Dynamics of an ice particle submerged in water

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Modelling aircraft icing
Ice particle in water

Ice particle in a water layer

Melting particle coated in a layer of water
Governing equations

underbody curve

\[ F = F_u(X) + h(T) + (X - \beta)\theta(T), \]

conservation of mass and momentum

\[ F_T + (uF)_X = 0, \]
\[ u_T + uu_X = -p_X, \quad 0 = -p_Y, \]

body motion equations

\[ M h_{TT} = \int_0^1 p \, dx, \quad I \theta_{TT} = \int_0^1 (x - \beta)p \, dx. \]

\[ \frac{\rho_B}{\rho_F} = O \left( \frac{L}{\Delta} \right)^2 \]

\[ M, I \ll 1 \]
Boundary conditions

\[ \frac{1}{2}u^2 + p = \frac{1}{2} \text{ at } X = 0 \]

\[ p = 0 \text{ at } X = 1 \]
Governing equations

To approach an $O(1)$ density ratio, assume:

$$M, I \ll 1$$

New governing equations:

$$F = F_u(X) + h(T) + (X - \beta)\theta(T),$$

$$F_T + (uF)_X = 0,$$

$$u_T + uu_X = -p_X, 0 = -p_Y$$

$$\int_0^1 p \, dx = \int_0^1 (x - \beta)p \, dx = 0.$$
Linear analysis for flat plate

\[ (F, h_C, \theta, u, p) = (1 + \delta F_1, 1 + \delta h_1, \delta \theta_1, 1 + \delta u_1, \delta p_1) + \ldots, \]

\[ (h_c = h - \beta \theta) \]

Linearised equations:

\[ F_1 = h_1(T) + X \theta_1(T), \]
\[ F_{1T} + F_{1X} + u_{1X} = 0, \]
\[ u_{1T} + u_{1X} = -p_{1X}, \]
\[ \int_0^1 p_1 \, dx = \int_0^1 x p_1 \, dx = 0, \]
\[ u_1 = -p_1 \text{ at } X = 0, \]
\[ p_1 = 0 \text{ at } X = 1. \]
Linear analysis for a flat plate

Linearised equations:

\[ (F, h_C, \theta, u, p) = (1 + \delta F_1, 1 + \delta h_1, \delta \theta_1, 1 + \delta u_1, \delta p_1) + \ldots, \]

\[ F_1 = h_1(T) + X \theta_1(T), \]
\[ F_1T + F_1X + u_1X = 0, \]
\[ u_1T + u_1X = -p_1X, \]
\[ \int_0^1 p_1 \, dx = \int_0^1 xp_1 \, dx = 0, \]
\[ u_1 = -p_1 \text{ at } X = 0, \]
\[ p_1 = 0 \text{ at } X = 1. \]

\[ h_1''/2 + \theta_1''/6 + A'_1 = \mathcal{R}, \]
\[ h_1''/6 + \theta_1''/24 + A'_1/2 = \mathcal{S}, \]
\[ h_1''/8 + \theta_1''/30 + A'_1/3 = \mathcal{T}. \]

\[ \mathcal{R} = -\theta'_1 - h'_1 - \theta_1 - A_1, \]
\[ \mathcal{S} = -\theta'_1/3 - h'_1/2 - \theta_1/2 - A_1, \]
\[ \mathcal{T} = -\theta'_1/4 - h'_1/3 - \theta_1/3 - A_1/2. \]
Figure 1: Results from solving the linearized system, showing convergence to two different steady states for the two different initial conditions, each with \( h_1' = -\theta_1 = \text{const} \) and \( A_1 = 0 \). The body may have either a negative velocity (left) or a positive velocity (right).
Full nonlinear system

\[ F_T + (uF)_X = 0, \]
\[ u_T + uu_X = -p_X, 0 = -p_Y \]
\[ \int_0^1 p \, dx = \int_0^1 (x - \beta)p \, dx = 0. \]
\[ \frac{1}{2}u^2 + p = \frac{1}{2} \text{ at } X = 0. \]
\[ p = 0 \text{ at } X = 1. \]

\[ u = -\frac{[h_C X + \frac{1}{2} \theta' X^2 + A(T)]}{D}, \quad \text{with } D = h_C + X\theta. \]

\[ p = -\int_0^X u_T \, dx - \frac{1}{2}u^2 + \frac{1}{2}. \]
Full nonlinear system

\[
\alpha_0 A' + \alpha_1 h''_C + \frac{1}{2} \alpha_2 \theta'' = e_4 - \frac{1}{2} + \beta_0, \\
\alpha_1 A' + \alpha_2 h''_C + \frac{1}{2} \alpha_3 \theta'' = e_4 - \frac{1}{2} b_4 + \beta_1, \\
\alpha_2 A' + \alpha_3 h''_C + \frac{1}{2} \alpha_4 \theta'' = e_4 - d_4 + \beta_2, \\
\alpha_i = \frac{1}{\theta} \log \left(1 + \frac{\theta}{h_C}\right),
\]

\[
\beta_i = \int_0^1 \frac{x^i}{D} \frac{1}{D} dx, \quad i = 0, 1, 2, 3, \ldots
\]

\[
b_4 = \int_0^1 \frac{(h'_C x + \theta' x^2/2 + A)^2}{D^2} dx,
\]

\[
d_4 = \int_0^1 \frac{x (h'_C x + \theta' x^2/2 + A)^2}{D^2} dx,
\]

\[
e_4 = \frac{1}{2} \frac{(h'_C + \theta'/2 + A)^2}{D^2},
\]

\[
D = h_C + X\theta \\
D_1 = h_C + \theta
\]
Leading edge collisions

In the limit of collision at the leading edge, i.e. \( h_C \to 0 \), the following asymptotic expansions apply:

\[
\begin{align*}
\alpha_0 A' + \alpha_1 h''_C + \frac{1}{2} \alpha_2 \theta'' &= e_4 - \frac{1}{2} + \beta_0, \\
\alpha_1 A' + \alpha_2 h''_C + \frac{1}{2} \alpha_3 \theta'' &= e_4 - \frac{1}{2} b_4 + \beta_1, \\
\alpha_2 A' + \alpha_3 h''_C + \frac{1}{2} \alpha_4 \theta'' &= e_4 - d_4 + \beta_2,
\end{align*}
\]

\[
\begin{align*}
\alpha_0 &= \sum_{j=0}^{i-1} \frac{h_C^j (-1)^j}{(i-j)! \theta^{i+1}} + \frac{h_C^j (-1)^j}{\theta^{i}} \alpha_0, \quad i \geq 1, \\
\alpha_0 &= \frac{1}{\theta} \log \left( 1 + \frac{\theta}{h_C} \right).
\end{align*}
\]

\[
\begin{align*}
A_\lambda \left[ A_\lambda + \theta_0 h_1 \left( 1 + \frac{1}{6} \log \left( \frac{\theta_0}{h_1} \right) \right) \right] &= 0, \\
A_1 &= -\frac{h_1^2}{\theta_0}.
\end{align*}
\]
$A_\lambda = 0$
$A_\lambda \neq 0$
Summary

• Adapted model of ice particles in air to address ice particles in water
• Found solutions to linearized problem for flat plate
• Found solutions to full non-linear problem for flat plate
• Asymptotically described leading edge collisions

Thank you!