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Finite element approximation of Hamilton–Jacobi–Bellman equations with nonlinear mixed boundary conditions

Bartosz Jaroszkowski
Department of Mathematics, University of Sussex, Falmer Campus, Brighton BN1 9QH, UK

and

Max Jensen*
Mathematics Department, University College London, 25 Gordon Street, London WC1H 0AY, UK
*Corresponding author: M.Jensen@sussex.ac.uk

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We show strong uniform convergence of monotone P1 finite element methods to the viscosity solution of isotropic parabolic Hamilton–Jacobi–Bellman equations with mixed boundary conditions on unstructured meshes and for possibly degenerate diffusions. Boundary operators can generally be discontinuous across face-boundaries and type changes. Robin-type boundary conditions are discretized via a lower Dini derivative. In time, the Bellman equation is approximated through IMEX schemes. Existence and uniqueness of numerical solutions follows through Howard's algorithm.

Keywords: finite element methods; Hamilton–Jacobi–Bellman equations; mixed boundary conditions; fully nonlinear equations; viscosity solutions.

1. Introduction

The value function of an optimal control problem is, under suitable assumptions, the solution of a Hamilton–Jacobi–Bellman (HJB) equation. In this work, we consider the numerical solution of HJB equations with mixed boundary conditions of the form:

$$-\partial_t v + \sup_{\alpha\in A}(L^\alpha \; v - f^\alpha) = 0 \qquad \text{in } [0,T)\times\Omega, \tag{1.1a}$$

$$-\partial_t v + \sup_{\alpha\in A}\big(L^\alpha_{\partial\Omega} v - g^\alpha\big) = 0 \qquad \text{on } [0,T)\times\partial\Omega_t, \tag{1.1b}$$

$$\sup_{\alpha\in A}\big(L^\alpha_{\partial\Omega} v - g^\alpha\big) = 0 \qquad \text{on } [0,T)\times\partial\Omega_R, \tag{1.1c}$$

$$v - g \quad = 0 \qquad \text{on } [0,T)\times\partial\Omega_D, \tag{1.1d}$$

$$v - v_T \;= 0 \qquad \text{on } \{T\}\times\overline{\Omega}. \tag{1.1e}$$

Here, L^α and $L^\alpha_{\partial\Omega}$ denote operators on the domain Ω and its boundary, respectively. The sets $\partial\Omega_t$, $\partial\Omega_R$ and $\partial\Omega_D$ form a decomposition of $\partial\Omega$. While leaving further details of the notation to the next section, it is already apparent how the basic fully nonlinear structure of the PDE operator, meaning the left-hand side of (1.1a), is mirrored in the Robin-type boundary conditions (1.1b) and (1.1c). But there is a crucial, additional complication of the boundary operators $L^\alpha_{\partial\Omega}$: they will in general depend on the full gradient

∇v and not just on the tangential gradient $\nabla_{\partial\Omega} v$, meaning that $L^{\alpha}_{\partial\Omega} v$ cannot be evaluated with knowledge of $v|_{\partial\Omega}$ only.

Recalling the connection between optimal control and HJB equations, Bellman-type equations as in (1.1b) naturally arise on sections $\partial\Omega_t$ of the boundary. Indeed, the boundary condition (1.1b) expresses the possibility of controlling the particle or agent on the boundary through processes that are implicitly described by the $L^{\alpha}_{\partial\Omega}$. In contrast, Dirichlet conditions (1.1d) are appropriate for those sections $\partial\Omega_D$ of $\partial\Omega$, where the possibility to control may cease in exchange for the reward or cost of g_D.

Boundary conditions of type (1.1c) arise from the Skorokhod control problem, which models particle reflection at the boundary (Lions, 1985; Kushner & Dupuis, 2001; Serea, 2003). Moreover, they have recently been used for the numerical solution of optimal transport problems in the setting of Monge–Ampère equations, where the transport boundary conditions are examined in Hamilton–Jacobi form (Benamou *et al.*, 2012; Kawecki, 2019).

For the authors, the problem of primary interest is the Heston model of financial interest rates with an uncertain market price of volatility risk (Jaroszkowski & Jensen, 2021). The Heston equation is most naturally posed on an unbounded domain, where already with a certain market price of volatility risk, it appears with mixed boundary terms corresponding to (1.1b), (1.1c) as well as (1.1d). All those types of boundary conditions remain when introducing uncertainty and when truncating the domain for the purposes of numerical approximation.

The aim of this work is to introduce a finite element method capable of computing approximations to viscosity solutions for the aforementioned problems. This paper extends results of Jensen & Smears (2013) with the inclusion of mixed, fully nonlinear boundary conditions. The presented method permits degenerate diffusions. Boundary operators may exhibit discontinuities across face boundaries and where the type of boundary condition changes.

A challenge for problems of this type is the discretization of the first-order directional derivatives in (1.1b) and (1.1c), which needs to be simultaneously consistent and monotone. On the one hand, establishing monotonicity with an artificial diffusion approximating the Laplace–Beltrami operator of $\partial\Omega$ would not be sufficient, because of the normal component in the directional derivatives of (1.1b) and (1.1c). On the other hand, an artificial diffusion approximating the Laplace operator of Ω would not vanish under refinement due to different scaling of boundary and domain terms, thus leading to an inconsistent method. Our formulation is based on the observation that lower Dini directional derivatives exist for all functions in the P1 approximation space, whenever the direction in question is in the tangent cone of Ω at the position of interest.

A benefit of the finite element approach is that, besides L^{∞}, also $L^2(H^1)$ convergence can be established on unstructured meshes, as was shown in Jensen & Smears (2013); Jensen (2017) for the Dirichlet problem. The $L^2(H^1)$ convergence is for instance important for the above-mentioned Heston model (Jaroszkowski & Jensen, 2021), as Delta hedging requires knowledge of partial derivatives of the value function.

The main contribution of this work are as follows:

1. the FEM discretization of HJB equations with mixed boundary conditions, which contain (nontangential) HJB operators on the boundary,
2. the formulation of a monotone scheme without time-derivative in part of the Robin-boundary,
3. the proof of consistency of the FEM method with viscosity boundary conditions, differing the from strong conditions posed in Jensen & Smears (2013); Jensen (2017),

4. the proof of numerical stability via consistency, allowing to derive the stability of the fully nonlinear problem through the analysis of a simpler, linear problem.

The numerical analysis of HJB equations with Neumann and Robin conditions encompasses only few works. First, results were provided by the finite difference community; we refer to the text book by Kushner & Dupuis (2001). More recently, the transport boundary conditions of optimal transport were in Benamou *et al.* (2012) approximated with a filtered wide stencil scheme. In Achdou & Falcone (2012), a nonlinear Neumann boundary operator is approximated by extending the boundary into a strip of positive thickness, allowing the boundary conditions to be treated like a PDE operator. Within the finite element setting, one line of research has developed around the approximation of Cordes solutions, in Gallistl (2019) with a mixed, nonconforming finite element method, while in Kawecki (2019) with a discontinuous Galerkin finite element method. Both Gallistl (2019) and Kawecki (2019) concentrate on the linear setting in nondivergence form. In the context of the related semi-Lagrangian methods, we refer to Carlini & Silva (2016) for the discretization of mixed problems. For a general review of the approximation of fully nonlinear equations with other types of boundary conditions, we refer to Feng *et al.* (2013); Neilan *et al.* (2017).

The structure of this article is as follows. In Section 2, we formulate the HJB problem with mixed boundary conditions. In Section 3, we define the numerical method. In Section 4, we prove monotonicity properties of the discretized operators. In Section 5, we show the existence and uniqueness of numerical solutions. In Sections 6 and 7, we establish consistency and stability, respectively, leading us to our main result of convergence in Section 8. Finally, we present numerical experiments in Section 9.

2. Mixed final time boundary value Bellman problem

In this section, we introduce time-dependent Bellman equations with mixed boundary conditions. We consider a polytopic domain $\varOmega \subset \mathbb{R}^d$ with $d \geqslant 2$, i.e., a bounded, connected, closed domain, whose interior is nonempty and whose boundary is formed of flat faces. We allow $\varOmega$ to be nonconvex. Let $\mathcal{F}_k$ denote set of open k-dimensional faces of $\varOmega$ contained in $\partial\varOmega$.

We consider Dirichlet and Robin boundary conditions on disjoint subsets of boundary $\partial\varOmega$. Additionally, the region of the Robin boundary conditions breaks into two parts, one with and one without time derivative, as is indicated in (1.1). We denote those three disjoint regions as $\partial\varOmega_D$, $\partial\varOmega_R$ and $\partial\varOmega_t$, respectively. Therefore $\partial\varOmega_D \cap \partial\varOmega_R \cap \partial\varOmega_t = \emptyset$ and $\partial\varOmega_D \cup \partial\varOmega_R \cup \partial\varOmega_t = \partial\varOmega$.

It is convenient to define the notion of a *generalized face* as the intersection of an $\omega' \in \mathcal{F}_{d-1}$ and a region linked to a boundary condition:

$$\mathcal{F} := \left\{\omega \subset \partial\varOmega \,:\, \omega = \omega' \cap \partial\varOmega_X \text{ where } \omega' \in \mathcal{F}_{d-1}, \partial\varOmega_X \in \{\partial\varOmega_D, \partial\varOmega_R, \partial\varOmega_t\}\right\}.$$

We assume that the boundary conditions are continuous on each ω; however, discontinuities across (generalized) face boundaries may occur. To make this statement precise, we introduce the normed space of piecewise continuous functions

$$PC(\partial\varOmega, \mathbb{R}^k) := \{g \in L^\infty(\partial\varOmega, \mathbb{R}^k) :\ g|_\omega \in C(\omega, \mathbb{R}^k) \quad \forall\, \omega \in \mathcal{F}\},$$

equipped with the $L^\infty(\partial\varOmega, \mathbb{R}^k)$ norm. If $k = 1$, we simply write $PC(\partial\varOmega)$. We denote the standard inner product of $L^2(\varOmega)$ and $L^2(\varOmega, \mathbb{R}^d)$ by $\langle\cdot,\cdot\rangle$.

Let A be a compact metric space, $\alpha \in A$, and let L^α be a linear operator of the following form:

$$L^\alpha : \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R} \to \mathbb{R},\ (x, q, p, s) \mapsto -a^\alpha(x)\, q - b^\alpha(x) \cdot p + c^\alpha(x)\, s.$$

The interpretation as differential operator follows with $q = \Delta w(x)$, $p = \nabla w(x)$ and $s = w(x)$ for $w \in C^2(\overline{\Omega})$. Frequently, we abbreviate $L^\alpha(x, \Delta w(x), \nabla w(x), w(x))$ by $L^\alpha w(x)$ and $x \mapsto L^\alpha w(x)$ by $L^\alpha w$.

For $\alpha \in A$, the Robin operators $L^\alpha_{\partial\Omega}$ are defined as

$$L^\alpha_{\partial\Omega} : \partial\Omega \times \mathbb{R}^d \times \mathbb{R} \to \mathbb{R},\ (x, p, s) \mapsto -b^\alpha_{\partial\Omega}(x) \cdot p + c^\alpha_{\partial\Omega}(x) s, \tag{2.1}$$

with $b^\alpha_{\partial\Omega} \in PC(\partial\Omega, \mathbb{R}^d)$, $c^\alpha_{\partial\Omega} \in PC(\partial\Omega)$. The abbreviations $L^\alpha_{\partial\Omega} w(x)$ and $L^\alpha_{\partial\Omega} w$ are used analogously to L^α.

We can now pose HJB problem, whose numerical solution is the subject of this paper:

$$-\partial_t v + \sup_{\alpha\in A}(L^\alpha\ \ v - f^\alpha) = 0 \qquad \text{in } [0, T) \times \Omega, \tag{2.2a}$$

$$-\partial_t v + \sup_{\alpha\in A}\big(L^\alpha_{\partial\Omega} v - g^\alpha\big) = 0 \qquad \text{on } [0, T) \times \partial\Omega_t, \tag{2.2b}$$

$$\sup_{\alpha\in A}\big(L^\alpha_{\partial\Omega} v - g^\alpha\big) = 0 \qquad \text{on } [0, T) \times \partial\Omega_R, \tag{2.2c}$$

$$v - g \quad = 0 \qquad \text{on } [0, T) \times \partial\Omega_D, \tag{2.2d}$$

$$v - v_T \ = 0 \qquad \text{on } \{T\} \times \overline{\Omega}, \tag{2.2e}$$

with $g \in C(\partial\Omega)$, $g^\alpha \in PC(\partial\Omega)$, $v_T \in C(\overline{\Omega})$ and $T \in (0, \infty)$. The suprema are applied pointwise. An interpretation of (2.2) in the context of optimal control is given in Appendix A. Observe that the data terms g^α of the Robin conditions are α-dependent, while the corresponding Dirichlet data g are not.

Assumption 2.1 We assume
(a) that the mapping

$$A \to C(\overline{\Omega}) \times C(\overline{\Omega}, \mathbb{R}^d) \times C(\overline{\Omega}) \times C(\overline{\Omega}) \times PC(\partial\Omega, \mathbb{R}^d) \times PC(\partial\Omega) \times PC(\partial\Omega),$$
$$\alpha \mapsto (a^\alpha, b^\alpha, c^\alpha, f^\alpha, b^\alpha_{\partial\Omega}, c^\alpha_{\partial\Omega}, g^\alpha),$$

is equicontinuous in $\alpha \in A$,
(b) that $a^\alpha(x) \geqslant 0$ for all $\alpha \in A$ so that all L^α are degenerate elliptic,
(c) similarly that the sign-conditions c^α_Ω, $c^\alpha_{\partial\Omega}$, v_T, g, $g^\alpha \geqslant 0$ hold,
(d) that v_T satisfies the Dirichlet boundary conditions on $\partial\Omega_D$.

It follows from equicontinuity that

$$\sup_{\alpha\in A} \| (a^\alpha, b^\alpha, c^\alpha, f^\alpha, b^\alpha_{\partial\Omega}, c^\alpha_{\partial\Omega}, g^\alpha) \|_\infty < \infty. \tag{2.3}$$

The assumption in (a) above that $v_T, g, g^\alpha \geqslant 0$ simplifies the statement of the weak discrete maximum principle (wDMP) (4.1), and the proof of Theorem 5.1 about the existence of numerical solutions. It should be noted, however, that boundary value problems not satisfying these conditions can often be reformulated by subtracting a function with suitable boundary values from the exact solution, so that these inequalities are known to hold.

It is useful to reformulate (2.2) more succinctly as $F(t,x,\Delta v(t,x),\nabla v(t,x),\partial_t v(t,x),v(t,x))=0$ with

$$F(t,x,q,p,r,s)=\begin{cases}-r+\sup_\alpha\big(L^\alpha(x,q,p,s)-f^\alpha(x)\big) & \text{if } (t,x)\in[0,T)\times\overline{\Omega},\\ -r+\sup_\alpha\big(L^\alpha_{\partial\Omega}(x,p,s)-g^\alpha(x)\big) & \text{if } (t,x)\in[0,T)\times\partial\Omega_t,\\ \sup_\alpha\big(L^\alpha_{\partial\Omega}(x,p,s)-g^\alpha(x)\big) & \text{if } (t,x)\in[0,T)\times\partial\Omega_R,\\ s-g(x) & \text{if } (t,x)\in[0,T)\times\partial\Omega_D,\\ s-v_T(x) & \text{if } (t,x)\in\{T\}\times\overline{\Omega}.\end{cases}$$

We conclude the section with a definition of a viscosity solution similar to the setting of Barles & Souganidis (1991), which will be used throughout the chapter. To this end, let us consider a bounded function $v:[0,T]\times\overline{\Omega}\to\mathbb{R}$ and its upper and lower semicontinuous envelopes, defined respectively as

$$v^*(t,x):=\limsup_{\substack{(s,y)\to(t,x)\\(s,y)\in[0,T]\times\overline{\Omega}}} v(s,y)$$

and

$$v_*(t,x):=\liminf_{\substack{(s,y)\to(t,x)\\(s,y)\in[0,T]\times\overline{\Omega}}} v(s,y).$$

We analogously extend the definition of lower and upper semicontinuous envelopes to F.

DEFINITION 2.2 A bounded function v is a viscosity supersolution (respectively, subsolution) of (2.2) if, for any test function $\psi\in C^2(\mathbb{R}\times\mathbb{R}^d)$,

$$F^*(t,x,\Delta\psi(t,x),\nabla\psi(t,x),\partial_t\psi(t,x),v_*(t,x))\geqslant 0,$$

(respectively,

$$F_*(t,x,\Delta\psi(t,x),\nabla\psi(t,x),\partial_t\psi(t,x),v^*(t,x))\leqslant 0,)$$

provided that $v_*-\psi$ attains a local minimum (respectively, $v^*-\psi$ attains a local maximum) at $(t,x)\in[0,T]\times\overline{\Omega}$. Finally, we call $v:[0,T]\times\overline{\Omega}\to\mathbb{R}$ a viscosity solution of (2.2) if it is simultaneously a viscosity sub- and supersolution of (2.2).

Here, the value of

$$F^*(t,x,\Delta\psi(t,x),\nabla\psi(t,x),\partial_t\psi(t,x),v_*(t,x))$$

should only depend on ψ's restriction to $[0,T]\times\overline{\Omega}$. However, generally for lower-dimensional faces $F\in\mathcal{F}_{d-2}$, one finds test functions $\psi,\phi\in C^2(\mathbb{R}\times\mathbb{R}^d)$ with $\psi|_{[0,T]\times\overline{\Omega}}=\phi|_{[0,T]\times\overline{\Omega}}$ such that $\nabla\psi(t,x)\neq\nabla\phi(t,x)$ for $x\in F$.

We therefore demand that the coefficient $b^\alpha_{\partial\Omega}(x)$ of (2.1) belongs to the tangent cone:

ASSUMPTION 2.3 For all $\alpha\in A$,

$$b^\alpha_{\partial\Omega}(x)\in K(x)\qquad\forall\alpha\in A, x\in\partial\Omega. \tag{2.4}$$

Here, owing to the polytopic nature of the domain, the tangent cone $K(x)$ is defined as

$$K(x):=\big\{x'\in\mathbb{R}^d\,\big|\,\exists\Lambda\in(0,\infty)\;\forall\lambda\in[0,\Lambda]:x+\lambda x'\in\overline{\Omega}\big\},$$

i.e., x' is in the cone if there is a line segment from x in the direction of x', which is contained in $\overline{\Omega}$.

Indeed, for $b^{\alpha}_{\partial\Omega}(x) \in K(x) \setminus \{0\}$, we observe how

$$- b^{\alpha}_{\partial\Omega}(x) \cdot \nabla\psi(t,x) = \partial_{-b^{\alpha}_{\partial\Omega}(x)}\psi(t,x) = \lim_{\substack{\lambda\to 0\\ \lambda>0}} \frac{\psi(t,x) - \psi(t,x+\lambda\, b^{\alpha}_{\partial\Omega}(x))}{\lambda} \tag{2.5}$$

is expressed only referring to ψ on $[0,T] \times \overline{\Omega}$ and thus independently of ψ's extension to $\mathbb{R} \times \mathbb{R}^d$. The limit on the right-hand side of (2.5) is known as the lower Dini derivative of ψ in direction $-b^{\alpha}_{\partial\Omega}(x)$. On smooth sections of the boundary and at outward pointing corners, the requirement (2.4) corresponds to an outflow condition, while at re-entrant corners, (2.4) may permit an inflow term. In that sense, (2.4) is less restrictive than oblique boundary conditions such as Crandall *et al.* (1992, (7.35)). We remark, however, that strengthened versions such as Crandall *et al.* (1992, (7.35)) may be necessary to ensure the existence of a comparison principle for the final time boundary value problem of the specific application of interest.

3. Numerical scheme

For the discretization of (2.2), we consider a sequence $V_i, i \in \mathbb{N}$, of piecewise linear, simplicial, shape-regular finite element spaces. Let $\mathcal{T}_i$ be the mesh corresponding to the finite element space V_i. The boundary mesh $\mathcal{B}_i$ consists of the $(d-1)$-dimensional faces F of elements $K \in \mathcal{T}_i$ with $F \subset \partial\Omega$. We make the assumption that $\mathcal{B}_i$ is subordinate to $\mathcal{F}$, i.e., every open $F \in \mathcal{B}_i$ is contained entirely in exactly one generalized face $\omega \in \mathcal{F}$.

Let $V_i^g \subset V_i$ be the affine subspace of functions that interpolate the Dirichlet boundary data on $\partial\Omega_D$, and $V_i^0 \subset V_i$ be the vector subspace of functions that interpolate 0 on $\partial\Omega_D$. The nodes of the finite element mesh are denoted by y_i^ℓ. Here, the index ℓ ranges over the nodes in the interior first, then the nodes on $\partial\Omega_t$, then $\partial\Omega_R$ and finally $\partial\Omega_D$. Therefore, $y_i^\ell \in \Omega$ for $\ell \leqslant N_i^{\Omega}$ for some $N_i^{\Omega} \in \mathbb{N}$ denoting number of interior nodes, $y_i^\ell \in \Omega \cup \partial\Omega_t$ for $\ell \leqslant N_i^t$ for some $N_i^t \in \mathbb{N}$ and, lastly, $y_i^\ell \in \Omega \cup \partial\Omega_t \cup \partial\Omega_R$ for $\ell \leqslant N_i := \dim V_i^g$. These nodes $y_i^\ell \in \Omega \cup \partial\Omega_t \cup \partial\Omega_R$ are called nonDirichlet nodes.

The associated hat functions $\phi_i^\ell \in V_i$ are chosen so that $\phi_i^\ell(y_i^\ell) = 1$, while $\phi_i^\ell(y_i^s) = 0$ for $\ell \neq s$. Set $\hat{\phi}_i^\ell := \phi_i^\ell / \|\phi_i^\ell\|_{L^1(\Omega)}$. Therefore, the ϕ_i^ℓ are normalized in the $L^\infty(\overline{\Omega})$ norm, while the $\hat{\phi}_i^\ell$ are normalized in the $L^1(\overline{\Omega})$ norm.

The mesh size, i.e., the largest diameter of an element, is denoted Δx_i. It is assumed that $\Delta x_i \to 0$ as $i \to \infty$. The uniform time step size is denoted h_i with the constraint that $T/h_i \in \mathbb{N}$. It is assumed that $h_i \to 0$ as $i \to \infty$. Let s_i^k be the kth time step at the refinement level i. Then, the set of time steps is $S_i := \{s_i^k : k = T/h_i, \dots, 0\}$.

We introduce the operator d_i, which approximates the time derivative on Ω and $\partial\Omega_t$, but that is 0 on the remaining boundary. More precisely, we let the ℓth entry of $d_i w(s_i^k, \cdot)$ be

$$(d_i w(s_i^k,\cdot))_\ell = \begin{cases} \frac{w(s_i^{k+1},y_i^\ell) - w(s_i^k,y_i^\ell)}{h_i} & \ell \leqslant N_i^t, \\ 0 & \text{otherwise}. \end{cases} \tag{3.1}$$

Observe how $(d_i w(s_i^k,\cdot))_\ell = 0$ for nodes $y_i^\ell \in \partial\Omega_R$ is consistent with the structure of (2.2c).

For each discretization of (2.2), we allow a splitting of L^α and $L^\alpha_{\partial\Omega}$ into an explicit and an implicit part. For each α and for each i, we introduce the explicit operator $E^\alpha_{\Omega,i}$ and the implicit operator $I^\alpha_{\Omega,i}$

such that

$$E^\alpha_{\Omega,i}: \; C^2(\overline{\Omega}) \to C(\overline{\Omega}), \; w \mapsto -\bar{a}^\alpha_{\Omega,i}\,\Delta w - \bar{b}^\alpha_{\Omega,i} \cdot \nabla w + \bar{c}^\alpha_{\Omega,i}\, w,$$

$$I^\alpha_{\Omega,i}: \; C^2(\overline{\Omega}) \to C(\overline{\Omega}), \; w \mapsto -\bar{\bar{a}}^\alpha_{\Omega,i}\,\Delta w - \bar{\bar{b}}^\alpha_{\Omega,i} \cdot \nabla w + \bar{\bar{c}}^\alpha_{\Omega,i}\, w,$$

where $\bar{a}^\alpha_{\Omega,i}, \bar{\bar{a}}^\alpha_{\Omega,i}, \bar{c}^\alpha_{\Omega,i}, \bar{\bar{c}}^\alpha_{\Omega,i} \in C(\overline{\Omega})$ and $\bar{b}^\alpha_{\Omega,i}, \bar{\bar{b}}^\alpha_{\Omega,i} \in C(\overline{\Omega}, \mathbb{R}^d)$. Assumption 3.1 below shows that the explicit and implicit operators are chosen such that $I^\alpha_{\Omega,i} + E^\alpha_{\Omega,i}$ approximates L^α. Analogously, we introduce non-negative $f^\alpha_i \in C(\overline{\Omega})$, which approximate f^α.

The discretizations of $E^\alpha_{\Omega,i}$ and $I^\alpha_{\Omega,i}$ are the mappings from V_i to $\mathbb{R}^{N_i}$, which are given by

$$(\mathsf{E}^\alpha_{\Omega,i} w)_\ell := \bar{a}^\alpha_{\Omega,i}(y^\ell_i)\langle \nabla w, \nabla \hat{\phi}^\ell_i \rangle + \langle -\bar{b}^\alpha_{\Omega,i} \cdot \nabla w + \bar{c}^\alpha_{\Omega,i}\, w, \hat{\phi}^\ell_i \rangle, \tag{3.2a}$$

$$(\mathsf{I}^\alpha_{\Omega,i} w)_\ell := \bar{\bar{a}}^\alpha_{\Omega,i}(y^\ell_i)\langle \nabla w, \nabla \hat{\phi}^\ell_i \rangle + \langle -\bar{\bar{b}}^\alpha_{\Omega,i} \cdot \nabla w + \bar{\bar{c}}^\alpha_{\Omega,i} w, \hat{\phi}^\ell_i \rangle, \tag{3.2b}$$

$$(\mathsf{F}^\alpha_{\Omega,i})_\ell := \langle f^\alpha_i, \hat{\phi}^\ell_i \rangle, \tag{3.2c}$$

where ℓ ranges over all internal nodes, i.e., $\ell \leqslant N^\Omega_i$. For boundary nodes $\ell > N^\Omega_i$, we set $(\mathsf{E}^\alpha_{\Omega,i} w)_\ell = (\mathsf{I}^\alpha_{\Omega,i} w)_\ell = (\mathsf{F}^\alpha_{\Omega,i})_\ell = 0$. Because of the scaling of the $\hat{\phi}^\ell_i$, integration-by-parts gives for smooth w and large i that $\langle \nabla w, \nabla \hat{\phi}^\ell_i \rangle \approx -\Delta w(x)$ if $x \approx y^\ell_i$ away from the boundary.

Similarly, we define operators $\mathsf{E}^\alpha_{\partial\Omega,i}$ and $\mathsf{I}^\alpha_{\partial\Omega,i}$ on the boundary to discretize $L^\alpha_{\partial\Omega}$ as the sum of an explicit and implicit part. Starting point is the observation that the directional derivative $\partial_{-b^\alpha_{\partial\Omega}} w$ is well-defined in the sense of (2.5) for functions $w \in V_i$, even though w is in general not differentiable.

For interior nodes with index $0 \leqslant \ell \leqslant N^\Omega_i$, we set $(\mathsf{E}^\alpha_{\partial\Omega,i} w)_\ell = (\mathsf{I}^\alpha_{\partial\Omega,i} w)_\ell = (\mathsf{F}^\alpha_{\partial\Omega,i})_\ell = 0$. More interestingly, for $N^\Omega_i < \ell \leqslant N_i$ ranging over the nodes of the Robin boundary conditions, we define the mappings from V_i to $\mathbb{R}^{N_i}$ by

$$(\mathsf{E}^\alpha_{\partial\Omega,i} w)_\ell := \partial_{-\bar{b}^\alpha_{\partial\Omega,i}(y^\ell_i)} w(y^\ell_i) + \bar{c}^\alpha_{\partial\Omega,i}(y^\ell_i)\, w(y^\ell_i), \tag{3.3a}$$

$$(\mathsf{I}^\alpha_{\partial\Omega,i} w)_\ell := \partial_{-\bar{\bar{b}}^\alpha_{\partial\Omega,i}(y^\ell_i)} w(y^\ell_i) + \bar{\bar{c}}^\alpha_{\partial\Omega,i}(y^\ell_i)\, w(y^\ell_i), \tag{3.3b}$$

$$(\mathsf{F}^\alpha_{\partial\Omega,i})_\ell := g^\alpha_i(y^\ell_i), \tag{3.3c}$$

where $\bar{c}^\alpha_{\partial\Omega,i}, \bar{\bar{c}}^\alpha_{\partial\Omega,i}, g^\alpha_i \in PC(\partial\Omega)$ and $\bar{b}^\alpha_{\partial\Omega,i}, \bar{\bar{b}}^\alpha_{\partial\Omega,i} \in PC(\partial\Omega, \mathbb{R}^d)$. Here, $\partial_{-\bar{b}^\alpha_{\partial\Omega,i}}$ and $\partial_{-\bar{\bar{b}}^\alpha_{\partial\Omega,i}}$ are understood as lower Dini derivatives as in (2.5).

On the Dirichlet boundary, the mappings $\mathsf{E}^\alpha_{\partial\Omega,i}$, $\mathsf{I}^\alpha_{\partial\Omega,i}$ and $\mathsf{F}^\alpha_{\partial\Omega,i}$ implement nodal interpolation. For $\ell > N_i$, we set

$$(\mathsf{E}^\alpha_{\partial\Omega,i} w)_\ell := 0, \tag{3.4a}$$

$$(\mathsf{I}^\alpha_{\partial\Omega,i} w)_\ell := w(y^\ell_i), \tag{3.4b}$$

$$(\mathsf{F}^\alpha_{\partial\Omega,i})_\ell := g(y^\ell_i). \tag{3.4c}$$

We assume a fully implicit discretization of the region $\partial\Omega_R$; additionally, suppose that $\bar{\bar{c}}^{\alpha}_{\partial\Omega,i}$ is chosen positive on $\partial\Omega_R$, even if $c^{\alpha}_{\partial\Omega} = 0$.

In summary, we require that the following assumption holds.

Assumption 3.1 (a) The coefficients satisfy

$$\lim_{i\to\infty} \sup_{\alpha\in A}\Big(\sup_{0\leqslant \ell\leqslant N_i} \big\| a^{\alpha} - \big(\bar{a}^{\alpha}_{\Omega,i}(y^{\ell}_i) + \bar{\bar{a}}^{\alpha}_{\Omega,i}(y^{\ell}_i)\big)\big\|_{L^{\infty}(\operatorname{supp}\hat{\phi}^{\ell}_i)} + \big\| b^{\alpha} - \big(\bar{b}^{\alpha}_{\Omega,i} + \bar{\bar{b}}^{\alpha}_{\Omega,i}\big)\big\|_{L^{\infty}(\Omega,\mathbb{R}^d)} + \big\| c^{\alpha} - \big(\bar{c}^{\alpha}_{\Omega,i} + \bar{\bar{c}}^{\alpha}_{\Omega,i}\big)\big\|_{L^{\infty}(\Omega)} + \big\| f^{\alpha} - f^{\alpha}_i\big\|_{L^{\infty}(\Omega)}\Big) = 0$$

and

$$\lim_{i\to\infty} \sup_{\alpha\in A}\Big(\big\| b^{\alpha}_{\partial\Omega} - \big(\bar{b}^{\alpha}_{\partial\Omega,i} + \bar{\bar{b}}^{\alpha}_{\partial\Omega,i}\big)\big\|_{L^{\infty}(\partial\Omega)} + \big\| c^{\alpha}_{\partial\Omega} - \big(\bar{c}^{\alpha}_{\partial\Omega,i} + \bar{\bar{c}}^{\alpha}_{\partial\Omega,i}\big)\big\|_{L^{\infty}(\partial\Omega)} + \big\| f^{\alpha} - f^{\alpha}_i\big\|_{L^{\infty}(\partial\Omega)} + \big\| g - g_i\big\|_{L^{\infty}(\partial\Omega)} + \big\| g^{\alpha} - g^{\alpha}_i\big\|_{L^{\infty}(\partial\Omega)}\Big) = 0.$$

(b) We require that the family

$$\{(\bar{a}^{\alpha}_{\Omega,i}, \bar{b}^{\alpha}_{\Omega,i}, \bar{c}^{\alpha}_{\Omega,i}, \bar{b}^{\alpha}_{\partial\Omega,i}, \bar{c}^{\alpha}_{\partial\Omega,i}, \bar{\bar{a}}^{\alpha}_{\Omega,i}, \bar{\bar{b}}^{\alpha}_{\Omega,i}, \bar{\bar{c}}^{\alpha}_{\Omega,i}, \bar{\bar{b}}^{\alpha}_{\partial\Omega,i}, \bar{\bar{c}}^{\alpha}_{\partial\Omega,i}, f^{\alpha}_i, g^{\alpha}_i)\}_{\alpha\in A}$$

is equicontinuous and depends continuously on α. We impose $\bar{a}^{\alpha}_{\Omega,i} = \bar{c}^{\alpha}_{\Omega,i} = \bar{c}^{\alpha}_{\partial\Omega,i} = 0 \in \mathbb{R}$ and $\bar{b}^{\alpha}_{\Omega,i} = \bar{b}^{\alpha}_{\partial\Omega,i} = 0 \in \mathbb{R}^d$, as well as $\bar{\bar{c}}^{\alpha}_{\partial\Omega,i} > 0$ on the restriction to $\partial\Omega_R$, $i \in \mathbb{N}$.

Part (a) of the assumption is a consistency statement: in the limit the coefficients of the numerical scheme should match those of the boundary value problem. The continuity condition of (b) is needed to prove the existence of numerical solutions through Howard's algorithm. The structure assumptions on the coefficients on $\partial\Omega_R$ are needed to establish monotonicity of the numerical scheme, accounting for the fact that there is no time derivative in (2.2c).

We define

$$\mathsf{E}^{\alpha}_i = \mathsf{E}^{\alpha}_{\Omega,i} + \mathsf{E}^{\alpha}_{\partial\Omega,i}, \quad \mathsf{I}^{\alpha}_i = \mathsf{I}^{\alpha}_{\Omega,i} + \mathsf{I}^{\alpha}_{\partial\Omega,i}, \quad \mathsf{F}^{\alpha}_i = \mathsf{F}^{\alpha}_{\Omega,i} + \mathsf{F}^{\alpha}_{\partial\Omega,i}.$$

We also use the notation I^{α}_i, E^{α}_i and F^{α}_i for the matrix representations of exactly these I^{α}_i, E^{α}_i and F^{α}_i with respect to the nodal basis $\{\phi^{\ell}_i\}_{\ell}$ for the trial functions. Moreover, we assume that the supremum operator is applied componentwise, i.e., $(\sup_{\alpha} v^{\alpha})_{\ell} = \sup_{\alpha} v^{\alpha}_{\ell}$ for $v \in \mathbb{R}^n$. The expression $a \lesssim b$ means that there exists a generic constant $C > 0$, independent of i and α, such that $a \leqslant Cb$. Relation $a \gtrsim b$ is defined analogously.

We can now state the numerical scheme used to approximate the solution of (2.2). We initialize the scheme by the nodal interpolation of v_T so that $v_i(T,\cdot) \in V^g_i$. Then, in order to find the numerical solution $v_i(s_i^k,\cdot) \in V_i^g$, we proceed inductively over the remaining timesteps $k \in \{T/h_i - 1, \ldots, 1, 0\}$:

$$-\frac{w(s_i^{k+1}, y_i^\ell) - w(s_i^k, y_i^\ell)}{h_i} + \sup_{\alpha\in A}\big(\mathsf{E}_i^\alpha v_i(s_i^{k+1},\cdot) + \mathsf{I}_i^\alpha v_i(s_i^k,\cdot) - \mathsf{F}_i^\alpha\big) = 0 \qquad \ell \leqslant N_i^t, \tag{3.5a}$$

$$\sup_{\alpha\in A}\big(\mathsf{E}_i^\alpha v_i(s_i^{k+1},\cdot) + \mathsf{I}_i^\alpha v_i(s_i^k,\cdot) - \mathsf{F}_i^\alpha\big) = 0 \qquad \text{otherwise.} \tag{3.5b}$$

Here (3.5a) corresponds to (2.2a) and (2.2b) while (3.5b) to (2.2c). This can be written more compactly with the d_i operator of (3.1) as

$$-d_i v_i(s_i^k,\cdot) + \sup_{\alpha\in A}\big(\mathsf{E}_i^\alpha v_i(s_i^{k+1},\cdot) + \mathsf{I}_i^\alpha v_i(s_i^k,\cdot) - \mathsf{F}_i^\alpha\big) = 0. \tag{3.6}$$

We also use an alternative formulation of the numerical scheme. The matrices $\mathsf{E}_i^{k,w}$, $\mathsf{I}_i^{k,w}$ and $\mathsf{F}_i^{k,w}$ are constructed row-wise out of the matrices E_i^α, I_i^α and F_i^α. More precisely, given a node y_i^ℓ, timestep s_i^k and a function $w(s_i^k,\cdot) \in H^1(\overline{\Omega})$, let $\hat\alpha$ be a maximizer of

$$\sup_{\alpha\in A}\big(\mathsf{E}_i^\alpha w(s_i^{k+1},\cdot) + \mathsf{I}_i^\alpha w(s_i^k,\cdot) - \mathsf{F}_i^\alpha\big)_\ell = 0.$$

Note that choice of $\hat\alpha$ is not necessarily unique; the analysis is valid for any choice of such $\hat\alpha$. We let the ℓth row of $\mathsf{E}_i^{k,w}$, $\mathsf{I}_i^{k,w}$ and $\mathsf{F}_i^{k,w}$ be equal to the ℓth row of $\mathsf{E}_i^{\hat\alpha}$, $\mathsf{I}_i^{\hat\alpha}$ and $\mathsf{F}_i^{\hat\alpha}$, respectively. In a nonambiguous case, we will omit explicit mention of k and simply write E_i^w, I_i^w and F_i^w. We can now reformulate (3.6) using the newly constructed operators. We initialize the scheme with the interpolant $v_i(T,\cdot)$. Then $v_i \in V_i^g$ for each $k \in \{T/h_i - 1, \ldots, 1, 0\}$ and for $0 \leqslant \ell \leqslant N_i^t$ solves

$$\Big((h_i\mathsf{I}_i^{k,v_i} + \mathsf{Id})\, v_i(s_i^k,\cdot) + (h_i\mathsf{E}_i^{k,v_i} - \mathsf{Id})\, v_i(s_i^{k+1},\cdot) - h_i\mathsf{F}_i^{k,v_i}\Big)_\ell = 0, \tag{3.7a}$$

and for each $k \in \{T/h_i - 1, \ldots, 1, 0\}$ and for $N_i^t < \ell$ solves

$$\Big(h_i\mathsf{I}_i^{k,v_i}\, v_i(s_i^k,\cdot) - h_i\mathsf{F}_i^{k,v_i}\Big)_\ell = 0, \tag{3.7b}$$

recalling the implicit discretization on $\partial\Omega_R \cup \partial\Omega_D$, enforced through (3.4) and Assumption 3.1.

For the sake of convenience, let us also introduce the operators $\hat{\mathsf{I}}_i^{k,v_i}$, $\hat{\mathsf{E}}_i^{k,v_i}$ and $\hat{\mathsf{F}}_i$, which combine spatial and temporal terms. The ℓth row of $\hat{\mathsf{I}}_i^{k,v_i}$, $\hat{\mathsf{E}}_i^{k,v_i}$ and $\hat{\mathsf{F}}_i$ is equal to that of $(h_i\mathsf{I}_i^{k,v_i} + \mathsf{Id})$, $(h_i\mathsf{E}_i^{k,v_i} - \mathsf{Id})$ and $h_i\mathsf{F}_i^{k,v_i}$, respectively, if $\leqslant \ell \leqslant N_i^t$. If $N_i^t < \ell$, the ℓth row is equal to $(h_i\mathsf{I}_i^{k,v_i})$, a zero vector and $h_i\mathsf{F}_i^{k,v_i}$, respectively. For a fixed control α, the operators $\hat{\mathsf{I}}_i^\alpha$, $\hat{\mathsf{E}}_i^\alpha$ and $\hat{\mathsf{F}}_i^\alpha$ are constructed in an analogous manner. Then for all timesteps s_i^k and each node y_i^ℓ solution v_i of (3.7) solves also:

$$\hat{\mathsf{I}}_i^{k,v_i}\, v_i(s_i^k, y_i^\ell) + \hat{\mathsf{E}}_i^{k,v_i}\, v_i(s_i^{k+1}, y_i^\ell) - \hat{\mathsf{F}}_i^{k,v_i} = 0. \tag{3.8}$$

REMARK 3.2 (a) To implement the lower Dini derivative $\partial_{-\bar{b}^{\alpha}_{\partial\Omega,i}(y_i^{\ell})} w(y_i^{\ell})$ in a computer code, we note that for $\lambda > 0$ sufficiently small there is an element $K \in \mathcal{T}_i$ whose closure contains both y_i^{ℓ} and $y_i^{\ell} + \lambda\, \bar{b}^{\alpha}_{\partial\Omega,i}(y_i^{\ell})$. Indeed, choosing λ such that $\lambda\, \bar{b}^{\alpha}_{\partial\Omega,i}(y_i^{\ell})$ is smaller than the smallest element edge diameter for all $N_i^{\Omega} < \ell \leqslant N_i$ achieves this. We then have

$$\partial_{-\bar{b}^{\alpha}_{\partial\Omega,i}(y_i^{\ell})} w(y_i^{\ell}) = \frac{w(y_i^{\ell}) - w(y_i^{\ell} + \lambda\, \bar{b}^{\alpha}_{\partial\Omega,i}(y_i^{\ell}))}{\lambda}.$$

Importantly, because $w \in V_i$ is affine on $\overline{K}$, even without taking a limit $\lambda \to 0$ as on the right-hand side of (2.5) the Dini derivative is obtained exactly.
(b) In Section 5, a practical algorithm is shown to compute maximizing controls $\hat{\alpha}$.

4. Monotonicity

In this section, we consider monotonicity properties of the discrete differential operators defined in the previous section. Monotonicity is crucial for proving the existence of a unique numerical solution of (3.6), as well as for establishing convergence to the viscosity solution.

DEFINITION 4.1 Let us consider $v \in V_i$ that has a local nonpositive minimum at a node y_i^{ℓ}. We say that an operator F satisfies a local monotonicity property (LMP) if for any such v, it follows that $(Fv)_{\ell} \leqslant 0$.

Additionally, the operator F satisfies the weak discrete maximum principle (wDMP) provided that, for any $v \in V_i$,

$$\big(Fv\big)_{\ell} \geqslant 0 \;\; \forall\, \ell \in \{1, \dots, N_i\} \quad \Longrightarrow \quad \min_{\Omega \cup \partial\Omega_R \cup \partial\Omega_t} v \geqslant \min\{\min_{\partial\Omega_D} v, 0\}. \tag{4.1}$$

The definition is a generalization of the wDMP used in Jensen & Smears (2013) to accommodate in (4.1) for Robin boundary conditions.

We now describe a method for choosing the artificial diffusion coefficients to impose the LMP on the matrices I_i^{α} and E_i^{α}. It is based on the assumption of strict acuteness on the mesh. Consider an element $K \in \mathcal{T}_i$ with diameter Δx_K. For a bounded function $g : \overline{\Omega} \to \mathbb{R}^d$, we define g's norm on the restriction to K as

$$|g|_K := \Big(\sum_{j=1}^{d} \big\| g_j \big\|^2_{L^{\infty}(K)}\Big)^{\frac{1}{2}}.$$

DEFINITION 4.2 Then, by strict acuteness of the meshes, we mean that there exists a $\theta \in (0, \frac{\pi}{2})$ such that the following holds:

$$\nabla\phi_i^{\ell} \cdot \nabla\phi_i^{m}\big|_K \leqslant -\sin(\theta)\, |\nabla\phi_i^{\ell}|_K\, |\nabla\phi_i^{m}|_K \qquad \forall \ell, m \leqslant N_i,\; \ell \neq m,\; \forall K \in \mathcal{T}_i. \tag{4.2}$$

We say that the family of meshes $\{\mathcal{T}_i\}_i$ is uniformly strictly acute if θ does not depend on i.

As discussed in Burman & Ern (2002), for $d = 2$ and $d = 3$, the angle θ can be interpreted geometrically as $\frac{\pi}{2}$ minus the largest angle between the pairs of $(d-1)$-dimensional faces of the element K.

4.1 *The LMP of* $\mathsf{E}^\alpha_{\Omega,i}$, $\mathsf{E}^\alpha_{\partial\Omega,i}$, $\mathsf{E}^\alpha_{\Omega,i}$ *and* $\mathsf{I}^\alpha_{\partial\Omega,i}$

Let the functions $\tilde{a}^\alpha_{\Omega,i}, \tilde{\tilde{a}}^\alpha_{\Omega,i}, \bar{c}^\alpha_{\Omega,i}, \bar{\bar{c}}^\alpha_{\Omega,i} \in C(\overline{\Omega})$ and $\bar{b}^\alpha_{\Omega,i}, \bar{\bar{b}}^\alpha_{\Omega,i} \in C(\overline{\Omega}, \mathbb{R}^d)$ be given. These functions may be chosen freely as long as Assumption 3.1 holds. Conceptually, $\tilde{a}^\alpha_{\Omega,i} + \tilde{\tilde{a}}^\alpha_{\Omega,i} \approx a^\alpha$ is the splitting of the second-order coefficients into explicit and implicit part *without* the addition of artificial diffusion. With the addition of artificial diffusion, the coefficients $\bar{a}^\alpha_i$ and $\bar{\bar{a}}^\alpha_i$ of Assumption 3.1 are obtained.

Indeed, as $\bar{b}^\alpha_{\Omega,i}, \bar{\bar{b}}^\alpha_{\Omega,i}, \bar{c}^\alpha_{\Omega,i}, \bar{\bar{c}}^\alpha_{\Omega,i}$ are bounded, we can select non-negative artificial diffusion coefficients $\bar{\nu}^{\alpha,\ell}_{\Omega,i}$ and $\bar{\bar{\nu}}^{\alpha,\ell}_{\Omega,i}$, so that we have for all interior nodes y^ℓ_i and mesh elements K with y^ℓ_i as vertex that

$$\begin{aligned} |\bar{b}^\alpha_{\Omega,i}|_K + \Delta x_K \|\bar{c}^\alpha_{\Omega,i}\|_{L^\infty(K)} &\leqslant \bar{\nu}^{\alpha,\ell}_{\Omega,i} \sin(\theta)\, |\nabla\hat{\phi}^\ell_i|_K \operatorname{vol}(K), \\ |\bar{\bar{b}}^\alpha_{\Omega,i}|_K + \Delta x_K \|\bar{\bar{c}}^\alpha_{\Omega,i}\|_{L^\infty(K)} &\leqslant \bar{\bar{\nu}}^{\alpha,\ell}_{\Omega,i} \sin(\theta)\, |\nabla\hat{\phi}^\ell_i|_K \operatorname{vol}(K). \end{aligned} \tag{4.3}$$

Now, choosing $\bar{a}^\alpha_{\Omega,i}, \bar{\bar{a}}^\alpha_{\Omega,i} \in C(\overline{\Omega})$ such that

$$\bar{a}^\alpha_{\Omega,i}(y^\ell_i) \geqslant \max\{\tilde{a}^\alpha_{\Omega,i}(y^\ell_i), \bar{\nu}^{\alpha,\ell}_{\Omega,i}\}, \qquad \bar{\bar{a}}^\alpha_{\Omega,i}(y^\ell_i) \geqslant \max\{\tilde{\tilde{a}}^\alpha_{\Omega,i}(y^\ell_i), \bar{\bar{\nu}}^{\alpha,\ell}_{\Omega,i}\}, \tag{4.4}$$

we obtain our splitting of L^α into implicit and explicit part.

Lemma 4.3 Suppose that the mesh $\mathcal{T}_i$ is strictly acute and that (4.3) holds. Then, $\mathsf{E}^\alpha_{\Omega,i}$ and $\mathsf{I}^\alpha_{\Omega,i}$ satisfy the LMP for all α.

Proof. The argument from Jensen & Smears (2013, Section 8) for the Dirichlet problem carries over unchanged for local minima at interior nodes of v from Definition 4.1. At boundary nodes, the LMP is trivially satisfied as $\mathsf{E}^\alpha_{\Omega,i}$ and $\mathsf{I}^\alpha_{\Omega,i}$ vanish there. □

We now turn to the monotonicity of the discrete boundary operators.

Lemma 4.4 The operators $\mathsf{E}^\alpha_{\partial\Omega,i}$ and $\mathsf{I}^\alpha_{\partial\Omega,i}$ satisfy the LMP for all α.

Proof. Let $w \in V_i$ have a local nonpositive minimum at a node $y^\ell_i \in \partial\Omega_t \cup \partial\Omega_R$. Then, we find for the lower Dini derivative $\partial_{-\bar{b}^\alpha_{\partial\Omega,i}(y^\ell_i)} w(y^\ell_i) \leqslant 0$. Also, $\bar{c}^\alpha_{\partial\Omega,i}(y^\ell_i)\, w(y^\ell_i) \leqslant 0$ because $c(y^\ell_i) \geqslant 0$. Hence, $\mathsf{E}^\alpha_{\partial\Omega,i}$ admits the LMP. The argument for $\mathsf{I}^\alpha_{\partial\Omega,i}$ is analogous. □

4.2 *Monotonicity properties of the* $\mathsf{E}^{k,w}_i$, $\hat{\mathsf{E}}^{k,w}_i$, $\mathsf{I}^{k,w}_i$ *and* $\hat{\mathsf{I}}^{k,w}_i$

Having examined the basic building blocks of the numerical scheme in the previous two subsections, we can now analyse the monotonicity properties of the derived operators $\hat{\mathsf{E}}^{k,w}_i$ and $\hat{\mathsf{I}}^{k,w}_i$ as they appear in formulation (3.8) of the scheme. We summarize the assumptions made so far in the selection of the artificial diffusion coefficients.

Assumption 4.5 Suppose that $\mathcal{T}_i$ is strictly acute and that (4.3) and (4.4) hold.

First, we examine the explicit terms.

Lemma 4.6 Consider a fixed $w : S_i \times \overline{\Omega} \to \mathbb{R}$ such that $w(s^k_i, \cdot) \in V_i$ for all $s^k_i \in S_i$. Then the operators $v \mapsto \mathsf{E}^{k,w}_i v$ satisfy the LMP and their matrix has nonpositive off-diagonal entries. For h_i small enough, $\hat{\mathsf{E}}^{k,w}_i$ is monotone, i.e., all entries of the matrix representation are nonpositive.

Proof. For any i and α, $\mathsf{E}_i^\alpha = \mathsf{E}_{\Omega,i}^\alpha + \mathsf{E}_{\partial\Omega,i}^\alpha$ satisfies the LMP because its summands do. Let us consider a $v \in V_i$ that has a local nonpositive minimum at a node y_i^ℓ. There is an $\alpha \in A$ such that $(\mathsf{E}_i^{k,w} v)_\ell = (\mathsf{E}_i^\alpha v)_\ell$. We know $(\mathsf{E}_i^\alpha v)_\ell \leqslant 0$ and therefore that $v \mapsto \mathsf{E}_i^{k,w} v$ satisfies the LMP.

For $j \neq \ell$, the hat function ϕ_i^j attains a nonpositive minimum at y_i^ℓ. Thus, by the LMP, we have that $(\mathsf{E}_i^{k,w} \phi_i^j)_\ell \leqslant 0$. Hence, all the off-diagonal entries of $\mathsf{E}_i^{k,w}$ are nonpositive.

Owing to Assumption 1, the discretization on $\partial\Omega_R$ is fully implicit. Thus the rows of $\mathsf{E}_i^{k,w}$ belonging to the discretization on $\partial\Omega_R$ contain only zeros. Similarly, the rows linked to $\partial\Omega_D$ vanish, see (3.4). All other rows include a term arising from the time derivative; their structure is $(h_i \mathsf{E}_i^{k,v_i} - \mathsf{Id})$. Therefore, if h_i is sufficiently small, then $\hat{\mathsf{E}}_i^{k,w}$ is monotone. □

Now we turn to the implicit terms.

LEMMA 4.7 Consider a fixed $w : S_i \times \overline{\Omega} \to \mathbb{R}$ such that $w(s_i^k, \cdot) \in V_i$ for all $s_i^k \in S_i$. Then the operators $v \mapsto \mathsf{I}_i^{k,w} v$ satisfy the LMP. Moreover, the $v \mapsto \hat{\mathsf{I}}_i^{k,w} v$ satisfy the wDMP and their matrix representation restricted to V_i are strictly diagonally dominant M-matrices.

Proof. Analogously to the proof of Lemma 4.6, the $v \mapsto \mathsf{I}_i^{k,w} v$ satisfy the LMP and their off-diagonal entries are nonpositive.

Before showing the wDMP, we verify strict diagonal dominance. By construction, $v \equiv -1$ attains a nonpositive local minimum at each node. Since $\mathsf{I}_i^{k,w}$ satisfies the LMP property, we have

$$0 \geqslant \left(\mathsf{I}_i^{k,w} v\right)_\ell = -\left(\mathsf{I}_i^{k,w}\right)_{\ell\ell} - \sum_{j\neq\ell} \left(\mathsf{I}_i^{k,w}\right)_{\ell j}. \tag{4.5}$$

As the off-diagonal entries of $\mathsf{I}_i^{k,w}$ are nonpositive, we conclude the weak diagonal dominance of the $\mathsf{I}_i^{k,w}$:

$$\left(\mathsf{I}_i^{k,w}\right)_{\ell\ell} - \sum_{j\neq\ell} \left| \left(\mathsf{I}_i^{k,w}\right)_{\ell j} \right| \geqslant 0.$$

The rows of $\hat{\mathsf{I}}_i^{k,w}$ that discretize on Ω and $\partial\Omega_t$ are equal to the respective rows of the strictly diagonally dominant matrix $h_i \mathsf{I}_i^{k,w} + \mathsf{Id}$.

By Assumption 3.1, we have $\bar{\bar{c}}_{\partial\Omega,i}^\alpha > 0$ on $\partial\Omega_R$. Then

$$0 > \left(\mathsf{I}_i^{k,w} v\right)_\ell = \left(\hat{\mathsf{I}}_i^{k,w} v\right)_\ell \quad \forall\, y_i^\ell \in \partial\Omega_R. \tag{4.6}$$

Using the same argument as above, but noting the strict inequality of (4.6) compared to (4.5), we conclude the strict diagonal dominance for rows linked to $\partial\Omega_R$. On $\partial\Omega_D$ the rows resemble an identity matrix, giving also strict diagonal dominance. It follows that the $\hat{\mathsf{I}}_i^{k,w}$ are invertible M-matrices because Berman & Plemmons (1994, Chapter 6, Theorem 2.3, (M_{35})) applies as $\hat{\mathsf{I}}_i^{k,w}$ is a Z-matrix.

Finally, consider a $v \in V_i$ with $\min_{\Omega\cup\partial\Omega_t\cup\partial\Omega_t} v < \min\{\min_{\partial\Omega_D} v, 0\}$. Let y_i^ℓ be a nonDirichlet node, where the negative, global minimum of v is attained. Since $\hat{\mathsf{I}}_i^{k,w}$ is a strictly diagonal dominant M-matrix, it follows that $(\hat{\mathsf{I}}_i^{k,w} v)_\ell < 0$. Hence, $\hat{\mathsf{I}}_i^{k,w}$ admits the wDMP. □

4.3 *Scaling of the artificial diffusion coefficients*

The purpose of this section is to investigate time-step restrictions that can arise because monotonicity of the scheme is required.

In order to achieve convergence of the numerical scheme, we expect the artificial diffusion coefficients $\bar{\nu}^{\alpha,\ell}_{\Omega,i}$, $\bar{\bar{\nu}}^{\alpha,\ell}_{\Omega,i}$ to vanish in the limit $i \to \infty$.

Ideally, from the point of view of consistency, one chooses the artificial diffusion coefficients as small as possible, i.e., such that (4.3) holds with equality. However, generally, in implementations of the algorithm quasi-optimally is more easily achieved than optimality, meaning

$$\bar{\nu}^{\alpha,\ell}_{\Omega,i} \lesssim \sup\Big\{ \frac{|\bar{b}^{\alpha}_{\Omega,i}|_K + \Delta x_K \|\bar{c}^{\alpha}_{\Omega,i}\|_{L^\infty(K)}}{\sin(\theta)\, |\nabla \hat{\phi}^{\ell}_i|_K \,\mathrm{vol}(K)} \;\Big|\; K \subset \mathrm{supp}\; \phi^{\ell}_i \Big\}, \tag{4.7}$$

$$\bar{\bar{\nu}}^{\alpha,\ell}_{\Omega,i} \lesssim \sup\Big\{ \frac{|\bar{\bar{b}}^{\alpha}_{\Omega,i}|_K + \Delta x_K \|\bar{\bar{c}}^{\alpha}_{\Omega,i}\|_{L^\infty(K)}}{\sin(\theta)\, |\nabla \hat{\phi}^{\ell}_i|_K \,\mathrm{vol}(K)} \;\Big|\; K \subset \mathrm{supp}\; \phi^{\ell}_i \Big\}. \tag{4.8}$$

For this reason, we suppose in the remainder of this subsection that the $\bar{\nu}^{\alpha,\ell}_{\Omega,i}$ are chosen quasi-optimally with regard to (4.3). We also suppose that (4.2) holds uniformly for some θ. Because of shape-regularity of the domain, one has $|\nabla \hat{\phi}^{\ell}_i|_K \,\mathrm{vol}(K) \gtrsim \frac{1}{\Delta x_K}$. We conclude that quasi-optimal artificial diffusion coefficients satisfy

$$\mathsf{O}(\bar{\nu}^{\alpha,\ell}_{\Omega,i}) = \mathsf{O}(\bar{\bar{\nu}}^{\alpha,\ell}_{\Omega,i}) = \Delta x_K. \tag{4.9}$$

We now turn our attention to the time step restrictions imposed through the quasi-optimality (4.7). Recall that, in order for the explicit operators to be monotone, we require all their entries in matrix representation to be nonpositive. This is satisfied trivially for nodes on $\partial\Omega_R \cup \partial\Omega_D$ where we use a fully implicit scheme. Therefore, let us consider nonpositivity of the diagonal terms of $h_i \mathsf{E}^{\alpha}_i - \mathsf{Id}$ on the complement $\Omega \cup \partial\Omega_t$. For $y^{\ell}_i \in \Omega$, this translates into the condition

$$1 \geqslant h_i \left(\bar{a}^{\alpha}_{\Omega,i}(y^{\ell}_i) \langle \nabla \phi^{\ell}_i, \nabla \hat{\phi}^{\ell}_i \rangle + \langle -\bar{b}^{\alpha}_{\Omega,i} \cdot \nabla \phi^{\ell}_i + \bar{c}^{\alpha}_{\Omega,i}\, \phi^{\ell}_i, \hat{\phi}^{\ell}_i \rangle \right)$$

and for $y^{\ell}_i \in \partial\Omega_t$

$$1 \geqslant h_i \left(\partial_{-\bar{b}^{\alpha}_{\partial\Omega,i}(y^{\ell}_i)} \phi^{\ell}_i(y^{\ell}_i) + \bar{c}^{\alpha}_{\partial\Omega,i}(y^{\ell}_i) \right).$$

Because

$$\begin{aligned} \langle \nabla \phi^{\ell}_i, \nabla \hat{\phi}^{\ell}_i \rangle &= \mathsf{O}\big((\Delta x_K)^{-2}\big), \\ \langle \nabla \phi^{\ell}_i, \hat{\phi}^{\ell}_i \rangle &= \mathsf{O}\big((\Delta x_K)^{-1}\big), \\ \langle \phi^{\ell}_i, \hat{\phi}^{\ell}_i \rangle &= \mathsf{O}(1), \\ \partial_{-\bar{b}^{\alpha}_{\partial\Omega,i}(y^{\ell}_i)} \phi^{\ell}_i(y^{\ell}_i) &= \mathsf{O}\big((\Delta x_K)^{-1}\big), \end{aligned}$$

we find $h_i = \mathsf{O}((\Delta x_K)^2)$ if $\|\bar{a}^{\alpha}_{\Omega,i}\|_\infty > 0$. Otherwise, if $\|\bar{b}^{\alpha}_{\Omega,i}\|_\infty > 0$ or $\|\bar{b}^{\alpha}_{\partial\Omega,i}\|_\infty > 0$, we have $h_i = \mathsf{O}(\Delta x_K)$, and if $\bar{a}^{\alpha}_{\Omega,i}$, $\bar{b}^{\alpha}_{\partial\Omega,i}$ and $\bar{b}^{\alpha}_{\Omega,i}$ vanish, but not $\bar{c}^{\alpha}_{\Omega,i}$ or $\bar{c}^{\alpha}_{\partial\Omega,i}$, then $h_i = \mathsf{O}(1)$. If also $\bar{c}^{\alpha}_{\Omega,i} = \bar{c}^{\alpha}_{\partial\Omega,i} = 0$, then there is no restriction on h_i, i.e., fully implicit discretizations are monotone for any $h_i > 0$.

5. Existence of numerical solutions

The discrete nonlinear problem (3.8) can be solved by a version of Howard's algorithm discussed in Bokanowski *et al.* (2009, Algorithm 1). We now present its formulation in our setting.

ALGORITHM 1. Given are timestep $k \in \{0, \dots, T/h_i - 1\}$, solution $v_i(s_i^{k+1}, \cdot) \in V_i$ at timestep $k+1$ and an (arbitrary) choice of $\alpha \in A$. Find $w_0 \in V_i$ such that

$$\hat{\mathsf{I}}_i^{\alpha} w_0 = \hat{\mathsf{F}}_i^{\alpha} - \hat{\mathsf{E}}_i^{\alpha} v_i(s_i^{k+1}, \cdot).$$

Inductively over $m \in \mathbb{N}$, compute $w_{m+1} \in V_i$ such that

$$\hat{\mathsf{I}}_i^{w_m} w_{m+1} = \hat{\mathsf{F}}_i^{w_m} - \hat{\mathsf{E}}_i^{w_m} v_i(s_i^{k+1}, \cdot). \tag{5.1}$$

To show the convergence of the sequence $(w_m)_m$ to the solution of (3.6), we appeal to an auxiliary problem: for some fixed control $\alpha \in A$, we consider the linear evolution problem associated to it. More precisely, we define $v_i^{\alpha} : S_i \to V_i$ to be such that $v_i^{\alpha}(T, \cdot) = v_i(T, \cdot)$, the interpolant of v_T, and for each $k \in \{0, \dots, T/h_i - 1\}$

$$\hat{\mathsf{I}}_i^{\alpha} v_i^{\alpha}(s_i^k, \cdot) + \hat{\mathsf{E}}_i^{\alpha} v_i^{\alpha}(s_i^{k+1}, \cdot) - \hat{\mathsf{F}}_i^{\alpha} = 0. \tag{5.2}$$

Notice that v_i^{α} is well-defined due to the invertibility of $\hat{\mathsf{I}}_i^{\alpha}$.

THEOREM 5.1 Suppose that Assumptions 2.1 (b) and (c), 2.3, 3.1 (b) and 4.5 hold. There exists a unique numerical solution $v_i : S_i \to V_i$ that solves (3.6) and (3.8). Algorithm 1, provided with the inputs k, $v_i(s_i^{k+1}, \cdot)$ and α, generates a sequence $(w_m)_m$ that converges superlinearly to $v_i(s_i^k, \cdot)$ as $m \to \infty$. Moreover, $0 \leqslant v_i \leqslant v_i^{\alpha}$ for all $\alpha \in A$.

Proof. For a fixed timestep k, the superlinear convergence of Algorithm 1 to the unique solution $v_i(s_i^k, \cdot)$ is shown in Bokanowski *et al.* (2009, Theorem 2.1) under Assumptions (H1) and (H2) stated therein. Condition (H1) requires the inverse positivity of the operators $\hat{\mathsf{I}}_i^{w_m}$, which holds because, according to Lemma 4.7, every $\hat{\mathsf{I}}_i^{w_m}$ is a nonsingular M-matrix. Condition (H2) requires that $\alpha \in A \mapsto -\hat{\mathsf{I}}_i^{\alpha}$ and $\alpha \in A \mapsto \hat{\mathsf{E}}_i^{\alpha} v_i(s_i^{k+1}, \cdot) - \hat{\mathsf{F}}_i^{\alpha}$ are continuous, which follows from Assumption 3.1. Induction over timesteps k gives existence and uniqueness of the solution v_i.

We now show that $v_i \geqslant 0$ on $S_i \times \overline{\Omega}$ by induction over k. First, we notice that $v_i(T, \cdot) \geqslant 0$ because we assumed that $v_T \geqslant 0$ on $\overline{\Omega}$, and the same holds for its interpolant. Let us assume $v_i(s_i^{k+1}, \cdot) \geqslant 0$ on $\overline{\Omega}$ for some $s_i^{k+1} \in S_i$. Due to the LMP, all entries of $\hat{\mathsf{E}}_i^{v_i}$ are nonpositive and by assumption all entries of $\hat{\mathsf{F}}_i^{v_i}$ are non-negative. Therefore, using (3.8), we have that

$$\hat{\mathsf{I}}_i^{v_i} v_i(s_i^k, \cdot) = -\hat{\mathsf{E}}_i^{v_i} v_i(s_i^{k+1}, \cdot) + \hat{\mathsf{F}}_i^{v_i} \geqslant 0.$$

We conclude that $v_i(s_i^k, \cdot) \geqslant 0$ on $\overline{\Omega}$ due to the inverse positivity of $\hat{\mathsf{I}}_i^{v_i}$.

We now prove the last statement, namely $v_i \leqslant v_i^\alpha$ for all $\alpha \in A$, by induction over k. Consider any $\alpha \in A$. At time T, both v_i and v_i^α interpolate v_T and hence are equal. Let us assume that for some $k \leqslant T/h_i - 1$, $v_i(s_i^{k+1}, \cdot) \leqslant v_i^\alpha(s_i^{k+1}, \cdot)$. From (3.6),

$$\hat{\mathsf{I}}_i^\alpha v_i(s_i^k, \cdot) \leqslant \hat{\mathsf{F}}_i^\alpha - \hat{\mathsf{E}}_i^\alpha v_i(s_i^{k+1}, \cdot).$$

Now subtracting (5.2) from the above inequality, together with the monotonicity of $\hat{\mathsf{E}}_i^\alpha$, gives

$$\hat{\mathsf{I}}_i^\alpha \left(v_i(s_i^k, \cdot) - v_i^\alpha(s_i^k, \cdot) \right) \leqslant \hat{\mathsf{E}}_i^\alpha \left(v_i^\alpha(s_i^{k+1}, \cdot) - v_i(s_i^{k+1}, \cdot) \right) \leqslant 0.$$

Using the inverse positivity of $\hat{\mathsf{I}}_i^\alpha$ gives us $v_i(s_i^k, \cdot) - v_i^\alpha(s_i^k, \cdot) \leqslant 0$ on $\overline{\Omega}$, as required. □

6. Consistency

We will assume existence of an elliptic projection P_i, described in Jensen & Smears (2013), with the properties required in the following assumption.

ASSUMPTION 6.1 There are linear mappings $P_i : C(H^1(\Omega)) \to V_i$ satisfying for all interior hat functions $\hat{\phi}_i^\ell$, $\ell \leqslant N_i^\Omega$,

$$\langle \nabla P_i w, \nabla \hat{\phi}_i^\ell \rangle = \langle \nabla w, \nabla \hat{\phi}_i^\ell \rangle. \tag{6.1}$$

There is a constant $C \geqslant 0$ such that for every $w \in C^\infty(\mathbb{R}^d)$ and $i \in \mathbb{N}$,

$$\left\| P_i w \right\|_{W^{1,\infty}(\Omega)} \leqslant C \left\| w \right\|_{W^{1,\infty}(\Omega)} \quad \text{and} \quad \lim_{i\to\infty} \left\| P_i w - w \right\|_{W^{1,\infty}(\Omega)} = 0. \tag{6.2}$$

To state consistency, it is convenient to abbreviate the operator of the numerical scheme as

$$F_i\, w(s_i^k, y_i^\ell) := \hat{\mathsf{I}}_i^{k,w}\, w(s_i^k, y_i^\ell) + \hat{\mathsf{E}}_i^{k,w}\, w(s_i^{k+1}, y_i^\ell) - \hat{\mathsf{F}}_i \tag{6.3}$$

for $w(s_i^k, \cdot) \in V_i$. Note that, while F_i is the discrete operator approximating the continuous operator F defined in Section 2, the notationally similar $\hat{\mathsf{F}}_i^\alpha$ represents the approximation of f^α, g^α and g as explained at the end of Section 3.

THEOREM 6.2 Suppose that Assumptions 2.1, 2.3, 3.1, 4.5 and 6.1 hold. Let $\psi \in C^2(\mathbb{R} \times \mathbb{R}^d)$, $s_i^{k(i)} \to t \in [0, T)$ and $y_i^{\ell(i)} \to x \in \overline{\Omega}$ as $i \to \infty$. Here, $s_i^{k(i)}$ is a time step and $y_i^{\ell(i)}$ a node of the ith refinement. Then

$$\limsup_{i\to\infty} F_i P_i \psi(s_i^{k(i)}, y_i^{\ell(i)}) \leqslant F^*(t, x, \Delta\psi(t,x), \nabla\psi(t,x), \partial_t\psi(t,x), \psi(t,x)), \tag{6.4}$$

and

$$\liminf_{i\to\infty} F_i P_i \psi(s_i^{k(i)}, y_i^{\ell(i)}) \geqslant F_*(t, x, \Delta\psi(t,x), \nabla\psi(t,x), \partial_t\psi(t,x), \psi(t,x)). \tag{6.5}$$

Proof. We prove (6.4). The result for (6.5) follows analogously. For ease of notation, the dependence of k and ℓ on i is made implicit.

Step 1: Standard finite difference bounds ensure that if $y_i^\ell \in \Omega \cup \partial\Omega_t$, then

$$\lim_{i\to\infty} d_i P_i \psi(s_i^k, y_i^\ell) = \partial_t \psi(t,x). \tag{6.6}$$

Otherwise, if $y_i^\ell \in \partial\Omega_R \cup \partial\Omega_D$, then

$$\lim_{i\to\infty} d_i P_i \psi(s_i^k, y_i^\ell) = 0. \tag{6.7}$$

Step 2: It is shown in Jensen & Smears (2013, Section 4) that if $y_i^\ell \in \Omega$, then

$$\lim_{i\to\infty} \Big(\mathsf{E}_i^\alpha P_i \psi(s_i^\ell, \cdot) + \mathsf{I}_i^\alpha P_i \psi(s_i^k, \cdot) - \mathsf{F}_i^\alpha \Big)_\ell = L^\alpha \psi(t,x) - f^\alpha(x), \tag{6.8}$$

where convergence to the limit is uniform over all $\alpha \in A$. We remark that the orthogonality (6.1) is used in this step.

Step 3: Now suppose that $y_i^\ell \in \partial\Omega_D$. Then, it follows from (6.2) that

$$\lim_{i\to\infty} F_i P_i \psi(s_i^k, y_i^\ell) = \psi(t,x) - g(x). \tag{6.9}$$

Step 4: Let $y_i^\ell \in \partial\Omega_t \cup \partial\Omega_R$. Just like the continuous operators, the corresponding first-order terms of the discrete Robin operators employ the lower Dini derivative, giving consistency directly. Thus, with

$$\lim_{i\to\infty} \left| c_{\partial\Omega}^\alpha P_i \psi(t,\cdot) - \bar{\bar{c}}_{\partial\Omega,i}^\alpha P_i \psi(s_i^k, \cdot) - \bar{c}_{\partial\Omega,i}^\alpha P_i \psi(s_i^{k+1}, \cdot) \right| = 0, \tag{6.10}$$

using Assumptions 3.1 and 6.1, we conclude that if $y_i^\ell \in \partial\Omega_t \cup \partial\Omega_R$, then

$$\lim_{i\to\infty} \Big(\mathsf{E}_i^\alpha P_i \psi(s_i^\ell, \cdot) + \mathsf{I}_i^\alpha P_i \psi(s_i^k, \cdot) - \mathsf{F}_i^\alpha \Big)_\ell = L_{\partial\Omega}^\alpha \psi(t,x) - g^\alpha(x). \tag{6.11}$$

Step 5: Consider the sequence $\{(s_i^k, y_i^\ell)\}_i$ as specified in the statement of the theorem, in particular with $y_i^\ell \in \overline{\Omega}$. We decompose $\{(s_i^k, y_i^\ell)\}_i$ into the subsequences of the (s_i^k, y_i^ℓ) where y_i^ℓ belongs to Ω, $\partial\Omega_t$, $\partial\Omega_R$ and $\partial\Omega_D$, respectively. Then the conclusions of Steps 1 to 4 above may be applied to the individual subsequences. □

7. Stability

In this section, we present a lemma that ensures L^∞ stability of the numerical scheme (3.8). The stability statement goes back to the boundedness of a supersolution of the continuous linear problem for a fixed

α. Let

$$F^\alpha(t,x,q,p,r,s) = \begin{cases} -r + L^\alpha(x,q,p,s) - f^\alpha(x) & \text{if } (t,x) \in [0,T) \times \overline{\Omega}, \\ -r + L^\alpha_{\partial\Omega}(x,p,s) - g^\alpha(x) & \text{if } (t,x) \in [0,T) \times \partial\Omega_t, \\ L^\alpha_{\partial\Omega}(x,p,s) - g^\alpha(x) & \text{if } (t,x) \in [0,T) \times \partial\Omega_R, \\ s - g(x) & \text{if } (t,x) \in [0,T) \times \partial\Omega_D, \\ s - v_T(x) & \text{if } (t,x) \in \{T\} \times \overline{\Omega}. \end{cases}$$

ASSUMPTION 7.1 There exists an $\alpha \in A$ and a $w(t,x) \in C^2(\mathbb{R} \times \mathbb{R}^d)$, which is a strict supersolution of the associated linear problem. More precisely, there is an $\varepsilon > 0$ such that

$$F^\alpha(t,x,\Delta w(t,x), \nabla w(t,x), \partial_t w(t,x), w(t,x)) \geqslant \varepsilon$$

on $[0,T] \times \overline{\Omega}$.

The assumption is essentially fulfilled if the linear equation

$$F^\alpha(t,x,\Delta\psi(t,x), \nabla\psi(t,x), \partial_t\psi(t,x), \psi(t,x)) = 2\varepsilon \tag{7.1}$$

is a well-posed problem in a suitable sense. For example, ψ may be a weak solution of (7.1) as in Oleĭnik & Radkevič (1973, Chapter 1), which admits a bounded extension to $\mathbb{R} \times \mathbb{R}^d$. In such cases, one may pass to a strict supersolution in $C^2(\mathbb{R} \times \mathbb{R}^d)$ through mollification.

LEMMA 7.2 Let α be as in Assumption 7.1, $s_i^{k(i)} \to t \in [0,T)$ and $y_i^{\ell(i)} \to x \in \overline{\Omega}$ as $i \to \infty$. Here, $s_i^{k(i)}$ is a time step and $y_i^{\ell(i)}$ a node of the ith refinement. Then,

$$\liminf_{i\to\infty} \left[\hat{\mathsf{I}}_i^\alpha P_i w(s_i^k, y_i^\ell) + \hat{\mathsf{E}}_i^\alpha P_i w(s_i^{k+1}, y_i^\ell) - (\hat{\mathsf{F}}_i^\alpha)_\ell \right] \geqslant \varepsilon.$$

Proof. We use Theorem 6.2 for a singleton control set $A = \{\alpha\}$ to cover the linear case. The result now follows because

$$(F^\alpha)_*(x, \Delta w(t,x), \nabla w(t,x), \partial_t w(t,x), w(t,x)) \geqslant \varepsilon,$$

owing to Assumption 7.1. □

An unusual feature of the stability proof is that it uses the consistency assumptions.

THEOREM 7.3 Suppose that Assumptions 2.1, 2.3, 3.1, 4.5, 6.1 and 7.1 hold. The numerical solutions v_i are uniformly bounded in the L^∞ norm. More precisely, there exists a finite constant $C > 0$ such that

$$\|v_i\|_{L^\infty(S_i \times \overline{\Omega})} \leqslant C \qquad \forall\, i \in \mathbb{N}. \tag{7.2}$$

Proof. Recall the solution v_i^α of the linear problem (5.2). We define

$$w_i^k := P_i w^\alpha(s_i^k, \cdot),$$
$$\tilde{v}_i^k := w_i^k - v_i^\alpha(s_i^k, \cdot).$$

It is convenient to set

$$e_i^k := \hat{\mathsf{I}}_i^\alpha \tilde{v}_i^k + \hat{\mathsf{E}}_i^\alpha \tilde{v}_i^{k+1} = \hat{\mathsf{I}}_i^\alpha w_i^k + \hat{\mathsf{E}}_i^\alpha w_i^{k+1} - \hat{\mathsf{F}}_i^\alpha. \tag{7.3}$$

Because of Assumptions 6.1 and 7.1, the w_i are uniformly bounded in the L^∞ norm. Moreover, $0 \leqslant v_i \leqslant v_i^\alpha$ due to Theorem 5.1. Thus, the statement of the theorem is proved once we demonstrate that the v_i^α are bounded from above independently of i. This is equivalent to showing a lower bound for the $\tilde{v}_i^k$.

It follows from Lemma 7.2 that $e^k \geqslant 0$ for i larger than some constant M. It is trivial that there exists a constant C such that the inequality of (7.2) holds for all $i \leqslant M$. We therefore may assume w.l.o.g. that $e^k \geqslant 0$ throughout. By possibly modifying w through the addition of a positive constant, we can assume that

$$w_i^{T/h_i} = P_i w(T, \cdot) \geqslant \|v_T(T, \cdot)\|_{L^\infty(\Omega)},$$

while maintaining the supersolution property of w because $c_\Omega^\alpha, c_{\partial\Omega}^\alpha \geqslant 0$. This implies $\tilde{v}_i^{T/h_i} \geqslant 0$, i.e., the non-negativity at the final time.

Now suppose $\tilde{v}_i^{k+1} \geqslant 0$. Then

$$\tilde{v}_i^k = (\hat{\mathsf{I}}_i^\alpha)^{-1}\big(e_i^k - \hat{\mathsf{E}}_i^\alpha \tilde{v}_i^{k+1}\big) \geqslant 0,$$

because also $(\hat{\mathsf{I}}_i^\alpha)^{-1} \geqslant 0$ and $-\hat{\mathsf{E}}_i^\alpha \tilde{v}_i^{k+1} \geqslant 0$. Now induction in k completes the proof. □

8. Uniform convergence

Analogously to the envelopes of functions introduced in Section 2, we define envelopes of the numerical solutions as follows:

$$\overline{v}(t,x) = \limsup_{i\to\infty} \sup_{(s_i^k, y_i^\ell)\to(t,x)} v_i(s_i^k, y_i^\ell), \qquad \underline{v}(t,x) = \liminf_{i\to\infty} \inf_{(s_i^k, y_i^\ell)\to(t,x)} v_i(s_i^k, y_i^\ell),$$

where limits are taken over all sequences of nodes in $[0,T] \times \overline{\Omega}$, which converge to $(t,x) \in [0,T] \times \overline{\Omega}$. Owing to Theorem 7.3, $\overline{v}$ and $\underline{v}$ attain finite values. By construction, $\overline{v}$ is upper and $\underline{v}$ lower semicontinuous and $\underline{v} \leqslant \overline{v}$.

THEOREM 8.1 Suppose that Assumptions 2.1, 2.3, 3.1, 4.5, 6.1 and 7.1 hold. The function $\overline{v}$ is a viscosity subsolution and $\underline{v}$ is a viscosity supersolution.

Proof. Step 1 ($\overline{v}$ is a subsolution). To show that $\overline{v}$ is a viscosity subsolution, suppose that $w \in C^\infty(\mathbb{R} \times \mathbb{R}^d)$ is a test function such that $\overline{v} - w$ has a strict local maximum at $(s,y) \in (0,T) \times \overline{\Omega}$, with $\overline{v}(s,y) = w(s,y)$. Note that (s,y) may be on the boundary. Consider a closed neighbourhood $B := \big\{(t,x) \in (0,T) \times \overline{\Omega} \ : \ |t-s| + |x-y| \leqslant \delta\big\}$ with $\delta > 0$ such that

$$\overline{v}(s,y) - w(s,y) > \overline{v}(t,x) - w(t,x) \quad \forall (t,x) \in B \setminus (s,y).$$

Choose i sufficiently large for B to contain nodes. As in Jensen & Smears (2013), we choose a sequence of nodes $\{(s^k_{i(j)}, y^\ell_{i(j)})\}_j$ that maximize $v_i(s^\kappa_{i(j)}, y^\lambda_{i(j)}) - P_i w(s^\kappa_{i(j)}, y^\lambda_{i(j)})$ among all nodes $(s^\kappa_{i(j)}, y^\lambda_{i(j)}) \in B$ and converge to (s, y). It follows that

$$v_i(s^k_i, y^\ell_i) - P_i w(s^k_i, y^\ell_i) \to \overline{v}(s, y) - w(s, y) = 0. \tag{8.1}$$

Moreover, because of $(s^k_i, y^\ell_i) \to (s, y)$, the neighbours of the (s^k_i, y^ℓ_i) eventually also belong to B: for i sufficiently large, we have $(s^\kappa_i, y^\lambda_i) \in B$ if $\kappa \in \{k, k+1\}$ and $y^\lambda_i \in \operatorname{supp} \hat{\phi}^\ell_i$, in which case

$$\begin{aligned} v_i(s^\kappa_i, y^\lambda_i) - P_i w(s^\kappa_i, y^\lambda_i) &\leqslant v_i(s^k_i, y^\ell_i) - P_i w(s^k_i, y^\ell_i) \\ &\Leftrightarrow\ P_i w(s^\kappa_i, y^\lambda_i) + \mu_i \geqslant v_i(s^\kappa_i, y^\lambda_i), \end{aligned}$$

with $\mu_i = v_i(s^k_i, y^\ell_i) - P_i w(s^k_i, y^\ell_i)$, and $\mu_i \to 0$ as $i \to \infty$ because of (8.1).

Recall that the matrices E^α_i have nonzero off diagonal entries $\left(\mathsf{E}^\alpha_i\right)_{\ell\lambda}$ only if $y^\lambda_i \in \operatorname{supp} \hat{\phi}^\ell_i$ and that $v_i(s^{k+1}_i, \cdot) \leqslant P_i w(s^{k+1}_i, \cdot) + \mu_i$ on $\operatorname{supp} \hat{\phi}^\ell_i$. Therefore, monotonicity of $h_i \mathsf{E}^\alpha_i - \mathsf{Id}$ for all $\alpha \in A$ implies that

$$\left((h_i \mathsf{E}^\alpha_i - \mathsf{Id}) \left[P_i w(s^{k+1}_i, \cdot) + \mu_i\right]\right)_\ell \leqslant \left((h_i \mathsf{E}^\alpha_i - \mathsf{Id}) v_i(s^{k+1}_i, \cdot)\right)_\ell.$$

Applying the LMP and linearity of I^α_i to $P_i w(s^k_i, \cdot) + \mu_i - v_i(s^k_i, \cdot)$, which has a nonpositive local minimum at y^ℓ_i, yields

$$\left((h_i \mathsf{I}^\alpha_i + \mathsf{Id}) \left[P_i w(s^k_i, \cdot) + \mu_i\right]\right)_\ell \leqslant \left((h_i \mathsf{I}^\alpha_i + \mathsf{Id}) v_i(s^k_i, \cdot)\right)_\ell.$$

From the definition of the scheme, with $\gamma := \sup_{\alpha,i} \|\bar{c}^\alpha_i + \bar{\bar{c}}^\alpha_i\|_\infty$,

$$\begin{aligned} 0 &= -d_i v_i(s^k_i, y^\ell_i) + \sup_{\alpha\in A} \left(\mathsf{E}^\alpha_i v_i(s^{k+1}_i, \cdot) + \mathsf{I}^\alpha_i v_i(s^k_i, \cdot) - \mathsf{F}^\alpha_i\right)_\ell \\ &\geqslant -d_i \left(P_i w(s^k_i, y^\ell_i) + \mu_i\right) + \sup_{\alpha\in A} \left(\mathsf{E}^\alpha_i \left(P_i w(s^{k+1}_i, \cdot) + \mu_i\right) + \mathsf{I}^\alpha_i \left(P_i w(s^k_i, \cdot) + \mu_i\right) - \mathsf{F}^\alpha_i\right)_\ell \\ &= -d_i P_i w(s^k_i, y^\ell_i) + \sup_{\alpha\in A} \left[\left(\mathsf{E}^\alpha_i P_i w(s^{k+1}_i, \cdot) + \mathsf{I}^\alpha_i P_i w(s^k_i, \cdot) - \mathsf{F}^\alpha_i\right)_\ell + \mu_i \langle \bar{c}^\alpha_i + \bar{\bar{c}}^\alpha_i \rangle \hat{\phi}^\ell_i\right] \\ &\geqslant -d_i P_i w(s^k_i, y^\ell_i) + \sup_{\alpha\in A} \left(\mathsf{E}^\alpha_i P_i w(s^{k+1}_i, \cdot) + \mathsf{I}^\alpha_i P_i w(s^k_i, \cdot) - \mathsf{F}^\alpha_i\right)_\ell - \gamma \left|\mu_i\right| \\ &= F_i P_i w(s^k_i, y^\ell_i) - \gamma \left|\mu_i\right|. \end{aligned} \tag{8.2}$$

For a fixed i, evaluating $F_i P_i w(s^k_i, y^\ell_i)$ may involve a boundary operator even if (s, y) is internal and vice versa may involve the PDE operator, even if (s, y) belongs to the boundary. Referring to the semicontinuous envelope F_*, it now follows from (8.2), $\lim_i \mu_i = 0$ and Theorem 6.2 that

$$\begin{aligned}0 &\geqslant \liminf_{i\to\infty} F_i P_i w(s_i^k, y_i^\ell)\\ &\geqslant F_*(t, y, \Delta w(s,y), \nabla w(s,y), \partial_t w(s,y), \overline{v}(s,y)).\end{aligned}$$

Therefore, $\overline{v}$ is a viscosity subsolution.

Step 2 ($\underline{v}$ is a supersolution). Arguments similar to those above show that $\underline{v}$ is a viscosity supersolution, where the principal change to the proof is that one considers $w \in C^\infty(\mathbb{R} \times \mathbb{R}^d)$ such that $\underline{v} - w$ has a strict local minimum at some $(s, y) \in (0, T) \times \Omega$ with $\underline{v}(s, y) = w(s, y)$. With analogous notation, the last line in (8.2) corresponds to

$$0 \leqslant -d_i P_i w(s_i^k, y_i^\ell) + \sup_{\alpha\in A} \left(\mathsf{E}_i^\alpha P_i w(s_i^{k+1}, \cdot) + \mathsf{I}_i^\alpha P_i w(s_i^k, \cdot) - \mathsf{F}_i^\alpha\right)_\ell + \gamma\,|\mu_i|,$$

i.e., there is a slight asymmetry in the argument due to the last sign in (8.2). Nevertheless, it is then deduced that

$$0 \leqslant F^*(t, x, \Delta w(t,x), \nabla w(t,x), \partial_t w(t,x), \underline{v}(t,x)).$$

Thus, $\underline{v}$ is a viscosity supersolution. □

The above proof is an adaptation of the Barles–Souganidis argument (Barles & Souganidis, 1991) to the finite element setting, in line with that in Jensen & Smears (2013), but differing in the treatment of the boundary conditions.

ASSUMPTION 8.2 Let $\overline{v}$ be a lower semicontinuous supersolution and $\underline{v}$ be an upper semicontinuous subsolution. Then, $\underline{v} \leqslant \overline{v}$.

THEOREM 8.3 Suppose that Assumptions 2.1, 2.3, 3.1, 4.5, 6.1, 7.1 and 8.2 hold. One has $\underline{v} = \overline{v} = v$, where v is the unique viscosity solution with $v(T, \cdot) = v_T$. Furthermore,

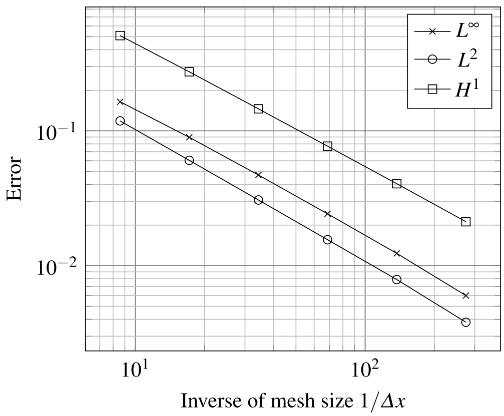
$$\lim_{i\to\infty} \|v_i - v\|_{L^\infty((0,T)\times\Omega)} = 0. \tag{8.3}$$

Proof. Follows as in the proof of Theorem 6.2 in Jensen & Smears (2013). □

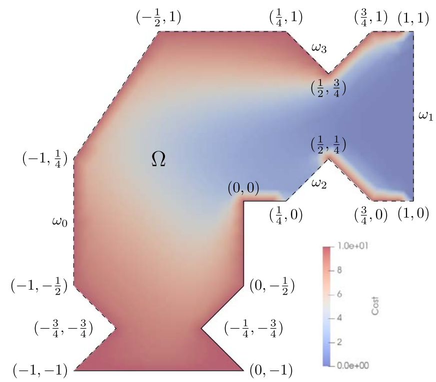
9. Numerical experiments

The first experiment investigates rates of convergence for a known smooth solution. The remaining experiments examine the approximation of solutions with singularities near type changes of boundary conditions, as well as the solution behaviour in the vicinity of nonlinear boundary conditions. The code is available from the public repository (Jaroszkowski, 2021) under the GNU Lesser General Public License.

Experiment 1 (Rates for smooth known solution): We consider a final time boundary value problem on the square domain $\Omega = [-1, 1]^2$ with Robin conditions on the right face $\partial\Omega_t = \{1\} \times (-1, 1)$ and Dirichlet conditions on the remaining faces $\partial\Omega_D = \partial\Omega \setminus \partial\Omega_t$. We have the control



(a) Approximation error of Experiment 1



(b) Value function of the Skorokhod problem of Experiment 2 for a coarse mesh size $\Delta x \approx 0.12$

FIG. 1.

set $A = [0, 1]$ and the final time $T = 1$ for the system

$$\begin{aligned} -\partial_t v + \sup_{\alpha \in A}\Big(-(\alpha + |x|^2/2)\Delta v + x v_x - f^\alpha\Big) &= 0 && \text{in } [0,T)\times\Omega, \\ -\partial_t v + \sup_{\alpha\in A}\big(\alpha v_x - g^\alpha\big) &= 0 && \text{on } [0,T)\times\partial\Omega_t, \\ v &= 0 && \text{on } [0,T)\times\partial\Omega_D, \\ v - (1-x^2)(1-y^2) &= 0 && \text{on } \{T\}\times\overline{\Omega}. \end{aligned} \tag{9.1}$$

We choose g^α and f^α such that

$$v(x,y,t) := t(1-x^2)(1-y^2) + (1-t)\sin(\pi x)\cos\left(\frac{\pi y}{2}\right)$$

is the exact solution of (9.1).

The artificial diffusion coefficients are selected quasi-optimally, cf. Section 4.3. The time dependent Robin boundary condition is treated fully explicitly. The time step size is chosen to ensure monotonicity while permitting a large time step, leading to $O(h_i) = O(\Delta x_i)$. The L^2, H^1 and L^∞ errors at time $t = 0$, presented also in Fig. 1a, obey in essence the same rates as those observed previously (Jensen & Smears, 2013) with Dirichlet conditions and $O(h_i) = O(\Delta x_i)$ scaling:

Δx	#DoFs	L^2	Rate	L^∞	Rate	H^1	Rate
0.1165	233	1.186e–1	0.98	1.645e–1	0.92	5.089e–1	0.93
0.0583	881	6.044e–2	0.98	8.960e–2	0.95	2.743e–1	0.94
0.0291	3425	3.072e–2	0.99	4.706e–2	0.97	1.457e–1	0.95
0.0146	13505	1.558e–2	0.99	2.426e–2	0.98	7.696e–2	0.95
0.0073	53633	7.894e–3	1.04	1.234e–2	1.03	4.058e–2	0.96
0.0036	213761	3.806e–3		6.006e–3		2.120e–2	

Experiment 2 (Skorokhod problem): The second numerical experiment is set on a nonconvex, less regular domain, which is depicted in Fig. 1b. The stochastic controlled process is subject to a terminal cost of 10 everywhere apart from $\overline{\omega_1}$ where it is 0. There is no running cost. On ω_2, the particle is transported through a Skorokhod reflection independently of the angle of incidence in the direction of the inner normal vector. Ultimately, a particle can avoid penalization only by reaching $\overline{\omega_1}$ before the terminal time $T = 1$. On Ω, the particle may only choose between an upwards drift and drift to the right:

$$-\partial_t v + \sup_\alpha \left(-a^\alpha \Delta v - b^\alpha \cdot \nabla v\right) = 0 \qquad \text{in } [0,T) \times \Omega, \tag{9.2a}$$

$$-b_{\partial\Omega} \cdot \nabla v = 0 \qquad \text{on } [0,T) \times \omega_2 \cup \left\{\left(\tfrac{1}{4}, 0\right)\right\}, \tag{9.2b}$$

$$v = 0 \qquad \text{on } [0,T) \times \overline{\omega_1}, \tag{9.2c}$$

$$v = 10 \qquad \text{on } [0,T) \times \omega_0 \cup \overline{\omega_3} \cup \left\{\left(\tfrac{1}{2}, \tfrac{1}{4}\right)\right\}, \tag{9.2d}$$

$$v = v_T \qquad \text{on } \{T\} \times \overline{\Omega}, \tag{9.2e}$$

where $a^\alpha = 0.1(1 - x_2)\alpha$ and $b^\alpha = (-2\alpha, 2(\alpha - 1))^T$ for $\alpha \in \{0, 1\}$. Moreover, $b_{\partial\Omega} = (1, -1)^T$ and

$$v_T(x) = \begin{cases} 10 & x \in \overline{\Omega} \setminus \omega_1, \\ 0 & x \in \omega_1. \end{cases} \tag{9.3}$$

Hence, when drifting to the right, the particle is exposed to Brownian noise, while the equation is degenerate when the upward drift is selected. The numerical operators are given by

$$(\mathsf{E}^\alpha_{\Omega,i} v)_\ell := \bar{\nu}^{\alpha,\ell}_{\Omega,i} \langle \nabla v, \nabla \hat{\phi}^\ell_i \rangle + \langle -b^\alpha \cdot \nabla v, \hat{\phi}^\ell_i \rangle, \tag{9.4a}$$

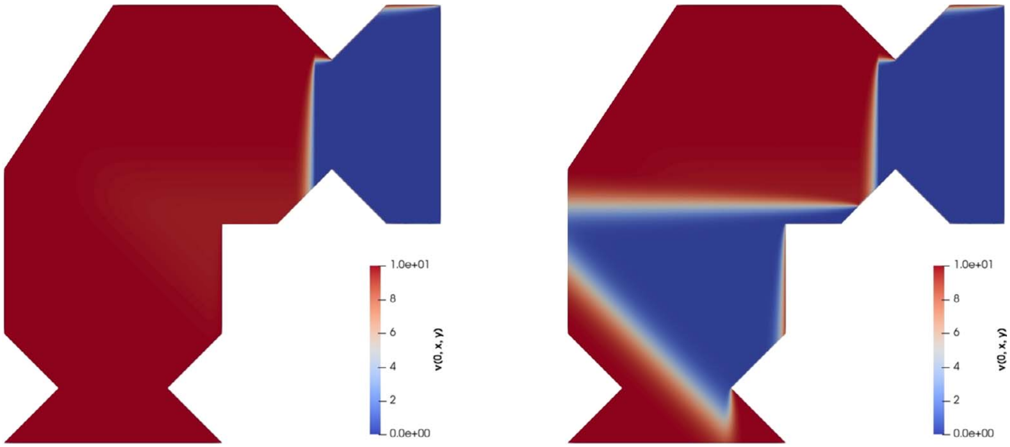
$$(\mathsf{I}^\alpha_{\Omega,i} v)_\ell := \max\left(a^\alpha - \bar{\nu}^{\alpha,\ell}_{\Omega,i}, 0\right) \langle \nabla v, \nabla \hat{\phi}^\ell_i \rangle, \tag{9.4b}$$

$$(\mathsf{E}^\alpha_{\partial\Omega,i} v)_\ell := 0, \tag{9.4c}$$

$$(\mathsf{I}^\alpha_{\partial\Omega,i} v)_\ell := \frac{v(t, x_1, x_2) - v(t, x_1 - \lambda, x_2 + \lambda)}{\lambda}. \tag{9.4d}$$

Figure 1b shows the approximation v_i at time $t = 0$ on a coarse mesh. Notice how the introduction of a Skorokhod-type boundary gives the particle starting in the vicinity of ω_2 a high probability of reaching the penalty-free exit zone $\overline{\omega_1}$. The node at $(\frac{1}{2}, \frac{1}{4})$ already belongs to the Dirichlet boundary. We observe that the penalty of 10 in the boundary segment between ω_1 and ω_2 leads to a layer-like behaviour of the solution of only one element thickness. The related numerical experiments on finer meshes depicted in Fig. 2 also exhibit this aspect of the numerical solution.

Experiment 3 (Internal barrier and nonlinear boundary conditions): The third experiment is an adaptation of the previous one to examine the effect of nonlinear boundary conditions. We break the adaptation into two parts.



(a) Skorokhod problem (b) Nonlinear boundary condition

FIG. 2. Linear and nonlinear boundary conditions on ω_2 with $\Delta x = 0.0035$.

Part (a) (Internal barrier): The experiment is identical to the previous one with exception of the drift terms on Ω:

$$b^\alpha(x) = \begin{cases} \big(-2\alpha, 2(\alpha-1)\big)^T & : |x - \frac{3}{8}| > \frac{1}{20}, \\ \big(0, 2(\alpha-1)\big)^T & : \text{otherwise.} \end{cases}$$

This means that the strip of all x with $|x - \frac{3}{8}| \leqslant \frac{1}{20}$ acts as barrier in Ω: within this strip, there is no process with drift to the right. The only way a particle can cross the strip from the left to the right in order to avoid penalization is by adopting $\alpha = 1$. In this case, the particle *might* cross the barrier by means of diffusion; however, there is no drift term to aid the crossing.

The construction results in a value function that at first sight resembles a piecewise constant function. It is close to 10 left of the barrier as the particle is unlikely to reach the penalty-free exit zone ω_1. It is mostly close to 0 right of the barrier; however, reaches 10 at Dirichlet boundary conditions on ω_0 as already indicated in Experiment 2. At the barrier, there is an internal layer arising from the possibility of crossing owing to diffusion.

This description of the value function is matched entirely by the numerical computations with the scheme of this paper: see Fig. 2a, where the internal layer, as well as the boundary conditions right of the barrier are well resolved.

Part (b) (Nonlinear boundary condition): Now, the boundary condition on ω_2 is replaced by a nonlinear operator that corresponds to the choice between the previously used Skorokhod reflection and instantaneous transport along the boundary towards the right. In other words, (9.2b) is replaced by

$$\sup\{-b^0_{\partial\Omega} \cdot \nabla v, -b^1_{\partial\Omega} \cdot \nabla v\} = 0 \quad \text{on } [0, T) \times \omega_2 \cup \left\{\left(\tfrac{1}{4}, 0\right)\right\},$$

with $b^0_{\partial\Omega} = (1, -1)$ and $b^1_{\partial\Omega} = (-1, -1)$.

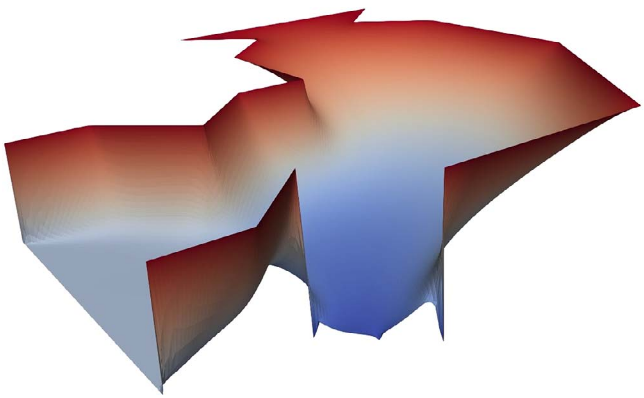


FIG. 3. Value function with nonlinear boundary condition on ω_3.

This modification has a striking impact on the behaviour of the controlled system. Now, an optimally controlled particle may drift onto the boundary segment ω_2 left of the barrier to be then transported on the boundary past the barrier. Right of the barrier the control is changed to $b^0_{\partial\Omega}$ in order to reflect the particle into Ω to avoid the penalty at the point $(\frac{1}{2}, \frac{1}{4})$ at the end of ω_2.

An approximation of the resulting value function at time 0 is shown in Fig. 2b, which illustrates how this time regions left of the barrier have a value function close to zero, since particles located there can now reach the penalty-free exit zone ω_1. Indeed, when computing the solutions for earlier times $t < 0$, one observes further growth of the region with cost close to zero as more time is available to arrive at ω_1 before termination.

Experiment 4 (Reflection vs. termination): We consider a final time boundary value problem on the same domain, but now with a nonlinear boundary condition corresponding to a choice between a Skorokhod reflection and termination of the process in exchange for an oscillatory cost g^α:

$$\begin{aligned}
-\partial_t v + \sup_\alpha \left(-a^\alpha \Delta v - b^\alpha \cdot \nabla v\right) &= 0 && \text{in } [0,T) \times \Omega,\\
\sup_\alpha \left(-b^\alpha_{\partial\Omega} \cdot \nabla v + c^\alpha_{\partial\Omega} v - g^\alpha\right) &= 0 && \text{on } [0,T) \times \omega_3 \cup \left\{\left(\tfrac{1}{4}, 1\right)\right\},\\
v &= 0 && \text{on } [0,T) \times \overline{\omega_1},\\
v &= 10 && \text{on } [0,T) \times \omega_0 \cup \overline{\omega_2} \cup \left\{\left(\tfrac{1}{2}, \tfrac{3}{4}\right)\right\},\\
v &= v_T && \text{on } \{T\} \times \overline{\Omega},
\end{aligned}$$

where $\alpha \in \{0, 1\}$, v_T as in (9.3) and

$$\begin{aligned}
a^\alpha &= 0.2(1 - x_2)(1 - \alpha) + 0.2(1 - x_1)\alpha,\\
b^\alpha &= (-2\alpha, 2(\alpha - 1))^T, \quad b^\alpha_{\partial\Omega} = (\alpha, \alpha)^T,\\
c^\alpha &= (1 - \alpha), \quad g^\alpha = -(10\cos(160 x_1/\pi + 4)(1 - \alpha)).
\end{aligned}$$

Note that, compared to the two previous examples, the Robin-type boundary has moved to ω_3, while ω_2 is part of the Dirichlet region with value 10. Both operators L^α now have regions of degeneracy, when either x_1 or x_2 is near 1.

The PDE operator on Ω is discretized according to (9.4a)–(9.4b), while the Robin operators are approximated implicitly, consistent with Assumption 3.1. The behaviour of the value function v in the vicinity of ω_3 is depicted in Fig. 3. One observes how troughs of g^0 are attained by the value function, while near peaks of g^0 the numerical scheme switches to the reflection principle. Overall, the experiment demonstrates how the framework of the paper, not only allows us to approximate nonlinear Robin conditions, but also incorporates a nonlinear switching between Robin conditions on the one hand and Dirichlet conditions on the other hand.

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Appendix A. Interpretation of mixed boundary conditions

We briefly sketch in a simplified setting how mixed boundary conditions can arise from an underlying optimal control problem. For a start, we assume here that the solution v of (2.2) is smooth. In optimal control formulations, the coefficients c^α and $c^\alpha_{\partial\Omega}$ are typically either 0 or they all coincide with some constant in order to model a discounting of cost. We shall assume the former.

We consider a particle, or agent, which occupies the state $\mathbf{x}(t) \in \overline{\Omega}$ at time $t \in [0, T]$. Its movements are described by the following rules:

1. Suppose $\mathbf{x}(t) \in \Omega$, $t < T$ and the control $\alpha \in A$ is selected. Then the particle's immanent movement is described by the SDE
$$\mathrm{d}\mathbf{x} = b^\alpha(\mathbf{x})\,\mathrm{d}t + \sqrt{2\,a^\alpha}\,\mathrm{d}W, \tag{A.1}$$
where W is a d-dimensional Brownian motion. While the particle follows (A.1), it is subject to the cost $f^\alpha\,\mathrm{d}t$.

2. Suppose $\mathbf{x}(t) \in \partial\Omega^\circ_D$, $t < T$ and the control $\alpha \in A$ is selected. Here, $\partial\Omega^\circ_D$ refers to the interior of $\partial\Omega_D$ relative to $\partial\Omega$. Then, in the context of viscosity boundary conditions, the particle may either follow (A.1) at the running cost $f^\alpha\,\mathrm{d}t$ or terminate its movement at a cost of $g(\mathbf{x}(t))$. If instead pointwise Dirichlet conditions were imposed in Definition 2.2, then the boundary conditions would correspond to the guarantee that the particle terminates its movement.

3. Suppose $\mathbf{x}(t) \in \partial\Omega^\circ_R$, $t < T$ and the control $\alpha \in A$ is selected. Then the Skorokhod reflection principle may apply. Indeed, as described in (Kushner & Dupuis, 2001, Sections 1.4, 3.1.3, …), upon reaching the boundary, the particle may instantaneously be transported a distance $b^\alpha_{\partial\Omega}\,\delta$, where $\delta > 0$ is small. Alternatively, because of the context of viscosity boundary conditions, the particle may continue to follow (A.1). To remain within the scope of Kushner & Dupuis (2001), we assume $g^\alpha = 0$ on $\partial\Omega_R$.

4. Suppose $\mathbf{x}(t) \in \partial\Omega^\circ_t$, $t < T$ and the control $\alpha \in A$ is selected. Then the particle may either move according to
$$\mathrm{d}\mathbf{x} = b^\alpha_{\partial\Omega}(\mathbf{x})\,\mathrm{d}t, \tag{A.2}$$
with running cost $g^\alpha\,\mathrm{d}t$ or according to (A.1) with running cost $f^\alpha\,\mathrm{d}t$.

5. Suppose $t = T$ and the particle's movement has not yet terminated at a Dirichlet boundary, then the final time cost $v_T(\mathbf{x}(T))$ incurs.

6. Suppose that $\mathbf{x}(t) \in \partial(\partial\Omega_D) \cup \partial(\partial\Omega_R) \cup \partial(\partial\Omega_t)$, $t < T$. Here, the outer ∂ of $\partial(\partial\Omega_X)$ refers to the boundary of $\partial\Omega_X$ relative to $\partial\Omega$ where $X \in \{D, R, t\}$. Then the particle's behaviour may be

selected from multiple of the above scenarios. For example, if $\mathbf{x}(t) \in \partial(\partial\Omega_D) \cup \partial(\partial\Omega_R)$, then the particle movement may terminate, the particle may be reflected or it may be transported according to (A.1).

All these possible scenarios occur when representing uncertain market price of volatility risk in a Heston model found in Jaroszkowski & Jensen (2021).

We now link the above description of the particle by means of the value function to the HJB final time boundary value problem (2.2). Let $\boldsymbol{\alpha} : [0, T] \to A$ represent a choice of controls for each time $s \in [0, T]$. Similarly, let $\xi_\Omega, \xi_{\partial\Omega_D}, \xi_{\partial\Omega_R}, \xi_{\partial\Omega_t} : [0, T] \times \overline{\Omega} \to \{0, 1\}$ be indicator functions such that $\operatorname{supp} \xi_{\partial\Omega_X} \subset \overline{\partial\Omega_X}$ for $X \in \{D, R, t\}$, where $\overline{\partial\Omega_X}$ is the closure of $\partial\Omega_X$ relative to $\partial\Omega$. Furthermore,

$$\xi_\Omega + \xi_{\partial\Omega_D} + \xi_{\partial\Omega_R} + \xi_{\partial\Omega_t} \equiv 1.$$

Where $\xi_\Omega = 1$ the particle path $\mathbf{x}$ obeys (A.1), where $\xi_{\partial\Omega_D} = 1$ the particle terminates, where $\xi_{\partial\Omega_R} = 1$ the particle is reflected and where $\xi_{\partial\Omega_t} = 1$ the particle follows (A.2). Since the particle terminates where $\xi_{\partial\Omega_D} = 1$, we requite $\xi_{\partial\Omega_D}(s_1, x) = 1 \Rightarrow \xi_{\partial\Omega_D}(s_2, x) = 1$ for $s_1 \leqslant s_2$. The value function v at (t, x) is the smallest cost realized among all possible choices for $\boldsymbol{\alpha}$ and $\boldsymbol{\xi} = (\xi_\Omega, \xi_{\partial\Omega_D}, \xi_{\partial\Omega_R}, \xi_{\partial\Omega_t})$:

$$v(t, x) = \inf_{\boldsymbol{\alpha},\boldsymbol{\xi}} \mathbf{E}_{xt}\Big(\int_t^\tau \xi_\Omega(\mathbf{x}(s)) f^{\boldsymbol{\alpha}(s)}(\mathbf{x}(s)) + \xi_{\partial\Omega_t}(\mathbf{x}(s))\, g^{\boldsymbol{\alpha}(s)}(\mathbf{x}(s))\, \mathrm{d}s$$
$$+ \xi_{\partial\Omega_D}(\mathbf{x}(\tau))\, g(\mathbf{x}(\tau)) + \xi_\Omega(\mathbf{x}(\tau))\, v_T(\mathbf{x}(\tau))\Big),$$

where τ is the exit time from $[0, T) \times \Omega$ of $\mathbf{x}$ and $\mathbf{E}_{xt}$ the expectation conditional to $\mathbf{x}(t) = x$.

We fix some $\boldsymbol{\alpha}, \boldsymbol{\xi}$, which are not necessarily optimal. Suppose that $\mathbf{x}(s) \in \operatorname{supp} \xi_{\partial\Omega_t}$ for a short duration $[t, t + \varepsilon) \ni s$. Then,

$$g^{\boldsymbol{\alpha}(t)}(\mathbf{x}(t)) = \lim_{h\to 0} \frac{1}{h} \int_t^{t+h} g^{\boldsymbol{\alpha}(s)}(\mathbf{x}(s))\, \mathrm{d}s \geqslant -\lim_{h\to 0} \frac{v(t + h, \mathbf{x}(t + h)) - v(t, x)}{h}$$
$$= -\partial_t v(t, \mathbf{x}) - \nabla v(t, \mathbf{x}) \cdot \dot{\mathbf{x}} = -\partial_t v(t, \mathbf{x}) - \nabla v(t, \mathbf{x}) \cdot b^\alpha_{\partial\Omega}(\mathbf{x}).$$

Here, the first equality follows from continuity, the inequality from the dynamic programming principle, the second equality from the chain rule and the third from (A.2).

Now, suppose that $\mathbf{x}(s) \subset \operatorname{supp} \xi_\Omega$ for a short duration $[t, t + \varepsilon) \ni s$. Then, by a similar argument, detailed in Kushner & Dupuis (2001), one finds

$$f^{\boldsymbol{\alpha}(t)}(\mathbf{x}(t)) \geqslant -\partial_t v(t, \mathbf{x}) - \nabla v(t, \mathbf{x}) \cdot b^\alpha(\mathbf{x}) - a^\alpha \nabla v(t, \mathbf{x}).$$

On $\operatorname{supp} \xi_{\partial\Omega_D}$, we find $g(\mathbf{x}(t)) \geqslant v(t, \mathbf{x}(t))$ as the minimal cost cannot be more than the cost of termination.

Suppose now that the particle is located at $x \in \operatorname{supp} \xi_{\partial\Omega_R}$. It cannot be more beneficial for the particle to be at $x + b^\alpha_{\partial\Omega}\lambda$, as it will immanently be transported there. Thus, $v(t, x) \leqslant v(t, x + b^\alpha_{\partial\Omega}\lambda)$, and therefore, with $\lambda \to 0$,

$$-\, b^\alpha_{\partial\Omega}(x) \cdot \nabla v(t, x) \leqslant 0.$$

When and wherever the choice of $\boldsymbol{\alpha}, \boldsymbol{\xi}$ is optimal, the respective above inequality turns into an equality. With the compactness of A and the continuous dependence of the coefficients on α, such optimal controls exist. Therefore, taking suprema over A, one obtains that the value function solves (2.2), at least conceptually, with boundary conditions in the viscosity sense. We refer here to the viscosity sense because the use of semicontinuous envelopes in Definition 1 is interpreted as permitting (A.1) as transport law on all of the closure $\overline{\Omega}$, and as offering at the interfaces between boundary regions multiple boundary operators for the choice of the optimal strategy, like indicated in scenario 6 of the above list. We note that the choice between (A.1) and the various boundary operators will in general be subject to some delicate restrictions, arising from the sub- and superjets. At the boundary, these jets are increased in size compared to their counterparts in the domain interior (Crandall *et al.*, 1992, Remark 2.7).