An Inquiry into the TUI Design Space for Parent-Child Math Engagement at Home

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ABSTRACT

Preschoolers' early-math development is vital for their later math and academic achievement. Tangible user interfaces (TUI) may support early math as they feature physical objects imperative to math development and multimedia to support engagement. As a potentially meaningful context for TUIs, developmental studies highlight the need to support the home math environment (HME) that covers math-related interactions among parents and children. Therefore, we focus on HME as a design space that has not been investigated in TUI literature. We conducted an observational study involving physical-object based math activities and semi-structured interviews with 13 parent-child dyads. Our findings revealed the multifaceted nature of the HME, where children's agency is valued and providing lasting materials is challenging. Also, we realized that parents juggled their child's demands and the object-based physical activity at once. By reflecting on these findings, we propose design directions for supporting the home-math environment with TUIs.

CCS CONCEPTS

• **Human-centered computing** → Interaction design; Interaction design process and methods; User centered design.

KEYWORDS

Young children, Math, Tangible user interfaces, Physical objects, Child-tangible interaction, Educational technologies

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1 INTRODUCTION

Preschool-aged children's (ages 3-5) early math development is vital for their later academic skills [38]. To support their development, using physical manipulatives is a ubiquitous and beneficial practice [24, 40]. Tangible user interfaces (TUI) embed computation into physical objects and surroundings, offering customized cues, self-directed discovery, and playful learning [26, 45, 56]. Regardless, TUIs for early math development are scarce [31, 57].

Designing TUIs for early childhood demand careful consideration of the age group's developmental needs and the educational concept at hand [5]. The child-tangible interaction framework [3] and developmentally situated design cards [10] offer guidance to the designers in such efforts. On the other hand, deciding on the context of where and with whom the interfaces will be used early on is vital in shaping TUIs. Most TUIs for young children are situated in formal environments like schools and kindergartens [57]. Recent developmental studies highlight the home math environment (HME) as a significant factor in young children's math development [23]. This environment covers interactions between parents and children around math (e.g., math games and utterances.) Previous studies have not explored HME as a design space to reveal the constraints and the opportunities for TUIs [13].

In this paper, we share our research that opens up the math-TUI design space in the home environment. Thirteen parent-child dyads (Mage of children= 4.2 years) participated in our research. In the first part, the observational study, the dyads followed an activity on magnitude understanding, an essential base for math development rarely targeted by parents in HME [20, 51]. The activity featured physical objects and a booklet to see how dyads naturally engage with math activities without the findings being affected by a technology choice (i.e., implications of I/O couplings, digital feedback choices). Similar approaches are being pursued for young children's spatial understanding and joint book reading practices to inform future interface designs [9, 67]. In this sense, our observations helped us elicit dyads' points of struggle with object-based activities and indicate how a TUI may complement these interactions. In the second part of the study, we conducted semi-structured interviews with parents to gain insights into dyads HME needs (Figure 1).

Our findings revealed that parents juggled their child's demands and the object-based physical activity at once. We also uncovered that the HME manifested as indirect activities, where children drove their math development, and parents struggled with providing engaging activities. Informed by these, we offer three design directions: (i) support HME through guided play with TUIs while considering the parent's needs, (ii) leave room for adjustments to support engagement and agency, and (iii) consider affordability in the TUI setup. We reveal clues on how TUIs can be designed to support parent-child math engagement at home, contrasting with approaches that report new technologies or finished products that

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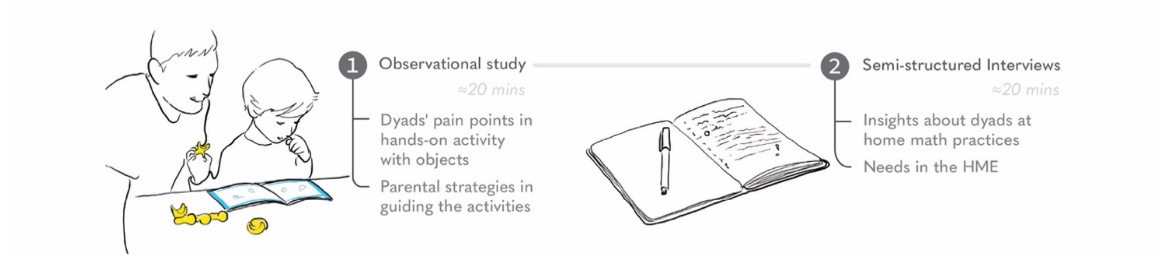

Figure 1: A snapshot our research procedure and intended outcomes in informing the HME design space for TUIs.

do not point at overarching knowledge [66]. Our work is an initial step for building knowledge on developing TUIs for the HME through design directions. While some of our directions are specific to math, some are overarching directions applicable for various domains of early childhood TUIs. The latter is also valuable as most TUI research focuses on school settings, lacking dyads' insights at informal settings like the home environment [57].

2 BACKGROUND

2.1 Early math development and the home math environment

Everyday life involves a broad range of mathematical knowledge, from splitting a bill among friends (division) to deciding which check-out to choose based on people's basket contents (magnitude comparison). In addition, math proficiency is an essential professional skill for many careers in STEM (science, technology, engineering, math) disciplines [38]. Research reveals that math development before primary school influences later math skills and even other academic skills like reading [38, 42]. Therefore, supporting math development before formal schooling is vital [6].

A recent meta-analysis reveals that the home math environment (e.g., parent-child math interactions) correlates with children's math attainment [23]. Home math environment (HME) reinforces learning through social interaction with caregivers, pushing children's abilities beyond what they can attain on their own [69]. HME prevails in two broad categories; direct and indirect math activities [23]. Direct activities entail explicit math activities such as comparing numbers, arithmetic operations, and playing math games. Indirect activities feature math incidentally during activities such as cooking. Direct math activities contribute more to children's math development as it provides the specificity required for young children's learning [15]. Another important facade in HME is parent math-talk, which refers to parents' use of math input like numbers, comparisons, etc., in direct or indirect math activities or daily exchanges [25]. Math-talk also contributes to children's math development [71]. Overall, research stresses the need to support HME, given its influence on young children's math development [23].

2.1.1 Magnitude understanding. Research reveals that parents may struggle with what math concepts their children should learn or how to support them [20]. Further, engaging with concepts other than counting can be challenging for parents [20]. Such a math concept is magnitude understanding, a foundational ability related to young children's later math achievements and a range of skills like spatial abilities [38, 50]. Magnitude understanding entails nonsymbolic 'more-less' comparisons amongst objects. Young children's magnitude understanding is affected by the objects' physical properties and spatial arrangements [41, 51]. For example, children may think two watermelons are more than three cherries due to their different volumes. In overcoming this confusion, research reveals providing children with diverse cases of a category helps them grasp the parameters of that category [29]. Yet research pursuing this endeavor for magnitude understanding is few [12]. Therefore, we assert that supporting magnitude understanding, a skill parents may overlook, is crucial in supporting young children's math development. Our HME exploration takes this topic as a case study.

2.1.2 Supporting the home math environment. Current HME intervention studies have mainly focused on tablet-based interventions, extensive parent training [52, 63] and low-cost board games supervised by researchers [60]. However, it is noted that without researcher supervision, the effects of the intervention diminished [61]. Therefore, digital technologies to support parents in HME appear valuable.

On the other hand, studies comparing the impact of traditional physical math games (e.g., chutes and ladders) and tablet-based ones on parent-child interactions assert that the traditional games were richer in math talk [59, 68]. This is because tokens in traditional games cause more talk-aloud during interactions, and the additional images in apps distract the dyads [68]. Another takeaway is the need to provide parents with specific roles or explicit instructions [36, 59, 71]. Previous research also revealed the need to support parents in broaching math and science topics while engaging with STEM media like apps and videos [34]. In sum, studies aiming to support HME used tablet interfaces while suggesting the importance of supporting parents and the role of physical interactions.

2.2 Tangible user interfaces for math development

Complementing physical object-based math practices with technology has been underexplored [31, 57]. On the other hand, physical objects are central to mathematics thinking and education [40]. Physical objects represent abstract concepts in a concrete channel, helping young children comprehend concepts [47]. For example, by distributing objects between two people, children experience the meaning of 'equal division' and place this information in the real world. According to the embodied cognition view, thinking and understanding emerge through real-life sensorimotor interactions [30]. As children engage in hands-on physical manipulation, their memories of these concepts can improve [48]. Further, tactile information of the objects may help focus attention [43]. All in all, physical manipulatives are crucial tools for young children's math development.

Tangible User Interfaces (TUI) amplify physical artifacts and environments with the computing paradigm [37]. One of the main affordances of TUIs for educational purposes is feedback and guidance[56]. Moreover, multimedia (e.g., text, images, sounds) support motivational game-like mechanisms and playfulness that boost learning gains [26, 45]. However, more research exploring TUIs' effectiveness on learning is needed [72]. In contrast to object-basedx education benefits, math-oriented technologies converge into tablet-based applications [31].

TUIs for math development usually target ages five and up, except for [3, 12], and focus on symbolic math skills (e.g., counting, arithmetic operations). Among the math-TUI examples, Tangible Ten A snapshot our research procedure and intended outcomes in informing the HME design space for TUIs.s employs an interactive surface with blocks to practice partner number concepts with 6-year-olds [55]. CETA targets additive composition and the number line representation (order of numbers on a linear line) of 5- to 6-year-olds. [44]. ARMath aims children between the ages 5-to-8 to practice arithmetic with everyday objects with the help of a mobile augmented reality system [39]. BlackBlocks features cubes with a sensing surface that draws a reaction from a nearby screen for children ages 4 to 8 to practice arithmetic operations. [2]. MaR-T targets children between the ages of 3 and 5 to practice magnitude comparisons within a projection-based mixed reality setup [12]. This is the only TUI designed for nonsymbolic math skills.

On the other hand, math TUIs for young children are deployed in preschools [12, 44, 55], museums [39], or the usage context is not explicitly stated [2]. Overall, math-TUIs targeting nonsymbolic skills (e.g., magnitude understanding) are rare for children under five. TUIs within the home math environment is also an unexplored area of research.

3 METHODOLOGY

3.1 Procedure

Our research had two steps and took about forty-five minutes to complete, with each step lasting around 20 minutes.

I. Observational study: The parents were presented with the activity booklet that featured one warm-up activity and two magnitude-related activities. The researcher introduced the prompts in the booklets that encouraged magnitude-understanding related talk and asked the parents to use these. Once the parents skimmed through the activities, the dyads were given the physical objects, and followed the activities offered in the booklet.

II. Semi-structured interview with the parents: After the observational study, the parents were asked questions about their math practices at home, the objects at their children's disposal, etc. This step followed the observational study so the parents could warm up with a math activity and more easily recall their at-home math practices.

3.2 Study Design

3.2.1 Observational Study. Our observational research aims to understand dyads' needs and points of struggle in a hands-on math activity to guide the design of math-TUIs. In designing TUIs, the Child-Tangible Interaction framework offers designers five dimensions (e.g., space for action, perceptual, behavioral, and semantic mappings) to consider catering to children's cognitive development. Alternatively, developmentally situated design cards inform designers about children's cognitive development between ages two to four [10] and over five. [11]. While these are foundational tools in designing systems for children, empowering the primary users in the design process before building interfaces is crucial. On the other hand, most design approaches for early childhood have user tests with children using finished prototypes [64].

Yet, young children's (under the age of 5) active involvement in the design practices is a developing practice. A recent review reveals that the number of research pursuing this approach is scarce, with limited successful attempts [64]. Young children may lose the design goal [35] or interest in the design process [46]. Observational methods appear suitable for this age group since it reveals the behaviors that signal their opinions [9, 32]. This method defines three requirements for eliciting deeper insights from children: (i) using age-appropriate hands-on tools, (ii) structured tasks, and (ii) collaboration with adults. Following previous work, we designed our hands-on tools for magnitude understanding, created structured tasks, and included parents in the task [9].

We refrained from providing a technological component to our setup as it may yield I/O related insights or modality-driven findings. Instead, we sought to extract how dyads engage with a hands-on activity and learn their at-home practices to inspire and inform interfaces. Similar approaches are being pursued for spatial understanding and joint book reading practices [9, 67]. Our next steps will follow the footsteps of previous research and inquire about the role of feedback and modality in children's behaviors within our TUI setup [3, 5, 33].

Physical objects: In this paper, we use magnitude understanding as a case study to understand the broader HME and to inform our overarching design goal of building an interface for this ability. For the age-appropriate tools, we designed our own set of objects with six different forms (Figure 2). This was motivated by the lack of tools for magnitude understanding, with traditional manipulatives on counting or fraction activities [40]. Therefore, from developmental literature, we elicited the design requirements to support magnitude understanding: (i) nonfigurative, (ii) non-salient, (iii) spatially configurable, and (iv) various physical forms. The details of these parameters are explained below:

Nonfigurative: For educational objects, research recommends using nonfigurative objects to support generalizability [21]. Prior experiences with an object (e.g., a car-shaped object) make it difficult for children to switch between two functions. Therefore, refraining from resemblances in object design is critical.

Non-salient: Developmental research suggests refraining from distracting features like color (bright-saturated) or surface properties (glitter) in educational objects [54]. These features draw attention to themselves and eclipse learning.

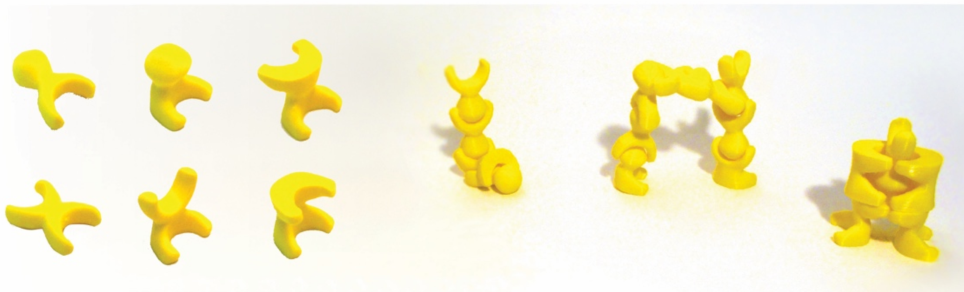

Figure 2: Designed objects used in the study. Left: Forms within the object set. Right: Examples of possible objects configurations.

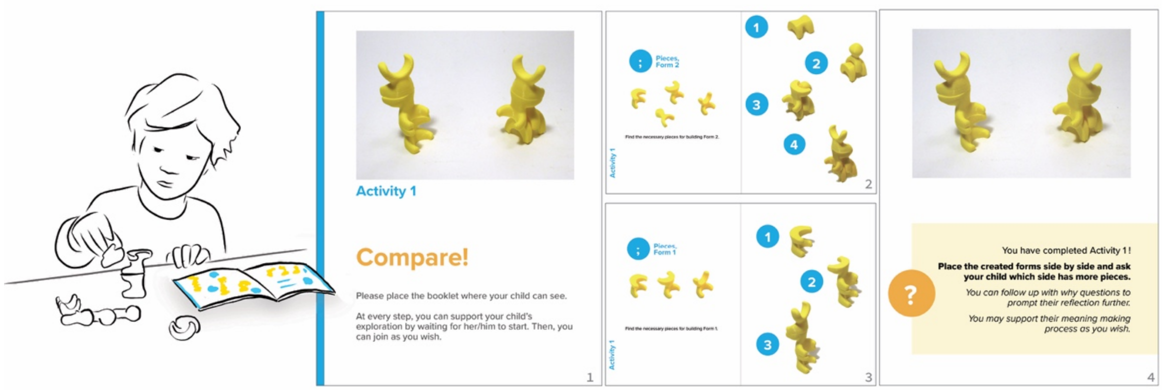

Figure 3: An example activity from the booklet. Far right: Prompts provided to the parents.

Spatially-configurable: Magnitude understanding is affected by the continuous physical properties of the objects (i.e., volume, shape, spatial arrangements), as explained earlier [50]. In this sense, a physical object set that offers different arrangements can help practice magnitude understanding.

Various physical forms: Providing children with multiple cases of a category helps them grasp its parameters [29]. Providing various physical forms (i.e., different shapes and sizes) may support magnitude understanding.

We modeled the forms using Rhinoceros 5.0 and 3d-printed them using PLA – a safe material for children's use (Figure 2). The dyads received an object set consisting of 18 pieces.

Activity booklet: Our activity booklet that features magnitude comparison activities, ranging from challenging to easy (Figure 3). We define challenging based on comparison ratios (e.g., 3:4 is difficult, 1:2 easy) [53]. We aimed to observe parents' guidance strategies and children's reactions to various difficulty levels to inform the TUI design's feedback requirements. In defining the parent's involvement in the task, we created the activities to be taken on together, given the influence of dyadic interactions on math development [23]. The activity content was largely inspired by LEGO DUPLO activity booklets tailored for ages 2-7, which we adapted to feature magnitude comparison tasks [41, 73].

The booklets feature warm-up suggestions to the parents for getting familiar with the objects before the structured tasks. In the activities, step-by-step constructions of two object configurations are provided, which are then compared in terms of their amount. For this comparison, there were prompts for the parents to initiate magnitude understanding talk (Figure 3, right). Their involvement helped us glimpse their scaffolding input that helped dyads complete the tasks. Scaffolding refers to the help a more skilled peer or adult provides that helps push a child's development [69]. Therefore, this informs us on ways to build the TUI to meet the dyads' requirements.

3.2.2 Semi-structured interviews. Following the observational study, we conducted semi-structured interviews with parents. The children were present during the interview and could join the conversation as they pleased. The goal of the interview was to gauge the home-math environment, structured or unstructured math activities at home, the parent's and the children's relationship with mathematics. The interview enabled us to understand them better to inform the math-TUI design space at home.

3.3 Participants and setting

Ethical approval for this study was obtained from the Koç University's Committee of Human Research. Thirteen parent-child dyads

Figure 4: Various study locations. Left: outdoor, middle: Meeting room, right: A dyad's home.

(Mage: 51 months, SD=4.71, gender distribution: 6F, 7M) were recruited via the Koç University database. All parents were university graduates, with eleven mothers and two fathers participating in the study. Before the observational study, the researcher introduced herself to the children and stated her goal was to understand their insights to build a math game. All children responded positively, and the parents signed the informed consent form. Furthermore, we stated that they could quit the session if they desired. Only one session was stopped halfway through as the child wanted to stop playing, yet we completed the interview with the parent. An Amazon gift card (35 TL) and a participation certificate were given to the dyads for their participation.

We conducted our study in diverse settings due to Covid-19 (Figure 4). We wanted to conduct the study outdoors (N=5) or in controlled spaces that provided distance and ventilation (N=4). Yet, due to weather conditions and transportation difficulties, some studies were conducted at the dyad's home (N=4). The setting differences are an unavoidable limitation. Yet, we believe that the studies conducted outside and at children's houses were more naturalistic than those conducted in controlled spaces, where children were timid.

3.3.1 *Pilot Study.*

We conducted two pilot studies with dyads (Mage of children= 50 months) to deduce if the objects were easy to handle (deducing 3d print-related defects) and whether the activity booklets were clear to understand. After, we refined the procedure into its final form, as presented earlier. The data of the pilot study participants are used in the analysis since the procedure did not change significantly.

The first version of the activity booklets featured three activities (two easy, one challenging activity). However, in the pilot studies, we observed that the amount of time dedicated to each activity was longer than expected, and children became uneasy. Therefore, we removed one easy activity from the booklets. Another problem was about the legibility of the activities, both in the sizes of the fonts and the figures. Also, the background in the figures obscured the visibility of the objects, as noted by the pilot study participants. Therefore, we improved this aspect of the activity booklets.

3.4 Measures and Analysis

The dataset consisted of observational study video recordings, audio recordings of semi-structured interviews, and the principal researcher's field notes. All data was analyzed by the principal researcher and three external researchers.

The video recordings of the observational study were analyzed qualitatively using BORIS software [27]. Three video data were viewed repeatedly to arrive at the initial codes in line with the aim of the observational study, revealing pain points in hands-on math activities and parental strategies in guiding the activities. The initial codes were discussed and refined (e.g., version 1: child confused by the booklet -> v.2 child losing track of the activity booklet steps; v.1 parent imitating sounds -> v.2 parent role-playing with objects to engage the child) applied across the dataset. The overarching categories of these codes resulted in: parent's activity scaffolding techniques, parents' and children's points of struggle in the activity, and parental strategies for managing disengagement.

The semi-structured interviews were transcribed and analyzed using reflexive thematic analysis (RTA) [17, 18]. The researchers met regularly through Zoom and using the Miro platform. RTA is an iterative process in which we followed an inductive approach. The analysis unfolded in six phases, as outlined by Braun and Clarke [16]. First, researchers familiarized themselves with the data through transcripts. Second, the researchers formulated, discussed, and refined initial codes. In RTA, the primary goal is not to ensure consensus among coders but to arrive at various interpretations of the data and to undertake "reflective and thoughtful engagement with their data and their reflexive and thoughtful engagement with the analytic process" [17]. The final set yielded six major code categories: daily routines, preferred play activities at home, children's favorite toys and cartoons, parents' educational beliefs, home-math environment, and educational materials. Third, early themes were generated from these general categories that encapsulated the patterns of meaning. To explain, the researchers sense-checked the patterns they found across the interview data and discussed its multiple interpretations [19]. Fourth, the candidate themes were reviewed so that they answered our research interest while being true to the data. Each theme was named and refined in the fifth step to ensure its distinct focus and depth. Lastly, the lead researcher weaved the themes with the relevant literature in the write-up process.

4 FINDINGS

We present our findings under two major categories. First, we report the results from the observational study that reveal the needs in our hands-on math activity under two themes: Parents (i) juggling with the child's demands and the activity at once and (ii) prompting hands-on reflection. Then, we report our themes from interviews that help shape the considerations for situating math-TUIs in the

Figure 5: Snapshots from the observational study where parents are helping children recall the activity step they are doing.

home-math environment. These are: (i) the home math environment deliberately manifesting as indirect activities, (ii) children driving their math development at home, and (iii) the challenges in engaging with activities that last.

4.1 Findings from the observational study

4.1.1 Parents juggling with the demands of the child and the activity at once. We observed that parents were tackling both maintaining the child's engagement and the demands of the activity at once. Some children were quickly disengaged from the magnitude understanding activity. We observed that parents had different strategies for re-engaging them in such instances. One parent started looking at the activities and doing them while remarking how exciting it was. This enticing strategy lured the child back into the activity. Other parents tied the activities to narratives incorporating their favorite cartoons or friends. Some parents facing their child's disengagement shifted the activity into bodily play with objects. One parent challenged her child to pick up objects from the ground without dropping them; another encouraged his emerging bodily play that involved running around with the objects like a game of pass the torch. While the latter did not tie the bodily activity with the magnitude understanding activity, this provided the children an exciting break.

An issue we observed was parents' struggles in keeping up with the directives of the booklets. Four participating parents forgot one of the prompts in the booklet due to their children's unease when transitioning from one activity to another. To explain, when one activity was completed, the children immediately questioned 'what was next.' This rushed some of the parents into flipping the page to the next activity, which made them forget to use the prompts. Some parents resorted to counting activities if they missed the directives from the booklet. Further, we realized that children demanded constant attention from their parents in the activity steps, constantly checking to see if they were watching. As a result, some children were frustrated when the parents attended the booklet rather than watching the child's actions.

4.1.2 Parents prompting hands-on reflection. One of our research interests was to observe where children needed help in an object-based magnitude understanding activity and how the parents provided support. We observed that most participating parents scaffolded the magnitude understanding reflections like a peer (e.g., asking questions) instead of giving direct instructions.

Some participating children needed their parent's help shifting back and forth between the activity steps on the booklet and the physical implementation itself. To illustrate, the booklet featured step-by-step activity illustrations that the dyads followed. The children first found the necessary object on the table and then executed the step in each activity. On the other hand, four children forgot what to do as they found the necessary object and started looking at the booklet again. In such situations, the parents pointed at their steps and affirmed their actions (Figure 5).

Another support was during comparison tasks. To explain, the activity booklets featured prompts that made parents initiate magnitude understanding related dialog with simple questions like 'which side has more objects?'. In answering these questions, four of the children gave the wrong answer. Some parents rephrased the questions and prompted them to count the objects. This led the children to reconsider their answers hands-on., and find the correct answer One parent managed this wrong answer by justifying the child's response first, then asking to reconsider together:

> "Well, you are right, actually. That side does look like it has more objects because it is tall! Shall we look over them once again, just to be sure?"

When the children struggled to answer, the parents acted like a peer who did not know the answer. They compared the objects together. The tangibility of the physical objects made the comparisons easy to follow and participate for the children. The dyads took apart the objects one by one, sorted and compared them.

4.2 Findings from the interviews

4.2.1 Home math environment deliberately manifesting as indirect activities. Dyads reports of the home math environment were divided into two major categories; direct math activities that specifically target math development (e.g., using math tools like an abacus) and indirect daily activities that incidentally feature math (e.g., following cooking instructions together). We found that the direct math activities at home were non-existent, whereas indirect math activities that casually included math-related exchanges were more common. We explain the underlying reasons for these tendencies in detail below.

Interviews with parents showed that they do not deliberately buy math-focused tools (e.g., math board games, abacus, etc.). Instead, parents reported using attention training sets, education kits, or educational activity booklets that sometimes feature mathematical activities. Two parents said that their library had math-focused stories which they sometimes read. We also gathered that math activities were not prioritized in daily routines. Amongst the staple activities in daily routines, joint reading, drawing, puzzle making, and playing with toys came forth. None of the children used digital technologies for educational purposes at home. Most parents refrained from mobile technologies (i.e., tablets, mobile phones) as they believed they harmed their children's development and caused temper tantrums.

Parents stated their various reasons behind sidelining direct math activities at home. Some parents considered doing mathematical activities with their children as a formal, sit-down activity and mentioned how they could not find the time or effort to partake in them. On the other hand, many parents were not keen on doing mathematical activities with their children at home (8/13). Two parents, specifically distant from mathematical activities, remarked about their role in the child's development, not being a teacher but a playmate and a source of care:

> "Well, that (mathematics) is school's business. If I do that at home, our play relationship would be affected negatively. So, I refrain from doing math activities at home."

Most parents explained that learning math was 'a natural process' that did not require activities. Another aspect of the home-math environment we explored was the indirect math activities, which included math-related utterances in daily speech or informal activities [25]. When asked about math-talk, seven parents said they must do it 'unconsciously.' The other parents said they casually blend math with play, such as counting toys. On the other hand, two parents said they deliberately integrate math into joint cooking activities, featuring math terms like half-glass milk, two eggs, etc. These are the same parents that specifically avoided direct math activities, which conflicted with their parental role.

Overall, we realized that the dyads did not practice direct math activities. Some parents refrained from engaging in direct math activities because they believed it conflicted with their role as a parent. Others stated their difficulties in accommodating such activities in their busy schedules. Another reason for not engaging in math-related activities was the belief that math development was natural and did not require deliberate effort. However, we also see an indirect use of math manifested as math-talk during informal activities by half of the participating parents.

4.2.2 Children driving their math development at home. Interviews revealed a child-initiated math interest rather than a parent-initiated one, aligning with the previous theme. All parents stated that their children were positive towards math. Five parents shared that their children ask math-related questions daily. The children's math talk was a wake-up call for some parents to use math-related:

> "Our son recently started doing math operations. The other day, his father said that there were two more children in the apartment. He suddenly said, 'So there are three children in the apartment,' counting himself in. He's adding numbers! I mean, I must be using math talk casually in everyday activities, but after he became interested, I started doing it differently and more frequently."

Parents noted their surprise when the younger child's math skills prosper with the older sibling's influence. Two parents' accounts included children eavesdropping on advanced math topics discussed with the older sibling and absorbing that information.

Participating parents highlighted the importance of children's agency in choosing the content and time of the activities. One parent contrasted it with the tasks they engaged with earlier in the observational study, where children were 'imposed' to engage in a task. She mentioned how they placed educational materials within the child's reach, which motivated the child to engage in educational activities.

> "Our style at home is not like that (referring to the activity earlier). We forced him to do the activities, and he got bored. . . Normally, we place educational tools at the locations he can reach. He does it when he wants."

In this theme, we reflected that even though parents do not exert deliberate effort in integrating math into their daily practices, children may show interest or surprise their parents in their math abilities. Further, the importance of making the child lead the time of the math activities was noted.

4.2.3 The challenges in engaging with activities that last. To understand our users better, we asked questions about their daily routines, children's favorite activities, toys, etc. This revealed that keeping up with children's pace and conserving their engagement in activities is difficult for the parents. Also, keeping the parents engaged in such child-oriented activities was another pain point.

> "Finding ways to entertain my daughter at home is very difficult. She constantly wants to do an activity. But the activities last for a very short time, 10- 15 minutes. She always wants something new."

Five parents remarked how children's favorite toys, cartoons, and interests were evolving daily. There is a similar pattern in educational materials (e.g., sets, books, apps) as parents mentioned their short life span in children's routines. This was due to two reasons. First, the materials lose their challenge due to children's rapid cognitive development. One parent mentioned her effort to duplicate and build on the activity materials to keep up with this.

> "(regarding educational tools at home) We have coloring books, we do matching activities. . . But I started to create these materials myself, like on a piece of paper with stickers or drawing them. These are activities found in many books, but he uses them up so quickly, so I resorted to building my content. . .."

Second, some children were noted to have a brief attention span or were usually bored during sit-down activities. Three parents mentioned their efforts to incorporate full-body activities for the latter. For example:

> "We write numbers in the hallway using electric tape. He matches the numbers on the wall with the balloons we give him. Once he aligns all the balloons, he sees his name appear. The other face of the balloon has his initials. He likes embodied games that spike his curiosity rather than sit-down activities."

While sharing many activities during their daily routine was considered positive for the parents, two added that they were sometimes personally bored during activities.

> "We used to do activity booklets together every day as a part of our routine. She enjoyed doing them. But I got quite bored with the content and avoided them for a while."

Overall, we uncovered a pain point of the participating parents: children's immediate consumption of activities and materials at their disposal. While this insight is not specific to math, it is an overarching consideration in the Discussion.

5 DISCUSSION

Our goal in this paper was to inquire about the design space for tangible user interfaces (TUI) for the home math environment (HME). The observational study captured dyads' struggles and parents' scaffolding strategies in the object-based activities. In addition, the interviews provided an account of the HME, namely the parents' beliefs, struggles in featuring math at home, and children's math-related interactions with their parents. Compiling these, we first discuss the HME as a challenging environment and provide three design directions for supporting math at home with TUIs.

5.1 Barriers to HME design space

The interviews revealed that indirect math (e.g., math talk in daily activities) was shared among our participating dyads, supporting previous research [23]. Participating parents refrained from direct math activities for three main reasons:

- Some parents believe that math development is a natural process that does not demand deliberate effort. This is in line with the literature that states parental belief around the importance of math is related to how frequently they do math activities with their children [49].
- Parents' have difficulties accommodating direct math activities in their busy schedules. They asserted that math activities demanded time and effort from the parent who juggles everyday chores. A similar struggle was echoed in a study discovering the design space of parent-child reading [67].
- Another group of parents stated that doing math activities at home conflicts with their role as a parent, which is reported as not a teacher but a source of care and play.

While situating math-TUIs at home has several barriers, we assert that it is a worthwhile effort. HME with parental involvement supports the development of children's early math skills before school entry [6]. Especially considering its effect on the significant differences among children before formal schooling and its lasting impact, supporting math at home early is beneficial [6, 42]. We also suggest that participating parents are not that distant from supporting their children's math development at home. Almost all parents report using math talk in their play and everyday activities (e.g., counting toys, using measurement terms during cooking). A natural question is, 'Why not design a TUI that manifests as an in-direct math support tool?' Research reveals that it may not suffice in supporting young children's math development. Parental engagement in direct math activities in the early years (3-5) is more fruitful than in-direct math, as children build foundational math knowledge that benefits from explicit instruction [23]. Considering this, we constructed the following design direction.

5.2 Design Direction 1: Support the home-math environment through guided play with TUIs while considering the parent's needs

Given the barriers mentioned above to incorporating direct math activities at home (e.g., parental beliefs on math development, busy schedules, etc.), we suggest math-TUIs embody a guided play setup. While play in learning is a popular method, there are various forms of play ranging from free-unstructured play to guided play[70]. The latter is more effective in children's learning. Guided play supports playfulness and goal-directedness at the same time[36,70]. The child takes the lead, and an experienced partner structures the explorations (e.g., asking questions, highlighting certain aspects of children's explorations.) Given this nature, we reckon that such a play activity does not necessarily make the parent a teacher, which was another reason for avoiding direct math activities.

Encouraging guided play with TUIs has practical benefits as well. Busy parents might not need to carve up extra time to engage with the TUI since play was reported as a staple part of every dyad's daily routine. The TUI learning activities may be built around a narrative or used in role-playing setups, typical approaches for boosting playfulness. On the other hand, an imperative to consider in a guided play TUI setup is supporting the way parents give guidance, discussed below.

5.2.1 Support parents in providing developmentally relevant math guidance and make children's progress visible. The use of prompts (e.g., questions to be asked by the parents to increase active learning or to initiate reflection in children when they give wrong answers) is a common method that the parent adopts in guided play activities. However, in the case of magnitude understanding, supporting the parent in giving developmentally appropriate guidance to their children may be an essential contribution of technology. In the observational study, we realized that dyads often counted the objects rather than doing the magnitude understanding activities suggested in the booklet. This partially supports the literature on parents' tendency to favor counting activities over others [20]. Therefore, our selection of magnitude understanding as a case study further demonstrates the need to provide parents with guidance in math subjects that are not practiced in current TUI research [12]. We further discuss this in the next section.

On the other hand, our interview findings suggest that parents are not always aware of their children's current math skills or cannot keep up with their fast development with educational materials. Many parents noted how activity booklets were used up quickly or lost their challenge due to their child's rapid cognitive development. Some parents recalled their surprise when their younger child answered questions directed at the older one. According to the social learning theory, the advanced topics inferred by the older sibling can push children's zone of proximal development (ZPD), which is the progress a child may demonstrate with guidance [69]. The positive influence of older siblings on young ones' development is widely shown [7]. On the other hand, these findings echo previous work that parents may struggle with what math concepts their children should be learning or how to support them [20].

Like the older sibling effect, the TUI can gauge the child's level and provide math prompts to the parents within the child's ZPD.

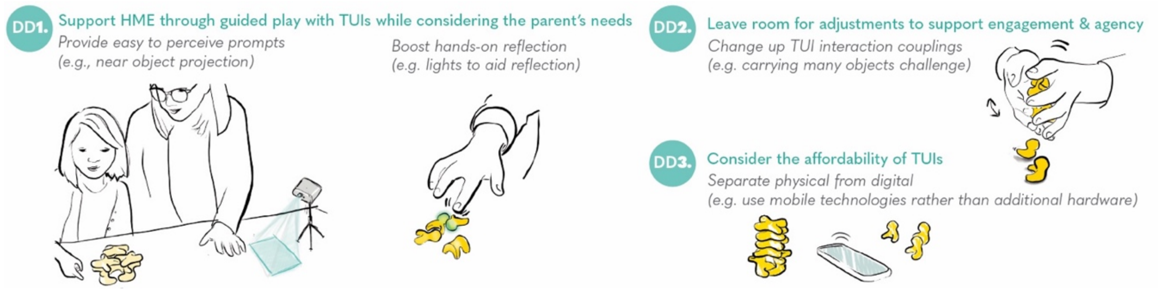

Figure 6: Examples of ways to achieve the design directions

This adaptation to the child's development is one of the premises of digital technologies [45]. Since parents may not know their children's math level, adaptation is essential for math TUIs. However, while adapting the math level, showing the children's progress to the parents is needed. Interviews revealed that when parents witnessed their children's math talk, this encouraged them to use more math-talk in their daily exchanges. These exchanges significantly influence children's math development and should be encouraged [23, 69]. To that end, children's progress can be visible to the parents by sending bite-size information via a mobile application. To elaborate, the app can feature progress tracking and suggestions on mathematical language and indirect math activities to be pursued in daily routines based on children's level. Again, this information needs to be bite-size, given the busy routines reported by the parents. We also think mobile apps are suitable for these efforts due to their widespread use and ease of access from the parents' end.

In sum, if the TUI made children's levels more apparent to the parents, it may encourage parents to broach relevant math topics in their daily utterances and support the children's math development beyond the TUI play session [25].

5.2.2 Consider parents' attentional load while providing math-related prompts. The previous heading outlined the math guidance content given to the parents. How it should be given to the parents is equally as important. We realized a critical issue in the observational study: parents attended to many things at once, ranging from children's engagement to activity demands. This imposed some challenges for parents in providing the suggested magnitude understanding prompts [20]. Previous research reports increased math talk during a tablet game when parents were provided prompts before the study [71]. However, in our study, some participating parents forgot to integrate the prompts presented earlier by the researcher, which were also written in the activity booklets. Unlike the tablet study, the abundance of attentional demands from the children, the booklet, and the object-based physical activity may have strained the parents. The retainment and manipulation of various presented information may have forced parents to prioritize their children's attention demands over the activity prompts [22]. We suggest that, beyond our case, this struggle should be considered in designing technologies for parent-child use.

Providing the parents with hard-to-miss prompts remains an open question. Considering the location of the digital representation in the TUI setup could be a vital issue. Our physical activity setup had two focal points: the booklet and the objects. In this sense, it closely resembled a discrete interaction setup, where the tangible interaction and the digital representation occur in separate locations (e.g., actions on physical objects corresponding to a change in a nearby tablet)[3]. These diverse focus points are more challenging for children in the activities [3]. Our observations also supported this, as children had difficulty shifting from the activity step they saw on the booklet to the physical objects due to their developing working memory [22]. We suggest that co-located and embedded TUIs may pose less demand to both the children and the parents as these use the representation space in close vicinity or the tangible itself [3].

On the other hand, an important point to consider is the dual audience that the prompts will target in the math-TUI. While the play directives target both parties, math-specific prompts must only be present for the parents. This latter guidance should not distract the child from the joint activity and endorse minimal effort from the parent. A recent study revealed that hearing parents prefer near-object projection as feedback location when providing sign language input to their deaf children during joint toy play [8]. This close vicinity projection was considered less intrusive and easy to perceive during play, compared to tablets and wearable solutions (i.e., smart glass and smartwatch). While this suggests that near object projection may be suitable for the math-TUI setup (Figure 6), there are several issues. First, current projection-based systems depend on lighting conditions and have fixed locations [8, 12]. Further, during our observational study, parents and children sat next to each other and shared the same field of vision. To that end, when presenting the projection-based play prompts to the parent, it is likely that the child will also see it and become distracted. We base this assumption to our observation that revealed children's frequent calls for attention even when the parents briefly glanced at the one to two sentence prompts in the booklets.

This need to be discreet yet accessible when providing prompts is an intriguing research inquiry that demands further investigation. Such research would support dyadic interactions across various technologies, such as joint reading practices and engagement with STEM media that commonly use prompts for parents [34, 67]. Current TUI research has rarely focused on the home environment, nor the needs that dyads experience in hands-on activities. We assert that our insights are valuable for this field[57].

5.2.3 Boost hands-on reflection. For providing guidance, our observations revealed that parents resort to hands-on reflection with the objects rather than giving didactic explanations when guiding their children's meaning-making process. The physical objects enabled the comparisons to be easy to follow and participate for the children, in line with the embodied cognition theory that asserts thinking is facilitated by bodily actions [30]. We affirm that our object's forms further supported these hands-on comparisons since their design was easy to handle and configure structures. The dyads took apart the objects one by one, sorted and compared them. TUI setups can further support the parents in providing hands-on guidance [3]. Making objects easy to assemble and disassemble could be a design direction, as apparent in our object set. Also, designing digital augmentations (i.e., color, sound) could be sought out to support the dyad's hands-on reflections. To illustrate, when the dyads are discussing the amounts of objects during guided play activities, the TUI can enter a 'reflection mode' upon the parent's initiation. In this mode, the touched objects could light up in different colors so that the amounts are more highlighted to aid children's processing of the magnitudes (Figure 6).

5.3 Design Direction 2: Leave room for adjustments to support engagement and agency

We realized that keeping up with children's pace and conserving their engagement is an important consideration. Parents remarked their difficulties in providing materials for their children (e.g., books, activities) that managed to sustain their attention. We noticed a similar pattern in the observational studies in which some children quickly disengaged from the math activity. In managing this, parents adapted the activities to their children's interests. Engagement is critical for learning as it supports children to stay on task [36]. While ways to engage children in activities is a much-explored topic in research, we underline that achieving engagement beyond the novelty effect remains a challenge. Many work in child-computer interaction report new technologies or finished products that do not address the afterlife of 'fun' tools [66].Therefore, fostering room for adjustments may provide a more sustainable life span for TUIs in the HME. We exemplify three ways to achieve this below. Given that TUI research with children below the age of 5 is scarce, we suggest that our suggestions, even though not math specific, are valuable for the TUI field.

5.3.1 Change up the TUI interaction couplings. Several ways can be explored to support a longer lifespan of the TUIs at home. TUIs provide responsive interactions and extrinsic motivation with feedback to prolong engagement[3,56]. Alternatively, intrinsic motivation can be sustained through curiosity. Fostering several types of interactions with the tangibles may break the monotony and spike children's curiosity [12, 65]. For example, as one activity would focus on math activities that involve stacking, the other can feature flipping or carrying many objects at once (Figure 6). Activities that involve more bodily actions that might help engage children who struggle with sit-down activities as well.

5.3.2 Use ambiguous physical objects for various play narratives. Another way to address children's ever-changing interests and maintain engagement could be by changing the narratives with the physical objects. To accommodate the physicals in various narratives, the design of the forms may refrain from inducing specific resemblances. The non-figurativeness design consideration is also vital for learning materials design to prevent function-fixedness that eclipses children to use the objects (e.g., toy cars or figurines) for educational purposes [21]. We observed a positive influence of our form's ambiguity as dyads adapted them in various narratives accompanying the activities.

5.3.3 Allow dyads to shape play narratives. Children's agency is vital for their motivation and learning gains, as evidenced in literature and highlighted in our interview findings [70]. Therefore, we assert that empowering them in shaping the play narratives could be frutiful. While featuring tangential game features is a common approach to appeal to children in educative activities, research cautions that it should not overshadow the learning experience [36]. Specifically for math, the TUI activity should be presented to the children as a 'math tool' rather than a toy to increase their link to the mathematical aspect of the activity [24]. Therefore, designers may provide dyads with options to shape the narrative that accompanies the TUI while maintaining the learning activities' necessities.

We also stress that this narrative adaptation should induce minimal effort since parents' busy schedule imposes a barrier to engaging in math activities at home. Allowing parents to select these parameters is essential too. Our interviews revealed that parents get bored with the activities designed for children's learning. A similar need in designing for a dual audience has been highlighted in previous research with apps [1, 67]. Therefore, supporting parents' engagement is also essential in the uptake of the TUI setup.

5.4 Design Direction 3: Consider the affordability of TUIs

Tangible user interfaces come in many forms, ranging from simple physical objects accompanied by discrete representation technologies such as tablets to embedded tangibles containing the digital parts themselves. An essential consideration in HME is the affordability of the TUI. It is noted that the TUI field is often hard to scale for the general population as it usually requires specialized hardware [4]. Providing affordable, easy to access solutions is vital for creating equal opportunities for children's development. This is especially important for math, where the differences in children's math development before formal education are partly caused by access to educational materials [6].

We suggest separating the digital components from the physical objects. To illustrate, the objects can be built with child-friendly materials such as wood or PLA. Or, forms for math education can be made open-source for parents to 3D print or manufacture in their preferred way. These objects can feature fiducial markers or NFC tags that are durable and accessible recognition methods. In such a setup, affordable mobile technologies like phones and tablets may track the objects and provide audio and visual feedback [58]. The literature on parental beliefs about mobile devices' effect on young children is widely researched, revealing parents' mixed feelings [28, 62]. We should note that many participating parents believed mobile phones were harmful to their children. Therefore, for our participating parents, these smart devices may be more acceptable as sensing agents and guide parents with written feedback as discussed in design direction 1. It may also provide audio feedback to immerse the children further, rather than being the highlight of the interaction.

While sturdy embedded tangibles (i.e., Sphero [74]) can provide additional features such as movement, or mixed-reality setups can provide visual augmentations, these approaches are yet costly. Separating digitals and physicals can further support TUIs to evolve with children's development. For example, the physical objects would change based on the math topic (e.g., objects with numerals to practice cardinality), and mobile technologies would augment the interactions accordingly (e.g., announcing the numerals via speakers). The designers can actively seek further reflection on how to create low-cost TUIs for young children's math development.

6 LIMITATIONS

Our study provides initial, descriptive insights about dyads' home math environment and observations of object-based physical activity. The directions will be refined and reflect a holistic design space with the making and evaluation of the prospective TUI [13]. On the other hand, research can extend the sample size and the makeup to include participants from various countries to further the extensibility of the findings. Due to Covid-19, our study was conducted in multiple locations, which is a limitation. We realized that studies at home yielded the most naturalistic interactions. To that end, future research may conduct these studies at home to reveal deep insights, using cultural probes [14] to capture situated experiences in the diverse home math environment.

7 CONCLUSION

We probed the tangible user interface (TUI) design space of the home-math environment (HME), which is an essential setting for young children's math development [23]. We (i) observed dyads' requirements and needs in the math-TUI design via physical-object based math activity and (ii) gained insights about dyads at home math practices and their needs through semi-structured interviews. Informed by our studies, we offered three design directions, ranging from the form of the TUI as a guided-play setup to suggestions during the activities like supporting the parents and ways to keep the TUI within the challenging HME by providing adaptability and affordability. While some of our directions are specific to math, some are overarching directions that may be applicable for various domains of TUIs for early childhood. We assert that our home and dyadic interaction specific insights are of particular value for TUI research, a domain where needs in the informal (i.e., home) environments with parental involvement are scarcely researched [57].

Our work acts as the first step into the HME as a design space. Given its importance on young children's development, we urge researchers to pursue gauging how TUIs may support parent-child math engagement at home. Many work in child-computer interaction is critiqued for reporting 'fun' new technologies or finished products that do not point at overarching knowledge [66]. We conjecture that our design suggestions and reflexive accounts of our object-based activity, without being bound to a specific technology, point to directions to shape math TUIs at home and provide insights applicable for future studies in this vital domain.

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