Payment Networks in the Presence of a Central Counterparty Clearing

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I, Haotian Gao, confirm that the work presented in this thesis is my own. Where information has been derived from other sources, I confirm that this has been indicated in the work.

Abstract

The establishment of Central Counterparty Clearing houses (CCPs) is a way to reduce the impact of counterparty risk, therefore strengthening the whole financial system. In this thesis, I focus on liquidity shortfalls arising from the failure to meet variation margin calls with the presence of a CCP. I generate several random network models as hypothetical interbank networks, then use numerical simulation to reach the results.

- Using the Erdős-Rényi network with independent exposures, I find that the shortfall is not necessarily minimum when all contracts are centrally cleared. Instead, the optimal value of shortfall exists with a combination of centrally and bilaterally clear. In this point of view, increasing the percentage of centrally cleared contracts is not always optimal.

- When using the Erdős-Rényi network with perfectly correlated exposures, I prove stronger results using mathematical derivation that the aggregate shortfall is decreasing with the fraction of centrally cleared contracts, indicating that the second-step netting provided by the CCP betters off payment efficiency. In this set-up, shortfall can never increase as percentage increases, and it only starts to decrease after a certain fraction of centrally cleared contracts is achieved.

- For power-law degree distributed networks and networks with non-neutral assortativity, I find similar relationships but higher value of shortfalls for both scale-free networks and disassortative networks. These properties provide a closer match to real interbank networks, therefore further enhanced my result.

Impact Statement

This thesis contributes both to academia and industry. From the academic point of view, it extends existing literature on liquidity shortfalls, payment efficiency associated with the introductions of central clearing, and interbank payment network topologies. First of all, it extends the Eisenberg-Noe clearing vector algorithm to the case of central clearing. Despite the algorithm has been extended in several directions, it is by my knowledge the first time it is incorporated with the CCP. Inspired by the updated version of EN-algorithm, I am able to further investigate effects after introducing a CCP into interbank networks. This thesis is innovative because it provides new findings on the effect of central clering on payment efficiency. It is commonly believed that the central clearing typically benefits interbank payment systems as increasing opportunities in multi-lateral netting reducing total exposures. It has been proved in this thesis that this is not always the case in some particular network structures. This provides a potential future direction to identify optimal levels of central clearing.

The findings of this thesis are also of importance for regulators and policymakers, as they provide a better understanding of the effects of central clearing on payment systems, and of its relationship with the network structure of such systems. The existence of an optimal value of central clearing may for instance inform future regulation on clearing of derivatives. My analyses of the impact of network properties on payment systems efficiency may also inform regulators on which policies and incentives may be put in place to drive the system towards a re-organisation of the interbank network to improve its robustness when facing shocks.

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Contents

Chapter 1

Introduction

1.1 Background and Motivation

In finance, systemic risk refers to the risk that the whole financial system breaks down. Rather than focusing on the risk of individual institutions, systemic risk studies the aggregate properties of the systems.

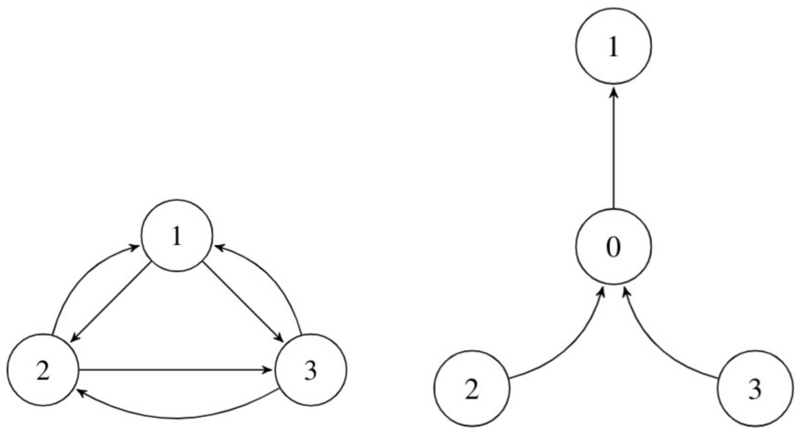
Systemic risk arises because of interconnections between banks. Banks are for instance connected with each other through bilateral exposures, for example, loans. One can imagine that bank A owes money to bank B, and bank B owes money to bank C. If bank A defaults on its loans, bank B would suffer a loss, hence it may default on its loans to C. It can be seen that the default of bank A at the beginning leads to default of the others. Bank A is therefore called basic default and bank B and C are called contagious default, which means that their defaults are the result of the default of bank A. The interbank system can be modeled as a network, where financial institutions are represented by nodes and their exposures, or i.e. connections, by edges. Such a financial network is directed (because an exposure goes from a borrower to a lender) and weighted (because exposures have different sizes).

Since the financial crisis in 2008, central banks around the world have established a number of measures to mitigate systemic risk, especially after seeing the impact of the failure of large institutions in the US. For instance, after the fall of Lehman Brothers, regulators realised that the default of large institutions could lead to contagion defaults across the system, which made them understand that some institutions are "too big to fail". Therefore authorities finally decided to support large institutions from failing to avoid further impact to the industry.

Regulators have implemented many improvements towards financial system, among which two major changes are closely related to this thesis (Duffie, 2018). The first is collateralisation in derivatives contract, where counterparties in a derivative contract exchange collateral to initiate the contract (initial margin) and to cover risk of mark-to-market losses (variation margin). In this way, there are two sources of risk, the one is counterparty risk associated with these two institutions, i.e. the risk that either party defaults on its contractual obligations, the other is liquidity risk, i.e. the risk that either party is not able to source the collateral required to fulfil payment obligations arising from margins. The second is the widely usage of Central Counterparty Clearinghouses (CCPs) to clear payment obligations (Powell, 2015). According to Bank for International Settlements (2021), in notional amounts and as of June 2021, almost 78% of interest rate derivatives, more than 60% of credit derivatives, and around 4% of foreign exchange derivatives are centrally cleared.

A CCP is a financial institution formed to facilitate trades between banks. It performs two main functions as an intermediary in a transaction: clearing and settlements. Rather than having banks settle their bilateral obligations, the CCP collects from each bank the net obligations and distributes them to the counterparties. An illustrative figure can be seen in Fig.1.1. Consider a simple example where the exposures of node 1 to 2, 2 to 3 and 3 to 1 are 10,10 and 8 respectively. If no CCP is present, all transactions go through as normal. When a CCP is introduced, because of the netting effect it brings, node 1 only pays 2 units of cash to the CCP, which are then routed to node 3, while node 2 remains passive as its net obligations are zero.

The CCP stands between two clearing firms to reduce the risk of a member firm failing to honor its trade settlement obligation. The purpose of establishing a CCP is to reduce systemic risk through a regulated payment system. CCPs became widely used after the financial crisis in 2008, especially for over-the-counter (OTC) derivatives. For instance, in the EU specific classes of OTC interest rate derivatives must be cleared though a CCP (Alfranseder et al., 2018). In practice, the payment to a CCP is netted, hence reducing the total amount of exposures in the system. However, on the other hand, regulators require all clearing members to clear the payments to the CCP before any other transactions can take place. Let us consider a simple scenario on a daily basis: without the CCP, all transactions are cleared simultaneously without any seniority structure at the end of the day. After introducing the CCP, payments to the CCP must be settled first, for instance, at the beginning of the trading day. However, some institutions might need to receive payments from other counterparties to be able to pay the CCP, but such payments will only be received at the end of the day. Therefore, to fulfill their obligation to the CCP, these institutions need to borrow additional funds, thus introducing an inefficiency in the system. Hence, there are two opposite effects due to the CCP, and their aggregate effect depends on their interplay.



(a) payment without CCP **(b)** payment with CCP

Figure 1.1: Simple illustration of how bilateral clearing are reformed to central clearing, where node 0 acts as the CCP.

In this thesis, I model the payment flows as demand for liquid assets from variation margin (VM) calls, i.e. rise in the margin requirement as a result of large fluctuation on the IM collateral, then investigate how the demand for liquid assets from variation margin payments[1] changes with the fraction of notional that is centrally cleared. I calculate the demand for liquid assets of one institution with its liquidity

[1]Variation margins are typically settled in cash, see e.g. International Swaps and Derivatives Association (2017).

shortfall, i.e. the amount of cash that it has to source to make payments both on centrally cleared and bilateral contracts and that it is not able to cover either with their cash buffers or with incoming VM payments. In order to quantify liquidity shortfalls I build a network model for clearing payments on both centrally cleared and bilateral obligations, similarly to Amini et al. (2016) and Cui et al. (2018).

1.2 Objectives

In this thesis, I am trying to find out the effects of central clearing on liquidity shortfall. To the best of my knowledge, most previous studies focused on difference between fully bilaterally clear and fully centrally clear (Amini et al., 2020; Ahn et al., 2011) systems, or partially centrally clear and fully centrally clear (Amini et al., 2016) systems. In my thesis, I will investigate the behaviour over the whole range between fully bilaterally clear and fully centrally clear. On top of this, I will provide results including decomposition of the liquidity shortfall, the critical value of centrally cleared threshold where the CCP becomes effective, and the relationship between connectivity and payment efficiency. At the end, I will provide relevant implications and suggestions to policy-makers according to results I derived.

In order to keep the comparison as fair as possible, I eliminate as many inefficiency aspects as possible, thus only left with the most critical properties in studying the effect of the CCP. For instance, since multiple CCPs can increase risk exposures (Duffie and Zhu, 2011), I only include one CCP; as the CCP itself can also become a source of risk (Hull, 2012), I assume the CCP can never default and record a shortfall; there is no bankruptcy cost (Rogers and Veraart, 2013) or devaluation in liquidating their assets, i.e. overlapping portfolio and fire sale (Caccioli et al., 2014; Shleifer and Vishny, 2011), both of which act as significant factors in weaken the system. Hence, the key objective falls into comparing different level of central clearing without other negative effects.

1.3 Results and Contributions

The main results of the thesis are as follows: First, I find that, in certain cases, the dependence of the aggregate shortfall on centrally cleared percentage is U-shaped.

In those cases there exists an optimal value of such percentage between 0 and 1 for which the aggregate shortfall is at its minimum, meaning that it is neither optimal to centrally clear all contracts nor to centrally clear no contract. The emergence of a U-shaped relationship can be understood in terms of the tradeoff between two competing effects. On the one hand, as the percentage increases, VM payment obligations decrease due to the multilateral netting performed by the CCP. On the other hand, driven by the reduction in realised payments on bilateral contracts, also total realised payments decline. When the aggregate shortfall has a minimum, for small values of centrally cleared percentage, VM payment obligations decrease at a faster pace than payments, while, for large values of the percentage, the opposite happens. When the network of counterparties is too densely interconnected the U-shaped relationship disappears as VM payment obligations decrease at a faster pace than payments for all values of the percentage, suggesting that gains from multilateral netting always dominate the reduction in realised bilateral payments. Amini et al. (2016) found instead that it is always optimal to centrally clear all contracts. Albeit they considered a more complex model, they also made the restrictive assumption that all institutions have exactly the same underlying counterparties on centrally cleared and bilateral contracts.[2] In this special case, I prove a stronger result: that the aggregate shortfall is weakly decreasing with the fraction of notional that is centrally cleared. This means that, in this case, increasing the fraction of centrally cleared notional is always (weakly) beneficial.

Second, I prove that, when all institutions have centrally cleared and bilateral contracts with exactly the same underlying counterparties, there is a critical threshold of percentage of payments that are cleared through the CCP that discriminates between two regions: If the percentage is smaller than the threshold, efficiency is constant. If the percentage is larger than the threshold, efficiency increases. The threshold is the smallest liquidity ratio across all institutions, defined as cash buffer divided by net obligations when no contract is centrally cleared. As a consequence, central clearing becomes beneficial when smaller percent of transactions are cen-

[2]For each centrally cleared contract the CCP interposes itself between two institutions. We call those institutions the *underlying* counterparties of the contract.

trally cleared and when cash buffers are smaller, or when net obligations are larger, i.e. in the cases in which liquidity shortfalls are likely to be larger.

Third, I find that the critical threshold increases as the average degree of the network increases. This implies that increasing the connectivity of the network has an effect similar to introducing a CCP. This is because increasing the number of connections in the network effectively reduces the size of interbank exposures.

Finally, I study how different network topologies react to introducing a CCP. In previous studies, the model I consider is the Erdős-Rényi network and treat as benchmark. However, as mostly observed in real world, interbank payment networks tend to exhibit power-law degree distribution and negative assortativity. Study on these properties provides further extensions and act as a supplementary to previous chapters. The main finding is that for these types of networks shortfall becomes larger than before, indicating that the system becomes less efficient. The main reason behind should be inefficiency on low degree nodes.

1.4 Thesis Structure

This thesis is organised as follows: Chapter 2 contains the literature review to explain the background knowledge used and previous research related to the current work. Chapter 3 explains the methodology used. Chapter 4 describes the payment algorithm with the presence of a CCP using EN clearing vector. In chapter 5, I present and discuss results related to independent exposures where total amount of obligation related to bilateral and CCP is different. Chapter 6 provides another particular set-up where obligations in chapter 5 are same. Chapter 7 extends the result in chapter 6 with different network models which are closer to empirical study. Concluding remarks will be given in Chapter 8.

Chapter 2

Literature Review

2.1 Network Model

In general, networks are a mathematical tool to model systems with pairwise interactions, and they have a variety of applications in the natural and social sciences. Examples of applications in biology include the modeling of predator-prey interactions in food webs (Pascual and Dunne, 2006) and the ingredient-flavour network (Ahn et al., 2011). In computer science, the internet is a network of physical connections between computers, mobile phones and other devices, where devices are nodes and connections are edges (Newman, 2018). As for data science, one of the most important application is community structure detection (Newman, 2006; Girvan and Newman, 2002) in complex systems. In this thesis, I will focus on the application of networks in finance, in particular, to the study of systemic risk.

In this thesis, I use random networks to carry out simulations on hypothetical banking system. The first random network I will look at is the Erdős-Rényi network (Erdos and Renyi, 1959), where each pair of nodes is connected with a fixed probability. The Erdős-Rényi network is one of the simplest random network model and will be used as the benchmark case throughout this thesis. By definition, the binomial distribution measures the number of succeed trails under fixed success rate p, therefore the degree distribution should be characterised as such distribution where the probability of connecting two particular nodes is fixed with constant (Newman et al., 2001).

Because the degree distribution decays exponentially fast, the Erdős-Rényi network is considered a prototype of systems with homogeneous connectivity. However, it has been shown that real networks have heavy-tailed degree distribution (Albert and Barabási, 2002), where the degree of nodes spans several orders of magnitude. In Albert et al. (1999) and Huberman and Adamic (1999), the authors studied the example of world-wide web, then found that the heavy-tail index ranges between 2 and 3. In order to model this type of connectivity patterns, Barabási and Albert (1999) introduced the scale-free network, which captures the power-law property on degree distribution. From empirical study, the Austrian interbank network follows power law degree distribution 2 (Boss et al., 2004a). The monetary transaction data in Japan also indicated that the network is described as power-law degree distribution and therefore statistical self-similarity (Inaoka et al., 2004). Moreover, Soramäki et al. (2007) studied the data from US commercial banks and determined the degree distribution follows scale-free property with coefficient slightly above 2.1. Therefore, I will generalise networks with power-law degree distribution in the second experiment.

In addition, most networks in reality exhibit non-neutral assortativity. For example, social network are positive assortative because highly connected people tend to connect with other highly connected people (Capocci et al., 2003; Newman, 2002). Plenty of literature indicated that interbank networks exhibits disassortativity. For instance, Bech and Atalay (2010) analysed data from federal fund market and found that small banks are more likely to lend to larger banks that borrow from many institutions. In Iori et al. (2008), the authors found that the Italian overnight money market exhibits disassortative mixing, of which there clearly existed a few hubs connected to a large number of peripheral banks with only a few links.

Several researches exist to study effects of different network structures on financial contagion. Lenzu and Tedeschi (2012) developed network structures where links are formed based on agents' performance via a fitness mechanism. The authors therefore were able to generate networks with different topology. Their main finding is that scale-free networks are more vulnerable and less resilient than random networks when facing random attacks. In Grilli et al. (2015), the authors analysed the relationship between interbank connectivity and contagion. One key contribution is that it considers different components in the economy, including goods, credit and interbank market. Their major finding is that increasing connectivity can worsen the system and increasing the chance of financial contagion. When looking into effects on goods markets, especially business cycle, they found that interbank connectivity does not have positive effects on economic growth. Another study worth to mention is Caccioli et al. (2012). In that paper, the authors studied the behaviour of contagion probability given the failure of a random bank. They considered heterogeneous degree distributions, heterogeneous distribution of assets and interbank degree correlations. They found that power-law degree distributed networks are more resilient to the failure of a random bank but more fragile to the failure of a high-degree bank. A power-law distribution of assets leads to inefficiency in diversification thus make the system more prone to contagion event. Disassortativity also enhances the stability of the system.

2.2 Systemic Risk and Financial Contagion

Systemic risk is the risk that the financial systems collapses. After 2008 it became clear that this risk is endogenous to the system, meaning that it emerges from the interactions between different financial institutions. The financial system can be modeled using networks to represent interactions between financial institutions (Haldane and May, 2011; Langsam and Fouque, 2013). In particular, networks are used to study financial contagion, which refers to the situation in which the default of an institution is triggered by the default of another institution.

Upper (2011) summarised a detailed list of different contagion channels, from asset side to liability side, such as counterparty default, overlapping portfolios and funding contagion. Counterparty default contagion describes the direct impact of the default an institution on its counterparties. For instance in interbank lending networks, the default of a bank causes losses to its creditors, who may in turn default and cause losses to their creditors and so on. Upper (2011) suggested that direct bank exposures do not contribute significantly to contagion risk, however Caccioli et al. (2015) argued that the consequence of bank failure can be amplified by overlapping portfolio contagion.

Overlapping portfolio contagion occurs because banks hold common assets. If bank A and bank B invest in the same asset and bank A liquidates its portfolio, the assets will be devalued because of market impact, which will cause bank B to suffer a loss. Bank B may then be forced to liquidate its assets, causing their further devaluation, and so on. Huang et al. (2013) considered this type of contagion dynamic as a bipartite network. By using data collected during the Financial Crisis 2008-2011, their model successfully identified a significant portion of fail banks during that period. In Caccioli et al. (2014), a similar bipartite network was used to test the effect of different parameters, for instance leverage, diversification (shocks on asset value) and types of shocks (shocks on asset value or a particular bank), on stabilities.

The presence of an asset common to all banks is also considered in Cifuentes et al. (2005), who argued that, in volatile market conditions, significant impact on asset prices can severely disrupt institutions holding the same asset. In this scenario, even large capital buffers may be insufficient to cover sudden losses, so liquidity buffer becomes more efficient to preserve the financial stability. In Caccioli et al. (2015) the authors pointed out that counterparty default together with overlapping portfolios trigger financial contagion. Later, Banwo et al. (2016) extended the model in Caccioli et al. (2014) to allow for power-law distributed degrees and asset sizes.

Funding refers to the ability to raise funds in the market. Funds can be originated in two ways: liquidating asset or borrowing from the market. If a bank cannot raise enough cash to make its payments, it would default, disturbing the whole system. In fact, fire sale can be triggered by a funding need. Tressel (2010) considered a complete funding shock scenario, where the funding cost is increased by 500bps. As a result, banks have to cut their foreign claims to maintain the leverage level (capital asset ratio). Deleveraging can be amplified together with fire sale triggered by funding shocks. If the banks cannot access to substitute funding sources when the interbank market becomes impaired, losses accumulated in the system, and become a channel of contagion.

To quantify losses caused by direct contagion, the most widely used methods are the Furfine default algorithm (Furfine, 2003) and DebtRank (Battiston et al., 2012). Furfine (2003) presented a study of the US system and showed that, although the impact of contagion on the whole banking system is limited, the initial default of one of the largest banks leads to the default of other banks. He also showed that illiquidity contagion – which refers to the situation in which a bank stops lending to other banks, who in turn will stop lending to their counterparties, and so on – is greater than default contagion. Similar result has also been found in Wells (2004) using UK banking data. An extensive review can be found in Upper (2011), where the author summarised the algorithm, assessed its assumptions and discussed its applications.

DebtRank is a quantity that measures the systemic importance of the nodes in the system. Battiston et al. (2012) analysed data from US financial institutions and proved that each node in a group of institutions became systemically important at the peak of the crisis. They argued that apart from 'too big to fail', regulators should also pay attention to 'too central to fail'. Then the dynamics was further generalised by allowing for shock propagation and amplification, meaning that after the initial shock, such distress can be transmitted throughout the system (Bardoscia et al., 2015). One of the important finding illustrated that the capability to amplify the shock depends only on the largest eigenvalue of the interbank leverage matrix. To validate the algorithm, Bardoscia et al. (2015) applied the model to European banks. The results showed that the most dangerous banks are also the most vulnerable ones, therefore should be paid more attention from regulators.

2.3 Eisenberg-Noe Clearing Vector

In this thesis I will focus on clearing of interbank payments. Let us consider an interbank network where each bank has to make payments to other banks. In a network context, the amount of money a bank is able to pay depends on the amount of cash it receives from others as payments. Eisenberg and Noe (2001) introduced an algorithm to compute clearing vectors in a self-consistent way (a clearing vector is a vector that in entry i contains the total payment carried out by bank i once taking into account the amount of cash received by other banks). Eisenberg and Noe (2001) proved that clearing vectors exist, and that they are unique under mild market and regulatory conditions.

Since established, the model has been extended several times. Elsinger (2009) introduced cross shareholding and outside borrowing in his study, incorporating these properties with debt seniority. By adjusting the EN-algorithm, the author proposed a new method to determine the debt and equity value of the financial institutions. Furthermore, injecting money from outside investor can increase the value of claims by more than the money supplied. The author assumed no bankruptcy cost, which was thereafter studied by Rogers and Veraart (2013). Rogers and Veraart (2013) gave a more realistic set-up as the bank cannot call in all its loans in face value at default. Clearing vectors still exist but are no longer unique. The greatest clearing vector can be found by allowing all banks to fail in succession until only one solvent bank remains. Then the authors also analysed the effect of rescue consortia. In contrast to Elsinger (2009), under non-zero bankruptcy cost, there are situations where consortia have the incentive and means to rescue failing banks. A recent research by Kusnetsov and Veraart (2019) introduced multiple maturities in the liabilities. The authors introduced two approaches to clearing at first maturity date: the functional approach which is similar to EN-algorithm and algorithm approach which extends the functional approach.

The empirical study on EN-algorithm also attracts attention of researchers. Boss et al. (2004b) carried out empirical analysis on the Austrian interbank market. They concluded that the domestic banking system is relatively stable as the default of a bank is unlikely to spread over the network. Only a small percentage of default can be classified as contagious, while a vast majority of default comes from macroeconomics shocks. In the international point of view, Kanno (2015) provided an empirical study based on global banking data. The author proved that in theory, there are a few contagious default triggered by fundamental default before and after the financial crisis. Stress tests indicated that major banks in the system can theoretically cause 1-6 contagious defaults. Another notable contribution comes from that the author provides a real example to estimate the bilateral interbank exposures using RAS algorithm (Censor et al., 1997).

2.4 Central Clearing

In this thesis, I will mainly study the effect of incorporating CCPs into the EN-algorithm. By definition and regulatory requirement, CCPs collect collateral to cover both the current (variation margin) and potential (initial margin) payments. The purpose of introducing CCPs into the system is to reduce the risk that one institution cannot fulfill its payment. The rationale has been justified in many researches. According to Duffie and Zhu (2011), within one particular asset class, adding multiple CCPs will always reduce netting efficiency, hence increasing risk exposures. When clearing several products, it is more efficient to clear everything in aggregate rather than having different CCPs. However, the reduction can only be achieved if the number of participants is sufficiently large. Similarly, Cont and Kokholm (2014) showed that the highest exposure reduction is reached if one CCP clears all asset classes as in Duffie and Zhu (2011). On the other hand, this situation would make the CCP faces high concentration of systemic risk and operational risk.

In contrast to Duffie and Zhu (2011), Cont and Kokholm (2014) found the opposite conclusion if the model parameters are more realistic, taking into account differences in riskiness and correlation across asset classes. Despite the positive aspects, as the size of the CCP gets larger and more institutions are exposed to the CCP, economists are worried that CCPs may become a source of systemic risk and instability. In Heath et al. (2016), a series of extreme scenarios was modeled, and the results demonstrated that there would be circumstances in which CCPs would exhaust their collateral fund. Such situation is extremely rare and the systemic risk can be well managed as long as the CCP maintains its financial resources in line with regulatory standards. Other studies related to this issue include Bank of International Settlement (2018) and Alfranseder et al. (2018).

2.5 Liquidity Shortfall with Central Clearing

There are several studies directly related to liquidity shortfall in the central clearing context and closely related to my work. To the best of my knowledge, Cui et al. (2018) is the only study that considers sequencing of payments. Their model is similar to mine where payments to the CCP are settled at beginning with bilateral payments afterwards. The major difference is that they allowed the CCP to record a shortfall while I assume the CCP always meets its obligation. They found that under the partial clearance with the CCP, shortfalls of all institutions are smaller than fully bilateral clearing. Another study related to this is Amini et al. (2016), with some notable differences. They assumed that institutions were required to sell illiquid assets when facing liquidity shortfalls, whereas I do not make specific assumptions on remedial actions that institutions might take and only record shortfalls; they accounted for default fund contributions; they allowed for heterogeneous values of the central clearing percentage that depend on the pair of counterparties, whereas I only consider the case of such percentage equals for all pairs of counterparties; they did not consider the sequencing of payments. They concluded that full central clearing always leads to weakly smaller shortfalls than the situation in which contracts are only partially centrally cleared. Amini et al. (2020) extended the model of Amini et al. (2016) to the case in which institutions also have liabilities to end users, whereas Ahn (2020) derived conditions that make central clearing beneficial for all institutions. I notice that these two papers (Amini et al., 2020; Ahn, 2020) only compared the case of fully centrally cleared framework to fully bilaterally cleared framework, while Amini et al. (2016) considered the difference between partial centrally cleared to full centrally cleared. In contrast, in this thesis, I also look into the effect within partial centrally cleared, where I increase (decrease) the proportion of centrally cleared transactions.

So far I have reviewed definitions and properties of random network models and EN clearing vector, I will then use equations and algorithm to illustrate how to incorporate these in my research. I will explain properties of different random network models and show how to generate them using algorithm. For the EN clearing payment, I will show how to calculate it analytically in special case, and use fixed point algorithm in a general set-up.

Chapter 3

Methodology

In this chapter, I explain in detail several properties and models used throughout this thesis. More specifically, I start from properties and quantities of networks, then introduce the random network models I use in my analysis, and the Eisenberg-Noe clearing payment algorithm. I also cover step-by-step algorithms to generate different random networks and to compute clearing payment vectors.

3.1 Network Theory and Network Properties

Banking systems can be modeled as networks. Networks are a way to represent how the individual components of a complex system interact. They represent systems composed by a discrete set of objects (e.g. financial institutions) and links connecting pairs of these objects (e.g. interactions, or exposures between institutions). In this thesis I use hypothetical banking systems to model the interbank network. Such hypothetical banking systems are generated from random networks.

I first introduce a few simple definitions and quantities that can be used to characterise the topology of a network. Many other quantities have been introduced in the literature. Here I focus only on those that will be used in this thesis.

Let us denote by N the number of nodes and M the average degree of the network. Let us also define the adjacency matrix as the matrix with elements A_{ij} equal to 1 if there is a link pointing i and j and 0 otherwise.

Types of network: There are mainly two types of networks that are observed: directed and undirected networks. In directed networks, links have direction pointing one node to the other, while for undirected networks, edges do not have direction so they are treated as bi-directional. This means that, in directed network, A connects to B does not necessarily mean B connects to A, but in undirected networks, A and B are both connected to each other. In this thesis, I only consider directed networks, as the direction indicates payment obligation from one bank to the other.

In some networks, not all edges are created equal. Links are often assigned with weights that make them different in strength, intensity or capacity. One simple example is the amount of traffic flows between two points. In this thesis, I use weighted networks to illustrate the amount of payments between nodes. Once I generate A_{ij}, I assign different values on links to get L_{ij}, which is the liability matrix.

Degree: The degree k_i of node i is the number of links of that node. For directed networks, it can be further split to in-degree and out-degree. Obviously, in-degree measures how many incoming links, and out-degree measures how many out-going links. In this thesis, I treat in-coming links as assets and out-going links as liabilities. Furthermore, the degree distribution measures the probability distribution of network degrees. Newman et al. (2001) reviewed three types of degree distribution: Poisson-distributed, exponentially distributed and power-law distributed. The Erdős-Rényi network follows a binomial distributed degree distribution, which can be approximated to Poisson distribution for large number of nodes by central limit theorem. In this thesis, I will mainly look at the Erdős-Rényi network and the scale-free network, which exhibits pow-law degree distribution.

The literature shows that the degree distribution strongly affects the properties of networks, such as their robustness to attacks or the outcome of dynamical processes taking place on them. In Albert et al. (2000) and Crucitti et al. (2004), they all found that scale-free networks are robust to random attacks but vulnerable to target attacks. Reason behind is that because of the power-law property, the majority of nodes is only connected with a few nodes, but those small-degree nodes are more likely to be selected in random attacks. Therefore, the consequence is regarded as manageable. However, for targeted attacks, it is more likely to damage the most connected nodes. Hence, it becomes more vulnerable in this case.

Assortativity: Assortativity measures the tendency of nodes to be connected with nodes with similar degree. Positive assortativity means that high degree nodes tend to connect with high degree nodes and vice versa, while negative assortativity means that high degree nodes tend to connect with low degree nodes. For example, social networks are assortative because highly connected people tend to connect with other highly connected people (Newman, 2002; Capocci et al., 2003). In contrast, empirical studies have shown that interbank networks are disassortative, where banks with few connections tend to form links with highly connected banks (Silva et al., 2016; Hurd, 2016). Assortativity can be measured using the Pearson coefficient of the degree of neighboring nodes:

$$r = \frac{\sum_{ij}(A_{ij} - \frac{1}{2E}k_i k_j)k_i k_j}{\sum_{ij}(A_{ij}\delta_{ij} - \frac{k_i k_j}{2E})k_i k_j}, \tag{3.1}$$

$$\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j, \end{cases}$$

where E represents the total number of edges. The coefficient ranges from -1 to 1. A negative value of r means that there is a negative correlation between the degree of a node and that of its neighbors, hence the network is disassortative. In contrast, a positive value of r implies a positive correlation between the degrees of neighboring nodes, signaling that the network is assortative.

Clustering: While assortativity looks at pairs of nodes, clustering describes the neighbourhood of a node by looking at triangles. If i is connected with both j and l, how likely is j to be connected with l? The clustering coefficient can be used to measure this effect. The clustering of node i can be measured as the ratio between the number of links exiting between the neighbors of node i (i.e. the number of triangles node i belongs to) and the maximum number of links that could possibly exist between the neighbors of i:

$$C_i = \frac{a_i}{k_i(k_i-1)/2} \tag{3.2}$$

where a_i is the number of actual links between neighbors of i, and k_i is the degree of i.

Both assortativity and clustering are defined for undirected networks, but that can still be used to analyse directed networks by ignoring the direction of links in the network. In the case of assortativity, this means considering the total degree of nodes rather than in or out-degree of nodes. In the case of clustering, this means considering all triadic motifs.

3.2 The Erdős-Rényi Network

The random network that is considered first in this thesis is the Erdős-Rényi network (Erdos and Renyi, 1959). It is a random network where any pair of nodes are connected with equal probability p, independently of other edges. I will use in the following the $G(N,p)$ ensemble, which consists of networks with N nodes and linking probability p. If M is the average degree of the network, it follows that $p = \frac{M}{N-1}$ (Newman et al., 2001). Thus the degree distribution follows binominal and can be expressed as (Bollobás, 1981):

$$P(k) = \binom{N-1}{k} p^k (1-p)^{N-1-k} \simeq \frac{M^k e^{-M}}{k!}, \tag{3.3}$$

where the last expression is valid for large values of N, and it shows that in this regime the degree distribution can be approximated as a Poisson distribution with parameter M. In order to ensure the system is well-connected, I only consider the case where the giant component exists. According to Erdos and Renyi (1959), the giant component for the Erdős-Rényi model exists only for average degree greater or equal to 1, otherwise the network may consist of several tree-typed sub-graphs. The situation is similar for directed networks, see Dorogovtsev et al. (2001) for example. Payment networks have been shown in literature (e.g. Soramäki et al.

(2007)) to display a giant component.

The Erdős-Rényi network is neutral in terms of assortativity, since any pair of nodes is connected with the same probability independently of their degree. The expected number of links between neighbors of node i is $\frac{pk_i(k_i-1)}{2}$, while the number of possible links between them is $\frac{k_i(k_i-1)}{2}$. Therefore the average clustering is p. Since $p = M/(N-1)$, for a sparse network in which the average degree is fixed the clustering coefficient tends to zero as $N \to \infty$.

Algorithm 1: Generating the Erdős-Rényi random network

input : N, M

output: A ($N \times N$ adjacency matrix)

$p \leftarrow \frac{M}{N-1}$

$r \leftarrow$ rand()

for $i,\ j \leftarrow 1$ *to* N **do**

  if $i = j$ **then**

    $A(i,j) \leftarrow 0$

  else

    $A(i,j) \leftarrow$Logic Value$(r < d)$

From the properties discussed above, it turns out that the Erdős-Rényi network provides a poor approximation to real interbank networks. In fact, the latter usually display heavy tails in the degree distribution, are disassortative and have a higher clustering coefficient (Boss et al., 2004a; Soramäki et al., 2007).

In the following, I will then extend the analysis to random networks that share these properties of real networks. In particular, I will consider the scale-free network as an example of a network with a heavy-tailed degree distribution, and I will consider Erdős-Rényi networks with non-neutral assortativity.

Before discussing the scale-free model, I briefly mention the configuration model, which is a natural generalization of the Erdős-Rényi network. In the Erdős-Rényi network the average degree M is fixed while leaving all other variables random. In general, other quantities apart from M can also be fixed. This is the idea of the configuration model, where, rather than fixing only the average degree, the degree of each node can be fixed. In order to generate networks from the configuration model, each node is assigned a number of half-edges corresponding to its degree. Half-edges are then randomly matched between nodes. Using the configuration model, one can generate networks with any given degree distribution by drawing the number of half-edges from the target distribution, while leaving all the rest random.

3.3 The Scale-free Network

Empirical studies (Albert et al., 1999; Huberman and Adamic, 1999) have shown that the degree distribution of many real-world networks is right skewed and heavy-tailed, in particular for interbank payment networks (see Chapter 2 for details). Therefore the Erdős-Rényi model is a poor approximation to real networks. A simple network to capture heavy-tailed degree distribution is using power-law distribution, where the degree distribution function should be characterised as:

$$P(k) \sim k^{-\gamma} \text{ with } k > k_{min} \tag{3.4}$$

Networks with power-law degree distributions are called scale-free. k_{min} is the lower bound of the degree, and γ is the tail index controlling the heavy-tailness, where higher γ indicates the heavy-tail effect is less pronounced. In this thesis, the scale free network is generated using the fitness model (Caldarelli et al., 2002), where two vertices are linked with some probability function $f(h_i, h_j)$ under some 'fitness' parameter h. One key advantage of this model is that it considers the fitness of both vertices. I assign equal weights on the fitness of two vertices and normalise it. Therefore, $f(h_i, h_j)$ should be expressed as:

$$f(h_i, h_j) = \frac{h_i h_j}{\sum_k h_k} \tag{3.5}$$

Clearly, since h_i are drawn from power-law distribution, the degree distribution of the network is also power law with the same exponent γ (Caldarelli et al., 2002). In fact, the Erdős-Rényi model is a particular example of above algorithm where the probability function f is taken as a constant. In order to achieve this, I consider the following algorithm.

Algorithm 2: Generate the Erdős-Rényi random network

---

input : N, k_{min}, γ

output: A ($N \times N$ adjacency matrix)

$h \leftarrow \vec{N+1}$ // $N \times 1 (N+1)$ vector

for $i \leftarrow 1$ *to* N **do**

  while $h(i) > \sqrt{k_{min} N}$ **do**

    $h(i) = k_{min} \times (1 - rand())^{\frac{1}{1-\gamma}}$

for $i,\ j \leftarrow 1$ *to* N **do**

  if $rand() < \frac{h(i) \times h(j)}{sum(h)}$ *AND* $i \neq j$ **then**

    $A(i, j) \leftarrow 1$

In Fig.3.1, I present two plots to illustrate how the fitness model works and can produce power-law degree distributed networks. I select a large sample size where the network consists of 10000 nodes, with $\gamma = 2.5$ and $k_{min} = 200$ as heavy-tail is more pronounced for large sample. On the left panel, I show the histogram on values of K. It seems that power law property exists after $K = 500$. Then on the right panel, I plot the log-log graph on the density function of K, together with the theoretical line for same parameters. I observe in the middle part between 500 and 2800 where the power-law property is most significantly pronounced, the theoretical line is nearly parallel to the empirical line. The parallelity implies that these two lines have same gradient, therefore same γ. The only difference in the position occurs because of the constant term in front of the empirical pdf to ensure it has probability 1 over the whole range.

As an example given in empirical study, Garlaschelli and Loffredo (2004) provided a test on world trade web, which models the trade relationship between countries. The authors implemented a fitness model of which the fitness of each vertex (country) depends on its current relative GDP. The probability function f is inspired from Maslov et al. (2004) to preserve the degree sequence with real networks. Garlaschelli and Loffredo (2004) concluded that the fitness model could be used to generate random networks to fit real observations.

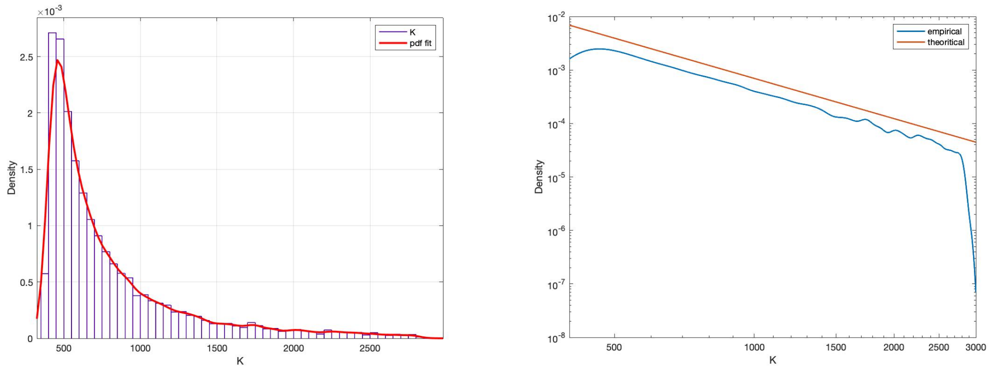


Figure 3.1: Example to verify the fitness model works to produce power-law distributed degree distribution

3.4 Assortativity

I generate networks with assortativity between -1 and 1 using the Monte-Carlo link rewiring process developed by Noh (2007). In particular, consider the cost function (or network Hamiltonian) defined as:

$$H(G) = -\frac{J}{2}\sum_{i,j=1}^{N} A_{ij}k_i k_j, \tag{3.6}$$

where A_{ij} are the elements of the adjacency matrix, k is the degree and J is a control parameter that allows to determine the level of assortativity in the network (positive assortativity corresponds to positive values of J and vice versa).

Starting from the Erdős-Rényi network, one can randomly pick two links between node pairs and propose to swap them, if the selected link has not been selected. The networks before and after the swapping are denoted by G and G'. The swap is accepted with probability 1 if $H(G') < H(G)$, or with probability $e^{H(G)-H(G')}$ if $H(G') > H(G)$. The key idea here is to maintain the objective function $H(G)$ at a stable level. Notice that the edge rewiring process preserves the degree distribution of the network but adjusts the degree correlation. Therefore, using such algorithm, I can generate networks with any value of assortativity but the same degree distribution of Erdős-Rényi networks. The following algorithm gives an example of applying Noh's algorithm using Matlab.

Algorithm 3: Generate random networks with different assortativity

input : A, J, T (number of MC simulations)

output: assortA

assortA $\leftarrow A$

while $T \neq 0$ **do**

  $A \leftarrow$ assortA

  $K \leftarrow sum(assortA)^T + sum(assortA, 2)$

  [i,j] $\leftarrow$ find(assortA) // all nodes with link connected

  link $\leftarrow$ [i j] // combine it into a single matrix

  $link' \leftarrow$ link

  check $\leftarrow 0$

  while *check=0* **do**

    r $\leftarrow$ randperm(length(link)) // randomise all combinations

    pick $\leftarrow$ link(r(1:2),:) // pick first two random links

    check $\leftarrow 1$

    if *pick(1,1)=pick(2,2)* ***OR*** *pick(2,1)=pick(1,2)* **then**

      check $\leftarrow 0$ // exclude self-link

    if *ismember([pick(1,1) pick(2,2)],link,'rows')=1* ***OR*** *ismember([pick(2,1) pick(1,2)],link,'rows')=1* **then**

      check $\leftarrow 0$ // exclude already existed link

  $pick([34]) \leftarrow pick([43])$ // rewire the selected edge

  $link'(r(1:2),:) \leftarrow pick$

  $G \leftarrow$ digraph($link'(:,1)$,$link'(:,2)$) // fit new edge into network

  $A' \leftarrow$ adjacency(G)

  $K' \leftarrow$ sum$(A')^T$+sum(A',2)

  $H \leftarrow -\frac{J}{2}\times$ sum(assortA $\times K. \times K$)

  $H' \leftarrow -\frac{J}{2}\times$ sum($A' \times K'. \times K'$)

  $p \leftarrow \min\left(1, e^{-H'+H}\right)$

  if $rand() < p$ **then**

    assortA=A'

  else

    assortA=A

  $T \leftarrow T - 1$

One important property to test is the number of Monte-Carlo simulations, which should be determined at an appropriate level. If I run the algorithm too many times it would be unnecessarily time-consuming; if the simulation is far more than enough, I cannot achieve required results. Clearly, this number depends on size of the network and degree. I therefore perform the analysis in Fig.3.2, which plots the relationship between assortativity (r) and number of link rewiring steps (T). It can be seen that assortativity becomes stable at around 10000, where coefficients reach stationary state at around ± 0.85 for $J = \pm 1$, indicating that this will be a reasonable number for analysis in the following chapters. Therefore, I will choose 10000 as desired number of simulations.

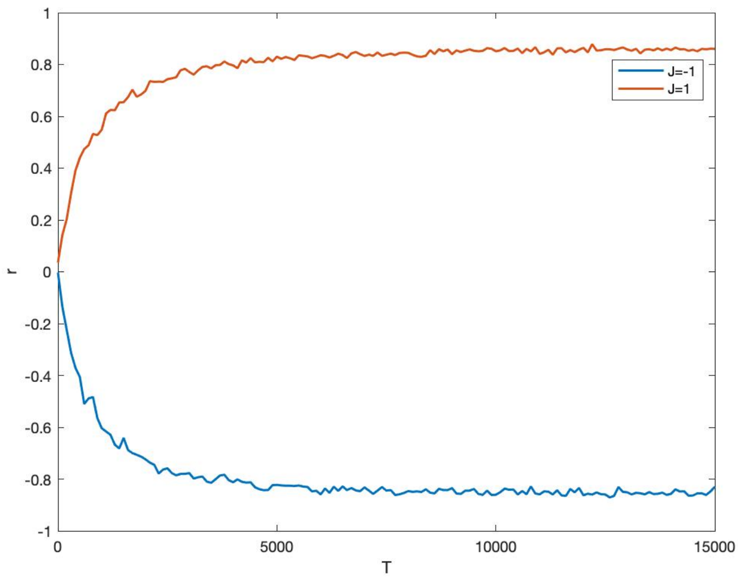


Figure 3.2: Analysis on assortativity against number of MC steps. Starting from the Erdős-Rényi network with $N = 100$ and $M = 5$. $J = \pm 1$ controls the level of assortativity.

In Fig.3.3, I give a simple illustration on how Noh (2007)'s algorithm works. Algorithm to calculate assortativity coefficient is constructed using steps in Newman (2002). I generate an Erdős-Rényi network first with $N = 100$ and $M = 5$, then I perform the link rewiring process for 10000 times. Fig.3.3 clearly shows that r is monotonic increasing with J, indicating that the edge rewiring process works successfully in constructing networks with different assortativity, and J controls the parameter of level of assortativity.

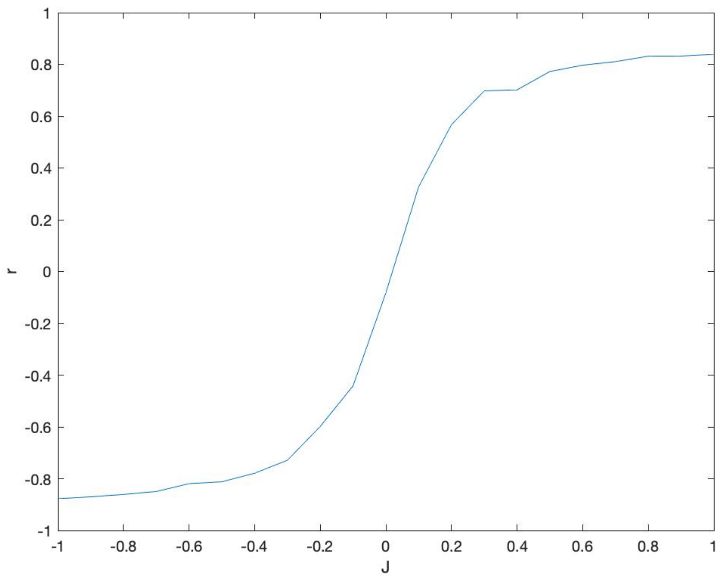


Figure 3.3: Plot of assortativity (r) against control parameter (J) to illustrate the link rewiring process works with $N = 100$ and $M = 5$.

So far, I have explained all random network models I will use in the following chapters. In Table 3.1 I summarise the properties of the different random networks described before, together with their advantages and disadvantages. Since the Erdős-Rényi model is one of the simplest ones without any special properties, I use it as a benchmark in Chapters 5 and 6. On the other hand, because it hardly preserves observed features of real interbank networks, I further generalise my analysis to other two models in Chapter 7. Both the scale-free network and networks with non-neutral assortativity are widely observed in interbank payment networks through empirical studies. However, they do have limitations to consider. For the scale-free network, I have to ensure the sample size (number of nodes) is large enough for heavy-tailness to exist. Moreover, this type of networks also tends to exhibit negative assortativity for small networks (see detailed explanations in Sec.7.3). As for Noh's assortativity network, the most noticeable drawback is the time complexity in executing the algorithm, because it requires more than 10000 iterations of MC simulation, especially when the network gets large as in Sec.7.3.

	Degree distribution	Assortativity	Advantage	Disadvantage
Erdős-Rényi network	Binomial	Neutral	Simple and straightforward	Poorly match real networks
Scale-free network	Heavy-tail	Neutral or disassortative	Match empirical study	Can be disassortative for small network
Assortativity network	Same as the original	Any assortativity level	Match empirical study	Time inefficiency

Table 3.1: Summarty of different random network models

3.5 Eisenberg-Noe Algorithm

This thesis will follow the algorithm proposed by Eisenberg and Noe (Eisenberg and Noe, 2001) and extend it to a more general framework that accounts for the presence of a CCP. In the EN-algorithm, the system consists of N banks, and each bank has exposures to the others, as well as operating cashflow which is given exogenously. The model relies on three fundamental assumptions, which will be used throughout this thesis: (i) limited liability–the bank never pays more than it has; (ii) proportionality of payment–in case of default, a bank pays each counterparty according to the proportion of total exposure associated with it; (iii) priority of payments–interbank exposures have the highest priority and should be paid first.

This section starts reviewing the idea of clearing payment vectors and introducing relevant parameters. Let L_{ij} be the exposure of node i to j. $\bar{p}_i$ represents the total nominal obligation of i to all other nodes in the network, and is defined as

$$\bar{p}_i = \sum_j L_{ij} \tag{3.7}$$

The relative liability matrix Π_{ij} is then defined as the obligation of i to j relative to the total obligation of i:

$$\Pi_{ij} = \begin{cases} \frac{L_{ij}}{\bar{p}_i} & \bar{p}_i > 0 \\ 0 & otherwise \end{cases} \tag{3.8}$$

Therefore, according to the assumption of proportional payments, the total amount received by node i from its counterparties equals:

$$\sum_j \Pi_{ji} p_j^*, \tag{3.9}$$

where I denote by p_j^* the amount of money actually paid by j (this will be equal to $\bar{p}_j$ if j can fulfill its obligations, or smaller if j defaults on its payments).

This implies that the total cash flow to node i is equal to

$$\sum_j \Pi_{ji} p_j^* + e_i \tag{3.10}$$

where e_i is an exogenous income from the operating activity of i and is given exogenously.

Considering now the assumption of limited liability, I can express the payment of node i as:

$$p_i^* = \min\left(\bar{p}_i, \sum_j \Pi_{ji} p_j^* + e_i\right) \tag{3.11}$$

Eisenberg and Noe (2001) proved that there exist a largest and a smallest clearing vector and that they are equal to each other under mild regularity conditions. In particular, a sufficient condition for the solution of Eq.3.11 to be unique is that all nodes have positive cashflows. One way to solve this problem is using numerical simulation when the network is large and complex. In general, a solution of the clearing equations can be found by iterating the following map until a fix point is reached:

$$p_i^{(n)} = \min\left\{\bar{p}_i, \sum_j \Pi_{ji} p_j^{(n-1)} + e_i\right\} \; \forall\, i = 1, \ldots, N, \tag{3.12}$$

where $p_i^{(n)}$ denotes the value of p_i at iteration n. A common starting guess is to set

$p_i^0 = \bar{p}_i$, which would find the greatest clearing vector as fixed point. The following algorithm provides an illustration on how to find the clearing vector using Matlab.

Algorithm 4: Eisenberg-Noe clearing vector

---

input : N, L, e

output: p^*

$\varepsilon \leftarrow 0.001$

$\bar{p} \leftarrow sum(L, 2)$ `// sum over columns of each row`

$\Pi \leftarrow L./\bar{p}[i]$

$p \leftarrow \bar{p}$

for $i \leftarrow 1$ *to* N **do**

  $p^*[i] \leftarrow \min(\bar{p}[i], sum(\Pi[:][i]. \times \bar{p}[i]) + e[i])$

while $p[i] - p^*[i] > \varepsilon$ **do**

  $p \leftarrow p^*$

  for $i \leftarrow 1$ *to* N **do**

    $p^*[i] \leftarrow \min(\bar{p}[i], sum(\Pi[:][i]. \times p[i]) + e[i])$

In the following, I provide a simple example to present how to calculate the clearing vector. The clearing vector will be found in two ways: one in analytical form, followed by the numerical solution.

Example 1. Consider a system consisting of 4 nodes and defined by the following liability matrix and vector of external cashflows

$$L = \begin{bmatrix} 0 & 7 & 1 & 1 \\ 3 & 0 & 3 & 3 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}, e = \begin{bmatrix} 1 \\ 2 \\ 2 \\ 2 \end{bmatrix}$$

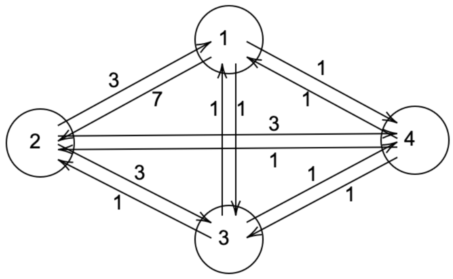


Figure 3.4: Network illustration of this example

The sketch plot in Fig.3.4 presents a weighted directed network to give readers idea about how the network looks like. By Eq.3.7, total obligation of each node should be found by summing up each rows:

$$\bar{p} = \begin{bmatrix} 7+1+1=9 \\ 3+3+3=9 \\ 1+1+1=3 \\ 1+1+1=3 \end{bmatrix}$$

Then according to Eq.3.8, the relative liability matrix Π_{ij} should be expressed as:

$$\Pi = \begin{bmatrix} 0 & \frac{L_{12}}{\bar{p_1}} & \frac{L_{13}}{\bar{p_1}} & \frac{L_{14}}{\bar{p_1}} \\ \frac{L_{21}}{\bar{p_2}} & 0 & \frac{L_{23}}{\bar{p_2}} & \frac{L_{24}}{\bar{p_2}} \\ \frac{L_{31}}{\bar{p_3}} & \frac{L_{32}}{\bar{p_3}} & 0 & \frac{L_{34}}{\bar{p_3}} \\ \frac{L_{41}}{\bar{p_4}} & \frac{L_{42}}{\bar{p_4}} & \frac{L_{43}}{\bar{p_4}} & 0 \end{bmatrix} = \begin{bmatrix} 0 & \frac{7}{9} & \frac{1}{9} & \frac{1}{9} \\ \frac{1}{3} & 0 & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \end{bmatrix}$$

I start noticing that node 3 and node 4 never default. This is because their states do not depend on payments from nodes 1 and 2. For instance, regarding node 3, it has obligation $\bar{p_3} = 3$ and $e_3 = 2$. If it receives \$1 from node 4, then it is sufficient to pay its \$3 obligation, and vice versa for node 4.

Then I turn my attention to nodes 1 and 2. Since the clearing payments can never exceed payment obligations (Eq.3.11) I have that:

$$p_1^* = \min\left(\bar{p_1}, \Pi_{21} p_2^* + \Pi_{31} p_3^* + \Pi_{41} p_4^* + e_1\right)$$

$$
\begin{aligned}
&\leq \Pi_{21}\bar{p}_2 + \Pi_{31}\bar{p}_3 + \Pi_{41}\bar{p}_4 + e_1 \\
&= \frac{1}{3} \times 9 + \frac{1}{3} \times 9 + 1 + 1 + 1 = 6,
\end{aligned}
$$

where the inequality follows because the latter expression is computed considering the highest amount node 1 can receive from its counterparties. Clearly node 1 will default under this set-up as it can pay out at most \$6. Then as a result, the payment of node 2 should follow:

$$
\begin{aligned}
p_2^* &= \min\left(\bar{p}_2, \Pi_{12}p_1^* + \Pi_{32}p_3^* + \Pi_{42}p_4^* + e_2\right) \\
&\leq \Pi_{12}p_1^* + \Pi_{32}\bar{p}_3 + \Pi_{42}\bar{p}_4 + e_2 \\
&= \frac{7}{9} \times 6 + 1 + 1 + 2 = \frac{26}{3}
\end{aligned}
$$

Node 2 also defaults as a result of the default of node 1, financial contagion happens. Since I have discussed that node 3 and 4 always pay out full obligations, the remaining work will become determine appropriate values of p_1^* and p_2^*. From Π I can see node 1 receives $\frac{1}{3}$ of node 2's payment, and node 2 receives $\frac{7}{9}$ of node 1's payment, I can summarise the relationship in following equations:

$$
\begin{cases}
p_1^* = \frac{1}{3}p_2^* + 3 \\
p_2^* = \frac{7}{9}p_1^* + 4
\end{cases}
$$

By solving this linear system, I eventually get $p_1^* = 5.85$ and $p_2^* = 8.55$. Therefore, the full clearing vector should be:

$$
p^* = \begin{bmatrix} 5.85 \\ 8.55 \\ 3 \\ 3 \end{bmatrix}
$$

In the following I will present how Eq.3.12 works. In the first iteration I assume that there is no default in the system and everyone pay out $\bar{p}$.

$$\begin{cases} p_1^{(1)} = \min\left(\bar{p_1}, \sum_j \Pi_{j1}\bar{p_j} + e_1\right) = \min\left(9, \frac{1}{3}\times 9 + \frac{1}{3}\times 3 + \frac{1}{3}\times 3 + 1\right) = 6 \\ p_2^{(1)} = \min\left(\bar{p_2}, \sum_j \Pi_{j2}\bar{p_j} + e_2\right) = \min\left(9, \frac{7}{9}\times 9 + \frac{1}{3}\times 3 + \frac{1}{3}\times 3 + 2\right) = 9 \\ p_3^{(1)} = \min\left(\bar{p_3}, \sum_j \Pi_{j3}\bar{p_j} + e_3\right) = \min\left(3, \frac{1}{9}\times 9 + \frac{1}{3}\times 9 + \frac{1}{3}\times 3 + 2\right) = 3 \\ p_4^{(1)} = \min\left(\bar{p_4}, \sum_j \Pi_{j4}\bar{p_j} + e_4\right) = \min\left(3, \frac{1}{9}\times 9 + \frac{1}{3}\times 9 + \frac{1}{3}\times 3 + 2\right) = 3 \end{cases}$$

Notice that node 1 defaults as it pays out smaller than its obligation. As a consequence, other nodes can only receive the updated amount (\$6) rather than the original amount (\$9) from node 1. Therefore, I should implement Eq.3.12 again:

$$\begin{cases} p_1^{(2)} = \min\left(\bar{p_1}, \sum_j \Pi_{j1}p_j^{(1)} + e_1\right) = \min\left(9, \frac{1}{3}\times 9 + \frac{1}{3}\times 3 + \frac{1}{3}\times 3 + 1\right) = 6 \\ p_2^{(2)} = \min\left(\bar{p_2}, \sum_j \Pi_{j2}p_j^{(1)} + e_2\right) = \min\left(9, \frac{7}{9}\times 6 + \frac{1}{3}\times 3 + \frac{1}{3}\times 3 + 2\right) = \frac{26}{3} \\ p_3^{(2)} = \min\left(\bar{p_3}, \sum_j \Pi_{j3}p_j^{(1)} + e_3\right) = \min\left(3, \frac{1}{9}\times 6 + \frac{1}{3}\times 9 + \frac{1}{3}\times 3 + 2\right) = 3 \\ p_4^{(2)} = \min\left(\bar{p_4}, \sum_j \Pi_{j4}p_j^{(1)} + e_4\right) = \min\left(3, \frac{1}{9}\times 6 + \frac{1}{3}\times 9 + \frac{1}{3}\times 3 + 2\right) = 3 \end{cases}$$

Notice that node 2 defaults as a consequence of node 1's default. This is primarily because it receives few incoming payment from node 1. This is what I called financial contagion, where the default of one institution is triggered by the default of the other institution. Therefore, the fixed point is not reached as there is new default agent in the system. I will implement Eq.3.12 further:

$$\begin{cases} p_1^{(3)} = \min\left(\bar{p_1}, \sum_j \Pi_{j1}p_j^{(2)} + e_1\right) = \min\left(9, \frac{1}{3}\times \frac{26}{3} + \frac{1}{3}\times 3 + \frac{1}{3}\times 3 + 1\right) = \frac{53}{9} \\ p_2^{(3)} = \min\left(\bar{p_2}, \sum_j \Pi_{j2}p_j^{(2)} + e_2\right) = \min\left(9, \frac{7}{9}\times 6 + \frac{1}{3}\times 3 + \frac{1}{3}\times 3 + 2\right) = \frac{26}{3} \\ p_3^{(3)} = \min\left(\bar{p_3}, \sum_j \Pi_{j3}p_j^{(2)} + e_3\right) = \min\left(3, \frac{1}{9}\times 6 + \frac{1}{3}\times \frac{26}{3} + \frac{1}{3}\times 3 + 2\right) = 3 \\ p_4^{(3)} = \min\left(\bar{p_4}, \sum_j \Pi_{j4}p_j^{(2)} + e_4\right) = \min\left(3, \frac{1}{9}\times 6 + \frac{1}{3}\times \frac{26}{3} + \frac{1}{3}\times 3 + 2\right) = 3 \end{cases}$$

One good thing to point out that at this time step, there is no new defaults. However, because payment of node 1 changes again, I need to further iterate the algorithm. By proceeding with further iterations, the changes between two iterations become smaller. It seems that the value is converging. When calculating the payment using recursion, I eventually find that the clearing vector should be:

$$p^* = \begin{bmatrix} 5.85 \\ 8.55 \\ 3 \\ 3 \end{bmatrix}$$

Clearly the numerical method provides a more general approach to find the clearing vector, especially when the network is large and an analytical solution is difficult to find. In the following of this thesis, I will perform all analysis using this fixed point algorithm.

In this chapter, I have described all fundamental models and methods I will be using in the following of this thesis. In the next chapter, I will further explain the clearing mechanism on how to apply the EN algorithm in the presence of a CCP.

Chapter 4

Clearing Mechanism

I consider a system of N financial institutions, which I henceforth refer to banks for simplicity. Between bank i and j, there are several derivative contracts. I then assume that, following a shock, the value of some contracts between i and j changes and that, as a consequence, VM calls must be posted. In this thesis, I do not investigate the specific nature of the shock, I only focus on the effect in terms of margin calls. The liability matrix $\mathbf{L_{ij}}$ is similar to what Eisenberg and Noe (2001) did where i is the borrower and j is the lender, and in this thesis, it represents the system after the shock. Here I do not consider IMs, which is often referred to margin requirements prior to the contract is settled. It is usually calculated using risk models, among which the most popular and widely-used ones are described as procyclical: margin requirements are lower at bull market and higher at bear market (Murphy et al., 2014). In this thesis, because I assume the CCP always pays out its full obligation, it is therefore equivalent to say that IMs are always sufficient for the CCP to fill the gap in case any clearing member records a shortfall.

VM obligations are settled following a specific sequencing of payments that follows market protocols (Bardoscia et al., 2021). First, banks pay their VM obligations to the CCP. Next, the CCP pays its VM obligations to banks. Finally, banks settle the bilateral VM obligations. At the end of each of these three payment rounds, the institutions that are not able to pay their VM obligations in full record a shortfall, which is the central quantity of this analysis.

In practice institutions might take a mix of remedial actions to source the cash needed to cover their shortfalls, such as borrowing on the repo market or selling illiquid assets. Here I do not make specific assumptions on which remedial actions are taken by institutions, and therefore I do not quantify the downstream impact on interest rates in funding markets or on asset prices, for instance fire sales. To that effect one could use the demand for cash, which is the input of my model. However, I do assume that institutions take *some* remedial actions if they do not have enough cash to cover their payment obligations.

There are in summary three types of clearing markets according to my framework: fully bilaterally cleared market, fully centrally cleared market and mixed market.

4.1 Fully Bilaterally Cleared Market

This is the most common-seen mechanism before implementing CCPs. I denote with $\mathbf{L}^b$ the matrix of gross bilateral VM obligations, prior to any netting. Both L^b_{ij} and L^b_{ji} can be strictly larger than zero. Banks do not have VM obligations to themselves, i.e. $L^b_{ii} = 0$, for all i.

In this case VM obligations are netted independently for each pair of institutions. By denoting with L_{ij} the net VM obligation that i owes to j, I have:

$$L_{ij} = (L^b_{ij} - L^b_{ji})^+ , \tag{4.1}$$

where $(\dots)^+$ is the positive part. Clearly, if $L_{ij} > 0$, then $L_{ji} = 0$, and vice versa, i.e. when i owes to j, j does not owe to i, and vice versa.

4.2 Fully Centrally Cleared Market

For each centrally cleared contract, the CCP interposes itself between two banks, say i and j. I denote with $\mathbf{L}^c$ the matrix of gross centrally cleared VM obligations between underlying counterparties, i.e. L^c_{ij} is the VM obligation that i owes to the CCP arising from centrally cleared contracts for which the CCP has interposed between i and j, prior to any netting. Even though all VM payment obligations on centrally cleared contracts are to be paid to the CCP, or to be received from the CCP, for brevity I often refer to L^c_{ij} as to the gross VM payment obligation on centrally cleared contracts that i owes to j. In general, since some contracts are "in-the-money" for i and some for j, I have that both L^c_{ij} and L^c_{ji} can be strictly larger than zero. I assume that banks do not have VM obligations to themselves, meaning that $L^c_{ii} = 0$, for all i.

The CCP performs multilateral netting. This means that, for each bank, the CCP offsets VM obligations due to be paid to and received from all other banks. If bank i's net VM obligations $\sum_j (L^c_{ij} - L^c_{ji})$ are positive, i has a VM obligation to the CCP. Otherwise, the CCP has a VM obligation to bank i. By denoting with $\bar{p}_{i\to CCP}$ the net VM obligation that i owes to the CCP and with $\bar{p}_{CCP\to i}$ the net VM obligation that the CCP owes to i, I have:

$$\bar{p}_{i\to CCP} = \left(\sum_j L^c_{ij} - L^c_{ji}\right)^+ \tag{4.2a}$$

$$\bar{p}_{CCP\to i} = \left(\sum_j L^c_{ji} - L^c_{ij}\right)^+ , \tag{4.2b}$$

If $\bar{p}_{i\to CCP} > 0$, then $\bar{p}_{CCP\to i} = 0$, and vice versa, i.e. when i owes to the CCP, the CCP does not owe to i, and vice versa.

4.3 Mixed market

My main objective is to compare financial systems in which the amount of contracts that are centrally cleared varies. To this end I assume that the gross VM obligations L^{tot}_{ij} that i owes to j equal to the convex combination:

$$L^{tot}_{ij} = \alpha L^c_{ij} + (1-\alpha) L^b_{ij}, \tag{4.3}$$

with $0 \leq \alpha \leq 1$. $\alpha = 0$ corresponds to the case of the fully bilateral market, whereas $\alpha = 1$ to the case of the fully centrally cleared market. Eq. 4.3 allows to interpolate between these two cases with a single and easily interpretable parameter. When $0 < \alpha < 1$, the gross centrally cleared VM obligations of i to j equal to αL^c_{ij}, while the gross bilateral VM obligations of i to j equal to $(1-\alpha) L^b_{ij}$.

Net VM obligations are obtained from Eq.4.2 and Eq.4.1 by performing the substitutions $L^c_{ij} \to \alpha L^c_{ij}$ and $L^b_{ij} \to (1-\alpha)L^b_{ij}$:

$$\bar{p}_{i\to CCP} = \alpha\left(\sum_j L^c_{ij} - L^c_{ji}\right)^+ \tag{4.4a}$$

$$\bar{p}_{CCP\to i} = \alpha\left(\sum_j L^c_{ji} - L^c_{ij}\right)^+ \tag{4.4b}$$

$$\bar{p}_{ij} = (1-\alpha)(L^b_{ij} - L^b_{ji})^+ . \tag{4.4c}$$

To summarise, this scenario can be defined as:

Definition 1 (Mixed clearing system)**.** *Let N (the number of banks) be a strictly positive integer and let:*

- $\mathbf{L}^b$ *(the matrix of gross VM bilateral obligations) and* $\mathbf{L}^c$ *(the matrix of gross VM centrally cleared obligations) be two $n \times n$ matrices such that $L^b_{ij} \geq 0$, $L^c_{ii} \geq 0$, and $L^b_{ii} = L^c_{ii} = 0$, for all i and j.*

- $\mathbf{e}^{(1)}$ *(the cash endowments at the beginning of the first stage) be a vector of length N, such that $e_i \geq 0$, for all i.*

- $\alpha \in [0,1]$ *(the parameter interpolating between centrally cleared and bilateral gross VM obligations).*

Then the tuple $S(\mathbf{L}^b, \mathbf{L}^c, \mathbf{e}^{(1)}, \alpha)$ is a mixed clearing system.

The mixed clearing system defined above has total obligation as in Eq.4.3 and net CCP obligations as in Eq.4.2. However, I restrict the model to financial systems in which, for each bank i, total gross VM obligations L^{tot}_i are independent of α. This allows me to avoid any bias due to the difference in levels of bilateral and centrally cleared VM obligations and to focus on their relative importance. From Eq. (4.3) I have:

$$
\begin{aligned}
L_i^{tot} = \sum_j L_{ij}^{tot} &= \alpha \sum_j L_{ij}^c + (1-\alpha) \sum_j L_{ij}^b \\
&= \sum_j L_{ij}^b + \alpha \left(\sum_j L_{ij}^c - \sum_j L_{ij}^b \right) ,
\end{aligned}
\tag{4.5}
$$

which shows that this is possible if and only if $\sum_j L_{ij}^c = \sum_j L_{ij}^b$, for all i. In this case, the total gross VM obligations of bank i that are centrally cleared equal to αL_i^{tot}, whereas the bilateral ones equal to $(1-\alpha)L_i^{tot}$. Therefore, another clearing system is introduced:

Definition 2 (Balanced clearing system)**.** *Let $S(\mathbf{L}^b, \mathbf{L}^c, \mathbf{e}^{(1)}, \alpha)$ be a mixed clearing system such that:*

$$
\sum_j L_{ij}^b = \sum_j L_{ij}^c ,
$$

for all i. Then the tuple $S(\mathbf{L}^b, \mathbf{L}^c, \mathbf{e}^{(1)}, \alpha)$ is a balanced clearing system.

In this thesis, I am trying to understand how α affects the clearing efficiency. In the following of this thesis, I will explore two situations: the general case of independent exposures where $\mathbf{L}^c \neq \mathbf{L}^b$ and the special case of perfectly correlated exposures where $\mathbf{L}^b = \mathbf{L}^c = \mathbf{L}$. One way to justify perfectly correlated exposures is the following. I start from an initial state in which all contracts are bilateral ($\alpha = 0$) and the matrix of gross bilateral VM obligations is $\mathbf{L}^b = \mathbf{L}$, and I further assume that all banks novate contracts by replacing new contracts to the updated fraction with all their counterparties corresponding to a fraction α of their notional *at the same time*. At the end of this process the matrix of gross centrally cleared VM obligations would be $\alpha\mathbf{L}$ and the matrix of gross bilateral VM obligations would be $(1-\alpha)\mathbf{L}$, resulting in perfectly correlated exposures. Moreover, if one wants to compare results at α with results at $\alpha' = \alpha + \Delta\alpha$, one has to assume again that all banks novate a fraction $\Delta\alpha$ of notional of contracts with all their counterparties at the same time. Therefore, it is convenient to introduce the following shorthand notation.

Definition 3 (Clearing system with perfectly correlated exposures)**.** *Let* $S(\mathbf{L}^b, \mathbf{L}^c, \mathbf{e}^{(1)}, \alpha)$ *be a mixed clearing system such that:*

$$\mathbf{L}^b = \mathbf{L}^c = \mathbf{L}\,.$$

Then the tuple $S(\mathbf{L}, \mathbf{e}^{(1)}, \alpha)$ *is a clearing system with perfectly correlated exposures.*

4.4 Payments

I assume that each bank i is initially endowed with $e_i^{(1)}$ units of cash. In the first payment round banks pay the CCP. Since they have not received any other payment, banks can only rely on their initial cash endowment. If this is larger than their VM obligation to the CCP, banks immediately pay the CCP in full. Otherwise, without taking any further action, they can pay only up to their initial cash endowment:

$$p_{i\to CCP} = \min(e_i^{(1)}, \bar{p}_{i\to CCP})\,. \tag{4.6}$$

The shortfall of bank i at the end of the first round, i.e. the shortfall recorded by i on its centrally cleared VM obligations, is denoted with s_i^c and is defined as the difference between the VM obligation and payment that i can make without taking further actions:

$$\begin{aligned} s_i^c &= \bar{p}_{i\to CCP} - p_{i\to CCP} \\ &= (\bar{p}_{i\to CCP} - e_i^{(1)})^+\,. \end{aligned} \tag{4.7}$$

Importantly, I assume that banks that record a shortfall in the first round do take some actions to source the corresponding amount of cash. This means that, after taking action, they are able to pay the CCP in full. By doing so they use both their initial endowment and the additional amount of cash sourced. Hence, their cash at the beginning of the second round is equal to zero. Banks that do not record a shortfall in the first round do not take further actions. As a consequence, their cash at the beginning of the second round is their initial cash endowment minus the payment made in the first round, i.e. their VM obligation to the CCP. Putting both cases together I have:

$$e_i^{(2)} = e_i^{(1)} - p_{i\to CCP}\,. \tag{4.8}$$

In the second payment round the CCP pays the banks. Notice that the CCP has a perfectly matched trading book, meaning that for each incoming VM obligation from a bank there is a matching outgoing VM obligation to another bank. Therefore, the CCP's total outgoing VM obligation is equal to its total incoming VM obligations:

$$\sum_i \bar{p}_{i\to CCP} = \sum_i \bar{p}_{CCP\to i}\,. \tag{4.9}$$

Since all banks have paid the CCP in full (after taking actions), at the end of the first round the CCP has received a total amount of cash equal to $\sum_i \bar{p}_{i\to CCP}$. This means that the CCP always has enough cash to pay all banks in full, regardless of its initial cash endowment and without taking further actions. Therefore, no shortfalls are recorded in the second round. Banks' cash at the beginning of the third round is equal to their cash at the beginning of the second round plus the payments received from the CCP:

$$\begin{aligned} e_i^{(3)} &= e_i^{(2)} + \bar{p}_{CCP\to i} \\ &= e_i^{(1)} - p_{i\to CCP} + \bar{p}_{CCP\to i}\,. \end{aligned} \tag{4.10}$$

In the third payment round banks settle their bilateral VM obligations by using the Eisenberg and Noe model as in Eq.3.11, which allows me to compute clearing payments under the assumptions of i) limited liabilities, ii) proportionality of payments and iii) priority of debt over equity. I denote the total bilateral VM obligations of bank i with $\bar{p}_i$:

$$\bar{p}_i = \sum_j L_{ij} = (1-\alpha)\sum_j (L^b_{ij} - L^b_{ji})^+ \tag{4.11}$$

and the relative liability matrix should follow as in Eq.3.8:

$$\Pi_{ij} = \begin{cases} \frac{L_{ij}}{\bar{p}_i} = \frac{(1-\alpha)(L^b_{ij}-L^b_{ji})^+}{(1-\alpha)\sum_k (L^b_{ik}-L^b_{ki})^+} = \frac{(L^b_{ij}-L^b_{ji})^+}{\sum_k (L^b_{ik}-L^b_{ki})^+} & \text{if } \bar{p}_i > 0 \\ 0 & \text{otherwise} \end{cases}, \tag{4.12}$$

According to the EN model in Eq.3.11, the clearing payment of bank i is determined by the equilibrium condition:

$$p_i^* = \min\left(\bar{p}_i, e_i^{(3)} + \sum_j \Pi_{ji} p_j^*\right), \tag{4.13}$$

while individual payments are $L = \Pi_{ij} p_i$. Also in this round p_i is the payment that bank i can make without taking any further actions. It is worth to point out that in Eisenberg and Noe (2001), Eq.4.13 has unique solution only if $e_i^{(3)} > 0, \forall i$. However, because after first two sequences of payments, the value of $e_i^{(3)}$ cannot be guaranteed to be larger than 0. Here I focus on the greatest solution using the algorithm presented in Algo.4. The idea is that the sequence of payments is monotonic non-increasing so can be used to track following payment rounds. Start from assuming all firms are able to make full payments in the first iteration, their might exist some firms record liquidity shortfalls even though all other counterparties pay out in full. In this case, firms that are able to make full payments previously might not be able to do so in the following iterations, while firms record shortfalls at beginning still cannot make higher payments compare to before.

The shortfall of bank i at the end of the third round, i.e. the shortfall recorded by i on its bilateral VM obligations, is denoted with s_i^b and is defined as the difference between its bilateral VM obligations and its clearing payment without taking any further actions:

$$s_i^b = \bar{p}_i - p_i^*. \tag{4.14}$$

The shortfall of bank i is simply defined as the sum of shortfalls recorded in the first and third payment round, i.e. on both centrally cleared and bilateral VM obligations:

$$s_i = s_i^c + s_i^b. \tag{4.15}$$

When I will compare quantities for different values of α I explicitly indicate their dependence on α, e.g. $s_i(\alpha)$. Since the shortfall measures how much external cash required to fulfil payment obligations, I would conclude that for lower value of shortfall, the payment system is more efficient.

L^b_{ij}	L^c_{ij}	Π_{ij}	α
Bilateral liability matrix	Central liability matrix	Relative liability	Percentage of central clearing
e_i	$\bar{p}_i$	$\bar{p}_{i\to CCP}$	p^*_i
Exogenous cashflow	Bilateral total obligation	Central clearing obligation	Bilateral clearing payment
$p_{i\to CCP}$	s^b_i	s^c_i	s_i
Payment to the CCP	Shortfall bilateral	Shortfall central payment	Total shortfall

Table 4.1: Summary of key variables

The table above summarises all essential variables I will use in the following chapters. In this chapter, I have explained three different clearing mechanism and introduced the liquidity shortfall. In the following I will focus on the mixed system in which payments are partially centrally cleared and partially bilateral cleared. I will study the behaviour of the liquidity shortfall as a function of different parameters, and compare results related to different interbank payment topologies.

Chapter 5

Independent Exposures

I present results in the following three chapters under three frameworks: Erdős-Rényi networks with independent exposures, Erdős-Rényi networks with perfectly correlated exposures, and scale-free and/or non-neutral assortativity networks with perfectly correlated exposures. One may argue that, after introducing the CCP, the network structure would be different from the original one. Here, I only consider the netting effect imposed by the CCP and its further impact on bilateral payments. Hence, the remaining network on bilateral payments still follows the original one. In this chapter I discuss the general case of independent exposures where $\mathbf{L}^b \neq \mathbf{L}^c$, i.e. exposures are not necessarily correlated and indeed no further assumption is made on the correlation between $\mathbf{L}^c$ and $\mathbf{L}^b$. I generate two independent liability matrices with same parameters to ensure they do not interplay with each other. While some results on liquidity shortfalls from centrally cleared obligations can be proved, liquidity shortfalls from bilateral (and therefore also from total) obligations need to be investigated numerically in the general case.

5.1 General Result

I start by noting that, for all banks, net centrally cleared VM obligations are increasing in α (see Eq.4.4a). From Eq.4.6:

$$p_{i \to CCP} = \begin{cases} \alpha \left(\sum_j L^c_{ij} - L^c_{ji} \right)^+ & \text{for } \alpha \leq \frac{e_i^{(1)}}{\left(\sum_j L^c_{ij} - L^c_{ji}\right)^+} \\ e_i^{(1)} & \text{for } \alpha > \frac{e_i^{(1)}}{\left(\sum_j L^c_{ij} - L^c_{ji}\right)^+} \end{cases} \tag{5.1}$$

meaning that payments of all banks to the CCP increase linearly in α up to $\alpha = e_i^{(1)} / \left(\sum_j L_{ij}^c - L_{ji}^c\right)^+$ and that they saturate to $e_i^{(1)}$ for larger values of α. As a consequence, the shortfall towards the CCP should be expressed as:

$$s_i^c = \begin{cases} 0 & \text{for } \alpha \leq \frac{e_i^{(1)}}{\left(\sum_j L_{ij}^c - L_{ji}^c\right)^+} \\ \alpha \left(\sum_j L_{ij}^c - L_{ji}^c\right)^+ - e_i^{(1)} & \text{for } \alpha > \frac{e_i^{(1)}}{\left(\sum_j L_{ij}^c - L_{ji}^c\right)^+}. \end{cases} \tag{5.2}$$

Those observations are summarised in the following proposition.

Proposition 1. *Let $S(\mathbf{L}^c, \mathbf{L}^b, \mathbf{e}^{(1)})$ be a mixed clearing system. For all banks, net centrally cleared VM obligations, (sequenced) payments to the CCP, and shortfalls on centrally cleared VM obligations are non-decreasing functions of α, the fraction of centrally cleared notional.*

From Eq.4.11 it shows that for all banks total bilateral VM obligations are decreasing in α. However, from Eq.4.10, Eq.4.4a, and Eq.4.6 I have:

$$\begin{aligned} e_i^{(3)} &= e_i^{(1)} - \min\left(e_i^{(1)}, \alpha\left(\sum_j L_{ij}^c - L_{ji}^c\right)^+\right) + \alpha\left(\sum_j L_{ij}^c - L_{ji}^c\right)^- \\ &= \begin{cases} e_i^{(1)} - \min\left(e_i^{(1)}, \alpha\left(\sum_j L_{ij}^c - L_{ji}^c\right)^+\right) & \text{for } \sum_j L_{ij}^c - L_{ji}^c \geq 0 \\ e_i^{(1)} + \alpha\left(\sum_j L_{ij}^c - L_{ji}^c\right)^- & \text{for } \sum_j L_{ij}^c - L_{ji}^c < 0. \end{cases} \end{aligned} \tag{5.3}$$

If bank i has positive net centrally cleared VM obligations, i.e. $\sum_j L_{ij}^c - L_{ji}^c \geq 0$, it will not have any incoming payments from the CCP. Therefore, the cash left is:

$$e_i^{(3)} = \begin{cases} e_i^{(1)} - \alpha\left(\sum_j L_{ij}^c - L_{ji}^c\right) & \text{for } \alpha \leq \frac{e_i^{(1)}}{\left(\sum_j L_{ij}^c - L_{ji}^c\right)} \\ 0 & \text{for } \alpha > \frac{e_i^{(1)}}{\left(\sum_j L_{ij}^c - L_{ji}^c\right)}, \end{cases} \tag{5.4}$$

meaning that $e_i^{(3)}$ is non-increasing with α. Instead, if bank i has negative net centrally cleared VM obligations, i.e. $\sum_j L^c_{ij} - L^c_{ji} < 0$ I have:

$$\begin{aligned} e_i^{(3)} &= e_i^{(1)} - \alpha\Big(\sum_j L^c_{ij} - L^c_{ji}\Big) \\ &= e_i^{(1)} + \alpha\Big|\sum_j L^c_{ij} - L^c_{ji}\Big|, \end{aligned} \tag{5.5}$$

meaning that $e_i^{(3)}$ is increasing with α. The fact that $e_i^{(3)}$, the cash available to pay VM bilateral obligations, is non-increasing with α for some banks and increasing with α for others prevents from applying results on comparative statics of payments in the EN model (Eisenberg and Noe, 2001). Therefore, it is not straightforward to derive the behaviour of bilateral payments and shortfalls.

In the following example, I construct a framework to present how I clear interbank system with independent exposures, where $\mathbf{L}^b \neq \mathbf{L}^c$ are given randomly. Different from Example 1, I introduce two step netting as described before, where the bilateral liability is netted first, then the CCP performs multi-lateral netting on aggregate exposures. In addition, I also consider the payment sequencing where payments towards the CCP have highest priority thus should be cleared first. After that, bilateral obligations are settled using remaining cashflows.

Example 2. Consider the following set-up with liability matrices and cashflow are given by:

$$L^b = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & \frac{3}{2} & \frac{3}{2} \\ 3 & 3 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}, L^c = \begin{bmatrix} 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{4}{3} & 0 & \frac{1}{3} & \frac{1}{3} \\ 2 & 2 & 0 & 0 \\ 5 & 5 & 0 & 0 \end{bmatrix}, e = \begin{bmatrix} 3 \\ 5 \\ 1 \\ 4 \end{bmatrix}$$

Then liability matrices after netting should be found as:

$$L^b - L^{b'} = \left(\begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & \frac{3}{2} & \frac{3}{2} \\ 3 & 3 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 3 & 0 \\ 1 & 0 & 3 & 0 \\ 1 & \frac{3}{2} & 0 & 0 \\ 1 & \frac{3}{2} & 3 & 0 \end{bmatrix}\right)^+ = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & \frac{3}{2} \\ 2 & \frac{3}{2} & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$L^c - L^{c\prime} = \left(\begin{bmatrix} 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{4}{3} & 0 & \frac{1}{3} & \frac{1}{3} \\ 2 & 2 & 0 & 0 \\ 5 & 5 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & \frac{4}{3} & 2 & 5 \\ \frac{1}{3} & 0 & 2 & 5 \\ \frac{1}{3} & \frac{1}{3} & 0 & 0 \\ \frac{1}{3} & \frac{1}{3} & 0 & 0 \end{bmatrix} \right)^+ = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ \frac{5}{3} & \frac{5}{3} & 0 & 0 \\ \frac{14}{3} & \frac{14}{3} & 0 & 0 \end{bmatrix}$$

Consider the case of $\alpha = 0.1$ first. It means that I clearer 10% transactions centrally and 90% transactions bilaterally. The liabiliy matrices should follow:

$$0.9\left(L^b - L^{b\prime}\right) = 0.9 \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & \frac{3}{2} \\ 2 & \frac{3}{2} & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & \frac{9}{10} & 0 & \frac{9}{10} \\ 0 & 0 & 0 & \frac{27}{20} \\ \frac{9}{5} & \frac{27}{20} & 0 & \frac{27}{10} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$0.1\left(L^c - L^{c\prime}\right) = 0.1 \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ \frac{5}{3} & \frac{5}{3} & 0 & 0 \\ \frac{14}{3} & \frac{14}{3} & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ \frac{1}{10} & 0 & 0 & 0 \\ \frac{1}{6} & \frac{1}{6} & 0 & 0 \\ \frac{7}{15} & \frac{7}{15} & 0 & 0 \end{bmatrix}$$

Referring to Eq.4.4a, the CCP performs second step netting, where aggregate exposures on L^c are netted. Then obligations towards the CCP should be:

$$\begin{cases} \bar{p}_{1\to CCP} = \alpha\left(\sum_j L^c_{1j} - L^c_{j1}\right)^+ = 0.1\left(0 - \left(\frac{1}{10} + \frac{1}{6} + \frac{7}{15}\right)\right)^+ = 0 \\ \bar{p}_{2\to CCP} = \alpha\left(\sum_j L^c_{2j} - L^c_{j2}\right)^+ = 0.1\left(\frac{1}{10} - \left(\frac{1}{6} + \frac{7}{15}\right)\right)^+ = 0 \\ \bar{p}_{3\to CCP} = \alpha\left(\sum_j L^c_{3j} - L^c_{j3}\right)^+ = 0.1\left(\frac{1}{6} + \frac{1}{6} - 0\right)^+ = \frac{1}{3} \\ \bar{p}_{4\to CCP} = \alpha\left(\sum_j L^c_{4j} - L^c_{j4}\right)^+ = 0.1\left(\frac{7}{15} + \frac{7}{15} - 0\right)^+ = \frac{14}{15} \end{cases}$$

Using Eq.4.11, the bilateral obligations between banks can be found as $\bar{p}_i = \sum_j 0.9\left(L^b_{ij} - L^b_{ji}\right)$:

$$\bar{p}_i = \begin{bmatrix} \frac{9}{10} + \frac{9}{10} \\ \frac{27}{20} \\ \frac{9}{5} + \frac{27}{20} + \frac{27}{10} \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{9}{5} \\ \frac{27}{20} \\ \frac{117}{25} \\ 0 \end{bmatrix}$$

Referring to Eq.4.6, payments towards the CCP should be found as:

$$p_{i\to CCP} = \begin{bmatrix} \min(e_1^{(1)}, \bar{p}_{1\to CCP}) \\ \min(e_2^{(1)}, \bar{p}_{2\to CCP}) \\ \min(e_3^{(1)}, \bar{p}_{3\to CCP}) \\ \min(e_4^{(1)}, \bar{p}_{4\to CCP}) \end{bmatrix} = \begin{bmatrix} \min(3,0) \\ \min(5,0) \\ \min(1,\frac{1}{3}) \\ \min(4,\frac{14}{15}) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \frac{1}{3} \\ \frac{14}{15} \end{bmatrix}$$

In the above round, all banks are able to pay the CCP in full. Therefore I record 0 shortfall towards the CCP. In the next payment step, I pay attention to money received from the CCP. Notice that because nodes 3 and 4 are net payer towards the CCP, so they do not receive anything. According to Eq.4.4b, such quantity should be calculated as:

$$\begin{cases} p_{CCP\to 1} = \alpha\left(\sum_j L_{j1}^c - L_{1j}^c\right)^+ = \frac{1}{10} + \frac{1}{6} + \frac{7}{15} - 0 = \frac{11}{15} \\ p_{CCP\to 2} = \alpha\left(\sum_j L_{j2}^c - L_{2j}^c\right)^+ = \frac{1}{6} + \frac{7}{15} - \frac{1}{10} = \frac{8}{15} \\ p_{CCP\to 3} = 0 \\ p_{CCP\to 4} = 0 \end{cases}$$

Fig.5.1 below shows how the networks looks like with the presence of a CCP at middle. All weights have undertaken two-step netting.

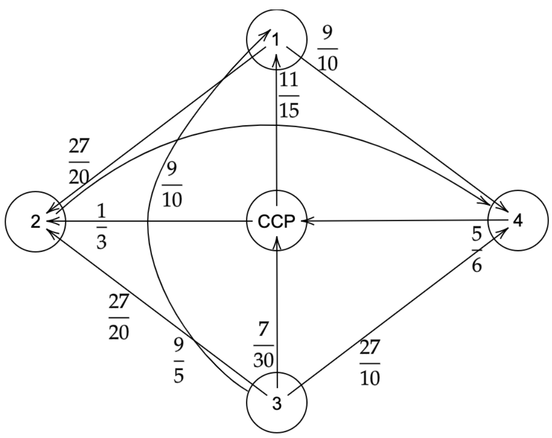


Figure 5.1: Network illustration of this example

So far, I have already derived all payments associated with the CCP. After the previous two steps, the cash left for each bank should follow Eq.4.10 thus given by:

$$e_i^{(3)} = \begin{bmatrix} 3-0+\frac{11}{15} \\ 5-0+\frac{8}{15} \\ 1-\frac{1}{3}+0 \\ 4-\frac{14}{15}+0 \end{bmatrix} = \begin{bmatrix} \frac{56}{15} \\ \frac{83}{15} \\ \frac{2}{3} \\ \frac{46}{15} \end{bmatrix}$$

The remaining of the calculation process becomes exactly the same as in Example 1 with liability matrix $0.9\left(L^c - L^{c\prime}\right)$ and $e_i^{(3)}$. At the end, the clearing vector and the corresponding bilateral shortfall are given by:

$$p_i^* = \begin{bmatrix} 1.8 \\ 1.35 \\ 0.67 \\ 0 \end{bmatrix}, s_i^b(0.1) = \begin{bmatrix} 0 \\ 0 \\ 5.18 \\ 0 \end{bmatrix} = s_i(0.1)$$

The calculation process for other values of α is the same as the case presented so is omitted here. In this way, I only present results of several essential quantities in tables below. Quantities include obligations, actual payments and shortfalls related to the CCP (Table 5.1), bilateral obligations, bilateral clearing vectors and corresponding shortfalls (Table 5.2) and aggregate shortfall (Table 5.3).

α	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
$\bar{p}_{1\to CCP}$	0	0	0	0	0	0	0	0	0
$\bar{p}_{2\to CCP}$	0	0	0	0	0	0	0	0	0
$\bar{p}_{3\to CCP}$	0.67	1	1.33	1.67	2	2.33	2.67	3	3.33
$\bar{p}_{4\to CCP}$	1.87	2.8	3.73	4.67	5.6	6.53	7.47	8.4	9.33
$\sum_i \bar{p}_{i\to CCP}$	2.54	3.8	5.06	6.34	7.6	8.86	10.14	11.4	12.66
$p_{1\to CCP}$	0	0	0	0	0	0	0	0	0
$p_{2\to CCP}$	0	0	0	0	0	0	0	0	0
$p_{3\to CCP}$	0.67	1	1	1	1	1	1	1	1
$p_{4\to CCP}$	1.87	2.8	3.73	4	4	4	4	4	4
$\sum_i p_{i\to CCP}$	2.54	3.8	4.73	5	5	5	5	5	5
$\sum_i s_i^c(\alpha)$	0	0	0.33	1.34	2.6	3.86	5.14	6.4	7.66

Table 5.1: Payments associated with the CCP for different values of α

Table 5.1 presents all payments related to the CCP, starting from $\alpha = 0.2$ to $\alpha = 1$ with increment 0.1. It is worth to point out that at $\alpha = 0.4$, node 3 begins to record shortfalls towards the CCP. Then as α getting larger, both node 3 and 4 are cannot make full payments to the CCP thus exhaust their cash at $\alpha = 0.5$. The total value $\sum_i s_i^c(\alpha)$ corresponds to Prop.1 that it is a non-decreasing function of α.

α	0.2	0.3	0.3	0.5	0.6	0.7	0.8	0.9	1
$\bar{p}_1$	1.6	1.4	1.2	1	0.8	0.6	0.4	0.2	0
$\bar{p}_2$	1.2	1.05	0.9	0.75	0.6	0.45	0.3	0.15	0
$\bar{p}_3$	5.2	4.55	3.9	3.25	2.6	1.95	1.3	0.65	0
$\bar{p}_4$	0	0	0	0	0	0	0	0	0
$\sum_i \bar{p}_i$	8	7	6	5	4	3	2	1	0
p_1^*	1.6	1.4	1.2	1	0.8	0.6	0.4	0.2	0
p_2^*	1.2	1.05	0.9	0.75	0.6	0.45	0.3	0.15	0
p_3^*	0.33	0	0	0	0	0	0	0	0
p_4^*	0	0	0	0	0	0	0	0	0
$\sum_i p_i^*$	3.13	2.45	2.1	1.75	1.4	1.05	0.7	0.35	0
$s_i^b(1-\alpha)$	4.87	4.55	3.9	3.25	2.6	1.95	1.3	0.65	0

Table 5.2: Payments related to bilateral exposures at different level of α

In Table 5.2, I summarise results related to bilateral payments. Together with Table 5.1, I can see nodes 1 and 2 never record shortfalls neither bilaterally nor centrally. Not surprisingly, bilateral obligation, clearing payment and bilateral shortfall are non-increasing w.r.t. α.

α	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
$s_i(\alpha)$	4.87	4.55	4.23	4.59	5.2	5.81	6.44	7.05	7.66

Table 5.3: Aggregate shortfall at different α

As it can be seen, the aggregate shortfall is non-monotonic in terms of α. It achieves smallest value at around 0.4. In the following sections, I will present deep analysis into the behaviour.

5.2 Simulation Study

In order to overcome the difficulty of characterising bilateral shortfalls analytically, in this section I perform numerical experiments. In a nutshell: I generate networks of random VM obligations, simulate the three stages of the payment algorithm as described in Section 4.4, and compute liquidity shortfalls. I first focus on the case in which $\mathbf{L}^c$ and $\mathbf{L}^b$ are independent, which refer to as the case of *independent exposures*. The main result is that increasing the fraction of centrally cleared notional α is not always beneficial, in the sense that it does not necessarily lead to smaller aggregate liquidity shortfalls.

Conceptually the generation of random VM obligations consists of two steps. First, generating the network of counterparties, i.e. for each bank i generating the set of banks j owes to. Second, generating the amounts of the individual VM obligations. In the language of network theory, in the first step I generate the topology of the network of VM obligations, while in the second step I generate its weights. Formally, I write centrally and bilateral VM obligations as follows:

$$L^c_{ij} = A^c_{ij} w^c_{ij} \tag{5.6a}$$

$$L^b_{ij} = A^b_{ij} w^b_{ij}, \tag{5.6b}$$

where $A^c_{ij} \in \{0,1\}$ and $w^c_{ij} > 0$, for all i and j. If $A^c_{ij} = 1$, i has a centrally cleared VM obligation to j of amount w^c_{ij}. The same applies to bilateral VM obligations. $\mathbf{A}^c$ and $\mathbf{A}^b$ are the adjacency matrices for centrally cleared and bilaterally cleared VM obligations. Moreover, within both centrally and bilaterally cleared networks, I assume that all counterparty relationships exist independently of each other with probability p as given in Sec. 3.2. The larger M and the larger p, therefore the denser the network.

As regards the weights, for all banks i I generate total obligations $L^{tot}_i = \sum_j L^c_{ij} = \sum_j L^b_{ij}$ and partition those uniformly across its counterparties:

$$w^c_{ij} = \frac{L^{tot}_i}{\sum_j A^c_{ij}} \tag{5.7a}$$

$$w^b_{ij} = \frac{L^{tot}_i}{\sum_j A^b_{ij}}, \tag{5.7b}$$

so that

$$L^c_{ij} = \begin{cases} \frac{L^{tot}_i}{\sum_j A^c_{ij}} & \text{for } A^c_{ij} = 1 \\ 0 & \text{for } A^c_{ij} = 0 \end{cases} \tag{5.8a}$$

$$L^b_{ij} = \begin{cases} \frac{L^{tot}_i}{\sum_j A^b_{ij}} & \text{for } A^b_{ij} = 1 \\ 0 & \text{for } A^b_{ij} = 0\,. \end{cases} \tag{5.8b}$$

Notice that, since the two adjacency matrices $\mathbf{A}^c$ and $\mathbf{A}^b$ are different, also the two VM obligation matrices $\mathbf{L}^c$ and $\mathbf{L}^b$ are different. This simplified set-up allows to express the variability of VM obligations in terms of the density M and the parameters of the distribution of total obligations.

5.2.1 Homogeneous VM Obligations

I start from the case in which VM obligations are the same for all banks, i.e. $L^{tot}_i = L$, for all i. In Fig.5.2 I show the aggregate shortfall $\sum_i s_i$ over average of 100 realizations of $\mathbf{L}^c$ and $\mathbf{L}^b$. In this specific set-up, $N = 100$ and M changes from 5 to 8, meaning that each bank has, on average, centrally cleared VM obligations and bilateral obligations towards those number of banks. Moreover, $e = 15$, meaning that the initial cash endowment of each bank is equal to 15. For all values of M the aggregate shortfall starts to decrease as α increases, but after having reached a minimum at $\alpha_{\min}$, it starts to increase again. This means that there is an optimal value of α at which the aggregate liquidity shortfall is minimal. Both the position of the minimum and its value are variable and depend on the individual realisation of VM obligations. For example, for the realisations in Figure 5.2, $\alpha_{\min}$ ranges from about 0.4 to about 0.5, while the aggregate shortfall at the minimum ranges from about 600 to about 1100.

In Figure 5.3 I focus on one individual realisation of VM obligations. In the top panel I show not only the aggregate shortfall, but also the aggregate centrally cleared and bilateral shortfalls. The minimum in the aggregate shortfall emerges as the result of two competing trends: the aggregate centrally cleared shortfall increases with α while the aggregate normalised bilateral shortfall decreases with it.

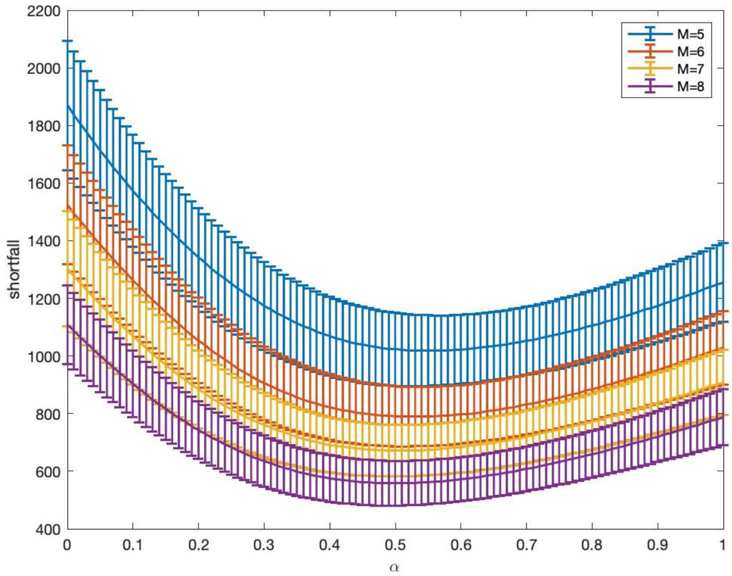


Figure 5.2: Aggregate shortfall for averaging over 100 realisations with errorbar over 1 standard deviation of VM obligations at different levels of M. $N = 100$, $e^{(1)} = 15$, and $L_i^{tot} = 100$, for all i.

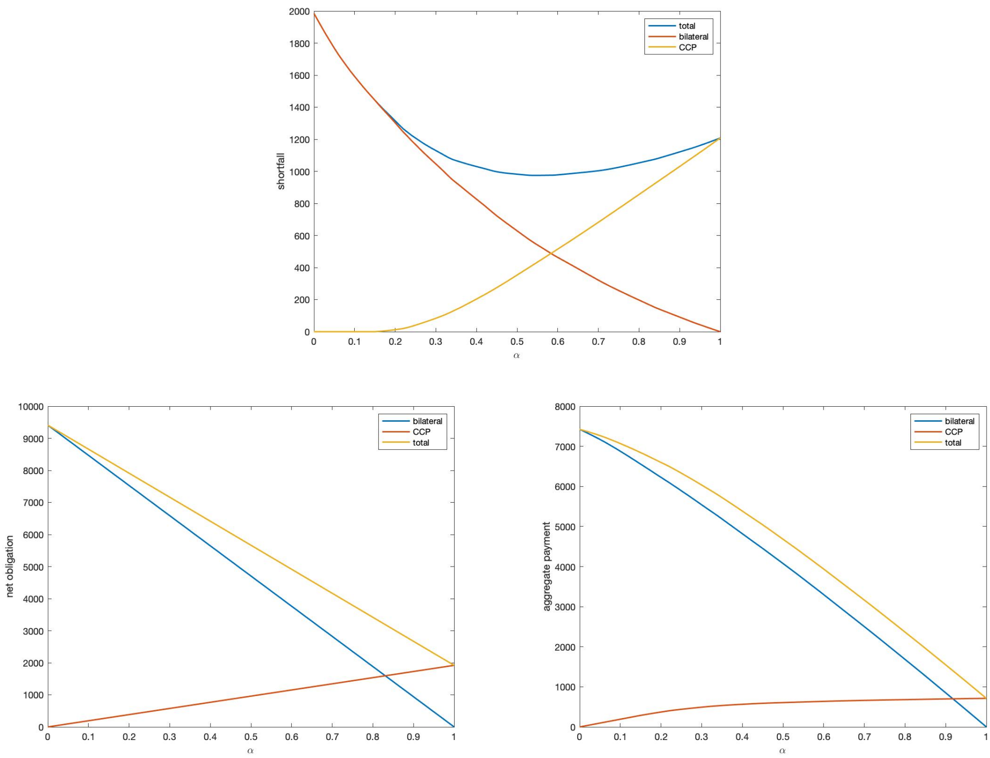


Figure 5.3: Decomposition of aggregate shortfall (top panel), net VM obligation (bottom left panel), and aggregate payments (bottom right panel) for one realisation of VM obligations. $N = 100$, $M = 5$, $L_i^{tot} = 100$ and $e_i^{(1)} = 15$.

From the bottom left panel of Figure 5.3, it can be seen that net centrally cleared VM obligations increase with α (as expected from Proposition 1) and net bilateral VM obligations decrease with α (from Eq.4.11). Overall, total net VM obligations decrease with α. Since total gross VM obligations do not depend on α, this means that netting is more and more efficient as α increases, indicating that multilateral netting (on centrally cleared VM obligations) dominates bilateral netting more and more. From the bottom right panel of Figure 5.3, the aggregate payment on centrally cleared contracts increases with α (as expected from Proposition 1). However, the aggregate payment on bilateral contracts decreases with α more, resulting in an overall trend of the aggregate payment on all contracts that is decreasing with α. Therefore, I conclude that the U-shaped behaviour of aggregate shortfall is the result of the tradeoff between the increased netting opportunities provided by central clearing, as α increases, and the reduction in bilateral payments.

The trends of both VM obligations and payments displayed in Fig.5.3 for an individual realisation are confirmed for all realisations in our sample. In particular, total net VM obligations decrease strictly with α. Again, since net centrally cleared VM obligations $\sum_i \bar{p}_{i\to CCP}$ increase with α (see Proposition 1), this means that net bilateral VM obligations $\sum_i \bar{p}_i$ decrease *more*. In other words, as α increases, net bilateral VM obligations are replaced by smaller centrally cleared VM obligations due to the multilateral netting performed by the CCP. Similarly, total payments $\sum_i p_{i\to CCP} + \sum_i p_i$ decrease strictly with α and, since payments on centrally cleared obligations $\sum_i p_{i\to CCP}$ increase with α (see Proposition 1), this means that payments on bilateral VM obligations $\sum_i p_i$ decrease *more*. Remember that

$$\begin{aligned}\sum_i s_i &= \sum_i s_i^c + s_i^b \\ &= \sum_i \bar{p}_{i\to CCP} + \sum_i \bar{p}_i - \sum_i p_{i\to CCP} - \sum_i p_i\end{aligned} \tag{5.9}$$

For values of α to the left of the minimum the aggregate shortfall decreases because VM obligations (the first two terms) decrease faster than payments (the last two terms). For values of α to the right of the minimum, the opposite is true. In this region the aggregate shortfall increases because even though VM payment obligations decrease, they decrease at a slower pace than payments. In turn, as discussed above, payments decrease because bilateral payments decrease at a faster pace than the increase in payments to the CCP. Therefore, the increased efficiency of multilateral netting is more than compensated by the reduction in bilateral payments.

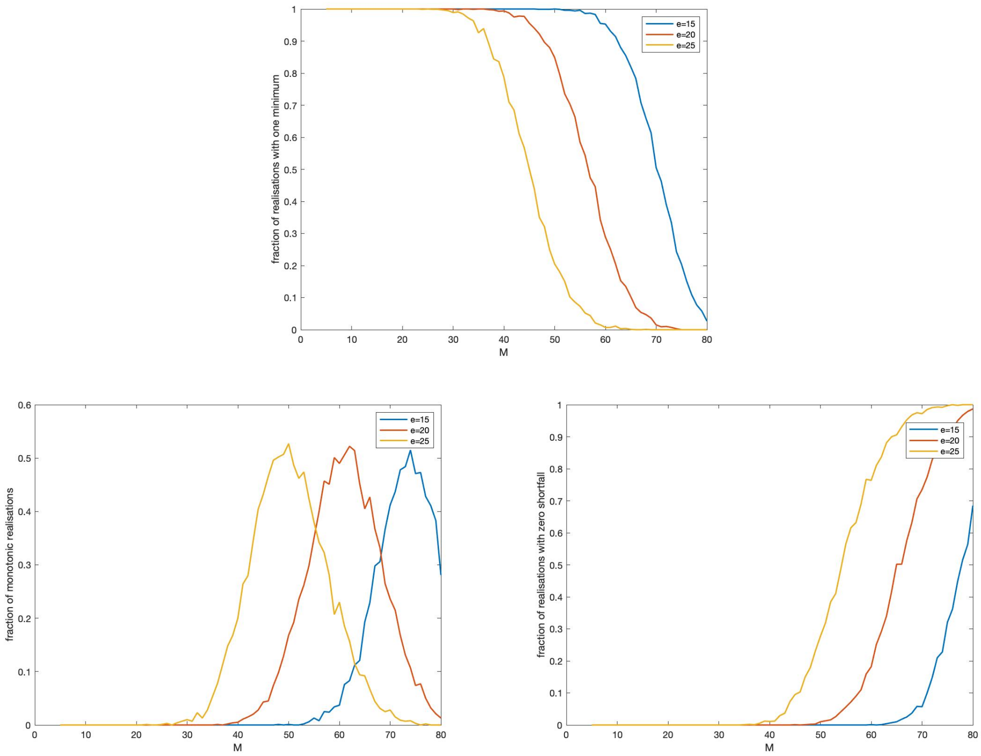


Figure 5.4: Fraction of realisations in the which aggregate shortfall achieves a minimum with respect to α (top panel), is monotonic with respect to α (bottom left) or in which is equal to zero for all values of α (bottom right). $N = 100$, 1000 realisations, $e_i^{(1)} = e$ and $L_i^{tot} = 100$ for all i.

Does this minimum always exists for different parameters? Intuitively from Fig.5.2 the answer is no, as it can be noticed that theses 4 lines get flatter for higher M. I then look at the statistical properties of a large sample of 1 000 realisations. Fig.5.4 studies the behaviour of the system as M is varied over a wider range than the values considered previously. More specifically, M is varied between 5 and 80. First, I check how frequently the aggregate shortfall has one minimum with respect to α. From top panel of Fig.5.4 I can see that, for a given value of total gross VM obligation L, the fraction of realisations with a minimum is equal to one for smaller values of the connectivity M, that it decreases with M, and that for sufficiently large values of the density it is equal to zero. Similarly, for a fixed value of density M, the fraction of realisations with a minimum is decreasing with total exogenous cashflow e. Therefore, I find that, in the region of the parameter space in which the connectivity and exogenous cashflow are sufficiently low, the aggregate liquidity shortfall has a minimum with respect to the fraction of centrally cleared notional.

What about the realisations in which the aggregate shortfall has no minimum with α? From Fig.5.4 it can be seen that, by increasing the connectivity of the network, realisations with one minimum decrease and monotonic (non-increasing) realisations start to increase. In these realisations, VM obligations decrease faster than payments for all values of α (see Eq.5.9), suggesting that gains from multilateral netting dominate the reduction in bilateral payments. By further increasing the connectivity of the network, also monotonic (non-increasing) realisations decrease and realisations in which the aggregate shortfall is zero for all values of α start to appear. Eventually, for even larger connectivity, in most realisations the aggregate shortfall is zero for all values of α.

In the left panel of Fig.5.5 I show the mean value of α_{min} across network realisations, in order to understand how it depends on the parameters. When minimum shortfall is achieved in a continuous region of α, the median of that interval is taken. In order to provide meaningful result, only parameter combinations provide reasonable fraction (more than 5%) of non-zero shortfalls are included. α_{min} is broadly around 50%, and largely does not depend either on the density M or on the exogenous cashflow e. I stress that the variability of α_{min} is substantial, pointing to a strong dependence of α_{min} on the specific realisation of the network of VM obligations.

Next, I look at the shortfall at α_{min}. From the right panel of Fig.5.5 it can be seen that it is decreasing with M and increasing with e. Intuitively, as M increases the average number of counterparties increases, which leads to more opportunities for bilateral netting. Hence, one can expect shortfalls to be generally smaller for larger value of M. As the total exogenous cashflow e increases, shortfalls become smaller when the total liability is constant.

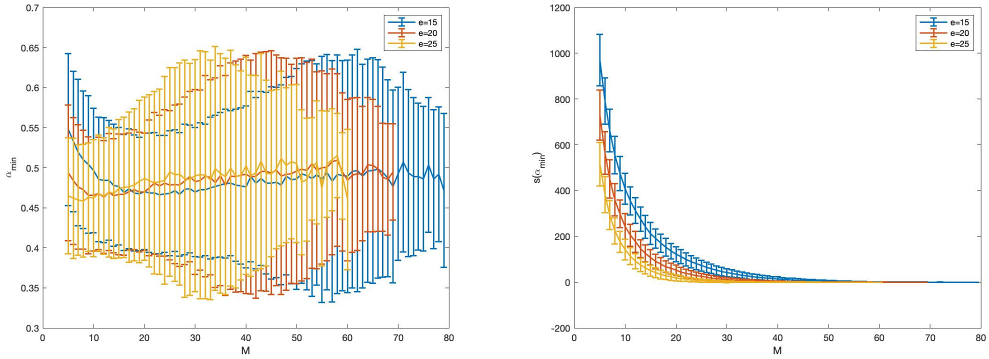


Figure 5.5: Mean value of $\alpha_{\min}$ (left panel) and mean aggregate shortfall at $\alpha_{\min}$ (right panel) for several values of M and e with errorbar. $N = 100$, 1000 realisations, and $e_i^{(1)} = e$, for all i.

Looking at Fig.5.2, 5.4 and 5.5, it is also worth to point out that increasing connectivity can stabilise the system. For instance, in Fig.5.2, the shortfall is getting smaller for higher M. Then in the bottom right panel of Fig.5.4, there exist more realisations with zero shortfall when M increases. Moreover, in Fig.5.5, $s(\alpha_{min})$ keeps decreasing as M increases. All the above figures confirm that the payment efficiency is improved with higher connectivity in this framework. In the literature, this is called risk-sharing (Cabrales et al., 2017). In my research, I show that, at least if M is larger than 5, risk sharing always benefits the system as it increases interconnections between banks. In this point of view, more linkage of one bank to others can disperse exposure and eventually reduce the exposure toward the shock.

Finally, in Figure 5.6 I show the relative improvement of being at $\alpha_{\min}$ compared to the fully centrally cleared setting ($\alpha = 1$) or to the fully bilateral setting ($\alpha = 0$), i.e. $\frac{\sum_i s_i(1) - \sum_i s_i(\alpha_{\min})}{\sum_i s_i(1)}$ and $\frac{\sum_i s_i(0) - \sum_i s_i(\alpha_{\min})}{\sum_i s_i(0)}$. I use relative value in this analysis because absolute values of improvements are obviously decreasing as shortfall get smaller for higher M, while it can never drop below 0. I can see that the relative improvement compared to both $\alpha = 1$ and $\alpha = 0$ increases with the total exogenous cashflow e and increases with the density M. Mean relative improvements with respect to $\alpha = 1$ are economically significant, ranging from around 20% to almost 100%. Similarly, mean relative improvements compared to $\alpha = 0$, ranging from around 50% to almost 100%.

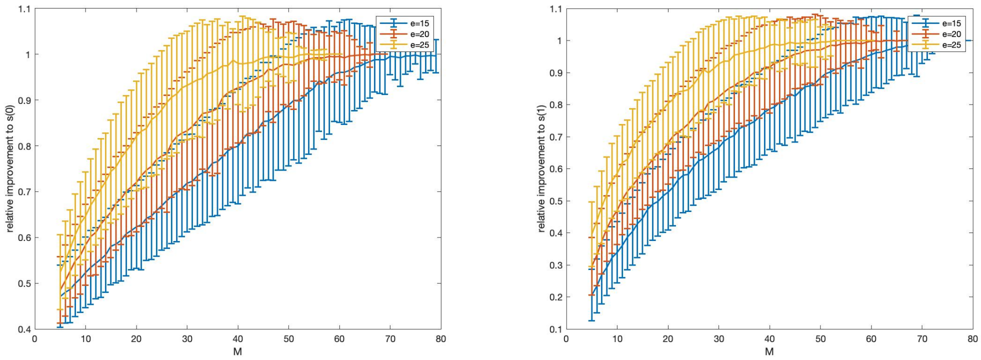


Figure 5.6: Relative improvements of being at $\alpha_{\min}$ compared to the fully bilateral cleared setting (left panel) or to the fully centrally cleared setting (right panel). 1 000 realisations, and $L_i^{tot} = 100$.

5.2.2 Heterogeneous VM Obligations

I now turn my attention towards the case in which total gross VM obligations L_i^{tot} are not equal for all banks. The purpose of this analysis is to figure out how my model performs and subsequent consequences under different scenarios. I draw L_i^{tot} from exponential random variables. To maintain the study as fair as possible, I set it has 100 mean:

$$L_i^{tot} \sim \exp\left(\frac{1}{100}\right). \tag{5.10}$$

While the exponential distribution is not representative of real systems (Boss et al., 2004a), it is the simplest extension to the previous set-up that allows me to check the robustness of previous results with respect to the relaxation of the homogeneity of the system.

In Fig.5.7, I plot the relationship between shortfall and α under this scenario. Comparing to Fig.5.2, it is obvious that values in shortfall become larger. In the case of heterogeneous VM obligations, all banks are not uniformly exposed to the failure of their counterparties. If one counterparty with particularly large liability exposures failed, its creditors would suffer higher loss with respect to the case of uniform exposures (Caccioli et al., 2012). Such explanations are further justified in Fig.5.8, as it can be seen in the left panel net obligations do not change too much

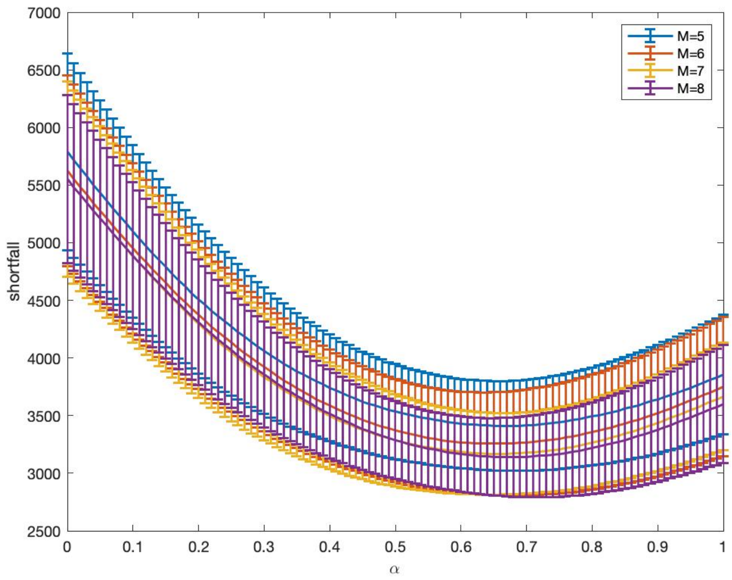


Figure 5.7: Aggregate shortfall for averaging over 100 realisations with errorbar over 1 standard deviation for heterogeneous liabilities. $N = 100$, $e^{(1)} = 15$, and $\overline{L_i^{tot}} = 100$, i.i.d. exponentially distributed for all i

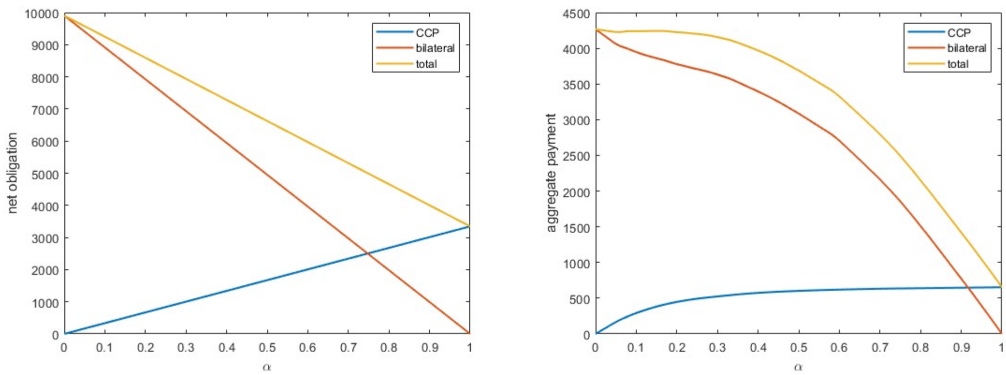


Figure 5.8: Decomposition of net VM obligation (left panel), and aggregate payments (right panel) for one realisation of VM obligations. $N = 100$, $M = 5$, $e_i^{(1)} = 15$, and $\overline{L_i^{tot}} = 100$, i.i.d.exponentially distributed for all i.

compare to the bottom left panel in Fig.5.3, but in the right panel total payments become significantly smaller. Also in the case of heterogeneous VM obligations, both total net VM obligations and total payments decrease strictly with α, confirming that the existence of the minimum in aggregate shortfall is due to the tradeoff between the increasing efficiency of multilateral netting and the reduction in bilateral payments with α.

In Fig.5.9 I check the fraction of realisations where the aggregate shortfall has one minimum w.r.t. α. Surprisingly, in contrast to homogeneous case, this variable mainly maintain at 1 with only a few variations, regardless of the values of M and e. Realisations without a minimum appear to be outlier cases. I can conclude from this that an optimal value of α always exists. Therefore, it is more possible for the CCP to be effective in reducing aggregate shorfall of the system.

In summarise, shortfalls of heterogeneous VM obligations increase compare to the homogeneous case. This is mainly driven by different exposures of banks. On the other hand, the presence of the CCP becomes more effective than homogeneous case in terms of the optimal value of α_{min} exists in almost all realisations.

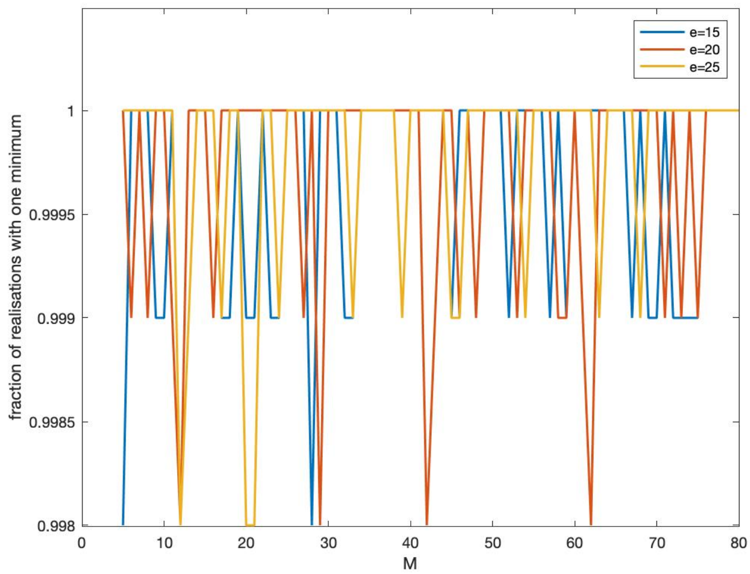


Figure 5.9: Fraction of realisations in which the aggregate shortfall has one minimum with respect to α. $N = 100$, 1000 realisations, and $\overline{L_i^{tot}} = 100$, i.i.d.exponentially distributed for all i.

5.3 The Role of Payment Sequencing

As I explained in Chapter 4, payments associated with the CCP follow a particular sequencing — banks pay the CCP first, then the CCP pays banks, and only afterwards banks settle bilateral obligations among themselves. Such sequencing brings some frictions into the system, thus one may wonder whether it increases inefficiency of the payment system, as banks have a smaller cashflow after the first round of payments. For example, imagine that one bank is a net payer to the CCP and a net receiver from bilateral counterparties. If payments were *not* sequenced, that bank could redirect the payments received from bilateral counterparties to the CCP. Instead, if payments are sequenced and its cash buffer is not sufficient to cover the payment obligation due to the CCP, that bank has to source the gap in order to be able to pay the CCP. I have already discussed how increasing the fraction of notional that is centrally cleared generates two competing forces — VM payment obligations decrease, but (bilateral) payments also decrease — and how the minimum in aggregate shortfall results from the tradeoff between those. Increasing the fraction of notional that is centrally cleared would also increase the payments subject to the temporal constraints, and that would be why bilateral payments would decrease. In the following, I will relax the assumption on payment sequencing and investigate whether it is a source of payment inefficiency.

In this section I show that, in most cases, this is not the case and that in practice payment sequencing plays only a limited role in the existence of an optimal fraction of centrally cleared notional. To this end I compare the results of the simulations above, in which payments are sequenced, with analogous simulations in which payments are not sequenced and take place in a single round. This means that all payments occur in the third round of the payment algorithm described in Section 4.4, i.e. by using the Eisenberg and Noe model for all payment obligations, in which case CCP is treated as a normal node with only multi-lateral netting. In the version of the model with payment sequencing the CCP is always able to pay its obligations to banks in full. Therefore, in order to keep the comparison as fair as possible, in the section I assign a very large cash buffer to the CCP, say 1000, so that also in this case it is always able to pay its obligations in full.

When VM obligations are homogeneous, I find in Fig.5.10 that the non-monotonicity still exists with one minimum, using same parameters as in Fig.5.2. In Fig.5.11 the fraction of realisations with one minimum is almost the same compare to the original model. As an initial conclusion, payment sequencing appears to

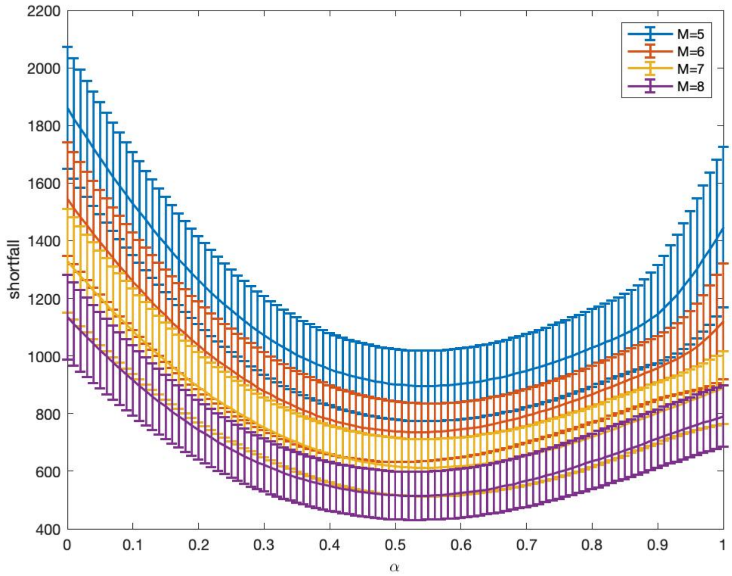


Figure 5.10: Aggregate shortfall without payment sequencing at different levels of M. $N = 100$, 100 realisations, and $L_i^{tot} = 100$, and $e_i^{(1)} = 15$.

have no impact on existing the minimum. However, it could impact the value of the aggregate shortfall at the minimum. In Fig.5.12 I show the cumulative distribution (across all realisations and specific parameters) of the difference between aggregate shortfalls at $\alpha_{\min}$ with and without payment sequencing. Positive (negative) values indicate that shortfalls are larger with payment sequencing. I find that, on the left panel of homogeneous VM obligations, more than 99% of the realisations leads to larger aggregate shortfalls at $\alpha_{\min}$ when introducing the sequence. However, considering the magnitude of total exposures (NL^{tot}) is of 10^4, the mean and median of such difference under 20, and the 95th percentile less than 40, such difference can be treated as small and insignificant.

For heterogeneous VM obligations, I find that the fraction of realisations with a minimum is exactly the same with and without payment sequencing. In both cases, such minimum always exists. When I checking the difference at minimum, as in the right panel of Fig.5.12, it gets difficult to compare which one is larger, as the value on x-axis ranges from large negative to large positive, and the probability of negative values goes beyond 0.6.

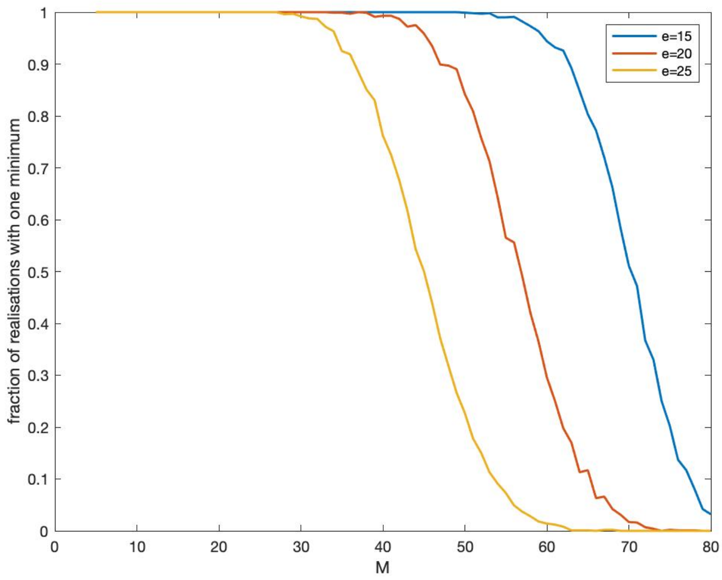


Figure 5.11: Fraction of realisations without payment sequencing. $N = 100$, 1000 realisations, and $L_i^{tot} = 100$ for all i.

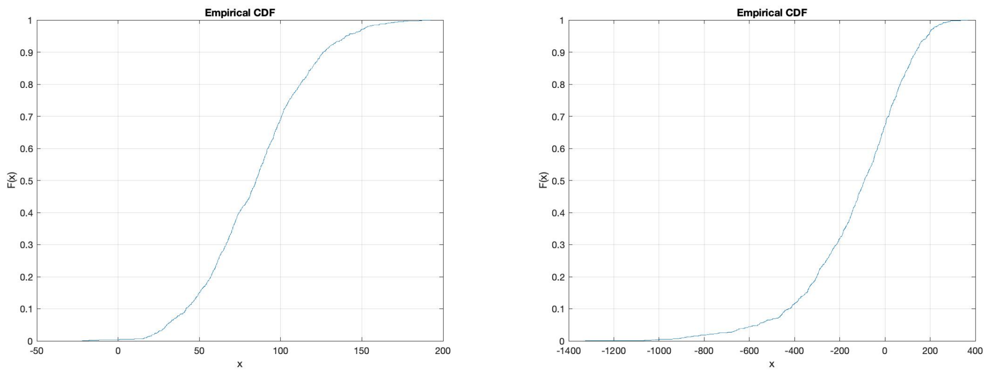


Figure 5.12: Cumulative distribution of the difference between shortfalls with and without payment sequencing for homogeneous VM obligations (left panel) and heterogeneous VM obligations (right panel). $N = 100$, 1000 realisations, $L_i^{tot} = 100$, and $e_i^{(1)} = 15$ for all i, and $M = 5$.

Overall, those results show that payment sequencing plays a minor role in the existence of an optimal value of centrally cleared notional, while for heterogeneous VM obligations such difference becomes slightly more significant. The relationship between α and shortfall, together with the fraction of realisations with a minimum, do not have significant difference in terms of payment sequencing. The only difference exists when I look at the difference at α^*. This suggests that the reduction in bilateral payments that is responsible for the existence of the minimum in aggregate shortfall might be due to the change in the topology of the network of obligations that occurs when the fraction of centrally cleared notional increases.

Chapter 6

Perfectly Correlated Exposures

In the following I discuss results under perfectly correlated exposures $\mathbf{L}^c = \mathbf{L}^b = \mathbf{L}$, when moving from α to $\alpha' = \alpha + \Delta\alpha$. This corresponds to all banks novating with a fraction $\Delta\alpha$ of their notional simultaneously with all their counterparties. This case is interesting because its behaviour is different from the case considered in chapter 5, and it allows me to prove some statements of liquidity shortfalls of individual banks and the behaviour of liquidity shortfalls is very different from the case $\mathbf{L}^c \neq \mathbf{L}^b$. In this chapter I often refer to quantities at a certain value of α. For example, with $s_i(\alpha)$ I denote the total shortfall of bank i at α, and $s_i^b(\alpha)$ denote the shortfall of bank i on bilateral VM obligations at α (but the fraction of VM obligations that are bilaterally cleared at α is $1-\alpha$). To begin with, I start from a simple example to illustrate how this payment framework works.

Example 3. Consider the network used in Example 1 again. L, Π and e are the same. The case of $\alpha = 0$ are exactly the same, so I start from considering $\alpha = 0.1$ and novating the percentage with step 10%. I introduce bilateral netting to L first and record all variables in the following are based on bilateral netting:

$$(L-L')^+ = \left(\begin{bmatrix} 0 & 7 & 1 & 1 \\ 3 & 0 & 3 & 3 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 3 & 1 & 1 \\ 7 & 0 & 1 & 1 \\ 1 & 3 & 0 & 1 \\ 1 & 3 & 1 & 0 \end{bmatrix} \right)^+ = \begin{bmatrix} 0 & 4 & 0 & 0 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

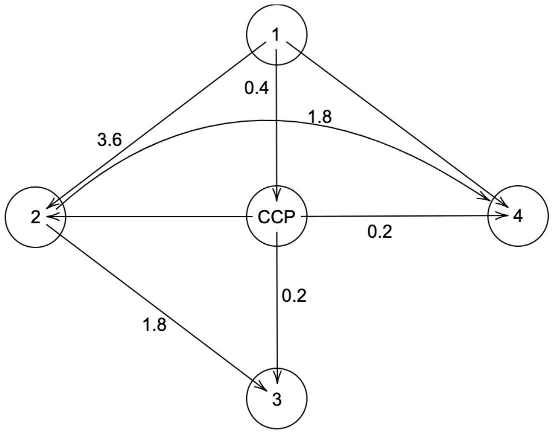


Figure 6.1: Network illustration of this example

After bilateral netting, only net payers between banks have positive obligations. Under perfectly correlated exposures, $\mathbf{L}^c = \mathbf{L}^b = \mathbf{L}$. I split the liability matrix to bilateral and central, which account for 90% and 10% of $\mathbf{L}$ respectively:

$$0.9\left(L^b - L^{b'}\right) = 0.9 \times \begin{bmatrix} 0 & 4 & 0 & 0 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 3.6 & 0 & 0 \\ 0 & 0 & 1.8 & 1.8 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$0.1\left(L^c - L^{c'}\right) = 0.1 \times \begin{bmatrix} 0 & 4 & 0 & 0 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0.4 & 0 & 0 \\ 0 & 0 & 0.2 & 0.2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The majority of calculation process follows the same as in Example 2 so I only present the most important parts. Refer to Eq.4.4a and Eq.4.11, bilateral and central olibations are:

$$\bar{p}_{i\to CCP} = \begin{bmatrix} 0.4 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \bar{p}_i = \begin{bmatrix} 3.6 \\ 3.6 \\ 0 \\ 0 \end{bmatrix}$$

respectively. The sketch map of this example is presented in Fig.6.1. According to Eq.4.6, payments from banks to the CCP are:

$$p_{i\to CCP} = \begin{bmatrix} \min(1,0.4) \\ \min(2,0) \\ \min(2,0) \\ \min(2,0) \end{bmatrix} = \begin{bmatrix} 0.4 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

There is no default towards the CCP at this stage, so $s_i^c(0.1) = 0, \forall i$. In the second payment round, I consider payments from the CCP to each bank. Since node 1 is a net payer towards the CCP, it does not receive any money. Node 2 nets all aggregate obligations so it does not have any exposures with the CCP. Eventually, $p_{CCP\to i}$ becomes:

$$p_{CCP\to i} = \begin{bmatrix} 0 \\ 0 \\ 0.2 \\ 0.2 \end{bmatrix}$$

Following Eq.5.3 the money left by each bank, $e_i^{(3)}$, equals to:

$$e_i^{(3)} = \begin{bmatrix} 0.6 \\ 2 \\ 2.2 \\ 2.2 \end{bmatrix}$$

To compute clearing payments and bilateral shortfall, it follows the same procedure as in Example 1 and 2 with liability matrix $0.9\left(L^b - L^{b'}\right)$ and external cash $e_i^{(3)}$. Hence, the clearing vector p^* and bilateral shortfall $s^b(0.1)$ are:

$$p^* = \begin{bmatrix} 0.6 \\ 2.6 \\ 0 \\ 0 \end{bmatrix}, s_i^b(0.1) = \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

The remaining calculation process should be the same as the case of $\alpha = 0.1$. I then summarise remaining results to table below. Different from tables in Example 2, here for simplicity, I only present results related to shortfalls, which is the key quantity I want to study.

α	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
$s_i^c(\alpha)$	0	0.2	0.6	1	1.4	1.8	2.2	2.6	3
$s_i^b(\alpha)$	4	3.6	2.8	2	1.6	1.2	0.8	0.4	0
$s_i(\alpha)$	4	3.8	3.4	3	3	3	3	3	3

Table 6.1: Summary of shortfall values

If I take into consideration of $\alpha = 0$ and $\alpha = 0.1$, one of the most noticeable result is that the aggregate shortfall is constant until $\alpha = 0.2$, after which point it starts to decrease. This potentially provides evidence on the fact of when the CCP starts to become effective in reducing shortfalls. I will take a further look into relevant properties.

In the following sections, I will present main results of this chapter, starting from theoretical proofs, follow by numerical solutions.

6.1 Theoretical Result

The first result I present compares shortfalls in a fully bilateral setting with shortfalls in a fully centrally cleared setting.

Theorem 1. *Let the tuple $(\mathbf{L}, \mathbf{e}^{(1)})$ be a family of clearing systems with perfectly correlated exposures. For all banks, the shortfall in the fully bilateral setting is larger than or equal to the shortfall in the fully centrally cleared setting:*

$$s_i(0) \geq s_i(1) \quad \forall i. \tag{6.1}$$

Proof. By dividing banks into two groups, banks i with net obligations smaller than or equal to zero ($\sum_j L_{ij} - L_{ji} \leq 0$) and banks i with net obligations larger than zero ($\sum_j L_{ij} - L_{ji} > 0$). This property clearly does not depend on α. For banks such that $\sum_j L_{ij} - L_{ji} \leq 0$, from Eq.4.4a and Eq.4.7 it gives that: $s_i(1) = [(\sum_j L_{ij} -$

$L_{ji})^+ - e_i^{(1)}]^+ = [-e_i^{(1)}]^+ = 0 \leq s_i(0)$. So, for all banks i that have net obligations smaller than or equal to zero the shortfall for $\alpha = 0$ is larger than or equal to the shortfall for $\alpha = 1$. For banks i with net obligations larger than zero I have: $s_i(1) = [(\sum_j L_{ij} - L_{ji})^+ - e_i^{(1)}]^+ = [(\sum_j L_{ij} - L_{ji}) - e_i^{(1)}]^+$. For $\alpha = 0$ the shortfall is only on bilateral contracts. Moreover, as no payment has been made in the first two rounds, $e_i^{(3)} = e_i^{(1)}$. If $s_i(0) > 0$, i.e. if $\bar{p}_i > p_i^*$, I have that $p_i^* = e_i^{(1)} + \sum_j \Pi_{ji} p_j^*$. If $s_i(0) \leq 0$, i.e. if $p_i^* = \bar{p}_i$, I have that $\bar{p}_i \leq e_i^{(1)} + \sum_j \Pi_{ji} p_j^*$. Therefore, I can write:

$$\begin{aligned} s_i(0) = \bar{p}_i - p_i^* &= \left[\bar{p}_i - e_i^{(1)} - \sum_j \Pi_{ji} p_j^*\right]^+ \\ &= \left[\sum_j \left(L_{ij} - L_{ji}\right)^+ - e_i^{(1)} - \sum_j \Pi_{ji} \bar{p}_j\right]^+ \\ &\geq \left[\sum_j \left(L_{ij} - L_{ji}\right)^+ - e_i^{(1)} - \sum_j \left(L_{ji} - L_{ij}\right)^+\right]^+ \\ &= \left[\sum_j \left(L_{ij} - L_{ji}\right)^+ - e_i^{(1)} - \sum_j \left(L_{ij} - L_{ji}\right)^-\right]^+ \\ &= \left[\sum_j \left(L_{ij} - L_{ji}\right) - e_i^{(1)}\right]^+ \\ &= s_i(1)\,. \end{aligned} \tag{6.2}$$

Therefore, also for all banks i that have net obligations larger than zero the shortfall for $\alpha = 0$ is larger than or equal to the shortfall for $\alpha = 1$. □

Theorem 1 formalises the intuition according to which a system in which all VM obligations are centrally cleared is more efficient (in the sense that liquidity shortfalls are smaller) than a system in which all VM obligations are bilateral.

It is worth to point out that under my framework, total shortfall and payment can be split into several stages and the aggregate values equal to the summation of values in each stage. For simplicity, I present a proof on two-stage version so the multi-stage will follow by further divide the stage into two.

Lemma 1 (Splitting of central clearing in two sub-stages)**.** *Let $\bar{p}_{i\to CCP}$ and $\bar{p}_{CCP\to i}$, for all i, be two vectors of net obligations to and from the CCP, defined as in Eq.4.2,*

$\mathbf{e} \geq 0$ *a vector of cash endowments, and* $\beta \in [0,1]$. *Let us introduce quantities in the first sub-stage:*

$$\bar{p}^{(1)}_{i\to CCP} = \beta\, \bar{p}_{i\to CCP} \tag{6.3a}$$

$$\bar{p}^{(1)}_{CCP\to i} = \beta\, \bar{p}_{CCP\to i} \tag{6.3b}$$

$$e^{(1)}_i = e_i\,, \tag{6.3c}$$

with payments $p^{(1)}_{i\to CCP}$ *and shortfalls* $s^{(1)}_i$ *defined as in Eq.4.6 and Eq.4.7, for all i. Let us introduce quantities in the second sub-stage:*

$$\bar{p}^{(2)}_{i\to CCP} = (1-\beta)\bar{p}_{i\to CCP} \tag{6.4a}$$

$$\bar{p}^{(2)}_{CCP\to i} = (1-\beta)\bar{p}_{CCP\to i} \tag{6.4b}$$

$$e^{(2)}_i = e^{(1)}_i - p^{(1)}_{i\to CCP} + \bar{p}^{(1)}_{CCP\to i}\,, \tag{6.4c}$$

with payments $p^{(2)}_{i\to CCP}$ *and shortfalls* $s^{(2)}_i$ *defined as in Eq.4.6 and Eq.4.7, for all i. Finally, let us introduce cash endowment at the end of the second sub-stage:*

$$e^{(3)}_i = e^{(2)}_i - p^{(2)}_{i\to CCP} + \bar{p}^{(2)}_{CCP\to i}\,, \tag{6.5}$$

for all i. Then I have that:

$$p_{i\to CCP} = p^{(1)}_{i\to CCP} + p^{(2)}_{i\to CCP} \tag{6.6}$$

$$s^c_i = s^{c\,(1)}_i + s^{c\,(2)}_i \tag{6.7}$$

$$e^{(3)}_i = e_i - p_{i\to CCP} + \bar{p}_{CCP\to i}\,, \tag{6.8}$$

for all i.

Proof. Let us focus on bank i, as central clearing proceeds independently for each bank. First, I consider the case $\sum_j L_{ij} - L_{ji} > 0$, which implies that $\bar{p}_{i\to CCP} > 0$, $\bar{p}^{(1)}_{i\to CCP} > 0$, $\bar{p}^{(2)}_{i\to CCP} > 0$, and $\bar{p}_{CCP\to i} = \bar{p}^{(1)}_{CCP\to i} = \bar{p}^{(2)}_{CCP\to i} = 0$. Therefore, from by adding $p^{(1)}_{i\to CCP}$ on both sides of Eq.4.6 for $p^{(2)}_{i\to CCP}$ I have:

$$
\begin{aligned}
p^{(1)}_{i\to CCP}+p^{(2)}_{i\to CCP} &= \min\left[e^{(2)}_i+p^{(1)}_{i\to CCP},\bar{p}^{(2)}_{i\to CCP}+p^{(1)}_{i\to CCP}\right] \\
&= \min\left[e^{(1)}_i-p^{(1)}_{i\to CCP}+\bar{p}^{(1)}_{CCP\to i}+p^{(1)}_{i\to CCP},\bar{p}^{(2)}_{i\to CCP}+p^{(1)}_{i\to CCP}\right] \\
&= \min\left[e^{(1)}_i,\bar{p}^{(2)}_{i\to CCP}+p^{(1)}_{i\to CCP}\right]. \qquad (6.9)
\end{aligned}
$$

In the sub-case in which $e^{(1)}_i \geq \bar{p}^{(1)}_{i\to CCP}$ I have that $p^{(1)}_{i\to CCP}=\bar{p}^{(1)}_{i\to CCP}$ and Eq.6.9 reads:

$$
p^{(1)}_{i\to CCP}+p^{(2)}_{i\to CCP} = \min\left[e^{(1)}_i,\bar{p}^{(1)}_{i\to CCP}+\bar{p}^{(2)}_{i\to CCP}\right]. \qquad (6.10)
$$

In the sub-case in which $e^{(1)}_i < \bar{p}^{(1)}_{i\to CCP}$, I have that $p^{(1)}_{i\to CCP}=e^{(1)}_i$ and Eq.6.9 reads:

$$
\begin{aligned}
p^{(1)}_{i\to CCP}+p^{(2)}_{i\to CCP} &= \min\left[e^{(1)}_i,e^{(1)}_i+\bar{p}^{(2)}_{i\to CCP}\right] \\
&= \min\left[e^{(1)}_i,\bar{p}^{(1)}_{i\to CCP}+\bar{p}^{(2)}_{i\to CCP}\right].
\end{aligned} \qquad (6.11)
$$

Second, I consider the case $\sum_j L_{ij}-L_{ji}\leq 0$, which implies that $\bar{p}_{i\to CCP}=\bar{p}^{(1)}_{i\to CCP}=\bar{p}^{(2)}_{i\to CCP}=0$. Since there are no obligations to the CCP, also all payments to the CCP are equal to zero: $p_{i\to CCP}=p^{(1)}_{i\to CCP}=p^{(2)}_{i\to CCP}=0$. This concludes the proof that $p_{i\to CCP}=p^{(1)}_{i\to CCP}+p^{(2)}_{i\to CCP}$ in all cases.

For shortfalls I have that:

$$
\begin{aligned}
s^c_i &= \bar{p}_{i\to CCP}-p_{i\to CCP} \\
&= \bar{p}^{(1)}_{i\to CCP}+\bar{p}^{(2)}_{i\to CCP}-p^{(1)}_{i\to CCP}-p^{(1)}_{i\to CCP} \\
&= s^{c\,(1)}_i+s^{c\,(2)}_i.
\end{aligned} \qquad (6.12)
$$

Finally, I have:

$$
\begin{aligned}
e^{(3)}_i &= e^{(2)}_i-p^{(2)}_{i\to CCP}+\bar{p}^{(2)}_{CCP\to i} \\
&= e^{(1)}_i-p^{(1)}_{i\to CCP}+\bar{p}^{(1)}_{CCP\to i}-p^{(2)}_{i\to CCP}+\bar{p}^{(2)}_{CCP\to i} \\
&= e_i-p_{i\to CCP}+\bar{p}_{CCP\to i}.
\end{aligned} \qquad (6.13)
$$

□

The next result generalises Theorem 1 by showing that a system in which all VM obligations are centrally cleared is more efficient, in the sense that liquidity shortfalls are smaller, than a system in which only a fraction of all VM obligations are centrally cleared. Amini et al. (2020) and Ahn (2020) proved similar results in richer models. Amini et al. (2020) introduced liquidation costs, end users, and default fund contributions. In Ahn (2020) there are bankruptcy costs and institutions can have external liabilities.

Theorem 2. *Let the tuple $(\mathbf{L}, \mathbf{e}^{(1)})$ be a family of clearing systems with perfectly correlated exposures and let $\alpha \in [0,1)$. For all banks, the shortfall at $\alpha < 1$ is larger than or equal to the shortfall in the fully centrally cleared setting:*

$$s_i(\alpha) \geq s_i(1) \quad \forall i\,. \tag{6.14}$$

Proof. At $\alpha = 1$ shortfalls from bilateral obligations are equal to zero, hence: $s_i(1) = s_i^c(1)$. At $\alpha = 1$ centrally cleared obligations are cleared in two sub-stages. Since $\alpha \in [0,1)$ is an arbitrary number, I can always find such value which the system clears centrally obligations corresponding to a fraction α of notional in the first stage, while it clears centrally cleared obligations corresponding to a fraction $1-\alpha$ of notional in the second stage (see Chapter 4.4). I denote the shortfalls that bank i records in those two sub-stages with $s_i^c(1|\alpha)$ and $s_i^c(1|1-\alpha)$. Therefore, by using Lemma 1 it gives:

$$s_i(\alpha) = s_i^c(\alpha) + s_i^b(1-\alpha) \tag{6.15a}$$

$$s_i(1) = s_i^c(1|\alpha) + s_i^c(1|1-\alpha)\,. \tag{6.15b}$$

The first observation is that $s_i^c(\alpha) = s_i^c(1|\alpha)$, for all i. This descends from Eq.4.7 because the cash endowment is equal to $e_i^{(1)}$ in both cases and net VM obligations are equal to $\bar{p}_{i\to CCP} = \alpha(\sum L_{ij} - L_{ji})^+$ in both cases (as the same fraction of notional α is centrally cleared). From Eq.4.10 it descends that also the cash endowment *after this stage* is the same.

The second observation is that $s_i^b(1-\alpha) \geq s_i^c(1|1-\alpha)$, for all i. Cash endowments are the same in both cases because they are the cash endowments at the end of the previous stage. The matrix of *gross* VM obligations is $(1-\alpha)\mathbf{L}$ in both cases, but at α those cleared fully bilaterally, while at $\alpha = 1$ those are fully centrally cleared. Therefore, by using Theorem 1 I have that $s_i^b(1-\alpha) \geq s_i^c(1|1-\alpha)$, which implies $s_i(\alpha) \geq s_i(1)$. $\square$

Notice that the EN clearing vector found in Eq.3.11 can also be split into two stages as the case on centrally cleared obligations. The relationship between total value and partial value also follows summation relationship.

Lemma 2 (Splitting of Eisenberg and Noe in two sub-stages)**.** *Let L_{ij}, for all i and j be a matrix of obligations defined as in Eq.4.1, $\mathbf{e} \geq 0$ a vector of cash endowments, and $\beta \in [0,1]$. Quantities in the first sub-stage as following:*

$$L_{ij}^{(1)} = \beta L_{ij} \tag{6.16a}$$

$$e_i^{(1)} = e_i\,, \tag{6.16b}$$

with total obligations $\bar{p}_i^{(1)}$ defined as in Eq.4.11, relative liability matrix $\Pi_{ij}^{(1)}$ defined as in Eq.4.12, payments $p_i^{(1)*}$ defined as in Eq.4.13 and shortfalls $s_i^{b\,(1)}$ defined as in Eq.4.14, for all i and j. Quantities in the second sub-stage as following:

$$\bar{p}_i^{(2)} = \bar{p}_i - p_i^{(1)*} \tag{6.17a}$$

$$\Pi_{ij}^{(2)} = \Pi_{ij}^{(1)} \tag{6.17b}$$

$$L_{ij}^{(2)} = \Pi_{ij}^{(2)}\bar{p}_i^{(2)} \tag{6.17c}$$

$$e_i^{(2)} = e_i^{(1)} - p_i^{(1)*} + \sum_j \Pi_{ji}^{(1)} p_j^{(1)*}\,, \tag{6.17d}$$

with payments $p_i^{(2)*}$ defined as in Eq.4.13, and shortfalls $s_i^{b\,(2)}$ defined as in Eq.4.14, for all i and j. Then I have that:

$$L_{ij}^{(2)} \geq (1-\beta)L_{ij} \tag{6.18}$$

$$p_i^* = p_i^{(1)*} + p_i^{(2)*} \tag{6.19}$$

$$s_i^b = s_i^{b\,(1)} + s_i^{b\,(2)}\,, \tag{6.20}$$

for all i and j.

Proof. In order to prove Eq.6.18 I note that in the case $\beta = 0$, $L_{ij} = L_{ij}^{(2)}$, as the clearing reduces to the second sub-stage. In the case $\beta > 0$ I have:

$$\begin{aligned}
L_{ij}^{(2)} &= \Pi_{ij}^{(2)} \bar{p}_i^{(2)} \\
&= \Pi_{ij}^{(2)} \left(\bar{p}_i - p_i^{(1)*}\right) \\
&\geq \Pi_{ij}^{(2)} \left(\bar{p}_i - \bar{p}_i^{(1)}\right) \\
&= \Pi_{ij}^{(2)} \left(\bar{p}_i - \beta\,\bar{p}_i\right) \\
&= (1-\beta)\Pi_{ij}\bar{p}_i \\
&= (1-\beta)L_{ij}\,,
\end{aligned} \tag{6.21}$$

for all i and j.

From Eq.6.17 I note that $L_{ij} = L_{ij}^{(1)} + L_{ij}^{(2)}$, for all i. Using Eq.4.14, this means that it is sufficient to prove $p_i^* = p_i^{(1)*} + p_i^{(2)*}$, for all i. The case $\beta = 0$ is easy to prove, as the clearing reduces to the second sub-stage. Let us then focus on the case in which $\beta > 0$. I start by observing that $\bar{p}_i = 0 \Leftrightarrow \bar{p}_i^{(1)} = 0$, meaning that the zero entries of the matrix Π coincide with the zero entries of the matrix $\Pi^{(1)}$. As regards non-zero entries, I have:

$$\Pi_{ij}^{(1)} = \frac{L_{ij}^{(1)}}{\bar{p}_i^{(1)}} = \frac{\beta L_{ij}}{\beta\,\bar{p}_i} = \frac{L_{ij}}{\bar{p}_i} = \Pi_{ij}\,, \tag{6.22}$$

so that $\Pi = \Pi^{(1)}$. From the definition of $p_i^{(2)*}$ I have:

$$\begin{aligned}
p_i^{(2)*} &= \min\left(\bar{p}_i^{(2)}, e_i^{(2)} + \sum_j \Pi_{ji}^{(2)} p_j^{(2)*}\right) \\
&= \min\left(\bar{p}_i - p_i^{(1)*}, e_i^{(1)} - p_i^{(1)*} + \sum_j \Pi_{ji}^{(1)} p_j^{(1)*} + \sum_j \Pi_{ji}^{(2)} p_j^{(2)*}\right)
\end{aligned} \tag{6.23}$$

$$= \min\left(\bar{p}_i - p_i^{(1)*}, e_i^{(1)} - p_i^{(1)*} + \sum_j \Pi_{ji}(p_j^{(1)*} + p_j^{(2)*})\right),$$

for all i, or:

$$p_i^{(1)*} + p_i^{(2)*} = \min\left(\bar{p}_i, e_i + \sum_j \Pi_{ji}(p_j^{(1)*} + p_j^{(2)*})\right), \tag{6.24}$$

for all i. Since:

$$p_i^* = \min\left(\bar{p}_i, e_i + \sum_j \Pi_{ji} p_j^*\right), \tag{6.25}$$

for all i, by taking the least solution of both the previous equations, I have that $p_i^* = p_i^{(1)*} + p_i^{(2)*}$, for all i. □

To the best of my knowledge, previous studies compared fully centrally cleared markets either with fully bilateral markets (Amini et al., 2020; Ahn, 2020) (as in Theorem 1) or with mixed markets (Amini et al., 2016) (as in Theorem 2). While those results allow to gauge what happens when I move to a fully centrally cleared market, they do not tell anything about the case in which the fraction of notional that is centrally cleared increases (or decreases). To this end I now compare two different mixed markets corresponding to two fractions of centrally cleared notional, e.g. α_1 and α_2.

In the following theorems and proofs, I will split centrally and bilaterally cleared obligations in the mixed markets. Regardless how payments are split, remember that the bilateral clearing never begins before the central clearing completes.

Theorem 3. *Let the tuple* $(\mathbf{L}, \mathbf{e}^{(1)})$ *be a family of clearing systems with perfectly correlated exposures and let* $\alpha_1, \alpha_2 \in [0,1]$*, with* $\alpha_1 \leq \alpha_2$*. The aggregate shortfall is a decreasing function of* α*:*

$$\sum_i s_i(\alpha_1) \geq \sum_i s_i(\alpha_2). \tag{6.26}$$

Proof. Let $\alpha_1 \leq \alpha_2$. The strategy I follow here is based on a specific decomposition of shortfalls at α_1 and α_2. Both at α_1 and α_2 I have

$$s_i(\alpha_1) = s_i^c(\alpha_1) + s_i^b(\alpha_1) \tag{6.27a}$$

$$s_i(\alpha_2) = s_i^c(\alpha_2) + s_i^b(\alpha_2)\,. \tag{6.27b}$$

At α_1, a fraction $1-\alpha_1$ of notional is in bilateral obligations, which yield the shortfall $s_i^b(\alpha_1)$. I further split the stage in which bilateral obligations are cleared in two sub-stages. In the first sub-stage I clear bilateral obligations corresponding to a fraction $\alpha_2-\alpha_1$ of notional, while in the second sub-stage I clear the residual bilateral obligations (see Lemma 2). In the second sub-stage, obligations will be *larger than or equal to* obligations corresponding to a fraction $1-\alpha_2$ of notional, which are the bilateral obligations cleared at α_2. I denote the shortfalls that bank i records in those two sub-stages with $s_i^b(\alpha_1|\alpha_2-\alpha_1)$ and $s_i^b(\alpha_1|\geq 1-\alpha_2)$. Similarly, at α_2 a fraction α_2 of notional is in centrally cleared obligations, which yields the shortfall $s_i^c(\alpha_2)$. Here I clear centrally cleared obligations in two sub-stages. In the first sub-stage I clear centrally cleared obligations corresponding to a fraction α_1 of notional, while in the second sub-stage I clear centrally cleared obligations corresponding to a fraction $\alpha_2-\alpha_1$ of notional (see Lemma 1). I denote the shortfalls that bank i records in those two sub-stages with $s_i^c(\alpha_2|\alpha_1)$ and $s_i^c(\alpha_2|\alpha_2-\alpha_1)$. By using Lemmas 1 and 2 I can rewrite Eq.6.27 as:

$$s_i(\alpha_1) = s_i^c(\alpha_1) + s_i^b(\alpha_1|\alpha_2-\alpha_1) + s_i^b(\alpha_1|\geq 1-\alpha_2) \tag{6.28a}$$

$$s_i(\alpha_2) = s_i^c(\alpha_2|\alpha_1) + s_i^c(\alpha_2|\alpha_2-\alpha_1) + s_i^b(\alpha_2)\,. \tag{6.28b}$$

The first observation is that $s_i^c(\alpha_1) = s_i^c(\alpha_2|\alpha_1)$, for all i. This descends from Eq.4.7 because the cash endowment is equal to $e_i^{(1)}$ in both cases and net VM obligations are equal to $\bar{p}_{i\rightarrow CCP} = \alpha_1(\sum L_{ij} - L_{ji})^+$ in both cases (as I centrally clear the same fraction of notional α_1). From Eq.4.10 it descends that also the cash endowment *after this stage* is the same, let us denote it with $\tilde{\mathbf{e}}$.

The second observation is that $s_i^b(\alpha_1|\alpha_2-\alpha_1) \geq s_i^c(\alpha_2|\alpha_2-\alpha_1)$, for all i. Cash endowments are the same in both cases because are the cash endowments at the end of the previous stage. The matrix of *gross* VM obligations is $(\alpha_2-\alpha_1)\mathbf{L}$ in both cases, but at α_1 those cleared fully bilaterally, while at α_2 those are fully centrally cleared. Therefore, by using Theorem 1 I have that $s_i^b(\alpha_1|\alpha_2-\alpha_1) \geq s_i^c(\alpha_2|\alpha_2-\alpha_1)$.

The final step is to compare $\sum_i s_i^b(\alpha_1| \geq 1-\alpha_2)$ with $\sum_i s_i^b(\alpha_2)$. Both at α_1 and α_2 obligations are cleared bilaterally. At α_2 I clear bilaterally the matrix of *gross* VM obligations $\mathbf{L}'' = (1-\alpha_2)\mathbf{L}$, corresponding to *net* VM obligations $L''_{ij} = (1-\alpha_2)L_{ij}$, for all i and j. At α_2 the cash available is the cash after the second sub-stage of central clearing (see Lemma 1), which I denote with $\mathbf{e}''$. At α_1 I clear bilaterally *net* VM obligations $L'_{ij} \geq (1-\alpha_2)L_{ij}$. To see this, it is sufficient to use Eq.6.18 in Lemma 2 and noting that at α_1 I clear bilaterally a fraction of notional $1-\alpha_1$. At α_1 the cash available is the cash after the first sub-stage of bilateral clearing (see Lemma 2), which I denote with $\mathbf{e}'$. I now show that $e''_i \geq e'_i$, for all i. In order to see this, let us remind that:

$$e''_i = \left[\tilde{e}_i - (\alpha_2-\alpha_1)\Big(\sum_j L_{ij} - L_{ji}\Big)^+\right]^+ + (\alpha_2-\alpha_1)\Big(\sum_j L_{ij} - L_{ji}\Big)^- . \tag{6.29}$$

If $\sum_j L_{ij} - L_{ji} \geq 0$, then $\Big(\sum_j L_{ij} - L_{ji}\Big)^+ = \sum_j L_{ij} - L_{ji}$ and $\Big(\sum_j L_{ij} - L_{ji}\Big)^- = 0$, so that:

$$e''_i = \left[\tilde{e}_i - (\alpha_2-\alpha_1)\Big(\sum_j L_{ij} - L_{ji}\Big)\right]^+ . \tag{6.30}$$

If $\sum_j L_{ij} - L_{ji} < 0$, then $\Big(\sum_j L_{ij} - L_{ji}\Big)^+ = 0$ and $\Big(\sum_j L_{ij} - L_{ji}\Big)^- = -\Big(\sum_j L_{ij} - L_{ji}\Big) > 0$, meaning that

$$\begin{aligned} e''_i &= \tilde{e}_i - (\alpha_2-\alpha_1)\Big(\sum_j L_{ij} - L_{ji}\Big) \\ &= \left[\tilde{e}_i - (\alpha_2-\alpha_1)\sum_j \left(L_{ij} - L_{ji}\right)\right]^+ , \end{aligned} \tag{6.31}$$

which then holds in all cases. Instead:

$$e_i' = \tilde{e}_i - p_i^*(\alpha_1|\alpha_2-\alpha_1) + \sum_j \Pi_{ji} p_j^*(\alpha_1|\alpha_2-\alpha_1)\,, \tag{6.32}$$

where with $p_i^*(\alpha_1|\alpha_2-\alpha_1)$ I denote the payment made by bank i in the first sub-stage of bilateral clearing at α_1. Now, payments in the first sub-stage at α_1 are:

$$p_i^*(\alpha_1|\alpha_2-\alpha_1) = \min\left[\bar{p}_i(\alpha_1|\alpha_2-\alpha_1), \tilde{e}_i + \sum_j \Pi_{ji} p_j^*(\alpha_1|\alpha_2-\alpha_1)\right]\,, \tag{6.33}$$

where with $\bar{p}_i(\alpha_1|\alpha_2-\alpha_1)$ I denote the net VM obligation of bank i in the first sub-stage of bilateral clearing at α_1. This leaves us with two cases. Either: $p_i^*(\alpha_1|\alpha_2-\alpha_1) = \tilde{e}_i + \sum_j \Pi_{ji} p_j^*(\alpha_1|\alpha_2-\alpha_1)$, and therefore $e_i' = 0$, which immediately implies $e_i' \le e_i''$. Or: $p_i^*(\alpha_1|\alpha_2-\alpha_1) = \bar{p}_i(\alpha_1|\alpha_2-\alpha_1)$, and therefore:

$$\begin{aligned}
e_i' &= \tilde{e}_i - p_i^*(\alpha_1|\alpha_2-\alpha_1) + \sum_j \Pi_{ji} p_j^*(\alpha_1|\alpha_2-\alpha_1) \\
&\le \tilde{e}_i - \bar{p}_i(\alpha_1|\alpha_2-\alpha_1) + \sum_j \Pi_{ji} \bar{p}_j(\alpha_1|\alpha_2-\alpha_1) \\
&\le \tilde{e}_i - (\alpha_2-\alpha_1)\sum_j \left(L_{ij}-L_{ji}\right)^+ + (\alpha_2-\alpha_1)\sum_j \left(L_{ji}-L_{ij}\right)^+ \\
&= \tilde{e}_i - (\alpha_2-\alpha_1)\sum_j \left(L_{ij}-L_{ji}\right)^+ + (\alpha_2-\alpha_1)\sum_j \left(L_{ij}-L_{ji}\right)^- \\
&= \tilde{e}_i - (\alpha_2-\alpha_1)\sum_j \left(L_{ij}-L_{ji}\right) \\
&\le e_i''\,.
\end{aligned} \tag{6.34}$$

To summarise, $\sum_i s_i^b(\alpha_1| \ge 1-\alpha_2)$ is the aggregate shortfall resulting from the least solution of the Eisenberg and Noe algorithm with obligations $\mathbf{L}'$ and cash $\mathbf{e}'$, while $\sum_i s_i^b(\alpha_2)$ is the aggregate shortfall resulting from the least solution of the Eisenberg and Noe algorithm with obligations $\mathbf{L}''$ and cash $\mathbf{e}''$. In both case the matrix of relative liabilities is Π. Moreover, $\mathbf{L}' \ge \mathbf{L}''$ and $\mathbf{e}' \le \mathbf{e}''$. Since the least solution of the Eisenberg and Noe algorithm (and payments therefore shortfalls) depend only on obligations and cash (as the matrix of relative liabilities is Π in all cases), for the remainder of the proof I will denote with $p_i(\mathbf{e},\mathbf{L})$ the payment of bank i when cash is $\mathbf{e}$ and obligations are $\mathbf{L}$, and analogously for shortfalls. I have: $s_i^b(\alpha_1|\geq 1-\alpha_2) = s_i^b(\mathbf{e}',\mathbf{L}')$, and $s_i^b(\alpha_2) = s_i^b(\mathbf{e}'',\mathbf{L}'')$, for all i. By using Lemma 5 in Eisenberg and Noe (2001) I have that:

$$\sum_i |\bar{p}_i' - \bar{p}_i''| \geq \sum_i |p_i(\mathbf{e}',\mathbf{L}') - p_i(\mathbf{e}',\mathbf{L}'')| \tag{6.35}$$

but from Lemma 5 in Eisenberg and Noe (2001) I also have that $p_i(\mathbf{e}',\mathbf{L}') \geq p_i(\mathbf{e}',\mathbf{L}'')$, for all i. Since $\mathbf{L}' \geq \mathbf{L}''$, I have:

$$\sum_i \bar{p}_i' - \bar{p}_i'' \geq \sum_i p_i(\mathbf{e}',\mathbf{L}') - p_i(\mathbf{e}',\mathbf{L}'') \tag{6.36}$$

or, by re-arranging terms:

$$\begin{aligned} \sum_i \bar{p}_i' - p_i(\mathbf{e}',\mathbf{L}') &\geq \sum_i \bar{p}_i'' - p_i(\mathbf{e}',\mathbf{L}'') \\ \sum_i s_i^b(\mathbf{e}',\mathbf{L}') &\geq \sum_i s_i^b(\mathbf{e}',\mathbf{L}''). \end{aligned} \tag{6.37}$$

Moreover, using again Lemma 5 in Eisenberg and Noe (2001), since $\mathbf{e}'' \geq \mathbf{e}'$, I have that $p_i(\mathbf{e}'',\mathbf{L}'') \geq p_i(\mathbf{e}',\mathbf{L}'')$, for all i, or:

$$s_i^b(\mathbf{e}'',\mathbf{L}'') = \bar{p}_i'' - p_i(\mathbf{e}'',\mathbf{L}'') \leq \bar{p}_i'' - p_i(\mathbf{e}',\mathbf{L}'') = s_i^b(\mathbf{e}',\mathbf{L}''), \tag{6.38}$$

for all i. Therefore, from Eq.6.37 I have:

$$\sum_i s_i^b(\mathbf{e}',\mathbf{L}') \geq \sum_i s_i^b(\mathbf{e}',\mathbf{L}'') \geq \sum_i s_i^b(\mathbf{e}'',\mathbf{L}'') \tag{6.39}$$

or, by remembering the definitions:

$$\sum_i s_i^b(\alpha_1|\geq 1-\alpha_2) \geq \sum_i s_i^b(\alpha_2), \tag{6.40}$$

which concludes the proof. □

Theorem 3 shows that, for perfectly correlated exposures, increasing the fraction of centrally cleared notional from α to $\alpha' = \alpha + \Delta\alpha$, is always (weakly) beneficial in aggregate, independently of the starting fraction of notional α that is centrally cleared and of the additional fraction of notional $\Delta\alpha$ that becomes centrally cleared. In other words, for perfectly correlated exposures, in aggregate there are no unintended consequences of increasing the fraction of notional that is centrally cleared.

Theorem 3 indicates that increasing the fraction of centrally cleared notional is weakly beneficial in aggregate. But it does not exclude that, at least in some interval, increasing the fraction of centrally cleared notional is not strictly beneficial. That is, it does not exclude that, in some interval, the aggregate shortfall does not decrease as the fraction of centrally cleared notional increases. I find that this is indeed the case. More precisely, shortfalls of all banks do not decrease in the interval $[0, \alpha^*)$, where α^* is a critical value that depends of cash buffers and net obligations of individual banks.

Theorem 4. *Let $\mathscr{S}(\mathbf{L}, \mathbf{e}^{(1)})$ be a family of clearing systems with perfectly correlated exposures and let:*

$$\alpha^* = \min_i \frac{e_i^{(1)}}{\left(\sum_j L_{ij} - L_{ji}\right)^+}. \tag{6.41}$$

Then, for all $\alpha < \alpha^*$, $s_i(\alpha)$ is independent of α, i.e. $s_i(0) = s_i(\alpha)$, for all i.

Proof. Let us start by observing that:

$$e_i^{(3)} = \left[e_i^{(1)} - \alpha\left(\sum_j L_{ij} - L_{ji}\right)^+\right]^+ + \alpha\left(\sum_j L_{ij} - L_{ji}\right)^-. \tag{6.42}$$

If $\sum_j L_{ij} - L_{ji} \geq 0$, then $\left(\sum_j L_{ij} - L_{ji}\right)^+ = \sum_j L_{ij} - L_{ji}$ and $\left(\sum_j L_{ij} - L_{ji}\right)^- = 0$. Therefore, for $\alpha \leq \alpha^*$:

$$e_i^{(3)} = e_i^{(1)} - \alpha \sum_j \left(L_{ij} - L_{ji}\right). \tag{6.43}$$

If $\sum_j L_{ij} - L_{ji} < 0$, then $\left(\sum_j L_{ij} - L_{ji}\right)^+ = 0$ and $\left(\sum_j L_{ij} - L_{ji}\right)^- = -\Big(\sum_j L_{ij} -$

$L_{ji}\Big) > 0$, meaning that:

$$e_i^{(3)} = e_i^{(1)} - \alpha \sum_j \left(L_{ij} - L_{ji} \right). \tag{6.44}$$

which then holds for all i and for $\alpha \leq \alpha^*$.

For the remainder of the proof, in order to make our notation more compact I introduce:

$$b_{ij} = L_{ij} - L_{ji} \tag{6.45}$$

and I briefly note that $b_{ij} = b_{ij}^+ - b_{ij}^-$ and that $b_{ij}^+ = b_{ji}^-$. In order to prove that shortfalls do not depend on α I will check that all the terms that multiply α (which I refer to as the α terms) are equal to zero. I use the symbol $\overset{\alpha}{\simeq}$ to indicate that I are keeping only the α terms or the terms that may contain α.

Let us denote with $\mathscr{S}$ the set of banks that do not default in the bilateral round and with $\mathscr{D}$ the set of banks that default in the bilateral round. All banks in $\mathscr{S}$ pay in full and have zero shortfall. The realized payments of banks in $\mathscr{D}$ are:

$$\begin{aligned} p_i^* &= e_i^{(3)} + \sum_j \Pi_{ji} p_j^* \\ &= e_i^{(1)} - \alpha \sum_j b_{ij} + \sum_{j \in \mathscr{S}} \Pi_{ji} \bar{p}_j + \sum_{j \in \mathscr{D}} \Pi_{ji} p_j^* \\ &= e_i^{(1)} - \alpha \sum_{j \in \mathscr{S}} b_{ij} - \alpha \sum_{j \in \mathscr{D}} b_{ij} + (1-\alpha) \sum_{j \in \mathscr{S}} b_{ji}^+ + \sum_{j \in \mathscr{D}} \Pi_{ji} p_j^*, \end{aligned} \tag{6.46}$$

while their shortfall is:

$$\begin{aligned} s_i &= (1-\alpha) \sum_j b_{ij}^+ - p_i^* \\ &= (1-\alpha) \sum_{j \in \mathscr{S}} b_{ij}^+ + (1-\alpha) \sum_{j \in \mathscr{D}} b_{ij}^+ - p_i^* \\ &\overset{\alpha}{\simeq} -\alpha \sum_{j \in \mathscr{S}} b_{ij}^+ - \alpha \sum_{j \in \mathscr{D}} b_{ij}^+ + \alpha \sum_{j \in \mathscr{S}} b_{ij} + \alpha \sum_{j \in \mathscr{D}} b_{ij} + \alpha \sum_{j \in \mathscr{S}} b_{ji}^+ - \sum_{j \in \mathscr{D}} \Pi_{ji} p_j^* \\ &= \alpha \sum_{j \in \mathscr{S}} \left(-b_{ij}^+ + b_{ij} + b_{ji}^+ \right) + \alpha \sum_{j \in \mathscr{D}} \left(b_{ij} - b_{ij}^+ \right) - \sum_{j \in \mathscr{D}} \Pi_{ji} p_j^* \end{aligned} \tag{6.47}$$

$$= \alpha \sum_{j\in\mathscr{S}} \left(b_{ij} - (b^{+}_{ij} - b^{-}_{ij})\right) - \alpha \sum_{j\in\mathscr{D}} b^{-}_{ij} - \sum_{j\in\mathscr{D}} \Pi_{ji} p^{*}_{j}$$
$$= -\alpha \sum_{j\in\mathscr{D}} b^{-}_{ij} - \sum_{j\in\mathscr{D}} \Pi_{ji} p^{*}_{j}.$$

As a consequence, I are left to prove that all the α terms in:

$$-s_i \overset{\alpha}{\simeq} \alpha \sum_{j\in\mathscr{D}} b^{-}_{ij} + \sum_{j\in\mathscr{D}} \Pi_{ji} p^{*}_{j} \tag{6.48}$$

sum to zero. To this effect let us re-write Eq.6.48 as:

$$\begin{aligned}
-s_i &\overset{\alpha}{\simeq} \alpha \sum_{j\in\mathscr{D}} b^{-}_{ij} + \sum_{j\in\mathscr{D}} \Pi_{ji} p^{*}_{j} \\
&= \alpha \sum_{j\in\mathscr{D}} b^{-}_{ij} + \sum_{j\in\mathscr{D}} \frac{b^{+}_{ji}}{\sum_k b^{+}_{jk}} p^{*}_{j} \\
&= \alpha \sum_{j\in\mathscr{D}} b^{-}_{ij} + \sum_{j\in\mathscr{D}} \frac{b^{-}_{ij}}{\sum_k b^{+}_{jk}} p^{*}_{j} \\
&= \sum_{j\in\mathscr{D}} b^{-}_{ij} \left(\alpha + \frac{p^{*}_{j}}{\sum_k b^{+}_{jk}}\right)
\end{aligned} \tag{6.49}$$

Since all js in the summation above are in $\mathscr{D}$, I can use Eq.6.46 and keep only the α terms:

$$\begin{aligned}
p^{*}_{j} &\overset{\alpha}{\simeq} -\alpha \sum_{k\in\mathscr{S}} b_{jk} - \alpha \sum_{k\in\mathscr{D}} b_{jk} - \alpha \sum_{k\in\mathscr{S}} b^{+}_{kj} + \sum_{k\in\mathscr{D}} \Pi_{kj} p^{*}_{k} \\
&= -\alpha \sum_{k\in\mathscr{S}} \left(b_{jk} + b^{-}_{jk}\right) - \alpha \sum_{k\in\mathscr{D}} b_{jk} + \sum_{k\in\mathscr{D}} \Pi_{kj} p^{*}_{k} \\
&= -\alpha \sum_{k\in\mathscr{S}} b^{+}_{jk} - \alpha \sum_{k\in\mathscr{D}} b_{jk} + \sum_{k\in\mathscr{D}} \Pi_{kj} p^{*}_{k} \\
&= -\alpha \sum_{k\in\mathscr{S}} b^{+}_{jk} - \alpha \sum_{k\in\mathscr{D}} b^{+}_{jk} + \alpha \sum_{k\in\mathscr{D}} b^{-}_{jk} + \sum_{k\in\mathscr{D}} \Pi_{kj} p^{*}_{k} \\
&= -\alpha \sum_{k} b^{+}_{jk} + \alpha \sum_{k\in\mathscr{D}} b^{-}_{jk} + \sum_{k\in\mathscr{D}} \Pi_{kj} p^{*}_{k},
\end{aligned} \tag{6.50}$$

which I can now plug into Eq.6.49:

$$
\begin{aligned}
-s_i \overset{\alpha}{\simeq} & \sum_{j\in\mathscr{D}} b_{ij}^- \left[\alpha + \frac{1}{\sum_k b_{jk}^+}\left(-\alpha\sum_k b_{jk}^+ + \alpha\sum_{k\in\mathscr{D}} b_{jk}^- + \sum_{k\in\mathscr{D}} \Pi_{kj} p_k^*\right)\right] \\
&= \sum_{j\in\mathscr{D}} \frac{b_{ij}^-}{\sum_k b_{jk}^+}\left(\alpha\sum_{k\in\mathscr{D}} b_{jk}^- + \sum_{k\in\mathscr{D}} \Pi_{kj} p_k^*\right) \\
&= \sum_{j\in\mathscr{D}} \frac{b_{ji}^+}{\sum_k b_{jk}^+}\left(\alpha\sum_{k\in\mathscr{D}} b_{jk}^- + \sum_{k\in\mathscr{D}} \Pi_{kj} p_k^*\right) \\
&= \sum_{j\in\mathscr{D}} \Pi_{ji}\left(\alpha\sum_{k\in\mathscr{D}} b_{jk}^- + \sum_{k\in\mathscr{D}} \Pi_{kj} p_k^*\right),
\end{aligned}
\tag{6.51}
$$

where the coefficients $\frac{b_{ij}^-}{\sum_k b_{jk}^+}$ do not depend on α, while the terms in parentheses have the same form of Eq.6.48. The important observation here is that, when computing the α terms for the shortfall of banks in $\mathscr{D}$, I are left with only with summations over banks in $\mathscr{D}$. If I proceed iteratively by plugging in the analogous of Eq.6.50 for p_k^* I arrive at an analogous expression where $\sum_{j\in\mathscr{D}}\Pi_{ji}$ is replaced by $\sum_{j,k\in\mathscr{D}}\Pi_{kj}\Pi_{ji}$ and the terms in parentheses correspond to the neighbors of k that are in $\mathscr{D}$. Eventually, if I keep iterating this procedure, the only terms left correspond to cycles of banks in $\mathscr{D}$. (This immediately implies that, if I only have two defaulted banks, shortfalls do not depend on α, as bilateral netting means that I cannot have 2-cycles.) Therefore, if I denote with $\mathscr{C}_i$ the set of cycles of i and with ℓ_c the length of the cycle c, I have:

$$
-s_i \overset{\alpha}{\simeq} \sum_{c\in\mathscr{C}_i} \Pi_{ij_{\ell_c}} \ldots \Pi_{j_2 j_1}\Pi_{j_1 i}\left(\alpha\sum_{j_1\in\mathscr{D}} b_{ij_1}^- + \sum_{j_1\in\mathscr{D}} \Pi_{j_1 i} p_{j_1}^*\right). \tag{6.52}
$$

I can now go through each cycle an arbitrary number of times, say n_c for cycle c:

$$
-s_i \overset{\alpha}{\simeq} \sum_{c\in\mathscr{C}_i} \left(\Pi_{ij_{\ell_c}} \ldots \Pi_{j_2 j_1}\Pi_{j_1 i}\right)^{n_c}\left(\alpha\sum_{j_1\in\mathscr{D}} b_{ij_1}^- + \sum_{j_1\in\mathscr{D}} \Pi_{j_1 i} p_{j_1}^*\right). \tag{6.53}
$$

The term in parentheses does not depend on n_c and is finite (realized payments) cannot exceed the payment obligations. On the other hand, as long as one of the $\Pi_{j_m,j_{m+1}}$ is strictly smaller than one, in the limit $n_c\to\infty$ I are left with no α terms. If all $\Pi_{j_m,j_{m+1}}$ are equal to zero, it means that i is part of an isolated cycle (i.e.

a closed chain in which all banks do not have obligations to any other bank) in which all banks are in $\mathscr{D}$, which is the only case that is left to prove. However, in the Eisenberg and Noe algorithm, as long as $e_i^{(3)}$ is strictly larger than zero for all i, there cannot be a closed cycle of banks in $\mathscr{D}$. In fact, banks will make partial payments and at least the link with the smallest payment obligation will disappear. This means that I are only requiring that $e_i^{(3)} > 0$, for all i, which is true as long as $\alpha < \alpha^*$. □

6.2 Homogeneous VM obligations

So far I have been able to derive some general qualitative properties of how shortfalls depend on α in the case of perfectly correlated exposures. That said, understanding whether α^* is a tight bound (i.e. whether aggregate shortfalls start decreasing at α^* or for α strictly larger than α^*) and characterising the dependence of α^* on the model parameters still requires me to perform numerical experiments. Hence, I generate a random ensemble of networks of VM obligations and, on each network of the ensemble, I simulate the payment algorithm in Section 4.4. I fix the number of institutions $N = 100$ and $L_i^{tot} = 100$, and the total VM exposure is evenly distributed to its counterparties. I further assume that all counterparty relationships exist independently with probability p (i.e. that networks have a Erdős-Rényi topology). I explore the effect of different level of connectivity by varying p, which is equivalent to varying average degree M. The exogenous cashflow $e_i^{(1)}$ is another variable I am interested in, so its relationship with the shortfall is also examined.

Fig.6.2 shows the behavior of the aggregate shortfall as a function of α for different values of the network density and exogenous cashflow. From the figure, it can clearly be seen the functional behavior discussed in Sec.6.1, i.e. the facts that the shortfall is non-increasing in α and that it is constant until a critical value. The figure also shows that increasing the density of the network leads to an overall reduction of the shortfall. This is due to the fact that increasing connectivity while keeping other variables constant leads to an increased level of bilateral netting. The explanation is further verified in Fig.6.3, where the blue area (bilateral netting) is larger in the right panel than the left. Finally, the shortfall increases if total cashflow is increased.

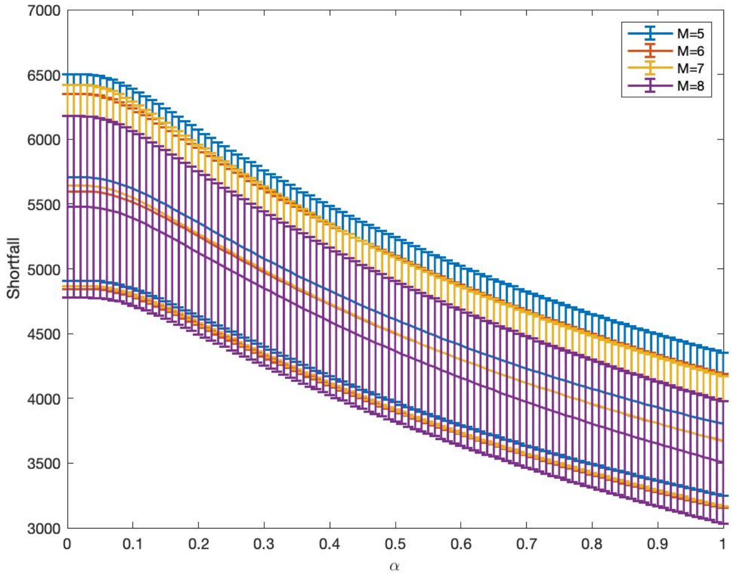


Figure 6.2: Average over 100 realisations of shortfall against α at $e^{(1)} = 15$ with errorbar.

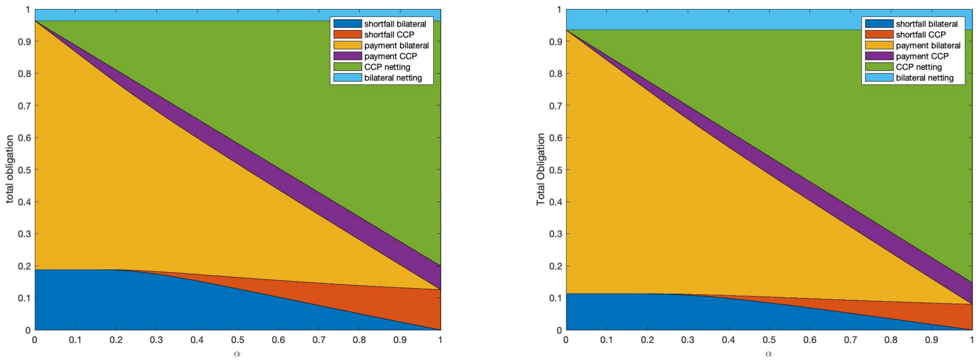


Figure 6.3: Decomposition of total obligations in netting, payments, and shortfalls with $e_i^{(1)} = 15$ at $M = 5$ (left panel) and $M = 8$ (right panel) over 100 realisation

In Figure 6.4 I show the behavior of α^* (left panel) and of the shortfall $s(\alpha^*)$ recorded at $\alpha = \alpha^*$ (right panel) as a function the average degree M for different values of the exogenous cashflow. Those small fluctuations that can be seen in the curves because I round α^* to the nearest percentage. It can be seen that, regardless of the value of $e^{(1)}$, α^* increases with the network connectivity, while $s(\alpha^*)$

decreases. This means that, the denser the network is, the larger the fraction of notional that should be centrally cleared before central clearing becomes beneficial, and the smaller the shortfall at α^*. This is not surprising as the larger number of counterparty relationships in a denser network yield more bilateral netting opportunities. Therefore, all else equal, a larger fraction of notional must be centrally cleared before I can see any benefit of central clearing.

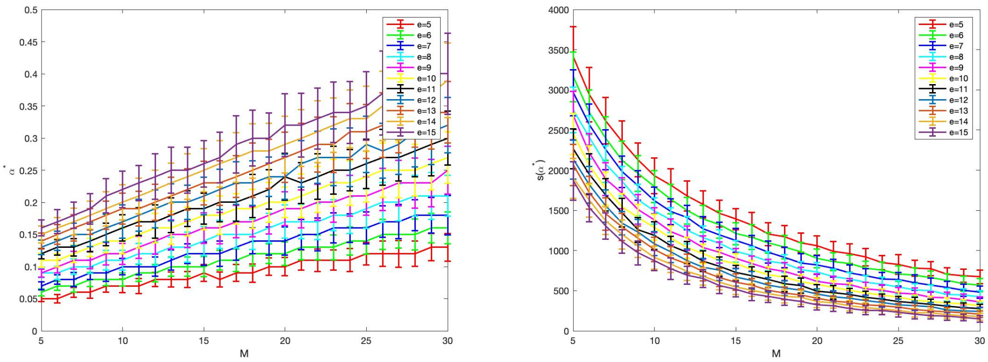


Figure 6.4: Critical fraction α^* of centrally cleared notional after which the aggregate shortfall starts start to decline (left panel) and aggregate shortfall at α^* (right panel). All points are averaged over 100 realisation of the network of obligations. All institutions have the same obligations that are distributed uniformly across their counterparties.

6.3 Heterogeneous VM obligations

I now relax the assumption of homogeneous total gross VM obligations across banks. More specifically, I consider the total gross VM obligations of each bank to be drawn from an exponential distribution with mean $L_i^{tot} = 100$, but I still consider the obligations of an institution to be uniformly distributed across its counterparties.

Results are reported in Fig.6.5 and 6.6. On the LHS of Fig.6.6 the variation occurs because I round the result to the nearest percentage. The most noticeable difference with respect to the homogeneous case is a much weaker dependency of α^* on the network density. Refer to Fig.6.5, the shortfall appears overall larger with respect to the homogeneous case, but α^* is now smaller. This suggest that, while a system with heterogeneous total gross VM obligations is less efficient (in the sense that shortfalls are larger), it is easier for central clearing to be beneficial, as the critical fraction of obligations that should be centrally cleared for the shortfall to start decreasing is lower with respect to the benchmark of a system in which total gross VM obligations are homogeneous.

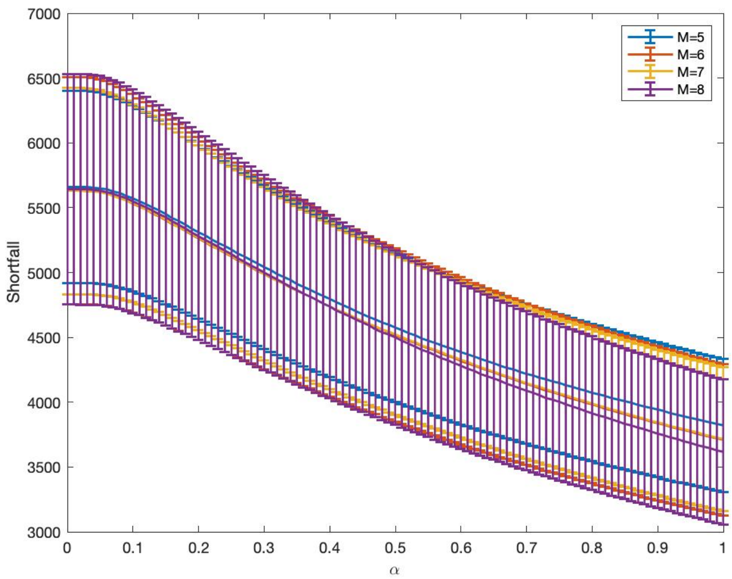


Figure 6.5: Shortfall of heterogeneous exposures against α at different level of M at $e^{(1)} = 15$ with errorbar. All points are averaged over 100 realisation of the network of obligations.

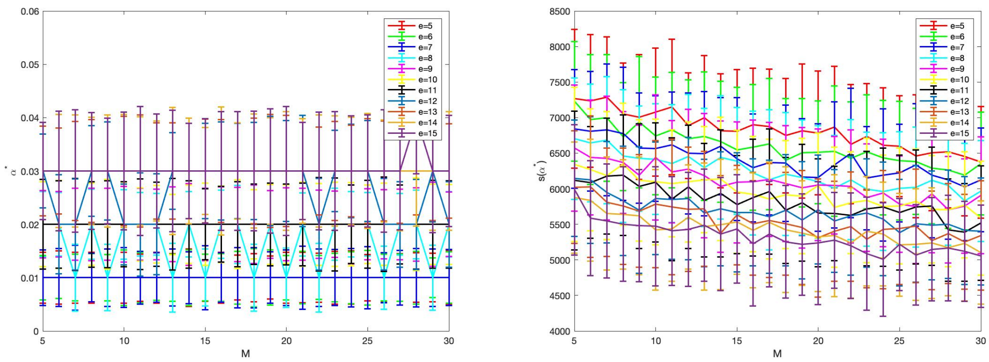


Figure 6.6: Critical fraction α^* of centrally cleared notional after which the aggregate shortfall starts start to decline (left panel) and aggregate shortfall at α^* (right panel). All points are averaged over 100 realisation with errorbar of the network of obligations. All institutions have the same obligations that are distributed uniformly across their counterparties.

Overall, in this chapter, I have analysed both analytically and numerically on the behaviour of central clearing to the liquidity shortfalls. The most noticeable result is that the non-increasing relationship between shortfall and α. On top of this, the CCP only becomes effective after a certain threshold of α. These findings will have implications for regulators to understand the impact of central clearing and thereby design the optimal clearing mechanism.

Chapter 7

Networks with Different Topology

In previous chapters, I have generalised the EN algorithm to account for the presence of a CCP and study the effect of central clearing. I proved general statements that are valid for generic networks and carried out analysis on hypothetical banking systems for which the clearing vectors are found by solving numerical the EN algorithm. However, the the Erdős-Rényi network is based on assumptions which are rarely observed in real financial networks. Therefore, in this chapter I will study networks with more realistic features to understand how they affect the efficiency of central clearing.

Previous studies (see Chapter 2) have proved that interbank networks mainly follow a heavy-tailed degree distribution, which means they are characterised by the presence of highly connected nodes (hubs) (Boss et al., 2004a; Inaoka et al., 2004; Soramäki et al., 2007). A typical example of heavy-tailed degree distribution is a power-law. Therefore, in the following I consider the networks with power law degree distribution as described in Eq.3.4 with appropriate level of γ and k_{min}.

The second generalisation relates to network assortativity, which measures the degree correlation. Positive assortativity means that high degree nodes tend to connect with high degree nodes and vice versa, while negative assortativity means that high degree nodes tend to connect with low degree nodes. It has been widely observed in reality (see Chapter 2) that interbank networks tend to be disassortative (Bech and Atalay, 2010; Iori et al., 2008). Such phenomenon can also be explained as small institutions are more likely to carry out business with large institutions instead of other small market players.

In the following I will present results related to above-mentioned two properties. In particular, the first experiment will be generating scale-free networks with similar average degree to the benchmark case. In the second experiment, I will keep the degree distribution the same as Erdős-Rényi ones then use the link rewiring process provided by Noh (2007) to generate networks with different assortativity. I consider a network structure similar to that of Chapter 6, i.e. with perfectly correlated exposures, as I proved stronger results analytically under this framework, and this is the case that is typically considered in the literature (Amini et al., 2016). In both experiments, I will investigate relationships between shortfall and α, and compare the different behaviour between those extended frameworks to the benchmark framework.

7.1 Scale-free Network

I am interested in understanding how the efficiency of scale-free networks compares to that of Erdős-Rényi networks, and whether it is easier or harder for a CCP to be effective. The rationale behind this question is the fact that scale-free networks are characterized by a hub-and-spoke structure that is more centralized with respect to Erdős-Rényi networks. In this section I carry out extensive numerical simulations to study the behaviour of liquidity shortfall under the scale-free network. Recall that the power-law degree distribution as I explained in Section 3.3 is the following:

$$P(k) \sim k^{-\gamma} \text{ with } k > k_{min}. \tag{7.1}$$

Different from all other sections where I generate networks using $N = 100$, in this section I generate scale-free networks with $N = 1000$. The reason falls behind that I need large sample to make heavy-tail exists. Accordingly, the average degree M now becomes $10\times$ as before, to ensure the probability of a link exists stays as close as the benchmark. The first step is to determine appropriate values for k_{min} and γ. It has been shown in literature that γ varies between 2.1 and 3 (De Masi and Gallegati, 2012; Soramäki et al., 2007) across different interbank networks, therefore the initial value used in the following simulation is $\gamma = 2.5$. To match the average degree of the Erdős-Rényi network, I run simulations on different values of k_{min} to infer the mapping between k_{min} and the average degree M. The appropriate value of k_{min} can then be selected as the one for which the average degree is the closest to the desired one. In detail, for each selected k_{min}, I generate 100 random scale-free networks and record the average degree. Then I select the one which produces closest level of desired average degree appropriate k_{min}. The corresponding k_{min} of values that are used in the following analysis is summarised in the following table:

γ	2.5	2.5	2.5	2.5	2.8	3	3.1	3.4	3.5	3.7	4
M	50	60	70	80	50	50	50	50	50	50	50
k_{min}	26	32	38	45	28	29	30	32	32	33	34

Table 7.1: Summarise all values of k_{min} that are used in the following analysis

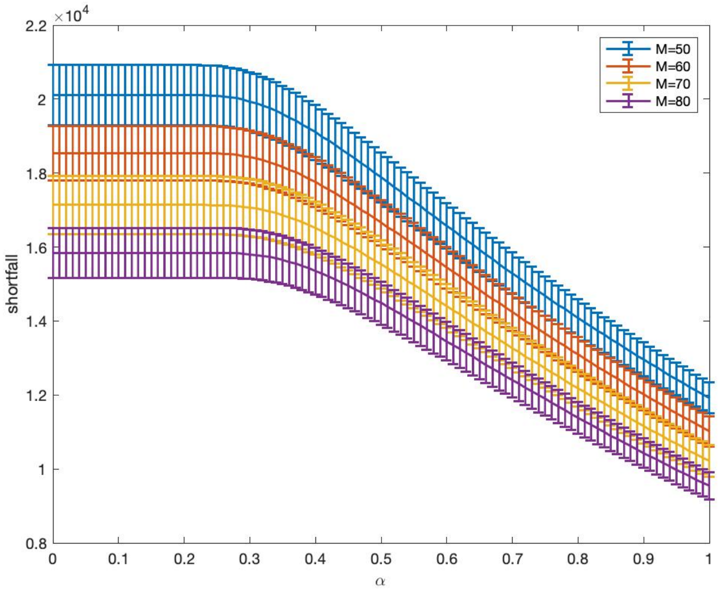


Figure 7.1: Shortfall against α for scale-free network at different level of M with $e^{(1)} = 15$ with errorbar. All points are averaged over 100 realisation of the network of obligations.

I run numerical simulations similar to those carry out in Chapter 6 and I compare results for those two sets of simulations. I consider perfectly correlated exposures and homogeneous VM obligations, i.e. $L_i^{tot} = L = 100$, for all i. Similarly to Sec.6.1 and observed in Sec.6.2, Fig.7.1 shows the relationship between shortfall and fraction of notions centrally cleared for different values of M and e. As for the case of Erdős-Rényi network I observe shortfall is decreasing against α and there is a threshold effect where the shortfall is constant for $\alpha < \alpha^*$, and increasing connectivity and cashflow improve the efficiency of the system.

The left panel in Fig.7.2 shows the performance of the scale free network compares to that of the Erdős-Rényi case. The relationship between shortfall and α is the same across these four experiments. However, the figure makes it clear that the scale-free network is less efficient than the Erdős-Rényi network and that increasing γ reduces the shortfall. As γ controls the level of heavy-tailness in the degree distribution, I conclude a more centralised system is adverse to the payment efficiency. In contrast, the system should be more robust if exposures are spreaded across more institutions.

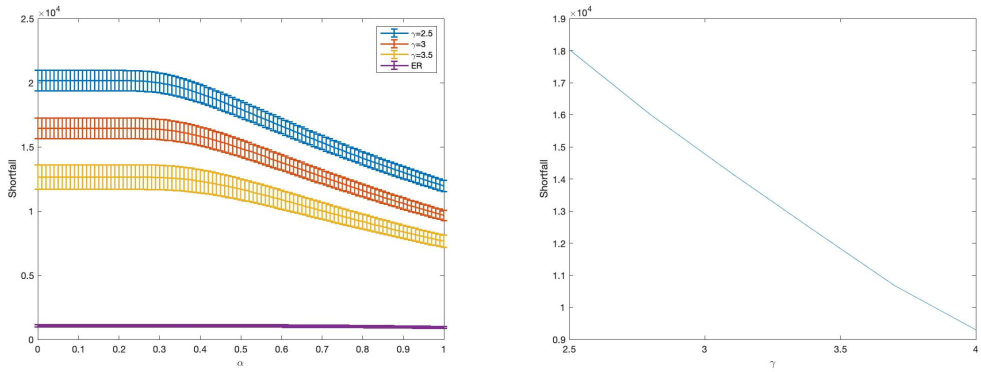


Figure 7.2: Compare scale-free networks with Erdős-Rényi network (left panel) and shortfall at different level of γ (right panel). Averaging over 100 realisations with $N = 1000$, $M = 5$, $e_i^{(1)} = 15$ and $L_i^{tot} = 100$.

An intuitive explanation of this phenomenon is the following: low-degree nodes are poorly connected, but in my experimental setting they have the same liabilities as the high-connected nodes. High-degree nodes have incoming links from several different counterparties. In contrast, low-degree nodes only have a few counterparites that provide interbank assets to them. Therefore, high-degree nodes have higher interbank assets than low-degree nodes do. The shortfall of a high-degree node is therefore lower than that of low-degree node. From Fig.7.3, I can see that CCP netting gets smaller for smaller γ, from which I conclude that the CCP is less effective in reducing shortfalls for scale-free networks. Recall Eq.4.4a that the CCP nets all transactions in aggregate level. Because interbank assets of low-degree nodes are smaller, the CCP nets less transactions, pushing total shortfall up.

In order to further verify this, I now focus on quantities of individual nodes instead of aggregate level. First, one should remember that all results and relationships in Sec.6.1 hold for both aggregate and individual level. After checking interbank assets of individual banks and comparing them with their degree, I do recognise that low-degree nodes exhibit lower interbank assets. Further, it seems that these nodes also experience higher shortfalls.

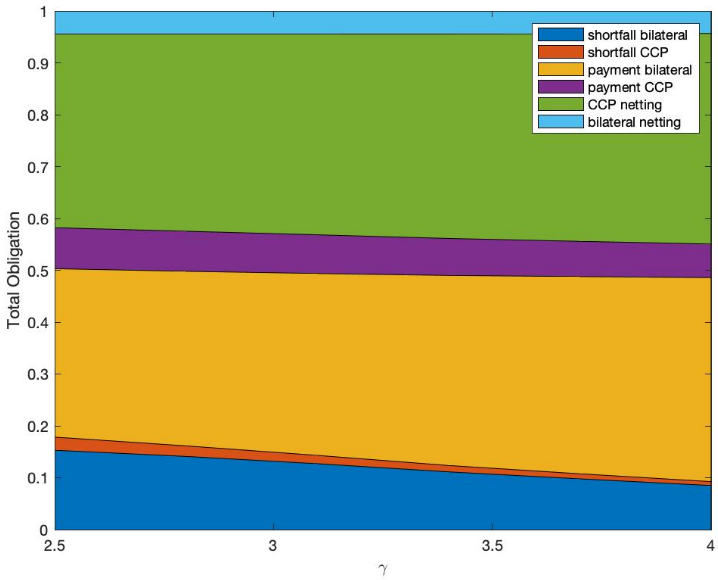


Figure 7.3: Decomposition of total obligations as six components at different level of γ. $N = 1000$, $e_i^{(1)} = 15$, $L_i^{tot} = 100$, $M = 50$ and corresponding k_{min} can be found in Table 7.1.

I use a regression model where the dependent variable is the shortfall of each node i and the exogenous variables are degree of each node 'deg' and a dummy variable 'ER', which takes 1 if node i is the Erdős-Rényi network or 0 if it is the scale-free network. The equation should be specified as:

$$SF_i = \beta_0 + \beta_1 deg_i + \beta_2 ER_i + \varepsilon_i \tag{7.2}$$

In Eq.7.2, β_0 is the intercept, β_1 is the coefficient associated with the degree, and β_2 is the coefficient of the dummy variable, which should be interpreted as the relationship to reference category (scale free network in this case). ε_i is simply the residual term. Table 7.2 shows that β_2 is significantly negative, indicating the Erdős-Rényi network has a lower shortfall compared to the scale-free network. β_1 is also significantly negative, confirming that lower degree nodes have higher shortfalls. This is because, as previously explained, low-degree nodes tend to have smaller interbank assets, with constant cash and interbank liabilities, and they are therefore more likely to record a higher shortfall. This, together with the fact that there are many nodes with low-degree in the scale-free network (see left panel Fig.3.1 for example), make the aggregate shortfall higher than that of the benchmark.

	Estimate	SE	t-stat	p-value
(Intercept)	53.5665	0.35184	152.2454	0
degree	-2.7682	0.028928	-95.6936	0
ER	-9.9124	0.28165	-35.1936	2.58×10^{-263}

Table 7.2: Regression results of Eq.7.2

As it can be seen in the left panel of Fig.7.2, the plateau also exists for small values of α, indicating that there also exists a threshold α^* after which the CCP begins to be effective. In Thm.4, I have already proved that α^* only depends on e and $\bar{p}_i$ as long as $\mathbf{L}^c = \mathbf{L}^b$. Therefore, properties related to α^* still holds and the effectiveness of the CCP remains the same compare to the Erdős-Rényi case.

7.2 Network Assortativity

In this section, I study the effect of assortativity on payment efficiency. To this end, I start from the Erdős-Rényi network and construct networks with different level of assortativity by rewiring edges following the algorithm in Noh (2007). The link rewiring process can generate networks with different assortativity, at the same time

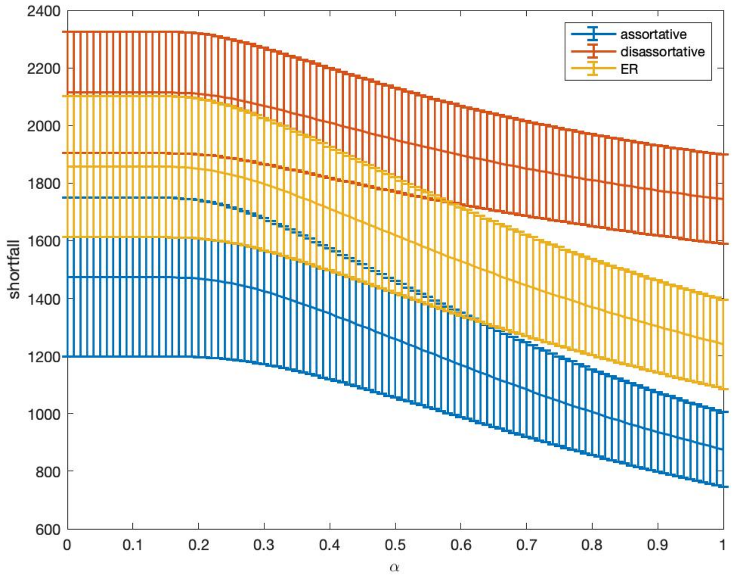


Figure 7.4: Compare Erdős-Rényi with (dis)assortative networks over 100 resolutions. $N = 100$, $M = 5$, $e = 15$ and $L_i^{tot} = 100$.

maintaining the degree of each node. It therefore allows me to study the effect at different degree correlation within a given degree distribution.

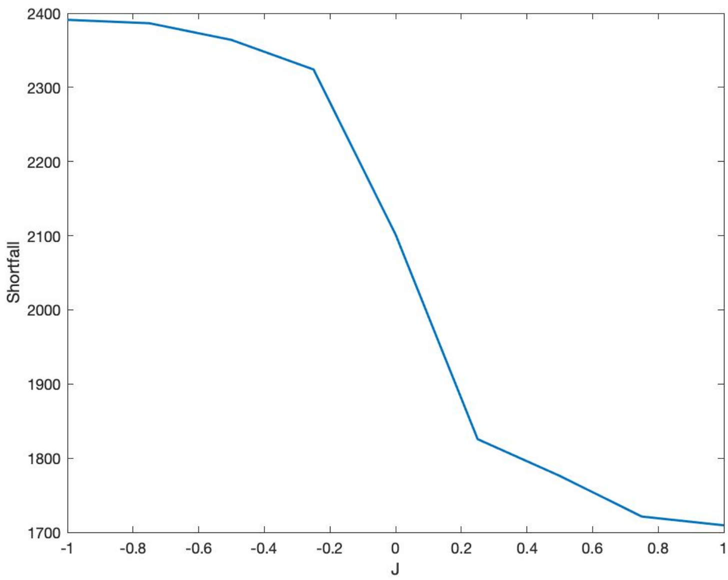


Figure 7.5: Relationship of J against shortfall at $\alpha = 0.5$. The network consists of $N = 100$ nodes and has average degree $M = 5$. Each node has initial cashflow $e_i^{(1)} = 15$.

In Fig.7.4, three series of shortfall values are calculated with Erdős-Rényi networks, assortative networks ($J = 1$) and disassortative networks ($J = -1$). It can be seen that the disassortative network has the highest value of shortfall, while the assortative network has the smallest. The behaviour of the shortfall as a function of α is similar across all models, and all properties found and proved in Chapter 6 are preserved.

In Fig.7.5, the value of shortfall is measured as a function of assortativity level at $\alpha = 0.5$, which is chosen to ensure it is greater than the threshold α^*. The key finding is that the shortfall is a decreasing function of assortativity. In this thesis, such observed phenomenon can be explained using the network structure. Because I have already proved in Table 7.2 that the degree has negative relationship with the shortfall, the most critical components lie with low-degree nodes. Assortativity only takes into account the degree correlation. Let us consider the case that the network is disassortative, and recall that by assumption all liabilities are fixed at 100 and distributed evenly across counterparties. Since low-degree nodes are connected with high-degree nodes, the liabilities of high-degree nodes are spread to many counterparites, therefore the amount of incoming assets of a low-degree node becomes smaller. On the other hand, for assortative networks, low-degree nodes are connected with low-degree nodes, total liabilities are concentrated to only a few counterparties.

7.3 Scale-Free Networks and Assortativity

What will happen if I combine scale-free with assortativity? In the first place, I should point out that these two properties are not independent and sometimes incorporating with each other. The scale-free network is assortative neutral only if the number of nodes getting large (Newman, 2002), and a small tail parameter ($\gamma < 3$) makes the scale free network disassortative (Fricke et al., 2013). That is the reason why I choose larger network with $N = 1000$ in Sec.7.1. When initially checking the assortativity of random scale-free networks, I find it has magnitude of -10^{-3}. This value is considered to be negligible in terms of assortativity. In the following, I consider networks with both scale-free and assortative properties to study how the model performs under this scenario.

I use a network with $N = 1000$ and $M = 50$, so $k_{min} = 26$ refer to Table7.1. To produce networks with different assortativity, I use Noh (2007)'s algorithm again with $J = \pm 1$. Because total degree has increased, I choose $T = 150000$. The regression model I use is characterised as:

$$SF_i = \beta_0 + \beta_1 deg_i + \beta_2 SF_i + \beta_3 disassortative_i + \varepsilon_i \tag{7.3}$$

In Eq.7.3, the dependent variable is the shortfall of each node i and the exogenous variables are degree of each node 'deg', a dummy variable 'SF', which takes 1 if node i is the scale-free network or 0 if it is the Erdős-Rényi network, and a dummy variable 'disassortative' which takes 1 if node i is disassortative or 0 if it is assortative. In this case, I only need to produce 4 combinations: scale-free & assortative, scale-free & disassortative, Erdős-Rényi & assortative, and Erdős-Rényi & disassortative. β_0 is the intercept, β_1 is the coefficient associated with the degree, β_2 is the coefficient of the SF dummy variable, which should be interpreted as the relationship to reference category (Erdős-Rényi network in this case), and β_3 is the coefficient of the disassortativity dummy variable, which should be interpreted as the relationship to the reference category (assortative network in this case). ε_i is simply the residual term.

	Estimate	SE	t-stat	p-value
(Intercept)	12.329	0.18186	67.7923	0
degree	-0.17853	0.001496	-119.3398	0
SF	13.9113	0.11913	116.7738	0
disassortative	13.542	0.11913	113.67	0

Table 7.3: Regression result of Eq.7.3. All coefficients are statistically significant and consistent with previous results.

As it can be seen in Table7.3, β_1 is still significantly negative, verifying the fact that low-degree nodes contribute more to the shortfall. Both β_2 and β_3 are significantly positive, which is consistent with previous sections. The inefficiency related to these two properties mainly comes from the above-average number of low-degree nodes which are shortage in interbank assets as explained before. Hence, scale-free and disassortative together can further drive up the shortfall, meaning that in reality, the interbank network might be more fragile than the benchmark of the Erdős-Rényi network in terms of payment shortfalls. The R^2 of the regression is 0.506, indicating that the regression provides a good fit, so the result is considered to be reliable.

In this chapter, I have studied the effect of heavy-tail in the degree distribution and network (dis)assortativity on liquidity shortfalls in payment networks. Both properties have been empirically observed in real interbank networks, and I showed that they both contribute to increasing liquidity shortfalls, thereby making the system less efficient w.r.t the case of homogeneous networks with positive assortativity. As a result, a suggestion for regulators would be encouraging interbank connections to avoid the existence of a large number of peripheral banks which only carry out business with a small number of counterparties.

Chapter 8

Conclusions

This thesis studies clearing in presence of CCPs. To this end, it generalises the Eisenberg-Noe algorithm for the computation of clearing vectors. Central clearing is one of the pillars of the approach undertaken by regulators after the Global Financial Crisis to improve financial stability. However, in both academia and industry, researchers are still debating whether there exist any unintended consequences when introducing CCPs and extending their scope.

In this thesis I focus on liquidity shortfalls from the failure of variation margin calls. I consider only one CCP and that all asset classes are cleared within it to remove potential inefficiency related to multiple CCPs (Duffie and Zhu, 2011). I also assume the CCP always pays its obligations in full without any delays. In short, I find that increasing the fraction of central clearing (α) does not necessarily lead to smaller liquidity shortfalls.

Introducing the CCP has two main effects. First, all payments that occur through the CCP are netted. This reduces the amount of money that needs to be exchanged in the system, thus increasing its efficiency through lower liquidity shortfall. Second, the presence of a CCP introduces a sequencing of payments: Each institution needs first to pay the CCP (e.g. at the beginning of the day), then bilateral transactions can be settled (e.g. at the end of the day). Such sequencing can bring some inefficiency, as some institutions who would be able to pay the CCP only after receiving bilateral payments, will need to borrow from the outside to meet their commitment to the CCP. I have studied both of these two effects to figure out how the other overcome the other in aggregate level.

To study the effect of central clearing, I generalise the EN algorithm to allow for the computation of clearing vectors in presence of a CCP. The algorithm takes into consideration both the sequencing constraint and the netting effect induced by the CCP. After that, I carry out a numerical analysis with hypothetical interbanking systems.

I find that, at least when the underlying counterparties on centrally cleared and bilateral contracts are not exactly the same (more precisely when centrally cleared and bilateral exposures are independent), the aggregate shortfall is not minimal when all contracts are centrally cleared (or when all contracts are bilateral). Indeed, unless the network of counterparties is too interconnected or unless the exogenous cashflows are sufficiently large, there exists a non-trivial optimal fraction of centrally cleared notional for which the aggregate liquidity shortfall is minimal. This suggests that increasing the scope for central clearing might not necessarily reduce the aggregate demand for collateral. In fact, it is true that increasing the fraction of centrally cleared notional leads to a reduction in net VM payment obligations due to the multilateral netting performed by the CCP. However, it also leads to the reduction in realised bilateral payments. When the fraction of centrally cleared notional is small the first effect prevails, but when it is large the second effect becomes dominant. Furthermore, the reduction in realised bilateral payments does not appear to be driven by the temporal constraints due to the sequencing of centrally cleared and bilateral payments, suggesting that it might be rather linked to the change in the underlying topology of the network of counterparties that occurs when the fraction of centrally cleared notional increases.

I then study the behaviour of the optimal fraction under different combinations of M and e. I can see that the fraction of realisations with a minimum is non-increasing w.r.t. M, and for larger values of e, the fraction drops to 0 faster than realisations with smaller e. I would therefore conclude that the central clearing will be more efficient in reaching an optimal shortfall for systems with low connectivity and exogenous cashflow. On the other hand, the fraction of realisations with 0 shortfall increases for higher average degree, and the fraction reaches 1 faster for higher exogenous cash. These two results match with each other on the fact that having a CCP becomes less necessary when the network is widely connected and each node has sufficient cashflow.

When centrally cleared and bilateral exposures are perfectly correlated, and therefore the underlying counterparties on centrally cleared and bilateral contracts are exactly the same, increasing the fraction of centrally cleared notional is always weakly beneficial, in the sense that the the aggregate liquidity shortfall weakly decreases with the fraction of centrally cleared notional. However, the aggregate shortfall starts decreasing only if a sufficiently large fraction of notional is centrally cleared. As a consequence, in this case, introducing central clearing might not reduce the demand for collateral, unless its scope is sufficiently large.

Different distributions of aggregate liabilities are checked. In particular, I extend the homogeneous exposures where all banks have the same liabilities to the case of heterogeneous exposures, where the total liability for each bank is drawn from exponential distribution. In case heterogeneous total liabilities, α and shortfall are observed to exhibit an relationship similar to the case of homogeneous exposures. However, the system becomes less efficient as total shortfall is getting larger in this case. Another key difference falls within the fraction of realisations with a minimum. For heterogeneous liabilities, the fraction of realisations with a minimum has no conclusive relationship w.r.t M and e as the minimum always exists in almost all realisations. This phenomenon indicates that in this scenario, having a CCP is more efficient in reducing liquidity shortfall over the whole system. When total exposures are perfectly correlated, the relationship between α and shortfall is similar between homogeneous and heterogeneous liabilities, while total shortfall gets larger for heterogeneous case. The most notable difference comes from the value of α^*, at which point the CCP begins to be effective. I find weaker dependency of α^* against connectivity and cashflow. This suggests that, it is easier for central clearing to be beneficial, as the critical fraction of obligations that should be centrally cleared for the shortfall to start decreasing is lower with respect to the benchmark of a system in which total gross VM obligations are homogeneous.

I also study the role of payment sequencing. In practice, transactions associated with the CCPs are organised in a specific order, where payments towards the CCP should be paid first, then banks receive payments from the CCP, after that bilateral transactions can be settled. One may argue that this phenomenon could result in the minimum shortfall exists as for higher values of α, as a bank might have fewer cashflow to settle transactions in the third round so the shortfall goes up after certain level of α. I find, however, such payment sequencing plays only little role for the minimum to occur. After checking the scenario that no payment sequencing involved, I find such minimum still exists. In short word, this clearly implies that payment sequencing has no effect on the minimum. When checking fraction of realisations that the minimum occurs, it still does not have significant difference. Such results indicate that the existence of the minimum in aggregate shortfall may mainly arise because of the change in network topology when novating bilateral obligations to central obligation.

I then extend results from the Erdős-Rényi random network to other different network topologies. Two other topologies studied are heavy-tailed degree distribution and network assortativity. Those two frameworks are widely observed and examined in literature (see previous). For the scale-free network, I consider the fitness model to construct networks with power-law degree distribution (Caldarelli et al., 2002). To generate non-neutral assortativity networks, I use the link rewiring algorithm developed by Noh (2007). These two properties provide more accurate and realistic hypothetical interbank networks.

For the scale-free network, liquidity shortfall exhibits higher value compare to the Erdős-Rényi benchmark case. At the same time, under perfectly correlated exposures, the relationship of shortfall against α stays similar with plateau at beginning and monotonic decreasing after α_c. Because under homogeneous distributed VM assumption, low-degree nodes have fewer incoming links but same outgoing links, as a result contribute more shortfall to the system. Above justification is verified using regression in Table 7.2, which I can see high degree nodes tend to experience lower shortfall, and the Erdős-Rényi model tends to produce lower shortfall.

Regarding networks with different assortativity, assortative networks have lower shortfall than disassortative networks. This can be explained using the specific network structure, where the liabilities of high-degree nodes are spread to many counterparites, the amount of incoming assets of a low-degree node therefore becomes smaller. For all network frameworks I studied, as long as exposures are perfectly correlated, I still find the fully centrally cleared network has shortfall no larger than the case in fully bilaterally cleared network, indicating that having a CCP is at least not worse off the system.

This thesis contributes to the literature and policy-making perspectives on the design of derivatives markets, in particular, the proportion of notional to be centrally cleared to gain the most maximised effects. Inspired by the finding of Duffie and Zhu (2011) that it is always more efficient to clear all asset classes within one CCP, I consider a framework with only one CCP presents. Despite the fact that central clearing is becoming one of the pillars of the approach undertaken by regulators after the Global Financial Crisis to enhance financial stability, it is still very much debated whether introducing central clearing or extending its scope can have unintended consequence (Pirrong, 2011; Ghamami and Glasserman, 2017; Bellia et al., 2019; Berndsen, 2020; Menkveld and Vuillemey, 2021). The key finding falls into the fact that it is not always optimal to increase the fraction of central clearing, despite it has been proved in the literature that increases in multi-lateral netting is the major source in reducing interbank exposures (Cont and Kokholm, 2014). For regulators and policy-makers, this study provides a potential direction in figuring out the percentage of central clearing required to achieve the most efficient clearing mechanism. In some cases, not all contracts should be centrally cleared and an optimal level of liquidity shortfall can be achieved before that point. On the other hand, for the other network structures, it is never inadequate to increase the percentage of centrally cleared contracts, but the advantage only occurs after a certain threshold.

This thesis can be extended in several ways. Regarding the network topology, it has been proved in empirical study that interbank networks follow low level of clustering (see Eq.3.2) (Boss et al., 2004a). Therefore, how the liquidity shortfall behaves should be further examined in this framework. The other direction could be to relax the assumption that CCP never defaults. In adverse market conditions, the value of IMs can become extremely volatile so the speed of VM calls might not meet the large fluctuation in asset prices. In addition, when all asset classes are cleared with in single CCP, and the percentage of central clearing keeps increasing, it can impose higher operational risk on the CCP as it faces greater workload. Further possible way is adjusting $e_i^{(1)}$ from exogenous cashflow to other liquidity/illiquidity assets, then incorporating stochastic terms in asset values (Barucca et al., 2020) and fire sales on asset. At the end, this model should also be tested using empirical data to assess its stability in real payment systems and the scope of their improvements.

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