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Conceptualising TPACK Within Mathematics Education: Teachers' Strategies for Capitalising on Transitions Within and Beyond Dynamic Geometry Software

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Abstract

This article investigates the knowledge arising in mathematics teachers' planning of how to manage transitions within and beyond dynamic geometry environments in the topic of circle theorems. The notion of situated abstraction is used to elaborate the central TPACK construct within mathematics education and address previous criticisms of the framework, specifically to clarify the distinction between the central construct and the dyadic constructs. Four case-study teachers each participated in a semi-structured interview based upon a pre-configured GeoGebra file. The teachers were asked to demonstrate how they would use the GeoGebra file to introduce students to the circle theorem that the angle at the centre of the circle, subtended by an arc, is double the angle at the circumference subtended by the same arc. The visual and audio aspects of the GeoGebra interviews were recorded and the TPACK framework used to analyse teachers' knowledge arising in the four interviews. The central TPACK construct is illustrated with examples of teachers' strategies for capitalising on transitions within and beyond dynamic geometry environments for the purposes of teaching circle theorems and contrasted with the dyadic construct of TCK. The utility of the theoretical elaboration of the TPACK construct within mathematics education is demonstrated and implications discussed.

Keywords Situated abstraction · TPACK · Dynamic geometry · Transparency

This article investigates the knowledge arising in mathematics teachers' planning of how to manage transitions within and beyond dynamic geometry environments for the purposes of teaching circle theorems. The notion of *situated abstraction* (Noss & Hoyles, 1996a) provides an overarching perspective for conceptualising mathematical knowledge. That is, 'abstract' mathematical knowledge can be thought of

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as generalisations arising from and yet embedded within specific mathematical contexts or 'situations'. The domain of geometry is a useful example for considering mathematical knowledge in terms of situated abstraction, since drawings or images provide specific contexts from which students may abstract the general properties of geometric figures, such as circle theorems. Nevertheless, the figural properties (Fischbein, 1993) of geometric objects entail that such abstractions remain situated to some extent in the visual images or drawings used to represent them.

Transitions are thus defined as events that provide potential opportunities for students to create situated abstractions by generalising or abstracting mathematical meaning from specific contexts. For example, transitions within dynamic geometry environments are exemplified by events, such as moving between specific configurations of the same mathematical figure through dragging, providing potential opportunities to abstract the geometrical properties of the figure through identifying variants and invariants. Transitions beyond dynamic geometry environments are exemplified by events where it is possible to compare (implicitly or explicitly) the affordances and constraints of representing a mathematical figure in a dynamic geometry environment with a static environment, such as 'paper-and-pencil' or even a dynamic geometry software reduced to a static environment to imitate a textbook-style diagram, in order to support generalisation across these specific technological contexts. In this sense, transitions beyond dynamic geometry environments are similar to synergy events (Mariotti & Montone, 2020), although they are not necessarily defined with respect to a specific 'duo of artefacts' (p. 112).

In this study, I assert that the knowledge revealed in teachers' articulation of how they plan to manage such transitions is captured by the central construct of the technological pedagogical and content knowledge (TPACK) framework (Mishra & Koehler, 2006). The aim of this study is to offer a theoretical elaboration of the central TPACK construct, clarifying the nature and exemplifying the content of this construct within mathematics education. TPACK is a popular and influential framework used extensively in designing teacher education (Voogt et al., 2013; Willermark, 2018), including mathematics teacher education (Bowers & Stephens, 2011; Niess et al., 2009), and in attempts to create measures of teacher knowledge to evaluate the effectiveness of such education (Baier & Kunter, 2020; Kaplon-Schilis & Lyublinskaya, 2020). However, theoretical difficulties continue to be highlighted in the conceptualisation of the TPACK framework, in terms of distinguishing the individual constructs (Ruthven, 2014), how they interact with each other (Graham, 2011) and the underdeveloped role of context in the framework (Rosenberg & Koehler, 2015), including subject-specific contexts (Willermark, 2018). These theoretical difficulties underline the need to clarify the nature of the central TPACK construct and how it interacts with other constructs within the context of mathematics education.

The use of dynamic geometry environments in teaching circle theorems has been relatively well-researched (Bozkurt & Ruthven, 2017; Komatsu & Jones, 2020; Mavani et al., 2018; Ruthven et al., 2008), showing that this is a potentially fertile context for investigating mathematics teachers' knowledge in planning how to manage transitions within and beyond dynamic geometry environments. In particular, research reveals how affordances and constraints of dynamic geometry environments can give rise to contingent moments (Kayali & Biza, 2021; Rowland et al., 2005) that may trigger teachers to plan how to manage transitions. For example, Ruthven et al. (2008) identify the appearance of rounding errors and issues with angle measurement as challenges in handling dynamic geometry environments which prompted teachers to either conceal or capitalise on such issues. Similarly, Komatsu and Jones (2020) identify ways in which teachers capitalise on affordances both of dynamic geometry environments and of physical tools to support students in proving circle theorems. These findings provide a basis for beginning to operationalise the constructs of the TPACK framework, in relation to teachers' knowledge of how to manage transitions within and beyond dynamic geometry environments for the purposes of teaching circle theorems.

The next section sets out the theoretical background for the study, elaborating the central TPACK construct within mathematics education. The methodology for the study is then presented, explaining how semi-structured interviews were conducted with four case-study teachers, based upon a pre-configured *GeoGebra* file on the topic of circle theorems to address the following question: What is the nature and content of the central TPACK construct in relation to teachers' knowledge of how to manage transitions within and beyond dynamic geometry environments for the purposes of teaching circle theorems? The subsequent analysis section identifies examples of TCK and TPACK revealed through teachers' planning of how to manage transitions within and beyond dynamic geometry environments for the purposes of teaching circle theorems, demonstrating the utility of the theoretical elaboration of the TPACK construct. The discussion and conclusion set out how the findings are useful in specifying the TPACK framework within mathematics education and implications are discussed.

Theoretical Background

In keeping with the overarching perspective for this article, mathematical knowledge for teaching is viewed as a *situated abstraction* (Noss & Hoyles, 1996a), that is, 'abstract' mathematical knowledge arising from and embedded within situations that occur in the context of teaching. A broad view is taken of teaching to include activities such as planning and professional conversations with colleagues, as well as classroom teaching. The notion of situated abstraction has been used to conceptualise the mathematics used in various professions such as nursing (Noss et al., 2002) and banking (Noss & Hoyles, 1996b). In particular, Bednarz and Proulx (2017) also use situated abstraction as part of their conceptualisation of teachers' *professional mathematics*, though not in the context of teaching mathematics with technology. This perspective is useful for the purposes of this study because it focuses on mathematical knowledge situated in the professional context of teaching. Moreover, situated abstraction originally arose as a means of expressing the mathematical knowledge students developed as they worked in computer-based environments (Noss & Hoyles, 1996a). Hence, the notion seems particularly well-suited to investigating mathematical knowledge for teaching using dynamic geometry environments.

Conceptualising the nature of the TPACK framework in terms of situated abstraction addresses the theoretical difficulties identified with the framework. Firstly, viewing teacher knowledge as situated in their professional contexts is commensurate with the TPACK framework (Mishra & Koehler, 2006). Secondly, by focusing on mathematical knowledge in teaching, situated abstraction provides a means of contextualising the TPACK framework within mathematics education, addressing the underdeveloped role of subject-specific contexts within the framework (Willermark, 2018). Third, and finally, understanding TPACK in terms of situated abstraction clarifies the distinction between the central construct and the dyadic constructs, such as TCK (technological content knowledge), within mathematics education.

One of the underlying theoretical issues is whether the central construct should be interpreted from an integrative perspective (Graham, 2011), where TPACK represents the use of the distinct domains of pedagogical, content and technological knowledge in combination, or a transformative perspective (Graham, 2011), where TPACK represents a new domain of synthesised knowledge. For example, taking an integrative perspective would suggest that the distinction between TCK and TPACK is in terms of pedagogic knowledge. Instead, viewing TPACK as a situated abstraction entails a transformative perspective where the central construct is distinguished from TCK by mathematical knowledge situated in the context of teaching with technology. Graham (2011) argues a transformative perspective is closer to Mishra and Koehler's (2006) original description of TPACK, building on Shulman's (1987) description of PCK as knowledge which enables teachers to transform content knowledge into forms that are pedagogically powerful. This article focuses on the dyadic construct of TCK as a means of investigating, by comparison, whether conceptualising the framework in terms of situated abstraction produces a meaningful way of distinguishing between TCK and the central TPACK construct, as opposed to an integrative perspective, for example.

In terms of situated abstraction, then, the TCK construct is defined as abstract mathematical knowledge arising from a transition event situated in a specific technological context. For the purposes of this study, the TCK construct is defined as abstract mathematical knowledge, relating to circle theorems, situated in a dynamic geometry environment. An example of TCK arising from a transition event *within a dynamic geometry environment* would be identifying (or abstracting) the doubling relationship between the angle at the circumference and the angle at the centre by observing co-variation in these angles, whilst moving between configurations of a geometric figure representing the 'angle at the centre' circle theorem through dragging. An example of TCK arising from a transition event *beyond a dynamic geometry environment* would be abstracting the mathematical constraints and affordances (Greeno, 1998) of the software, such as how angles are defined and measured, from moves between configurations of the 'angle at the centre' theorem that produce anomalies in the doubling relationship such as the appearance of rounding errors and other issues with angle measurement (Ruthven et al., 2008).

Constraints and affordances are governed by mathematical rules embedded in the design of the dynamic geometry environment, constraining the user to obey these rules e.g. by imposing an explicit order in constructing geometric figures (Jones, 2000). This mathematical *rigidity* may be fruitful in supporting the user to appreciate and explore the rules embedded. Of course, static environments, such as paper-and-pencil, also have constraints and affordances. For example, an affordance is that a static environment is relatively *flexible*, allowing the user to draw mathematical figures without the necessity of obeying a specific set of construction rules. Static environments are also relatively flexible in another sense: they allow the user to switch between imagining an ‘ideal’ mathematical world where measurement is exact and a real-world environment complete with rounding errors.

In dynamic environments, imagining an ideal mathematical world may be more difficult, since rounding errors are immediately present on dragging. Identifying constraints and affordances of a dynamic geometry environment is abstract mathematical knowledge, because it requires recognising the mathematical rules that underpin that software. At the same time, these rules may be specific to the software and so such knowledge is situated in the particular technological context. In addition to abstracting mathematical knowledge, these examples of TCK entail an implicit or explicit recognition of the transition events themselves through identifying what has changed and what has stayed the same in moving between configurations by dragging. Implicit or explicit recognition of a transition event also indicates the situatedness of TCK within the dynamic geometry environment, depending on the specific configurations that enabled the abstraction of mathematical knowledge and how these configurations arose through dragging.

The TPACK construct is defined as abstract mathematical knowledge, arising from transition events situated in the context of teaching using a specific technology. TPACK consists of strategies to capitalise on transition events with the purpose of enabling students to generalise mathematical knowledge from the specific technological contexts for themselves. Having such a strategy depends upon having TCK to begin with, that is, being able to identify the transition event implicitly or explicitly and the mathematical knowledge for students potentially arising from the event. In addition, having such a strategy entails making decisions about how to use transition events to enable students to generalise mathematical knowledge from the specific technological contexts for themselves.

Hence, for the purposes of this study, TPACK is understood to be a transformation of TCK by further abstracting mathematical knowledge situated in the context of teaching circle theorems using dynamic geometry environments. For example, observing that the doubling relationship holds between configurations of a geometric figure representing the circle theorem was noted earlier as TCK. Building upon this, an example of TPACK arising from transition events within a dynamic geometry environment would be articulating a teaching strategy capitalising upon dynamic imagery in order to highlight for students the specificity and the generality of the angle at the centre theorem.

In addition to observing that the doubling relationship holds, articulating a strategy to capitalise on dynamic imagery means making decisions about how to control mathematical variation, to highlight the specificity and the generality of the angle at the centre theorem to enable students to appreciate the doubling relationship themselves. Controlling mathematical variation depends on abstract mathematical knowledge of the connections between circle theorems or ‘special cases’ of the angle at the centre theorem. However, the purpose and means of controlling mathematical variation through identifying dragging sequences that enable students to appreciate the doubling relationship is particular to using dynamic geometry software for teaching circle theorems. Hence, TPACK seems at once abstract and situated within the context of teaching using the specific software.

Moreover, Ruthven et al. (2008) distinguished two distinct teaching strategies for managing rounding errors and angle measurement in dynamic geometry environments: either concealing or capitalising on such issues. Both strategies show TCK in terms of recognising the mathematical constraints and affordances of the technology. Concealing rounding errors and angle measurement is a way of avoiding further consideration of such issues from a teaching perspective. However, seeking to capitalise on rounding errors and angle measurement might mean making additional decisions about how to realise the potential of such transition events to support students' generalisation beyond the dynamic geometry environment, by comparing these constraints and affordances with those of static environments. In this case, TPACK would be distinguished from TCK by a display of additional mathematical knowledge that seems abstract, in the sense that it is general knowledge about mathematical constraints and affordances across dynamic and static environments. Yet, TPACK also appears situated within the context of teaching using the software, because such knowledge is triggered by the teachers' recognition of the mathematical rigidity of *GeoGebra* and how it might support their students' learning.

Defining TCK and TPACK in this way enables the identification of the knowledge teachers have and that underpins the strategies they plan to use. However, viewing knowledge as situated means identifying a lack of knowledge is not possible, only that such knowledge did not emerge in a given context. As a result, identifying TPACK may help explain why teachers plan to use certain strategies, but cannot explain why other strategies are not planned for. Instead, Adler's (1999) dilemma of transparency provides a means of explaining why teachers plan to conceal or capitalise on transitions within and beyond dynamic geometry environments that is commensurate with the perspective adopted in this study (see, for example, Hennessy et al., 2005; Noss & Hoyles, 1996b; Noss et al., 2007). In particular, Adler's dilemma of transparency helps to explain why the nature of TPACK as knowledge that is both abstract and situated is important for understanding why some teachers may be able to articulate strategies for capitalising on transition events where others were not.

When TPACK is seen as abstract mathematical knowledge, the dynamic geometry environment appears invisible, meaning that the teacher can 'see through' the software to the mathematical content and hence articulate a strategy for capitalising on transition events to teach circle theorems. When TPACK is seen as situated knowledge, the dynamic geometry environment has to be visible to the teacher for them to 'see' the constraints and affordances of the software and how they might be exploited for the purposes of teaching circle theorems. For example, 'seeing' how angles and geometric rules are defined in the dynamic geometry environment is necessary to use these constraints to reflect upon which angles and geometric conditions are specified in the circle theorem. Alternatively, the dynamic geometry environment may be too visible, so that the teacher cannot 'see past' the specific technological context to elicit ways of supporting students to identify the mathematical generalities of circle theorems. Finally, if the dynamic geometry environment is too invisible, then the teacher may assume the mathematical generality of the circle theorem is obvious to students without paying attention to the specificities of how the theorem is situated in the specific technological context.

Methodology

A qualitative approach consisting of semi-structured interviews based upon a pre-configured *GeoGebra* file on the topic of circle theorems was selected to provide rich data (Given, 2008), with which to investigate knowledge arising in mathematics teachers' planning of how to manage transitions within and beyond dynamic geometry environments. The interviews were conducted with four teachers selected as critical cases (Miles et al., 2020), in the sense that they represented technology enthusiasts to ensure TPACK would arise in the interview discussion. In addition, variation in case selection aimed to expose teachers' knowledge through comparing contrasting cases, serving to make more TPACK more 'visible', by highlighting the absences (things left unsaid), as well as the presences (things made explicit), (Hoyles et al., 1999; Venkatakrishnan, 2004).

For this reason, two dimensions of variation for case selection were chosen to select cases that would provide such purposive contrasts, based on factors known from prior research to be associated with teachers' digital technology use. Details of these dimensions of variation and how teachers were selected to provide critical and contrasting cases are explained in the following paragraph. A *GeoGebra* file was used as a stimulus in the interview because the software provides a free-to-use, open-source, dynamic mathematics environment that incorporates geometry (Hohenwarter & Preiner, 2007) and is well-known by many secondary mathematics teachers in England (Jones et al., 2009).

Four teachers were selected from participants in a survey of secondary school mathematics teachers' use of ICT (Information and Communication Technology, $n = 183$) and who agreed to be contacted as case-study teachers. A description of the full survey instrument, its development and the survey sample is presented in Bretscher (2014). Teachers were considered for selection as case studies if they self-reported as being confident with ICT, and were therefore likely to be technology enthusiasts, displaying TCK and TPACK. The four case study teachers—Robert, Michael, Edward and Anne—were chosen along two dimensions of variation likely to be associated with TPACK, based upon their responses to survey items. Firstly, the case-study teachers were chosen based on a measure of their orientation to mathematics pedagogy in general, not limited to ICT use. Teachers' orientation towards mathematics pedagogy, in terms of whether they see learning as a teacher- or student-directed process, has long been associated with digital technology use (e.g. Kaput, 1992; Thurm & Barzel, 2022).

Mathematics pedagogy was modelled as a continuum between a 'teacher-centred' versus a 'student-centred' orientation towards teaching mathematics, based on Pampaka et al. (2012), with lower scores indicating a more student-centred approach. The measure of mathematics pedagogy is fully reported in Bretscher (2021); summary statistics (mean = 0.17, standard deviation = 0.52) are provided here for brevity. As a result, two of the most student-centred survey respondents (Robert, -1.01; Anne, -0.50) were chosen and two of the most teacher-centred ones (Michael, 1.01; Edward, 0.74) in their orientation to mathematics pedagogy.

Secondly, the case-study teachers were chosen based on the level of support for ICT in their school. Stein et al. (2007) highlight school and department culture and support for teachers in terms of professional development as factors influencing technology use. Items underlying the level of school support are reported in Bretscher (2014). Scores on these items were aggregated to provide a mean school score for ICT support, with higher scores on these indicating a greater degree of support: summary statistics of the school scores (mean = 3.64, s.d. = 0.44) are provided here for brevity. The case study teachers were therefore chosen, so that one of the student-centred teachers came from a school more supportive of ICT use (Robert, 3.92) and one from a less supportive school (Anne, 3.66) and, similarly, for the teacher-centred teachers (Michael, 3.83; Edward, 3.04).

A further case-study context was provided by demographic data from the survey and background information collected during the interviews as follows: Robert had 4–6 years of teaching experience, was the most technologically proficient of the four case study teachers with a background in computer engineering and had used *GeoGebra* to design innovative activities for classroom use (see Bretscher, 2017). Anne had 10–15 years of teaching experience, but was the least technologically proficient and confident of the four case-study teachers. Edward had 2–3 years teaching experience, achieved a first-class grade in his undergraduate degree in mathematics (in this sense, he had the strongest mathematical background of the four teachers) and had some experience of using *GeoGebra* for classroom teaching. Michael had 2–3 years teaching experience and appeared to be the least confident of the teachers in relation to his own subject knowledge. Michael and Anne were both aware of *GeoGebra* and Michael noted he had previously used a resource similar to the *GeoGebra* file employed in the interview in his own teaching.

The interviews provided a common situation to compare teachers' knowledge and contained questions designed to cover each of the dyadic and triadic categories of the TPACK framework (see Supplementary Information). The case-study teachers were prompted to show and discuss how they would use Diagram 1 (D1) presented in the *GeoGebra* file (see Fig. 1), in order to demonstrate that the angle at the centre of the circle, subtended by an arc, is double the angle at the circumference subtended by the same arc.

D1 was designed to be similar to resources found on a Web search. Circle theorems were chosen since it is a topic which is commonly identified with the use of dynamic geometry environments (Ruthven et al., 2008). It was therefore reasonable to assume that the case-study teachers would be familiar with technological resources similar to D1 and might even have previously used such resources in their own teaching. Thus, they would be likely to show some mathematical knowledge for teaching circle theorems using the *GeoGebra* file, even if they were unfamiliar with the particular software. The second and third diagrams were inspired by Küchemann (2003) and designed to be unusual by comparison.

These diagrams could be manipulated to produce a soft construction (Laborde, 2005) of the angle at the centre theorem, but could also be disrupted to produce non-examples of the theorem, and were included to provide both a context that would challenge the subject knowledge of and some interest to a technologically proficient and mathematically confident secondary mathematics teacher. In addition, the topic

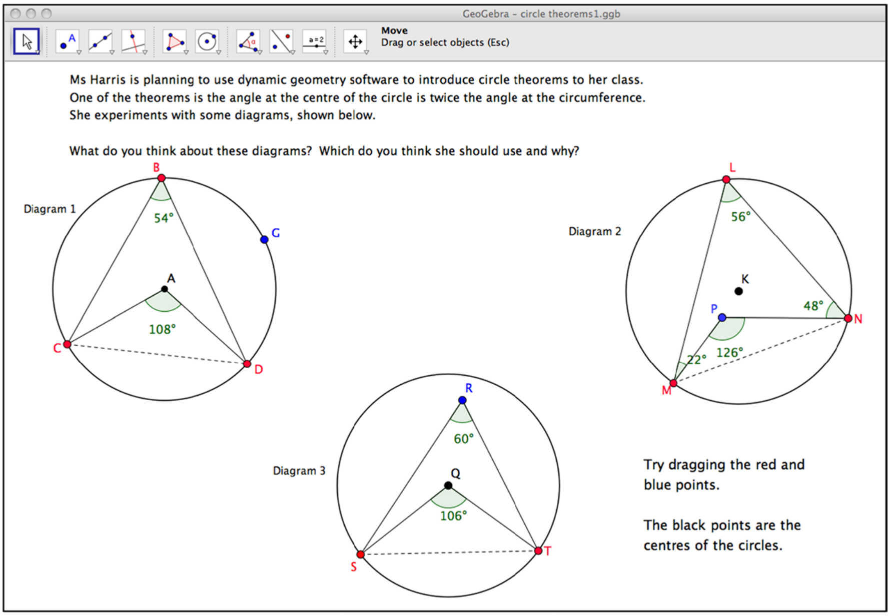


Fig. 1 The *GeoGebra* interview file on circle theorems

of circle theorems is at the apex of geometry in the compulsory English mathematics curriculum: hence, it provided a potentially challenging context even for experienced teachers. The semi-structuring of the interview allowed some flexibility to respond to events during the interview, whilst maintaining an overall structure that would allow for and facilitate comparison. Both the visual and audio aspects of the *GeoGebra* interviews were recorded and analysed.

The *GeoGebra* interviews were transcribed and then a narrative was written as a means of co-ordinating the visual data with the interview transcript. Writing the narrative meant viewing the video at different speeds by breaking it down into different grain-sizes of interval. For example, watching the discussion of the first *GeoGebra* diagram all the way through, without stopping, gave a sense of key moments and the general flow of the interview. It was then possible to zoom in, watching short sequences of the video, to write the narrative, paying closer attention to key moments and, at times, watching the video stop/start to co-ordinate better the case study teacher's manipulation of *GeoGebra* with the interview transcript. Zooming out again to watch longer intervals provided a means of checking whether the narrative gave a valid portrayal of the key moments and general flow of the interview.

Brief quotations from the interview transcript were included as a means of linking what the teachers did with what they said. For example, configurations of the angle in the centre theorem mentioned in the narrative were those that were both elicited through dragging and identified verbally by the case-study teacher. The narratives were coded individually using the constructs of the TPACK framework and then compared across codes and cases. Examples of TCK were identified where a teacher abstracted mathematical knowledge (either relating to a circle theorem or to the mathematical constraints and affordances of *GeoGebra*) from a transition event within or beyond the dynamic geometry environment and made an implicit or explicit reference to that transition event either through their dragging or verbal commentary. Examples of TPACK were identified where, in addition to showing TCK, a teacher articulated a strategy to capitalise on transition events for the purpose of supporting students to abstract mathematical knowledge in relation to circle theorems across contexts.

Analysis

The analysis section identifies examples of TCK and TPACK revealed through teachers' planning of how to manage transitions within and beyond dynamic geometry environments for the purposes of teaching circle theorems. Four examples of TCK are identified that arose during the interviews where teachers abstracted mathematical knowledge from a transition event and made implicit or explicit references to transitions within or beyond the dynamic geometry environment. Arising from transitions within the dynamic geometry environment, teachers showed TCK by identifying the 'angle at the centre' theorem held across different configurations of the circle theorem diagram that arise instantaneously through dragging. Examples of TCK arising from transitions beyond the dynamic geometry environment, include teachers knowing how angles are defined and measured in *GeoGebra*; knowing about rounding errors; knowing about issues of dependency.

These examples are intended as being indicative rather than an attempt to provide an exhaustive list of the types of TCK that arose during the case study teachers' interaction with the *GeoGebra* file in interview. In three of these TCK examples, at least one of the teachers also articulated a strategy for capitalising on these transitions, thereby providing an example of TPACK. However, none of the teachers articulated a strategy for capitalising on angle measurement in *GeoGebra*. Of course, this does not mean the teachers did not know a strategy for managing angle measurement, but just that such a strategy did not arise during the interviews. Instead, drawing on Ruthven et al. (2008), these authors imagine a strategy for capitalising on how angles are measured in *GeoGebra* for the purpose of teaching circle theorems to provide an example of TPACK. These examples are further analysed to explain why some teachers were able to articulate strategies for capitalising on transition events while others were not, using Adler's (1999) dilemma of transparency.

Transitions Within GeoGebra: Managing Dynamic Imagery

All four teachers demonstrated TCK by moving points B, C and D to different positions around the circumference and noting that the relationship between the angle at the centre and the angle at the circumference still held. More specifically, the teachers each made implicit or explicit references to transitions within the dynamic environment by distinguishing between different configurations of D1. For example, Michael began by briefly dragging B on the major arc CD, before continuing his exploration of D1 by dragging points C and D upwards, to produce the configuration shown in Fig. 2c. He distinguished between the configurations in Fig. 2a and c by referring to them as ‘arrowhead’ and ‘quadrilateral’ configurations respectively, remarking that the rule still applies, hesitating only to point out rounding errors:

> M: Yeah, I’d probably then eventually move [C and D] up like that to form like a quadrilateral, so that the rule still applies. Well, it was an arrowhead before, wasn’t it, but now it’s a reflex angle on the outside of a quadrilateral, but it still applies just. I guess that’s a rounding [error].

Instead, Edward, Robert and Anne instead began by dragging point B on the major arc CD, maintaining the arrowhead configuration, before dragging B onto the minor arc to produce the convex quadrilateral configuration shown in Fig. 2b. Initially, moving B onto the minor segment seemed to disrupt the teachers’ knowledge of the angle at the centre theorem because the ‘incorrect’ angle at the centre is shown.

For example, Robert said, ‘I can’t remember what happens if I bring it over here’, before dragging B onto the minor arc. He noted subsequently that previously he had prevented B from being dragged onto the minor arc, thus barring the convex quadrilateral configuration, ‘Because on diagrams I’ve had in the past I’ve forced it to just lie on the major arc’. Anne struggled with whether the angle at the centre theorem still held when B moves onto the minor segment, saying, ‘the rule has not fallen apart … it hasn’t fallen apart here, in that, yeah, the rule has fallen apart a bit, hasn’t it?’ Whilst Edward was convinced that the theorem still holds when B is positioned on the minor arc, he referred to this configuration as a ‘complication’. Despite some initial confusion, all three teachers resolved the circle theorem still held. Hence, each of the teachers made reference to transitions within the dynamic environment by distinguishing between the arrowhead and convex quadrilateral configurations whilst recognising that the circle theorem still held. In doing so, they showed TCK in terms of an understanding that the dynamic imagery of *GeoGebra* is a means of representing geometric relationships.

Beyond this, the teachers demonstrated TPACK by articulating strategies for capitalising on transitions between configurations to build mathematical meaning in three ways: they managed the dynamic imagery of *GeoGebra* to make connections

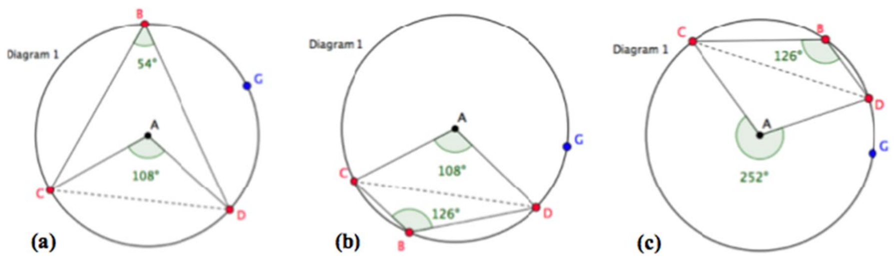


Fig. 2 **a** The ‘arrowhead’ configuration; **b** the ‘convex quadrilateral’ configuration produced by dragging B onto the minor arc; **c** the convex quadrilateral configuration produced by dragging points C and D upwards

between different circle theorems, to stimulate reasoning about particular instances of the theorem and to demonstrate the generality of the circle theorem. Each of these strategies requires knowing how to control mathematical variation for the purposes of teaching circle theorems. In addition to TCK, as in the example given in the theoretical background section, controlling mathematical variation appears to depend on abstract mathematical knowledge of the connections between circle theorems or 'special cases' of the angle at the centre theorem, beyond simply observing the doubling relationship. However, TPACK seems at once situated since the purpose and means of controlling mathematical variation through dragging sequences, potentially enabling students to appreciate the generality of the doubling relationship, is particular to the specific context of using *GeoGebra* for teaching circle theorems. The three strategies are now discussed in relation to the dragging sequence planned by Edward shown below in Fig. 3.

Robert and Edward both articulated strategies for capitalising on transitions to make connections between different circle theorems. Starting from an arrowhead configuration, Edward stated he would drag C and D until CD formed a diameter (see Fig. 3e) to show the theorem that the angle in a semi-circle is right is a consequence of the angle at the centre theorem:

> E: I'd do the thing 'Look what happens …', so it's all fine up till now, and then ... the diameter still works. […] So that's kind of nice cos you see immediately that the angle in the semicircle is right as a consequence of the angle at the centre is twice the angle at the circumference.

Similarly, Robert stated he would make this connection by dragging C and D:

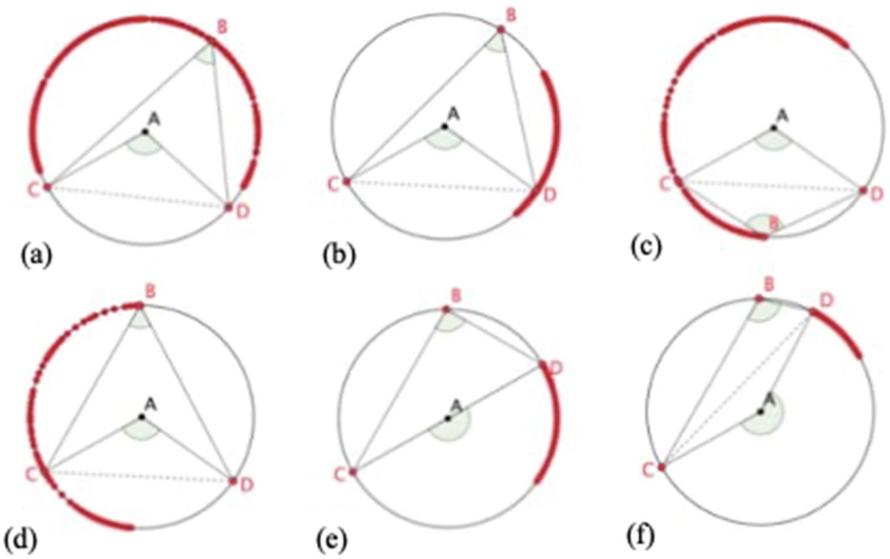


Fig. 3 An indication of Edward's sequence of configurations (the trace gives a sense of how he dragged points B, C and D)

R: I'd probably want to show you know linking into ... what's it called theorem of Thales? ... the angle in a semi-circle is just really a special case of the angle at the circumference being half the angle at the centre.

In addition, Robert said maintaining the arrowhead configuration by dragging point B (see Fig. 3a) made a useful connection with the theorem that angles in the same segment are equal.

Robert, Anne and Edward articulated strategies for capitalising on transitions to stimulate reasoning about particular instances of the theorem. Transitioning from Fig. 3b–c, Robert and Anne both stated they would introduce the convex quadrilateral configuration with the incorrect angle at the centre (Fig. 3c) to stimulate their students' reasoning about whether the angle at the centre theorem still held. Recalling that this configuration led Anne to question whether the circle theorem had broken down, the software initially appeared too visible for Anne, preventing her from generating a strategy to capitalise on the transition event. Nevertheless, after a brief struggle, she was able to 'see through' the apparently anomalous case of the convex quadrilateral configuration showing the incorrect angle at the centre to identify an opportunity to stimulate her students to reason mathematically. Edward also said he would introduce this configuration, but would demonstrate the reasoning himself and only once he was convinced students had grasped the theorem in relation to the arrowhead configuration.

By contrast, Michael did not distinguish between the convex quadrilateral configurations showing the correct (Fig. 3c) and incorrect (Fig. 3f) angles at the centre. When the configuration in Fig. 3c did arise later on in his dragging, he simply stated, 'Well, you know I mentioned that one earlier didn't I?', referring back to the convex quadrilateral configuration showing the correct angle at the centre which had arisen first through his dragging of C and D. Here, the software seemed invisible, because Michael could 'see through' the configuration with the incorrect angle at the centre, recognising it as similar to a configuration he generated previously through dragging. However, the software seems too invisible to Michael: he accepts uncritically the configuration with the incorrect angle at the centre, without provoking consideration of why the incorrect angle is showing and whether such a configuration might cause confusion for students.

Robert and Michael articulated a strategy for capitalising on transitions to demonstrate the generality of the circle theorem. As noted earlier, Michael stated he would first drag C and D upwards in a smooth transition from Fig. 3d–f to demonstrate that the angle at the centre theorem holds for both the arrowhead and convex quadrilateral configurations. Similarly, transitioning between Fig. 3d–f, Robert stated he would demonstrate the theorem holds for all angles between 0° and 360° by showing 'a few generic examples of this angle [at the centre] being less than 180, this angle being more than 180'. By contrast, Edward found the configuration shown in Fig. 3f unexpected and troubling, disrupting his understanding of how the angles in the circle theorem were defined. In his initial discussion of D1, Edward had assumed the angles are defined as being subtended by the chord CD:

E: The chord C and D, joining C and D […] subtends an angle of 108 at the centre and 54 at the circumference, […] so what it shows is the angle subtended at the circumference by chord CD is always twice the angle at the centre, irrespective of where B is.

The configuration in Fig. 3f led Edward to question his previous definition of the central angle as being subtended by the chord CD:

> E: Um ... so ... let's take an example ... so ninety-four doubled is one hundred and eighty-eight, so it's still true that ... so that angle is twice that angle. But uh ... how do you know it was that angle ... so the computer is kind of showing you the right angle for what it's working for isn't it? But in words, how do you explain what that angle is, it's not really the angle that chord CD is subtending at the centre is it? Because it's that ... chord CD is subtending that angle at the centre, so suddenly you have to say it's the other angle, the reflex angle at the centre that's subtending.

Instead of capitalising on this transition, Edward opted for concealment, stating he would not let complications of this nature occur in his initial demonstration of the angle at the centre theorem. In this case, *GeoGebra* was too visible for Edward. The presence of the chord appeared to prevent Edward from recognising the correct definition of the angles in the theorem as being subtended from the same arc, leading him to conceal rather than capitalise on the transition event.

Transitions Beyond the Software: Exploiting Angle Definition

Two of the teachers, Michael and Edward, questioned how angles are defined and measured in *GeoGebra* prompted by unexpected configurations of D1 appearing during dragging, displaying the 'incorrect' angle at the centre. In *GeoGebra*, the angle measured at the centre in D1 is defined by specifying the ordered triad of points CAD and measured anticlockwise from the line segment AC to the line segment AD. D1 had been designed so that, whilst the angle at the centre could become reflex, the angle measured at the circumference was constrained to be less than 180° whatever the relative position of points C and D. Hence, the 'correct' angle at the circumference in relation to the circle theorem was always displayed, however some configurations of D1 displayed the 'incorrect' angle at the centre, as discussed in the previous section. After experimenting by dragging points C and D and, some prompting by the interviewer, each teacher concluded the angle measured at the centre was dependent on the relative position of points C and D. For Edward, the software's definition and measurement of angles was a source of frustration, appearing idiosyncratic in the way D1 'flipped' between displaying the correct and the incorrect angle at the centre. He argued:

> E: this is sort of a function of how the software works isn't it, rather than a ... I that bringing out anything useful mathematically that ... that's just a bit annoying the way it does that, isn't it?

His frustration with angle definition in the software led him to suggest that, for proof, he would prefer a static environment: 'I'd project this on the whiteboard […] and then *mark on the angles that I want* [emphasis added]'.

The ‘flipping’ of D1 between configurations showing the correct and the incorrect angle at the centre is a transition beyond the dynamic geometry environment, because it is an event with potential to compare the affordances and constraints of the software with static environments to build mathematical meaning. Both Michael and Edward show TCK by demonstrating an understanding of angle measurement in *GeoGebra*, thus showing an appreciation of the mathematical constraints of the dynamic geometry environment and indirectly recognising the transition event.

Furthermore, Edward reflected explicitly on whether there is potential to build mathematical meaning from the way angles are defined in *GeoGebra*, thereby making a direct reference to the transition event. Edward went on to compare implicitly the mathematical constraints and affordances of the dynamic geometry environment with a static projection on a whiteboard. However, his frustration with the software led him to conclude that there was ultimately nothing mathematically useful in the way *GeoGebra* measures angles, dismissing the transition event as ‘just a bit annoying’. Again, for Edward, the software appears to be too visible, his knowledge too situated in the specific software context, since his frustration prevents him from seeing past the constraints of *GeoGebra*, with regard to angle definition to abstract how they could be exploited for teaching circle theorems across contexts.

Neither teacher showed TPACK by articulating a strategy for capitalising on their insight into how *GeoGebra* measures angles. However, drawing on Ruthven et al. (2008), it is possible to imagine a strategy for using the way *GeoGebra* defines and measures angles to build understanding of the circle theorem. The variation in whether the ‘correct’ or ‘incorrect’ angle displayed in D1 provides a potential stimulus for discussing how the angles in the circle theorem are defined precisely in a full statement of the theorem. Clearly such a definition is far from obvious given Edward’s confusion, described in previous sub-section, about whether the angles are subtended from the chord. Instead, the angle at the circumference and centre are correctly defined as being subtended from the same arc. Arriving at a precise definition of the angles in the circle theorem through a discussion of how angles are defined in D1 might support students to identify the angles involved in the circle theorem in other contexts. Such a discussion might be especially useful in supporting students apply circle theorems in static environments where the precise definition of the angles may be overlooked. As Edward indicates, one of the affordances of static environments is that the relevant angles of the circle theorem may simply be marked on a diagram with a brief stroke of a pen, without needing to consider explicitly how they are defined. Using the way *GeoGebra* defines and measures angles to stimulate a discussion of how the angles in the circle theorem are defined precisely is a strategy that capitalises on a transition beyond the dynamic geometry environment.

Having such a strategy relies on knowing how to define the specific angles required in the circle theorem, in addition to knowing how angles are defined in the software. Knowing how to define the specific angles required in the circle theorem is mathematical and abstract in that it holds across contexts. Yet, identifying a strategy to exploit issues of angle definition for the purposes of teaching circle theorems also seems to arise from, and hence be situated in, the particular ways in which angles are defined in *GeoGebra*. For example, the ‘flipping’ of D1 between configurations showing the correct and the incorrect angle at the centre makes issues of angle definition salient, creating an urgency to explain how angles are defined in circle theorem. In this case, the situatedness of TPACK is clearly interrelated with the notion of transparency. The software needs to be visible for the teacher to 'see' the specific constraints that underpin angle definition in *GeoGebra* and so recognise an opportunity to create a need to reflect on how angles are defined in the circle theorem. At the same, time the software needs to be invisible so that the teacher can 'see through' the specific constraints of the software, to identify similarities with the way angles are defined in a static environment, and so support students to find a definition for the angles in the circle theorem that holds across contexts.

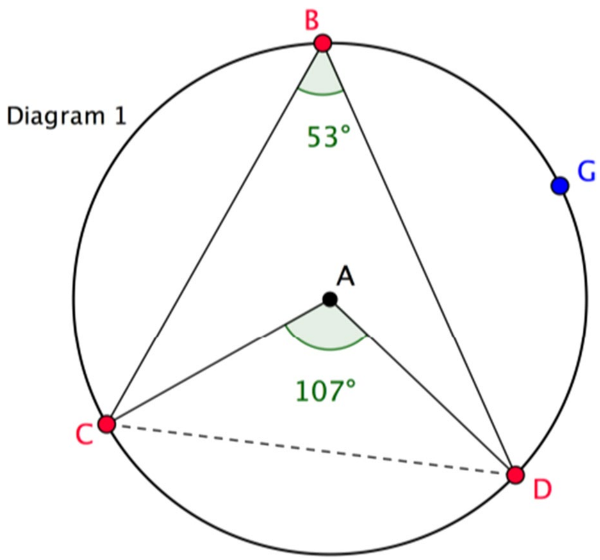
Transitions Beyond the Software: Managing Rounding Errors

At some point during the discussion of the three diagrams, each of the four teachers appreciated that *GeoGebra* displayed rounding errors in measuring the angles at the centre and circumference of the circle (see Fig. 4).

Each of the teachers noted the apparent breakdown in the doubling relationship between the angle at the circumference and the angle at the centre, but appreciated this was a result of rounding errors in the software's measurement, rather than a counter-example that might lead them to reject the theorem. In particular, Edward expressed deep frustration with the presence of rounding errors:

> E: Cos it's built for a purpose this [software]. The rounding really gets in the way of what you're trying to show.
> E: [later in the interview] I'd definitely mention it because sometimes it doesn't seem to work does it? ... it was one degree out ... but I really see it as a hindrance to learning what's going on. I'd just, I'd have to keep saying 'Look, within rounding error this result is ...', sort of, it's much less convincing ...

Fig. 4 Rounding errors in angle measurement



Each of the teachers show TCK by recognising that rounding errors do not provide a counter-example to the theorem, thus showing an appreciation of the mathematical constraints of the dynamic geometry environment and indirectly recognising the transition event to a static environment e.g. paper-and-pencil. Again, Edward made a direct reference to the transition event by reflecting explicitly on the potential to build mathematical meaning from rounding errors. However, although he stated he would draw his students' attention to the existence of rounding errors, Edward's frustration with the software led him to conclude that rounding errors that there was nothing to be gained from doing so. Edward's frustration again shows the software is too visible for him, being unable in the interview to see past the constraints of *GeoGebra* with regard to rounding errors to appreciate how they could be exploited for teaching circle theorems.

The other three teachers articulated two distinct strategies for capitalising on rounding errors to build mathematical meaning. Firstly, Robert explained that he would deliberately introduce rounding errors as a possible instance of the conjectured relationship breaking down, thereby indicating the 'limitations of the accuracy of computers and calculators and why we do need to prove things'. In particular, he stated recognising the limitations of the computer provides a stimulus and support for transitioning to a static environment to address proof: 'if I was actually going into formal proof, I would probably be, I might have this diagram, but I think it would probably be static from that point forward'. Here, Robert shows (tacit) recognition that a static environment provides the flexibility to imagine an ideal mathematical world, where proof exists, compared to the relative rigidity of a dynamic environment, where rounding errors are inescapable. He appears to 'see through' the issue of rounding errors in the software to argue that empirical testing is always limited, hence there is a need for proof, and that transitioning to a static environment enables proof because an ideal mathematical world can more easily be imagined.

Secondly, both Michael and Anne suggested that rounding errors would provide an opportunity to link circle theorems with topics on the accuracy of measurement in other contexts, beyond the software. Michael suggested:

> M: It brings up a wider point of accuracy I guess and how everything is measured to varying degrees of accuracy and the importance of accuracy, because if you're not accurate then the theorems won't work. I guess if you're kind of sloppy in your angle measuring then you won't be able to prove anything.

Similarly, Anne suggested linking the issue of rounding errors to the topic of upper and lower bounds of measurement, asking pupils within what bounds the angle could have been, given that it had been rounded to a certain degree:

> A: Yeah, I would discuss it with pupils you know, the numbers were rounded. Yeah. I would discuss it with them, and depending on if we've done ... what is it called ... depending on if we've done bounds of measure, I could just bring it up, what could it [the angle] have been.

In contrast with Robert's strategy, Michael and Anne 'see through' the rounding errors in the software to identify similar limitations in measurement across dynamic and static contexts, enabling a connection to be made with topics relating to accuracy of measurement.

Robert, Michael and Anne demonstrated TPACK by recognising rounding errors as a transition event and articulating a strategy to capitalise on the potential of this transition to build mathematical meaning beyond the dynamic geometry environment. Robert's strategy of using rounding errors to stimulate proof relies on knowing real-world limitations and (at least implicitly) distinguishing circle theorems and proof as belonging to another ideal world of mathematics. Michael and Anne's strategy of linking circle theorems with topics on the accuracy of measurement relies on making a mathematical connection across topics and knowing that rounding errors occur in any real-world environment. In both cases, such knowledge of mathematical affordances and constraints seems generalised across dynamic and static environments. However, rounding errors appear more salient when using *GeoGebra* for teaching circle theorems—surprising even, given Edward's expression of frustration—since the mathematical 'neatness' of figures in *GeoGebra*, that obey geometric rules when dragged, invites the user to imagine ideal mathematical objects and yet rounding errors are inescapable. Whereas rounding errors were easily dismissed by the teachers in relation to their own appreciation of the circle theorem, they each recognised a need to manage the appearance of rounding errors when using the software with students. Hence, the knowledge that highlighting rounding errors in the dynamic environment can stimulate proof or provide a link to topics on accuracy of measurement appears to be situated in the context of using the software for the purposes of teaching circle theorems.

Transitions Beyond the Software: Highlighting Issues of Dependency

Each of the case-study teachers demonstrated some understanding of the issue of dependency, when asked directly about the difference between the red and blue points in relation to all three diagrams. For example, they all noted that the red points are restricted to move on the circumference, whereas the blue points P and R, in D2 and D3, respectively, may be dragged freely. Robert and Edward drew attention towards issues of dependency in D1 without prompting. Initially, describing D1, Edward implicitly referred to issues of dependency, noting, 'So, in this diagram, you've got three red points on the circumference'. With D2, he was more explicit, noting that 'L's confined to the circumference', then after further dragging, he generalised that all red points in the *GeoGebra* file are confined to move on the circumference but that point P is 'pretty free form'.

Robert's initial unprompted comments about D1 indicated that he was aware of issues of dependency or the 'rules of construction' in GeoGebra. He noted that the points B, C and D 'are presumably all fixed to the circumference of the circle'. Similarly, with D2, before dragging the diagram, he immediately said, 'I guess the big difference is P is no longer fixed'. Hence, Robert and Edward show TCK by recognising that how dependencies underpin the construction of the circle theorem diagrams. Specifically, they recognise the transition from a fixed point at A in D1 to a free form point at P in D2, thus showing an appreciation of the mathematical affordances of the dynamic geometry environment.

Robert and Edward's opening comments regarding D1, describing the construction of the diagram, draw attention to the positioning of the points B, C and D. Their description implied that the positioning of the B, C and D on the circumference is a critical feature of this diagram. Thus, implicitly, they suggested any relationship that appears to hold between the angle at the centre and the angle at the circumference is conditional on the positioning of B, C and D on the circumference. In relation to D2, Robert went further to suggest exploring dependencies in GeoGebra could help expose geometric relationships implied (or not) in paper-and-pencil examination questions:

> R: Coming back to this idea of things looking the way they do and what they actually are could be a useful discussion and you know if nothing else just to reinforce … the kind of 'diagram is not drawn accurately'-type exam technique.

Here, Robert demonstrates TPACK by articulating a strategy for capitalising on issues of dependency to expose the rules that underpin geometric figures, and thereby build mathematical meaning beyond the dynamic geometry environment. Having strategies to exploit issues of dependency in *GeoGebra* relies on knowing the necessary and sufficient geometric conditions that underpin the circle theorem, beyond recognising how dependencies underpin the construction of the circle theorem diagrams. Again, such knowledge is mathematical and abstract in that it holds across contexts. Yet knowing how to exploit issues of dependency for teaching circle theorems also seems to arise from, and hence be situated in, the particular ways in which geometric rules are defined in *GeoGebra*. As with issues of angle definition, the software needs to be visible for the teacher to 'see' issues of dependency in *GeoGebra* and so recognise an opportunity to reflect on the necessary and sufficient geometric conditions of circle theorems. At the same time, the software needs to be invisible, so that the teacher can 'see through' the specific constraints of the software, in order to make comparisons with constraints and affordances in other environments, and so support students to identify conditions for the circle theorem that hold across contexts.

Discussion

This article makes a theoretical contribution by identifying Noss and Hoyles' (1996a) notion of situated abstraction as a productive perspective within which to view the TPACK framework. Situated abstraction provides a means of contextualising the TPACK framework within mathematics education, addressing the underdeveloped role of subject-specific contexts within the framework (Willermark, 2018) by focusing on mathematical knowledge in teaching. Moreover, understanding the TPACK framework in terms of situated abstraction clarifies the distinction between the central construct and the dyadic constructs, such as TCK, within mathematics education.

In this study, TCK was exemplified by abstracting the doubling relationship in the angle of the centre theorem or the mathematical constraints of the software from transitions between configurations of a circle theorem elicited by dragging. TPACK consists of strategies to capitalise on transition events with the purpose of enabling students to generalise mathematical knowledge from the specific technological contexts for themselves. Such strategies require not only recognising the mathematical knowledge arising from a transition event, but, in addition, knowing how to use the software to support students' generalisation. TPACK was exemplified by knowing how to control mathematical variation by dragging to enable students to abstract the doubling relationship in the angle of the centre theorem within GeoGebra.

In this case, the teacher needed to know how to control mathematical variation by dragging in addition to recognising the doubling relationship themselves. Knowing how to control mathematical variation is abstract mathematical knowledge, but, at the same time, situated in the context of teaching using *GeoGebra* by the purpose and means of controlling variation through dragging that enables students to appreciate the doubling relationship. TPACK was also exemplified by knowing how to use mathematical constraints to enable students to abstract the properties of a circle theorem beyond *GeoGebra*. In this second case, the teacher not only needed to know the mathematical constraints of *GeoGebra* but how they compare with the constraints and affordances of other, static environments.

Again, generalising mathematical constraints across technological contexts seems to be abstract mathematical knowledge, yet it appears simultaneously situated, since issues such as rounding errors and angle definition seem more salient in the context of teaching using *GeoGebra*. In each case, that is, for strategies capitalising on transition events both within and beyond the dynamic geometry environment, the TPACK construct was shown to be distinguished from TCK by additional (abstract) mathematical knowledge which is simultaneously situated in the context of teaching using technology.

This article makes a further theoretical contribution by using Adler's (1999) dilemma of transparency to explain why teachers' knowledge might enable them or not to capitalise on transitions within and beyond dynamic geometry environments. When TPACK was seen as abstract mathematical knowledge, the dynamic geometry environment appeared invisible, meaning that the teachers could 'see through' anomalous configurations, rounding errors or issues of angle measurement and hence articulate a strategy for capitalising on transition events to teach circle theorems. When TPACK was seen as situated knowledge, the dynamic geometry environment had to be visible to the teacher for them to 'see' the constraints and affordances of the software and how they might be exploited for the purposes of teaching circle theorems.

An implication for teacher education is that teachers need to be supported to manage the dilemma of transparency. That is, teacher educators need to make affordances and constraints of technologies explicit or 'visible' to teachers. In addition, teacher educators need to support teachers to 'see through' the affordances and constraints of dynamic geometry environments to identify strategies that capitalise on transitions within and beyond dynamic geometry environments, so that the technology becomes 'invisible' and just another resource for teaching mathematics.

This article makes an empirical contribution by specifying the central TPACK construct within mathematics education with examples of strategies that teachers draw upon in planning to capitalise on transitions within and beyond dynamic geometry environments in the topic of circle theorems. Bowers and Stephens (2011, p. 290) suggest that the central TPACK construct may be an ‘empty set’, in terms of teachable knowledge or skills for teaching mathematics using technology. More positively, the analysis presented in this article identified several strategies articulated by teachers for capitalising on transitions within and beyond dynamic geometry environments, which exemplify the central TPACK construct for the topic of circle theorems. Similar strategies for capitalising on anomalies caused by rounding errors (Bozkurt & Ruthven, 2017; Ruthven et al., 2008) and angle measurement (Mavani et al., 2018; Ruthven et al., 2008) to build mathematical meaning in the topic of circle theorems have been identified in previous research combining classroom observation with post-lesson interviews.

The present study suggests that resource-based interviews may be sufficient to elicit teachers planned strategies for capitalising on transitions within and beyond dynamic geometry environments to specify the TPACK framework further. However, understanding knowledge as situated entails recognising the limitations of this study in terms of the specific context in which it was carried out. Thus, the findings may be contingent on the knowledge and experiences of teaching mathematics within the English secondary school system of the researcher and case-study teachers, as well as the design of the *GeoGebra* file upon which the interview was based. Again, these limitations point to a need for similar qualitative research to validate the findings in other contexts, including classroom contexts, and quantitative research to test these findings at scale. Finally, the analysis was restricted to a comparison of the TCK and TPACK constructs, so it remains to be seen whether this conceptualisation is productive in distinguishing the other framework constructs.

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Data Availability The datasets generated and analysed during the study are not publicly available since they consist of qualitative data that could identify research participants.

Declarations

Conflict of Interest The author declares no competing interests.

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