Wave Interaction with a Floating Finite Rectangular Plate in a Channel

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HIGHLIGHTS

- An analytical scheme is developed for wave-finite floating plate interaction in a channel.
- The expansion of velocity potential for fluid beneath the plate is properly constructed for matching.
- Various edge types and their combinations can be easily implemented.

1 INTRODUCTION

The interaction of water waves with floating bodies in a channel or long tank is a significant hydrodynamic problem with considerable applications such as model testing in towing and wave tanks, the safety of ship navigating in channels, renewable energy harvesting, and modelling an icy water environment in a tank/flume (e.g., [1], [2]). The present work investigates the interaction between surface waves with non-rigid bodies by considering a finite plate or ice cover in a channel. A finite rectangular plate is floating on the water in an infinite long channel of a rectangular section, with two parallel edges constrained with channel banks, while the other two parallel edges are in contact with the open water. An analytical scheme is developed to solve the coupled problem of flow motion and plate vibration, which enables us to consider the effect of various types of edge conditions, such as the commonly used ones, including clamped, free, simply supported, and elastically supported, and their combinations.

2 MATHEMATICAL MODELLING AND SOLUTION PROCEDURE

A Cartesian coordinate system O-*xyz* is established with the origin located in the centre of the undisturbed plate cover. The *x*- and *y*-axes are along the longitudinal and transverse directions of the channel, respectively, while its z-axis is pointing upwards. The water depth is denoted as H, and the width of the channel is $2b$. The width and length of the rectangular plate are $2b$ and $2l$, respectively. The velocity potential is applied to the fluid flow and the Kirchhoff-Love plate theory to the elastic plate vibration. For the former, it assumes that fluids are incompressible and inviscid, and their motion is irrotational. When the incoming wave of amplitude A_0 is sinusoidal with time, the velocity potential and the deflection of the plate may be respectively written as $\Phi(x,y,z,t) = \mathrm{Re}\{A_0\phi(x,y,z)\, e^{\mathrm{i}\omega t}\}$ and $W(x,y,t) = \mathrm{Re}\{A_0 w(x,y)\, e^{\mathrm{i}\omega t}\}$, and the problem can then be solved in the frequency domain.

The velocity potential $\phi(x,y,z)$ satisfies the Laplace equation throughout the fluid domain as

$$\phi_{xx} + \phi_{yy} + \phi_{zz} = 0. \tag{1}$$

The deflection $w(x,y)$ of the plate is governed by the dynamic equation

$$L\nabla^4 w - \omega^2 \rho_e h\, w = p, \tag{2}$$

where $L = Eh^3/[12(1-\nu^2)]$ is the flexural rigidity, with h, ρ_e and ν being the thickness, density, and Poisson's ratio of the plate, respectively. $m_e = \rho_e h$ then refers to the mass per unit area of the plate. The pressure term on the right-hand side of (2) can be given as

$$p = -\rho g w - \mathrm{i}\rho\omega\, \phi|_{z=0}. \tag{3}$$

At the interface of fluid and the plate, the linearized kinematic condition can be given as

$$\mathrm{i}\omega w = \phi_z|_{z=0}. \qquad |x| < l \tag{4}$$

Besides, the linearized free surface boundary condition can be given as

$$\phi_z - \gamma\phi = 0|_{z=0}, \qquad |x| > l \tag{5}$$

where $\gamma = \omega^2/g$ and g is the acceleration due to gravity. The surfaces at the bottom and channel walls are assumed to be rigid, and the impermeable boundary condition yields

$$\phi_y\big|_{y=\pm b} = \phi_z|_{z=-H} = 0. \tag{6}$$

At far field $x = \pm\infty$, the radiation condition ensures that the waves are outgoing. In addition, edge conditions are very important for plate problems. Three common conditions for a curved shape can be written as [3]

$$w = 0, \qquad w_n = 0; \tag{7}$$

for the clamped edge,

$$\nabla^2 w = (1-\nu)\left[\frac{\partial^2}{\partial s^2} + \frac{1}{\mathcal{R}}\frac{\partial}{\partial n}\right] w, \quad \frac{\partial}{\partial n}[\nabla^2 w] = -(1-\nu)\frac{\partial}{\partial s}\left[\frac{\partial^2}{\partial s \partial n} - \frac{1}{\mathcal{R}}\frac{\partial}{\partial s}\right] w; \tag{8}$$

for the free edge (e.g., Eqs. (A1) and (A2) in [4]), and

$$w = 0, \quad \nabla^2 w = (1-\nu)\left[\frac{\partial^2}{\partial s^2} + \frac{1}{\mathcal{R}}\frac{\partial}{\partial n}\right] w. \tag{9}$$

for the simply supported edge. $\boldsymbol{n}$ and $\boldsymbol{s}$ in the above equations are the normal and circumferential unit vectors of the edge, and $\mathcal{R}$ is the radius of curvature.

The fluid domain can be divided into three sub-domains, namely, the upstream open water domain ($\Omega_1: -\infty < x < -l, -H \le z \le 0$), plate-covered water domain ($\Omega_2: -l \le x \le l, -H \le z \le 0$) and downstream open water domain ($\Omega_3: l < x < +\infty, -H \le z \le 0$). In Ω_1, the corresponding total velocity potential can be written as the sum of incident wave potential and diffracted wave potential, or $\phi_1 = \phi_I + \phi_d$, with

$$\phi_I(x,y,z) = -\frac{g}{\mathrm{i}\omega} \times \frac{\cosh k_0(z+H)}{\cosh k_0 H} \times e^{-\mathrm{i}k_0 x}, \tag{10}$$

and

$$\phi_d = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} a_{nm} \frac{\cosh k_m(z+H)}{\cosh k_m H} \cos\beta_n(y+b)\, e^{\mathrm{i}\tau_{nm}x}, \tag{11}$$

where $\beta_n = n\pi/2b$, k_m are solutions of the dispersion equation $k \tanh kH = \gamma$, including both real one k_0 and imaginary ones k_m, $m > 0$, and $\tau_{nm}^2 = k_m^2 - \beta_n^2$ is obtained from the Laplace equation. It is noted that for $k_0^2 > \beta_n^2$, τ_{n0} is a real number which should be taken as positive to ensure the travelling wave is outgoing; while for $k_0^2 < \beta_n^2$, or $m > 0$, τ_{nm} is the pure imaginary number which should be taken as negative to ensure that it is an evanescent wave. Similarly, in Ω_3, the velocity potential can be written as

$$\phi_3 = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} c_{nm} \frac{\cosh k_m(z+H)}{\cosh k_m H} \cos\beta_n(y+b)\, e^{-\mathrm{i}\tau_{nm}x}. \tag{12}$$

In the plate-covered water domain Ω_2, the velocity potential can be expanded as

$$\phi_2 = \chi + \varphi + \psi. \tag{13}$$

We have

$$\chi = P_{00} + \frac{Q_{00}}{l}x + \mathcal{P}_{00}[x^2 - (z+H)^2], \tag{14}$$

as a particular solution obtained from taking the average of flux out on each boundary and treating it separately, and $\mathcal{P}_{00}$ is related to the average on $z = 0$. Besides, φ can be expressed as

$$\varphi = \sum_{j=0}^{\infty}\sum_{n=0}^{\infty} \varepsilon_{nj}\mathcal{P}_{nj} \cos\alpha_j(x+l)\cos\beta_n(y+b)\frac{\cosh \kappa_{nj}(z+H)}{\cosh \kappa_{nj}H}, \tag{15}$$

based on the mode expansions horizontally, with $\varepsilon_{00} = 0$ and $\varepsilon_{nj} = 1$ otherwise, and $\alpha_j = \frac{j\pi}{2l}$, $\hbar_{nj}^2 = \alpha_j^2 + \beta_n^2$. In addition, based on the vertical mode expansion, ψ can be expressed as

$$\psi = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \varepsilon_{nm} \left[\frac{P_{nm}\cosh\lambda_{nm}(x+l) + Q_{nm}\cosh\lambda_{nm}(x-l)}{\cosh 2\lambda_{nm} l}\right] \cos\beta_n(y+b)\cos K_m(z+H), \quad (16)$$

where $K_m = \frac{m\pi}{H}$ and $\lambda_{nm}^2 = \beta_n^2 + K_m^2$ from the Laplace equation.

The unknown coefficients in Eqs. (11) to (16) can be obtained by establishing and solving the system of linear equations obtained from the dynamic and kinematic boundary conditions, the matching conditions at the interfaces of neighbouring domains, and the plate edge conditions. The procedures are introduced briefly below: As $\partial\varphi/\partial x = 0$ at $x = \mp l$, thus we have $\partial(\psi + \chi)/\partial x = \partial\phi_1/\partial x$ at $x = -l$ and $\partial(\psi + \chi)/\partial x = \partial\phi_3/\partial x$ at $x = l$. Based on the orthogonality of the cosine functions for y and $\cosh k_m(z + H)$ for z, we can get the expressions of a_{nm} and c_{nm} in (11) and (12) as unknown coefficients in (14) and (16). Then, from the continuity condition of the potentials at the interfaces of different fluid domains, we can have $\phi_2 = \phi_1$ at $x = -l$ and $\phi_2 = \phi_3$ at $x = l$. Multiplying both sides of these two equations with $\cos\beta_n(y + b)\cosh k_m(z + H) / \cosh k_m H$ and integrating with respect to y from $-b$ to b and z from $-H$ to 0, respectively; using the orthogonality of these functions and further using the expressions of a_{nm} and c_{nm}, we can eliminate a_{nm} and c_{nm}, and obtain the linear equations for $\mathcal{P}_{nj}, P_{nm}$ and Q_{nm} for $n, m = 0,1,2, \ldots$. In addition, we can further impose the conditions on the interface of the plate cover and the upper surface of the fluid domain. We may further expand the deflection of the rectangular plate as the summation of a double cosine series and four additional terms in each direction [5]

$$w = \sum_{n=0}^{\infty}\sum_{j=0}^{\infty} C_{nj}\cos\alpha_j(x+l)\cos\beta_n(y+b) + \sum_{n=0}^{\infty}\cos\beta_n(y+b)\left[\sum_{i=1}^{4} d_n^{(i)}\left(\frac{x}{l}\right)^i\right]$$
$$+\sum_{j=0}^{\infty}\cos\alpha_j(x+l)\left[\sum_{i=1}^{4} c_j^{(i)}\left(\frac{y}{b}\right)^i\right]. \quad (17)$$

Based on (17), $\nabla^4 w$ can also be expanded as double cosine series. When doing this, derivatives should be applied to x^i and y^i first before they are expanded into the cosine series to ensure convergence [5]. Substituting (13), (17) and the expansion of $\nabla^4 w$ into (2) and (4), and using the orthogonality of trigonometric functions, linear equations can be obtained. To close the problem, the conditions at the four edges of the plate, e.g., (7)-(9), are also imposed on w in (17). With all the above, a linear system of equations with the unknowns $C_{nj}, \mathcal{P}_{nj}, P_{nm}, Q_{nm}, \boldsymbol{c_j} = \left[c_j^{(1)}, c_j^{(2)}, c_j^{(3)}, c_j^{(4)}\right]$ and $\boldsymbol{d_n} = \left[d_n^{(1)}, d_n^{(2)}, d_n^{(3)}, d_n^{(4)}\right]$. For practical computation, infinite series needs to be truncated to finite terms, e.g., N for n, J for j, and M for m. This gives

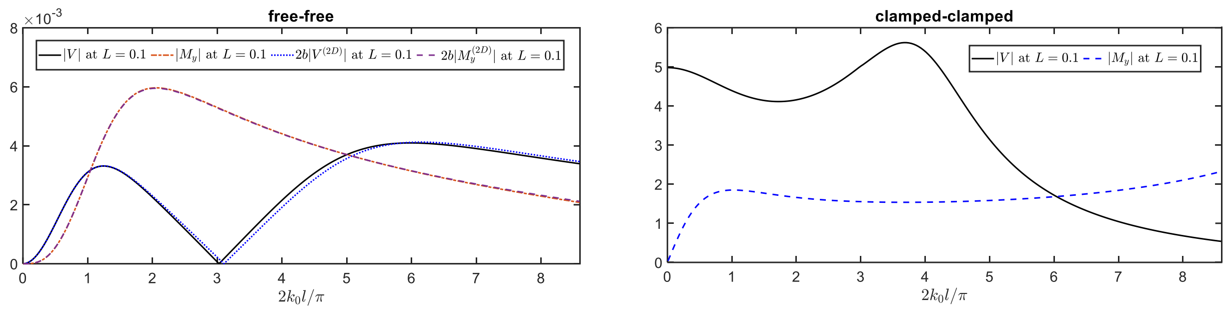
$$\boldsymbol{A}\cdot\boldsymbol{x} = \boldsymbol{B}, \quad (18)$$

where $\boldsymbol{x}$ is a column vector containing $2NM + 4(N + J) + 2NJ$ unknowns, $\boldsymbol{A}$ is a square matrix of size $2NM + 4(N + J) + 2NJ$. $\boldsymbol{B}$ is the column vector of the same size as $\boldsymbol{x}$, depending on the incident wave. It is worth noting that using the linear equations obtained from dynamic and kinematic equations, we can further eliminate C_{nj} and $\mathcal{P}_{nj}$ from $\boldsymbol{x}$ to reduce the computation. Once unknown coefficients have been solved, results such as hydrodynamic loads on the plate, plate deformation, and the reflection/transmission coefficients can be worked out.

3 RESULTS AND ANALYSIS

In the following analysis, the two parallel edges in contact with the open water ($x = \pm l$) can be assumed to be free, while the edges along the channel banks ($y = \pm b$) can be set as free, clamped or simply supported, etc. In Fig. 1, graphs of the modulus of the total vertical force $|V|$ and the moment of y-axis

$|M_y|$ versus $2k_0 l/\pi$ are displayed for different edge conditions at $L = 0.1$. The results of the 2D case, $|V^{(2D)}|$ and $|M_y^{(2D)}|$, are also provided for comparison. The curves of the hydrodynamic forces on the plate for the free-free and clamped-clamped edge cases at the channel walls are displayed in Figs 1(a) and 1(b), respectively. In the case of the free-free edges, there is only a slight difference between the results and those obtained for the 2D case, suggesting that the 3D effect may not be significant. However, when the plate edges are clamped to the channel walls, the 3D effect becomes more noticeable. This observation is further supported by the graphs depicting the modulus of plate deformation induced by incident waves at these two different edge types along the channel banks. In Fig. 2, the cases of $k_0 l/\pi = 1$ and $L = 0.1$ are considered, where the wavelength equals the plate length. Other parameters are the same as those in Fig. 1.



(a) (b)

Figure 1. Modulus of the total vertical force $|V|$ and the moment of the y-axis $|M_y|$ against $2k_0 l/\pi$ at both free and both clamped edge conditions along tank walls. ($l = 1.5, b = 1, H = 1, L = 0.1, m_e = 1 \times 10^{-3}, \nu = 0.3$)

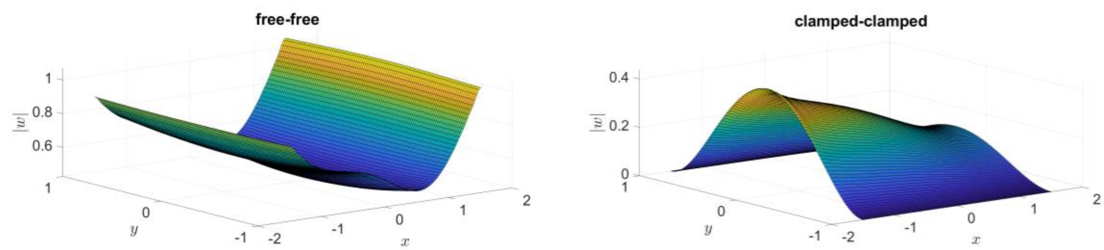


Figure 2. Mudulus of the deformation of plate cover induced by the incoming wave at $k_0 l/\pi = 1$ and $L = 0.1$.

4 CONCLUSIONS

The abstract provides an efficient solution procedure for the interaction between water waves with a finite rectangular plate cover in a channel as a follow-up to a previous work [1] where an ice/elastic cover fully covers the channel. The results illustrate the importance of edge constraints to this problem, which the present scheme can conveniently investigate. Further results/analysis will be provided during the workshop.

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