Appendix A

Proof of Proposition 1 (Equation 5 with cost function 4)

The proof follows from Proposition 2 by replacing the cost functions of Proposition 2, and the geometric sum D_n by arithmetic cost functions and an arithmetic sum, since the case enumerations for Propositions 1 and 2 are exactly the same. Although not central to the argument, the replacement cost functions are as follows.

For n-non-service players $P_2,\ldots,P_n$ with values $v_2-\varepsilon,\ldots,v_n-\varepsilon$ define a cost function by replacing the geometric term $d^{|C|}$ in (6) by the arithmetic term d.

$v(C \cup P_r) = v(C) + v(P_r) - dv_r - dm = v(C) + v_r - \varepsilon - dv_r - dm$ where $0 < d < 1$ is a constant factor and m is the median of $v_2,\ldots,v_n$. (A1)

Similarly, the Service player P_1 has value $(n-1)\varepsilon$, and its cost function is derived by replacing the geometric term $d^{|C|}$ in (6a) by the arithmetic term d.

$$v(C \cup P_1) = v(C) + (n-1)\varepsilon + (n-1)dm \qquad \text{(A1a)}$$

The proof then uses result B8, with the replacement $D_n \rightarrow d + d + \ldots + d = (n-1)d$,

For the non-service cases, $2 \le r \le n$

$$SH(n,r) = v_r - \varepsilon - (m + v_r)D_n/n \rightarrow v_r - \varepsilon - (m + v_r)(n-1)d/n = v_r - \varepsilon - dv_r(1 - 1/n) - dm(1-1/n). \qquad \text{(A2)}$$

Similarly, for the service case

$$SH(n,1) = (n-1)\varepsilon + (n-1)mD_n/n \rightarrow (n-1)\varepsilon + (n-1)m(n-1)d/n = (n-1)\varepsilon + (n-1)dm(1-1/n). \qquad \text{(A2a)}$$

This completes the proof of *Proposition 1*. Equations A2 and A2a correspond to (5).

Appendix B

Proof of *Proposition 2* (Equation 7 with cost function 6)

We consider enumerations for four cases.

Case 1: P_1 is first in the permutation: allocation is to the service P_1

P_1 is first $(n-1)!$ times out of $n!$, and the marginal allocation to P_1 each time is $(n-1)\varepsilon$ (from Equation 7). The total marginal allocation for this case is

$$M_{(1)}(P_1) = (n-1)! \times (n-1)\varepsilon \qquad \text{(B2)}$$

Case 2: P_1 is not first in the permutation: allocation is to the service P_1

P_1 is not first $[n! - (n-1)!]$ times. When P_1 joins the coalition, P_1 receives a marginal allocation $(n-1)\varepsilon$. Additionally there are $(n-1)!$ diversification cases of each of the following, to be added:

$(n-1)md$, $(n-1)md^2$, ..., $(n-1)md^{n-1}$.

Therefore the total marginal allocation for this case is (with $D_n = d + d^2 + ... + d^{n-1}$)

$$M_{(2)}(P_1) = [n! - (n-1)!] \times (n-1)\varepsilon + (n-1)!\,(n-1)m(d + d^2 + ... + d^{n-1})$$

$$= [n! - (n-1)!] \times (n-1)\varepsilon + (n-1)!\,(n-1)mD_n \qquad \text{(B3)}$$

Case 3: P_r (a non-service) is first in the permutation: allocation is to P_r

P_r is first in $(n-1)!$ cases, each with marginal allocation $v_r - \varepsilon$. There is no diversification. The total marginal allocation for this case is

$$M_{(3)}(P_r) = (n-1)! \times (v_r - \varepsilon) \qquad \text{(B4)}$$

Case 4: P_r is not first in the permutation: allocation is to P_r

P_r is not first in $[n! - (n-1)!]$ cases. When P_r joins the coalition, P_r receives a marginal allocation $(v_r - \varepsilon)$ in all of those cases. Additionally there are $(n-1)!$ diversification cases of each of the following, to be subtracted:

$v_r d$, $v_r d^2$, ..., $v_r d^{n-1}$,

md, md^2, ..., md^{n-1}.

The total marginal allocation for this case is then (with $D_n = d + d^2 + ... + d^{n-1}$ as in Case 2)

$$M_{(4)}(P_r) = [n! - (n-1)!] \times (v_r - \varepsilon) - (n-1)!(v_r + m)(d + d^2 + ... + d^{n-1})$$

$$= [n! - (n-1)!] \times (v_r - \varepsilon) - (n-1)!(v_r + m)D_n \qquad \text{(B5)}$$

By symmetry, all non-service players can be analysed in the same way and have the results that follow the same pattern.

The total marginal allocation for P_1, $M(P_1)$ is the sum of the marginal in (B2) and (B3).

$$M(P_1) = M_{(1)}(P_1) + M_{(2)}(P_1)$$

$$= (n-1)!\times(n-1)\varepsilon + [n! - (n-1)!]\times(n-1)\varepsilon + (n-1)!\,(n-1)mD_n$$

$$= n!\times(n-1)\varepsilon + (n-1)m(n-1)!D_n \qquad \text{(B6)}$$

The total marginal allocation for P_r, $M(P_r)$ is the sum of the marginal in (B4) and (B5).

$$M(P_r) = M_{(3)}(P_r) + M_{(4)}(P_r)$$

$$= (n-1)!\times(v_r - \varepsilon) + [n! - (n-1)!]\times(v_r - \varepsilon) - (n-1)!(v_r + m)D_n$$

$$= n!\times(v_r - \varepsilon) - (v_r + m)D_n(n-1)! \qquad \text{(B7)}$$

The final stage in the proof is to calculate the mean marginal allocation by dividing (B6) and (B7) by the total number of permutations, $n!$

$$SH(n,1) = (n-1)\varepsilon + (n-1)mD_n/n \qquad \text{(B8a)}$$

$$SH(n,r) = v_r - \varepsilon - (v_r + m)D_n/n \qquad (2 \leq r \leq n) \qquad \text{(B8)}$$

This completes the proof of *Proposition 2*, and Equations (B8, B8a) correspond to (7).

Appendix C

Summary of the LDA algorithm (Frachot et al 2001)

The algorithm was designed to estimate value-at-risk (VaR) in the context of operational risk losses. Therefore we use to the term ‘loss’ in this appendix, although, in principle, the data can originate from elsewhere. LDA is a Monte Carlo process with *T* trials. Given a list of *N* losses covering a period *Y* years, the algorithm requires a pre-calculated severity distribution *D* for the losses (typically fat-tailed such a Lognormal).

Algorithm LDA(*T*):

1. Calculate frequency $f = N/Y$
2. For trial *t* from 1 to *T* do
    a. Generate a loss number *n* from a Poisson(*f*) distribution. This represents an annual number of losses
    b. Generate a random sample of size *n* from *D*
    c. Calculate the sum *S*(*t*) of the elements in the random sample in the previous step
3. End_For
4. Calculate VaR = the 99.9 percentile of the *S*(*t*)