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INVERSE PROBLEMS FOR SUBDIFFUSION FROM OBSERVATION AT AN UNKNOWN TERMINAL TIME*

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Abstract. Inverse problems of recovering space-dependent parameters, e.g., initial condition, space-dependent source, or potential coefficient in a subdiffusion model from the terminal observation have been extensively studied in recent years. However, all existing studies have assumed that the terminal time at which one takes the observation is exactly known. In this work, we present uniqueness and stability results for three canonical inverse problems, e.g., backward problem, inverse source, and inverse potential problems from the terminal observation at an unknown time. The subdiffusive nature of the problem indicates that one can simultaneously determine the terminal time and space-dependent parameter. The analysis is based on explicit solution representations, asymptotic behavior of the Mittag–Leffler function, and mild regularity conditions on the problem data. Further, we present several one- and two-dimensional numerical experiments to illustrate the feasibility of the approach.

Key words. backward subdiffusion, inverse source problem, inverse potential problem, subdiffusion, unknown terminal time, uniqueness, stability

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1. Introduction.

Let $\Omega \subset \mathbb{R}^d$ $(d=1,2,3)$ be an open bounded smooth domain with a boundary $\partial\Omega$. Consider the following initial-boundary value problem with $\alpha \in (0,1)$ for the subdiffusion model:

$$\begin{cases} \partial_t^\alpha u - \Delta u + qu = f & \text{in } \Omega \times (0,\infty), \\ u = 0 & \text{on } \partial\Omega \times (0,\infty), \\ u(0) = u_0 & \text{in } \Omega, \end{cases} \tag{1.1}$$

where $T>0$ is a fixed final time, $f \in L^\infty(0,T;L^2(\Omega))$ and $u_0 \in L^2(\Omega)$ are given source term and initial data, respectively, the nonnegative function $q \in L^\infty(\Omega)$ is a spatially dependent potential, and Δ denotes the Laplace operator in space. The notation $\partial_t^\alpha u(t)$ denotes the Djrbashian–Caputo fractional derivative in time t of order $\alpha \in (0,1)$ (see [10, p. 70] or [4, section 2.3.2]),

$$\partial_t^\alpha u(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} u'(s) \mathrm{d}s, \tag{1.2}$$

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where $\Gamma(z)$ is the Gamma function defined by $\Gamma(z) = \int_0^\infty s^{z-1}e^{-s}\mathrm{d}s$, for $\Re(z) > 0$. Note that the fractional derivative $\partial_t^\alpha u$ recovers the first-order derivative $u'(t)$ as $\alpha \to 1^-$ if u is sufficiently smooth. Thus the model (1.1) is a fractional analogue of the classical parabolic equation.

The model (1.1) arises naturally in the study of anomalously slow diffusion processes, which encompasses a broad range of important applications in engineering, physics, and biology. The list of successful applications includes thermal diffusion in fractal media [21], dispersion in heterogeneous aquifer [1], ion dispersion in column experiments [2], and protein transport in membranes [11], to name just a few. Thus its mathematical theory has received immense attention in recent years; see the monographs [12, 4] for detailed discussions on the solution theory. Related inverse problems have also been extensively studied [7, 17, 15]. The surveys [17, 15] cover many inverse source problems and coefficient identification problems, respectively.

The observation $g(x) = u(x,T)$, $x \in \Omega$, at a terminal time T is a popular choice for the measurement data in practice. There is extensive literature on inverse problems using terminal data, e.g., backward sudiffusion [22, 28], inverse source problem [7, 17, 3], and inverse potential problem [27, 9, 8, 26], where the references are rather incomplete but we refer to the reviews [7, 17, 15] for further references. Notably, several uniqueness and stability results have been proved. For example, backward subdiffusion is only mildly ill-posed, and enjoys (conditional) Lipschitz stability [22, Theorem 4.1]; cf. (2.9) below. In all of these existing studies, the terminal time T at which one collects the measurement has always been assumed to be fully known. Nonetheless, in practice, the terminal time T might be known only imprecisely. Therefore, it is natural to ask whether one can still recover some information about the concerned parameter(s). The missing knowledge of T introduces additional technical challenges since the associated forward map is not fully known then. In this work, we address this question in the affirmative both theoretically and numerically, and study the inverse problem of identifying one of the following three parameters: (i) initial condition u_0, (ii) space-dependent source component ψ, and (iii) space-dependent potential q, from the observation $u(T)$ at an unknown terminal time T.

For each inverse problem, we shall establish the unique recovery of the space-dependent parameter and the terminal time T simultaneously from the terminal observation, as well as conditional stability estimates, under suitable a priori regularity assumptions on the initial data u_0 and the source f; see Theorems 2.3, 3.3, and 4.6 for the precise statements. The analysis relies heavily on explicit solution representations via Mittag–Leffler functions (see, e.g., [22], [4, section 6.2]). The essence of the argument is that the regularity difference leads to distinct decay behavior of the Fourier coefficients of u_0 and f. This combined with distinct polynomial decay behavior of Mittag–Leffler function $E_{\alpha,1}(z)$ (on the negative real axis) allows unique determining of the terminal time T. Note that the polynomial decay holds only for $E_{\alpha,1}(z)$ with a order $\alpha \in (0,1)$, and does not hold in the integer case (i.e., $\alpha = 1$). Thus, the unique determination of T does not hold for normal diffusion. Once the terminal time T is determined, the unique determination of the space-dependent parameter follows. The proof of the stability results relies on smoothing properties of the solution operators. In addition, we present several numerical experiments to illustrate the feasibility of numerical recovery. The numerical reconstructions are obtained using the Levenberg–Marquadt method [13, 19]. Numerically, by choosing the hyperparameters in the method properly, both space-dependent parameter and terminal time can be accurately recovered. To the best of our knowledge, this work presents the first uniqueness and stability results for inverse problems from terminal data at an unknown time.

The rest of this paper is organized as follows. In section 2, we present uniqueness and stability results for the backward problem, which are then extended to the inverse source problem in section 3. In section 4, we discuss the inverse potential problem, which requires several new technical estimates on the solution regularity and asymptotic decay. Finally, some numerical results for one- and two-dimensional problems are given in section 5. Throughout, we denote by $u=u(v)$ and $\tilde u=u(\tilde v)$ the solutions to problem (1.1) with the space dependent parameter v and $\tilde v$, respectively. We often write a function $f(x,t):\Omega\times(0,T)\to\mathbb{R}$ as $f(t)$ a vector-valued function on $(0,T)$. The notation c denotes a generic constant which may differ at each occurrence, but it is always independent of the concerned parameter and terminal time T.

2. Backward problem.

First, we investigate the backward problem (BP): recover the initial data $u_0=u(0)$ from the solution profile $u(T)$ to problem (1.1) at an unknown terminal time T.

2.1. Solution representation.

First, we recall the solution representation for problem (1.1), which plays a key role in the analysis below. For any $s\geq 0$, we denote by $\dot H^s(\Omega)\subset L^2(\Omega)$ the Hilbert space induced by the norm:

$$\|v\|_{\dot H^s(\Omega)}=\Big(\sum_{j=1}^\infty \lambda_j^s(v,\varphi_j)^2\Big)^{\frac12}, \tag{2.1}$$

with $\{\lambda_j\}_{j=1}^\infty$ and $\{\varphi_j\}_{j=1}^\infty$ being, respectively, the eigenvalues (with multiplicity counted) and eigenfunctions of the operator $A=-\Delta+qI$ on the domain Ω with a zero Dirichlet boundary condition. Then $\{\varphi_j\}_{j=1}^\infty$ can be taken to form an orthonormal basis in $L^2(\Omega)$. Further, $\|v\|_{\dot H^0(\Omega)}$ is the norm in $L^2(\Omega)$, $\|v\|_{\dot H^1(\Omega)}$ is the norm in $H_0^1(\Omega)$, and $\|v\|_{\dot H^2(\Omega)}=\|\Delta v\|_{L^2(\Omega)}$ is equivalent to the norm in $H^2(\Omega)\cap H_0^1(\Omega)$ [25, section 3.1]. For $s>0$, $\dot H^{-s}(\Omega)$ denotes the dual space of $\dot H^s(\Omega)$. Throughout, $(\cdot,\cdot)$ denotes both duality pairing between $\dot H^{-s}(\Omega)$ and $\dot H^s(\Omega)$ and the $L^2(\Omega)$ inner product.

Now we represent the solution u to problem (1.1) using the eigenpairs $\{(\lambda_j,\varphi_j)\}_{j=1}^\infty$, following [22] and [4, section 6.2]. Specifically, we define two solution operators $F(t)$ and $E(t)$ by

$$F(t)v=\sum_{j=1}^\infty E_{\alpha,1}(-\lambda_jt^\alpha)(v,\varphi_j)\varphi_j \quad\text{and}\quad E(t)v=\sum_{j=1}^\infty t^{\alpha-1}E_{\alpha,\alpha}(-\lambda_jt^\alpha)(v,\varphi_j)\varphi_j, \tag{2.2}$$

where $E_{\alpha,\beta}(z)$ is the Mittag–Leffler function defined by (see [10, pp. 40–45], [4, section 3.1])

$$E_{\alpha,\beta}(z)=\sum_{k=0}^\infty\frac{z^k}{\Gamma(k\alpha+\beta)}\quad\forall z\in\mathbb{C}.$$

Then the solution u of problem (1.1) can be written as

$$u(t)=F(t)u_0+\int_0^t E(t-s)f(s)\mathrm{d}s. \tag{2.3}$$

The function $E_{\alpha,\beta}(z)$ generalizes the exponential function e^z. The following decay estimates of $E_{\alpha,\beta}(z)$ are crucial in the analysis below; see e.g., [10, equation (1.8.28), p. 43] and [4, Theorem 3.2] for the first estimate, and [24, Theorem 4] or [4, Theorem 3.6] for the second estimate.

LEMMA 2.1. *Let $\alpha \in (0,2)$, $\beta \in \mathbb{R}$, and $\varphi \in (\frac{\alpha\pi}{2}, \min(\pi, \alpha\pi))$, and $N \in \mathbb{N}$. Then for $\varphi \le |\arg z| \le \pi$ with $|z| \to \infty$:*

$$E_{\alpha,\beta}(z) = -\sum_{k=1}^{N} \frac{1}{\Gamma(\beta - \alpha k)} \frac{1}{z^k} + O\Big(\frac{1}{z^{N+1}}\Big).$$

For $0 < \alpha_0 < \alpha < \alpha_1 < 1$, there exist constants c_0, $c_1 > 0$ depending only on α_0 and α_1 such that

$$c_0(1-x)^{-1} \le E_{\alpha,1}(x) \le c_1(1-x)^{-1} \quad \forall x \le 0.$$

2.2. Uniqueness and stability. Now we study uniqueness and stability for BP, when the source f is time-independent, i.e., $f(x,t) \equiv f(x)$. The key idea in proving uniqueness is to distinguish decay rates of Fourier coefficients of the initial data u_0 (with respect to the eigenfunctions $\{\varphi_j\}_{j=1}^\infty$) and the source f. We use the set $\mathbb{S}_\gamma$, $\gamma \in [-1, \infty)$, defined by

$$\mathbb{S}_\gamma = \Big\{ v \in \dot H^{-2}(\Omega) : \lim_{n\to\infty} \lambda_n^\gamma |(v, \varphi_n)| = 0 \Big\}. \tag{2.4}$$

Clearly, for any $\gamma \ge 0$, $\dot H^\gamma(\Omega) \subset \mathbb{S}_{\frac{\gamma}{2}}$. If $v \in \dot H^{-2}(\Omega) \setminus \mathbb{S}_\gamma$, the sequence $\{\lambda_n^\gamma |(v,\varphi_n)|\}_{n=1}^\infty$ contains a subsequence that is bounded away from zero, i.e., there exists $c_v > 0$ and a sequence $\{n_\ell\}_{\ell=1}^\infty$ such that $\lim_{\ell\to\infty} n_\ell = \infty$ and $\lambda_{n_\ell}^\gamma |(v, \varphi_{n_\ell})| \ge c_v$, for all $\ell \in \mathbb{N}$.

THEOREM 2.2. *Let $f \in \dot H^{-2}(\Omega) \setminus \mathbb{S}_\gamma$ for some $\gamma \ge 0$. If BP has two solutions (T, u_0) and $(\widetilde T, \widetilde u_0)$ in the set $\mathbb{R}_+ \times \mathbb{S}_{\gamma+1}$ with the data $u(T)$ and $\tilde u(\tilde T)$, respectively, then $T = \widetilde T$ and $u_0 = \widetilde u_0$.*

Proof. Using the solution representation (2.3) and noting the identity

$$\frac{\mathrm{d}}{\mathrm{d}t} E_{\alpha,1}(-\lambda_n t^\alpha) = -\lambda_n t^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n t^\alpha), \tag{2.5}$$

since f is time-independent, the solution u to problem (1.1) can be written as

$$u(x,t) = \sum_{n=1}^\infty \Big[E_{\alpha,1}(-\lambda_n t^\alpha)(u_0, \varphi_n) + \frac{1 - E_{\alpha,1}(-\lambda_n t^\alpha)}{\lambda_n} (f, \varphi_n) \Big] \varphi_n(x).$$

Let $\mathbb{K} = \{k \in \mathbb{N} : (f, \varphi_n) \ne 0\}$, which under the condition $f \in \dot H^{-2}(\Omega) \setminus \mathbb{S}_\gamma$ satisfies $|\mathbb{K}| = \infty$. For any $n \in \mathbb{K}$, taking inner product (or duality pairing) with $\frac{\lambda_n \varphi_n}{(f,\varphi_n)}$ on both sides of the identity gives

$$\frac{\lambda_n (u(t), \varphi_n)}{(f, \varphi_n)} = \lambda_n E_{\alpha,1}(-\lambda_n t^\alpha) \frac{(u_0, \varphi_n)}{(f, \varphi_n)} + 1 - E_{\alpha,1}(-\lambda_n t^\alpha).$$

Then setting $t = T$ and rearranging the terms lead to

$$\lambda_n \left(1 - \frac{\lambda_n (u(T), \varphi_n)}{(f, \varphi_n)}\right) = -\lambda_n E_{\alpha,1}(-\lambda_n T^\alpha) \frac{\lambda_n (u_0, \varphi_n)}{(f, \varphi_n)} + \lambda_n E_{\alpha,1}(-\lambda_n T^\alpha). \tag{2.6}$$

By assumption, $f \in \dot H^{-2}(\Omega) \setminus \mathbb{S}_\gamma$ and $u_0 \in \mathbb{S}_{\gamma+1}$, and hence we deduce

$$\lim_{n\in\mathbb{K}, n\to\infty} \frac{\lambda_n (u_0, \varphi_n)}{(f, \varphi_n)} = \lim_{n\in\mathbb{K}, n\to\infty} \frac{\lambda_n^{\gamma+1} (u_0, \varphi_n)}{\lambda_n^\gamma (f, \varphi_n)} = 0. \tag{2.7}$$

Then, by letting $n \to \infty$ and $t = T$, the relation (2.6) implies

$$\lim_{n\in\mathbb{K},\ n\to\infty} \lambda_n \left(1 - \frac{\lambda_n (u(T), \varphi_n)}{(f, \varphi_n)}\right) = \lim_{n\in\mathbb{K},\ n\to\infty} \lambda_n E_{\alpha,1}(-\lambda_n T^\alpha) = \frac{1}{\Gamma(1-\alpha) T^\alpha}. \tag{2.8}$$

The last identity follows from the fact that $\lambda_n(q)\to\infty$ and the asymptotics of $E_{\alpha,1}(z)$ in Lemma 2.1. Note that $\frac{1}{\Gamma(1-\alpha)T^\alpha}$ is strictly decreasing in the time T. Hence, T can be uniquely determined from the data $u(T)$. Finally, the unique determination of u_0 follows from [22, Theorem 4.1]. ∎

Remark 2.1. The validity of Theorem 2.2 relies crucially on the regularity difference between the initial data u_0 and source f, so that the limit (2.7) holds.

Remark 2.2. Theorem 2.2 shows the unique determination of the terminal time T in problem (1.1) from the observation $u(T)$. This interesting phenomenon is due to the distinct asymptotic behavior of Mittag–Leffler functions and different smoothness of the initial data u_0 and source f. It sharply contrasts with the backward problem of normal diffusion ($\alpha=1$): analogous to (2.8),

$$\lim_{n\in\mathbb{K},\ n\to\infty}\lambda_n\left(1-\frac{\lambda_n(u(x,T),\varphi_n)}{(f,\varphi_n)}\right)=\lim_{n\in\mathbb{K},\ n\to\infty}\lambda_n e^{-\lambda_n T}=0,$$

and thus it is impossible to determine both the terminal time T and initial data u_0 from the data $u(T)$. This shows one distinct feature of anomalously slow diffusion processes, when compared with standard diffusion.

Next, we establish a stability estimate for BP with an approximately given T. When the terminal time T is exactly given, it recovers the following well-known estimate [22, Theorem 4.1] (or [4, Theorem 6.28]):

$$\|u_0-\tilde u_0\|_{L^2(\Omega)}\le c\|u(T)-\tilde u(T)\|_{H^2(\Omega)}. \tag{2.9}$$

THEOREM 2.3. *Let u_0 and $\tilde u_0$ be the solutions of BP with observations $u(T)$ and $\tilde u(\tilde T)$ with $T<\tilde T$, respectively. Then the following conditional stability estimate holds:*

$$\|u_0-\tilde u_0\|_{L^2(\Omega)}\le c\big(1+c\tilde T^\alpha\big)\big(\|A(u(T)-u(\tilde T))\|_{L^2(\Omega)}+c|T-\tilde T|T^{-1-\alpha}\|u_0-A^{-1}f\|_{L^2(\Omega)}\big).$$

Proof. By the solution representation (2.3) and the identity (2.5), we have

$$u(T)=F(T)u_0+A^{-1}(I-F(T))f \quad\text{and}\quad \tilde u(\tilde T)=F(\tilde T)\tilde u_0+A^{-1}(I-F(\tilde T))f.$$

Subtracting these two identities leads to

$$\begin{aligned}u(T)-\tilde u(\tilde T)&=\big(F(T)u_0-F(\tilde T)\tilde u_0\big)+A^{-1}\big(F(\tilde T)-F(T)\big)f\\&=F(\tilde T)(u_0-\tilde u_0)+\big(F(T)-F(\tilde T)\big)(u_0-A^{-1}f).\end{aligned}$$

Consequently, we arrive at

$$u_0-\tilde u_0=F(\tilde T)^{-1}\big(u(T)-\tilde u(\tilde T)\big)-F(\tilde T)^{-1}\big(F(T)-F(\tilde T)\big)(u_0-A^{-1}f).$$

Next, we bound the two terms separately. For any $v\in L^2(\Omega)$, by Lemma 2.1, we derive

$$\begin{aligned}\|A^{-1}F(\tilde T)^{-1}v\|^2_{L^2(\Omega)}&=\sum_{n=1}^\infty\lambda_n^{-2}E_{\alpha,1}(-\lambda_n\tilde T^\alpha)^{-2}(v,\varphi_n)^2\\&\le c\sum_{n=1}^\infty\Big(\frac{1+\lambda_n\tilde T^\alpha}{\lambda_n}\Big)^2(v,\varphi_n)^2\le c\big(1+\tilde T^\alpha\big)^2\|v\|^2_{L^2(\Omega)}.\end{aligned}$$

Next, we bound $F(\tilde{T})^{-1}(F(T)-F(\tilde{T})) = A^{-1}F(\tilde{T})^{-1}\int_T^{\tilde{T}} AF'(s)\,\mathrm{d}s$. Note that for any $s\in(T,\tilde{T})$, there holds [4, Theorem 6.4]

$$\|AF'(s)v\|_{L^2(\Omega)} \le cs^{-1-\alpha}\|v\|_{L^2(\Omega)} \le cT^{-1-\alpha}\|v\|_{L^2(\Omega)}.$$

This estimate implies

$$\|A(F(T)-F(\tilde{T}))v\|_{L^2(\Omega)} \le cT^{-1-\alpha}|\tilde{T}-T|\|v\|_{L^2(\Omega)}. \tag{2.10}$$

Consequently, we obtain

$$\begin{aligned}\|F(\tilde{T})^{-1}(F(T)-F(\tilde{T}))v\|_{L^2(\Omega)} &\le \|A^{-1}F(\tilde{T})\|\|A(F(T)-F(\tilde{T}))v\|_{L^2(\Omega)}\\ &\le c\big(1+c\tilde{T}^{\alpha}\big)|T-\tilde{T}|T^{-1-\alpha}\|v\|_{L^2(\Omega)}.\end{aligned}$$

Then the desired result follows immediately from the preceding estimates. ∎

The next result bounds the terminal time T in terms of data perturbation.

COROLLARY 2.4. *Let $f\in \dot{H}^{-2}(\Omega)\setminus\mathbb{S}_\gamma$ for some $\gamma\ge 0$. Let (T,u_0), $(\tilde{T},\tilde{u}_0)\in\mathbb{R}_+\times\mathbb{S}_{\gamma+1}$ be the solutions of* BP *with observations $u(T)$ and $\tilde{u}(\tilde{T})$, respectively. Then there holds*

$$|T-\tilde{T}| \le c\min(\Lambda,\tilde{\Lambda})^{-\frac{1}{\alpha}-1}|\Lambda-\tilde{\Lambda}|,$$

with the quantities Λ and $\widetilde{\Lambda}$, respectively, given by

$$\Lambda = \lim_{n\in\mathbb{K},\ n\to\infty}\lambda_n\left(1-\frac{\lambda_n(u(T),\varphi_n)}{(f,\varphi_n)}\right) \quad \text{and} \quad \tilde{\Lambda} = \lim_{n\in\mathbb{K},\ n\to\infty}\lambda_n\left(1-\frac{\lambda_n(\tilde{u}(\tilde{T}),\varphi_n)}{(f,\varphi_n)}\right).$$

In particular, for $\Lambda<\tilde{\Lambda}$, there holds

$$\|u_0-\tilde{u}_0\|_{L^2(\Omega)} \le c\big(1+c\tilde{\Lambda}^{-1}\big)\big(\|A(u(T)-\tilde{u}(\tilde{T}))\|_{L^2(\Omega)} + c|\Lambda-\tilde{\Lambda}|\|u_0-A^{-1}f\|_{L^2(\Omega)}\big).$$

Proof. It follows from the relation (2.8) that

$$|T-\tilde{T}| = |\Lambda^{-\frac{1}{\alpha}}\Gamma(1-\alpha)^{-\frac{1}{\alpha}} - \tilde{\Lambda}^{-\frac{1}{\alpha}}\Gamma(1-\alpha)^{-\frac{1}{\alpha}}| \le \Gamma(1-\alpha)^{-\frac{1}{\alpha}}\min(\Lambda,\tilde{\Lambda})^{-\frac{1}{\alpha}-1}|\Lambda-\tilde{\Lambda}|.$$

The assertion follows from Theorem 2.3 and the identities $T=\Lambda^{-\frac{1}{\alpha}}\Gamma(1-\alpha)^{-\frac{1}{\alpha}}$ and $\tilde{T}=\tilde{\Lambda}^{-\frac{1}{\alpha}}\Gamma(1-\alpha)^{-\frac{1}{\alpha}}$. ∎

3. Inverse source problem. Now we extend the argument in section 2 to an inverse source problem of recovering the space dependent component from the data $u(T)$. Following the standard setup for inverse source problems [17], we assume that the source $f(x,t)$ is separable and satisfies

$$f(x,t) = g(t)\psi(x), \quad \text{with } g\in L^\infty(0,T),\ g\ge c_g>0, \text{ and } \psi\in\dot{H}^{-1}(\Omega). \tag{3.1}$$

Then we consider the following inverse source problem (ISP): determine the spatially dependent source component $\psi(x)$ from the solution profile $u(T)$ at a later but unknown time T.

First, we give an intermediate result.

LEMMA 3.1. *Let $G(T):=\int_0^T E(s)g(T-s)\,\mathrm{d}s$. Then under condition* (3.1), *G is invertible and*

$$\|A^{-1}G(T)^{-1}v\|_{L^2(\Omega)} \le c_g^{-1}(1-E_{\alpha,1}(-\lambda_1T^\alpha))^{-1}\|v\|_{L^2(\Omega)}.$$

Proof. Note that the function $E_{\alpha,\alpha}(-t)>0$ for all $0\le t<\infty$ (since it is completely monotone and analytic) (see, e.g., [23], [20], or [4, Corollary 3.3]). This, the condition $g\ge c_g>0$, and the identity (2.5) imply

$$\int_0^T s^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n s^\alpha)g(T-s)\,\mathrm{d}s\ge c_g\int_0^T s^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n s^\alpha)\mathrm{d}s=c_g\lambda_n^{-1}(1-E_{\alpha,1}(-\lambda_n T^\alpha)).$$

This implies the invertibility of the operator $G(T)$:

$$G(T)^{-1}v=\sum_{n=1}^\infty\frac{(v,\varphi_n)}{\int_0^T s^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n s^\alpha)g(T-s)\,\mathrm{d}s}\varphi_n.$$

Consequently, for any $v\in L^2(\Omega)$, we have

$$\begin{aligned}\|A^{-1}G(T)^{-1}v\|_{L^2(\Omega)}^2&=\sum_{n=1}^\infty\Big[\frac{(v,\varphi_n)}{\lambda_n\int_0^T s^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n s^\alpha)g(T-s)\,\mathrm{d}s}\Big]^2\\ &\le\sum_{n=1}^\infty\Big[\frac{(v,\varphi_n)}{c_g\lambda_n\int_0^T s^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n s^\alpha)\,\mathrm{d}s}\Big]^2=\sum_{n=1}^\infty\Big[\frac{(v,\varphi_n)}{c_g(1-E_{\alpha,1}(-\lambda_n T^\alpha))}\Big]^2\\ &\le\sum_{n=1}^\infty\Big[\frac{(v,\varphi_n)}{(1-E_{\alpha,1}(-\lambda_1 T^\alpha))c_g}\Big]^2=c_g^{-2}(1-E_{\alpha,1}(-\lambda_1T^\alpha))^{-2}\|v\|_{L^2(\Omega)}^2,\end{aligned}$$

where in the last inequality we have used the monotonicity of $E_{\alpha,1}(-t)\in(0,1]$ for $t\ge 0$. ∎

The next result gives the unique determination of the source ψ and terminal time T.

THEOREM 3.2. *Let $u_0\in L^2(\Omega)\setminus\mathbb{S}_\gamma$ for some $\gamma\ge0$. If* ISP *has two solutions (T,ψ) and $(\tilde T,\tilde\psi)$ in the set $\mathbb{R}_+\times\mathbb{S}_\gamma$ from the observations $u(T)$ and $\tilde u(\tilde T)$, respectively, then $T=\tilde T$ and $\psi=\tilde\psi$.*

Proof. Using the representation (2.3) and the separability assumption (3.1), we have

$$u(t)=\sum_{n=0}^\infty\Big[E_{\alpha,1}(-\lambda_n t^\alpha)(u_0,\varphi_n)+\int_0^t s^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n s^\alpha)g(t-s)\,\mathrm{d}s\,(\psi,\varphi_n)\Big]\varphi_n. \tag{3.2}$$

Define $\mathbb{K}=\{k\in\mathbb{N}:(u_0,\varphi_n)\ne0\}$. For any $n\in\mathbb{K}$, taking inner product (or duality pairing) with $\frac{\lambda_n\varphi_n}{(u_0,\varphi_n)}$ on both sides of the identity (3.2) and setting $t=T$, we have

$$\frac{\lambda_n(u(T),\varphi_n)}{(u_0,\varphi_n)}=\lambda_nE_{\alpha,1}(-\lambda_nT^\alpha)+\int_0^T s^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n s^\alpha)g(T-s)\,\mathrm{d}s\frac{\lambda_n(\psi,\varphi_n)}{(u_0,\varphi_n)}.$$

By assumption, $g\in L^\infty(0,T)$ and $E_{\alpha,\alpha}(-t)>0$ for all $\infty>t\ge0$ (since it is completely monotone and analytic) [4, Corollary 3.3], we have

$$\begin{aligned}\Big|\int_0^T s^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n s^\alpha)g(T-s)\,\mathrm{d}s\Big|&\le\int_0^T s^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n s^\alpha)\,\mathrm{d}s\|g\|_{L^\infty(0,T)}\\ &=\lambda_n^{-1}[1-E_{\alpha,1}(-\lambda_nT^\alpha)]\|g\|_{L^\infty(0,T)}\\ &\le\lambda_n^{-1}\|g\|_{L^\infty(0,T)},\end{aligned}$$

where we have used the identity (2.5) and the inequality $E_{\alpha,1}(-t)\in[0,1]$ for all $t\geq 0$. Since $\psi\in\mathbb{S}_\gamma$ and $u_0\in L^2(\Omega)\backslash\mathbb{S}_\gamma$, we have

$$\begin{aligned}\lim_{n\in\mathbb{K},\ n\to\infty}&\left|\int_0^T s^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n s^\alpha)g(T-s)\,\mathrm{d}s\frac{\lambda_n(\psi,\varphi_n)}{(u_0,\varphi_n)}\right|\\ &\leq \lim_{n\in\mathbb{K},\ n\to\infty}\|g\|_{L^\infty(0,T)}\left|\frac{(\psi,\varphi_n)}{(u_0,\varphi_n)}\right|=0.\end{aligned}$$

Consequently, we have

$$\lim_{n\in\mathbb{K},\ n\to\infty}\frac{\lambda_n(u(T),\varphi_n)}{(u_0,\varphi_n)}=\lim_{n\in\mathbb{K},\ n\to\infty}\lambda_n E_{\alpha,1}(-\lambda_n T^\alpha)=\frac{1}{\Gamma(1-\alpha)T^\alpha}. \tag{3.3}$$

Note that the function $\frac{1}{\Gamma(1-\alpha)T^\alpha}$ is strictly decreasing in T. Hence, the terminal time T is uniquely determined by $u(T)$. Finally, the uniqueness of ψ follows from the representation $\psi=G(T)^{-1}u(T)-G(T)^{-1}F(T)u_0$ (with $G(T)=\int_0^T E(s)g(T-s)\,\mathrm{d}s$), and Lemma 3.1. ∎

Remark 3.1. Note that $\frac{1}{\Gamma(1-\alpha)T^\alpha}$ is also decreasing with respect to α if T is sufficiently large [3, Lemma 4]. So the data $u(T)$ can uniquely determine the order α if T is a priori known. See some related arguments in [3, 16] for the inverse source problem with an unknown order α.

The next theorem gives a stability result for recovering the source ψ.

THEOREM 3.3. *Fix $T_0>0$. Let ψ and $\tilde\psi$ be the solutions of ISP with the data $u(T)$ and $\tilde u(\tilde T)$ with $T_0\leq T<\tilde T$, respectively. Then for $g\in C^2[0,T]$, there holds*

$$\begin{aligned}\|\psi-\tilde\psi\|_{L^2(\Omega)}\leq c\big(&\|A(u(T)-\tilde u(\tilde T))\|_{L^2(\Omega)}+|T-\tilde T|T^{-1-\alpha}\|u_0\|_{L^2(\Omega)}\\ &+|T-\tilde T|(1+T^{-1}+T)\|\tilde\psi\|_{L^2(\Omega)}\big).\end{aligned}$$

Proof. It follows from the solution representation (2.3) that

$$u(T)=F(T)u_0+\int_0^T E(s)g(T-s)\,\mathrm{d}s\,\psi\quad\text{and}\quad \tilde u(\tilde T)=F(\tilde T)u_0+\int_0^{\tilde T}E(s)g(\tilde T-s)\,\mathrm{d}s\,\tilde\psi.$$

Then subtracting these two identities leads to

$$\begin{aligned}u(T)-\tilde u(\tilde T)&=\big(F(T)-F(\tilde T)\big)u_0+\int_0^T E(s)g(T-s)\,\mathrm{d}s\,(\psi-\tilde\psi)\\ &\quad+\int_0^T E(s)[g(T-s)-g(\tilde T-s)]\,\mathrm{d}s\tilde\psi-\int_T^{\tilde T}E(s)g(\tilde T-s)\,\mathrm{d}s\tilde\psi.\end{aligned}$$

Therefore, with $G(T):=\int_0^T E(s)g(T-s)\,\mathrm{d}s$, we arrive at

$$\begin{aligned}\psi-\tilde\psi&=G(T)^{-1}(u(T)-\tilde u(\tilde T))+G(T)^{-1}\big(F(\tilde T)-F(T)\big)u_0\\ &\quad+G(T)^{-1}\int_0^T E(s)[g(\tilde T-s)-g(T-s)]\,\mathrm{d}s\tilde\psi\\ &\quad+G(T)^{-1}\int_T^{\tilde T}E(s)g(\tilde T-s)\,\mathrm{d}s\tilde\psi=\sum_{j=1}^4\mathrm{I}_j.\end{aligned}$$

Next, we bound the four terms I_j separately. First, by Lemma 3.1,

$$\|\mathrm{I}_1\|_{L^2(\Omega)} \le c_g^{-1}(1 - E_{\alpha,1}(-\lambda_1 T_0^\alpha))^{-1}\|A(u(T) - \tilde{u}(\tilde{T}))\|_{L^2(\Omega)}.$$

Second, by the estimate (2.10), we can bound the term I_2 by

$$\|\mathrm{I}_2\|_{L^2(\Omega)} \le c\|A(F(\tilde{T}) - F(T))u_0\|_{L^2(\Omega)} \le c|\tilde{T} - T|T^{-1-\alpha}\|u_0\|_{L^2(\Omega)}.$$

Next, we bound the third term I_3. It follows directly from integration by parts and the identities $F'(t) = -AE(t)$ and $F(0) = I$ [4, Lemmas 6.2 and 6.3] that

$$\begin{aligned}
\int_0^T E(s)g(T-s)\,\mathrm{d}s &= A^{-1}\big(g(T)I - g(0)F(T)\big) - A^{-1}\int_0^T F(s)g'(T-s)\,\mathrm{d}s,\\
\int_0^T E(s)g(\tilde{T}-s)\,\mathrm{d}s &= A^{-1}\big(g(\tilde{T})I - g(\tilde{T}-T)F(T)\big) - A^{-1}\int_0^T F(s)g'(\tilde{T}-s)\,\mathrm{d}s.
\end{aligned}$$

Since $g \in C^2[0,T]$, we have

$$\begin{aligned}
\|\mathrm{I}_3\|_{L^2(\Omega)} &\le c\Big\|A\int_0^T E(s)(g(T-s) - g(\tilde{T}-s))\,\mathrm{d}s\tilde{\psi}\Big\|_{L^2(\Omega)}\\
&\le c\Big(|g(T) - g(\tilde{T})| + |g(\tilde{T}-T) - g(0)| + \int_0^T |g'(\tilde{T}-s) - g'(T-s)|\,\mathrm{d}s\Big)\|\tilde{\psi}\|_{L^2(\Omega)}\\
&\le c|\tilde{T} - T|(1+T)\|g\|_{C^2[0,\tilde{T}]}\|\tilde{\psi}\|_{L^2(\Omega)}.
\end{aligned}$$

Finally, for the term I_4, the estimate $\|AE(s)\| \le cs^{-1}$ [4, Theorem 6.4] yields

$$\begin{aligned}
\|\mathrm{I}_4\|_{L^2(\Omega)} &\le c\Big\|\int_T^{\tilde{T}} AE(s)g(\tilde{T}-s)\,\mathrm{d}s\tilde{\psi}\Big\|_{L^2(\Omega)}\\
&\le c\|g\|_{C[0,\tilde{T}]}\int_T^{\tilde{T}} s^{-1}\,\mathrm{d}s\|\tilde{\psi}\|_{L^2(\Omega)} \le c|\tilde{T} - T|T^{-1}\|g\|_{C[0,\tilde{T}]}\|\tilde{\psi}\|_{L^2(\Omega)}.
\end{aligned}$$

The preceding four estimates together complete the proof of the theorem. ∎

The next result bounds the terminal time T for perturbed data. The proof is identical with that for Corollary 2.4, and hence it is omitted.

COROLLARY 3.4. *Fix $T_0 > 0$. Let $u_0 \in L^2(\Omega)\setminus\mathbb{S}_\gamma$ for some $\gamma \ge 0$, and (T,ψ), $(\tilde{T},\tilde{\psi}) \in \mathbb{R}_+ \times \mathbb{S}_\gamma$ with $T, \tilde{T} \ge T_0$ be the solutions of ISP with observations $u(T)$ and $\tilde{u}(\tilde{T})$, respectively. Then the following estimate holds:*

$$|T - \tilde{T}| \le \Gamma(1-\alpha)^{-\frac{1}{\alpha}}\min(\Lambda,\tilde{\Lambda})^{-\frac{1}{\alpha}-1}|\Lambda - \tilde{\Lambda}|,$$

with the scalars Λ and $\tilde{\Lambda}$, respectively, given by

$$\Lambda = \lim_{n\in\mathbb{K},\ n\to\infty}\frac{\lambda_n(u(T),\varphi_n)}{(u_0,\varphi_n)} \quad \text{and} \quad \tilde{\Lambda} = \lim_{n\in\mathbb{K},\ n\to\infty}\frac{\lambda_n(\tilde{u}(\tilde{T}),\varphi_n)}{(u_0,\varphi_n)}.$$

In particular, for $\Lambda < \tilde{\Lambda}$, there holds $\|\psi - \tilde{\psi}\|_{L^2(\Omega)} \le c\big(\|A(u(T) - \tilde{u}(\tilde{T}))\|_{L^2(\Omega)} + |\Lambda - \tilde{\Lambda}|\Lambda^{-\frac{1}{\alpha}-1}(1 + \Lambda^{\frac{1}{\alpha}} + \Lambda^{-\frac{1}{\alpha}})\|\tilde{\psi}\|_{L^2(\Omega)}\big)$.

4. Inverse potential problem. In this section, we discuss the identification of the potential q in the model (1.1) from the observation $u(T)$, at an unknown terminal time T. Specifically, consider the domain $\Omega = (0,1)$ and a nonzero Dirichlet boundary condition:

$$\begin{cases} \partial_t^\alpha u - \partial_{xx} u + qu = f & \text{in } \Omega \times (0,T], \\ u(0,t) = a_0, u(1,t) = a_1 & \text{on } (0,T], \\ u(0) = u_0 & \text{in } \Omega, \end{cases} \tag{4.1}$$

where the functions $f > 0$ and $u_0 > 0$ are given spatially dependent source and initial data, respectively, and a_0 and a_1 are positive constants. Throughout, the potential q belongs to the following admissible set $\mathcal{A} = \{q \in L^\infty(\Omega) : 0 \le q \le c_0\}$. The inverse potential problem (IPP) is to recover the potential $q \in \mathcal{A}$ from the observation $u(T)$, for an unknown time T.

Similar to the discussions in section 2, let A_q be the realization of the elliptic operator $-\partial_{xx} + q(x)I$ in $L^2(\Omega)$, with its domain $\mathrm{Dom}(A_q)$ given by $\mathrm{Dom}(A_q) := \{v \in L^2(\Omega) : -\partial_{xx} v + qv \in L^2(\Omega) \text{ and } v(0) = v(1) = 0 \text{ in } \partial\Omega\}$. Let $\{(\lambda_n(q), \varphi_n(q))\}_{n=1}^\infty$ be the eigenpairs of A_q, which is not known for an unknown q. Note that for any $q \in \mathcal{A}$, the set $\{\varphi_n(q)\}_{n=1}^\infty$ can be chosen to form a complete (orthonormal) basis of the space $L^2(\Omega)$. Note that for any $q \in \mathcal{A}$, the eigenvalues $\lambda_n(q)$ and eigenfunctions $\varphi_n(q)$ satisfy the following asymptotics [14, section 2, Chapter 1]:

$$\lambda_n(q) = n^2\pi^2 + O(1) \quad \text{and} \quad \varphi_n(x;q) = \sqrt{2}\sin(n\pi x) + O(n^{-1}). \tag{4.2}$$

Further, for any $v \in H_0^1(\Omega) \cap H^2(\Omega)$ and $q \in \mathcal{A}$, the following two-sided inequality holds:

$$c_1 \|v\|_{H^2(\Omega)} \le \|A_q v\|_{L^2(\Omega)} + \|v\|_{L^2(\Omega)} \le c_2 \|v\|_{H^2(\Omega)}, \tag{4.3}$$

with constants c_1 and c_2 independent of q. Next, we define a function $\phi_q \in H^2(\Omega)$ satisfying

$$\begin{cases} -\partial_{xx}\phi + q\phi = 0 & \text{in } \Omega = (0,1), \\ \phi_q(0) = a_0, \quad \phi_q(1) = a_1. \end{cases}$$

It is easy to see that $\phi_0(x) = a_0(1-x) + a_1 x$. Then the solution u of problem (4.1) is given by

$$u(t) = F_q(t)u_0 + (I - F_q(t))\phi_q + (I - F_q(t))A_q^{-1} f, \tag{4.4}$$

where E_q and F_q denote the solution operators (cf. (2.2)) for the elliptic operator A_q, and the subscript q explicitly indicates the dependence on the potential q.

Next, we show the unique recovery of the terminal time T. Like before, the key is to distinguish the decay rates of Fourier coefficients of u_0 and f with respect to the eigenfunctions $\{\varphi_j(q)\}_{j=1}^\infty$.

THEOREM 4.1. *Let $u_0 \in L^2(\Omega)\backslash \dot H^{s'}(\Omega)$, with $s' \in (0, \frac{1}{2})$, and $f \in \dot H^s(\Omega)$ and $q \in \mathcal{A} \cap \dot H^s(\Omega)$ with $s \in (s' + \frac{1}{2}, 1)$. Then in IPP, the terminal time T is uniquely determined by the data $u(T)$.*

Proof. Let $\bar u_0 = u_0 - \phi_0$. Since $u_0 \in L^2(\Omega)\backslash \dot H^{s'}(\Omega)$, by the asymptotics (4.2), we obtain

$$\sum_{n=1}^\infty \lambda_n(q)^{s'} (\bar u_0, \sin(n\pi x))^2 = \infty. \tag{4.5}$$

Then we claim that for $\bar{s} = \frac{1}{4} + \frac{s'}{2} + \frac{\epsilon}{4} < \frac{s}{2}$, with a small $\epsilon \in (0, 2s - 2s' - 1)$, there holds $\bar{u}_0 \notin \mathbb{S}_{\bar{s}}$. Indeed, assuming the contrary, i.e., $\bar{u}_0 \in \mathbb{S}_{\bar{s}}$, the definition of $\mathbb{S}_{\bar{s}}$ in (2.4) implies $\lim_{n\to\infty} \lambda_n(q)^{\bar{s}} |(\bar{u}_0, \sin(n\pi x))| = 0$, and thus the sequence $\{\lambda_n(q)^{\bar{s}} |(\bar{u}_0, \sin(n\pi x))|\}_{n=1}^{\infty}$ is uniformly bounded. This and the asymptotics (4.2) lead to

$$\sum_{n=1}^{\infty} \lambda_n(q)^{s'} (\bar{u}_0, \sin(n\pi x))^2 = \sum_{n=1}^{\infty} \lambda_n(q)^{2\bar{s}} (\bar{u}_0, \sin(n\pi x))^2 \lambda_n(q)^{s'-2\bar{s}} \leq c \sum_{n=1}^{\infty} \lambda_n(q)^{s'-2\bar{s}} \leq c \sum_{n=1}^{\infty} n^{-1-\epsilon} < \infty.$$

This contradicts the identity (4.5), and hence the desired claim follows. The claim $\bar{u}_0 \notin \mathbb{S}_{\bar{s}}$ and the asymptotics (4.2) imply that there exists a constant $c_* > 0$, for any $N > 0$, we can find $n > N$ such that $n^{2\bar{s}} |(\bar{u}_0, \sin(n\pi x))| \geq c_*$. Let $\mathbb{K} = \{n \in \mathbb{N} : n^{2\bar{s}} |(\bar{u}_0, \sin(n\pi x))| \geq c_*\}$. Then we have $|\mathbb{K}| = \infty$. By the asymptotics (4.2), we may assume that $n^{2\bar{s}} |(\bar{u}_0, \varphi_n(q))| \geq \sqrt{2} c_*/2$ for $n \in \mathbb{K}$, and hence $(\bar{u}_0, \varphi_n(q)) \neq 0$. Meanwhile, it follows directly from (2.3) that the solution $u(t)$ satisfies

$$u(t) - \phi_q = \sum_{n=0}^{\infty} \left[E_{\alpha,1}(-\lambda_n(q) t^\alpha)(u_0 - \phi_q, \varphi_n(q)) + \big(1 - E_{\alpha,1}(-\lambda_n(q) t^\alpha)\big) \frac{(f, \varphi_n(q))}{\lambda_n(q)} \right] \varphi_n(q). \tag{4.6}$$

By taking inner product with $\varphi_n(q)$, $n \in \mathbb{K}$, on both sides of the identity (4.6), we obtain

$$\begin{aligned} \frac{\lambda_n(q)(u(t) - \phi_0, \varphi_n(q))}{(\bar{u}_0, \varphi_n(q))} &= \lambda_n(q) E_{\alpha,1}(-\lambda_n(q) t^\alpha) + \big(1 - E_{\alpha,1}(-\lambda_n(q) t^\alpha)\big) \frac{(f, \varphi_n(q))}{(\bar{u}_0, \varphi_n(q))} \\ &\quad + \big(1 - E_{\alpha,1}(-\lambda_n(q) t^\alpha)\big) \frac{\lambda_n(q)(\phi_q - \phi_0, \varphi_n(q))}{(\bar{u}_0, \varphi_n(q))} := \sum_{i=1}^{3} \mathrm{I}_i. \end{aligned} \tag{4.7}$$

We analyze the three terms I_i, $i = 1, 2, 3$, separately. Since $E_{\alpha,1}(-\lambda_n t^\alpha) \in (0, 1]$ for $t \geq 0$, the regularity condition $f \in \dot{H}^s(\Omega)$ implies $\lim_{n\to\infty} n^s |(f, \varphi_n(q))| = 0$. This and the condition $s > 2\bar{s}$ yield $0 \leq n^{2\bar{s}} |(f, \varphi_n(q))| \leq n^s |(f, \varphi_n(q))| \to 0$, as $n \to \infty$. Hence

$$\lim_{n\in\mathbb{K},\, n\to\infty} |\mathrm{I}_2| \leq \lim_{n\in\mathbb{K},\, n\to\infty} \left| \frac{n^{2\bar{s}} (f, \varphi_n(q))}{n^{2\bar{s}} (\bar{u}_0, \varphi_n(q))} \right| = 0.$$

By the definitions of the eigenpairs $(\lambda_n(q), \varphi_n(q))$ and ϕ_q and ϕ_0 and integration by parts, we have

$$\lambda_n(q)(\phi_q - \phi_0, \varphi_n(q)) = (\phi_q - \phi_0, -\partial_{xx} \varphi_n(q) + q \varphi_n(q)) = -(q\phi_0, \varphi_n(q)).$$

Moreover, since $q \in \dot{H}^s(\Omega)$ with some $s \in (\frac{1}{2}, 1)$ and $\phi_0 \in C^\infty(\overline{\Omega})$, we have $q\phi_0 \in \dot{H}^s(\Omega)$. Hence,

$$\lim_{n\in\mathbb{K},\, n\to\infty} |\mathrm{I}_3| \leq \lim_{n\in\mathbb{K},\, n\to\infty} \left| \frac{(q\phi_0, \varphi_n(q))}{(\bar{u}_0, \varphi_n(q))} \right| = \lim_{n\in\mathbb{K},\, n\to\infty} \left| \frac{n^{2\bar{s}} (q\phi_0, \varphi_n(q))}{n^{2\bar{s}} (\bar{u}_0, \varphi_n(q))} \right| = 0. \tag{4.8}$$

Now letting $n \to \infty$ and setting $t = T$, the relation (4.7) and the asymptotics of $E_{\alpha,1}(z)$ imply

$$\lim_{n\in\mathbb{K},\ n\to\infty} \frac{\lambda_n(q)(u(T) - \phi_0, \varphi_n(q))}{(\bar{u}_0, \varphi_n(q))} = \lim_{n\in\mathbb{K},\ n\to\infty} \lambda_n(q) E_{\alpha,1}(-\lambda_n(q) T^\alpha) = \frac{1}{\Gamma(1-\alpha) T^\alpha}, \tag{4.9}$$

since $\lambda_n(q) \to \infty$; cf. (4.2). Note that the function $\frac{1}{\Gamma(1-\alpha)T^\alpha}$ is decreasing in T and independent of q. Last, we show that the left-hand side of (4.9) can actually be computed independently of $q \in \dot{H}^s(\Omega) \cap \mathcal{A}$, using the asymptotics (4.2). Indeed, since $u_0 \in L^2(\Omega) \backslash \dot{H}^{s'}(\Omega)$, $f \in \dot{H}^s(\Omega)$, and $q \in \dot{H}^s(\Omega) \cap \mathcal{A}$ with $s \in (s' + \frac{1}{2}, 1)$ and $s' \in (0, \frac{1}{2})$, we have $A(q)u(T) \in L^2(\Omega) \backslash \dot{H}^{s'}(\Omega)$ [22]. This and the assumption $q \in \dot{H}^s(\Omega)$ imply $qu(T) \in \dot{H}^s(\Omega)$, and hence $\partial_{xx} u(T) \in L^2(\Omega) \backslash \dot{H}^{s'}(\Omega)$. Meanwhile, integration by parts twice yields

$$\lambda_n(q)(u(T) - \phi_0, \varphi_n(q)) = -(\partial_{xx} u(T), \varphi_n(q)) + (q(u(T) - \phi_0), \varphi_n(q)).$$

Noting the fact that $q(u(T) - \phi_0) \in \dot{H}^s(\Omega)$ and then repeating the argument for (4.8) yield

$$\lim_{n \in \mathbb{K},\ n \to \infty} \frac{(q(u(T) - \phi_0), \varphi_n(q))}{(\bar{u}_0, \varphi_n(q))} = 0.$$

Consequently, we derive

$$\lim_{n \in \mathbb{K},\ n \to \infty} \frac{\lambda_n(q)(u(T) - \phi_0, \varphi_n(q))}{(\bar{u}_0, \varphi_n(q))} = \lim_{n \in \mathbb{K},\ n \to \infty} \frac{-(\partial_{xx} u(T), \varphi_n(q))}{(\bar{u}_0, \varphi_n(q))}. \tag{4.10}$$

Now using the asymptotics (4.2) again, we obtain

$$\begin{aligned} \frac{-(\partial_{xx} u(T), \varphi_n(q))}{(\bar{u}_0, \varphi_n(q))} &= \frac{-(\partial_{xx} u(T), \sqrt{2}\sin(n\pi x)) + O(n^{-1}) \cdot \|\partial_{xx} u(T)\|_{L^1(\Omega)}}{(\bar{u}_0, \sqrt{2}\sin(n\pi x)) + O(n^{-1}) \cdot \|\bar{u}_0\|_{L^1(\Omega)}} \\ &= \frac{-n^{2\bar{s}}(\partial_{xx} u(T), \sqrt{2}\sin(n\pi x)) + O(n^{2\bar{s}-1}) \cdot \|\partial_{xx} u(T)\|_{L^1(\Omega)}}{n^{2\bar{s}}(\bar{u}_0, \sqrt{2}\sin(n\pi x)) + O(n^{2\bar{s}-1}) \cdot \|\bar{u}_0\|_{L^1(\Omega)}}. \end{aligned} \tag{4.11}$$

Since $n^{2\bar{s}}|(\bar{u}_0, \sin(n\pi x))| \geq c_*$ for all $n \in \mathbb{K}$, by the condition $2\bar{s} - 1 < 0$, we obtain

$$\begin{aligned} &\lim_{n \in \mathbb{K},\ n \to \infty} \frac{-(\partial_{xx} u(T), \varphi_n(q))}{(\bar{u}_0, \varphi_n(q))} \\ &= \lim_{n \in \mathbb{K},\ n \to \infty} \frac{-n^{2\bar{s}}(\partial_{xx} u(T), \sqrt{2}\sin(n\pi x)) + O(n^{2\bar{s}-1}) \cdot \|\partial_{xx} u(T)\|_{L^1(\Omega)}}{n^{2\bar{s}}(\bar{u}_0, \sqrt{2}\sin(n\pi x)) + O(n^{2\bar{s}-1}) \cdot \|\bar{u}_0\|_{L^1(\Omega)}} \\ &= \lim_{n \in \mathbb{K},\ n \to \infty} \frac{-(\partial_{xx} u(T), \sin(n\pi x))}{(\bar{u}_0, \sin(n\pi x))}. \end{aligned}$$

Therefore, the terminal time T is uniquely determined by the observation $u(T)$. ◻

Remark 4.1. The independence of the limit in (4.9) on the potential q relies on the asymptotics (4.2). This seems valid only in the one-dimensional case, and it represents the main obstacle for the extension to the multidimensional case. Nonetheless, the rest of the analysis does not use the estimate (4.2), and all the remaining results hold also for the multidimensional case.

Next, we determine the potential $q \in \mathcal{A}$ from $u(T)$. First, we give useful smoothing properties of the operators F_q and E_q. The notation $\|\cdot\|$ denotes the operator norm on $L^2(\Omega)$.

LEMMA 4.2. *For $q \in \mathcal{A}$, there exists a $c > 0$ independent of q and t such that for any $s = 0, 1$ and $\ell = 0, 1$,*

$$\|F_q(t)\| + t^{1-\alpha}\|E_q(t)\| \leq c \min(1, t^{-\alpha}) \quad \textit{and} \quad \|A_q^s F_q^{(\ell)}(t)\| \leq c t^{-\alpha-\ell}.$$

Proof. The estimates follow from [4, Theorem 6.4]. By [4, Theorem 6.4(iv)] and Lemma 2.1,

$$\|E_q(t)\| \le t^{\alpha-1}E_{\alpha,\alpha}(-\lambda_1(q)t^\alpha) \le ct^{\alpha-1}\min(1,t^{-\alpha}).$$

The bound on $F_q(t)$ follows similarly. For the second estimate, the case $s=1$, $\ell=0,1$, the assertion is contained in [4, Theorem 6.4(iii)], and the case $s=0, \ell=0$ is direct from the first estimate. The remaining case $s=0, \ell=1$ follows from Lemma 2.1 (noting $1/\Gamma(0)=0$):

$$\begin{aligned}\|F_q'(t)v\|_{L^2(\Omega)}^2 = \|A_qE_q(t)v\|_{L^2(\Omega)}^2 &= \sum_{n=1}^\infty \lambda_n(q)^2 t^{2\alpha-2}E_{\alpha,\alpha}(-\lambda_n(q)t^\alpha)^2(v,\varphi_n(q))^2\\ &\le ct^{2\alpha-2}\sum_{n=1}^\infty \frac{\lambda_n^2}{(1+(\lambda_n t^\alpha)^2)^2}(v,\varphi_n(q))^2 \le ct^{-2-2\alpha}\|v\|_{L^2(\Omega)}^2.\end{aligned}$$

Combining these assertions completes the proof of the lemma. ∎

The next lemma gives a priori estimate on the solution u to problem (4.1).

LEMMA 4.3. *Let $u_0, f \in L^2(\Omega)$ and $q \in \mathcal{A}$, and let u be the solution to problem* (4.1). *Then there exists $c>0$ independent of q and t such that*

$$\|\partial_t u(t)\|_{H^2(\Omega)} \le ct^{-\alpha-1} \quad \text{and} \quad \|\partial_t^\alpha u(t)\|_{L^2(\Omega)} + \|u(t)\|_{H^2(\Omega)} \le c(1+t^{-\alpha}).$$

Proof. The proof employs (4.3) and Lemma 4.2. First, by (4.4), we have $\partial_t u(t) = F_q'(t)(u_0-\phi_q-A_q^{-1}f)$. Then from Lemma 4.2 and the norm equivalence (4.3), the first estimate follows:

$$\|\partial_t u(t)\|_{H^2(\Omega)} \le c(\|A_qF_q'(t)\| + \|F_q'(t)\|)\|u_0-\phi_q-A_q^{-1}f\|_{L^2(\Omega)} \le ct^{-\alpha-1}.$$

Similarly, by (4.4) and Lemma 4.2, there holds that

$$\begin{aligned}\|u(t)\|_{L^2(\Omega)} &\le \|F_q(t)u_0 + (I-F_q(t))\phi_q + (I-F_q(t))A_q^{-1}f\|_{L^2(\Omega)} \le c,\\ \|u(t)\|_{H^2(\Omega)} &\le \|F_q(t)(u_0-\phi_q) + (I-F_q(t))A_q^{-1}f\|_{\dot H^2(\Omega)} + \|\phi_q\|_{H^2(\Omega)}\\ &\le c\min(1,t^{-\alpha})\|u_0-\phi_q\|_{L^2(\Omega)} + \|f\|_{L^2(\Omega)} + \|\phi_q\|_{H^2(\Omega)} \le c(1+t^{-\alpha}).\end{aligned}$$

Last, we bound $\partial_t^\alpha u$ using the identity $\partial_t^\alpha u(t) = -A_qF_q(t)(u_0+\phi_q-A_q^{-1}f)$ and Lemma 4.2. ∎

For any $q\in\mathcal{A}$, we denote the solution u to problem (4.1) by $u(q)$. The next lemma provides a crucial a priori estimate. Like before, we denote by u and $\tilde u$ to be $u(q)$ and $u(\tilde q)$ below.

LEMMA 4.4. *Let $T_*>0$ be fixed, and let $q,\tilde q\in\mathcal{A}$. Then there exists $c>0$ independent of q, $\tilde q$ and t such that for any $t\ge T_*>0$,*

$$\|\partial_t^\alpha(u-\tilde u)(t)\|_{H^2(\Omega)} \le ct^{-\alpha}\|q-\tilde q\|_{L^2(\Omega)}.$$

Proof. Let $w=u-\tilde u$. Then w solves

$$\begin{cases}\partial_t^\alpha w - \Delta w + qw = (\tilde q - q)\tilde u & \text{in } \Omega\times(0,T],\\ w=0 & \text{on } \partial\Omega\times(0,T],\\ w(0)=0 & \text{in } \Omega.\end{cases} \tag{4.12}$$

The representation (2.3) implies $w(t) = \int_0^t E_q(t-s)(\tilde q - q)\tilde u(s)\,\mathrm{d}s$. The governing equation for w and the identities $\partial_t^\alpha F_q(t) = -A_q F_q(t)$ and $A_q E_q(t) = -F_q'(t)$ [4, Lemma 6.3] lead to

$$\begin{aligned}\partial_t^\alpha w(t) &= -A_q \int_0^t E_q(t-s)(\tilde q - q)\tilde u(s)\,\mathrm{d}s + (\tilde q - q)\tilde u(t)\\ &= \int_0^t F_q'(t-s)(\tilde q - q)\tilde u(s)\,\mathrm{d}s + (\tilde q - q)\tilde u(t) = \partial_t \int_0^t F_q(t-s)(\tilde q - q)\tilde u(s)\,\mathrm{d}s.\end{aligned}$$

Let $\phi(t) = \int_0^t F_q(t-s)(\tilde q - q)\tilde u(s)\,\mathrm{d}s$. Sobolev embedding theorem and Lemmas 4.2 and 4.3 imply

$$\begin{aligned}\|A_q\phi(t)\|_{L^2(\Omega)} &\le \|\tilde q - q\|_{L^2(\Omega)} \int_0^t \|A_q F_q(t-s)\|\, \|\tilde u(s)\|_{L^\infty(\Omega)}\,\mathrm{d}s\\ &\le c\|\tilde q - q\|_{L^2(\Omega)} \int_0^t (t-s)^{-\alpha}\|\tilde u(s)\|_{H^2(\Omega)}\,\mathrm{d}s\\ &\le c(t^{1-\alpha} + t^{1-2\alpha})\|\tilde q - q\|_{L^2(\Omega)} \le c_{T_*} t^{1-\alpha}\|\tilde q - q\|_{L^2(\Omega)}.\end{aligned}$$

Next, the identity $tA_q\phi(t) = \int_0^t (t-s)A_qF_q(t-s)(\tilde q - q)\tilde u(s)\,\mathrm{d}s + \int_0^t A_qF_q(s)(\tilde q - q)(t-s)\tilde u(t-s)\,\mathrm{d}s$ yields

$$\begin{aligned}\partial_t(tA(q_1)\phi(t)) &= \int_0^t \left[(t-s)A_qF_q'(t-s) + A_qF_q(t-s)\right](\tilde q - q)\tilde u(s)\,\mathrm{d}s\\ &\quad + \int_0^t A_qF_q(s)(\tilde q - q)\left[\tilde u(t-s) + (t-s)\tilde u'(t-s)\right]\mathrm{d}s =: \mathrm{I}_1 + \mathrm{I}_2.\end{aligned}$$

Next, we bound I_1 and I_2. First, we bound the term I_1 by Lemmas 4.2 and 4.3:

$$\begin{aligned}\|\mathrm{I}_1\|_{L^2(\Omega)} &\le \int_0^t \left[(t-s)\|A_qF_q'(t-s)\| + \|A_qF_q(t-s)\|\right]\|(\tilde q - q)\tilde u(s)\|_{L^2(\Omega)}\,\mathrm{d}s\\ &\le c\|\tilde q - q\|_{L^2(\Omega)} \int_0^t (t-s)^{-\alpha}\|\tilde u(s)\|_{L^\infty(\Omega)}\,\mathrm{d}s \le c_{T_*} t^{1-\alpha}\|\tilde q - q\|_{L^2(\Omega)}.\end{aligned}$$

Similarly, by Lemma 4.2, Sobolev embedding theorem, and Lemma 4.3, the term I_2 is bounded by

$$\begin{aligned}\|\mathrm{I}_2\|_{L^2(\Omega)} &\le \int_0^t \|A_qF_q(s)\|\, \|\tilde q - q\|_{L^2(\Omega)}\left[(t-s)\|\tilde u'(t-s)\|_{L^\infty(\Omega)} + \|\tilde u(t-s)\|_{L^\infty(\Omega)}\right]\mathrm{d}s\\ &\le c\|\tilde q - q\|_{L^2(\Omega)} \int_0^t s^{-\alpha}\left[(t-s)\|\tilde u'(t-s)\|_{H^2(\Omega)} + \|\tilde u(t-s)\|_{H^2(\Omega)}\right]\mathrm{d}s\\ &\le c\|\tilde q - q\|_{L^2(\Omega)} \int_0^t s^{-\alpha}((t-s)^{-\alpha} + 1)\,\mathrm{d}s \le c_{T_*} t^{1-\alpha}\|\tilde q - q\|_{L^2(\Omega)}.\end{aligned}$$

Then the triangle inequality yields that for any $t > 0$, there holds

$$t\|A_q\phi'(t)\|_{L^2(\Omega)} \le \|(tA_q\phi(t))'\|_{L^2(\Omega)} + \|A_q\phi(t)\|_{L^2(\Omega)} \le c_{T^*} t^{1-\alpha}\|\tilde q - q\|_{L^2(\Omega)}.$$

Now the desired inequality follows directly. This completes the proof of the lemma. ∎

Next, we give a stability result. It improves a known result [27, 26] by relaxing the regularity assumption.

THEOREM 4.5. *Let $u_0, f \in L^2(\Omega)$, with $u_0, f \geq m$ a.e. in Ω and $a_0, a_1 \geq m$ for some $m > 0$. Then for $q, \tilde{q} \in \mathcal{A}$, and a sufficiently large T, there exists $c > 0$ independent of q, $\tilde{q}$, and T such that*

$$\|q - \tilde{q}\|_{L^2(\Omega)} \leq c\|u(T) - \tilde{u}(T)\|_{H^2(\Omega)}.$$

Proof. It follows from (4.1) that q can be expressed as

$$q = [u(T)]^{-1}(f - \partial_t^\alpha u(T) + \partial_{xx} u(T)). \tag{4.13}$$

Then we split the difference $q - \tilde{q}$ into

$$\begin{aligned} q - \tilde{q} &= f\frac{\tilde{u}(T) - u(T)}{u(T)\tilde{u}(T)} + \frac{u(T)\partial_t^\alpha \tilde{u}(T) - \tilde{u}(T)\partial_t^\alpha u(T)}{u(T)\tilde{u}(T)} + \frac{\tilde{u}(T)\partial_{xx} u(T) - u(T)\partial_{xx}\tilde{u}(T)}{u(T)\tilde{u}(T)} \\ &= \sum_{i=1}^{3} \mathrm{I}_i. \end{aligned}$$

By the maximum principle of time-fractional diffusion [18], we deduce $u(T) \geq m > 0$. This and the standard Sobolev embedding $H^2(\Omega) \hookrightarrow L^\infty(\Omega)$ (for $d = 1, 2, 3$) imply

$$\|\mathrm{I}_1\|_{L^2(\Omega)} \leq m^{-2}\|f\|_{L^2(\Omega)}\|\tilde{u}(T) - u(T)\|_{H^2(\Omega)}.$$

By Lemma 4.3 and Sobolev embedding theorem, we have the a priori bound $\|u(T)\|_{L^\infty(\Omega)} + \|\partial_t^\alpha u(T)\|_{L^2(\Omega)} \leq c_{T^*}$ for $T \geq T^*$. This and Lemma 4.4 lead to

$$\begin{aligned} \|\mathrm{I}_2\|_{L^2(\Omega)} &\leq c\big(\|u(T)\|_{L^\infty(\Omega)}\|\partial_t^\alpha(\tilde{u}(T) - u(T))\|_{L^2(\Omega)} \\ &\qquad + \|\partial_t^\alpha u(T)\|_{L^2(\Omega)}\|u(T) - \tilde{u}(T)\|_{L^\infty(\Omega)}\big) \\ &\leq c\big(T^{-\alpha}\|q - \tilde{q}\|_{L^2(\Omega)} + \|u(T) - \tilde{u}(T)\|_{H^2(\Omega)}\big), \\ \|\mathrm{I}_3\|_{L^2(\Omega)} &\leq c\big(\|u(T)\|_{L^\infty(\Omega)}\|\partial_{xx}(\tilde{u}(T) - u(T))\|_{L^2(\Omega)} \\ &\qquad + \|\partial_{xx} u(T)\|_{L^2(\Omega)}\|u(T) - \tilde{u}(T)\|_{L^\infty(\Omega)}\big) \\ &\leq c\|u(T) - \tilde{u}(T)\|_{H^2(\Omega)}. \end{aligned}$$

Then for sufficiently large T, we have $\|q-\tilde{q}\|_{L^2(\Omega)} \leq c(1-cT^{-\alpha})^{-1}\|u(T)-\tilde{u}(T)\|_{H^2(\Omega)}$. This completes the proof of the theorem. ∎

The next stability estimate is the main result of this section.

THEOREM 4.6. *Let $u_0, f \in L^2(\Omega)$, with $u_0, f \geq m$ a.e. in Ω and $a_0, a_1 \geq m$ for some $m > 0$. Then for $q, \tilde{q} \in \mathcal{A}$, and $T_0 \leq T \leq \tilde{T}$ with sufficiently large T_0, there exists c independent of q, $\tilde{q}$, T and $\tilde{T}$ such that*

$$\|q - \tilde{q}\|_{L^2(\Omega)} \leq C\big(\|u(T) - \tilde{u}(\tilde{T})\|_{H^2(\Omega)} + T^{-\alpha-1}|T - \tilde{T}|\big).$$

Proof. In view of the identity (4.13), we have the following splitting:

$$\begin{aligned} q - \tilde{q} &= \left(\frac{f - \partial_t^\alpha u(T) + \partial_{xx} u(T)}{u(T)} - \frac{f - \partial_t^\alpha \tilde{u}(T) + \partial_{xx}\tilde{u}(T)}{\tilde{u}(T)}\right) \\ &\quad \left(\frac{f - \partial_t^\alpha \tilde{u}(T) + \partial_{xx}\tilde{u}(T)}{\tilde{u}(T)} - \frac{f - \partial_t^\alpha \tilde{u}(\tilde{T}) + \partial_{xx}\tilde{u}(\tilde{T})}{\tilde{u}(\tilde{T})}\right) =: \mathrm{I}_1 + \mathrm{I}_2. \end{aligned}$$

Theorem 4.5 implies the following estimate on I_1:

$$\|\mathrm{I}_1\|_{L^2(\Omega)} \leq c\big(T^{-\alpha}\|q - \tilde{q}\|_{L^2(\Omega)} + \|u(T) - \tilde{u}(T)\|_{H^2(\Omega)}\big).$$

The triangle inequality and Lemma 4.3 with $\gamma=2$ lead to

$$\begin{aligned}\|u(T)-\tilde u(T)\|_{H^2(\Omega)} &\le \|u(T)-\tilde u(\tilde T)\|_{H^2(\Omega)}+\|\tilde u(T)-\tilde u(\tilde T)\|_{H^2(\Omega)}\\ &\le \|u(T)-\tilde u(\tilde T)\|_{H^2(\Omega)}+cT^{-\alpha-1}|\tilde T-T|,\end{aligned}$$

and consequently,

$$\|\mathrm{I}_1\|_{L^2(\Omega)}\le c\big(T^{-\alpha}\|q-\tilde q\|_{L^2(\Omega)}+\|u(T)-\tilde u(\tilde T)\|_{H^2(\Omega)}+cT^{-\alpha-1}|\tilde T-T|\big).$$

Next, to bound the term I_2, we rewrite

$$\begin{aligned}\mathrm{I}_2&=f\frac{\tilde u(\tilde T)-\tilde u(T)}{\tilde u(T)\tilde u(\tilde T)}+\frac{\tilde u(T)\partial_t^\alpha\tilde u(\tilde T)-\tilde u(\tilde T)\partial_t^\alpha\tilde u(T)}{\tilde u(T)\tilde u(\tilde T)}+\frac{\tilde u(\tilde T)\partial_{xx}\tilde u(T)-\tilde u(T)\partial_{xx}\tilde u(\tilde T)}{\tilde u(T)\tilde u(\tilde T)}\\ &=\sum_{i=1}^3\mathrm{I}_{2,i}.\end{aligned}$$

By the maximum principle [18], we have $u(T),\tilde u(\tilde T)\ge m>0$. This and Lemma 4.3 lead to

$$\|\mathrm{I}_{2,1}\|_{L^2(\Omega)}\le m^{-2}\|f\|_{L^2(\Omega)}\|\tilde u(\tilde T)-\tilde u(T)\|_{H^2(\Omega)}\le cT^{-\alpha-1}|\tilde T-T|.$$

Meanwhile, by the Sobolev embedding theorem and Lemma 4.3, we obtain

$$\begin{aligned}\|\mathrm{I}_{2,2}\|_{L^2(\Omega)}&\le c\big(\|\partial_t^\alpha\tilde u(\tilde T)\|_{L^2(\Omega)}\|\tilde u(T)-\tilde u(\tilde T)\|_{H^2(\Omega)}\\ &\qquad+\|\partial_t^\alpha\tilde u(\tilde T)-\partial_t^\alpha\tilde u(T)\|_{L^2(\Omega)}\|\tilde u(\tilde T)\|_{H^2(\Omega)}\big)\\ &\le c\big(\|\partial_t^\alpha\tilde u(\tilde T)\|_{L^2(\Omega)}+\|\tilde u(\tilde T)\|_{H^2(\Omega)}\big)\|\tilde u(T)-\tilde u(\tilde T)\|_{H^2(\Omega)}\le cT^{-\alpha-1}|\tilde T-T|,\\ \|\mathrm{I}_{2,3}\|_{L^2(\Omega)}&\le c\big(\|\partial_{xx}\tilde u(\tilde T)\|_{L^2(\Omega)}\|\tilde u(T)-\tilde u(\tilde T)\|_{H^2(\Omega)}\\ &\qquad+\|\partial_{xx}(\tilde u(\tilde T)-\tilde u(T))\|_{L^2(\Omega)}\|\tilde u(\tilde T)\|_{H^2(\Omega)}\big)\\ &\le c\big(\|\partial_t^\alpha\tilde u(\tilde T)\|_{L^2(\Omega)}+\|\tilde u(\tilde T)\|_{H^2(\Omega)}\big)\|\tilde u(T)-\tilde u(\tilde T)\|_{H^2(\Omega)}\le cT^{-\alpha-1}|\tilde T-T|.\end{aligned}$$

Combining the preceding estimates yields

$$\|q-\tilde q\|_{L^2(\Omega)}\le cT^{-\alpha}\|q-\tilde q\|_{L^2(\Omega)}+c\|u(T)-\tilde u(\tilde T)\|_{H^2(\Omega)}+cT^{-\alpha-1}|\tilde T-T|.$$

By choosing T_0 large enough such that $cT_0^{-\alpha}\le\frac12$, we deduce that for $\tilde T\ge T\ge T_0$, the desired estimate holds. ∎

The next corollary of Theorem 4.6 bounds the terminal time T for perturbed data.

Corollary 4.7. *Suppose that $u_0\in L^2(\Omega)\backslash\dot H^{s_1}(\Omega)$, $f\in\dot H^s(\Omega)$, and $q\in\mathcal A\cap H^s(\Omega)$, with $s\in(0,\frac12)$ and $s_1\in(0,s)$. Let (T,q), $(\tilde T,\tilde q)\in\mathbb R_+\times\mathcal A\cap\dot H^s(\Omega)$ be the solutions of* IPP *with observations $u(T)$ and $\tilde u(\tilde T)$, respectively. Then the following estimate holds:*

$$|T-\tilde T|\le\Gamma(1-\alpha)^{-\frac1\alpha}\min(\Lambda,\tilde\Lambda)^{-\frac1\alpha-1}|\Lambda-\tilde\Lambda|,$$

with the scalars Λ and $\tilde\Lambda$, respectively, given by

$$\Lambda=\lim_{n\in\mathbb K,\ n\to\infty}\frac{-(\partial_{xx}u(T),\sin(n\pi x))}{(u_0-\phi_0,\sin(n\pi x))}\quad\text{and}\quad\tilde\Lambda=\lim_{n\in\mathbb K,\ n\to\infty}\frac{-(\partial_{xx}\tilde u(\tilde T),\sin(n\pi x))}{(u_0-\phi_0,\sin(n\pi x))}.$$

In particular, for $\Lambda<\tilde\Lambda$, there holds $\|q-\tilde q\|_{L^2(\Omega)}\le c\big(\|u(T)-\tilde u(\tilde T)\|_{H^2(\Omega)}+|\Lambda-\tilde\Lambda|\big)$.

5. Numerical experiments and discussions. In this section we present numerical results to illustrate simultaneous recovery of a spatially dependent parameter and terminal time T.

5.1. Numerical algorithm. First, we describe a numerical algorithm for reconstruction. The inverse problems involve two parameters: unknown time T and space-dependent parameter v (u_0, ψ, or q). Since these two parameters have different influence on the measured data $u(T)$, standard iterative regularization methods, e.g., Landweber method and conjugate gradient method, do not work very well. We employ the Levenberg–Marquardt method [13, 19], which has been shown to be effective for solving related inverse problems [16]. Due to the ill-posedness of the inverse problems, early stopping is required in order to obtain good reconstructions.

Specifically, define a nonlinear operator $F:(v,T)\in L^2(\Omega)\times\mathbb{R}_+\to u(v)(x,T)\in L^2(\Omega)$, where $u(v)$ solves problem (1.1), with the parameter v. Let (v^0,T^0) be the initial guess of the unknowns (v,T). Now given the approximation (v^k,T^k), we find the next approximation (v^{k+1},T^{k+1}) by

$$(v^{k+1},T^{k+1}) = \arg\min J_k(v,T),$$

with the functional $J_k(v,T)$ at the kth iteration (based at (v^k,T^k)) given by

$$\begin{aligned} J_k(v,T) =& \frac{1}{2}\|F\left(v^k,T^k\right) - g^\delta + \partial_v F\left(v^k,T^k\right)\left(v-v^k\right) + \partial_T F\left(v^k,T^k\right)\left(T-T^k\right)\|_{L^2(\Omega)}^2 \\ &+ \frac{\gamma^k}{2}\|v-v^k\|_{L^2(\Omega)}^2 + \frac{\mu^k}{2}|T-T^k|^2, \end{aligned}$$

where $\gamma^k>0$ and $\mu^k>0$ are regularization parameters, and $\partial_v F(v^k,T^k)$ and $\partial_T F(v^k,T^k)$ are the derivatives of the forward map F in v and T, respectively. We employ two parameters since v and T influence the data $u(T)$ differently. The parameters γ and μ are often decreased geometrically with $\rho\in(0,1)$: $\gamma^{k+1}=\rho\gamma^k$ and $\mu^{k+1}=\rho\mu^k$. The derivative $\partial_v F(v,T)$ can be evaluate explicitly. For example, for BP, the (directional) derivative $w=\partial_v F(v,T)[h]$ (in the direction h) satisfies

$$\begin{cases} \partial_t^\alpha w - \Delta w + qw = 0 & \text{in } \Omega\times(0,T), \\ w = 0 & \text{on } \partial\Omega\times(0,T), \\ w(0) = h & \text{in } \Omega. \end{cases}$$

To approximate the derivative $\partial_T F(v,T)$, we use the finite difference $\partial_T F(v,T)\approx(\delta T)^{-1}(F(v,T+\delta T)-F(v,T))$, where δT is a small number, fixed at $\delta T = 1\times 10^{-3}$ below. Due to the quadratic structure of the functional $J_k(v,T)$, the increments $\delta v^k := v^{k+1}-v^k$ and $\delta T^k := T^{k+1}-T^k$ satisfy

$$\begin{bmatrix} J_v^*J_v+\mu^{k+1}I & J_v^*J_T \\ J_T^*J_v & J_T^*J_T+\gamma^{k+1} \end{bmatrix}\begin{bmatrix} \delta v^k \\ \delta T^k \end{bmatrix} = \begin{bmatrix} J_v^*(g^\delta - F(v^k,T^k)) \\ J_T^*(g^\delta - F(v^k,T^k)) \end{bmatrix},$$

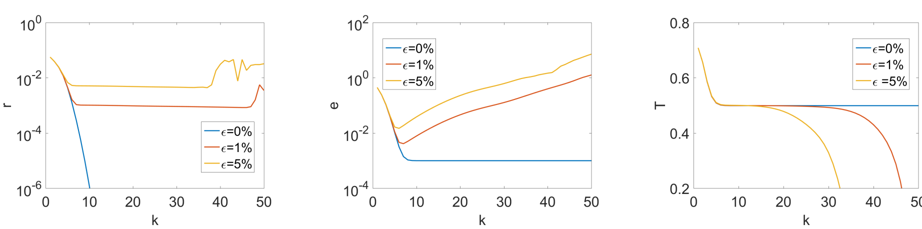
where $*$ denotes the adjoint operator, and $J_v=\partial_v F(v^k,T^k)$ and $J_T=\partial_T F(v^k,T^k)$.

5.2. Numerical illustrations. Now we present numerical results for the three problems, and with the domain $\Omega=(0,1)$ in one dimension and $\Omega=(0,1)^2$ in two dimensions, and the terminal time $T=0.5$. We discretize problem (1.1) using the Galerkin finite element method with continuous piecewise linear functions in space, and L1 approximation in time [5, 6]. The accuracy of a reconstruction $\hat{v}$ relative to the exact one $v^\dagger$ is measured by the $L^2(\Omega)$ error $e(\hat{v}) = \|\hat{v} - v^\dagger\|_{L^2(\Omega)}$. The residual $r(\hat{v})$ of the recovered tuple $(\hat{v}, \hat{T})$ is computed as $r(\hat{v}) = \|F(\hat{v}, \hat{T}) - g^\delta\|_{L^2(\Omega)}$. The exact data $g^\dagger$ is generated on a fine space-time mesh. The noisy data g^δ is generated from $g^\dagger$ by $g^\delta(x) = g^\dagger + \epsilon\|g^\dagger\|_{L^\infty(\Omega)}\xi(x)$, where $\xi(x)$ follows the standard Gaussian distribution, and $\epsilon > 0$ indicates the noise level.

The first example is BP, with $q \equiv 0$.

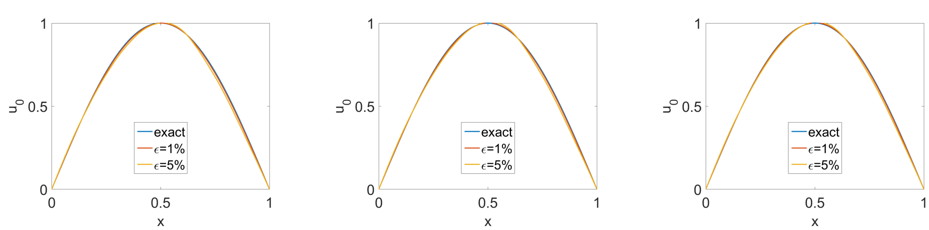
Example 5.1. (i) The source $f = \min(x, 1-x)$, and the unknown initial condition $u_0 = \sin(\pi x)$, and (ii) the diffusion coefficient $a = 1 + \sin(\pi x)y(1-y)$, the source $f = \min(x, 1-x)e^x \sin(2\pi y)$, and the unknown initial condition $u_0 = \sin(\pi x)\sin(\pi y)$.

The convergence of the Levenberg–Mardquardt method is shown in Figure 1. For the exact data, the residual r decreases rapidly to zero, and the error e also decreases steadily and eventually levels off at 1e-3 (due to the presence of discretization error). For noisy data, the method exhibits a typical semiconvergence behavior: the error e first decreases and then starts to increase rapidly afterwards, necessitating the use of early stopping. This behavior is also observed for the estimated terminal time T, but it appears to be more resilient to the iteration number k and it does not change much after a few extra iterations. Nonetheless, the estimate $\hat{T}$ will eventually drift away when the method is run for too many iterations. Exemplary reconstructions of the initial data u_0 are shown in Figures 2 and 3, which has the smallest $L^2(\Omega)$ error along the iteration trajectory; see Table 1 for the stopping index k^*. These plots show that the reconstructions are accurate for up to 5% noise in the data. See also Table 1 for quantitative results. Note that the accuracy e does not depend very much on the order α, and decreases as the noise level $\epsilon \to 0^+$. This agrees with the Lipschitz stability estimate in Theorem 2.3. However, the reconstructions for case (ii) tend to be less accurate than case (i), which is attributed to larger discretization errors in two dimensions.



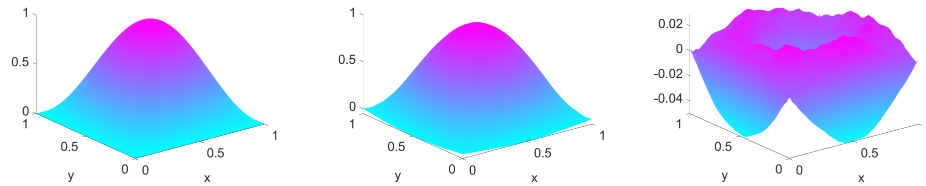
(a) residual r (b) error e (c) terminal time T

FIG. 1. *The convergence of the Levenberg–Marquadt method for Example* 5.1(i)*, for* $\alpha = 0.5$.



(a) $\alpha = 0.25$ (b) $\alpha = 0.50$ (c) $\alpha = 0.75$

FIG. 2. *The reconstructions of the initial condition* u_0 *for Example* 5.1(i).



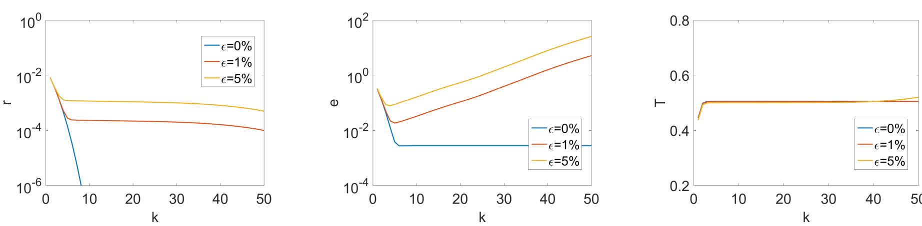
(a) exact (b) reconstruction (c) error

FIG. 3. *The reconstructions of the initial condition u_0 for Example* 5.1(ii) *with* $\alpha = 0.5$, $\epsilon = 5\%$.

TABLE 1

The numerical results (error e, stopping index k^*, and recovered $\hat{T}$) for Example 5.1. *For case* (i), $\gamma_0 =$ 1e-2, $\mu_0 = 2.7$e-3, 6.3e-3 *and* 1.3e-2 *for* $\alpha = 0.25$, 0.50 *and* 0.75, *respectively, and* $\rho = 0.80$; *for case* (ii), $\gamma_0 =$ 1e-3, $\mu_0 = 1.1$e-4, 2.7e-4, *and* 6e-4 *for* $\alpha = 0.25$, 0.50 *and* 0.75, *respectively, and* $\rho = 0.80$.

Case	α	0.25			0.50			0.75		
	ϵ	e	k^*	$\hat{T}$	e	k^*	$\hat{T}$	e	k^*	$\hat{T}$
(i)	0e-3	1.221e-3	16	0.496	9.948e-4	20	0.498	5.023e-4	31	0.500
	1e-3	1.455e-3	8	0.496	1.210e-3	8	0.499	7.217e-4	10	0.500
	5e-3	2.961e-3	7	0.496	2.468e-3	7	0.499	1.852e-3	9	0.500
	1e-2	4.793e-3	6	0.498	4.114e-3	7	0.499	3.275e-3	9	0.500
	2e-2	8.121e-3	6	0.498	6.872e-3	6	0.501	5.578e-3	8	0.502
	5e-2	1.675e-2	5	0.505	1.476e-2	6	0.502	1.230e-2	8	0.503
(ii)	0e-3	7.368e-2	32	0.496	7.414e-2	33	0.500	7.586e-2	36	0.498
	1e-3	7.574e-2	15	0.497	7.622e-2	17	0.500	7.799e-2	20	0.498
	5e-3	8.230e-2	11	0.497	8.315e-2	13	0.500	8.648e-2	17	0.498
	1e-2	9.119e-2	10	0.497	9.247e-2	11	0.500	9.737e-2	15	0.498
	3e-2	1.169e-1	6	0.497	1.186e-1	7	0.500	1.235e-1	7	0.499
	5e-2	1.281e-1	4	0.499	1.275e-1	4	0.503	1.300e-1	6	0.501



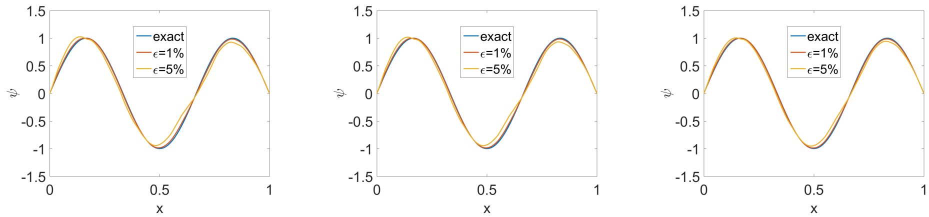
(a) residual r (b) error e (c) terminal time T

FIG. 4. *The convergence of the Levenberg–Marquadt method for Example* 5.2(i) *with* $\alpha = 0.5$.

The next example is about ISP, with $q \equiv 0$.

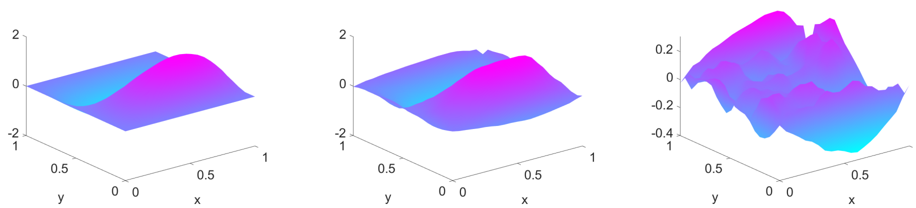
Example 5.2. (i) The initial condition $u_0 = \sin(2\pi x)$, and the unknown source $\psi(x) = \sin(3\pi x)$, and (ii) the known diffusion coefficient $a = 1 + \sin(\pi x)y(1-y)$, initial condition $u_0 = \sin(\pi x)\sin(\pi y)$, and the unknown source $\psi = 4x(1-x)e^x \sin(2\pi y)$.

The numerical results for Example 5.2 are shown in Figures 4, 5, and 6, and Table 2. The convergence plots in Figure 4 show the semiconvergence phenomenon, and with the chosen parameters, the method converges rapidly to an acceptable solution, and then the error e starts to increase shortly afterwards. Nonetheless, the estimate of T converges fairly fast (within five iterations), and it is also quite stable



(a) $\alpha = 0.25$ (b) $\alpha = 0.50$ (c) $\alpha = 0.75$

FIG. 5. *The reconstructions of the space-dependent source ψ for Example* 5.2(i).



(a) exact (b) reconstruction (c) pointwise error

FIG. 6. *The reconstructions of the space source component ψ for Example* 5.2(ii) *with* $\alpha = 0.5$, $\epsilon = 5\%$.

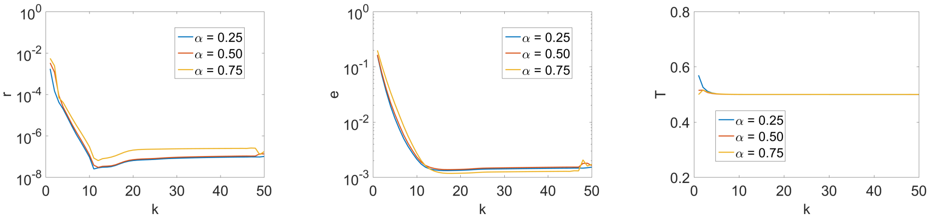
TABLE 2
The numerical results (error e, stopping index k^*, and recovered $\hat{T}$) for Example 5.2. *For case* (i), $\gamma_0 = $ 1e-4, $\mu_0 = $ 1e-8, 5e-8, *and* 1e-7 *for* $\alpha = 0.25, 0.50,$ *and* 0.75, *respectively, and* $\rho = 0.8$; *and for case* (ii), $\gamma_0 = $ 1e-4, $\mu_0 = $ 1e-9, *and* $\rho = 0.8$.

Case	α	0.25			0.50			0.75		
	ϵ	e	k^*	$\hat{T}$	e	k^*	$\hat{T}$	e	k^*	$\hat{T}$
(i)	0e-3	2.534e-3	6	0.507	2.709e-3	6	0.505	3.057e-3	6	0.506
	1e-3	3.402e-3	6	0.507	3.381e-3	6	0.504	3.417e-3	6	0.506
	5e-3	1.098e-2	5	0.506	1.000e-2	5	0.504	8.016e-3	6	0.505
	1e-2	2.035e-2	5	0.505	1.825e-2	5	0.503	1.410e-2	5	0.505
	2e-2	3.839e-2	4	0.503	3.488e-2	4	0.502	2.669e-2	5	0.504
	5e-2	8.668e-2	4	0.496	7.761e-2	4	0.499	5.944e-2	4	0.501
(ii)	0e-3	2.122e-1	35	0.514	2.121e-1	35	0.508	2.119e-1	35	0.509
	1e-3	2.173e-1	9	0.514	2.171e-1	9	0.508	2.166e-1	10	0.509
	5e-3	2.223e-1	4	0.515	2.218e-1	4	0.509	2.206e-1	5	0.509
	1e-2	2.276e-1	3	0.516	2.267e-1	3	0.509	2.250e-1	3	0.509
	3e-2	2.626e-1	3	0.522	2.576e-1	3	0.512	2.477e-1	3	0.512
	5e-2	2.843e-1	2	0.519	2.797e-1	2	0.507	2.707e-1	2	0.500

during the iteration. We obtain very accurate reconstructions for the noise level ϵ up to 5e-2. Like before, the reconstruction quality does not depend much on the order α; cf. Table 2 and Figure 5, concurring with the observations for `BP`. The latter also agrees with the fact that `ISP` enjoys similar stability as `BP`, as indicated by Theorems 2.3 and 3.3.

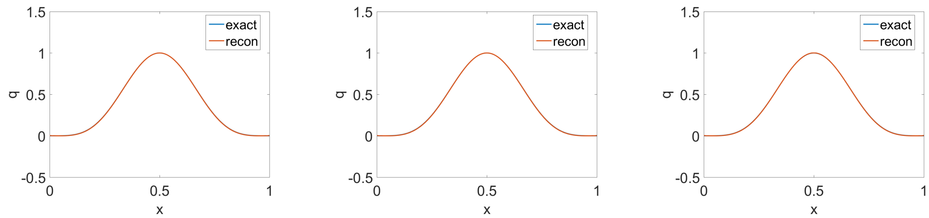
The last example is about IPP in one dimension.

Example 5.3. The source $f = |\sin(2\pi x)|$, initial condition $u_0 \equiv 1$, and a zero Dirichlet boundary condition. The unknown potential $q = \sin^4(\pi x)$.



(a) residual r (b) error e (c) terminal time T

FIG. 7. *The convergence of the Levenberg–Marquadt method for Example* 5.3 *with exact data.*



(a) $\alpha = 0.25$ (b) $\alpha = 0.50$ (c) $\alpha = 0.75$

FIG. 8. *The reconstructions of the space-dependent potential q for Example* 5.3 *with exact data.*

The parameter γ_0 is fixed at 1e-7 and μ_0 at 1e-8, and the decreasing factor ρ is set to 0.5. The numerical results are summarized in Figures 7 and 8. Note that we can obtain highly accurate reconstructions for exact data, with the $L^2(\Omega)$ error of the recovered potential $\hat{q}$ being 1.318e-3, 1.374e-3, and 1.174e-3 for $\alpha = 0.25$, 0.50, and 0.75, respectively. The estimated terminal time $T = 0.4996$, 0.4997, and 0.4998 for $\alpha = 0.25$, 0.50, and 0.75, respectively, are also fairly accurate. This clearly shows the feasibility of simultaneous recovery. However, for noisy data, the recovery is very challenging. Numerically, we observe that the singular value spectrum of the linearized forward operator has many tiny values (and hence we have to use a tiny value for the parameter γ_0), which precludes applying any realistic amount of noise to the data and renders the recovery from noisy data highly unstable.

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