Information and Inference: A Journal of the IMA (2022) **00**, 1–24
https://doi.org/DOI HERE

Output feedback backstepping control for nonlinear systems using an adaptive finite time sliding mode observer

JIEHUA FENG
College of New Energy, China University of Petroleum (East China), China, Qingdao

DONGYA ZHAO*
College of New Energy, China University of Petroleum (East China), China, Qingdao

XING-GANG YAN
School of Engineering & Digital Arts, University of Kent, United Kingdom, Canterbury

AND

SARAH K. SPURGEON
Department of Electronic & Electrical Engineering, University College London, United Kingdom, London
*Corresponding author: dyzhao@upc.edu.cn

[Received on Date Month Year; revised on Date Month Year; accepted on Date Month Year]

In this paper, a class of nonlinear systems in normal form is considered, which is composed of internal and external dynamics. An adaptive finite time sliding mode observer (AFTSMO) is first designed so that the system states, unmatched uncertain parameters and matched uncertainties can all be observed in finite time (FT). Then, the systematic backstepping design procedure is employed to develop a novel output feedback backstepping control (OFBC). The proposed OFBC method can stabilize the considered nonlinear systems despite the presence of nonlinear internal dynamics and unmatched uncertainties. A Lyapunov method is used to ensure that the closed-loop system is asymptotically stable. Two MATLAB simulation examples are used to demonstrate the method.

Keywords: Finite time sliding mode observer; Nonlinear systems; Output feedback backstepping control.

1. Introduction

In the systems and control area, increasing demands on system performance may require that information about the system states is available for control design and/or system monitoring. However, the measurement of state information may be challenging or costly in practice as in the case of reactant concentration in chemical systems for example (Besançon (2007); Clempner & Yu (2018)). It becomes of interest to develop effective methods to construct estimates of state variables that cannot be readily measured. One such approach is to develop a state observer. Such an observer will typically use the available information from the known system inputs and outputs and use this to obtain estimates of the unmeasurable states.

For many of the existing observer formulations, such as the H_∞ observer in Rastegari et al. (2019), L_∞ observer in Han et al. (2019), extended Luenberger observer in Zeitz (1987), only asymptotic convergence of the observer error may be achieved. Moreover, the majority of state observers cannot accommodate unmatched uncertainty well. Such uncertainty appears in many practical systems. This motivates the current study which seeks to design a FT state observer while considering the presence of unmatched uncertainty.

FT observers have received much attention in the literature and some interesting results have been developed, such as the second-order sliding mode observer in Davila et al. (2005) and the terminal sliding mode observer in Mousavi et al. (2019). A step by step sliding mode observer has been proposed in Daly & Wang (2009) for a class of integrator systems where the observer error can tend to zero in FT. Nevertheless, unmatched uncertainty has not been considered in this work. A novel FT dynamic parameter estimator has been designed to deal with unmatched parameters by using low pass filters in Na et al. (2015). This requires the system states to be available. In fact, when there is unmatched uncertainty present in a system, in order to recover the system state information in FT, the unmatched uncertainty must first be compensated in FT. The development of a FT observer in the presence of unmatched uncertainties in the considered system is thus particularly challenging.

A FT output feedback controller has been designed for a second-order system in Zhao et al. (2016). This can observe both the system unmatched uncertainty and state information in FT. The approach proposed in Zhao et al. (2016) has been extended to high-order systems in Zhao et al. (2018). Note that the nominal systems in Zhao et al. (2018) are required to be linear. In practice, many systems are nonlinear and it may be limiting to remove all the nonlinear behaviour from the system model for the purposes of analysis and design (see, e.g., Zhu (2021); Zhu et al. (2022)). Observer paradigms focussed on linear systems may not achieve satisfactory performance levels when applied to the physical nonlinear system. This motivates consideration of frameworks to develop FT observers for nonlinear systems.

The development of a FT observer from a nonlinear nominal system model has been considered and some important results have been obtained. By using the super-twisting method, a FT sliding mode observer has been designed for a class of nonlinear systems in Floquet & Barbot (2007). However, this observer imposes the requirement that the system can be accurately linearized and only matched uncertainty is considered. An adaptive sliding mode observer has been investigated for a class of systems subject to unmatched uncertainty in Yang et al. (2017), where the results are developed for a nominal system representation that is a series of integrators.

In this paper, a novel AFTSMO is designed for a class of nonlinear systems in the presence of unmatched uncertain parameters and matched lumped uncertainties. The proposed method does not necessitate that the considered systems are linear or linearizable. When compared with other FT observer approaches in Yang et al. (2017); Zhao et al. (2018), the considered system has a more general form and the nominal system is nonlinear, which extends existing results in terms of both potential practical application as well as providing a contribution to theoretical research. Compared with current FT state observers (see, e.g., Daly & Wang (2009); Slotine et al. (1986); Zhao et al. (2013)), the proposed method can estimate the unmatched parameters in FT in light of the proposed adaptive law while the matched uncertainties can also be estimated in FT. Compared with current FT parameter estimators (see, e.g., Kapetina et al. (2019); Na et al. (2015); Xing et al. (2019)), the proposed method can observe the external dynamics in FT by employing the sliding mode equivalent injection approach.

For this class of uncertain nonlinear systems with unmatched uncertainties, many of the existing control methods are based on state feedback, such as Wang et al. (2016); Yu & Wu (2012); Zhang et al. (2017), which may have limitations for practical applications. This motivates the study of observer based dynamic output feedback control for uncertain nonlinear systems. The observer approaches frequently used in these existing dynamic output feedback control methods cannot observe the system states and uncertain parameters simultaneously in FT as seen with the high-gain observer based robust output tracking control proposed in Yu et al. (2018) and the fuzzy state observer based adaptive robust control in Tong & Li (2010); Xu et al. (2013). In summary, when the uncertain systems considered are nonlinear and have unmatched uncertainties, difficulties frequently exist due to: (i). the need to deal with the nonlinear internal dynamics; (ii). the need to ensure that the system state can be observed in FT while the uncertain parameters can be observed in FT; (iii). the need to use the observed information to design the controller to ensure system stability and perform corresponding stability analysis.

In this paper, the observed system states and estimated uncertain parameters are compensated by the designed control using the step-by-step recursive backstepping technique. A robust OFBC is obtained using virtual Lyapunov-based control to enhance the system robustness and improve system performance. The main contributions of this paper are: (i). an AFTSMO is designed for a class of nonlinear systems where the system states, unmatched uncertain parameters and matched uncertainties can all be observed in FT; (ii). using the system output and the observed information, an OFBC is proposed to ensure the closed-loop system is asymptotically stable despite the presence of matched and unmatched uncertainties; (iii). a Lyapunov approach is used to address stability.

The paper is structured as follows. Section 2 formulates the problem and some basic assumptions are given, which will be used in the following sections. An AFTSMO is proposed for the external dynamics and an adaptive law is designed to estimate the uncertain parameters in Section 3. An OFBC is designed in Section 4. Section 5 uses two simulation examples to validate the designed approach while Section 6 presents the conclusions.

2. Problem formulation and basic assumptions

Consider the following Multiple-Input Multiple-Output (MIMO) nonlinear system:

$$\begin{aligned}
\dot{z}_1 &= A_1 z_1 + B_1\left(u_1 + \xi_1(t,z)\right) + \psi_1(t,z)\\
&\vdots\\
\dot{z}_l &= A_l z_l + B_l\left(u_l + \xi_l(t,z)\right) + \psi_l(t,z)\\
&\vdots\\
\dot{z}_m &= A_m z_m + B_m\left(u_m + \xi_m(t,z)\right) + \psi_m(t,z)\\
\dot{z}^b &= \omega\left(y, z^b\right) + \Theta(t,z)\\
y &= \left[z_{11}, \cdots, z_{l1}, \cdots, z_{m1}\right]^T
\end{aligned} \tag{2.1}$$

where $z := col\left(z^a, z^b\right) \in Z \in R^n$, $u := col\left(u_1, \cdots, u_m\right) \in R^m$, $y := col\left(y_1, \cdots, y_m\right) \in R^m$ with $y_l = z_{l1}$ and $l = 1, 2, \cdots, m$ represent the system state, input and output respectively and Z is a neighborhood of the origin. $z^a := col\left(z_1, \cdots, z_l, \cdots, z_m\right) \in R^{r_1+\cdots+r_l+\cdots+r_m} = R^r$ with $z_l := col\left(z_{l1}, z_{l2}, \cdots, z_{lr_l}\right) \in R^{r_l}$ represents the external dynamics and $z^b \in R^{n-r}$ represents the internal dynamics. $\xi_l \in R$, $\psi_l := col\left(\psi_{l1}, \cdots, \psi_{lr_l}\right) \in R^{r_l}$, $\Theta(t,z) \in R^{n-r}$, where all the nonlinear terms are smooth enough and

$$A_l = \begin{bmatrix} 0 & 1 & 0 & \vdots & 0 & 0\\ 0 & 0 & 1 & \vdots & 0 & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & 0 & \cdots & 0 & 1\\ 0 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}_{r_l \times r_l} \quad B_l = \begin{bmatrix} 0\\ 0\\ \vdots\\ 0\\ 1 \end{bmatrix}_{r_l \times 1} \tag{2.2}$$

It should be noted that ξ_l and ψ_{lr_l} denote the matched uncertainties, whereas $\psi_{l1}, \cdots, \psi_{l,r_l-1}$ and Θ denote the unmatched uncertainties for all $l = 1, 2, \cdots, m$.

Remark 1 *It should be noted that the uncertain nonlinear system (2.1) is in output feedback normal form. This can be obtained from a general affine nonlinear system by local coordinate transformation and feedback linearization as described in Feng et al. (2020); Isidori (2013). In addition, the internal dynamics z_b in (2.1) can also be viewed as the unmodeled dynamics or dynamic uncertainty (Jiang & Praly (1998)). System (2.1) has thus been extensively studied as in Jiang (1999); Jiang & Hill (1999); Xu et al. (2019), and many practical systems can be modeled in the form of (2.1), such as a simple pendulum (see, e.g., Jiang & Hill (1999)) and the field-controlled DC motor (see, e.g., Khalil & Grizzle (2002)).*

The following assumptions will be imposed on system (2.1).

Assumption 1 *(see, e.g., Yan et al. (2016)) There exists a C^1 function $V^b\left(t, z^b\right) : R \times R^{n-r} \mapsto R^+$ such that*

$$\begin{gathered} c_1 \left\| z^b \right\|^2 \leqslant V^b\left(t, z^b\right) \leqslant c_2 \left\| z^b \right\|^2 \\ \frac{\partial V^b}{\partial t} + \frac{\partial V^b}{\partial z^b} \omega\left(0, z^b\right) \leqslant -c_3 \left\| z^b \right\|^2 \\ \left\| \frac{\partial V^b}{\partial z^b} \right\| \leqslant c_4 \left\| z^b \right\| \end{gathered} \tag{2.3}$$

where $c_1, \cdots, c_4$ are positive constants. Meanwhile $\omega\left(y, z^b\right)$ is Lipschitz with respect to y and uniformly for z^b in the considered domain Z, that is, for any col $\left(y, z^b\right) \in$ Z and col $\left(\bar{y}, z^b\right) \in$ Z, there exists a nonnegative function $\mathscr{L}_\omega\left(z^b\right)$ such that

$$\left\| \omega\left(y, z^b\right) - \omega\left(\bar{y}, z^b\right) \right\| \leqslant \mathscr{L}_\omega\left(z^b\right) \|y - \bar{y}\| \tag{2.4}$$

Assumption 2 *There exist known nonnegative continuous functions $\Phi(t, z)$ and $\tau(t, z)$ such that*

$$\|\Theta(t, z)\| \leqslant \Phi(t, z) \|y\| + \tau(t, z) \left\| z^b \right\| \tag{2.5}$$

The objective of this paper is to propose an AFTSMO for the external dynamics of system (2.1) and an adaptive law to estimate the uncertain parameters. Then, for system (2.1), an OFBC will be designed such that the associated closed-loop system is stable. The structural block diagram of the proposed method is given in Fig. 1.

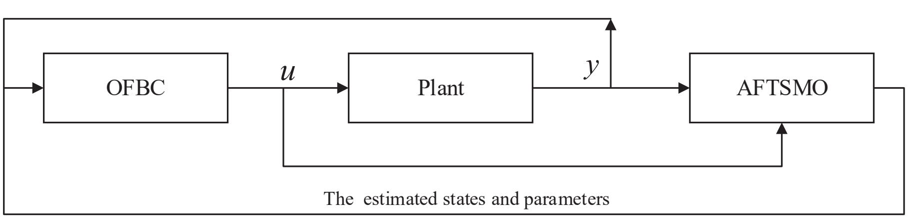


FIG. 1. The structural block diagram of the proposed method

3. An AFTSMO for the external dynamics

In this section, an AFTSMO is proposed for the external dynamics and an adaptive law is designed to estimate the uncertain parameters. It follows from (2.2) that the external dynamics z_a can be further described as:

$$\begin{gathered}\dot{z}_{l1} = z_{l2} + \psi_{l1}(t,z) \\ \dot{z}_{l2} = z_{l3} + \psi_{l2}(t,z) \\ \vdots \\ \dot{z}_{l(r_l-1)} = z_{lr_l} + \psi_{l(r_l-1)}(t,z) \\ \dot{z}_{lr_l} = u_l + \xi_l(t,z) + \psi_{lr_l}(t,z) \\ y_l = z_{l1}\end{gathered} \tag{3.1}$$

with $l = 1,2,\cdots,m$.

Assumption 3 *(see, e.g., Sun & Guo (2014); Wang et al. (2017)) The unmatched uncertainties ψ_{lj} with $j = 1,2,\cdots,r_l - 1$ satisfy*

$$\psi_{lj} = d_{lj}\left(\overline{z}_{lj}\right)\theta_{lj} \tag{3.2}$$

where $\overline{z}_{lj} := col\left(z_{l1}, z_{l2}, \cdots, z_{lj}\right)$, $d_{lj}\left(\overline{z}_{lj}\right) \in R^{1\times q_{lj}}$ is a known function, $\theta_{lj} \in R^{q_{lj}}$ is the uncertain parameter vector, and $d_{lj}\left(\overline{z}_{lj}\right)$ satisfies the persistently excited condition.

Assumption 4 *The matched uncertainties satisfy*

$$\left|\xi_l(t,z) + \psi_{lr_l}(t,z)\right| \le \rho_l(t,z) \tag{3.3}$$

where $\rho_l(t,z)$ is a known function.

As in Na et al. (2015), define filters as:

$$\begin{gathered}\phi_l \dot{\hat{z}}_{lj\text{-}f} + \hat{z}_{lj\text{-}f} = \hat{z}_{lj} \\ \phi_l \dot{\hat{z}}_{l(j+1)\text{-}f} + \hat{z}_{l(j+1)\text{-}f} = \hat{z}_{l(j+1)} \\ \phi_l \dot{d}_{lj\text{-}f} + d_{lj\text{-}f} = d_{lj}\left(\hat{\overline{z}}_{lj}\right)\end{gathered} \tag{3.4}$$

with initial conditions $\hat{z}_{lj\text{-}f} = 0, \hat{z}_{l(j+1)\text{-}f} = 0$ and $\hat{d}_{lj\text{-}f} = 0$ respectively, and where $\phi_l > 0$ are filter parameters.

Then it follows from (3.1) and (3.4) that

$$\dot{\hat{z}}_{lj\text{-}f} = \frac{\hat{z}_{lj} - \hat{z}_{lj\text{-}f}}{\phi_l} = \hat{z}_{l(j+1)\text{-}f} + d_{lj\text{-}f}\theta_{lj\text{-}f} - d_{lj\text{-}f}\zeta_{lj}(t) \tag{3.5}$$

where $\zeta_{lj}(t)$ represents the filter error caused by the observer.

Define corresponding auxiliary filters as:

$$\begin{aligned}\dot{p}_{l_j}(t) &= -\gamma_l p_{l_j} + d_{lj\text{-}f}^T d_{lj\text{-}f} \\ \dot{q}_{l_j}(t) &= -\gamma_l q_{l_j} + d_{lj\text{-}f}^T \left(\frac{\hat{z}_{lj} - \hat{z}_{lj\text{-}f}}{\phi_l} - \hat{z}_{l(j+1)\text{-}f} \right)\end{aligned} \tag{3.6}$$

with corresponding solutions

$$\begin{aligned}p_{l_j}(t) &= \int_0^t e^{-\gamma_l (t-r)} d_{lj\text{-}f}^T(r)\, d_{lj\text{-}f}(r)\, dr \\ q_{l_j}(t) &= \int_0^t e^{-\gamma_l (t-r)} d_{lj\text{-}f}^T(r) \left[\frac{\hat{z}_{lj}(r) - \hat{z}_{lj\text{-}f}(r)}{\phi_l} - \hat{z}_{l(j+1)\text{-}f}(r) \right] dr\end{aligned} \tag{3.7}$$

where $\gamma_l > 0$ are auxiliary filter parameters.

Lemma 1 *(see, e.g., Na et al. (2015)) If the regressor matrix $d_{lj}\left(\overline{z}_{lj}\right)$ satisfies the persistently excited condition, then there exists $T_{lj} > 0$ such that $p_{l_j}(t) > 0$ for $t > T_{lj}$.*

Then from Lemma 1 and (3.7),

$$\theta_{l_j}(t) = p_{lj}^{-1}(t)\, q_{l_j}(t) + \zeta_{lj}(t) \tag{3.8}$$

The adaptive observer for z_{lj} and adaptive law are designed as follows: when $j = 1$,

$$\begin{aligned}\dot{\hat{z}}_{l1} &= \hat{z}_{l2} + d_{l1}\left(\hat{\overline{z}}_{l1}\right) \hat{\theta}_{l1} + \alpha_{l1} \operatorname{sgn}\left(\tilde{z}_{l1}\right) \\ \dot{\hat{\theta}}_{l1} &= -\Gamma_{l1} \left\{ p_{l1}^T(t) \operatorname{sgn}\left(w_{l1}(t)\right) - d_{l1}\left(\hat{\overline{z}}_{l1}\right) \tilde{z}_{l1} \right\}\end{aligned} \tag{3.9}$$

and when $j = 2, 3, \cdots, r_l - 1$,

$$\begin{aligned}\dot{\hat{z}}_{lj} &= \hat{z}_{l(j+1)} + d_{lj}\left(\hat{\overline{z}}_{lj}\right) \hat{\theta}_{lj} + \alpha_{lj} \operatorname{sgn}\left(\widehat{z}_{lj} - \hat{z}_{lj}\right) \\ \dot{\hat{\theta}}_{lj} &= -\Gamma_{lj} \left\{ p_{lj}^T(t) \operatorname{sgn}\left(w_{lj}(t)\right) - d_{lj}\left(\hat{\overline{z}}_{lj}\right) \tilde{z}_{lj} \right\}\end{aligned} \tag{3.10}$$

where $\Gamma_{lj} \in R^{q_{lj} \times q_{lj}} > 0$ are constant diagonal gain matrices, $\alpha_{lj} > 0$ are design parameters, $w_{l_j}(t) = p_{l_j}(t)\, \hat{\theta}_{l_j}(t) - q_{lj}(t)$, $\tilde{z}_{lj} = z_{lj} - \hat{z}_{lj}$, $\tilde{\theta}_{lj} = \theta_{lj} - \hat{\theta}_{lj}$, and $\widehat{z}_{lj} = \left[\alpha_{l(j-1)} \operatorname{sgn}\left(\tilde{z}_{l(j-1)}\right)\right]_{eq} + \hat{z}_{lj}$.

The observer for z_{lr_l} is designed as:

$$\dot{\hat{z}}_{lr_l} = u_l + \alpha_{lr_l} \operatorname{sgn}\left(\widehat{z}_{lr_l} - \hat{z}_{lr_l}\right) \tag{3.11}$$

where $\widehat{z}_{lr_l} = \left[\alpha_{l(r_l-1)} \operatorname{sgn}\left(\tilde{z}_{l(r_l-1)}\right)\right]_{eq} + \hat{z}_{lr_l}$ and $\alpha_{lr_l} > \rho_l$.

It should be noted that $\left[\alpha_{l(j-1)} \operatorname{sgn}\left(\tilde{z}_{l(j-1)}\right)\right]_{eq}$ represent equivalent injections, which can be realized by passing the signal $\left[\alpha_{l(j-1)} \operatorname{sgn}\left(\tilde{z}_{l(j-1)}\right)\right]_{eq}$ through a low pass filter. The detailed explanation of the equivalent injection has been discussed in Haskara (1998).

Assumption 5 *(see, e.g., Zhao et al. (2018, 2016))*

$$\left| p_{lj}(t)\, \zeta_{lj}(t) \right| \le \left| p_{lj}(t)\, \tilde{\theta}_{lj}(t) \right| \tag{3.12}$$

for all $l = 1, 2, \cdots, m$ and $j = 1, 2, \cdots, r_l - 1$.

Remark 2 *Assumption 5 shows that the effect of the deviation ζ_{lj} is less than that of the parameter estimation error $\tilde{\theta}_{lj}$, which can be realized as long as the observer parameters are selected appropriately by the designer.*

The observer design process can be provided step by step in the following.

Step 1

At the first step, when $j=1$,

$$\begin{aligned}\dot{\hat{z}}_{l1} &= \hat{z}_{l2} + d_{l1}\left(\hat{\bar{z}}_{l1}\right)\hat{\theta}_{l1} + \alpha_{l1}\mathrm{sgn}\left(\tilde{z}_{l1}\right)\\ \dot{\hat{\theta}}_{l1} &= -\Gamma_{l1}\left\{p_{l1}^{T}\left(t\right)\mathrm{sgn}\left(w_{l1}\left(t\right)\right) - d_{l1}\left(\hat{\bar{z}}_{l1}\right)\tilde{z}_{l1}\right\}\end{aligned} \tag{3.13}$$

where $\tilde{z}_{l1} = z_{l1} - \hat{z}_{l1}$ and z_{l1} is measurable.

It follows that

$$\dot{\tilde{z}}_{l1} = \tilde{z}_{l2} + d_{l1}\left(\hat{\bar{z}}_{l1}\right)\tilde{\theta}_{l1} + \Delta d_{l1}\left(\bar{z}_{l1}\right)\theta_{l1} - \alpha_{l1}\mathrm{sgn}\left(\tilde{z}_{l1}\right) \tag{3.14}$$

where $\Delta d_{l1}\left(\bar{z}_{l1}\right) = d_{l1}\left(\bar{z}_{l1}\right) - d_{l1}\left(\hat{\bar{z}}_{l1}\right)$.

A Lyapunov function is chosen as:

$$V_{l1}^{o} = \frac{1}{2}\tilde{z}_{l1}^{2} + \frac{1}{2\Gamma_{l1}}\tilde{\theta}_{l1}^{T}\tilde{\theta}_{l1} \tag{3.15}$$

Differentiate (3.15) along (3.14):

$$\begin{aligned}\dot{V}_{l1}^{o} &= \tilde{z}_{l1}\left\{\tilde{z}_{l2} + d_{l1}\left(\hat{\bar{z}}_{l1}\right)\tilde{\theta}_{l1} + \Delta d_{l1}\left(\bar{z}_{l1}\right)\theta_{l1} - \alpha_{l1}\mathrm{sgn}\left(\tilde{z}_{l1}\right)\right\}\\ &\quad + \tilde{\theta}_{l1}^{T}\left\{p_{l1}^{T}\left(t\right)\mathrm{sgn}\left(w_{l1}\left(t\right)\right) - d_{l1}\left(\hat{\bar{z}}_{l1}\right)\tilde{z}_{l1}\right\}\\ &= -\alpha_{l1}\left|\tilde{z}_{l1}\right| + \tilde{z}_{l1}\left(\tilde{z}_{l2} + \Delta d_{l1}\left(\bar{z}_{l1}\right)\theta_{l1}\right) + \tilde{\theta}_{l1}^{T}p_{l1}^{T}\left(t\right)\mathrm{sgn}\left\{-p_{l1}\tilde{\theta}_{l1} + p_{l1}\zeta_{l1}\left(t\right)\right\}\end{aligned} \tag{3.16}$$

It follows that

$$\dot{V}_{l1}^{o} \le -\left|\tilde{z}_{l1}\right|\left\{\alpha_{l1} - \left|\tilde{z}_{l2} + \Delta d_{l1}\left(\bar{z}_{l1}\right)\theta_{l1}\right|\right\} - \left|p_{l1}\left(t\right)\tilde{\theta}_{l1}\right| \tag{3.17}$$

Choose a large enough α_{l1} so that

$$\alpha_{l1} - \left|\tilde{z}_{l2} + \Delta d_{l1}\left(\bar{z}_{l1}\right)\theta_{l1}\right| \ge \eta_{l1} \tag{3.18}$$

where $\eta_{l1} > 0$.

Then (3.17) can be described as:

$$\begin{aligned}\dot{V}_{l1}^{o} &\le -\eta_{l1}\left|\tilde{z}_{l1}\right| - \left|p_{l1}\left(t\right)\tilde{\theta}_{l1}\right|\\ &\le -c_{l11}\sqrt{\frac{1}{2}\tilde{z}_{l1}^{2}} - c_{l12}\sqrt{\frac{1}{2\Gamma_{1}}\tilde{\theta}_{l1}^{T}\tilde{\theta}_{l1}}\\ &\le -c_{l1}\sqrt{V_{l1}^{o}}\end{aligned} \tag{3.19}$$

where $\lambda_{\min}\left(p_{l1}\right) > \delta_{l1} > 0$, $c_{l11} = \sqrt{2}\eta_{l1}$, $c_{l12} = \delta_{l1}\sqrt{2/\lambda_{\max}\left(\Gamma_{1}^{-1}\right)}$, $c_{l1} = \min\left\{c_{l11}, c_{l12}\right\}$.

It follows from (3.19), $\tilde{z}_{l1}$ and $\tilde{\theta}_{l1}$ will converge to zero when $t \ge t_{l1} = \frac{2\sqrt{V_{l1}^{o}(0)}}{c_{l1}}$ where $V_{l1}^{o}(0)$ represents the initial value of V_{l1}^{o}.

According to (3.14), when $t \geq t_{l1}$, $\tilde{\theta}_{l1} = 0$, $\tilde{z}_{l1} = 0$, $\Delta d_{l1}(\bar{z}_{l1}) = 0$, the following equation holds:

$$\tilde{z}_{l2} = [\alpha_{l1}\mathrm{sgn}(\tilde{z}_{l1})]_{eq} \tag{3.20}$$

Step 2

At the second step, when $j = 2$,

$$\begin{aligned} \dot{\hat{z}}_{l2} &= \hat{z}_{l3} + d_{l2}(\hat{\bar{z}}_{l2})\hat{\theta}_{l2} + \alpha_{l2}\mathrm{sgn}\left(\widehat{z}_{l2} - \hat{z}_{l2}\right) \\ \dot{\hat{\theta}}_{l2} &= -\Gamma_{l2}\left\{p_{l2}^T(t)\,\mathrm{sgn}(w_{l2}(t)) - d_{l2}(\hat{\bar{z}}_{l2})\tilde{z}_{l2}\right\} \end{aligned} \tag{3.21}$$

where $\widehat{z}_{l2} = [\alpha_{l1}\mathrm{sgn}(\tilde{z}_{l1})]_{eq} + \hat{z}_{l2}$ and $\tilde{z}_{l2}$ has been given in (3.20).

It follows that

$$\dot{\tilde{z}}_{l2} = \tilde{z}_{l3} + d_{l2}(\hat{\bar{z}}_{l2})\tilde{\theta}_{l2} + \Delta d_{l2}(\bar{z}_{l2})\theta_{l2} - \alpha_{l2}\mathrm{sgn}(\tilde{z}_{l2}) \tag{3.22}$$

where $\Delta d_{l2}(\bar{z}_{l2}) = d_{l2}(\bar{z}_{l2}) - d_{l2}(\hat{\bar{z}}_{l2})$.

A Lyapunov function is chosen as:

$$V_{l2}^o = \frac{1}{2}\tilde{z}_{l2}^2 + \frac{1}{2\Gamma_{l2}}\tilde{\theta}_{l2}^T\tilde{\theta}_{l2} \tag{3.23}$$

By the similar analysis as given in Step 1, if α_{l2} is large enough:

$$\alpha_{l2} - |\tilde{z}_{l3} + \Delta d_{l2}(\bar{z}_{l2})\theta_{l2}| \geq \eta_{l2} \tag{3.24}$$

where $\eta_{l2} > 0$.

It follows from (3.24), $\tilde{z}_{l2}$ and $\tilde{\theta}_{l2}$ will converge to zero when $t \geq t_{l2} = \frac{2\sqrt{V_{l2}^o}}{c_{l2}}$ with $\lambda_{\min}(p_{l2}) > \delta_{l2} > 0$, $c_{l21} = \sqrt{2}\eta_{l2}$, $c_{l22} = \delta_{l2}\sqrt{2/\lambda_{\max}(\Gamma_2^{-1})}$, $c_{l2} = min\{c_{l21}, c_{l22}\}$.

According to (3.22), when $t \geq t_{l2}$, $\tilde{\theta}_{l2} = 0$, $\tilde{z}_{l2} = 0$, $\Delta d_{l2}(\bar{z}_{l2}) = 0$, the following equation holds:

$$\tilde{z}_{l3} = [\alpha_{l2}\mathrm{sgn}(\tilde{z}_{l2})]_{eq} \tag{3.25}$$

Step *i* $(i = 3 \sim r_l - 1)$

At the *i*-th step,

$$\begin{aligned} \dot{\hat{z}}_{li} &= \hat{z}_{li} + d_{li}(\hat{\bar{z}}_{li})\hat{\theta}_{li} + \alpha_{li}\mathrm{sgn}\left(\widehat{z}_{li} - \hat{z}_{li}\right) \\ \dot{\hat{\theta}}_{li} &= -\Gamma_{li}\left\{p_{li}^T(t)\,\mathrm{sgn}(w_{li}(t)) - d_{li}(\hat{\bar{z}}_{li})\tilde{z}_{li}\right\} \end{aligned} \tag{3.26}$$

where $\widehat{z}_{li} = \left[\alpha_{l(i-1)}\mathrm{sgn}\left(\tilde{z}_{l(i-1)}\right)\right]_{eq} + \hat{z}_{li}$ and $\tilde{z}_{li} = \left[\alpha_{l(i-1)}\mathrm{sgn}\left(\tilde{z}_{l(i-1)}\right)\right]_{eq}$.

It follows that

$$\dot{\tilde{z}}_{li} = \tilde{z}_{l(i+1)} + d_{li}(\hat{\bar{z}}_{li})\tilde{\theta}_{li} + \Delta d_{li}(\bar{z}_{li})\theta_{li} - \alpha_{li}\mathrm{sgn}(\tilde{z}_{li}) \tag{3.27}$$

where $\Delta d_{li}(\bar{z}_{li}) = d_{li}(\bar{z}_{li}) - d_{li}(\hat{\bar{z}}_{li})$.

A Lyapunov function is chosen as:

$$V_{li}^o = \frac{1}{2}\tilde{z}_{li}^2 + \frac{1}{2\Gamma_{li}}\tilde{\theta}_{li}^T\tilde{\theta}_{li} \tag{3.28}$$

By the similar analysis as given in Step 1-2, if α_{li} is large enough so that:

$$\alpha_{li} - \left|\tilde{z}_{l(i+1)} + \Delta d_{li}(\bar{z}_{li})\theta_{li}\right| \geq \eta_{li} \tag{3.29}$$

where $\eta_{li} > 0$.

It follows from (3.29), $\tilde{z}_{li}$ and $\tilde{\theta}_{li}$ will converge to zero when $t \geq t_{li} = \frac{2\sqrt{V_{li}^{o}}}{c_{li}}$ with $\lambda_{\min}(p_{li}) > \delta_{li} > 0$, $c_{li1} = \sqrt{2}\eta_{li}$, $c_{li2} = \delta_{li}\sqrt{2/\lambda_{\max}\left(\Gamma_{li}^{-1}\right)}$, $c_{li} = min\{c_{li1}, c_{li2}\}$.

According to (3.27), when $t \geq t_{li}$, $\tilde{\theta}_{li} = 0$, $\tilde{z}_{li} = 0$, $\Delta d_{li}(\bar{z}_{li}) = 0$, the following equation holds:

$$\tilde{z}_{l(i+1)} = [\alpha_{li}\text{sgn}(\tilde{z}_{li})]_{eq} \tag{3.30}$$

Step r_l

At the r_l step, the state observer is designed as follows:

$$\dot{\hat{z}}_{lr_l} = u_l + \alpha_{lr_l}\text{sgn}\left(\widehat{z}_{lr_l} - \hat{z}_{lr_l}\right) \tag{3.31}$$

where $\widehat{z}_{lr_l} = \left[\alpha_{l(r_l-1)}\text{sgn}\left(\tilde{z}_{l(r_l-1)}\right)\right]_{eq} + \hat{z}_{lr_l}$ and $\tilde{z}_{lr_l} = \left[\alpha_{l(r_l-1)}\text{sgn}\left(\tilde{z}_{l(r_l-1)}\right)\right]_{eq}$.

The observer error dynamic equation is given by:

$$\dot{\tilde{z}}_{lr_l} = \xi_l(t,z) + \psi_{lr_l}(t,z) - \alpha_{lr_l}\text{sgn}\left(\tilde{z}_{lr_l}\right) \tag{3.32}$$

A Lyapunov function is chosen as:

$$V_{lr_l}^{o} = \frac{1}{2}\tilde{z}_{lr_l}^{2} \tag{3.33}$$

Differentiate (3.33) along (3.32):

$$\begin{aligned} \dot{V}_{lr_l}^{o} &= \tilde{z}_{lr_l}\left\{\xi_l(t,z) + \psi_{lr_l}(t,z) - \alpha_{lr_l}\text{sgn}\left(\tilde{z}_{lr_l}\right)\right\} \\ &\leq -\left|\tilde{z}_{lr_l}\right|\left(\alpha_{lr_l} - \rho_l\right) \end{aligned} \tag{3.34}$$

If α_{lr_l} is large enough so that:

$$\alpha_{lr_l} - \rho_l \geq \eta_{lr_l} \tag{3.35}$$

where $\eta_{lr_l} > 0$.

It follows from (3.34), $\tilde{z}_{lr_l}$ will converge to zero when $t \geq t_{lr_l} = \frac{2\sqrt{V_{lr_l}^{0}}}{c_{lr_l}}$ with $c_{lr_l} = \sqrt{2}\eta_{lr_l}$.

Lemma 2 *Under Assumptions 1-5, the AFTSMO (3.9)-(3.11) can guarantee that the system external dynamics and uncertain parameters of system (3.1) can be observed in FT if* α_{lj} *is large enough. In addition, the matched uncertainties can be estimated in FT as:*

$$\hat{\xi}_l + \hat{\psi}_{lr_l} = \left[\alpha_{lr_l}\text{sgn}\left(\widehat{z}_{lr_l} - \hat{z}_{lr_l}\right)\right]_{eq} \tag{3.36}$$

Proof Define the following Lyapunov function:

$$\begin{aligned} V^o &= \sum_{l=1}^{m} \sum_{j=1}^{r_l} V_{lj}^o \\ &= \sum_{l=1}^{m} \sum_{j=1}^{r_l} \frac{1}{2} \tilde{z}_{lj}^2 + \sum_{l=1}^{m} \sum_{j=1}^{r_l - 1} \frac{1}{2\Gamma_{lj}} \tilde{\theta}_{lj}^T \tilde{\theta}_{lj} \end{aligned} \tag{3.37}$$

Differentiate (3.37):

$$\dot{V}^o = \sum_{l=1}^{m} \sum_{j=1}^{r_l} \dot{V}_{lj}^o \tag{3.38}$$

According to the above analysis, for all $l = 1, \cdots, m$ and $j = 1, \cdots, r_l$, when $t \geq \max t_{lj}$,

$$\dot{V}_{lj}^o \leq -c_{lj} \sqrt{V_{lj}^o} \tag{3.39}$$

Combine (3.38) and (3.39),

$$\begin{aligned} \dot{V}^o &\leq \sum_{l=1}^{m} \sum_{j=1}^{r_l} -c_{lj} \sqrt{V_{lj}^o} \\ &\leq -c \sum_{l=1}^{m} \sum_{j=1}^{r_l} \sqrt{V_{lj}^o} \end{aligned} \tag{3.40}$$

where $c = \min c_{lj}$.

It follows from the inequality $\sqrt{a_1} + \sqrt{a_2} + \cdots + \sqrt{a_n} \geq \sqrt{a_1 + a_2 + \cdots + a_n}$ where $a_1, a_2, \cdots, a_n$ are positive constants,

$$\sum_{l=1}^{m} \sum_{j=1}^{r_l} \sqrt{V_{lj}^o} \geq \sqrt{\sum_{l=1}^{m} \sum_{j=1}^{r_l} V_{lj}^o} = \sqrt{V^o} \tag{3.41}$$

Then from (3.40) and (3.41),

$$\dot{V}^o \leq -c\sqrt{V^o} \tag{3.42}$$

It follows from (3.42), when $t \geq \frac{2\sqrt{V^o(0)}}{c}$, $\tilde{z}_{lj}$ and $\tilde{\theta}_{lj}$ will converge to zero.

Meanwhile according to (3.32), when $t \geq t_{lr_l}$, $\tilde{z}_{lr_l} = 0$. Then, the matched uncertainties can be estimated from the following equivalent injection signal:

$$\hat{\xi}_l + \hat{\psi}_{lr_l} = \left[\alpha_{lr_l} \operatorname{sgn}\left(\tilde{z}_{lr_l}\right)\right]_{eq} \tag{3.43}$$

Hence, Lemma 2 follows. □

Remark 3 *So as to ensure that the system external dynamics can be observed in FT, the parameter estimator $\hat{\theta}_{lj}$ as shown in (3.9) and (3.10) has been designed to guarantee that the unmatched uncertain parameter estimation error $\tilde{\theta}_{lj}$ converge to zero in FT by using the adaptive method and the principle of the equivalent injection.*

4. Output feedback backstepping control

For system (3.1), it is necessary to design the following coordinate transformation:

$$\begin{aligned}\sigma_{l1} &= z_{l1} - \beta_{l1} \\ \sigma_{l2} &= \hat{z}_{l2} - \beta_{l2} \\ &\vdots \\ \sigma_{l(r_l-1)} &= \hat{z}_{l(r_l-1)} - \beta_{l(r_l-1)} \\ \sigma_{lr_l} &= \hat{z}_{lr_l} - \beta_{lr_l}\end{aligned} \tag{4.1}$$

where $\beta_{lj}\,(j=1,2,\cdots,r_l)$ will be provided in the subsequent analysis.

Step 1

Let β_{l1}= 0. Consider the σ_{l1} subsystem:

$$\begin{aligned}\dot{\sigma}_{l1} &= \dot{z}_{l1} - \dot{\beta}_{l1} \\ &= \tilde{z}_{l2} + \hat{z}_{l2} + d_{l1}\left(\overline{z}_{l1}\right)\theta_{l1}\end{aligned} \tag{4.2}$$

where $z_{l2} = \tilde{z}_{l2} + \hat{z}_{l2}$ has been used since z_{l2} is not available.

A Lyapunov function is chosen as:

$$V_{l1} = \frac{1}{2}\sigma_{l1}^2 \tag{4.3}$$

Differentiate (4.3):

$$\dot{V}_{l1} = \sigma_{l1}\left(\tilde{z}_{l2} + \hat{z}_{l2} + d_{l1}\left(\overline{z}_{l1}\right)\theta_{l1}\right) \tag{4.4}$$

Then let $\beta_{l2} = -d_{l1}\left(\hat{\overline{z}}_{l1}\right)\hat{\theta}_{l1} - k_{l1}\sigma_{l1}$, where $k_{l1} > 0$. It follows that:

$$\begin{aligned}\dot{V}_{l1} &= \sigma_{l1}\left(\tilde{z}_{l2} + \sigma_{l2} + \beta_{l2} d_{l1}\left(\overline{z}_{l1}\right)\theta_{l1}\right) \\ &= \sigma_{l1}\left(\tilde{z}_{l2} + \sigma_{l2} - d_{l1}\left(\hat{\overline{z}}_{l1}\right)\hat{\theta}_{l1} - k_{l1}\sigma_{l1} + d_{l1}\left(\overline{z}_{l1}\right)\theta_{l1}\right)\end{aligned} \tag{4.5}$$

According to (3.16)-(3.19), when $t \geq t_{l1}^0$, $\tilde{\theta}_{l1} = 0$, $\tilde{z}_{l1} = 0$, $\Delta d_{l1}\left(\overline{z}_{l1}\right) = 0$. Then (4.5) can be described by:

$$\dot{V}_{l1} = -k_{l1}\sigma_{l1}^2 + \sigma_{l1}\sigma_{l2} + \sigma_{l1}\tilde{z}_{l2} \tag{4.6}$$

Step 2

Consider the $(\sigma_{l1}, \sigma_{l2})$ subsystem. This can be described as:

$$\begin{cases}\dot{\sigma}_{l1} = -k_{l1}\sigma_{l1} + \sigma_{l2} + \tilde{z}_{l2} \\ \dot{\sigma}_{l2} = \dot{z}_{l2} - \dot{\tilde{z}}_{l2} - \dot{\beta}_{l2} \\ \qquad = z_{l3} + d_{l2}\left(\overline{z}_{l2}\right)\theta_{l2} - \dot{\tilde{z}}_{l2} - \dfrac{\partial \beta_{l2}}{\partial \hat{z}_{l1}}\dot{\hat{z}}_{l1}\end{cases} \tag{4.7}$$

A Lyapunov function is chosen as:

$$V_{l2} = V_{l1} + \frac{1}{2}\sigma_{l2}^2 \tag{4.8}$$

Differentiate (4.8):

$$
\begin{aligned}
\dot{V}_{l2} &= -k_{l1}\sigma_{l1}^2 + \sigma_{l1}\sigma_{l2} + \sigma_{l1}\tilde{z}_{l2} \\
&+ \sigma_{l2}\left\{\tilde{z}_{l3} + \hat{z}_{l3} + d_{l2}\left(\overline{z}_{l2}\right)\theta_{l2} - \dot{\hat{z}}_{l2} - \frac{\partial \beta_{l2}}{\partial \hat{z}_{l1}}\dot{\hat{z}}_{l1}\right\}
\end{aligned}
\tag{4.9}
$$

where $z_{l3} = \tilde{z}_{l3} + \hat{z}_{l3}$ has been used since z_{l3} is not available.

Let $\beta_{l3} = -\sigma_{l1} - d_{l2}\left(\hat{\overline{z}}_{l2}\right)\hat{\theta}_{l2} + \frac{\partial \beta_{l2}}{\partial \hat{z}_{l1}}\dot{\hat{z}}_{l1} - k_{l2}\sigma_{l2}$, where $k_{l2} > 0$.

Then (4.9) can be described as:

$$
\begin{aligned}
\dot{V}_{l2} &= -k_{l1}\sigma_{l1}^2 + \sigma_{l1}\sigma_{l2} + \sigma_{l1}\tilde{z}_{l2} \\
&\quad + \sigma_{l2}\left\{\tilde{z}_{l3} + \beta_{l3} + \sigma_{l3} + d_{l2}\left(\overline{z}_{l2}\right)\theta_{l2} - \dot{\hat{z}}_{l2} - \frac{\partial \beta_{l2}}{\partial \hat{z}_{l1}}\dot{\hat{z}}_{l1}\right\} \\
&= -k_{l1}\sigma_{l1}^2 + \sigma_{l1}\tilde{z}_{l2} \\
&\quad + \sigma_{l2}\left\{\tilde{z}_{l3} - d_{l2}\left(\hat{\overline{z}}_{l2}\right)\hat{\theta}_{l2} + \frac{\partial \beta_{l2}}{\partial \hat{z}_{l1}}\dot{\hat{z}}_{l1} - k_{l2}\sigma_{l2} + \sigma_{l3} + d_{l2}\left(\overline{z}_{l2}\right)\theta_{l2} - \dot{\hat{z}}_{l2} - \frac{\partial \beta_{l2}}{\partial \hat{z}_{l1}}\dot{\hat{z}}_{l1}\right\}
\end{aligned}
\tag{4.10}
$$

According to (3.23)-(3.24), when $t \geq t_{l2}^0$, $\tilde{\theta}_{l2} = 0$, $\tilde{z}_{l2} = 0$, $\Delta d_{l2}\left(\overline{z}_{l2}\right) = 0$. Then (4.10) becomes:

$$
\dot{V}_{l2} = -k_{l1}\sigma_{l1}^2 - k_{l2}\sigma_{l2}^2 + \sigma_{l2}\sigma_{l3} + \sigma_{l2}\tilde{z}_{l3} \tag{4.11}
$$

Step i $(3 \leq i \leq r_l - 1)$

Consider the $(\sigma_{l1}, \sigma_{l2}, \cdots, \sigma_{li})$ subsystem,

$$
\begin{aligned}
\dot{\sigma}_{li} &= \dot{z}_{li} - \dot{\hat{z}}_{li} - \dot{\beta}_{li} \\
&= z_{l(i+1)} + d_{li}\left(\overline{z}_{li}\right)\theta_{li} - \dot{\hat{z}}_{li} - \sum_{q=1}^{i-1}\frac{\partial \beta_{li}}{\partial \hat{z}_{lq}}\dot{\hat{z}}_{lq}
\end{aligned}
\tag{4.12}
$$

A Lyapunov function is chosen as:

$$
V_{li} = V_{l(i-1)} + \frac{1}{2}\sigma_{li}^2 \tag{4.13}
$$

Differentiate (4.13) along (4.12):

$$
\begin{aligned}
\dot{V}_{li} &= -\sum_{q=1}^{i-1} k_{lq}\sigma_{lq}^2 + \sigma_{l(i-1)}\sigma_{li} + \sigma_{l(i-1)}\tilde{z}_{li} \\
&+ \sigma_{li}\left\{\tilde{z}_{l(i+1)} + \hat{z}_{l(i+1)} + d_{li}\left(\overline{z}_{li}\right)\theta_{li} - \dot{\hat{z}}_{li} - \sum_{q=1}^{i-1}\frac{\partial \beta_{li}}{\partial \hat{z}_{lq}}\dot{\hat{z}}_{lq}\right\}
\end{aligned}
\tag{4.14}
$$

where $z_{l(i+1)} = \tilde{z}_{l(i+1)} + \hat{z}_{l(i+1)}$ has been used since $z_{l(i+1)}$ is not available.

Then let $\beta_{l(i+1)} = -\sigma_{l(i-1)} - k_{li}\sigma_{li} + \sum_{q=1}^{i-1}\frac{\partial \beta_{li}}{\partial \hat{z}_{lq}}\dot{\hat{z}}_{lq} - d_{li}\left(\hat{\overline{z}}_{li}\right)\hat{\theta}_{li}$, where $k_{li} > 0$.

Then,

$$\begin{aligned}\dot{V}_{li} &= -\sum_{q=1}^{i-1} k_{lq}\sigma_{lq}^2 + \sigma_{l(i-1)}\sigma_{li} + \sigma_{l(i-1)}\tilde{z}_{li} \\ &+ \sigma_{li}\left\{\tilde{z}_{l(i+1)} + \hat{z}_{l(i+1)} + d_{li}\left(\overline{z}_{li}\right)\theta_{li} - \dot{\hat{z}}_{li} - \sum_{q=1}^{i-1}\frac{\partial \beta_{li}}{\partial \hat{z}_{lq}}\dot{\hat{z}}_{lq}\right\} \\ &= -\sum_{q=1}^{i-1} k_{lq}\sigma_{lq}^2 + \sigma_{l(i-1)}\tilde{z}_{li} + \sigma_{li}\left\{\tilde{z}_{l(i+1)} + \sigma_{l(i+1)} - k_{li}\sigma_{li} - d_{li}\left(\hat{\overline{z}}_{li}\right)\hat{\theta}_{li} + d_{li}\left(\overline{z}_{li}\right)\theta_{li} - \dot{\hat{z}}_{li}\right\}\end{aligned} \tag{4.15}$$

According to (3.28)-(3.29), when $t \geq t_{li}^0$, $\tilde{\theta}_{li} = 0$, $\tilde{z}_{li} = 0$, $\Delta d_{li}\left(\overline{z}_{li}\right) = 0$, (4.15) becomes:

$$\begin{aligned}\dot{V}_{li} &= -\sum_{q=1}^{i-1} k_{lq}\sigma_{lq}^2 + \sigma_{li}\left\{\tilde{z}_{l(i+1)} + \sigma_{l(i+1)} - k_{li}\sigma_{li}\right\} \\ &= -\sum_{q=1}^{i} k_{lq}\sigma_{lq}^2 + \sigma_{li}\tilde{z}_{l(i+1)} + \sigma_{li}\sigma_{l(i+1)}\end{aligned} \tag{4.16}$$

Step r_l

Based on the above analysis,

$$\begin{aligned}\beta_{lr_l} &= -\sigma_{l(r_l-2)} - k_{l(r_l-1)}\sigma_{l(r_l-1)} + \sum_{q=1}^{r_l-2}\frac{\partial \beta_{l(r_l-1)}}{\partial \hat{z}_{lq}}\dot{\hat{z}}_{lq} - d_{l(r_l-1)}\left(\hat{\overline{z}}_{l(r_l-1)}\right)\hat{\theta}_{l(r_l-1)} \\ \dot{V}_{l(r_l-1)} &= -\sum_{q=1}^{r_l-1} k_{lq}\sigma_{lq}^2 + \sigma_{l(r_l-1)}\tilde{z}_{lr_l} + \sigma_{l(r_l-1)}\sigma_{lr_l}\end{aligned} \tag{4.17}$$

and

$$\begin{aligned}\dot{\sigma}_{lr_l} &= \dot{z}_{lr_l} - \dot{\hat{z}}_{lr_l} - \dot{\beta}_{lr_l} \\ &= u_l + \xi_l\left(t,z\right) + \psi_{lr_l}\left(t,z\right) - \dot{\hat{z}}_{lr_l} - \sum_{q=1}^{r_l-1}\frac{\partial \beta_{lr_l}}{\partial \hat{z}_{lq}}\dot{\hat{z}}_{lq}\end{aligned} \tag{4.18}$$

A Lyapunov function is chosen as:

$$V_{lr_l} = V_{l(r_l-1)} + \frac{1}{2}\sigma_{lr_l}^2 \tag{4.19}$$

Differentiate (4.19) along (4.18):

$$\begin{aligned}\dot{V}_{lr_l} &= -\sum_{q=1}^{r_l-1} k_{lq}\sigma_{lq}^2 + \sigma_{l(r_l-1)}\tilde{z}_{lr_l} + \sigma_{l(r_l-1)}\sigma_{lr_l} \\ &+ \sigma_{lr_l}\left\{u_l + \xi_l\left(t,z\right) + \psi_{lr_l}\left(t,z\right) - \dot{\hat{z}}_{lr_l} - \sum_{q=1}^{r_l-1}\frac{\partial \beta_{lr_l}}{\partial \hat{z}_{lq}}\dot{\hat{z}}_{lq}\right\}\end{aligned} \tag{4.20}$$

Then design the output feedback control:

$$u_l = -\left[\alpha_{lr_l}\operatorname{sgn}\left(\tilde{z}_{lr_l}\right)\right]_{eq} + \sum_{q=1}^{r_l-1}\frac{\partial \beta_{lr_l}}{\partial \hat{z}_{lq}}\dot{\hat{z}}_{lq} - \sigma_{l(r_l-1)} - k_{lr_l}\sigma_{lr_l} \tag{4.21}$$

where $k_{lr_l} > 0$.

It follows (4.20) and (4.21),

$$\dot{V}_{lr_l} = -\sum_{q=1}^{r_l} k_{lq}\sigma_{lq}^2 + \sigma_{l(r_l-1)}\tilde{z}_{lr_l} + \sigma_{lr_l}\left\{-\left[\alpha_{lr_l}\operatorname{sgn}\left(\tilde{z}_{lr_l}\right)\right]_{eq} + \xi_l(t,z) + \psi_{lr_l}(t,z) - \dot{\hat{z}}_{lr_l}\right\} \tag{4.22}$$

According to (3.34)-(3.35), when $t \geq t_{lr_l}^0$, $\tilde{z}_{lr_l} = 0$, (4.22) can be described by:

$$\dot{V}_{lr_l} = -\sum_{q=1}^{r_l} k_{lq}\sigma_{lq}^2 \tag{4.23}$$

Lemma 3 *Under Assumptions 1-5, the control (4.21) can guarantee that σ_{lq} tends to zero exponentially for all $l = 1,\cdots,m$, $q = 1,2,\cdots,r_l$.*

Proof Define the following Lyapunov function:

$$\bar{V} = \sum_{l=1}^{m} V_{lr_l} = \sum_{l=1}^{m}\sum_{q=1}^{r_l}\frac{1}{2}\sigma_{lq}^2 \tag{4.24}$$

Differentiate (4.24):

$$\dot{\bar{V}} = \sum_{l=1}^{m}\dot{V}_{lr_l} = -\sum_{l=1}^{m}\sum_{q=1}^{r_l} k_{lq}\sigma_{lq}^2 \tag{4.25}$$

It is obvious that

$$\begin{aligned}\dot{\bar{V}} &\leqslant -2\sum_{l=1}^{m} k_{lq}V_{lr_l}\\ &\leqslant -k\bar{V}\end{aligned} \tag{4.26}$$

where $k = 2\min k_{lq}$.

Hence, Lemma 3 follows. □

Theorem 1 *Under Assumptions 1-5, the closed-loop system employed by system (2.1), the AFTSMO (3.9)-(3.11) and the control (4.21) is asymptotically stable if $-c_3 + \frac{1}{2}c_4\mathscr{L}_\omega + \frac{1}{2}c_4\Phi + c_4\tau < 0$ and $\frac{1}{2}c_4\mathscr{L}_\omega + \frac{1}{2}c_4\Phi - k_1 < 0$ with $k_1 = 2\min k_{l1}$.*

Proof Define the following Lyapunov function:

$$V = V^b\left(t,z^b\right) + V^y(t,y) + V^o\left(\tilde{z}_a,\tilde{\theta}_{lj}\right) + \bar{V}\left(\bar{z}_a,\beta_{lq}\right) \tag{4.27}$$

where $l = 1,\cdots,m$, $q = 1,2,\cdots,r_l$, $j = 1,2,\cdots,r_l - 1$, $\tilde{z}_a := col\left(z_1,\cdots,z_l\cdots,z_m\right) - col\left(\hat{z}_1,\cdots,\hat{z}_l\cdots,\hat{z}_m\right)$ with $\hat{z}_l := col\left(\hat{z}_{l1},\hat{z}_{l2}\cdots,\hat{z}_{lr_l}\right)$, $\bar{z}_a := col\left(\bar{z}_1,\cdots,\bar{z}_l\cdots,\bar{z}_m\right)$ with $\bar{z}_l := col\left(z_{l1},\hat{z}_{l2}\cdots,\hat{z}_{lr_l}\right)$.

It follows from Lemma 2,

$$\dot{V}^o \leqslant -c\sqrt{V^o} \tag{4.28}$$

From (4.25) and the definition of σ_{lq} in (4.1),

$$\begin{aligned}\bar{\dot{V}} &= -\sum_{l=1}^{m}\sum_{q=2}^{r_l} k_{lq}\sigma_{lq}^2 - \sum_{l=1}^{m} k_{l1}\sigma_{l1}^2 \\ &\leqslant -\sum_{l=1}^{m}\sum_{q=2}^{r_l} k_{lq}\sigma_{lq}^2 - k_1\sum_{l=1}^{m}\sigma_{l1}^2 \\ &= -\sum_{l=1}^{m}\sum_{q=2}^{r_l} k_{lq}\sigma_{lq}^2 - k_1\|y\|^2\end{aligned} \tag{4.29}$$

It follows from (2.3)-(2.5) and (4.28)-(4.29),

$$\begin{aligned}&\dot{V} = \dot{V}^b + \dot{V}^y + \dot{V}^o + \bar{\dot{V}} \\ &\dot{V} = \frac{\partial V^b}{\partial t} + \frac{\partial V^b}{\partial z^b}\omega\left(0,z^b\right) + \frac{\partial V^b}{\partial z^b}\left[\omega\left(y,z^b\right) - \omega\left(0,z^b\right)\right] + \frac{\partial V^b}{\partial z^b}\Theta(t,z) + \bar{\dot{V}} + \dot{V}^o \\ &\leqslant -c_3\left\|z^b\right\|^2 + c_4\mathscr{L}_\omega\left\|z^b\right\|\|y\| + c_4\left\|z^b\right\|\left(\Phi\|y\| + \tau\left\|z^b\right\|\right) + \bar{\dot{V}} + \dot{V}^o \\ &\leqslant -c_3\left\|z^b\right\|^2 + c_4\mathscr{L}_\omega\left(\frac{1}{2}\left\|z^b\right\|^2 + \frac{1}{2}\|y\|^2\right) + c_4\Phi\left(\frac{1}{2}\left\|z^b\right\|^2 + \frac{1}{2}\|y\|^2\right) + c_4\tau\left\|z^b\right\|^2 \\ &-\sum_{l=1}^{m}\sum_{q=2}^{r_l} k_{lq}\sigma_{lq}^2 - k_1\|y\|^2 - c\sqrt{V^o} \\ &\leqslant \left(-c_3 + \frac{1}{2}c_4\mathscr{L}_\omega + \frac{1}{2}c_4\Phi + c_4\tau\right)\left\|z^b\right\|^2 + \left(\frac{1}{2}c_4\mathscr{L}_\omega + \frac{1}{2}c_4\Phi - k_1\right)\|y\|^2 \\ &-\sum_{l=1}^{m}\sum_{q=2}^{r_l} k_{lq}\sigma_{lq}^2 - c\sqrt{V^o}\end{aligned} \tag{4.30}$$

Hence, $\dot{V} < 0$ follows from the conditions that $-c_3 + \frac{1}{2}c_4\mathscr{L}_\omega + \frac{1}{2}c_4\Phi + c_4\tau < 0$ and $\frac{1}{2}c_4\mathscr{L}_\omega + \frac{1}{2}c_4\Phi - k_1 < 0$ and Theorem 1 holds. □

Remark 4 *For the uncertain nonlinear system with unmodeled dynamics, there have been other observer based backstepping control methods developed. In a number of contributions the systems are SISO such as Sui et al. (2021) and the unmatched uncertainties $\Theta(t,z)$ have not been considered such as in Chang et al. (2020); Jiang & Praly (1998). Furthermore, the closed-loop systems are uniformly bounded rather than asymptotically stable with existing observer based backstepping control methods such as Tong & Li (2010); Xu et al. (2013). Based on the designed AFTSMO, an OFBC has been designed for a class of MIMO nonlinear systems in this paper to guarantee the closed-loop system is asymptotically stable, despite the presence of matched and unmatched uncertainties.*

5. Simulation examples

Based on the MATLAB software, this section will test the effectiveness of the designed OFBC by two simulation examples.

Example 1:

Consider a nonlinear system described by:

$$\begin{aligned}
\dot{z}_1 &= A_1 z_1 + B_1 \left(u_1 + \xi_1 (t,z)\right) + \psi_1 (t,z) \\
\dot{z}_2 &= A_2 z_2 + B_2 \left(u_2 + \xi_2 (t,z)\right) + \psi_2 (t,z) \\
\dot{z}^b &= -4z^b + z^b z_{11} + \Theta (t,z) \\
y &= \begin{bmatrix} z_{11} & z_{21} \end{bmatrix}^T
\end{aligned} \tag{5.1}$$

where $z := col\left(z^a, z^b\right) \in Z = \left\{ \left(z^a, z^b\right) \middle| \left|z^b\right| < 5 \right\}$, $z^a := col\left(z_1, z_2\right)$, $z_1 := col\left(z_{11}, z_{12}\right)$, $z_2 := col\left(z_{21}, z_{22}\right)$, $A_1 = A_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $B_1 = B_2 = \begin{bmatrix} 0 & 1 \end{bmatrix}^T$.

The uncertainties satisfy

$$\Theta (t,z) = \|y\| + \left|z^b\right| \tag{5.2}$$

$$\xi_1 (t,z) = 0.1 \sin (t) \quad \xi_2 (t,z) = 0.2 \sin \left(z^b\right) \tag{5.3}$$

$$\psi_1 (t,z) = \begin{bmatrix} z_{11}\theta_{11} & 0 \end{bmatrix}^T \qquad \psi_2 (t,z) = \begin{bmatrix} z_{21}\theta_{21} & 0 \end{bmatrix}^T \tag{5.4}$$

where $\theta_{11} = -2$ and $\theta_{21} = 1$.

It is clear that Assumptions 1-2 are all satisfied with $\mathscr{L}_\omega = \left|z^b\right|, \Phi = 1, \tau = 1$ while V^b is chosen as $V^b\left(t, z^b\right) = \frac{1}{2}\left(z^b\right)^2$ with $c_3 = 4, c_4 = 1$.

The main parameters of the adaptive observer for z_1, z_2, θ_{11}, θ_{21} are given as:

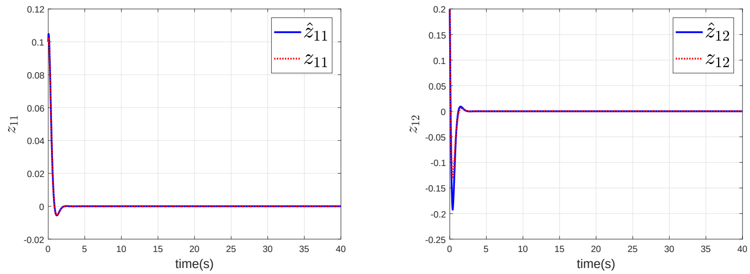
$$\phi_1 = 5, \phi_2 = 10, \gamma_1 = 1, \gamma_2 = 2, \Gamma_{11} = \Gamma_{12} = \Gamma_{21} = \Gamma_{22} = 5 \tag{5.5}$$

The main parameters of the control (4.21) are given as:

$$k_{11} = k_{12} = k_{21} = k_{22} = 8 \tag{5.6}$$

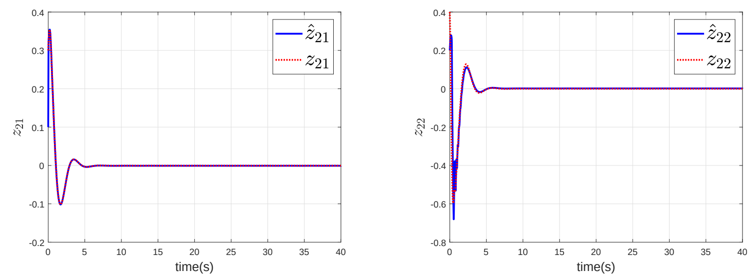
Then $k_1 = 8$ and by direct computation, Theorem 1 holds in $z \in Z$.

Note that $\cdot / (\|\cdot\| + \beta)$ is used to replace $\mathrm{sgn}(\cdot)$ to reduce the chattering with $\beta = 0.001$. Fig. 2 and Fig. 3 show the time response of z_1, z_2 and their estimates, respectively. According to Figs. 2-3, the proposed AFTSMO can observe the system external dynamics in FT. The parameter estimation is shown in Fig. 4. The estimates of the matched uncertainties are shown in Fig. 5. As shown in Figs. 4-5, the designed parameter estimator can construct the unmatched parameters and lumped matched uncertainties in FT. Fig. 6 shows the time response of z_b and u. From Figs. 2-6, the considered system is stabilized regardless of the matched and unmatched uncertainties. The simulation results demonstrate that the designed OFBC is effective.



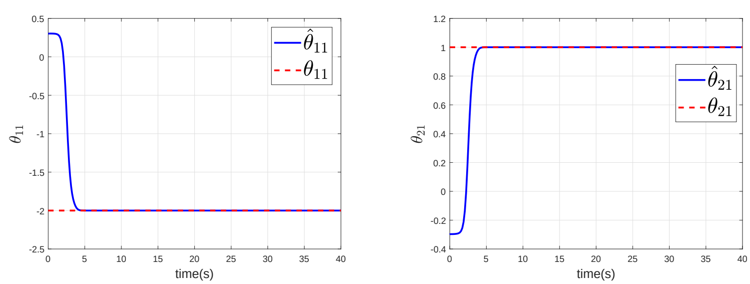
(a) The system state z_{11} and its estimate $\hat{z}_{11}$ (b) The system state z_{12} and its estimate $\hat{z}_{12}$

FIG. 2. The system states z_1 and their estimates



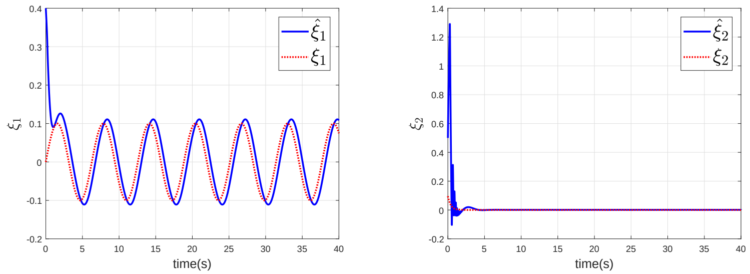
(a) The system state z_{21} and its estimate $\hat{z}_{21}$ (b) The system state z_{22} and its estimate $\hat{z}_{22}$

FIG. 3. The system states z_2 and their estimates



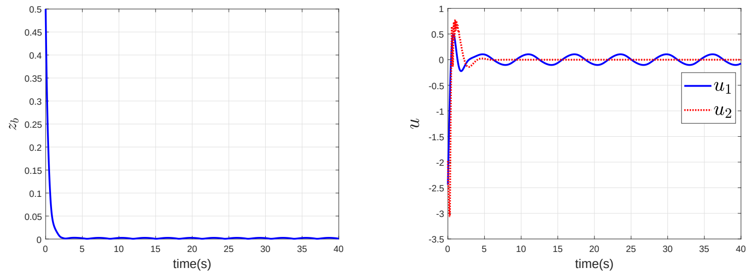
(a) Estimate of parameter θ_{11} and the actual value (b) Estimate of the parameter θ_{21} and the actual value

FIG. 4. Estimate of the parameters and their actual values



(a) The estimate of ξ_1 and its actual value (b) The estimate of ξ_2 and its actual value

FIG. 5. The estimates of the matched uncertainties and their actual values



(a) The time response of the system state z_b (b) The time response of the system control signals u

FIG. 6. The time response of z_b and u

To further test the proposed method, the observer-based adaptive FT tracking control proposed in Chang et al. (2020) will be compared with the proposed AFTSMO. Note that only a SISO system was considered in Chang et al. (2020), hence it is necessary to divide the system (5.1) into the following two subsystems:

$$\begin{aligned} \dot{z}_1 &= A_1 z_1 + B_1 (u_1 + \xi_1) + \psi_1 \\ \dot{z}^b &= -4z^b + z^b z_{11} + \Theta \\ y_1 &= z_{11} \end{aligned} \tag{5.7}$$

$$\begin{aligned} \dot{z}_2 &= A_2 z_2 + B_2 (u_2 + \xi_2) + \psi_2 \\ y_2 &= z_{12} \end{aligned} \tag{5.8}$$

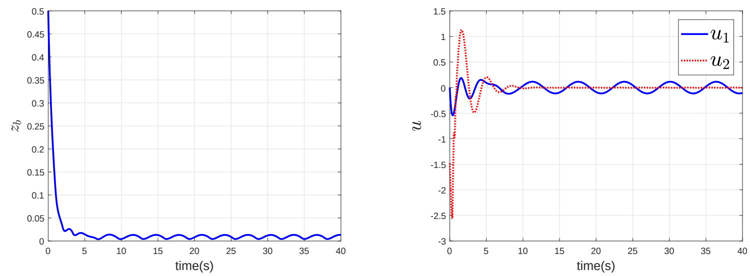
where $z_1 := col(z_{11}, z_{12})$, $z_2 := col(z_{21}, z_{22})$, $\Theta = |z^b| + |z_{11}|$ and $A_1, A_2, B_1, B_2, \xi_1, \xi_2, \psi_1, \psi_2$ are as for system (5.1).

The main control parameters for the method of Chang et al. (2020) are given by:

$$y_r = 0, \mu = 10, l = 30, c = 30, \lambda = 10, \eta = 0.99 \tag{5.9}$$

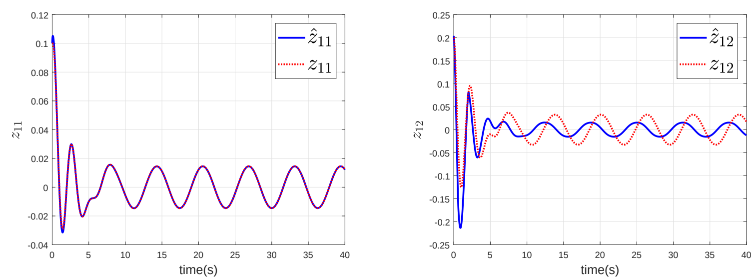
The time response of z_b and u are shown in Fig. 7 while Fig. 8 shows the time response of the system states z_1, z_2 and their estimates using the method proposed in Chang et al. (2020). It should be noted that the unmatched uncertainties $\Theta(t,z)$ have not been considered directly in Chang et al.

(2020) in contrast to the proposed method. Furthermore the results in Chang et al. (2020) show that the closed-loop system is uniformly bounded while the closed-loop system is asymptotically stable in this paper. Comparing Figs. 2-8, the proposed method has better performance as would be expected from the theoretical foundations.

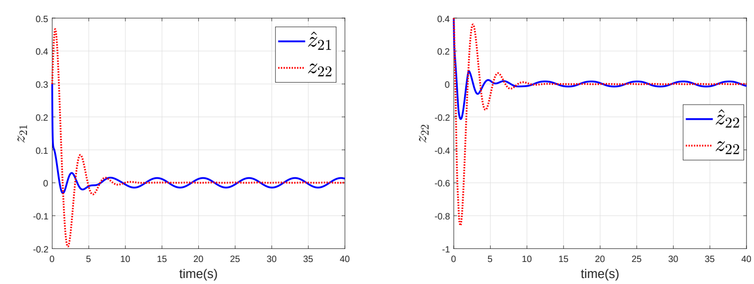


(a) The time response of the system state z_b (b) The time response of the system control signals u

FIG. 7. The time response of z_b and u using the method proposed in Chang et al. (2020)



(a) The time response of the system state z_{11} (b) The time response of the system state z_{12}



(c) The time response of the system state z_{21} (d) The time response of the system state z_{22}

FIG. 8. The time response of the system state z_1 and z_2 using the method proposed in Chang et al. (2020)

Example 2:

Consider a pendulum, whose dynamic equation is described in Jiang & Hill (1999) by

$$ml\ddot{\eta} = -mg\sin\eta - kl\dot{\eta} + \frac{1}{l}u + \Delta \tag{5.10}$$

where $u \in R$ represents the torque, $\eta \in R$ represents the anticlockwise angle, g represents the acceleration of gravity, and m, l, k represent the mass of bob, length of rod and coefficient of friction respectively. Note that the constant k is unknown and the angular velocity $\dot{\eta}$ is not available in the subsequent analysis. Δ denotes the unmodeled dynamics.

The objective is to develop an OFBC to stabilize system (5.10) at $\eta = \pi$. For convenience of the observer and control design, the following coordinate transformation is introduced:

$$\begin{aligned} z_1 &= ml^2(\eta - \pi) \\ z_2 &= ml^2\left(\dot{\eta} + \frac{k}{m}(\eta - \pi)\right) \end{aligned} \tag{5.11}$$

to bring the point $(\eta, \dot{\eta}) = (\pi, 0)$ into $(z_1, z_2) = (0, 0)$.

Suppose that the unmodeled dynamics is given by

$$\dot{z}^b = -5z^b + \left(z^b\right)^2 z_1 + \Theta(t, z), \qquad \Delta = 0.2\sin\left(z^b\right) \tag{5.12}$$

where $z := col\left(z_1, z_2, z^b\right) \in Z = \left\{ \left(z_1, z_2, z^b\right) \middle| \left(z^b\right)^2 < 7 \right\}$, $\Theta(t, z) = |z_1| + \left|z^b\right|$.

In the new z-coordinates, the pendulum equation (5.10) is written as

$$\begin{aligned} \dot{z}_1 &= z_2 + z_1\theta \\ \dot{z}_2 &= u + \xi \\ \dot{z}^b &= -5z^b + \left(z^b\right)^2 z_1 + \Theta(t, z) \end{aligned} \tag{5.13}$$

where $\theta = -\frac{k}{m}$ denotes the uncertain parameter, $\xi = mgl\sin\left(\frac{1}{ml^2}z_1\right) + 0.2l\sin\left(z^b\right)$ denotes the matched uncertainty, $\Theta(t, z)$ denotes the unmatched uncertainty.

It is clear that Assumptions 1-2 are all satisfied with $\mathscr{L}_\omega = \left(z^b\right)^2, \Phi = 1, \tau = 1$ while V^b is chosen as $V^b\left(t, z^b\right) = \frac{1}{2}\left(z^b\right)^2$ with $c_3 = 5, c_4 = 1$.

The main parameters of the adaptive observer for z_1, z_2, θ are given as:

$$\phi_1 = 3, \gamma_1 = 2, \Gamma_{11} = \Gamma_{12} = 5 \tag{5.14}$$

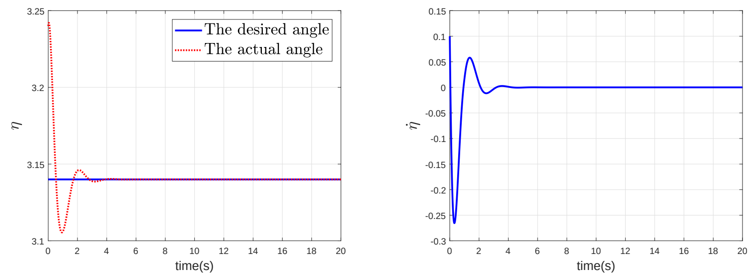
The main parameters of the control (4.21) are given as:

$$k_{11} = k_{12} = 4 \tag{5.15}$$

Then $k_1 = 4$ and by direct computation, Theorem 1 holds in $z \in Z$.

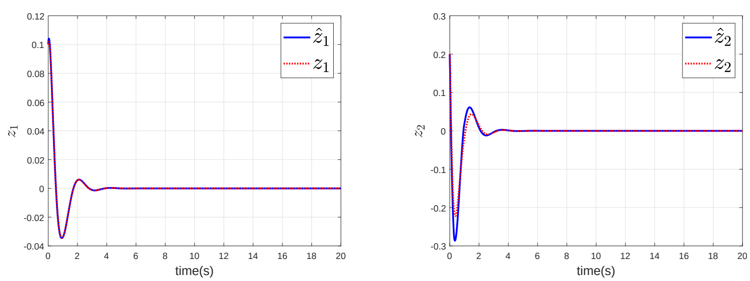
For simulation, let $l = g = 9.8$ with $m = k = g^{-2}$. Fig. 9 shows the time response of the anticlockwise angle and angular velocity. Fig. 9 illustrates that the pendulum can be stabilized at the desired angle by the proposed OFBC while the system shows good performance. Fig. 10 shows the time response of

z_1, z_2 and their estimates. Fig. 11 shows the time response of z_b and u. The estimates of the uncertain parameter and matched uncertainty are shown in Fig. 12. As shown in Figs. 9-12, the designed AFT-SMO can observe the system external dynamics in FT while the unmatched parameter and the lumped matched uncertainty also can be estimated in FT.



(a) The time response of the anticlockwise angle (b) The time response of the angular velocity

FIG. 9. The time response of the anticlockwise angle and angular velocity

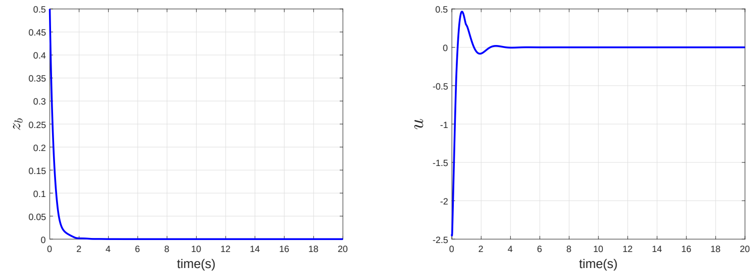


(a) The system state z_1 and its estimate $\hat{z}_1$ (b) The system state z_2 and its estimate $\hat{z}_2$

FIG. 10. The system states z_1, z_2 and their estimates

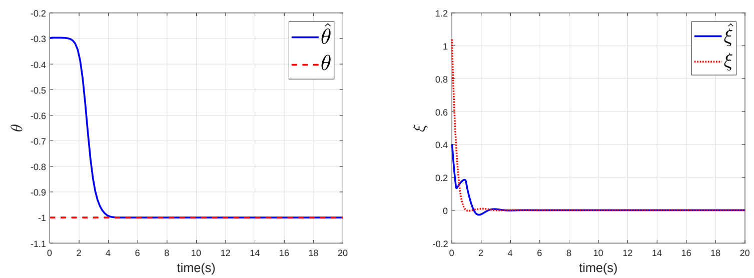
To further test the proposed method, the static sliding mode control proposed in Feng et al. (2020) will be compared with the proposed AFTSMO. Note that the method proposed in Feng et al. (2020) aims to deal with a class of nonlinear interconnected systems but is used here to stabilize a subsystem. The sliding function and controller are designed as:

$$
\begin{aligned}
s &= 2z_1 + z_2 \\
u &= \begin{bmatrix} 0 & -2 \end{bmatrix} \begin{bmatrix} z_1 & z_2 \end{bmatrix}^T - 0.01\text{sgn}(s)
\end{aligned} \tag{5.16}
$$



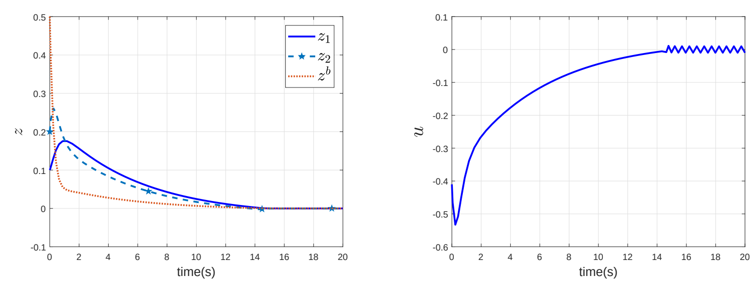
(a) The time response of the system state z_b (b) The time response of the system control signal u

FIG. 11. The time response of z_b and u



(a) Estimate of parameter θ and the actual value (b) Estimate of the matched uncertainty ξ and their actual value

FIG. 12. Estimate of the parameter, matched uncertainty and their actual values



(a) The time response of the system states z (b) The time response of the system control signals u

FIG. 13. The time response of system states and control input using the method proposed in Feng et al. (2020)

Fig. 13 shows the time response of system states and control input using the method proposed in Feng et al. (2020). As shown in Fig. 13, the the method proposed in Feng et al. (2020) can also stabilize the uncertain nonlinear system (5.13), but the system performance is poor. In addition, the system states are required to be measurable in Feng et al. (2020).

From the above analysis, it can be concluded that the designed OFBC shows strong robustness for a class of uncertain nonlinear systems. The observed system states and estimated uncertain parameters are effectively reconstructed by the AFTSMO in FT so that they can be compensated by the controller.

6. Conclusion

An OFBC has been designed for a class of nonlinear systems. In order to observe the external dynamics and unmatched uncertain parameters in FT, a novel AFTSMO is first proposed. By using the backstepping principle, a set of Constructive Lyapunov Functions is designed to obtain the desired control while the system stability is guaranteed. Numerical examples are used to demonstrate the effectiveness of the designed OFBC. Future work will study how to further relax the requirements on both the system model and unmatched uncertainties.

Acknowledgments

This work is partially supported by the National Nature Science Foundation of China under grant nos 61973315, 61473312.

References

Besançon, G. (2007) *Nonlinear observers and applications*. Springer, New York.

Chang, Y., Zhang, S., Alotaibi, N. D., & Alkhateeb, A. F. (2020) Observer-based adaptive finite-time tracking control for a class of switched nonlinear systems with unmodeled dynamics. *IEEE Access*, **8**, 204782–204790.

Clempner, J. B. & Yu, W. (2018) *New perspectives and applications of modern control theory*. Springer International Publishing.

Daly, J. M. & Wang, D. W. (2009) Output feedback sliding mode control in the presence of unknown disturbances. *Syst. Control Lett.*, **58**, 188–193.

Davila, J., Fridman, L., & Levant, A. (2005) Second-order sliding-mode observer for mechanical systems. *EEE Trans. Automat. Contr.*, **50**, 1785–1789.

Feng, J., Zhao, D., Yan, X.-G., & Spurgeon, S. K. (2020) Decentralized sliding mode control for a class of nonlinear interconnected systems by static state feedback. *Int. J. Robust Nonlinear Control*, **30**, 2152–2170.

Floquet, T. & Barbot, J.-P. (2007) Super twisting algorithm-based step-by-step sliding mode observers for nonlinear systems with unknown inputs. *Int. J. Syst. Sci.*, **38**, 803–815.

Han, W., Wang, Z., Shen, Y., & Qi, J. (2019). L_∞ observer for uncertain linear systems. *Asian J. Control*, **21**, 632–638.

Haskara, I. (1998) On sliding mode observers via equivalent control approach. *Int. J. Control*, **71**, 1051–1067.

Isidori, A. (2013) *Nonlinear control systems*. Springer Science & Business Media.

Jiang, Z.-P. (1999) A combined backstepping and small-gain approach to adaptive output feedback control. *Automatica*, **35**, 1131–1139.

Jiang, Z.-P. & Hill, D. J. (1999) A robust adaptive backstepping scheme for nonlinear systems with unmodeled dynamics. *IEEE Trans. Automat. Contr.*, **44**, 1705–1711.

Jiang, Z.-P. & Praly, L. (1998) Design of robust adaptive controllers for nonlinear systems with dynamic uncertainties. *Automatica*, **34**, 825–840.

Kapetina, M. N., Rapaić, M. R., Pisano, A., & Jeličić, Z. D. (2019) Adaptive parameter estimation in lti systems. *IEEE Trans. Automat. Contr.*, **64**, 4188–4195.

Khalil, H. K. & Grizzle, J. W. (2002) *Nonlinear systems*. Prentice hall Upper Saddle River, NJ.

Mousavi, M., Rahnavard, M., Ayati, M., & Hairi Yazdi, M. R. (2019) Terminal sliding mode observers for uncertain linear systems with matched disturbance. *Asian J. Control*, **21**, 377–386.

Na, J., Mahyuddin, M. N., Herrmann, G., Ren, X., & Barber, P. (2015) Robust adaptive finite-time parameter estimation and control for robotic systems. *Int. J. Robust Nonlinear Control*, **25**, 3045–3071.

Rastegari, A., Arefi, M. M., & Asemani, M. H. (2019) Robust h_∞ sliding mode observer-based fault-tolerant control for one-sided lipschitz nonlinear systems. *Asian J. Control*, **21**, 114–129.

Slotine, J. E., Hedrick, J. K., & Misawa, E. A. (1986) On sliding observers for nonlinear systems. In *1986 American Control Conference*, pp. 1794–1800.

Sui, S., Chen, C. L. P., & Tong, S. (2021) Event-trigger-based finite-time fuzzy adaptive control for stochastic nonlinear system with unmodeled dynamics. *IEEE Trans. Fuzzy Syst.*, **29**, 1914–1926.

Sun, H. & Guo, L. (2014) Composite adaptive disturbance observer based control and back-stepping method for nonlinear system with multiple mismatched disturbances. *J. Franklin Inst.*, **351**, 1027–1041.

Tong, S. & Li, Y. (2010) Observer-based fuzzy adaptive robust control of nonlinear systems with time delays and unmodeled dynamics. *Neurocomputing*, **74**, 369–378.

Wang, F., Zou, Q., & Zong, Q. (2017) Robust adaptive backstepping control for an uncertain nonlinear system with input constraint based on lyapunov redesign. *Int. J. Control Autom. Syst.*, **15**, 212–225.

Wang, H., Yang, H., Liu, X., Liu, L., & Li, S. (2016) Direct adaptive neural control of nonlinear strict-feedback systems with unmodeled dynamics using small-gain approach. *Int. J. Adapt. Control Signal Process.*, **30**, 906–927.

Xing, Y., Na, J., & Costa-Castelló, R. (2019) Real-time adaptive parameter estimation for a polymer electrolyte membrane fuel cell. *IEEE Trans Industr. Inform.*, **15**, 6048–6057.

Xu, B., Liu, X., Wang, H., & Zhou, Y. (2019) Event-triggered adaptive backstepping control for strict-feedback nonlinear systems with zero dynamics. *Complexity*, https://doi.org/10.1155/2019/7890968.

Xu, Y., Tong, S., & Li, Y. (2013) Observer-based fuzzy adaptive control of nonlinear systems with actuator faults and unmodeled dynamics. *Neural Comput Appl.*, **23**, 391–405.

Yan, X.-G., Spurgeon, S. K., Zhao, D., & Guo, L. (2016) Decentralised stabilisation of nonlinear time delay interconnected systems. *IFAC-PapersOnLine*, **49**, 152–157.

Yang, H., Wang, Y., & Yang, Y. (2017) Adaptive finite-time control for high-order nonlinear systems with mismatched disturbances. *Int. J. Adapt. Control Signal Process.*, **31**, 1296–1307.

Yu, J. & Wu, Y. (2012) Global set-point tracking control for a class of non-linear systems and its application in continuously stirred tank reactor systems. *IET Control Theory A.*, **6**, 1965–1971.

Yu, J.-B., Zhao, Y., & Wu, Y.-Q. (2018) Global robust output tracking control for a class of uncertain cascaded nonlinear systems. *Automatica*, **93**, 274–281.

Zeitz, M. (1987) The extended luenberger observer for nonlinear systems. *Syst. Control Lett.*, **9**, 149–156.

Zhang, T., Xia, M., & Yi, Y. (2017) Adaptive neural dynamic surface control of strict-feedback nonlinear systems with full state constraints and unmodeled dynamics. *Automatica*, **81**, 232–239.

Zhao, D., Li, S., & Zhu, Q. (2013) Output feedback terminal sliding mode control for a class of second order nonlinear systems. *Asian J. Control*, **15**, 237–247.

Zhao, D., Spurgeon, S. K., & Yan, X. (2018) An adaptive finite time sliding mode observer. *New Perspectives and Applications of Modern Control Theory: In Honor of Alexander S. Poznyak* (Clempner, J. B. & Yu, W. eds). Springer International Publishing, Cham, pp. 523–538

Zhao, D., Spurgeon, S. K., & Yan, X.-G. (2016) Adaptive output feedback finite time control for a class of second order nonlinear systems. *2016 14th International Workshop on Variable Structure Systems (VSS)*, IEEE, pp. 53–58.

Zhu, Q. (2021) Complete model-free sliding mode control (CMFSMC). *Sci Rep*, **11**, 22565.

Zhu, Q., Mobayen, S., Nemati, H., Zhang, J., & Wei, W. (2022) A new configuration of composite nonlinear feedback control for nonlinear systems with input saturation. *J VIB CONTROL*.