Isogeometric Analysis and Augmented Lagrangian Galerkin Least Squares Methods for Residual Minimization in Dual Norm

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Abstract

We explore how recent advances in Isogeometric analysis, Galerkin Least-Squares methods, and Augmented Lagrangian techniques can be applied to solve nonstandard problems, for which there is no classical stability theory, such as that provided by the Lax-Milgram lemma or the Banach-Necas-Babuska theorem. In particular, we consider continuation problems where a second-order partial differential equation with incomplete boundary data is solved given measurements of the solution on a subdomain of the computational domain. The use of higher regularity spline spaces leads to simplified formulations and potentially minimal multiplier space. We show that our formulation is inf-sup stable, and given appropriate a priori assumptions, we establish optimal order convergence.

1 Introduction

The advent of the Galerkin Least-Squares (GaLS) finite element method at the beginning of the 1980s was a significant advance in computational mechanics [8,34,35]. In particular, for problems in fluid mechanics, it has become a state-of-the-art tool. For problems in structural mechanics, the picture is less clear. In Hulbert and Hughes [36,38], a space-time finite element method for second-order hyperbolic partial differential equations (pde) was introduced using GaLS stabilization on the bulk residual and the jumps of the stresses over element faces and a discontinuous Galerkin method in time. It was, however, remarked that the stabilization did not improve the solution quality in computational examples, and it was later proved by French [30] and Johnson [39] independently that optimal error estimates could indeed be obtained without the stabilizing terms. Recently, however, the method of Hulbert and Hughes was reintroduced in the context of control problems for wave equations using space-time finite element methods [11]. The upshot here was that the stabilization used in the control context made it possible to leverage stability estimates designed specifically for control problems [4] and thereby obtain error estimates.

In this work, we are interested in exploring how recent advances in Isogeometric analysis [20], Galerkin Least-Squares methods, and Augmented Lagrangian techniques [14] can be applied also to nonstandard problems for which there is no classical stability theory, such as that provided by the Lax-Milgram lemma or the Banach-Necas-Babuska theorem. Instead, for example for certain control or data assimialtion problems stability can be proven under certain assumptions on the geometry and the data.

The problem, of that type, we consider here is to find $u \in H^1(\Omega)$ such that

$$\boxed{\mathcal{P}u = f \text{ in } H^{-1}(\Omega) \text{ and } u|_\omega = q} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^n$, $n = 2, 3, 4$, is a domain; $\omega \subset \bar{\Omega}$ is a subset of Ω, where $u = q$ with q known measured data; and $\mathcal{P} : H^1(\Omega) \to H^{-1}(\Omega) = [H^1_0(\Omega)]^*$ a second-order pde-operator. Depending on the given data, the problem can be well-posed or not. In the latter case the stability of the problem can be shown only using a priori assumptions of the existence of a solution in some space. This is known as *conditional stability.* Examples of relevant well-posed problems include indefinite problems, such as the Helmholtz equation or non-coercive convection-diffusion equations (for instance with non-solenoidal transport field). In the ill-posed case typical examples are elliptic problems where boundary data are unavailable on part of the boundary, but instead some other data set is at hand, for instance in the unique continuation problem the boundary conditions are unknown, but measurements of the solution are available in the bulk. Another related example is the elliptic Cauchy problem. Here the boundary data are not available on part of the boundary, but on the complement both Dirichlet and Neumann data are at hand. The former problem is a model problem for data assimilation and the latter problem typically appears in Electrocardiography

Typically, for ill-posed problems with conditional stability, an estimate of the form

$$\boxed{|u|_X \leq C(u)(\|u\|_{M(\omega)} + \|\mathcal{P}u\|_{H^{-1}(\Omega)})^\alpha, \quad \forall u \in H^1(\Omega)} \tag{1.2}$$

with $\alpha \in (0, 1)$ can be shown to hold (see, for instance, [1] for a detailed discussion in the context of the elliptic Cauchy problem). The $M(\omega)$-norm is the natural norm for the measurements. The coefficient α measures, in some sense, how ill-posed the problem is for the quantity measured in the X-seminorm. The estimate is conditional since it typically requires an a priori bound such as

$$C(u) \sim (\|u\|_{H^1(\Omega)} + \|\mathcal{P}u\|_{H^{-1}(\Omega)})^{1-\alpha} \lesssim 1 \tag{1.3}$$

Here and below we use the notation $a \lesssim b$ to denote $a \leq Cb$, where C is a positive constant and $a \sim b$ denotes $a \lesssim b$ and $b \lesssim a$. For ill-posed problems, there is no theory providing such a bound and it must therefore be an assumption on the exact solution. Of course, also well-posed second-order elliptic pdes satisfy estimates on the form (1.2), but without any a priori assumptions on the solution and with $\alpha = 1$. For instance, if $\mathcal{P} := \Delta + k^2\mathbb{I}$, $\omega = \partial\Omega$, and k^2 does not coincide with a Dirichlet-eigenvalue of the Laplace operator on Ω, then by the bounded inverse theorem there holds

$$\boxed{\|u\|_{H^1(\Omega)} \leq C_S(\|u\|_{H^{\frac{1}{2}}(\partial\Omega)} + \|\mathcal{P}u\|_{H^{-1}(\Omega)}), \quad \forall u \in H^1(\Omega)} \tag{1.4}$$

Observe that for homogeneous Dirichlet boundary conditions the boundary data term would vanish if we had chosen u in a space that satisfies the data constraint exactly, i.e., for $\omega = \partial\Omega$ and $q = 0$, $H^1_0(\Omega)$. In this framework, keeping the data term results in weak imposition of boundary conditions through a penalty similar to Nitsche's method [44].

In view of the stability estimate (1.2), a natural idea for solving such problems is to minimize the residual quantities implied by the right-hand side, and hence to find $u \in H^1(\Omega)$ that minimizes

$$J(v) := \frac{1}{2}\|v - q\|^2_{M(\omega)} + \frac{1}{2}\|\mathcal{P}v - f\|^2_{H^{-1}(\Omega)} \tag{1.5}$$

For well-posed problems, this approach was first considered in [6,7] and then in [9,17,19] for convection-diffusion type problems. It is also related to the discontinuous Petrov-Galerkin (DPG) method [23, 24, 41], with connections to earlier work on residual-free bubbles [28,32]. See [16] for a related approach in a discontinuous Galerkin setting.

For ill-posed problems, this least-squares minimization problem in negative norm has been proposed more recently in [18, 21]. When applied in a finite element framework, however, the implementation of the minimization problem in negative norm results in a delicate inf-sup condition, requiring a careful balancing of the trial and test spaces to ensure sufficient control of the pde-residual (for the DPG-method, this is related to the choice of the test space in the fully discrete scheme). This means that typically the test space (that approximates zero) has to be chosen larger than the trial space. In the case of ill-posed problems, it is not clear that stability can always be achieved in this way, for instance, when the equation has variable coefficients.

Another classical approach typically associated with the ill-posed problems of data assimilation is to use pde-constrained optimization to fit the data $u|_\omega = q$. This results in a Lagrangian on the form, $\mathcal{L} : H^1(\Omega) \times H^1_0(\Omega) \to \mathbb{R}$,

$$\mathcal{L}(u, z) := \frac{1}{2}\|u - q\|^2_{M(\omega)} + a(u, z) - \langle f, z\rangle_{H^{-1}(\Omega),H^1_0(\Omega)} \tag{1.6}$$

where $a(\cdot, \cdot)$ denotes the weak form of $\langle \mathcal{P}\cdot, \cdot\rangle_{H^{-1}(\Omega),H^1_0(\Omega)}$. Naive discretizations of (1.6) have poor stability, and for ill-posed problems, traditionally, Tikhonov regularizations are added. If the problem has (at least) some conditional stability, this is not always strictly necessary, as shown in [13]. Using a primal-dual type stabilized finite element method allows for stable discretizations of (1.6) using arbitrary order interpolation for the forward and dual equations [9]. Also for this problem it is possible to construct methods that are inf-sup stable [25, 42].

To obtain stable discretizations of (1.6) without any constraints on the test and trial spaces, both a stabilization of the Lagrange multiplier in the spirit of [2, 3], and an Augmented Lagrangian term of Galerkin Least-Squares type are needed. Indeed in [2,3] inf-sup stability is obtained by penalizing the distance between two approximations of the Lagrange multiplier, and here, the exact multiplier is zero, so any inner product applied to the multiplier has the same effect. Since the pde is the constraint, the least-squares type augmentation coincides with the classical GaLS term applied to second-order pde [31]. For finite element methods using C^0-approximation, typically, the solution gradient jump must also be penalized. In the space-time discretization of the wave equation, we recover exactly the stabilization of Hulbert-Hughes.

In this paper, we will show how Isogeometric Analysis [5] can be used in the Augmented Lagrangian Galerkin Least-Squares (AGaLS) framework leading to a powerful way of designing H^{-1}-residual minimizing methods, potentially with a minimal multiplier space. The rationale for using splines for the approximation space is their good approximation properties compared to the number of degrees of freedom [26], their good spectral properties [29, 37], and the C^1-regularity that allows us to eliminate the penalty term on the gradient that is present for finite element methods. Using the GaLS stabilization, on the other hand, allows us to use the smallest possible space for the adjoint variable, which approximates zero without losing stability. Indeed the only requirement for the multiplier space is that it has optimal approximation properties in the H^1-norm on a scale similar to that of the forward problem. The proposed approach increases the number of degrees of freedom compared to a standard Galerkin method. However, since the multiplier space can be chosen of such low order, the relative increase is much lower than is the case, for instance, in discontinuous Galerkin methods. We are indeed free to choose the test space independently of the trial space, which opens up possible new venues in the spirit of variational multiscale methods [33], but here on the adjoint side rather than the primal side, as already discussed in [17, 43].

There has recently been increased interest in the relation between the Augmented Lagrange Method and Galerkin Least-Squares stabilization prompted by the ideas behind Nitsche's method for imposing boundary conditions. The method we discuss below enters this framework, but here the constraint is the pde (on weak or strong form depending on the choice of spaces). The stabilization, therefore, coincides with the known Galerkin Least-Squares stabilization of the second-order operator, as mentioned above. For an overview of the relation between Galerkin Least-Squares methods and Augmented Lagrangian methods, we refer to [14]. For simplicity, we restrict the discussion below to linear partial differential operators with smooth coefficients in this contribution.

An outline of the paper is as follows. First, we introduce the discrete Augmented Lagrangian and its associated optimality system. In this context, we also define the stabilizing terms (Section 2). In the following Section 3, we discuss the stability of the methods and first show inf-sup stability in a particular seminorm when the Galerkin Least-Squares terms are included, and then we give an inf-sup result for the unstabilized method, relying on a "sufficiently fine" test space. Note, however, that inf-sup stability holds for the seminorm defining J in (1.5) and is therefore not related to the Lax-Milgram Lemma or the Banach-Necas-Babuska Theorem. Once such an inf-sup stability has been established, error estimates do not follow in a standard fashion. Instead, a stability estimate such as (1.2) or (1.4) must be at hand to ensure the triple seminorm bounds the error in a norm. Assuming such a stability of the pde problem, we then prove error estimates in Section 4. We conclude the theoretical investigations in Section 5 by giving examples of pde problems that enter the framework and showing the associated error estimates. Finally, in Section 6, we provide some numerical examples.

2 Discretization

2.1 The finite element spaces

Let $\mathcal{T}_V(\Omega)$ and $\mathcal{T}_W(\Omega)$ denote two different decompositions of Ω in elements K_V and K_W respectively, with mesh size parameters $h_V = \max_{K_V \in \mathcal{T}_V} \operatorname{diam}(K_V)$ and $h_W = \max_{K_W \in \mathcal{T}_W} \operatorname{diam}(K_W)$. By $\mathcal{F}(\mathcal{T})$ we denote the set of interior element faces in $\mathcal{T}$. We let $V_h \subset C^1(\Omega)$ denote an approximation space defined on the tesselation $\mathcal{T}_V$, with local polynomial degree less than or equal to p and W_h an approximation space on $\mathcal{T}_W$ to be defined below. We assume that there exist interpolation operators $\pi_V : H^1(\Omega) \to V_h$ and $\pi_W : H^1(\Omega) \to W_h$, such that

$$\|v - \pi_V v\|_{H^m(\Omega)} \leq C_{\pi_V} h_V^{k-m} \|v\|_{H^k(\Omega)}, \qquad 0 \leq m \leq k \leq p+1 \tag{2.1}$$

with $m = 0, 1, 2$, and

$$\|v - \pi_W v\|_\Omega \leq C_{\pi_W} h_W \|v\|_{H^1(\Omega)} \tag{2.2}$$

We note that the $H^2(\Omega)$ norm is well defined on V_h, since V_h is a piecewise polynomial C^1 space and that for π_W we only need approximation in H^1.

Remark 2.1. Note that C^1-piecewise polynomial spaces are typically constructed using tensor products of spline functions. We may also handle domains Ω with complex geometry by using trimmed or cut elements. See Remark 2.6 below for the necessary modifications of the method in the case of cut elements.

2.2 Discrete augmented Lagrangian and the optimality system

We consider the problem; find $u \in H^1(\Omega)$ satisfying (1.1). The discrete augmented version of (1.6) reads in this case $\mathcal{L} : V_h \times W_h \to \mathbb{R}$,

$$\mathcal{L}(v_h, \phi_h) := \frac{\gamma}{2}\|v_h - q\|^2_{M(\omega)} + a(v_h, \phi_h) - (f, \phi_h)_\Omega + \underbrace{\frac{1}{2}\|\tau^{\frac{1}{2}}(\mathcal{P}v_h - f)\|^2_\Omega}_{\text{residual control}} \underbrace{- \frac{1}{2}\|\phi_h\|^2_s}_{\text{inf-sup stability}} \quad (2.3)$$

where we have included the parameter γ that allows trading weight between enforcing the known data and the pde. Here $\|v\|^2_{M(\omega)} = (v, v)_{M(\omega)}$ is appropriately defined depending on ω, where we typically may take $(v, v)_{M(\omega)} = h_V^{-2r}(v, v)_\omega$ for a suitable choice of r, see the assumption (3.2) on the form $(\cdot, \cdot)_{M(\omega)}$ and Section 5 for examples. Further, we have added two additional terms to (1.6): to ensure sufficient control of the pde residual, we have augmented the Lagrangian with a Galerkin Least-Squares term, and to ensure inf-sup stability, we have added a stabilizing term $\|v\|^2_s = s(v, v)$ that we will define below. The GaLS-stabilization parameter τ is typically chosen as $\tau = \tau_0 h_W^2$ for some constant τ_0, since then $\|\tau^{\frac{1}{2}}(\mathcal{P}v_h - f)\|_\Omega \sim \|h(\mathcal{P}v_h - f)\|_\Omega$, which is a convenient discrete approximation of $\|\mathcal{P}v_h - f\|_{H^{-1}(\Omega)}$.

The Euler-Lagrange equations of (2.3) take the form: find $(u_h, z_h) \in V_h \times W_h$ such that for all $(v_h, \phi_h) \in V_h \times W_h$ there holds

$$a(u_h, \phi_h) - s(z_h, \phi_h) = (f, \phi_h)_\Omega \quad (2.4)$$

$$a(v_h, z_h) + (\tau\mathcal{P}u_h, \mathcal{P}v_h)_\Omega + \gamma(u_h, v_h)_{M(\omega)} = (f, \tau\mathcal{P}v_h)_\Omega + \gamma(q, v_h)_{M(\omega)} \quad (2.5)$$

Introducing the global form

$$A[(v_h, \phi_h), (w_h, \psi_h)] := a(v_h, \psi_h) + a(w_h, \phi_h) \quad (2.6)$$

$$- s(\phi_h, \psi_h) + (\tau\mathcal{P}v_h, \mathcal{P}w_h)_\Omega + \gamma(v_h, w_h)_{M(\omega)} \quad (2.7)$$

the formulation (2.4)–(2.5) may be written: find $(u_h, z_h) \in V_h \times W_h$ such that for all $(v_h, \phi_h) \in V_h \times W_h$ there holds

$$A[(u_h, z_h), (v_h, \phi_h)] = F_h[(v_h, \phi_h)] \quad (2.8)$$

with

$$F_h[(v_h, \phi_h)] := (f, \phi_h + \tau\mathcal{P}v_h)_\Omega + \gamma(q, v_h)_{M(\omega)} \quad (2.9)$$

2.3 The choice of the space W_h and the stabilizing form s

The key design criterion for the form s, with associated norm $\|\cdot\|_s$, is that there exists a norm $\|\cdot\|_*$ such that

$$a(v - v_h, \phi_h) \lesssim \|v - v_h\|_* \|\phi_h\|_s \quad (2.10)$$

and

$$\inf_{v_h \in W_h} \|v - v_h\|_* \lesssim h^p \|v\|_{H^{p+1}(\Omega)} \quad (2.11)$$

The inequality (2.10) must be proven in different ways depending on the operator $\mathcal{P}$ and we will give such results in Section 5 below.

We will consider two different choices of W_h. Either a C^0 space on $\mathcal{T}_W$ of degree $k \geq 1$ with functions that are zero on $\partial\Omega$, or the space of piecewise constant functions on $\mathcal{T}_W$. In both cases we define the form a by

$$a(v_h, \phi_h) := (\mathcal{P}v_h, \phi_h)_\Omega, \qquad v_h \in V_h, \phi_h \in W_h \tag{2.12}$$

that is we take the L^2-inner product of the strong form, which is well defined on V_h since $V_h \subset C^1$, and a test function in W_h. Using partial integration we note that for the C^0 space a is the standard bilinear form associated with $\mathcal{P}$, while for the piecewise case we get terms involving the jump in the test function on faces and a boundary term since the functions in W_h are not zero on the boundary. If we need to distinguish the two different choices of W_h, we add a subscript 0 to the bilinear form if the order of the functions in W_h is 0 and a subscript 1 if the order is larger than or equal to 1. We define the stabilization operator differently depending on the choice of W_h. More precisely, the operators s_i, $i = 0, 1$, are defined by

$$s_0(\phi_h, \psi_h) := \sum_{F \in \mathcal{F}(\mathcal{T}_W)} (h_W^{-1}[\phi_h], [\psi_h])_F + (h_W^{-1}\phi_h, \psi_h)_{\partial\Omega} \tag{2.13}$$

where $[y]|_F$ denotes the jump of y over the interior face F, and

$$s_1(\phi_h, \psi_h) := (\nabla\phi_h, \nabla\psi_h)_\Omega \tag{2.14}$$

In general we use the notation s for the two stabilizing forms and add the index when we need to be more specific. An important property of the stabilizing form s is that for all $v \in H_0^1(\Omega)$ there holds

$$\boxed{\|\pi_W v\|_s \leq C_s \|v\|_{H^1(\Omega)}} \tag{2.15}$$

where $\|\cdot\|_s$ is the norm associated with the form s. For s_1 this inequality is immediate by the stability of π_W and for s_0 it follows by using that $[v]_F = 0$ for all $v \in H_0^1(\Omega)$ and application of the elementwise trace inequality. If piecewise splines are used also for the multiplier space, the trace inequality from [27] can be applied.

Remark 2.2. The boundary condition on the W_h is needed to mimic control of the residual $\mathcal{P}u - f$ in the $H^{-1}\Omega) = [H_0^1(\Omega)]^*$ norm, since $a(v_h, \phi_h) - (f, \phi_h)_\Omega = (\mathcal{P}v_h - f, \phi_h)_\Omega$. In [22], page 46, it was shown in a related, but restricted to well-posed problems, situation that the boundary condition on W_h is necessary to have the equality between the inf-sup and the sup-inf.

Remark 2.3. We can take $\tau = 0$ above and get a stable and accurate method provided W_h is sufficiently fine compared to V_h, that is, h_W/h_V sufficiently small. Choosing $\tau > ch_W^2$, with a sufficiently large constant c, eliminates this mesh constraint while retaining the good properties of the H^{-1} minimization method. Observe that, from only minimizing the GaLS term, we cannot expect optimal results using the approach below since we bound the H^1-norm with the H^2-norm and hence lose one power of h_V. We derive these results in Sections 3 and 4.

Remark 2.4. Assume that $V_h \in H_0^1(\Omega)$ so that we consider a well-posed problem with strongly imposed boundary conditions and can omit the data term. For the above choices of s_i, equation (2.4) can be solved for z_h, and in the absence of the data term, we get $z_h = S^{-1}(\mathcal{P}u_h - f)$, where S^{-1} is the inverse of the discrete Laplacian defined by s_i. Injecting this into equation (2.5) we get

$$(\mathcal{P}u_h, (\tau + S^{-1})\mathcal{P}v_h)_\Omega = (f, (\tau + S^{-1})\mathcal{P}v_h)_\Omega \tag{2.16}$$

showing that the formulation is a combination of two discrete H^{-1} inner products on the residual: on the one hand, the h-weighted L^2-norm, and on the other hand, the approximate H^{-1}-norm. If h_W becomes small, S^{-1} is a good approximation of the inverse Laplacian, and therefore the L^2-contribution can be reduced or even omitted.

Remark 2.5. Note that both proposed stabilizations of the multiplier are consistent since the exact solution for the multiplier is zero in this context. It is indeed a stabilizer, similar in spirit to that in [2] where a least-squares stabilization is added on the distance between two realizations of the multiplier, just that in our case, the exact multiplier is known to be zero. In other situations, such as for instance in control problems, or if dual arguments must be applied, the stabilization of the adjoint equation must also be consistent and therefore residual based (see [9, 11]).

Remark 2.6. In [17] the method without stabilization was interpreted as a variational multiscale method. We see here that, similarly as in SUPG the Augmented Lagrangian version introduces a rough subgrid model that stabilizes the solution if the fine scales resolved by W_h are insufficient. This is also the intuitive way of understanding the stability analysis below.

Remark 2.7. In our exposition, we assume, for simplicity, that the meshes $\mathcal{T}$ are fitted to Ω, but the framework presented herein can easily be extended to the CutIGA framework by combining with the ideas from [10, 12, 15, 40]. To fix the ideas, let Ω_0 be some meshed domain with $\Omega \subset \Omega_0$ and define $\mathcal{T}_V$ and $\mathcal{T}_W$ be all the elements that intersect Ω in a suitable decomposition of Ω_0. We assume that the elements of the two partitions cover the same domain and denote this domain $\Omega_\mathcal{T}$. The only changes necessary to the above formulation are that

- The functions in V_h are defined through the extension technique proposed in [15] or suitable stabilization [40] is added.
- The stabilization term s_i should be integrated on the whole mesh domain $\Omega_\mathcal{T}$, but since the boundary condition in W_h on $\partial\Omega$ no longer can be imposed strongly, it must be added in the form of a penalty term. While s_0 already is defined this way, s_1 must be modified such that

$$s_1(\phi_h, \psi_h) := (\nabla\phi_h, \nabla\psi_h)_{\Omega_\mathcal{T}} + (h_W^{-1}\phi_h, \psi_h)_{\partial\Omega} \tag{2.17}$$

- Since the boundary condition in W_h is weakly enforced the form a is defined by (2.12), which for a second order operator $\mathcal{P}v = -\nabla\cdot(D\nabla v)$ gives

$$a(v_h, \phi_h) = (\mathcal{P}v_h, \phi_h)_\Omega = (D\nabla v_h, \nabla\phi_h)_\Omega - (n\cdot(D\nabla v_h), \phi_h)_{\partial\Omega} \tag{2.18}$$

With those modifications, the analysis herein carries over verbatim.

3 Stability

3.1 Inf-sup stability with GaLS-stabilization

We define the triple norm

$$|||(v,\phi)|||^2 := \gamma\|v\|^2_{M(\omega)} + \|\mathcal{P}v\|^2_{H^{-1}(\Omega)} + \|\tau^{\frac{1}{2}}\mathcal{P}v\|^2_\Omega + \|\phi\|^2_s \tag{3.1}$$

on $(H^1(\Omega)+V_h)\times(H^1_0(\Omega)+W_h)$. We note that the norm involves three terms which are present in the method and the additional control of $\|\mathcal{P}v\|_{H^{-1}(\Omega)}$. First we note that if there is a constant such that

$$\|v\|_{M(\omega)} \lesssim h_V^{-1}\|v\|_{L^2(\Omega)} + \|v\|_{H^1(\Omega)} \tag{3.2}$$

and $h_W \lesssim h_V$, then the interpolation error estimate

$$\boxed{|||(u-\pi_V u, 0)||| \lesssim h_V^p} \tag{3.3}$$

holds, since, with $\theta = u - \pi_V u$, we have the estimates

$$\begin{aligned} |||(\theta,0)|||^2 &\lesssim h_V^{-2}\|\theta\|^2_{L^2(\Omega)} + \|\theta\|^2_{H^1(\Omega)} + \|\mathcal{P}\theta\|^2_{H^{-1}(\Omega)} + \|h_W\mathcal{P}\theta\|^2_\Omega & (3.4)\\ &\lesssim h_V^{-2}\|\theta\|^2_{L^2(\Omega)} + \|\theta\|^2_{H^1(\Omega)} + \|\theta\|^2_{H^1(\Omega)} + h_W^2\|\theta\|^2_{H^2(\Omega)} & (3.5)\\ &\lesssim h_V^{-2}h_V^{2(p+1)} + h_V^{2p} + h_V^{2p} + h_W^2 h_V^{2(p-1)} & (3.6)\\ &\lesssim h_V^{2p} & (3.7)\end{aligned}$$

Here we used the estimate $\tau = \tau_0 h_W^2 \lesssim h_W^2 \lesssim h_V^2$, boundedness $\|\mathcal{P}\theta\|_{H^{-1}(\Omega)} \lesssim \|\theta\|_{H^1(\Omega)}$ of the operator $\mathcal{P}: H^1(\Omega) \to H^{-1}(\Omega)$, and the interpolation estimate (2.1) for π_V.

We shall next prove the following important inf-sup result.

Lemma 3.1. *There is a constant, proportional to τ_0, such that for all $(v_h,\phi_h)\in V_h\times W_h$,*

$$\boxed{|||(v_h,\phi_h)||| \lesssim \sup_{(w_h,\psi_h)\in V_h\times W_h} \frac{A[(v_h,\phi_h),(w_h,\psi_h)]}{|||(w_h,\psi_h)|||}} \tag{3.8}$$

Proof. We first note that the following positivity property holds

$$\boxed{\gamma\|v_h\|^2_{M(\omega)} + \|\tau^{\frac{1}{2}}\mathcal{P}v_h\|^2_\Omega + \|\phi_h\|^2_s = A[(v_h,\phi_h),(v_h,-\phi_h)]} \tag{3.9}$$

and hence to control the triple norm $|||(v_h,\phi_h)|||$, we only need to establish control of the H^{-1}-norm of $\mathcal{P}v_h$. To this end, consider the auxiliary problem

$$-\Delta\varphi + \varphi = \mathcal{P}v_h \quad \text{in } \Omega, \qquad \varphi = 0 \quad \text{on } \partial\Omega \tag{3.10}$$

with weak form: find $\varphi \in H^1_0(\Omega)$ such that

$$(\varphi, w)_{H^1(\Omega)} = (\mathcal{P}v_h, w)_\Omega \qquad \forall w \in H^1_0(\Omega) \tag{3.11}$$

where $(v,w)_{H^1(\Omega)} = (\nabla v, \nabla w)_\Omega + (v,w)_\Omega$ is the $H^1(\Omega$ inner product. By the Lax-Milgram lemma there is a unique solution $\varphi \in H^1_0(\Omega)$ to (3.11) and we also have the following identity

$$\|\varphi\|_{H^1(\Omega)} = \sup_{\substack{v\in H^1_0(\Omega)\\ \|v\|_{H^1(\Omega)}=1}} (\varphi, v)_{H^1(\Omega)} = \sup_{\substack{v\in H^1_0(\Omega)\\ \|v\|_{H^1(\Omega)}=1}} (\mathcal{P}v_h, v)_\Omega = \|\mathcal{P}v_h\|_{H^{-1}(\Omega)} \tag{3.12}$$

Setting $w = \varphi$ in (3.11) and using (3.12) give

$$(\mathcal{P}v_h, \varphi)_\Omega = (\varphi, \varphi)_{H^1(\Omega)} = \|\varphi\|^2_{H^1(\Omega)} = \|\mathcal{P}v_h\|^2_{H^{-1}(\Omega)} \tag{3.13}$$

Adding and subtracting an interpolant $\pi_W \varphi \in W_h$, we obtain

$$\begin{aligned}
\|\mathcal{P}v_h\|^2_{H^{-1}(\Omega)} &\leq (\mathcal{P}v_h, \varphi - \pi_W \varphi)_\Omega + (\mathcal{P}v_h, \pi_W \varphi)_\Omega & (3.14)\\
&\leq \|\mathcal{P}v_h\|_\Omega \|\varphi - \pi_W \varphi\|_\Omega + a(v_h, \pi_W \varphi) & (3.15)\\
&\leq \|\mathcal{P}v_h\|_\Omega C_{\pi_W} h_W \|\varphi\|_{H^1(\Omega)} + a(v_h, \pi_W \varphi) & (3.16)\\
&\leq \|\mathcal{P}v_h\|_\Omega C_{\pi_W} h_W \|\mathcal{P}v_h\|_{H^{-1}(\Omega)} + a(v_h, \pi_W \varphi) & (3.17)
\end{aligned}$$

where we used the Cauchy-Schwarz inequality, the interpolation estimate (2.2), and the identity (3.12). Using Young's inequality it finally follows that

$$\boxed{\|\mathcal{P}v_h\|^2_{H^{-1}(\Omega)} - C^2_{\pi_W} h^2_W \|\mathcal{P}v_h\|^2_\Omega \leq 2a(v_h, \pi_W \varphi)} \tag{3.18}$$

To prove the inf-sup condition (3.8), we note that taking $w_h = v_h$ and $\psi_h = -\phi_h + 2\zeta\pi_W\varphi$, with $\zeta > 0$ a parameter, and applying (3.9) followed by estimate (3.18) for the a-form and then (2.15) and (3.12) for the s-form, we obtain

$$\begin{aligned}
&A[(v_h, \phi_h), (v_h, -\phi_h + 2\zeta\pi_W\varphi)] & (3.19)\\
&= A[(v_h, \phi_h), (v_h, -\phi_h)] + A[(v_h, \phi_h), (0, 2\zeta\pi_W\varphi)] & (3.20)\\
&= \gamma\|v_h\|^2_{M(\omega)} + \|\tau^{\frac{1}{2}}\mathcal{P}v_h\|^2_\Omega + \|\phi_h\|^2_s + \zeta 2a(v_h, \pi_W\varphi) - \zeta s(\phi_h, 2\pi_W\varphi) & (3.21)\\
&\geq \gamma\|v_h\|^2_{M(\omega)} + \|\tau^{\frac{1}{2}}\mathcal{P}v_h\|^2_\Omega + \|\phi_h\|^2_s & (3.22)\\
&\qquad + \zeta(\|\mathcal{P}v_h\|^2_{H^{-1}(\Omega)} - C^2_{\pi_W} h^2_W \|\mathcal{P}v_h\|^2_\Omega) - 2\zeta C_s\|\phi_h\|_s\|\mathcal{P}v_h\|_{H^{-1}(\Omega)} & (3.23)\\
&\geq \gamma\|v_h\|^2_{M(\omega)} + (1 - \zeta C^2_{\pi_W} h^2_W \tau^{-1})\|\tau^{\frac{1}{2}}\mathcal{P}v_h\|^2_\Omega & (3.24)\\
&\qquad + (1 - 4\zeta C^2_s)\|\phi_h\|^2_s + \zeta\frac{1}{2}\|\mathcal{P}v_h\|^2_{H^{-1}(\Omega)} & (3.25)\\
&\geq (\min(1, 1 - \zeta C^2_{\pi_W} h^2_W/\tau, 1 - 4\zeta C^2_s, \zeta/2) |||(v_h, \phi_h)|||^2 & (3.26)\\
&\geq c_\zeta |||(v_h, \phi_h)|||^2 & (3.27)
\end{aligned}$$

Recalling that $\tau = \tau_0 h^2_W$ we get $1 - \zeta C^2_{\pi_W} h^2_W/\tau = 1 - \zeta C^2_{\pi_W}/\tau_0$ and we note that taking $\zeta = \min(\tau_0/(2C^2_{\pi_W}), 1/(8C^2_s)$ gives

$$\begin{aligned}
c_\zeta &= \min(1, 1 - \zeta C^2_{\pi_W} h^2_W/\tau, 1 - 4\zeta C^2_s, \zeta/2) & (3.28)\\
&= \min(1, 1/2, 1/2, \zeta/2) = \min(1/2, \tau_0/(4C^2_{\pi_W}), 1/(16C^2_s)) & (3.29)
\end{aligned}$$

We conclude that with hidden constant c_ζ^{-1} it holds

$$|||(v_h, \phi_h)|||^2 \lesssim A[(v_h, \phi_h), (v_h, -\phi_h + 2\pi_W\varphi)] \tag{3.30}$$

Finally, we also need to show that, for $w_h = v_h$ and $\psi_h = -\phi_h + 2\zeta\pi_W\varphi$,

$$|||(w_h, \psi_h)||| \lesssim |||(v_h, \phi_h)||| \tag{3.31}$$

Clearly, since $|||(w_h, 0)||| = |||(v_h, 0)||| \leq |||(v_h, \phi_h)|||$, we only need to show that $|||(0, \psi_h)||| \lesssim |||(v_h, \phi_h)|||$. To that end applying (2.15), and (3.12) yield

$$\begin{aligned}
|||(0, \psi_h)||| &= \| - \phi_h + 2\zeta\pi_W\varphi\|_s \leq \|\phi_h\|_s + 2\zeta\|\pi_W\varphi\|_s & (3.32)\\
&\lesssim \|\phi_h\|_s + \|\varphi\|_{H^1(\Omega)} \lesssim \|\phi_h\|_s + \|\mathcal{P}v_h\|_{H^{-1}(\Omega)} \lesssim |||(v_h, \phi_h)||| & (3.33)
\end{aligned}$$

We conclude that (3.31) holds. It then follows from (3.30) and (3.31) that the inf-sup stability (3.8) holds. ∎

Remark 3.1. Note that the triple norm of (3.1) is a norm only subject to a stability estimate such as (1.2) or (1.4). This means that the method is agnostic as to whether the problem is well-posed. We have stability in the residual norm, which will lead to optimal error estimates in this norm and then error estimates with whatever rate the physical stability allows in a second step. If the problem is ill-posed without any stability, the convergence of the residual norm says nothing about the accuracy of the solution. This is similar to the situation in a posteriori error estimation, where the residual is a good error estimator only if a good measure of the physical stability matching it to some goal quantity is available.

Remark 3.2. At least for $\mathcal{P}$ with constant coefficients, the spaces V_h and W_h may be chosen carefully so the Fortin interpolant exists, meaning that given $\varphi \in H_0^1(\Omega)$ there exists $\varphi_h \in W_h$ such that $\|\varphi_h\|_{H^1(\Omega)} \lesssim \|\varphi\|_{H^1(\Omega)}$ and $(\mathcal{P}v_h, \varphi)_\Omega = (\mathcal{P}v_h, \varphi_h)_\Omega$. For examples we refer to [21]. In that case one may take $\tau = 0$. If the partial differential equation has varying coefficients, it is unclear how to design the Fortin interpolant. In any case, it always requires $h_W < h_V$, meaning that the adjoint space that approximates zero has to be larger than the space V_h that must have approximation properties.

3.2 Inf-sup stability without GaLS-stabilization

The objective of this section is to show that for a problem with a stability of the type (1.4), provided W_h is sufficiently rich, the inf-sup condition is satisfied, without using any further information on V_h. To make the below argument work, we need to assume that the space V_h satisfies the following inverse inequality (see [5, Theorem 4.1], for the leading order term),

$$\|\mathcal{P}v_h\|_\Omega \le C_I h_V^{-1} \|v_h\|_{H^1(\Omega)} \tag{3.34}$$

To prove (3.8) we proceed as in (3.19)-(3.26), but with $\tau = 0$, which gives

$$\begin{aligned}
&A[(v_h, \phi_h), (v_h, -\phi_h + 2\zeta\pi_W\varphi)] && (3.35)\\
&= A[(v_h, \phi_h), (v_h, -\phi_h)] + A[(v_h, \phi_h), (0, 2\zeta\pi_W\varphi)] && (3.36)\\
&= \gamma\|v_h\|^2_{M(\omega)} + \|\phi_h\|_s^2 + \zeta 2a(v_h, \pi_W\varphi) - \zeta s(\phi_h, 2\pi_W\varphi) && (3.37)\\
&\ge \gamma\|v_h\|^2_{M(\omega)} + \|\phi_h\|_s^2 && (3.38)\\
&\qquad + \zeta(\|\mathcal{P}v_h\|^2_{H^{-1}(\Omega)} - C^2_{\pi_W} h_W^2 \|\mathcal{P}v_h\|^2_\Omega) - 2\zeta C_s \|\phi_h\|_s \|\mathcal{P}v_h\|_{H^{-1}(\Omega)} && (3.39)\\
&\ge \gamma\|v_h\|^2_{M(\omega)} - \zeta C^2_{\pi_W} h_W^2 \|\mathcal{P}v_h\|^2_\Omega + (1 - 4\zeta C_s^2)\|\phi_h\|_s^2 + \zeta\frac{1}{2}\|\mathcal{P}v_h\|^2_{H^{-1}(\Omega)} && (3.40)\\
&= \min(1, 1 - 4\zeta C_s^2, \zeta/2)|||(v_h, \phi_h)||| - \zeta C^2_{\pi_W} h_W^2 \|\mathcal{P}v_h\|^2_\Omega && (3.41)
\end{aligned}$$

We conclude that for appropriate choices of ζ there is a constant c_ζ, for instance we may take $\zeta = 1/(8C_s^2)$ which gives $c_\zeta = \min(1/2, 1/(16C_s^2))$, such that

$$c_\zeta |||(v_h, \phi_h)|||^2 - \zeta C^2_{\pi_W} h_W^2 \|\mathcal{P}v_h\|^2_\Omega \le A[(v_h, \phi_h), (v_h, -\phi_h + 2\zeta\pi_W\varphi)] \tag{3.42}$$

Now using the inverse inequality (3.34) in the second term on the left-hand side we see that

$$c_\zeta |||(v_h, \phi_h)|||^2 - \zeta C^2_{\pi_W} C_I^2 \left(h_W/h_V\right)^2 \|v_h\|^2_{H^1(\Omega)} \lesssim A[(v_h, \phi_h), (v_h, -\phi_h + 2\pi_W\varphi)] \tag{3.43}$$

Clearly, the first term on the left-hand side cannot control the second. However, if $\mathcal{P}$ satisfies a stability estimate of the type (1.4) we may conclude that the inf-sup condition

(3.8) holds in the following way. Using that $V_h \subset H^1(\Omega)$ it follows from (1.4), with homogeneous boundary conditions, that

$$\|v_h\|_{H^1(\Omega)} \leq C_S \|\mathcal{P} v_h\|_{H^{-1}(\Omega)} \leq C_S |||(v_h, 0)||| \leq C_S |||(v_h, \phi_h)||| \tag{3.44}$$

and hence

$$c_\zeta/2 |||(v_h, \phi_h)|||^2 + \left(c_\zeta/(2C_S^2) - \zeta C_{\pi_W}^2 C_I^2 \left(h_W/h_V\right)^2\right) \|v_h\|^2_{H^1(\Omega)} \tag{3.45}$$

$$\lesssim A[(u_h, \phi_h), (u_h, -\phi_h + 2\pi_W \varphi)] \tag{3.46}$$

Thus taking h_W sufficiently small, $2\zeta C_S^2 C_{\pi_W}^2 C_I^2 h_W^2 < c_\zeta h_V^2$, so that the second term on the left-hand side is positive we conclude that

$$c_\zeta/2 |||(v_h, \phi_h)|||^2 \lesssim A[(u_h, \phi_h), (u_h, -\phi_h + 2\pi_W \varphi)] \tag{3.47}$$

and thus, noting that (3.31) also holds with $\tau = 0$, it follows that the inf-sup condition (3.8) holds in the same way as above. Using the stability (3.44), we finally obtain

$$\boxed{C_S^{-1} \|v_h\|_{H^1(\Omega)} + |||(v_h, \phi_h)||| \lesssim \sup_{(w_h, \psi_h) \in V_h \times W_h} \frac{A[(v_h, \phi_h), (w_h, \psi_h)]}{|||(w_h, \psi_h)|||}} \tag{3.48}$$

Remark 3.3. Clearly, if the problem is such that either C_S or C_I is large, we may need to choose h_W very small to ensure stability and the factors C_S and C_I are not innocent even for well-posed problems. For Helmholtz-type equations C_S is typically proportional to some power of the wave number, which can be large. For multiscale problems on the form $\mathcal{P} v = -\nabla \cdot (\varepsilon(x) \nabla v)$, with $\varepsilon(x)$ some function with strong oscillation, we have $C_I \sim h_V \nabla \varepsilon$, which may be large.

4 Error estimates

We will first show an error estimate for the triple norm. Then for some different model cases, we will show how this translates to error estimates in a Sobolev norm. We have the following bound.

Theorem 4.1. *Let $u \in H^{2-k}(\Omega)$, $k = 0, 1$, be the solution of* (1.1) *and let $(u_h, z_h) \in V_h \times W_h$ be the corresponding approximation defined by* (2.8)*, using the adjoint space W_h of degree k. Then there is a constant such that*

$$|||(u - u_h, 0 - z_h)||| \lesssim \inf_{v_h \in V_h} (|||(u - v_h, 0)||| + \|u - v_h\|_*) \tag{4.1}$$

If, in addition, $u \in H^{p+1}(\Omega)$, with p the polynomial degree of V_h, then

$$|||(u - u_h, 0 - z_h)||| \lesssim h_V^p \|u\|_{H^{p+1}(\Omega)} \tag{4.2}$$

Proof. First we have $|||(u - u_h, 0 - z_h)||| = |||(u - u_h, z_h)|||$ and by the triangle inequality

$$|||(u - u_h, z_h)||| \leq |||(u - v_h, 0)||| + |||(v_h - u_h, z_h)||| \qquad \forall v_h \in V_h \tag{4.3}$$

By (3.8) we have

$$|||(u_h - v_h, z_h)||| \lesssim \sup_{(w_h, \psi_h) \in V_h \times W_h} \frac{A[(u_h - v_h, z_h), (w_h, \psi_h)]}{|||(w_h, \psi_h)|||} \tag{4.4}$$

By the consistency and the continuity (2.10) of the form a, there holds

$$A[(u_h - v_h, z_h), (w_h, \psi_h)] = A[(u - v_h, 0), (w_h, \psi_h)] \tag{4.5}$$
$$\lesssim \|u - v_h\|_* \|\psi_h\|_s + |||(u - v_h, 0)||| \; |||(w_h, \psi_h)||| \tag{4.6}$$

Using that $\|\psi_h\|_s \leq |||(w_h, \psi_h)|||$, we see that

$$|||(u_h - v_h, z_h)||| \lesssim \|u - v_h\|_* + |||(u - v_h, 0)||| \tag{4.7}$$

To conclude, we collect terms and use the fact that $v_h \in V_h$ is arbitrary,

$$|||(u - u_h, z_h)||| \lesssim \inf_{v \in V_h} |||(u - v_h, 0)||| + \|u - v_h\|_* \tag{4.8}$$

The second bound (4.2) is then a consequence of (3.3) and (2.11). ■

If, in addition, we have stability of the type (1.2) or (1.4), then Theorem 4.1 leads to an error estimate in the norm bounded by the stability estimate.

Corollary 4.1. *Assume that the assumptions of Theorem 4.1 hold. If* (1.4) *holds, then*

$$\|u - u_h\|_{H^1(\Omega)} \lesssim h_V^p \|u\|_{H^{p+1}(\Omega)} \tag{4.9}$$

If (1.2) *and* (1.3) *hold and, in addition,* $\|u_h\|_{H^1(\Omega)} \lesssim \|u\|_{H^{p+1}(\Omega)}$, *then*

$$\|u - u_h\|_{H^1(\Omega)} \lesssim h_V^{\alpha p} \|u\|_{H^{p+1}(\Omega)} \tag{4.10}$$

Proof. We only prove the second claim. The first follows using the same argument with $\alpha = 1$ and $|e|_X = \|e\|_{H^1(\Omega)}$. Let $e = u - u_h$. Using (1.2) and (1.3) we get

$$|e|_X \lesssim (\|e\|_{H^1(\Omega)} + \|\mathcal{P}e\|_{H^{-1}(\Omega)})^{1-\alpha} (\|e\|_{M(\omega)} + \|\mathcal{P}e\|_{H^{-1}(\Omega)})^{\alpha} \tag{4.11}$$

By the definition of the triple norm

$$|e|_X \lesssim (\|e\|_{H^1(\Omega)} + |||(e, 0)|||)^{1-\alpha} |||(e, 0)|||^{\alpha} \tag{4.12}$$

and using Theorem 4.1, followed by the bound on $\|u_h\|_{H^1(\Omega)}$ we see that

$$(\|e\|_{H^1(\Omega)} + |||(e, 0)|||)^{1-\alpha} \lesssim (\|u\|_{H^1(\Omega)} + \|u_h\|_{H^1(\Omega)} + h^p \|u\|_{H^{p+1}(\Omega)})^{1-\alpha} \tag{4.13}$$
$$\lesssim \|u\|_{H^{p+1}(\Omega)}^{1-\alpha} \tag{4.14}$$

and

$$|||(e, 0)|||^{\alpha} \lesssim h_V^{\alpha p} \|u\|_{H^{p+1}(\Omega)}^{\alpha} \tag{4.15}$$

The claim follows by injecting the two bounds in the right-hand side of (4.12). ■

Remark 4.1. The assumption $\|u_h\|_{H^1(\Omega)} \lesssim \|u\|_{H^{p+1}(\Omega)}$ can be checked a posteriori, simply by monitoring that $\|u_h\|_{H^1(\Omega)}$ remains bounded. It can also be imposed by adding an h_V^p-scaled, weakly consistent Tikhonov regularisation of the form

$$\frac{1}{2} h_V^{2p} \|u_h\|_{H^1(\Omega)}^2 \tag{4.16}$$

to the Lagrangian (1.6).

5 Applications of the theory

In this section, we will consider three different examples:

- Helmholtz equation with Dirichlet boundary conditions;
- Helmholtz equation with interior data;
- Data assimilation for the wave equation.

5.1 Helmholtz equation with Dirichlet boundary conditions

We consider a general Helmholtz equation with homogeneous Dirichlet boundary conditions. Let

$$\mathcal{P}u := -\nabla \cdot (D\nabla u) - k^2 u \tag{5.1}$$

where D is a symmetric positive definite matrix and

$$\|u\|_{M(\partial\Omega)} = \|h_V^{-r} u\|_{\partial\Omega}, \qquad 0 \leq r \leq 1/2 \tag{5.2}$$

We note that $M(\partial\Omega)$ satisfies (3.2). Assume that k^2 does not coincide with an eigenvalue of $-\nabla \cdot (D\nabla \cdot)$. Then the stability estimate (1.4) holds and by norm equivalence on discrete spaces,

$$\|u_h\|_{H^1(\Omega)} \lesssim |||(u_h, 0)||| \tag{5.3}$$

Thus the triple norm is a norm on V_h and the system (2.8) is invertible.

Proof of the properties (2.10) and (2.11). First assume that $W_h \subset C^0(\Omega)$, with functions that vanish on the domain boundary, then we may take

$$s(\varphi_h, \phi_h) = (\nabla \varphi_h, \nabla \phi_h)_\Omega, \qquad \|\varphi_h\|_s = \|\nabla \varphi_h\|_\Omega, \tag{5.4}$$

and

$$\|v\|_* := \|D^{\frac{1}{2}} \nabla v\|_\Omega + k^2 \|v\|_{H^{-1}(\Omega)} \tag{5.5}$$

To show (2.10) note that

$$a_1(v - v_h, \varphi_h) = (D\nabla(v - v_h), \nabla \varphi_h)_\Omega + k^2 (v - v_h, \varphi_h)_\Omega \tag{5.6}$$

Using the Cauchy-Schwarz inequality, we have for the first term on the right-hand side

$$(D\nabla(v - v_h), \nabla \varphi_h)_\Omega \leq \|D^{\frac{1}{2}} \nabla(v - v_h)\|_\Omega \|\varphi_h\|_s \tag{5.7}$$

For the second term we let $-\Delta z = v - v_h$, $z|_{\partial\Omega} = 0$, and

$$k^2 (v - v_h, \varphi_h)_\Omega = k^2 (\nabla z, \nabla \varphi_h)_\Omega \lesssim k^2 \|\nabla z\|_\Omega \|\varphi_h\|_s \lesssim k^2 \|v - v_h\|_{H^{-1}(\Omega)} \|\varphi_h\|_s \tag{5.8}$$

The continuity (2.10) follows by combining the two inequalities. It remains to show the bound (2.11) for some interpolant π_V. To this end, let π_V be defined by the L^2-projection on V_h. By standard estimates, we then have

$$\|D^{\frac{1}{2}} \nabla(v - \pi_V v)\|_\Omega \lesssim h_V^p \|u\|_{H^{p+1}(\Omega)} \tag{5.9}$$

For the second bound, we use that

$$\|u - \pi_V u\|_{H^{-1}(\Omega)} = \sup_{\substack{\varphi \in H^1_0(\Omega) \\ \|\varphi\|_{H^1(\Omega)}=1}} (u - \pi_V u, \varphi - \pi_V \varphi)_\Omega \lesssim h_V^{p+2} \|u\|_{H^{p+1}(\Omega)} \tag{5.10}$$

We conclude that

$$\boxed{\|u - \pi_V u\|_* \lesssim (1 + k^2 h_V^2) h_V^p \|u\|_{H^{p+1}(\Omega)}} \tag{5.11}$$

In the second case where W_h consists of a piecewise constant multiplier, we add the assumption that Ω is convex. Then we have

$$a_0(v - v_h, \varphi_h) = (\nabla \cdot (D \nabla (v - v_h)), \varphi_h)_\Omega + k^2 (v - v_h, \varphi_h)_\Omega \tag{5.12}$$

Introducing the same variable z as before, we may write

$$a_0(v - v_h, \varphi_h) = (\nabla \cdot (D \nabla (v - v_h) + k^2 \nabla z), \varphi_h)_\Omega \tag{5.13}$$

and after integration by parts and application of the Cauchy-Schwarz inequality

$$a_0(v - v_h, \varphi_h) \leq h_W^{\frac{1}{2}} (\|D \nabla (v - v_h) \cdot n\|_{\mathcal{F}(W_h)} + k^2 \|\nabla z \cdot n\|_{\mathcal{F}(W_h)}) s_0(\varphi_h, \varphi_h)^{\frac{1}{2}} \tag{5.14}$$

We see that continuity holds with the norm

$$\|v\|_* := h_W^{\frac{1}{2}} (\|D \nabla v \cdot n\|_{\mathcal{F}(W_h)} + k^2 \|\nabla z \cdot n\|_{\mathcal{F}(W_h)}) \tag{5.15}$$

where z is the function defined above. To prove the approximation properties, for $v_h = \pi_V v$, we use element trace inequalities to get

$$h_W^{\frac{1}{2}} (\|D \nabla (v - \pi_V v) \cdot n\|_{\mathcal{F}(W_h)} \lesssim \|\nabla (v - \pi_V v)\|_\Omega + h_W \|(v - \pi_V v)\|_{H^2(\Omega)} \tag{5.16}$$

which has the desired approximation properties and

$$h_W^{\frac{1}{2}} k^2 \|\nabla z \cdot n\|_{\mathcal{F}(W_h)} \lesssim k^2 (\|\nabla z\|_\Omega + h_W \|z\|_{H^2(\Omega)}) \tag{5.17}$$

Since Ω is convex we have $\|z\|_{H^2(\Omega)} \lesssim \|v - \pi_V v\|_\Omega$ and we conclude that

$$\|v - \pi_V v\|_* \lesssim \|\nabla (v - \pi_V v)\|_\Omega + h_W \|(v - \pi_V v)\|_{H^2(\Omega)} \tag{5.18}$$

$$+ k^2 (\|v - \pi_V v\|_{H^{-1}(\Omega)} + h_W \|v - \pi_V v\|_\Omega) \tag{5.19}$$

Assuming $h_W \lesssim h_V$ we conclude as before that

$$\boxed{\|v - \pi_V v\|_* \lesssim (1 + k^2 h_V^2) h_V^p \|v\|_{H^{p+1}(\Omega)}} \tag{5.20}$$

Error estimates. To derive error estimates, we note that since Theorem 4.1 holds we can combine with Corollary 4.1 to conclude that

$$\|u - u_h\|_{H^1(\Omega)} \lesssim C_S \inf_{v_h \in V_h} (|||(u - v_h, 0)||| + \|u - v_h\|_*) \tag{5.21}$$

$$\leq C_S (1 + k^2 h_V^2) h_V^p \|u\|_{H^{p+1}(\Omega)} \tag{5.22}$$

where we assumed $u \in H^{p+1}(\Omega)$. Here we simply applied the inequality (1.4) to the approximation error $u - u_h$ and applied Theorem 4.1 followed by Corollary 4.1 to the right hand side.

5.2 Helmholtz equation with interior data

Here we consider an ill-posed second-order elliptic problem where data is given in some subset ω of Ω and apply stability of the type (1.2). Let

$$\mathcal{P}u := -\nabla \cdot (D\nabla u) - k^2 u \tag{5.23}$$

and

$$\|u\|_{M(\omega)} := \|h_V^{-r} u\|_\omega, \quad 0 \leq r \leq 1, \quad \omega \subset \Omega \tag{5.24}$$

Then the stability (1.2) holds with $|u|_X = \|u\|_B$, where $\bar{B} \subset\subset \Omega$, in the left-hand side. Since the system matrix is square, uniqueness implies existence. If u_1 and u_2 are two solutions then the error $e = u_1 - u_2$ satisfies $\mathcal{P}e = 0$ and $e|_\omega = 0$ and since (1.2) holds for any subdomain B we can conclude that $e = 0$ and the system is invertible. The continuity (2.10) holds as in the previous example. Moreover, since Theorem 4.1 holds we conclude that, assuming $\|u_h\|_{H^1(\Omega)} \lesssim \|u\|_{H^{p+1}(\Omega)}$ and $u \in H^{p+1}(\Omega)$,

$$\|u - u_h\|_B \leq C h_V^{p\alpha} \|u\|_{H^{p+1}(\Omega)} \tag{5.25}$$

where the size of α depends on the Hausdorff distance from ∂B to $\partial\Omega$, going to one as the boundary of B approaches the boundary of $\partial\Omega$.

5.3 Data assimilation for the wave equation

Let $\Omega_d \subset \mathbb{R}^d$ be a domain and let $\Omega = \Omega_d \times [0,T]$ be the space-time domain associated with the time interval $[0,T]$. The wave operator is defined by

$$\mathcal{P}u = \partial_t^2 u - \Delta u \tag{5.26}$$

$u = 0$ on $\partial\Omega$ and the data norm is given by

$$\|u\|_{M(\omega)} := \|h^{-r} u\|_{L^2(\omega \times [0,T])}, \quad 0 \leq r \leq 1 \tag{5.27}$$

Observe that the initial datum is unknown. For this problem, it is known that provided the data set ω satisfies a certain condition, known as the geometric control condition [4], the following inequality similar to (3.43) holds for all $u \in V = \{v \in H^1(\Omega) : v|_{\partial\Omega_d} = 0\}$,

$$\sup_{t \in [0,T]} \|u(\cdot,t)\|_{L^2(\Omega_d)} + \|\partial_t u\|_{L^2(0,T;H^{-1}(\Omega_d))} \lesssim \|\mathcal{P}u\|_{H^{-1}(\Omega)} + \|u\|_{L^2(\omega \times [0,T])} \tag{5.28}$$

where $H^{-1}(\Omega) = [H_0^1(\Omega)]^*$ is the space-time weak norm.

In this case, the space-time method defined by (2.8), with V_h satisfying strong Dirichlet boundary conditions on $\partial\Omega_d$, admits a unique solution using arguments similar to the previous example. The continuity is also proved similarly by integration-by-parts in space and time, resulting in one term based on the space-time gradient. Details are left to the reader. Assuming that $u \in H^{p+1}(\Omega)$ the approximate solution satisfies the following error bound

$$\sup_{t \in [0,T]} \|(u - u_h)(\cdot,t)\|_{L^2(\Omega_d)} + \|\partial_t (u - u_h)\|_{L^2(0,T;H^{-1}(\Omega_d))} \lesssim h_V^p \|u\|_{H^{p+1}(\Omega)} \tag{5.29}$$

This bound is obtained by applying the inequality (5.28) to the approximation error $u - u_h$ and then bounding the right-hand side using the estimate of Theorem 4.1.

6 Numerical example

We consider a two-dimensional problem of the Helmholtz equation with interior data described in Section 5.2 above. We impose the boundary condition on W_h weakly via (2.17), and hence the form a for the Helmholtz operator is given by

$$a(v_h, \varphi_h) = (\mathcal{P}v_h, \varphi_h)_\Omega = (D\nabla v_h, \nabla\varphi_h)_\Omega - (n \cdot (D\nabla v_h), \varphi_h)_{\partial\Omega} - k^2(v_h, \varphi_h)_\Omega \tag{6.1}$$

Spaces and parameters. We let $V_h = W_h$ be the space of tensor product B-splines of degree $p = 2$ of maximum regularity on a quadrilateral mesh of size h, which means $V_h, W_h \subset C^1(\Omega)$. As penalty parameter for enforcing the interior data on ω, we use $\gamma = 100$, and as GaLS stabilization parameter, we use $\tau = 0.1h^2$.

Problem set-up. Let the domain be the unit square $\Omega = (0, 1)^2$ and the interior region where data is known be $\omega = ((0, 1) \times (0, t)) \cup ((0, t) \times (0, 1)) \cup ((0, 0.5) \times (1 - t, 1))$ where $t = 0.2$. This region is displayed on the meshes in Figure 1. For coefficients in the operator, we let D be the identity matrix and the wavenumber $k = 20$. We manufacture a problem with a known analytical solution by the ansatz,

$$u = \sin\left(k\frac{x + 3y}{\sqrt{20}}\right)\sin\left(k\frac{3x - y}{\sqrt{20}}\right) \tag{6.2}$$

which gives a zero loading $f = -\mathcal{P}u = 0$. The data on ω is taken from this analytical solution $q = u|_\omega$, and the $M(\omega)$-form is chosen as $(v, v)_{M(\omega)} = h^{-2}(v, v)_\omega$.

Numerical results. The geometric set-up, along with numerical solutions for $u_h \in V_h$ and the auxiliary field $z_h \in W_h$, are displayed for different meshes in Figure 1. By visual inspection, it is clear that the upper right parts where data is unavailable have the worst accuracy and that accuracy in those parts seems to improve as the mesh is refined.

The rule of thumb interpretation of the conditional stability estimate for the convex data domain is that the further inside the convex hull we measure the error, the closer to one we expect the order parameter α to be. On the other hand, as we measure the error in a domain approaching the boundary where data is unavailable, α goes to zero. If we consider the error over the whole domain, we no longer have Hölder stability and can only expect error convergence of order $O(|\log(h)|^{-\alpha})$, $0 < \alpha < 1$. To illustrate this effect, Figure 2 separately plots the error in the left half of the square domain and the right half. In the left half, we observe optimal convergence compared to interpolation, whereas in the right half, we get a severely capped convergence behavior of order approximately 0.36.

7 Conclusions

In this contribution, we have developed and analyzed a finite element method for approximating solutions to problems without standard stability theory, such as that provided by the Lax-Milgram lemma or the Banach-Necas-Babuska theorem. Our model problem is a continuation problem involving a second-order partial differential equation with incomplete boundary data with given measurements of the solution on a subdomain of the computational domain.

The method is based on an augmented Lagrangian Galerkin least squares formulation together with a C^1 finite element space for the primal variable and either a continuous or discontinuous space for the multiplier.

We show inf-sup stability with and without the Galerkin least squares stabilization term. In the latter case, the mesh size in the multiplier space must be sufficiently small compared to the primal space. In contrast, this requirement is not necessary for the stabilized case, leading to a more flexible and efficient method. We can then use the inf-sup stability and the approximation properties to derive error estimates in the energy norm. We derive error estimates in other norms by combing the energy norm error estimate with different stability assumptions on the underlying problem.

The C^1 space conveniently handles second-order derivatives, simplifying derivations, and there is no need to include additional stabilization involving jumps in gradients across faces. Often tensor products of spline functions are used to construct C^1 spaces, and in that case, it is convenient to be able to handle cut (trimmed) elements in the formulation, and we discuss this extension in an extended remark.

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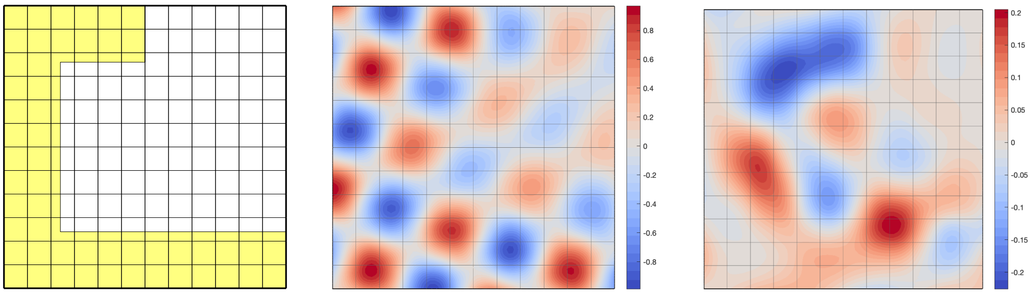
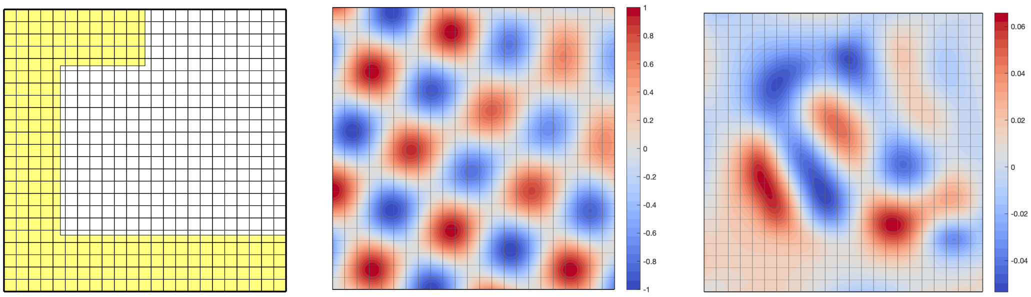
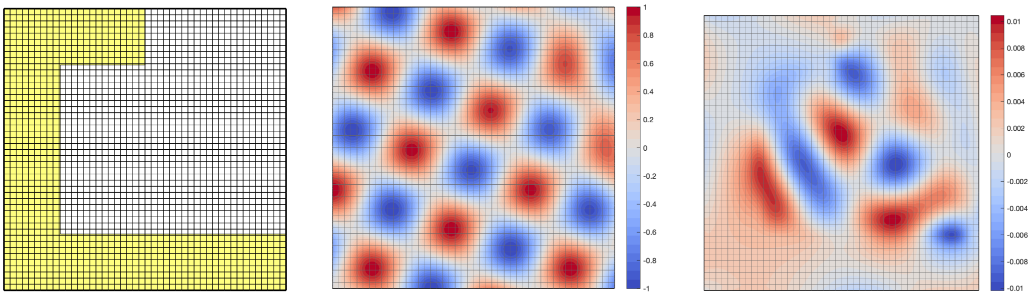


(a) $h_V = h_W = 2^{-2}$ (b) u_h (c) z_h

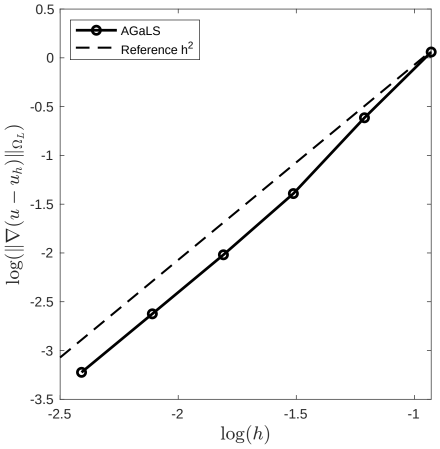


(d) $h_V = h_W = 2^{-3}$ (e) u_h (f) z_h

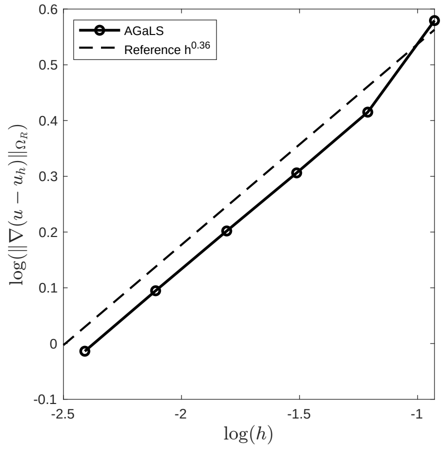


(g) $h_V = h_W = 2^{-4}$ (h) u_h (i) z_h

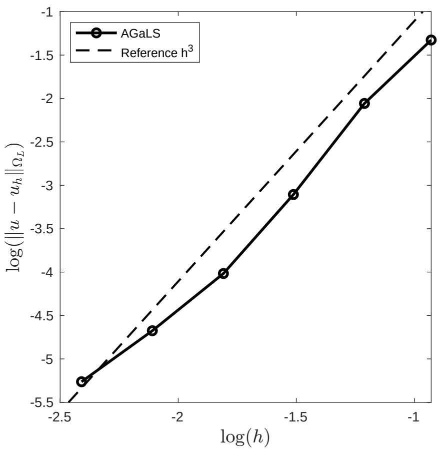
Figure 1: *Left:* Meshes with indicated region ω, where data is given. *Middle:* Numerical solution u_h where the upper right region without any boundary condition or given data can be noted to have worse accuracy than other regions. *Right:* Numerical solution for the Lagrangian field z_h.



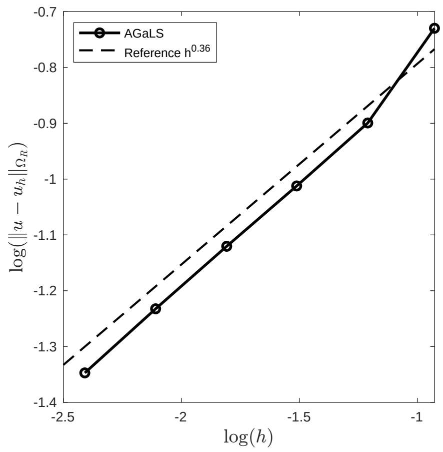
(a) Error on left half Ω_L



(b) Error on right half Ω_R



(c) Error on left half Ω_L



(d) Error on right half Ω_R

Figure 2: Errors in H^1-seminorm and in L^2-norm on the left half $\Omega_L = (0, 0.5) \times (0, 1)$ respectively on the right half $\Omega_R = (0.5, 1) \times (0, 1)$ of the domain $\Omega = (0, 1)^2$. While the errors on the left half converge at about rates expected from interpolation theory, the convergence rates in both H^1-seminorm and in L^2-norm on the right half seem capped at around $h^{0.36}$.