Swan modules over Laurent polynomials

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Abstract

Let $\Omega = P_{n,m}(\mathbb{Z}[C_p])$ and $A = P_{n,m}(\mathbb{Z})$ where p is a positive integer and $P_{n,m}(R)$ is the R-algebra $P_{n,m}(R) = R[t_1, t_1^{-1}, \ldots, t_n, t_n^{-1}] \otimes_R R[x_1, \ldots, x_m]$. A *Swan module* is an extension module of the form $0 \to I^{(k)} \to X \to A^{(k)} \to 0$ where I is the kernel of the augmentation homomorphism $\epsilon : \Omega \to A$. We show that, when p is prime, every such projective Swan module is free; this is false if p is not prime and $n + m > 0$. The proof relies on the fact that when R is the ring of algebraic integers in $\mathbb{Q}(\zeta_p)$ and $\mathbb{F}_p$ is the field with p elements then the canonical homomorphism $GL_k(P_{n,m}(R)) \twoheadrightarrow GL_k(P_{n,m}(\mathbb{F}_p))$ is surjective for all $k \geq 1$.

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Let Ω be an algebra over a commutative ring A, augmented by an A-algebra homomorphism $\epsilon : \Omega \to A$. By a *Swan module of rank k* we shall mean an extension module X of the form $0 \to I^{(k)} \to X \to A^{(k)} \to 0$ where $I = \mathrm{Ker}(\epsilon)$. The *augmentation exact sequence* $0 \to I \hookrightarrow \Omega \stackrel{\epsilon}{\to} A \to 0$ then shows that $\Omega^{(k)}$ is a Swan module of rank k. Given another such Swan module X' we write $X \cong_{\mathrm{Id}} X'$ when there exists a commutative diagram of Ω homomorphisms.

$$\begin{array}{ccccccccc} 0 & \to & I^{(k)} & \stackrel{i}{\to} & X & \stackrel{p}{\to} & A^{(k)} & \to & 0 \\ & & \downarrow \alpha & & \downarrow \widehat{\alpha} & & \downarrow \mathrm{Id} & & \\ 0 & \to & I^{(k)} & \stackrel{i'}{\to} & X' & \stackrel{p'}{\to} & A^{(k)} & \to & 0 \end{array}$$

in which α, and hence also $\widehat{\alpha}$, is an isomorphism.

If $\mathbb{A}$ is a commutative ring and n, m are non-negative integers we denote by $P_{n,m}(\mathbb{A})$ the $\mathbb{A}$-algebra $P_{n,m}(\mathbb{A}) = \mathbb{A}[t_1, t_1^{-1}, \ldots, t_n, t_n^{-1}] \otimes_{\mathbb{A}} \mathbb{A}[x_1, \ldots, x_m]$ where $\mathbb{A}[t_1, t_1^{-1}, \ldots, t_n, t_n^{-1}]$ is the algebra of Laurent polynomials in n commuting variables and $\mathbb{A}[x_1, \ldots, x_m]$ is the algebra of ordinary polynomials in m commuting variables. We allow the degenerate cases $n = 0$, $m = 0$ provided that $n + m > 0$. Now take $\Omega = P_{n,m}(\mathbb{Z}[C_p])$ where $C_p = \langle x \,|\, x^p = 1 \rangle$ is the finite cyclic group of order $p \geq 2$ and $A = P_{n,m}(\mathbb{Z})$. With the augmentation $\epsilon : \Omega \to A$ defined by the correspondence $x \mapsto 1$ we shall then prove:

(I) Let X be a projective Swan module of rank $k \geq 1$; if p is prime then $X \cong_{\mathrm{Id}} \Omega^{(k)}$.

We point out that, by the theorem of Montgomery and Uchida (cf. [14], Chapter 11), for each prime $p \geq 23$ there are projective modules induced from $\mathbb{Z}[C_p]$ which are not Swan modules and which are not free.

Our primary interest is in the cases $n \geq 1$ and $m = 0$ when we may identify Ω and A with the integral group rings $\Omega = \mathbb{Z}[T_n \times C_p]$, $A = \mathbb{Z}[T_n]$. Statement **(I)** is then best possible and may be compared with the original theorem of Swan ([10], Corollary 6.1) which corresponds to the (degenerate) cases $n = m = 0$, $k = 1$ of the present paper. In that case projective Swan modules are free for all finite cyclic groups whereas the statement of **(I)** is already false for $k = 1$ whenever p fails to be prime, as can be seen from a result of Bass and Murthy ([1], Theorem 8.10).

To prove **(I)** we consider a statement of independent interest. We denote by $E_k(\mathbb{A})$ the subgroup of $GL_k(\mathbb{A})$ generated by elementary matrices. If the ring homomorphism $f : \mathbb{B} \to \mathbb{A}$ is surjective the induced homomorphism $f_* : E_k(\mathbb{B}) \to E_k(\mathbb{A})$ is also surjective. It is natural to ask under what conditions the homomorphism $f_* : GL_k(\mathbb{B}) \to GL_k(\mathbb{A})$ is also surjective. In this connection, it has long been apparent that particular difficulties surround the study of GL_2 and its subgroups (cf [3], [12]). As an example, consider the groups

$$SL_k^{\pm}(\mathbb{F}[x_1, \ldots, x_m]) = \{X \in GL_k(\mathbb{F}[x_1, \ldots, x_m]) \mid \det(X) = \pm 1 \}$$

where $\mathbb{F}$ is a field. When $m = 1$, $SL_k^{\pm}(\mathbb{F}[x_1]) = E_k(\mathbb{F}[x_1])$. When $m \geq 2$, the following matrix (cf [3]) shows the corresponding statement is false for $k = 2$;

$$\begin{pmatrix} 1 + x_1 x_2 & x_2^2 \\ -x_1^2 & 1 - x_1 x_2 \end{pmatrix}.$$

However a remarkable theorem of Suslin [9] shows that if $k \geq 3$ then for all n, m,

$$SL_k^{\pm}(P_{n,m}(\mathbb{F})) = E_k(P_{n,m}(\mathbb{F})).$$

Let p be an odd prime, let R denote the ring of algebraic integers in the field $\mathbb{Q}(\zeta)$ where $\zeta = \exp(\frac{2\pi i}{p})$, let R_0 denote its real subring and let $\mathbb{F}_p$ be the field with p elements. If p is odd then $R_0 = \mathbb{Z}(\mu)$ where $\mu = \zeta + \zeta^{-1}$. The correspondence $\zeta \mapsto 1$ induces a surjective ring homomorphism $\nu : \mathbb{Z}(\zeta_p) \to \mathbb{F}_p$ under which μ maps to 2. As 2 generates the additive group of $\mathbb{F}_p$, ν restricts to a surjective ring homomorphism $\nu : R_0 \twoheadrightarrow \mathbb{F}_p$ and induces a surjective ring homomorphism $\nu : P_{n,m}(R_0) \twoheadrightarrow P_{n,m}(\mathbb{F}_p)$. If $p = 2$ then $R_0 = R = \mathbb{Z}$ and we still have a surjective ring homomorphism $\nu : P_{n,m}(R_0) \twoheadrightarrow P_{n,m}(\mathbb{F}_2)$. We shall prove

(II) The induced homomorphism $\nu_* : GL_k(P_{n,m}(R_0)) \twoheadrightarrow GL_k(P_{n,m}(\mathbb{F}_p))$ is surjective for all $k \geq 1$ and all n, m.

When $k \neq 2$, the statement **(II)** follows from the results of Higman [4] and Suslin [9]. In §6 we prove that $\nu_* : GL_2(P_{n,m}(R_0)) \twoheadrightarrow GL_2(P_{n,m}(\mathbb{F}_p))$ is also surjective for all n, m. As an immediate corollary we have:

(III) $\nu_* : GL_k(P_{n,m}(R)) \twoheadrightarrow GL_k(P_{n,m}(\mathbb{F}_p))$ is surjective for all $k \geq 1$ and all n, m.

As we shall see in §7, statement **(I)** above follows directly from **(III)**.

§1: The derived module category:

Let $\Omega = A[\Phi]$ be the group algebra of a finite group Φ over a commutative Noetherian integral domain A. We denote by $\mathcal{M}\mathrm{od}_\Omega$, $\mathcal{M}\mathrm{od}_A$ the categories of right Ω-modules and right A-modules respectively. The natural inclusion $i : A \to \Omega$ induces an *extension of scalars* functor

$$i_* : \mathcal{M}\mathrm{od}_A \to \mathcal{M}\mathrm{od}_\Omega \; ; \; i_*(M) = M \otimes_A \Omega.$$

There is a corresponding *restriction of scalars* functor $i^* : \mathcal{M}\mathrm{od}_\Omega \to \mathcal{M}\mathrm{od}_A$. By a *lattice* we shall mean a Ω-module M for which the A-module $i^*(M)$ is finitely generated and projective. For such lattices we have the following adjointness relations ([5], Chapters 4 and 5). The first of these is universal, namely:

$$\mathrm{Ext}^k_\Omega(i_*(N), M) \cong \mathrm{Ext}^k_A(N, i^*(M)).$$

As Φ is finite, we also have the Eckmann-Shapiro isomorphisms

$$\begin{cases} \mathrm{Hom}_A(i^*(M), N) \cong \mathrm{Hom}_\Omega(M, i_*(N)) \\[2ex] \mathrm{Ext}^k_A(i^*(M), N) \cong \mathrm{Ext}^k_\Omega(M, i_*(N)) \end{cases}$$

Observe that $i_*(A) = \Omega$. When M is a Ω-lattice then $\mathrm{Ext}^k_\Omega(M, \Omega) \cong \mathrm{Ext}^k_A(i^*(M), A)$. However, as $i^*(M)$ is projective over R then $\mathrm{Ext}^k_A(i^*(M), A) = 0$; hence:

(1.1) If M is a Ω-lattice then $\mathrm{Ext}^k_\Omega(M, \Omega) = 0$ for all $k \geq 1$.

If X is a Ω-lattice X we denote by $X^\bullet = \mathrm{Hom}_\Omega(X, \Omega)$ the conjugate dual of X on which Φ acts via $(\alpha \cdot g)(x) = \alpha(x \cdot g^{-1})$. Then $X^\bullet$ is also a Ω-lattice. We note that Ω lattices are reflexive; that is: $X^{\bullet\bullet} \cong X$.

If $f, g : M \to N$ are morphisms in $\mathcal{M}\mathrm{od}_\Omega$ we write '$f \approx g$' when $f - g$ can be written as a composition $f - g = \xi \circ \eta$ via a projective module P thus:

$$\begin{array}{ccc} M & \xrightarrow{\quad f-g \quad} & N \\ & {\scriptstyle\eta}\searrow \quad \nearrow{\scriptstyle\xi} & \\ & P & \end{array}$$

Then '$\approx$' is an equivalence relation compatible with composition; that is given Ω-homomorphisms $f, f' : M_0 \to M_1$, $g, g' : M_1 \to M_2$ we see that:

$$f \approx f' \text{ and } g \approx g' \implies g \circ f \approx g' \circ f'.$$

The *derived module category* $\mathcal{D}\mathrm{er}(\Omega)$ is quotient category of $\mathcal{M}\mathrm{od}_\Omega$ given by

$$\begin{cases} \mathrm{Hom}_{\mathcal{D}\mathrm{er}(\Omega)}(M, N) = \mathrm{Hom}_\Omega(M, N)/\langle M, N \rangle \\[2ex] \langle M, N \rangle = \{ f \in \mathrm{Hom}_\Omega(M, N) : f \approx 0 \} \end{cases}$$

As $\langle M, N \rangle$ is an A submodule of $\mathrm{Hom}_\Omega(M, N)$ then $\mathrm{Hom}_{\mathcal{D}\mathrm{er}}(M, N)$ has the natural structure of an A-module. We distinguish notationally between isomorphism in

$\mathcal{M}\text{od}_\Omega$, written as '$\cong_\Omega$' and isomorphism in $\mathcal{D}\text{er}(\Omega)$, written as '$\cong_{\mathcal{D}\text{er}}$'. For finitely generated Ω-modules the relationship between the two notions is ([5] p. 120) :

$$D \cong_{\mathcal{D}\text{er}} D' \iff D \oplus P' \cong_\Omega D' \oplus P$$

for some finitely generated projective Ω-modules P, P'. Given a Ω-lattice M we consider exact sequences in $\mathcal{M}\text{od}_\Omega$ thus

$$0 \to D \xrightarrow{i} P \xrightarrow{p} M \to 0$$

where P is finitely generated projective. As Ω is Noetherian then D is also finitely generated. Given another such exact sequence $0 \to D' \xrightarrow{i'} P' \xrightarrow{p'} M \to 0$ it follows from Schanuel's Lemma that $D \oplus P' \cong_\Omega D' \oplus P$ so that $D \cong_{\mathcal{D}\text{er}} D'$. We denote by $D_1(M)$ the isomorphism class in $\mathcal{D}\text{er}(\Omega)$ of any module D which occurs in an exact sequence of the above form. We may regard $D_1(M)$ as the *first derivative* of M. The correspondence $M \mapsto D_1(M)$ can be regarded as a functor as follows: given any such exact sequence and a Ω-homomorphism $f : M \to M$ the universal property of projective modules allows us to construct a commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & D & \xrightarrow{i} & P & \xrightarrow{p} & M & \to & 0 \\ & & \downarrow f_- & & \downarrow f_0 & & \downarrow f & & \\ 0 & \to & D & \xrightarrow{i} & P & \xrightarrow{p} & M & \to & 0. \end{array}$$

Moreover if $f' : M \to M$ is another morphism in $\mathcal{M}\text{od}_\Omega$ then :

(1.2) $$f \approx f' \implies f_- \approx f'_-.$$

Given $f \in \text{End}_\Omega(M)$ we denote by $\rho(f) = [f_-]$ the class of f_- in $\mathcal{D}\text{er}(\Omega)$. By (1.2), the correspondence $[f] \mapsto \rho(f) = [f_-]$ determines a ring hommoorphism $\rho : \text{End}_{\mathcal{D}\text{er}}(M) \xrightarrow{\simeq} \text{End}_{\mathcal{D}\text{er}}(D_1(M))$. When $\text{Ext}^1(M, \Omega) = 0$ then (cf [5], p.133) the conclusion of (1.2) is strengthed to:

(1.3) $$f \approx f' \iff f_- \approx f'_-.$$

With our underlying assumption that M is a Ω-lattice then

(1.4) $\rho : \text{End}_{\mathcal{D}\text{er}}(M) \xrightarrow{\simeq} \text{End}_{\mathcal{D}\text{er}}(D_1(M))$ is a ring isomorphism.

Given an exact sequence $\mathcal{E} = (0 \to D \xrightarrow{i} P \xrightarrow{p} M \to 0)$ and a Ω-homomorphism $\alpha : D \to N$ we construct the pushout diagram

(1.5) $$\begin{array}{c} \mathcal{E} \\ \downarrow \natural \\ \alpha_*(\mathcal{E}) \end{array} = \left(\begin{array}{ccccccccc} 0 & \to & D & \xrightarrow{i} & P & \xrightarrow{p} & M & \to & 0 \\ & & \downarrow \alpha & & \downarrow \natural & & \downarrow \text{Id} & & \\ 0 & \to & N & \xrightarrow{i} & \varinjlim(\alpha, i) & \xrightarrow{\pi} & M & \to & 0 \end{array} \right).$$

Then $\alpha_*(\mathcal{E}) = (0 \to N \xrightarrow{i} \lim_{\to}(\alpha, i) \xrightarrow{\pi} M \to 0)$ defines an extension class in $\mathrm{Ext}^1_\Omega(M, N)$. When P is projective the correspondence $\alpha \mapsto [\alpha_*(\mathcal{E})]$ defines a mapping $\delta : \mathrm{Hom}_{\mathcal{D}\mathrm{er}}(D, N) \to \mathrm{Ext}^1_\Omega(M, N)$. It follows from (1.1) that :

(1.6) $\quad \delta : \mathrm{Hom}_{\mathcal{D}\mathrm{er}}(D_1(M), N) \xrightarrow{\simeq} \mathrm{Ext}^1_\Omega(M, N)$ is an isomorphism.

The isomorphism of (1.6) is a *corepresentation formula* (cf [5], pp.109-114). Taking $N = D \in D_1(M)$ in (1.5), the resulting pushout extension has the following property which generalizes the recognition criterion for projective modules originally given by Swan ([10], Remark on p.279).

(1.7) $$\lim_{\to}(\alpha, i) \text{ is projective } \iff \alpha \in \mathrm{Aut}_{\mathcal{D}\mathrm{er}(\Omega)}(\mathrm{D}).$$

(For a proof of (1.7) see [5], Theorem 5.41, p.115).

Given the exact sequence $\mathcal{E}$ for any Ω-module N we have exact sequences for $k \geq 1$.

$$\mathrm{Ext}^k_\Omega(P, N) \xrightarrow{i^*} \mathrm{Ext}^k_\Omega(D_1(M), N) \xrightarrow{\delta} \mathrm{Ext}^{k+1}_\Omega(M, N) \xrightarrow{p^*} \mathrm{Ext}^{k+1}_\Omega(P, N).$$

As P is projective then $\mathrm{Ext}^k_\Omega(P, N) \cong \mathrm{Ext}^{k+1}_\Omega(P, N) = 0$ and we obtain the usual *dimension shifting* isomorphisms

(1.8) $$\mathrm{Ext}^{k+1}_\Omega(M, N) \cong \mathrm{Ext}^k_\Omega(D_1(M), N).$$

We may regard the corepresentation formula (1.6) as the degenerate case of (1.8) corresponding to the case $k = 0$. In particular, taking $N = D_1(M)$ in (1.6) we obtain a natural isomorphism $\delta : \mathrm{End}_{\mathcal{D}\mathrm{er}}(D_1(M)) \xrightarrow{\simeq} \mathrm{Ext}^1_\Omega(M, D_1(M))$ which, combined with (1.4) gives

(1.9) $\quad \natural = \rho^{-1} \circ \delta^{-1} : \mathrm{Ext}^1_\Omega(M, D_1(M)) \longrightarrow \mathrm{End}_{\mathcal{D}\mathrm{er}}(M)$ is an isomorphism.

We conclude this section by computing $\mathrm{End}_{\mathcal{D}\mathrm{er}}(A)$. As projective modules are direct summands of free modules it is enough to consider homomorphisms $f : A \to A$ which factor through $\Omega^{(n)}$. Let $\epsilon : \Omega \to A$ be the augmentation homomorphism, $\epsilon(x^r) = 1$. We note that $\mathrm{Hom}_\Omega(\Omega, A) \cong A$ generated by the augmentation homomorphism ϵ. Thus if $\xi : \Omega^{(n)} \to A$ is Ω linear then $\xi = (\xi_1\epsilon, \cdots, \xi_n\epsilon)$ for some $(\xi_1, \cdots, \xi_n) \in A^{(n)}$. Let $\epsilon^\bullet : A \to \Omega$ denote the Ω-dual of ϵ; then $\epsilon^\bullet(1) = \sum_{g \in \Phi} g$.

Then $\mathrm{Hom}_\Omega(A, \Omega) \cong A$ generated by $\epsilon^\bullet$. Hence if $\eta : A \to \Omega^{(n)}$ is Ω linear then $\eta = (\eta_1, \cdots, \eta_n)^t\epsilon^\bullet$ for some $(\eta_1, \cdots, \eta_n) \in A^{(n)}$. If $f : A \to A$ admits a factorization $f = \xi \circ \eta$ through the free module $\Omega^{(n)}$ thus $f(1) = (\sum_{r=1}^n \xi_r\eta_r)\epsilon \circ \epsilon^\bullet(1)$. However $\epsilon \circ \epsilon^\bullet(1) = |\Phi|$ so that

(1.10) $$\mathrm{End}_{\mathcal{D}\mathrm{er}}(A) \cong A/|\Phi|.$$

From (1.10) and (1.4) it follows that

(1.11) $$\mathrm{End}_{\mathcal{D}\mathrm{er}}(I^\bullet) \cong A/|\Phi|.$$

§2: Quasi-augmentations and generalized Swan modules:

Continuing with the notation of §1, the Eckmann-Shapiro Lemma implies that:

(2.1) If X is a Ω-lattice then $\mathrm{Ext}^k_\Omega(X,\Omega) \;=\; 0$ for all $k \geq 1$.

We have an augmentation homomorphism $\epsilon : \Omega \to A$ given by $\epsilon(g) = 1$ for all $g \in \Phi$. We denote by $\mathcal{E}$ the augmentation exact sequence

$$\mathcal{E} \;=\; (0 \to I \overset{i}{\hookrightarrow} \Omega \overset{\epsilon}{\to} A \to 0)$$

where $I \;=\; \mathrm{Ker}(\epsilon)$ and i is the inclusion. We generalize this as follows; an exact sequence $\mathcal{S} = (0 \to S_- \overset{i}{\to} \Omega \overset{p}{\to} S_+ \to 0)$ of Ω-lattices is called a *quasi-augmentation sequence* when S_+ S_- satisfy the condition $\mathrm{Hom}_\Omega(S_-, S_+) = 0$. In the above sequence $\mathcal{E}$ one sees easily that $\mathrm{Hom}_\Omega(A, I) \;=\; 0$; hence:

(2.2) $\mathcal{E}$ is a quasi-augmentation sequence.

Now fix a quasi-augmentation $\mathcal{S} = (0 \to S_- \overset{i}{\to} \Omega \overset{p}{\to} S_+ \to 0)$. If k is a positive integer we denote by $\mathcal{S}(k)$ the class of extensions of the form

$$\mathcal{X} \;=\; (0 \to S_-^{(k)} \xrightarrow{i_\mathcal{X}} X \xrightarrow{p_\mathcal{X}} S_+^{(k)} \to 0)$$

The module X defined by such an extension is called a *generalized Swan module* of rank k. There are a number of equivalence relations on $\mathcal{S}(k)$ to be considered:

Isomorphism : We write $\mathcal{X} \cong \mathcal{Y}$ when there is a commutative diagram

$$\begin{array}{ccccccc} 0 \to & S_-^{(n)} & \to & X & \to & S_+^{(n)} & \to 0 \\ & \downarrow f_- & & \downarrow f_0 & & \downarrow f_+ & \\ 0 \to & S_-^{(n)} & \to & Y & \to & S_+^{(n)} & \to 0 \end{array}$$

in $\mathcal{M}\mathrm{od}_\Omega$ in which f_- and f_+ are isomorphisms. We note that generalized Swan modules are rigid in the sense that the condition $\mathrm{Hom}_\Omega(S_-, S_+) = 0$ ensures that the defining exact sequence $\mathcal{X}$ is essentially unique (cf [5]. p.231); thus if Y is also a generalized Swan module defined by the exact sequence $\mathcal{Y}$ then

$$X \;\cong\; Y \iff \mathcal{X} \;\cong\; \mathcal{Y}.$$

There is a more refined relation on $\mathcal{S}(k)$, namely:

Congruence : We write $\mathcal{X} \equiv \mathcal{Y}$ when there is a commutative diagram of Ω-homomorphisms

$$\begin{array}{ccccccc} 0 \to & S_-^{(k)} & \to & X & \to & S_+^{(k)} & \to 0 \\ & \downarrow \mathrm{Id} & & \downarrow \nu & & \downarrow \mathrm{Id} & \\ 0 \to & S_-^{(k)} & \to & Y & \to & S_+^{(k)} & \to 0. \end{array}$$

In both relations the middle mapping $X \to Y$ is an isomorphism via the Five Lemma. Up to congruence such exact sequences are classified by $\mathrm{Ext}^1(S_+^{(k)}, S_-^{(k)})$. To allow for the coarser classification of isomorphism we consider the natural two-sided action of $\mathrm{Aut}_\Omega(S_-^{(k)}) \times \mathrm{Aut}_\Omega(S_+^{(k)})$ on $\mathrm{Ext}^1(S_+^{(k)}, S_-^{(k)})$:

$$\begin{array}{ccc} \mathrm{Aut}_\Omega(S_-^{(k)}) \times \mathrm{Ext}^1(S_+^{(k)}, S_-^{(k)}) \times \mathrm{Aut}_\Omega(S_+^{(k)}) & \to & \mathrm{Ext}^1(S_+^{(k)}, S_-^{(k)}) \\ (\alpha, \mathcal{X}, \beta) & \mapsto & \alpha_* \beta^*(\mathcal{X}) \end{array}$$

As is well known (cf [6] p.67) $\alpha_* \beta^*(\mathcal{X})$ is congruent to $\beta^* \alpha_*(\mathcal{X})$. We have the following Classification Theorem:

Theorem 2.3 : There is a $1-1$ correspondence

$$\left\{ \begin{array}{c} \text{Isomorphism classes of generalized Swan} \\ \text{modules of rank } k \text{ relative to } \mathcal{S} \end{array} \right\} \longleftrightarrow \mathrm{Aut}_\Omega(S_-^{(k)}) \backslash \mathrm{Ext}^1(S_+^{(k)}, S_-^{(k)}) / \mathrm{Aut}_\Omega(S_+^{(k)}).$$

Intermediate between isomorphism and congruence we have:

Isomorphism over $S_-^{(k)}$**:** We write $\mathcal{X} \ {}_{\mathrm{Id}}\!\cong \ \mathcal{Y}$ when there is a commutative diagram of Ω-homomorphisms

$$\begin{array}{ccccccccc} 0 \to & S_-^{(k)} & \to & X & \to & S_+^{(k)} & \to 0 \\ & \downarrow \mathrm{Id} & & \downarrow f_0 & & \downarrow f_+ & \\ 0 \to & S_-^{(k)} & \to & Y & \to & S_+^{(k)} & \to 0 \end{array}$$

in which f_+ is an isomorphism. We note that

(2.4) The equivalence classes under ' ${}_{\mathrm{Id}}\!\cong$ ' are in 1 - 1 correspondence with $\mathrm{Ext}^1(S_+^{(k)}, S_-^{(k)}) / \mathrm{Aut}(S_+^{(k)})$

Isomorphism over $S_+^{(k)}$**:** We write $\mathcal{X} \cong_{\mathrm{Id}} \mathcal{Y}$ when there is a commutative diagram of Ω-homomorphisms

$$\begin{array}{ccccccccc} 0 \to & S_-^{(k)} & \to & X & \to & S_+^{(k)} & \to 0 \\ & \downarrow f_- & & \downarrow f_0 & & \downarrow \mathrm{Id} & \\ 0 \to & S_-^{(k)} & \to & Y & \to & S_+^{(k)} & \to 0 \end{array}$$

in which f_- is an isomorphism. Likewise

(2.5) The equivalence classes under ' $\cong_{\mathrm{Id}}$ ' are in 1 - 1 correspondence with $\mathrm{Aut}(S_-^{(k)}) \backslash \mathrm{Ext}^1(S_+^{(k)}, S_-^{(k)})$

On dualising the augmentation sequence $\mathcal{E}$ we obtain the exact sequence

$$0 \to A^{\bullet} \overset{\epsilon^{\bullet}}{\hookrightarrow} \Omega^{\bullet} \overset{i^{\bullet}}{\to} I^{\bullet} \to \mathrm{Ext}^1_{\Omega}(A, \Omega).$$

As $\mathrm{Ext}^1_{\Omega}(A, \Omega) = 0$ then the exact sequence $\mathcal{E}^{\bullet} = (0 \to A^{\bullet} \overset{\epsilon^{\bullet}}{\hookrightarrow} \Omega^{\bullet} \overset{i^{\bullet}}{\to} I^{\bullet} \to 0)$ is exact. It is straightforward to see that $A^{\bullet} \cong A$ and $\Omega^{\bullet} \cong \Omega$ so we may write

(2.6) $$\mathcal{E}^{\bullet} = (0 \to A \overset{\epsilon^{\bullet}}{\hookrightarrow} \Omega \overset{i^{\bullet}}{\to} I^{\bullet} \to 0)$$

where $\epsilon^{\bullet} : A \to \Omega$ is given by $\epsilon^{\bullet}(1) = \sum_{g \in \Phi} g$. Again $\mathrm{Ext}^1_{\Omega}(I^{\bullet}, \Omega) = 0$ and by duality $\mathrm{Hom}_{\Omega}(I^{\bullet}, A) \cong \mathrm{Hom}_{\Omega}(A, I) = 0$. Hence we see also that:

(2.7) $\mathcal{E}^{\bullet}$ is a quasi-augmentation sequence.

Whilst it is possible to deal directly with $\mathcal{E}$ the fact that $I^{\bullet}$ has a natural ring structure usually makes it easier to work with the dual sequence $\mathcal{E}^{\bullet}$ as in the dual augmentation sequence $0 \to A \overset{\epsilon^{\bullet}}{\to} \Omega \overset{i^{\bullet}}{\to} I^{\bullet} \to 0$ we have $I^{\bullet} \cong \Omega/(\Sigma_x)$. Hence $I^{\bullet}$ has a natural ring structure for which $i^{\bullet} : \Omega \to I^{\bullet}$ is a ring homomorphism.

§3: Surjectivity when k = 1:

For any commutative ring B there is an obvious inclusion $B \subset P_{0,m}(B)$ which in turn induces an inclusion of unit groups $B^* \subset P_{0,m}(B)^*$. It is straightforward to see that:

(3.1) If B is a commutative integral domain then $P_{0,m}(B)^* = B^*$.

However, $P_{n,m}(A) \cong P_{0,m}(P_{n,0}(A))$. If A is an integral domain so is $P_{n,0}(A)$; hence :

(3.2) If A is a commutative integral domain then $P_{n,m}(A)^* = P_{n,0}(A)^*$.

Let $T = \underbrace{C_\infty \times \ldots \times C_\infty}_{n}$ denote the n-fold product of the infinite cyclic group C_∞ and let $A[T]$ denote its group algebra over the commutative ring A. By taking $t_1, \ldots, t_n$ to be the canonical generators of T we identify $P_{n,0}(A) = A[T]$. If $\alpha = (a_1, \ldots, a_n) \in \mathbb{Z}^{(n)}$ we define $\mathbf{t}^{\alpha} = t_1^{a_1} \ldots t_n^{a_n} \in T$. It follows from a theorem of Higman [4] that :

(3.3) If A is an integral domain then $A[T]^* = \{u \cdot \mathbf{t}^{\alpha} \mid u \in A^*, \alpha \in \mathbb{Z}^{(n)}\}$.

Combining (3.2) and (3.3) with the identification $P_{n,0}(A) = A[T]$ we see that:

(3.4) If A is an integral domain then $P_{n,m}(A)^* = \{u \cdot \mathbf{t}^{\alpha} \mid u \in A^*, \alpha \in \mathbb{Z}^{(n)}\}$.

As is well known (cf. [2], p.87), the induced map on units $\nu : R^* \twoheadrightarrow \mathbb{F}_p$ is surjective. When $p = 2$ then $R_0 = R$. When p is odd, then (cf. [14], Lemma 8.1, p.144), $R^* \cong R_0^* \times \langle \zeta \rangle$. As $\nu(\zeta) = 1$ then in either case:

(3.5) The induced homomorphism of unit groups $\nu : R_0^* \to \mathbb{F}_p^*$ is surjective.

Now let $\omega \in P_{n,m}(\mathbb{F}_p)^*$. As $\mathbb{F}_p$ is an integral domain we may write $\omega = u \cdot \mathbf{t}^\alpha$ where $u \in \mathbb{F}_p^*$ and $\alpha \in \mathbb{Z}^{(n)}$. By (3.5), choose $\widehat{u} \in R_0^*$ such that $\nu(\widehat{u}) = u$. Then $\widehat{u} \cdot \mathbf{t}^\alpha \in P_{n,m}(R_0)^*$ and $\nu(\widehat{u} \cdot \mathbf{t}^\alpha) = u \cdot \mathbf{t}^\alpha = \omega$. Hence ν induces a surjection of unit groups $\nu : P_{n,m}(R_0)^* \to P_{n,m}(\mathbb{F}_p)^*$. Otherwise expressed:

(3.6) $\qquad \nu_* : GL_1(P_{n,m}(R_0)) \to GL_1(P_{n,m}(\mathbb{F}_p))$ is surjective.

§4 : Surjectivity when $k \geq 3$:

In general, for any commutative ring $\mathbb{A}$, $GL_k(\mathbb{A})$ is a semidirect product

(4.1) $\qquad GL_k(\mathbb{A}) = SL_k(\mathbb{A}) \rtimes \mathbb{A}^*$

where $\mathbb{A}^*$ is imbedded in $GL_k(\mathbb{A})$ via the diagonal matrices

$$u \mapsto \begin{pmatrix} u & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}$$

and $SL_k(\mathbb{A}) = \{X \in GL_k(\mathbb{A}) \mid \det(X) = 1\}$. Let $\epsilon(i,j) \in M_k(\mathbb{A})$ denote the basic matrix $\epsilon(i,j)_{r,s} = \delta_{i,r}\delta_{j,s}$ We denote by $E_k(\mathbb{A})$ (cf. [7]) the subgroup of $GL_k(\mathbb{A})$ generated by the elementary transvections $E(i,j;\lambda) = I_k + \lambda\epsilon(i,j)$ where $i \neq j$ and $\lambda \in \mathbb{A}$, together with the diagonal matrices $\Delta(i,-1) = I_k - 2\epsilon(i,i)$ If (i,j) denotes the transposition which swaps i and j then the correponding permutation matrix can be expressed as $P(i,j) = \Delta(j,-1)E(i,j;1)E(j,i;-1)E(i,j;1)$. It follows that $E_k(\mathbb{A})$ also contains the group of $k \times k$ permutation matrices. Moreover

(4.2) $\qquad E_k(\mathbb{A}) \subset SL_k(\mathbb{A}) \rtimes \{\pm 1\}$.

A theorem Suslin [9] shows that:

(4.3) For any field $\mathbb{F}$, $E_k(P_{n,m}(\mathbb{F})) = SL_k(P_{n,m}(\mathbb{F})) \rtimes \{\pm 1\}$ when $k \geq 3$.

If $\psi : \mathbb{B} \to \mathbb{A}$ is a surjective ring homomorphism then the induced homomorphism $\psi : E_k(\mathbb{B}) \to E_k(\mathbb{A})$ is surjective for all $k \geq 2$. As $\nu : P_{n,m}(R_0)) \to P_{n,m}(\mathbb{F}_p)$ is surjective and $\mathbb{F}_p$ is a field then, by (4.3):

(4.4) $\qquad \nu : E_k(P_{n,m}(R_0)) \to SL_k(P_{n,m}(\mathbb{F}_p)) \rtimes \{\pm 1\}$ is surjective for $k \geq 3$.

It now follows from (3.6) and (4.1) that:

(4.5) ; $\qquad \nu_* : GL_k(P_{n,m}(R_0)) \to GL_k(P_{n,m}(\mathbb{F}_p))$ is surjective for $k \geq 3$.

§5 : The rings Ω and Ω_0:

Suppose given a fibre square of ring homomorphisms

$$\mathcal{F} = \left\{ \begin{array}{ccc} \Lambda & \xrightarrow{\pi_-} & \Lambda_- \\ \downarrow \pi_+ & & \downarrow \varphi_- \\ \Lambda_+ & \xrightarrow{\varphi_+} & \Lambda_0 . \end{array} \right.$$

which satisfies Milnor's condition [8] that at least one of φ_- , φ_+ is surjective, and let $\alpha \in GL_k(\Lambda_0)$. We denote by $\mathcal{L}(\alpha)$ the Λ-module X obtained as a fibre product

$$\begin{array}{ccc} \mathcal{L}(\alpha) & \longrightarrow & \Lambda_-^{(k)} \\ \downarrow & & \downarrow \\ \Lambda_+^{(k)} & \longrightarrow & \Lambda_0^{(k)}. \end{array}$$

by glueing $\Lambda_+^{(k)}$ and $\Lambda_-^{(k)}$ via α. Such a module is said to be *locally free of rank k with respect to* $\mathcal{F}$; when $\mathcal{F}$ is clear from context we omit 'with respect to $\mathcal{F}$'. Clearly any such locally free module is projective over Λ. Moreover, we note that

(5.1) $$\mathcal{L}(\mathrm{Id}_k) \cong \Lambda^{(k)}$$

Also, if $\alpha \in GL_k(\Lambda_0)$ then:

(5.2) $$\mathcal{L}(\alpha \oplus \mathrm{Id}_m) \cong \mathcal{L}(\alpha) \oplus \Lambda^{(m)}$$

Let $\natural : \mathbb{Z} \to \mathbb{F}_p$ be the canonical homomorphism and denote by Ω_0 the fibre product

(5.3) $$\left\{ \begin{array}{ccc} \Omega_0 & \stackrel{\pi_-}{\longrightarrow} & R_0 \\ \downarrow^{\pi_+} & & \downarrow \nu \\ \mathbb{Z} & \stackrel{\natural}{\longrightarrow} & \mathbb{F}_p\,. \end{array} \right.$$

Noting that $R_0 \otimes_{\mathbb{Z}} \mathbb{R} \cong \underbrace{\mathbb{R} \times \ldots \times \mathbb{R}}_{\frac{(p-1)}{2}}$ it follows that

(5.4) $$\Omega_0 \otimes_{\mathbb{Z}} \mathbb{R} \cong \underbrace{\mathbb{R} \times \ldots \times \mathbb{R}}_{\frac{(p+1)}{2}}.$$

In particular, Ω_0 satisfies the Eichler condition (cf. [11]) from which we see that:

(5.5) Every stably free Ω_0-module is free.

If A is a commutative ring we denote its Krull dimension by $\mathrm{Kdim}(A)$; then :

Proposition 5.6 : $\mathrm{Kdim}(\Omega_0) = 1$.

Proof : For a direct product we have $\mathrm{Kdim}(A_1 \times A_2) = \max\{\mathrm{Kdim}(A_1), \mathrm{Kdim}(A_2)\}$. If A is a Dedekind domain then $\mathrm{Kdim}(A) = 1$. Consequently $\mathrm{Kdim}(\mathbb{Z} \times R_0) = 1$. As Ω_0 is a subring of $\mathbb{Z} \times R_0$ then $\mathrm{Kdim}(\Omega_0) \leq 1$. However, Ω_0 has a subring isomorphic to $\mathbb{Z}$ so that $1 \leq \mathrm{Kdim}(\Omega_0)$, whence the conclusion. □

Define $\Omega = P_{n,m}(\Omega_0)$. Applying $P_{n,m}(-)$ to (5.3) we obtain another fibre square

(5.7)
$$\begin{cases} \Omega & \xrightarrow{\pi_-} & P_{n,m}(R_0) \\ \downarrow^{\pi_+} & & \downarrow \nu \\ P_{n,m}(\mathbb{Z}) & \xrightarrow{\natural} & P_{n,m}(\mathbb{F}_p)\,. \end{cases}$$

Note that (5.7) is a Milnor square as both $\natural$ and ν are surjective. Let $\alpha \in GL_k(P_{n,m}(\mathbb{F}_p))$ and denote by $\mathcal{L}(\alpha)$ the locally free module of rank k:

(5.8)
$$\begin{array}{ccc} \mathcal{L}(\alpha) & \xrightarrow{\pi_-} & P_{n,m}(R_0)^{(k)} \\ \downarrow^{\pi_+} & & \downarrow \nu \\ P_{n,m}(\mathbb{Z})^{(k)} & \xrightarrow{\natural} & P_{n,m}(\mathbb{F}_p)^{(k)}\,. \end{array}$$

We showed in **(4.5)** that $\nu_* : GL_k(P_{n,m}(R_0)) \to GL_k(P_{n,m}(\mathbb{F}_p))$ is surjective for $k \geq 3$. It now follows from Milnor's classification that:

(5.9) $$\mathcal{L}(\alpha) \;\cong\; \Omega^{(k)} \quad \text{for } k \geq 3.$$

Now consider the case $k = 2$; if $\alpha \in GL_2(P_{n,m}(\mathbb{F}_p))$ and $\mathrm{Id} \in GL_1(P_{n,m}(\mathbb{F}_p))$ then $\alpha \oplus \mathrm{Id} \in GL_3(P_{n,m}(\mathbb{F}_p))$ and so, by (5.1), (5.2) and (5.9):

(5.10) $$\mathcal{L}(\alpha) \oplus \Omega \;\cong\; \Omega^{(3)} \qquad \text{if } \alpha \in GL_2(P_{n,m}(\mathbb{F}_p)).$$

§6 : Surjectivity when $k = 2$:

We first improve on (5.10) as follows:

Theorem 6.1 : If $\alpha \in GL_2(P_{n,m}(\mathbb{F}_p))$ then $\mathcal{L}(\alpha) \cong \Omega^{(2)}$.

Proof : $\mathcal{L}(\alpha)$ is a projective module of rank 2 over $\Omega = P_{n,m}(\Omega_0)$. In particular, $\mathrm{rk}(\mathcal{L}(\alpha)) > \mathrm{Kdim}(\Omega_0)$. Moreover, by (5.10), $[\mathcal{L}(\alpha)] = 0 \in \widetilde{K}_0(\Omega)$. It now follows from a theorem of Swan [13] that $\mathcal{L}(\alpha)$ is induced from Ω_0; that is, there exists a projective module Q over Ω_0 such that $\mathcal{L}(\alpha) \cong i_*(Q)$ where $i : \Omega_0 \hookrightarrow \Omega$ is the canonical inclusion. Let $r : \Omega \to \Omega_0$ be the ring homomorphism uniquely specified by the assignments $r(t_i) = 1$ and $r(x_j) = 0$. Then $r \circ i = \mathrm{Id}_{\Omega_0}$. In particular, $r_*(\Omega) = \Omega_0$ and $r_*(\mathcal{L}(\alpha)) \cong Q$. Thus applying r_* to (5.10) we see that

$$Q \oplus \Omega_0 \;\cong\; \Omega_0^{(3)}.$$

It follows from (5.5) that $Q \cong \Omega_0^{(2)}$ and hence $\mathcal{L}(\alpha) \cong i_*(\Omega_0^{(2)}) = \Omega^{(2)}$. □

We arrive at statement **(II)** of the Introduction:

Theorem 6.2 : $\nu_* : GL_2(P_{n,m}(R_0)) \to GL_2(P_{n,m}(\mathbb{F}_p))$ is surjective.

Proof : Let $\alpha \in GL_2(P_{n,m}(\mathbb{F}_p))$. We claim that $\alpha \in \mathrm{Im}(\nu_*)$. Thus let $\mathcal{L}(\alpha)$ be the locally free Λ-module obtained by glueing via α

$$\begin{array}{ccc} \mathcal{L}(\alpha) & \longrightarrow & P_{n,m}(R_0)^{(2)} \\ \downarrow & & \downarrow \nu \\ P_{n,m}(\mathbb{Z})^{(2)} & \xrightarrow{\natural} & P_{n,m}(\mathbb{F}_p)^{(2)}\,. \end{array}$$

By (6.1), $\mathcal{L}(\alpha) \cong \Omega^{(2)}$. However, $\Omega^{(2)}$ is the locally free module of rank 2 obtained by glueing via $I_2 \in GL_2(P_{n,m}(\mathbb{F}_p))$ thus:

$$\mathcal{L}(I_2) \;=\; \left\{ \begin{array}{ccc} \Omega^{(2)} & \longrightarrow & P_{n,m}(R_0)^{(2)} \\ \downarrow & & \downarrow \nu \\ P_{n,m}(\mathbb{Z})^{(2)} & \xrightarrow{\natural} & P_{n,m}(\mathbb{F}_p)^{(2)}\,. \end{array} \right.$$

By Milnor's classification [8] there exist $\beta \in GL_2(P_{n,m}(\mathbb{Z}))$ and $\gamma \in GL_2(P_{n,m}(R_0))$ such that $\alpha = \natural_*(\beta) \cdot I_2 \cdot \nu_*(\gamma) = \natural_*(\beta) \cdot \nu_*(\gamma)$. However, if $j : \mathbb{Z} \hookrightarrow R_0$ is the canonical inclusion then the following diagram commutes

$$\begin{array}{ccc} \mathbb{Z} & \xrightarrow{\quad j \quad} & R_0 \\ & {\scriptstyle\natural}\searrow \quad \swarrow{\scriptstyle\nu} & \\ & \mathbb{F}_p & \end{array}$$

and induces a commutative diagram of group homomorphisms

$$\begin{array}{ccc} GL_2(P_{n,m}(\mathbb{Z})) & \xrightarrow{\quad j_* \quad} & GL_2(P_{n,m}(R_0)) \\ & {\scriptstyle\natural_*}\searrow \quad \swarrow{\scriptstyle\nu_*} & \\ & GL_2(P_{n,m}(\mathbb{F}_p)) & \end{array}$$

In particular, $\natural_*(\beta) = \nu_*(j_*(\beta))$, so $\alpha = \nu_*(j_*(\beta) \cdot \gamma) \in \mathrm{Im}(\nu_*)$ as claimed. □

Moreover, as the following diagram commutes

$$\begin{array}{ccc} GL_2(P_{n,m}(R_0)) & \hookrightarrow & GL_2(P_{n,m}(R)) \\ & {\scriptstyle\nu_*}\searrow \quad \swarrow{\scriptstyle\nu_*} & \\ & GL_2(P_{n,m}(\mathbb{F}_p)) & \end{array}$$

we obtain statement **(III)** of the Introduction, namely:

(6.3) $\nu_* : GL_k(P_{n,m}(R)) \twoheadrightarrow GL_k(P_{n,m}(\mathbb{F}_p))$ is surjective for all $k \geq 1$ and all n, m.

§7 : Locally free modules and stably free modules :

For the remainder of this paper, fixing a prime p, we apply the above considerations when $\Phi = C_p = \langle x \mid x^p = 1 \rangle$ is the cyclic group of prime order p. As in (5.7) we have a Milnor fibre square

$$(\mathfrak{S}) \qquad \begin{array}{ccc} P_{n,m}(\mathbb{Z}[C_p]) & \xrightarrow{\;i^\bullet\;} & P_{n,m}(\mathbb{Z}(\zeta_p)) \\ \downarrow \epsilon & & \downarrow \nu \\ P_{n,m}(\mathbb{Z}) & \xrightarrow{\;\natural\;} & P_{n,m}(\mathbb{F}_p) \end{array}$$

where $\Omega = P_{n,m}(\mathbb{Z}[C_p])$. We say the Ω-module X is *locally free of rank k with respect to* $\mathfrak{S}$ when X is obtained as a fibre product

$$\mathfrak{X}(\alpha) = \left\{ \begin{array}{ccc} X & \longrightarrow & P_{n,m}(\mathbb{Z}(\zeta_p))^{(k)} \\ \downarrow & & \downarrow \varphi \\ P_{n,m}(\mathbb{Z})^{(k)} & \xrightarrow{\natural} & P_{n,m}(\mathbb{F}_p)^{(k)} \end{array} \right.$$

by glueing via an element $\alpha \in GL_k(P_{n,m}(\mathbb{F}_p)))$. Clearly any such locally free module is projective over Ω. Moreover, with respect to the fibre square $\mathfrak{S}$, a locally free module X can equally be described as a dual projective Swan module; that is there is a bijective correspondence of isomorphism classes :

(7.1) $\left\{ \begin{array}{c} \text{locally free projective modules} \\ \text{of rank } k \text{ with respect to } \mathfrak{S} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{dual projective Swan} \\ \text{modules of rank } k \end{array} \right\}$

However, we showed in (6.3) above that:

(7.2) $\varphi_* : GL_k(P_{n,m}(\mathbb{Z}(\zeta_p))) \to GL_k(P_{n,m}(\mathbb{F}_p))$ is surjective for all $k \geq 1$.

Consequently, by Milnor's classification,

(7.3) If X is a locally free projective module with respect to $\mathfrak{S}$ then X is free.

Hence from (7.1) we see that:

(7.4) If X is a dual projective Swan module then X is free.

The statement **(I)** of the Introduction now follows from (7.4) on dualization.

References

[1] : H. Bass and M.P. Murthy ; Grothendieck groups and Picard groups of abelian group rings : Ann. of Math 86 (1967) 16-73.

[2] : B.J. Birch ; Cyclotomic fields and Kummer extensions : 'Algebraic Number Theory' (J.W.S. Cassells and A. Fröhlich eds), Academic Press (1967).

[3] : P.M. Cohn ; On the structure of the GL_2 of a ring: Publications Mathematiques IHES 30 (1966) 5-53.

[4] : D.G. Higman ; The units of group rings : Proc. London. Math. Soc. (2) 46 (1940) 231-248.

[5] : F.E.A. Johnson; Syzygies and homotopy theory. Springer Verlag, 2011.

[6] : S. MacLane ; Homology : Springer-Verlag, (1963).

[7] : J. Milnor ; Whitehead torsion : Bull. Amer. Math. Soc. 72 (1966), 358-426.

[8] : J. Milnor ; Introduction to Algebraic K-Theory : Ann. of Math. Studies vol 72 , Princeton University Press 1971

[9] : A.A. Suslin; On the structure of special linear groups over polynomial rings: Math USSR Izvestija 11 (1977) 221-238.

[10] : R.G. Swan ; Periodic resolutions for finite groups : Ann. of Math. 72 (1960) 267-291.

[11] : R.G. Swan ; K-Theory of finite groups and orders. (notes by E.G. Evans). Lecture Notes in Mathematics 149. Springer-Verlag 1970.

[12] : R.G. Swan ; Generators and relations for certain special linear groups. Advances in Math. 6 (1971) 1-77.

[13] : R.G. Swan ; Projective modules over Laurent polynomial rings. Trans. Amer. Math. Soc. 237 (1978) 111-120.

[14] : L.C. Washington; Introduction to cyclotomic fields. Graduate Texts in Mathematics vol 83, 2^{nd} edition. Springer-Verlag, 1997.

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