in the semantics of [5].

The following example exposes the difference in the semantics and illustrates the simulation mentioned above.

| $\mathcal{G}:$ | $S\left(\mathrm{Flip}\left\langle\frac{1}{2}\right\rangle\right) \leftarrow R(0)$ |
| :--- | :--- |
|  | $T\left(\operatorname{Flip}\left\langle\frac{1}{2}\right\rangle\right) \leftarrow R(0)$ |

(a) The program $\mathcal{G}$.

| $\mathcal{H}:$ | $A\left(\right.$ Flip $\left.\left\langle\frac{1}{2}\right\rangle\right) \leftarrow R(0)$ |
| :--- | :--- |
|  | $S(x) \leftarrow A(x)$ |
|  | $T(x) \leftarrow A(x)$ |

(b) The program $\mathcal{H}$.

Figure 12.: Exemplary simulation of the original semantics.

Under Bárány et al.s semantics, $\mathcal{G}$ has outcomes $\{R(0), S(0), T(0)\}$ and $\{R(0), S(1), T(1)\}$ with probability $1 / 2$ each, whereas our semantics yield the four possible outcomes $\{R(0), S(0), T(0)\},\{R(0), S(0), T(1)\},\{R(0), S(1), T(0)\}$ and $\{R(0), S(1), T(1)\}$, each with probability $1 / 4$. Yet, we can easily simulate the original semantics of $\mathcal{G}$ by pulling out the sampling to a separate rule, as in Figure 12b. This program has outcomes $\{R(0), A(0), S(0), T(0)\}$ and $\{R(0), A(1), S(1), T(1)\}$ with probability $1 / 2$ each. We can ignore the auxiliary predicate $A$ and restrict the resulting probabilistic database to the schema $\{R, S, T\}$ without changing the probabilities.

This simple argument can be generalized to arbitrary programs. We note that the original semantics also featured the use of "event expressions" that could be employed to reuse samples for the same parameter configuration. Such can be simulated by decomposing the rule, and
putting the event expression into a separate relation. We leave the details to the reader. Let us remark that it is similarly easy to simulate our semantics with that of Bárány et al. All our results would also hold starting from Bárány et al.'s semantics for discrete distributions, so it is really just a matter of taste which version the reader prefers.

Remark 6.4 (First-Order Equivalence). While the different semantics can simulate each other, the choice of Bárány et al.'s semantics was in particular made to obtain a decidable sufficient criterion for the "semantic equivalence" of two programs.

Following [5], two programs $\mathcal{G}$ and $\mathcal{G}^{\prime}$ are first-order equivalent, if the first-order theories defined by their collection of rules coincide ${ }^{10}$ For this, they interpret parameterized distributions as function symbols. Bárány et al. show that their notion of first-order equivalence is decidable for a simple syntactic class of programs, and that first-order equivalence of two programs entails that they produce the same output distribution under their semantics [5] Theorem 5.6 and 5.5] (this latter property is called semantic equivalence). In general, semantic equivalence is undecidable, even when there are no parameterized distributions at all [5, Theorem 5.6].

While the above definition of first-order equivalence feels natural and is in line with typical notions of first-order equivalence, the decision to interpret every occurrence of a parameterized distribution as another occurrence of the same function symbol is susceptible to debate. In particular, this treats a parameterized distribution just like a function, yet sampling repeatedly from a distribution without changing its parameters may well yield different outcomes. If the distribution is continuous, this even happens almost surely 11

The result from [5], that (their notion of) first-order equivalence implies semantic equivalence no longer holds with our semantics. One may easily verify that with our semantics, for example, that the program $\mathcal{G}_{0}$ from Example 1.1 is first-order equivalent to the program obtained by removing one of its rules. Yet, under our semantics, they produce different outputs.

We propose to adapt the notion of first-order equivalence to be tailored to our semantics. Recall that one major change we made was that we allowed rules to be present multiple times. When translating GDatalog to Datalog ${ }^{\exists}$ programs though, we introduced a distinction between the copies of a probabilistic rule, by providing (per rule) a unique new relation for storing the sample outcomes (cf. (3.4) in Section 3.3). Thus, we propose the following change in the notion of first-order independence: Call two GDatalog programs $\mathcal{G}$ and $\mathcal{G}^{\prime}$ first-order equivalent, if the first-order theories defined by their collection of rules coincide, where every occurrence of a parameterized distribution $\psi\langle\vec{p}\rangle$ is treated as an occurrence of a new distinguished function symbol. With this changed notion, copies of probabilistic rules cannot be (first-order equivalently) rewritten into single rules anymore.

With respect to this new notion of first-order equivalence it seems possible to transfer the results of [5] Section 5.1] to our semantics.

Remark 6.5 (Fairness). In [5], the authors restrict their scope to "fair" chase trees. This means that if for an intermediate instance $D$ it holds that $D \vDash \varphi_{b}(\vec{u})$ for some valuation $\vec{u}$, then eventually $D^{\prime} \models \varphi_{h}(\vec{u})$ for some later intermediate instance, on all paths. Note that by construction, any finite path of the chase tree that is ending in a leaf satisfies this fairness condition. Contrary

[^9]to [5], we do not impose the fairness condition on the infinite parts of the tree, as all infinite paths are collectively mapped to our error event $\perp$ anyway. The important point is that the (sub-)probability space obtained from the finite paths is independent of any fairness issue. $\triangleleft$

### 6.3. Termination Behavior

In Sections[4and5 we have constructed Markov processes for given GDatalog programs. Every point in the $i$ th component of the path space $\left(\mathbb{D}^{\omega}, \mathfrak{D}^{\otimes \omega}\right)$ can be seen as corresponding to a program configuration after $i$ steps and every path in $\left(\mathbb{D}^{\omega}, \mathfrak{D}^{\otimes \omega}\right)$ as a program run. A run is called terminating if it corresponds to a finite path in the respective chase tree. The program $\mathcal{G}$ is called terminating if all its runs terminate. The following result from the original paper trivially extends to our setting, with the notion of weak acyclicity remaining unchanged (see [5]).

Theorem 6.6 (cf. [5] Theorem 3.10]). Let $\mathcal{G}$ be $a$ GDatalog program. If $\mathcal{G}$ is weakly acyclic, then $\mathcal{G}$ is terminating.

That is, whenever there are no circular dependencies involving probabilistic rules, then all paths in any chase tree are finite. In general, the GDatalog program $\mathcal{G}$ terminates on input $D_{\text {in }}$ if and only if its existential version is terminating on $D_{\text {in }}$. Likewise, $\mathcal{G}$ terminates on every input probabilistic database, if its existential version terminates on all instances. There exists a lot of research surrounding the termination of existential Datalog programs, in particular regarding classes of programs (going way beyond the notion of weakly acyclicity) where termination is guaranteed, or at least decidable (see, for example, [17, 29]).

For probabilistic programs, termination is a more subtle notion. A program is said to be almost surely terminating if it terminates with probability one. If the program terminates in finitely many steps in expectation, it is called positively almost-sure terminating [9]. The hardness of these notions is studied in [43]. Active research in probabilistic programming is concerned with identifying sufficient criteria for determining (positive) almost-sure termination [14].

A more thorough investigation of, in particular, the probabilistic termination of generative Datalog programs is still open at this point.

## 7. The Full Probabilistic Programming Datalog Language

The GDatalog language is extended by constraints to obtain the full probabilistic-programming Datalog (PPDL) language [5] Section 5]. Formally, a PPDL program is a pair ( $\mathcal{G}, \Phi$ ) where $\Phi$ is some constraint specification, for example, a first order sentence over the database schema of $\mathcal{G}$. The semantics of the PPDL program is then given by conditioning the output $\mathcal{G}(\mathcal{D}):=$ $(\mathbb{D}, \mathfrak{D}, P)$ of $\mathcal{G}$ on the set of instances $\mathbb{D}_{\Phi}$ satisfying the constraint specification $\Phi$ (see [5, Definitions 5.1 and 5.3]). That is, the probability of an event $D$ is given as

$$
\frac{P\left(\boldsymbol{D} \cap \mathbb{D}_{\Phi}\right)}{P\left(\mathbb{D}_{\Phi}\right)} .
$$

There are multiple pitfalls to be aware of here. Conditioning the (sub-)probability space $\mathcal{G}(\mathcal{D})=(\mathbb{D}, \mathfrak{D}, P)$ in the suggested way requires not only that the set of instances $\mathbb{D}_{\Phi}$ satisfying $\Phi$ is measurable, but also that $P\left(\mathbb{D}_{\Phi}\right)>0$. If these two requirements are fulfilled, then the PPDL program returns a well-defined (sub-)probabilistic database.

As Bárány et al. note, the measurability of $\mathbb{D}_{\Phi}$ is already an issue in their setting of discrete distributions unless the program $\mathcal{G}$ is weakly acyclic [5, p. 22:18]. The good news is that in our framework of standard PDBs, the measurability of $\mathbb{D}_{\Phi}$ is constituted by Fact [2.6 as long as the constraint can be expressed by (for example) a relational algebra query.

Thus, we see the requirement of having $P\left(\mathbb{D}_{\Phi}\right)>0$ as the more delicate one. This discussion is bypassed in [5] by resorting to the weakly acyclic case, where the probability space gets discrete in their setup. For example, it might be reasonable to use constraints involving equality when designing PPDL programs. Alas, for example equality in $\mathbb{R}$ yields a set of Lebesgue measure 0 (namely the diagonal) in $\mathbb{R}^{2}$. From this, one can easily construct examples where $P\left(\mathbb{D}_{\Phi}\right)=0$ even though $\Phi$ is a very natural and typical kind of constraint. Trying to condition on events of measure 0 usually yields paradoxical results, as in the Borel-Kolmogorov paradox [46 p. 50 et seq.]. Still, this does not rule out sensible definitions of a probabilistic database conditioned on such a constraint. Work from the area of probabilistic programming however suggests that resolving the paradox is no trivial task [8 40].
Remark 7.1 (More Simulations). Bárány et al. [5] showed that their language version of PPDL can simulate Markov Logic Networks (MLNs) [58] and Probabilistic Context-Free Grammars (PCFGs) [7] (see [39]). Because we can simulate their language, we can simulate MLNs and PCFGs as well using the methods described in [5]. Since our language is more powerful, one might wonder whether these simulations can be reasonably extended.

There exist two extensions of MLNs that come to mind, one with infinitely many variables (Infinite MLNs [61]), and one with uncountable variable domains (although then finitely many variables; Hybrid MLNs [68]). Unfortunately, there is no easy correspondence between our language and these models.

- Hybrid MLNs describe distributions over a finite and fixed number of variables with uncountable domains, contrasting the unbounded size of our collections of facts. Moreover, the semantics of Hybrid MLNs only takes intervals of $\mathbb{R}$ into consideration.
- Infinite MLNs describe a probability space over countably (possibly countably infinitely) many variables all of which have (to our understanding) at most countable domains. While the resulting probability space is indeed uncountable, this contrasts our setting with a probability distribution over finite subsets of uncountable spaces. In particular, our continuous distributions seem to add no power here and, should a correspondence between infinite MLNs and Probabilistic Programming Datalog exist, then it is already exhibited by the discrete semantics of [5].

To our knowledge, there is no sensible generalization of PCFGs that involve uncountable spaces. Typically, the underlying domains in PCFGs are finite.

## 8. Conclusion

The original Probabilistic Programming Datalog language of Bárány, ten Cate, Kimelfeld, Olteanu and Vagena [5] is limited to discrete distributions. Given that continuous distributions appear in a variety of application scenarios for probabilistic databases (cf. [15, 21]), an extension with support for continuous distributions was noted as an open problem in [5]. In this paper, we developed such an extension.

Our key technical results are as follows:

1. The Generative Datalog language of [5] can be faithfully extended towards the support of continuous distributions, adding to its expressive power. Into this extended language, one may also incorporate conditioning under events of positive probability just as in [5].
2. We consider the semantics using a sequential and a parallel chase procedure, and show that the results of programs coincide under both kinds of approaches. In particular, outcomes do not depend on chase policies.
3. We may also use (sub-)probabilistic databases as inputs, and we obtain a well-defined output sub-probabilistic database as long as the probability of a finite computation is greater than 0 .

We summarize the technical developments in a high-level view of the (new) semantics. The generative part of a PPDL program can sample from probability distributions in order to generate new attribute values, and it can do so recursively. The semantics itself is described via chase trees, where different branches correspond to different samples. This becomes technically challenging with the introduction of continuous distributions, as nodes of the chase trees may now have uncountably many children. Exploiting the advanced machinery of probability and measure theory, we show that such uncountable chase trees are encodings of a Markov process of database instances. Associated with each Markov process is a probability measure on its paths, that is, on the paths of the chase tree. Each path in the tree that is starting in the root node corresponds to the process of building a database instance by adding facts one by one. Paths that end in leaf nodes then correspond to well-defined database instances. In cutting off infinite paths, and projecting the rest back to the represented database instances, we obtain a sub-probabilistic database "generated" by the program, that may afterwards be conditioned on satisfying a given set of constraints.

In addition to the added expressive power, we embed our semantics of PPDL into the Standard PDB framework of [37]. This makes the language compositional in the sense that it may use (sub-)probabilistic input databases, and produces (sub-)probabilistic output databases within the framework. We show the equivalence of sequential chase procedures with a notion of parallel chase, rendering the semantics quite robust.

## Future Work

The discrete version of PPDL can simulate every finite probabilistic database, for example, by using the MLN simulation. Thus, PPDL can be seen as a complete representation system for
finite PDBs. From the point of view of probabilistic databases, the most interesting question is how powerful PPDL is as a representation system for infinite PDBs. This is even unclear for the purely discrete version.

Moreover, PPDL raises a lot of natural questions in the overlap of probabilistic databases and probabilistic programming that remain unanswered. For the constraint part of a PPDL program, can we determine from the syntax of the program whether a particular constraint has measure 0 ? Are there sensible languages or fragments that avoid the issue of measure 0 conditioning? In the case of measure 0 conditioning, can we apply techniques from the probabilistic programming community in order to resolve the emergent problems in a reasonable way? And can we deal algorithmically with possibly infinite computation paths without resorting to mechanisms that ensure that all computations are finite?

That last question touches upon the challenging field of investigating the termination behavior of GDatalog programs in detail, and without restricting ourselves to a setting where all computations are finite. In general, whether the execution of a GDatalog program terminates may depend on the random choices that are made during the computation. That is, some paths in the chase tree may be terminating, while others are not. This is fine, as long as, for example the probability of the program terminating is 1 . It is an open research question to build a thorough understanding of the probabilistic termination of GDatalog programs.

Finally, it is not known whether (and if so, in which cases) the output probabilistic database of a generative Datalog program admits a concise representation allowing for the output to be queried effectively. At the moment, a generative Datalog program should be thought of as a representation of a probabilistic database itself, in particular with the option to have infinitely many possible worlds. In case of termination, we can always approximate the output through Monte Carlo sampling, by letting the program run multiple times. This can be extended to query answers by querying the samples. The properties of this option are yet to be explored.

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## A. Background Results from Measure Theory

This section is intended to extend Section 2.1 by some well-known results. They can accordingly be found in the literature [42, 62].

## A.1. Measurability of Functions and Sets

The following statement says that collections of measurable functions yield a function that is measurable with respect to the product $\sigma$-algebra.

Fact A. 1 ([42, Lemma 1.8, p. 5]). If $(\mathbb{X}, \mathfrak{X})$ and $\left(\mathbb{X}_{i}, \mathfrak{X}_{i}\right)$ are measurable spaces (for $i$ in some index set $I$ ) and $f_{i}: \mathbb{X} \rightarrow \mathbb{X}_{i}$ is measurable for all $i \in I$, then $f: \mathbb{X} \rightarrow \prod_{i \in I} \mathbb{X}_{i}: x \mapsto\left(f_{i}(x)\right)_{i \in I}$ is $\left(\mathfrak{X}, \bigotimes_{i \in I} \mathfrak{X}_{i}\right)$-measurable.

The next two results are concerned with the measurability of certain kinds of integration maps.

Fact A. 2 ([42, Lemma 1.41(i), p. 21]). Let $\mu$ be a stochastic kernel from $\mathbb{X}$ to $\mathbb{Y}$ and let $f: \mathbb{X} \times \mathbb{Y} \rightarrow$ $\mathbb{R}_{\geq 0}$ be measurable. Then

$$
\mathbb{X} \rightarrow \mathbb{R}_{\geq 0}: x \mapsto \int f(x, \cdot) d \mu(x, \cdot)
$$

is measurable.
Fact A. 3 ([42] Lemma 1.26, p. 14]). Let $(\mathbb{X}, \mathfrak{X})$ and $(\mathbb{Y}, \mathfrak{Y})$ be measurable spaces, $\mu$ a $\sigma$-finite measure on $\mathbb{X}$ and $f: \mathbb{X} \times \mathbb{Y} \rightarrow \mathbb{R}_{\geq 0}$ a measurable function. Then

1. the $y$-section $f(\cdot, y): \mathbb{X} \rightarrow \mathbb{R}_{\geq 0}$ of $f$ is $\left(\mathfrak{X}, \mathfrak{B o r}\left(\mathbb{R}_{\geq 0}\right)\right)$-measurable for all $y \in \mathbb{Y}$, and
2. the function $y \mapsto \int f(x, y) \mu(d x)$ is $(\mathfrak{Y}, \mathfrak{B o r}[0,1])$-measurable.

If we have a measurable function between two standard Borel spaces, then the image of measurable sets needs not to be measurable in general, the standard example perhaps being projection functions (see [62, Proposition 4.1.1 and Theorem 4.1.5]). Given certain conditions however, measurable sets have measurable images under measurable functions:

Fact A. 4 ([62, Theorem 4.5.4, p. 153]). Let $(\mathbb{X}, \mathfrak{X})$ and $(\mathbb{Y}, \mathfrak{Y})$ be standard Borel, $\boldsymbol{X} \in \mathfrak{X}$ and let $f: X \rightarrow \mathcal{Y}$ be an injective, $\left(\mathfrak{X} \upharpoonright_{X}, \mathfrak{Y}\right)$-measurable function. Then $f(X) \in \mathfrak{Y}$.

Finally, we come back to the multifunctions of Section2.1.4 and explicitly state the theorem of Kuratowski and Ryll-Nardzewski on the existence of measurable selections:

Fact A. 5 (Kuratowski and Ryll-Nardzewski [49], see [62, Theorem 5.2.1]). Let ( $\mathbb{X}, \mathfrak{X}$ ) be a measurable space and let $(\mathbf{Y}, \mathfrak{B o r}(\mathbb{Y}))$ be standard Borel. Then every closed-valued $\mathfrak{X}$-measurable multifunction $M: \mathbb{X} \rightrightarrows \mathbf{Y}$ has $a(\mathfrak{X}, \mathfrak{B o r}(\mathbb{Y}))$-measurable selection $s: \mathbb{X} \rightarrow \mathbb{Y}$.

## A.2. Identities for Integration

If $\mu$ is a measure and $f$ a measurable function, then $f \cdot \mu:=\nu$, defined by $v(X)=\int_{X} f d \mu$ is a measure. The following chain and substitution rules are the main tools to establish statements regarding the equality of transformed measures.

Fact A. 6 (Chain Rule, cf. [42, Lemma 1.23, p. 12]). Let $(\mathbb{X}, \mathfrak{X}, \mu)$ be a measure space and $f: \mathbb{X} \rightarrow$ $\mathbb{R}$ and $g: \mathbb{X} \rightarrow \mathbb{R}_{\geq 0}$ be measurable functions. Let $v:=f \cdot \mu$. Then, if either of the following integrals exists (i.e. is finite), it holds that

$$
\int_{\mathbf{X}} f \cdot g d \mu=\int_{\mathbf{X}} g d v
$$

Fact A. 7 (Substitution, cf. [42, Lemma 1.22, p.12]). Let (X, $\mathfrak{X})$ and (Y, $\mathfrak{Y})$ be measurable spaces and $\mu$ a measure on $(\mathbb{X}, \mathfrak{X})$. Let $f: \mathbb{X} \rightarrow \mathbb{Y}$ and $g: Y \rightarrow \mathbb{R}$ be measurable. Then, if either of the following integrals exists (i.e. is finite), it holds that

$$
\int_{\mathbf{X}} g \circ f d \mu=\int_{\mathbf{Y}} g d\left(\mu \circ f^{-1}\right)
$$

where $\mu \circ f^{-1}$ is the push-forward measure of $\mu$ along $f$ on (Y, $\left.\mathfrak{Y}\right)$.

## A.3. Existence of Markov Processes

Fact A. 8 (Existence of Markov Processes, Kolmogorov, cf. [42, Theorem 8.4]). Let ( $\mathbb{X}, \mathfrak{X}$ ) be a standard Borel space, $\mu_{0}$ a probability measure on $(\mathbb{X}, \mathfrak{X})$ and $\left(\mu_{i}\right)_{i \geq 1}$ a family of stochastic kernels $\mu_{i}: \mathbb{X} \times \mathfrak{X} \rightarrow[0,1]$ for $i \geq 1$. Then there exists a Markov process $\xi$ (with time scale $\mathcal{N}$ and paths in $\prod_{i=0}^{\infty} \mathbb{X}$ ) with initial distribution $\mu_{0}$ and transition kernels $\mu_{i}$.


[^0]:    ${ }^{1}$ Not to be confused with Kozen's Probabilistic Propositional Dynamic Logic [48].

[^1]:    ${ }^{2}$ In the examples of this section, we ignore any auxiliary relations that are introduced in the semantics. We will later see that this can always be done without introducing any problems (see Remark 4.15 in Section 4 .

[^2]:    ${ }^{3}$ The example itself is based on a famous example from [56].

[^3]:    ${ }^{4}$ Defining the output this way enables us to treat all input, output, and intermediate instances in the same space in our proofs. Alternatively, one could consider only the generated facts as the output but this is in the end just a matter of taste.

[^4]:    ${ }^{5}$ Units are only displayed for illustration, and not part of the tuples.

[^5]:    ${ }^{6}$ Note that $\vec{a}$ contains the full tuple of parameters (cf. 3.4. To be precise, we should write $\psi_{\varphi}\langle\vec{p}\rangle$ instead of $\psi_{\varphi}\langle\vec{a}\rangle$, where $\vec{p}$ is the projection of $\vec{a}$ to the parameter part. Again, all we need is that this transformation is measurable, but this is clearly the case, because $\vec{a} \mapsto \vec{p}$ is simply a projection between product measurable spaces. Alternatively, we could just let $\psi_{\varphi}$ be a version of the parameterized distribution that ignores those components of $\vec{a}$ that do not belong to the parameter. Either way, we just write $\psi_{\varphi}\langle\vec{a}\rangle$ and treat $\vec{a}$ as if it were the parameter tuple itself.

[^6]:    ${ }^{7}$ For the initial distributions, just a measure is needed. In particular, the problems of the Dirac distribution with respect to Fact 2.3 pointed out in Section 2.2 are irrelevant here.

[^7]:    ${ }^{8}$ This is not complicated to verify, but tedious. A proof of this can be found in [52 Lemma 6.4.14].

[^8]:    ${ }^{9}$ For details see [42 p. 21 \& Proposition 8.2].

[^9]:    ${ }^{10}$ To be precise, this notion is introduced for the full PPDL language (see Section 7 , not only the generative part in [5].
    ${ }^{11}$ With this, it is also clear that the first-order equivalence of [5] is a weaker notion than semantic equivalence.

