

Sliding Mode Control of Nonlinear Systems with Input Distribution Uncertainties

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Abstract—In this paper, a sliding mode control design method is developed for a class of fully nonlinear systems in generalized regular form, where both input distribution uncertainty and system uncertainties are considered. Based on the generalized regular form, a novel nonlinear sliding surface is designed and uniform ultimate stability of the corresponding sliding mode dynamics is analyzed. Then, under the assumption that the uncertainties are bounded by known nonlinear functions of the system states, a sliding mode controller is formulated to ensure that the dynamical system reaches the sliding surface in finite time even in the presence of the system uncertainties and input distribution uncertainties. Further, for the case of system uncertainty with unknown bound in parameterized form, an adaptive sliding mode controller is developed to drive the dynamical system to the sliding surface and maintain a sliding motion thereafter. The developed sliding mode controller is applied to a High Incidence Research Model (HIRM) aircraft model. Simulation results demonstrate that the developed methods are effective.

Index Terms—Sliding mode control, input distribution uncertainty, adaptive control, nonlinear systems.

I. INTRODUCTION

Sliding mode control employs discontinuous control signals to drive system trajectories to the pre-designed sliding surface in finite time and maintain a sliding motion on it thereafter (see [1]–[3]). This control method has been extensively applied to deal with matched and mismatched uncertainties/disturbances [4]–[9]. There is a class of uncertainties acting in the input channel, called input distribution uncertainties. Such uncertainties widely exist in the real world and usually arise due to modelling error, parameter variation and certain disturbances in the system input channel. It should be emphasised that the coupling between the input distribution uncertainty and the control signal occurs in a multiplicative way, which results in interaction between the uncertainty and the control signal. It follows that research on input distribution uncertainties is particularly difficult specifically in control design for nonlinear systems, and the corresponding results are very few (see [10]–[14]). In nearly all of the existing work, it is required that the uncertainties in the input distribution are matched, bounded by a constant or in a parameterized form. To deal with nonlinear uncertainties in the input distribution is challenging specifically when both the input distribution and the bounds on the input uncertainties are nonlinear.

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In sliding mode control, the equivalent control technique is a well known approach for analyzing the stability of the sliding motion (see [1], [3]). However, it is required that the exact sliding mode dynamics can be distinguished from the corresponding equivalent equations in order to reduce the conservatism which is usually very difficult for nonlinear systems [15]. Moreover, it becomes more difficult when the system involves input distribution uncertainties. For the well-known regular form based sliding mode control design, it is required that the system is either in a traditional regular form or can be transformed into such a form for controller design (see [1], [2], [16]). It is straightforward to find such a transformation matrix to transfer a linear system to the regular form but it is very difficult to obtain the associated diffeomorphism for nonlinear systems (see [17] and reference therein). This is particularly true for nonlinear systems with input distribution uncertainties as in this case the traditional definition of relative degree needs to be further modified. Although there are some results for linear systems with input distribution uncertainties, the corresponding studies for nonlinear systems are rarely found. This motivates the current investigation for nonlinear systems with nonlinear uncertainties in the input distribution.

This paper is focused on sliding mode control design for a class of fully nonlinear systems in a generalized regular form, where actuator and model uncertainties, specifically, uncertainty in the input distribution, are present. The main contributions of this paper can be summarized as follows:

- (i) A class of fully nonlinear systems with input distribution uncertainties is considered, which is in a generalized regular form. The generalized regular form includes the traditional regular form as a special case and thus the developed results can be applied to a wide class of systems.
- (ii) A novel nonlinear sliding surface is proposed and a new sliding mode controller to deal with the input distribution uncertainties is constructed to ensure that the resulting dynamical system reaches the sliding surface in finite time despite the presence of matched and mismatched uncertainties, which may be nonlinear functions of the state variables.
- (iii) For bounded system uncertainties in parameterized form with unknown parameters, an adaptive sliding mode approach is proposed. The properties of the update law including a detailed strict stability analysis are provided.

The paper is organized as follows: In Section II, the considered systems are presented, and the problem is then formulated. In Section III, the stability of the sliding mode is analyzed, and the sliding mode controller is designed. In Section IV, an adaptive sliding mode control scheme is proposed to deal with system uncertainties where the bounds are in parameterized form. In Section V, simulations of a HIRM aircraft are presented. Section VI concludes the paper.

II. SYSTEM DESCRIPTION AND ANALYSIS

Consider a class of nonlinear systems with uncertainties described by

$$\dot{x}_1(t) = f_1(x, t) + \Delta g_1(x, t)u(t) + \psi_1(x, t), \quad (1)$$

$$\dot{x}_2(t) = f_2(x, t) + (g_2(x, t) + \Delta g_2(x, t))u(t) + \psi_2(x, t), \quad (2)$$

where $x(t) = \text{col}(x_1, x_2) \in D \subset R^n$ with $x_1(t) \in D_1 \subset R^{n-m}$ and $x_2(t) \in D_2 \subset R^m$ is the system state (the domain $D = D_1 \times D_2$ is a neighbourhood of the origin) and $u(t) \in R^m$ is the system input. The vector functions $f_1(x, t) \in R^{n-m}$, $f_2(x, t) \in R^m$ and the matrix function $g_2(x, t) \in R^{m \times m}$ are known with $g_2(x, t)$ being nonsingular for $(x, t) \in D \times R^+$, where R^+ represents the set of all non-negative real numbers. The matrix functions $\Delta g_1(x, t) \in R^{(n-m) \times m}$, $\Delta g_2(x, t) \in R^{m \times m}$ and vector functions $\psi_1(x, t) \in R^{n-m}$, $\psi_2(x, t) \in R^m$ denote the uncertainties experienced by the system. It is assumed that all the nonlinear functions involved in the system are smooth enough for further analysis.

Remark 1: It should be noted that the methodology developed in this paper can be applied to all the nonlinear systems $\dot{x}(t) = f(x, t) + (g(x, t) + \Delta g(x, t))u(t) + \psi(x, t)$ with $x(t) \in R^n$ and $u(t) \in R^m$, which can be transformed to the form (1)-(2) by a diffeomorphism. For simplicity, this paper is focused on the system (1)-(2). If $\Delta g_1(x, t) = 0$, the system (1)-(2) becomes the standard regular form for nonlinear systems as previously studied in [17]. Thus, the proposed method for system (1)-(2) in this paper can also be used for nonlinear systems in the standard regular form. \square

Assumption 1: The uncertainties $\Delta g_1(x, t)$, $\Delta g_2(x, t)$, $\psi_1(x, t)$ and $\psi_2(x, t)$ in system (1)-(2) satisfy

$$\Delta g_1(x, t) = h_1(x, t)\Delta(x, t), \quad (3)$$

$$\Delta g_2(x, t) = h_2(x, t)\Delta(x, t), \quad \|\Delta(x, t)\| \leq \delta(x, t), \quad (4)$$

$$\|\psi(x, t)\| = \left\| [\psi_1(x, t), \psi_2(x, t)]^T \right\| \leq \eta(x, t), \quad (5)$$

where $h_1(x, t) \in R^{(n-m) \times r}$ and $h_2(x, t) \in R^{m \times r}$ are continuous, known and bounded in $x \in D$, $\Delta(x, t) \in R^{r \times m}$ is unknown, $\delta(x, t)$ and $\eta(x, t)$ are known continuous non-negative functions.

Remark 2: Assumption 1 describes the limitation on the uncertainties $\Delta g_1(x, t)$, $\Delta g_2(x, t)$, $\psi_1(x, t)$ and $\psi_2(x, t)$. The known matrices $h_1(x, t)$ and $h_2(x, t)$ in (3) and (4) are used to describe the structural distributions of the uncertainties $\Delta g_1(x, t)$ and $\Delta g_2(x, t)$, respectively, which are assumed to be bounded in $x \in D$. Assumption 1 implies that all the uncertainties are required to be bounded by known nonlinear functions. These bounds will be employed in the later system analysis and sliding mode control design to reduce conservatism and enhance robustness. \square

For system (1)-(2), it is assumed that a sliding function $\sigma(x)$ is designed. Then, the sliding surface is described by

$$S = \{x \in R^n \mid \sigma(x) = 0\}. \quad (6)$$

The concept of the generalised regular form is introduced as follows:

Definition 1 [22]: The system (1)-(2) is called the generalized regular form associated with the sliding surface (6) if

$$h_1(x, t)|_{x \in S} = 0, \quad (7)$$

where the function $h_1(\cdot)$ is given in (3) and S is the sliding surface defined in (6).

Using the above foundations, a set of conditions can now be developed and a sliding mode control designed such that the corresponding controlled system (1)-(2) is uniformly ultimately stable.

III. SLIDING MODE STABILITY ANALYSIS AND CONTROL DESIGN

In this section, assume that a sliding function $\sigma(x)$ exists to render the system (1)-(2) in the generalised regular form. The corresponding sliding mode dynamics for system (1)-(2) are derived and the stability of the sliding motion is analyzed. Then, a sliding mode control is designed to deal with the input distribution uncertainties caused by $\Delta(x, t)$ in (4) as well as the mismatched uncertainties $\psi_1(x, t)$ and $\psi_2(x, t)$, and drive the system (1)-(2) to the designed sliding surface.

A. Stability Analysis of the Sliding Mode

For the sliding mode analysis, the following assumption is imposed on the system (1)-(2).

Assumption 2: The system (1)-(2) has a generalised regular form associated with the sliding function $\sigma(x)$ described by

$$\sigma(x) =: Kx_2 - \gamma(x_1), \quad (8)$$

where $K \in R^{m \times m}$ is a nonsingular matrix to be designed, and the nonlinear function $\gamma(\cdot) \in R^m$ is continuously differentiable in the considered domain D_1 .

Remark 3: For sliding mode control, the sliding function $\sigma(x_1, x_2)$ should be designed to ensure that there exists a unique solution $x_2 = \Upsilon(x_1) : R^{n-m} \rightarrow R^m$ such that $\sigma(x_1, \Upsilon(x_1)) = 0$. Assumption 2 gives a form of the sliding function, which combined with Definition 1, helps to develop and analyse the sliding mode dynamics. It should be pointed out that the sliding function for the considered uncertain nonlinear system (1)-(2) can be a general nonlinear function, and in this case the implicit function theorem can be used to derive the required sliding function (8). \square

Under Assumption 2 and from Definition 1, the condition (7) holds with $\sigma(\cdot)$ defined in (8). Therefore, when the system (1)-(2) is constrained to the sliding surface (6), it follows from (8) that the subsystem (1) has the following form:

$$\dot{x}_1(t) = f_1^s(x_1, t) + \psi_1^s(x_1, t), \quad (9)$$

where $f_1^s(x_1, t) = f_1(x, t)|_{x_2=K^{-1}\gamma(x_1)}$ and $\psi_1^s(x_1, t) = \psi_1(x, t)|_{x_2=K^{-1}\gamma(x_1)}$ is the uncertain term.

Therefore, the system (9) is the sliding mode dynamics of system (1)-(2) corresponding to the sliding function (8). To undertake analysis of the sliding mode dynamics, the following assumption is introduced to guarantee that the nominal system (9) is uniformly ultimately bounded.

Assumption 3: There exists a continuously differentiable Lyapunov function $V(x_1, t) : D_1 \times R^+ \mapsto R$ satisfying

$$c_1(\|x_1\|) \leq V(x_1, t) \leq c_2(\|x_1\|), \quad (10)$$

and for $x_1 \in D_1$, $\forall \|x_1\| \geq \mu > 0$,

$$\frac{\partial V}{\partial t} + \left(\frac{\partial V}{\partial x_1} \right)^T f_1^s(x_1, t) \leq -W_1(x_1), \quad (11)$$

where the functions $c_1(\cdot)$ and $c_2(\cdot)$ are continuous class \mathcal{K} functions, μ is a positive constant, $W_1(x_1)$ is a continuous positive definite functions in D_1 , and $f_1^s(\cdot)$ is given in (9).

Recall the sliding mode dynamics (9), which includes mismatched uncertainty $\psi_1^s(x_1, t)$. From Assumption 1 and the definition of $\psi_1^s(x_1, t)$ in (9), it is straightforward to see that

$$\left\| \psi_1^s(x_1, t) \right\| \leq \eta_{\psi_1^s}(x_1, t), \quad (12)$$

where $\eta_{\psi_1^s}(x_1, t) = \eta(x, t)|_{x_2=K^{-1}\gamma(x_1)}$ is a known positive continuous function.

Then, the following result is ready to be presented.

Theorem 1: Under Assumptions 1-3, the solutions of the sliding mode dynamics (9) are uniformly ultimately bounded if there exists a continuous positive definite function $W_2 : D_1 \mapsto R^+$ such that in the considered domain D_1 , $\forall \|x_1\| \geq \mu > 0$,

$$W_1(x_1) - \left\| \left(\frac{\partial V}{\partial x_1} \right)^T \right\| \eta_{\psi_1^s}(x_1, t) \geq W_2(x_1), \quad (13)$$

for any $t \in R^+$, where $W_1(x_1)$ is defined in Assumption 3, and $\eta_{\psi_1^s}(x_1, t)$ satisfies (12).

Proof: Consider the Lyapunov candidate function $V(\cdot)$ satisfying Assumption 3 for system (9). From (11), the time derivative of $V(\cdot)$ along the trajectory of system (9) is given by:

$$\begin{aligned} \dot{V} &= \frac{\partial V}{\partial t} + \left(\frac{\partial V}{\partial x_1} \right)^T (f_1^s(x_1, t) + \psi_1^s(x_1, t)) \\ &\leq -W_1(x_1) + \left\| \left(\frac{\partial V}{\partial x_1} \right)^T \right\| \eta_{\psi_1^s}(x_1, t) \\ &\leq -W_2(x_1), \end{aligned} \quad (14)$$

where (12) and (13) are used above. So, $\forall \|x_1\| \geq \mu > 0$ and $t \in R^+$, $\dot{V} < 0$. Hence, the conclusion follows. \square

Remark 4: Assumption 3 shows the stability conditions of a general nonlinear system without uncertainty. It holds if the nonlinear system is uniformly ultimately bounded. For a linear system, Assumption 3 becomes necessary and sufficient to guarantee the equilibrium point $x = 0$ is uniformly ultimately bounded. Note that condition (11) only needs to hold for all $\|x_1\| \geq \mu > 0$ and $t \in R^+$ in D_1 . Furthermore, the bound of the solution $x_1(t)$ can be estimated as follows if the corresponding Lyapunov function is available: for the initial state $x_1(t_0)$ satisfying $\|x_1(t_0)\| \leq c_2^{-1}(c_1(r))$, where $r > 0$ ensures $B_r \subset D_1$ (B_r is a ball with radius r) and $\mu < c_2^{-1}(c_1(r))$, there exists $T \geq 0$ and a class \mathcal{K}_∞ function ρ such that the solutions $x_1(t)$ of the sliding mode dynamics (9) satisfy $\|x_1(t)\| \leq \rho(\|x_1(t_0)\|, t - t_0)$, for $t_0 \leq t \leq t_0 + T$, and $\|x_1(t)\| \leq c_2^{-1}(c_1(\mu))$, for $t > t_0 + T$, where $c_1(\cdot)$ and $c_2(\cdot)$ are defined in (10). This provides a way to estimate the ultimate bound of the steady state of the sliding motion governed by the dynamics (9). \square

B. Reachability Analysis

Theorem 1 above has guaranteed the uniform ultimate boundedness of the sliding mode dynamics (9). In this subsection, a sliding mode controller will be designed such that system (1)-(2) is driven to the sliding surface (6) in finite time. It should be noted that in the following control design, there is no specific requirement on the structure of the sliding function $\sigma(\cdot) \in R^m$ in (6).

For convenience, define the function matrices $g_\Delta(x, t)$, $f(x, t)$, $h(x, t)$ and $\psi(x, t)$ as

$$g_\Delta(x, t) = \frac{\partial \sigma}{\partial x_2} g_2(x, t), \quad f(x, t) = \begin{bmatrix} f_1(x, t) \\ f_2(x, t) \end{bmatrix}, \quad (15)$$

$$h(x, t) = \begin{bmatrix} h_1(x, t) \\ h_2(x, t) \end{bmatrix}, \quad \psi(x, t) = \begin{bmatrix} \psi_1(x, t) \\ \psi_2(x, t) \end{bmatrix}, \quad (16)$$

where $g_2(x, t)$, $f_1(x, t)$, $f_2(x, t)$, $h_1(x, t)$, $h_2(x, t)$, $\psi_1(x, t)$ and $\psi_2(x, t)$ are given in (1)-(2). The following assumption is required for sliding mode controller design.

Assumption 4: For $x \in D$ and $t \in R^+$, the function matrix $g_\Delta(x, t)$ defined in (15) is nonsingular and satisfies $1 - \Pi(x, t) \|g_\Delta^{-1}(x, t)\| > 0$ with $\Pi(x, t) = \left\| \frac{\partial \sigma}{\partial x} h(x, t) \right\| \delta(x, t)$, and $\delta(x, t)$ being given in (4).

Remark 5: Assumption 4 assumes that the input distribution matrices are nonsingular, which is commonly used in sliding mode control design to guarantee the existence of the controller. The term $1 - \Pi(x, t) \|g_\Delta^{-1}(x, t)\|$ contains the distribution matrices of the input and its corresponding uncertainty. Assumption 4 implies $\|g_\Delta(x, t)\| > \Pi(x, t)$, i.e., the norm of the input uncertain distribution matrix cannot be larger than that of the input distribution matrix. This condition guarantees that the designed controller is able to deal with the class of input distribution uncertainties which interact with the control signal. \square

For system (1)-(2), the control law is designed as

$$\begin{aligned} u(x, t) &= -g_\Delta^{-1}(x, t) \frac{\partial \sigma}{\partial x} f(x, t) \\ &\quad - g_\Delta^{-1}(x, t) \text{sgn}(\sigma(x, t)) \left(l(x, t) + \left\| \frac{\partial \sigma}{\partial x} \right\| \eta(x, t) \right), \end{aligned} \quad (17)$$

where $l(x, t)$ is a nonnegative, time-varying gain and $g_\Delta^{-1}(x, t)$ is the inverse matrix of $g_\Delta(x, t)$.

Theorem 2: Under Assumption 1 and Assumption 4, the sliding mode control in (17) drives the dynamical system (1)-(2) to the sliding surface (6) in finite time and maintains a sliding motion on it thereafter if the control gain $l(x, t)$ in (17) satisfies

$$\begin{aligned} l(x, t) &\geq \frac{1}{1 - \Pi(x, t) \|g_\Delta^{-1}(x, t)\|} \left(\epsilon + \Pi(x, t) \|g_\Delta^{-1}(x, t)\| \right. \\ &\quad \left. \times \left(\left\| \frac{\partial \sigma}{\partial x} f(x, t) \right\| + \left\| \frac{\partial \sigma}{\partial x} \right\| \eta(x, t) \right) \right), \end{aligned} \quad (18)$$

where $\delta(x, t)$ satisfies (4) and $\epsilon > 0$.

Proof: Substituting the controller (17) into (1)-(2), it can be seen that

$$\begin{aligned} &\sigma^T(x) \dot{\sigma}(x) \\ &\leq \sigma^T(x) \frac{\partial \sigma}{\partial x} h(x, t) \Delta(x, t) u(t) - l(x, t) \|\sigma(x)\| \\ &\quad + \sigma^T(x) \frac{\partial \sigma}{\partial x} \psi(x, t) - \|\sigma(x)\| \left\| \frac{\partial \sigma}{\partial x} \right\| \eta(x, t). \end{aligned} \quad (19)$$

From (5), it follows that $\sigma^T(x) \frac{\partial \sigma}{\partial x} \psi(x, t) - \|\sigma(x)\| \left\| \frac{\partial \sigma}{\partial x} \right\| \eta(x, t) \leq 0$. Then,

$$\begin{aligned} &\sigma^T(x) \dot{\sigma}(x) \\ &\leq \left\| \sigma^T(x) \frac{\partial \sigma}{\partial x} h(x, t) \Delta(x, t) g_\Delta^{-1}(x, t) \frac{\partial \sigma}{\partial x} f(x, t) \right\| \\ &\quad + \|\sigma(x)\| \left\| \frac{\partial \sigma}{\partial x} h(x, t) \Delta(x, t) g_\Delta^{-1}(x, t) \right\| l(x, t) \\ &\quad + \|\sigma(x)\| \left\| \frac{\partial \sigma}{\partial x} h(x, t) \Delta(x, t) g_\Delta^{-1}(x, t) \right\| \left\| \frac{\partial \sigma}{\partial x} \right\| \eta(x, t) \\ &\quad - l(x, t) \|\sigma(x)\| \\ &\leq \|\sigma(x)\| \left\| \frac{\partial \sigma}{\partial x} h(x, t) \right\| \delta(x, t) \|g_\Delta^{-1}(x, t)\| \left(\left\| \frac{\partial \sigma}{\partial x} f(x, t) \right\| \right. \\ &\quad \left. + l(x, t) + \left\| \frac{\partial \sigma}{\partial x} \right\| \eta(x, t) \right) - \|\sigma(x)\| l(x, t) \\ &= -\|\sigma(x)\| \left(l(x, t) - \left\| \frac{\partial \sigma}{\partial x} h(x, t) \right\| \delta(x, t) \|g_\Delta^{-1}(x, t)\| \right. \\ &\quad \left. \left(\left\| \frac{\partial \sigma}{\partial x} f(x, t) \right\| + \left\| \frac{\partial \sigma}{\partial x} \right\| \eta(x, t) + l(x, t) \right) \right). \end{aligned} \quad (20)$$

From (18), and the definition of $\Pi(x, t)$,

$$\begin{aligned} &\left(1 - \left\| \frac{\partial \sigma}{\partial x} h(x, t) \right\| \delta(x, t) \|g_\Delta^{-1}(x, t)\| \right) l(x, t) \\ &\geq \epsilon + \left\| \frac{\partial \sigma}{\partial x} h(x, t) \right\| \delta(x, t) \|g_\Delta^{-1}(x, t)\| \\ &\quad \times \left(\left\| \frac{\partial \sigma}{\partial x} f(x, t) \right\| + \left\| \frac{\partial \sigma}{\partial x} \right\| \eta(x, t) \right), \end{aligned} \quad (21)$$

which can be rewritten as

$$\begin{aligned} l(x, t) &- \left\| \frac{\partial \sigma}{\partial x} h(x, t) \right\| \delta(x, t) \|g_\Delta^{-1}(x, t)\| \left(l(x, t) \right. \\ &\quad \left. + \left\| \frac{\partial \sigma}{\partial x} f(x, t) \right\| + \left\| \frac{\partial \sigma}{\partial x} \right\| \eta(x, t) \right) \geq \epsilon. \end{aligned} \quad (22)$$

Substituting (22) into (20) yields

$$\sigma^T(x)\dot{\sigma}(x) \leq -\epsilon\|\sigma(x)\|. \quad (23)$$

Therefore, the reachability condition holds and hence the result follows from $\epsilon > 0$. ∇

Theorems 1 and 2 together show that the corresponding closed-loop system formed by applying the controller (17) to the dynamical system (1)-(2), is uniformly ultimately bounded. Therefore, the proposed sliding mode control scheme can guarantee the state $x(t)$ of the dynamical system (1)-(2) is uniformly ultimately bounded.

IV. UNKNOWN BOUNDEDNESS OF UNCERTAINTY $\psi(x, t)$

The case where the bound on the uncertainty $\psi(x, t)$ is in a parameterized form with unknown parameters, rather than a known nonnegative function, is now considered. First, the system (1)-(2) is rewritten in the following form:

$$\begin{aligned} \dot{x}_1(t) &= f_{11}(x_1, x_{21}, t) + f_{12}(x_1, x_{21}, t)x_{22}(t) \\ &\quad + \Delta g_1(x_1, x_{21}, t)u(t) + \psi_1(x_1, x_{21}, t), \end{aligned} \quad (24)$$

$$\begin{aligned} \dot{x}_{21}(t) &= f_{21}(x, t) + (g_{21}(x, t) + \Delta g_{21}(x, t))u(t) \\ &\quad + \psi_{21}(x, t), \end{aligned} \quad (25)$$

$$\begin{aligned} \dot{x}_{22}(t) &= f_{22}(x, t) + (g_{22}(x, t) + \Delta g_{22}(x, t))u(t) \\ &\quad + \psi_{22}(x, t), \end{aligned} \quad (26)$$

where $x(t) = \text{col}(x_1, x_2) \in D \subset R^n$ is the system state with $x_1(t) \in D_1 \subset R^{n-m}$, $x_2(t) = \text{col}(x_{21}, x_{22}) \in R^m$, $x_{21}(t) \in D_{21} \subset R^{m_1}$ and $x_{22}(t) \in D_{22} \subset R^{m_2}$, $m = m_1 + m_2$, (the domain $D = D_1 \times D_{21} \times D_{22}$ is a neighbourhood of the origin), $u(t) \in R^m$ is the system input. The vector functions $f_{11}(x_1, x_{21}, t)$, $f_{21}(x, t)$, $f_{22}(x, t)$, and the matrix functions $f_{12}(x_1, x_{21}, t)$, $g_{21}(x, t)$, $g_{22}(x, t)$ are known with $[g_{21}^T(x, t) \ g_{22}^T(x, t)]^T$ being nonsingular for $(x, t) \in D \times R^+$. The matrix functions $\Delta g_1(x_1, x_{21}, t)$, $\Delta g_{21}(x, t)$, $\Delta g_{22}(x, t)$ and vector functions $\psi_1(x_1, x_{21}, t)$, $\psi_{21}(x, t)$, $\psi_{22}(x, t)$ are the uncertainties, which satisfy the following assumption.

Assumption 5: The uncertainties $\Delta g_1(x_1, x_{21}, t)$, $\Delta g_{21}(x, t)$, $\Delta g_{22}(x, t)$, $\psi_1(x_1, x_{21}, t)$, $\psi_{21}(x, t)$ and $\psi_{22}(x, t)$ in (24)-(26) satisfy

$$\Delta g_1(x_1, x_{21}, t) = h_1(x_1, x_{21}, t)\Delta_1(x_1, x_{21}, t), \quad (27)$$

$$\|\Delta_1(x_1, x_{21}, t)\| \leq \delta_1(x_1, x_{21}, t), \quad (28)$$

$$\Delta g_{21}(x, t) = h_{21}(x, t)\Delta_2(x, t), \quad (29)$$

$$\Delta g_{22}(x, t) = h_{22}(x, t)\Delta_2(x, t), \quad \|\Delta_2(x, t)\| \leq \delta_2(x, t), \quad (30)$$

$$\|\psi_1(x_1, x_{21}, t)\| \leq \vartheta_1\varpi_1(x_1, x_{21}) + \zeta_1, \quad (31)$$

$$\|\psi_2(x, t)\| \triangleq \left\| [\psi_{21}^T(x, t), \psi_{22}^T(x, t)]^T \right\| \leq \vartheta_2\varpi_2(x) + \zeta_2, \quad (32)$$

where $h_1(x_1, x_{21}, t) \in R^{(n-m) \times r_1}$, $h_{21}(x, t) \in R^{m_1 \times r_2}$ and $h_{22}(x, t) \in R^{m_2 \times r_2}$ are known, $\Delta_1(x_1, x_{21}, t) \in R^{r_1 \times m}$ and $\Delta_2(x, t) \in R^{r_2 \times m}$ are unknown, $\delta_1(x_1, x_{21}, t)$, $\delta_2(x, t)$, $\varpi_1(x_1, x_{21})$ and $\varpi_2(x)$ are known continuous nonnegative functions and $\varpi_1(0, 0) = 0$ and $\varpi_2(0) = 0$, the parameters ϑ_1 , ϑ_2 , ζ_1 and ζ_2 are unknown positive constants.

Remark 6: The case where the bounds on the uncertainties involve unknown information is particularly challenging. In order to deal with the unknown bounds, Assumption 5 requires that the system uncertainties $\psi_1(x_1, x_{21}, t)$ and $\psi_2(x, t)$ are in the parameterized form (31)-(32) with unknown parameters. The considered system is thus separated into three subsystems (24)-(26) to facilitate the analysis and reduce conservatism. The conditions that the uncertainties satisfy for system (24)-(26) are similar to that of system (1)-(2), except for

the presence of the unknown parameters in (31) and (32), which relax the constraint conditions. \square

For the sliding mode analysis, the sliding function should be designed to make the system (24)-(26) have generalised regular form and ensure that the system can handle the uncertainty $\psi_1(x_1, x_{21}, t)$, simultaneously.

Assumption 6: The system (24)-(26) has a generalised regular form associated with the sliding function $\sigma(x) = [\sigma_1^T(x), \sigma_2^T(x)]^T$ described by

$$\sigma_1(x) =: K_1 x_{21} - \gamma_1(x_1), \quad (33)$$

$$\begin{aligned} \sigma_2(x) &=: K_2 x_{22} - \gamma_2(x_1) \\ &\quad + I_{m_2}(\hat{\vartheta}_1(t)\varpi_1(x_1, x_{21}) + \hat{\zeta}_1(t)), \end{aligned} \quad (34)$$

where $K_1 \in R^{m_1 \times m_1}$ and $K_2 \in R^{m_2 \times m_2}$ are nonsingular matrices to be designed, and the nonlinear functions $\gamma_1(\cdot) \in R^{m_1}$ and $\gamma_2(\cdot) \in R^{m_2}$ are continuously differentiable in the considered domain D_1 , $I_{m_2} = [1, 1, \dots, 1]^T \in R^{m_2}$, $\varpi_1(x_1, x_{21})$ is given in (31) and $\hat{\vartheta}_1(t)$ and $\hat{\zeta}_1(t)$ are designed nonnegative functions.

Remark 7: Similar to Assumption 2, the designed sliding functions (33)-(34) combined with Definition 1, can facilitate the development of an appropriate sliding mode dynamics. Moreover, the adaptive terms in Assumption 6 are introduced to deal with the unmatched uncertainty $\psi_1(x_1, x_{21}, t)$ in (24), which is a novel contribution for the presented sliding mode controller design. \square

A. Stability Analysis of the Sliding Mode

Under Assumption 6 and from Definition 1, the condition (7) holds with $\sigma(\cdot)$ defined in (33)-(34). When the system (24)-(26) is constrained on the sliding surface (6), it follows from (33) and (34) that the subsystem (24) has the following form:

$$\begin{aligned} \dot{x}_1(t) &= f_{11}^s(x_1, t) + f_{12}^s(x_1, t)K_2^{-1}(\gamma_2(x_1) \\ &\quad - I_{m_2}(\hat{\vartheta}_1(t)\varpi_1^s(x_1) + \hat{\zeta}_1(t))) + \psi_1^s(x_1, t), \end{aligned} \quad (35)$$

where $f_{11}^s(x_1, t) = f_{11}(x_1, x_{21}, t)|_{x_{21}=K_1^{-1}\gamma_1(x_1)}$,

$f_{12}^s(x_1, t) = f_{12}(x_1, x_{21}, t)|_{x_{21}=K_1^{-1}\gamma_1(x_1)}$,

$\psi_1^s(x_1, t) = \psi_1(x, t)|_{x_{21}=K_1^{-1}\gamma_1(x_1)}$,

and $\varpi_1^s(x_1) = \varpi_1(x_1, x_{21})|_{x_{21}=K_1^{-1}\gamma_1(x_1)}$.

From Assumption 6 and the definition of $\psi_1^s(x_1, t)$, it is straightforward to see that the mismatched uncertainty $\psi_1^s(x_1, t)$ in (35) satisfies

$$\|\psi_1^s(x_1, t)\| \leq \vartheta_1\varpi_1^s(x_1) + \zeta_1. \quad (36)$$

where ϑ_1 and ζ_1 are defined in (31) and $\varpi_1^s(x_1)$ is defined in (35).

Therefore, the system (35) is the sliding mode dynamics of the system (24)-(26) corresponding to the sliding function (33)-(34). To undertake an analysis of the sliding mode dynamics, the following assumption is introduced to guarantee that the nominal system (35) is uniformly ultimately bounded.

Assumption 7: There exists a continuously differentiable Lyapunov function $V(x_1, t) : D_1 \times R^+ \mapsto R$ satisfying

$$c_1(\|x_1\|) \leq V(x_1, t) \leq c_2(\|x_1\|), \quad (37)$$

and for $x_1 \in D_1$, $\forall \|x_1\| \geq \mu > 0$,

$$\begin{aligned} \frac{\partial V}{\partial t} + \left(\frac{\partial V}{\partial x_1} \right)^T &\left(f_{11}^s(x_1, t) + f_{12}^s(x_1, t)K_2^{-1}\gamma_2(x_1) \right) \\ &\leq -W_1(x_1), \end{aligned} \quad (38)$$

where the functions $c_1(\cdot)$ and $c_2(\cdot)$ are continuous class \mathcal{K} functions, μ is a positive constant, $W_1(x_1)$ is a continuous positive definite functions in D_1 , $f_{11}^s(\cdot)$, $f_{12}^s(\cdot)$ and K_2^{-1} are given in (35).

Under Assumption 7, the parameters $\hat{\vartheta}_1(t)$ and $\hat{\zeta}_1(t)$ in (34) are updated by

$$\dot{\hat{\vartheta}}_1(t) = \alpha_1 \left\| \left(\frac{\partial V}{\partial x_1} \right)^T \right\| \varpi_1^s(x_1), \quad \hat{\vartheta}_1(0) \geq 0, \quad (39)$$

$$\dot{\hat{\zeta}}_1(t) = \beta_1 \left\| \left(\frac{\partial V}{\partial x_1} \right)^T \right\|, \quad \hat{\zeta}_1(0) \geq 0, \quad (40)$$

where $V(\cdot)$ satisfies Assumption 7, $\alpha_1 > 0$ and $\beta_1 > 0$ are constants, and $\varpi_1^s(x_1)$ is given below (35).

Theorem 3: Under Assumptions 5-7, the solutions of the sliding mode dynamic (35) and the parameters updated by the adaptive laws (39)-(40) are uniformly ultimately bounded if there exists a continuous positive definite function $W_2 : D_1 \mapsto R^+$ such that in the considered domain D_1 and for any $t \in R^+$,

$$\left(\frac{\partial V}{\partial x_1} \right)^T f_{12}^s(x_1, t) K_2^{-1} I_{m_2} - \left\| \left(\frac{\partial V}{\partial x_1} \right)^T \right\| \geq W_2(x_1), \quad \forall \|x_1\| \geq \mu > 0. \quad (41)$$

Proof: For system (35), choose a Lyapunov function candidate as

$$V_s = V(x_1, t) + \frac{1}{2\alpha_1} \tilde{\vartheta}_1^2 + \frac{1}{2\beta_1} \tilde{\zeta}_1^2, \quad (42)$$

where $V(\cdot)$ satisfies Assumption 7, $\tilde{\vartheta}_1(t) = \vartheta_1 - \hat{\vartheta}_1(t)$ and $\tilde{\zeta}_1(t) = \zeta_1 - \hat{\zeta}_1(t)$.

From (38), the time derivative of $V_s(\cdot)$ along the trajectories of system (35) is given by

$$\begin{aligned} \dot{V}_s &= \frac{\partial V}{\partial t} + \left(\frac{\partial V}{\partial x_1} \right)^T (f_{11}^s(x_1, t) + f_{12}^s(x_1, t) K_2^{-1} (\gamma_2(x_1) \\ &\quad - I_{m_2} (\hat{\vartheta}_1(t) \varpi_1^s(x_1) + \hat{\zeta}_1(t)))) + \left(\frac{\partial V}{\partial x_1} \right)^T \psi_1^s(x_1, t) \\ &\quad + \frac{1}{\alpha_1} \tilde{\vartheta}_1(t) \dot{\tilde{\vartheta}}_1(t) + \frac{1}{\beta_1} \tilde{\zeta}_1(t) \dot{\tilde{\zeta}}_1(t) \\ &\leq -W_1(x_1) - \left(\frac{\partial V}{\partial x_1} \right)^T f_{12}^s(x_1, t) K_2^{-1} I_{m_2} (\hat{\vartheta}_1(t) \varpi_1^s(x_1) \\ &\quad + \hat{\zeta}_1(t)) + \left\| \left(\frac{\partial V}{\partial x_1} \right)^T \right\| (\hat{\vartheta}_1(t) \varpi_1^s(x_1) + \hat{\zeta}_1(t)) \\ &\quad + \left\| \left(\frac{\partial V}{\partial x_1} \right)^T \right\| (\tilde{\vartheta}_1(t) \varpi_1^s(x_1) + \tilde{\zeta}_1(t)) \\ &\quad + \frac{1}{\alpha_1} \tilde{\vartheta}_1(t) (-\dot{\tilde{\vartheta}}_1(t)) + \frac{1}{\beta_1} \tilde{\zeta}_1(t) (-\dot{\tilde{\zeta}}_1(t)), \end{aligned} \quad (43)$$

where the inequality (36), $\dot{\tilde{\vartheta}}_1(t) = -\dot{\hat{\vartheta}}_1(t)$ and $\dot{\tilde{\zeta}}_1(t) = -\dot{\hat{\zeta}}_1(t)$ are used.

With the adaptive laws (39)-(40), it follows that

$$\begin{aligned} \dot{V}_s &\leq -W_1(x_1) - \left(\left(\frac{\partial V}{\partial x_1} \right)^T f_{12}^s(x_1, t) K_2^{-1} I_{m_2} \right. \\ &\quad \left. - \left\| \left(\frac{\partial V}{\partial x_1} \right)^T \right\| \right) (\hat{\vartheta}_1(t) \varpi_1^s(x_1) + \hat{\zeta}_1(t)) \\ &\leq -W_1(x_1) - W_2(x_1) (\hat{\vartheta}_1(t) \varpi_1^s(x_1) + \hat{\zeta}_1(t)), \end{aligned} \quad (44)$$

where $\hat{\vartheta}_1(t) \varpi_1^s(x_1) + \hat{\zeta}_1(t) \geq 0$, due to $\hat{\vartheta}_1(t)$, $\varpi_1^s(x_1)$ and $\hat{\zeta}_1(t)$ being nonnegative. Then, it follows that $\dot{V}_s \leq 0$, which implies $\vartheta_1(t)$ and $\zeta_1(t)$ are bounded. As ϑ_1 and ζ_1 are constants, the definitions of $\tilde{\vartheta}_1(t)$ and $\tilde{\zeta}_1(t)$ ensure that $\hat{\vartheta}_1(t)$ and $\hat{\zeta}_1(t)$ are bounded. Hence, the conclusion follows. ∇

Remark 8: The adaptive laws (39)-(40) are designed to deal with the case when the bound on the unmatched uncertainty $\psi_1^s(x_1, t)$

in (35) has unknown parameters, which are specifically designed to have nonnegative right-hand sides to guarantee that the derived sliding motion is uniformly ultimately bounded. It is worth pointing out that, in this paper, it only needs that the adaptive parameters are bounded instead of converging to their real values. This can reduce the conservatism when compared with the case that the adaptive parameters converge to their real values asymptotically. \square

B. Reachability Analysis

For notational convenience, define the function matrices $g_n(x, t)$, $f_n(x, t)$, $h_n(x, t)$ and $\psi_n(x, t)$ as

$$g_n(x, t) = \begin{bmatrix} \frac{\partial \sigma_1}{\partial x_{21}} & \frac{\partial \sigma_1}{\partial x_{22}} \\ \frac{\partial \sigma_2}{\partial x_{21}} & \frac{\partial \sigma_2}{\partial x_{22}} \end{bmatrix} \begin{bmatrix} g_{21}(x, t) \\ g_{22}(x, t) \end{bmatrix}, \quad (45)$$

$$\psi_n(x, t) = \begin{bmatrix} \psi_1(x_1, x_{21}, t) \\ \psi_{21}(x, t) \\ \psi_{22}(x, t) \end{bmatrix}, \quad (46)$$

$$f_n(x, t) = \begin{bmatrix} f_{11}(x_1, x_{21}, t) + f_{12}(x_1, x_{21}, t) x_{22}(t) \\ f_{21}(x, t) \\ f_{22}(x, t) \end{bmatrix}, \quad (47)$$

$$h_n(x, t) = \begin{bmatrix} g_{11}(x_1, x_{21}, t) \\ g_{21}(x, t) \\ g_{22}(x, t) \end{bmatrix}, \quad (48)$$

$$\phi_n(x, t) = \begin{bmatrix} 0 \\ I_{m_2} (\dot{\hat{\vartheta}}_1(t) \varpi_1(x_1, x_{21}) + \dot{\hat{\zeta}}_1(t)) \end{bmatrix}, \quad (49)$$

where $g_{21}(x, t)$, $g_{22}(x, t)$, $f_{11}(x_1, x_{21}, t)$, $f_{12}(x_1, x_{21}, t)$, $f_{21}(x, t)$, $f_{22}(x, t)$, $h_1(x, t)$, $h_{21}(x, t)$, $h_{22}(x, t)$, $\psi_1(x_1, x_{21}, t)$, $\psi_{21}(x, t)$, and $\psi_{22}(x, t)$ are given in (24)-(26) and $\hat{\vartheta}_1(t)$ and $\hat{\zeta}_1(t)$ are given in (39) and (40), respectively. The following assumption is required for sliding mode controller design.

Assumption 8: The function matrix $g_n(x, t)$ defined in (45) is nonsingular for $x \in D$ and $t \in R^+$.

For the system (24)-(26) with the sliding function and the uncertainty satisfying Assumption 6 and (27)-(32), respectively, the controller can be redesigned as

$$u(x, t) = -g_n^{-1}(x, t) \left(\frac{\partial \sigma}{\partial x} f_n(x, t) + \phi_n(x, t) \right) - g_n^{-1}(x, t) \times \text{sgn}(\sigma(x, t)) (\bar{l}(x, t) + \Phi(x, t) \Theta(x, t)), \quad (50)$$

$$\Phi(x, t) = \left[\left\| \frac{\partial \sigma}{\partial x_1} \right\| \quad \left\| \frac{\partial \sigma}{\partial x_{21}} \right\| + \left\| \frac{\partial \sigma}{\partial x_{22}} \right\| \right], \quad (51)$$

$$\Theta(x, t) = \begin{bmatrix} \hat{\vartheta}_{n1}(t) \varpi_1(x_1, x_{21}) + \hat{\zeta}_{n1}(t) \\ \hat{\vartheta}_{n2}(t) \varpi_2(x) + \hat{\zeta}_{n2}(t) \end{bmatrix}, \quad (52)$$

where

$$\begin{aligned} \bar{l}(x, t) &\geq \frac{1}{1 - \Pi(x, t) \|g_n^{-1}(x, t)\|} \left(\epsilon + \Pi(x, t) \|g_n^{-1}(x, t)\| \right. \\ &\quad \left. \times \left(\left\| \frac{\partial \sigma}{\partial x} f_n(x, t) + \phi_n(x, t) \right\| + \Phi(x, t) \Theta(x, t) \right) \right), \end{aligned} \quad (53)$$

$$\begin{aligned} \Pi(x, t) &= \left\| \frac{\partial \sigma}{\partial x_1} h_1(x_1, x_{21}, t) \right\| \delta_{11}(x_1, x_{21}, t) \\ &\quad + \left(\left\| \frac{\partial \sigma}{\partial x_{21}} h_{21}(x, t) \right\| + \left\| \frac{\partial \sigma}{\partial x_{22}} h_{22}(x, t) \right\| \right) \delta_{22}(x, t), \end{aligned} \quad (54)$$

$$\dot{\hat{\vartheta}}_{n1}(t) = \bar{\gamma}_{n1} \|\sigma(x)\| \left\| \frac{\partial \sigma}{\partial x_1} \right\| \varpi_1(x_1, x_{21}), \quad (55)$$

$$\dot{\hat{\zeta}}_{n1}(t) = \bar{\gamma}_{n1} \|\sigma(x)\| \left\| \frac{\partial \sigma}{\partial x_1} \right\|, \quad (56)$$

$$\dot{\hat{\vartheta}}_{n2}(t) = \bar{\gamma}_{n2} \|\sigma(x)\| \left(\left\| \frac{\partial \sigma}{\partial x_{21}} \right\| + \left\| \frac{\partial \sigma}{\partial x_{22}} \right\| \right) \varpi_2(x), \quad (57)$$

$$\dot{\hat{\zeta}}_{n2}(t) = \bar{\gamma}_{n2} \|\sigma(x)\| \left(\left\| \frac{\partial \sigma}{\partial x_{21}} \right\| + \left\| \frac{\partial \sigma}{\partial x_{22}} \right\| \right), \quad (58)$$

with $\hat{\vartheta}_{n1}(0) \geq 0$, $\hat{\zeta}_{n1}(0) \geq 0$, $\hat{\vartheta}_{n2}(0) \geq 0$, $\hat{\zeta}_{n2}(0) \geq 0$, and $1 - \Pi(x, t) \|g_n^{-1}(x, t)\| > 0$ being nonnegative functions, $\epsilon > 0$, $\bar{\gamma}_{n1} > 0$, $\bar{\gamma}_{n2} > 0$, $\bar{\gamma}_{n1} > 0$ and $\bar{\gamma}_{n2} > 0$ being constants.

Then, the following result holds.

Theorem 4: Under Assumptions 5 and 8, the sliding mode control (50) with the adaptive laws (55)-(58) drives the dynamical system (24)-(26) to the sliding surface (33)-(34) and maintains a sliding motion on it thereafter if the control gain $\bar{l}(t)$ satisfies (53).

Proof: According to (33)-(34) and with the definitions (45)-(49), the derivative of $\sigma(x)$ is expressed as

$$\begin{aligned} \dot{\sigma}(x) &= \frac{\partial \sigma}{\partial x} f_n(x, t) + \frac{\partial \sigma}{\partial x} \psi_n(x, t) \\ &+ \left(g_n(x, t) + \frac{\partial \sigma}{\partial x} h_n(x, t) \right) u(t) + \phi_n(x, t). \end{aligned} \quad (59)$$

Choose a Lyapunov function candidate as

$$\begin{aligned} V_r &= \frac{1}{2} \sigma^T \sigma + \frac{1}{2\bar{\gamma}_{n1}} \tilde{\vartheta}_{n1}^2 + \frac{1}{2\bar{\gamma}_{n2}} \tilde{\vartheta}_{n2}^2 + \frac{1}{2\bar{\gamma}_{n1}} \tilde{\zeta}_{n1}^2 \\ &+ \frac{1}{2\bar{\gamma}_{n2}} \tilde{\zeta}_{n2}^2, \end{aligned} \quad (60)$$

where $\sigma(x) = [\sigma_1^T(x), \sigma_2^T(x)]^T$, $\tilde{\vartheta}_{n1}(t) = \vartheta_1 - \hat{\vartheta}_{n1}(t)$, $\tilde{\vartheta}_{n2}(t) = \vartheta_2 - \hat{\vartheta}_{n2}(t)$, $\tilde{\zeta}_{n1}(t) = \zeta_1 - \hat{\zeta}_{n1}(t)$, and $\tilde{\zeta}_{n2}(t) = \zeta_2 - \hat{\zeta}_{n2}(t)$.

Substituting the controller (50) into (59), the time derivative of V_r is given by

$$\begin{aligned} \dot{V}_r &= \sigma^T(x) \dot{\sigma}(x) + \frac{1}{\bar{\gamma}_{n1}} \tilde{\vartheta}_{n1}(t) \dot{\tilde{\vartheta}}_{n1}(t) + \frac{1}{\bar{\gamma}_{n2}} \tilde{\vartheta}_{n2}(t) \dot{\tilde{\vartheta}}_{n2}(t) \\ &+ \frac{1}{\bar{\gamma}_{n1}} \tilde{\zeta}_{n1}(t) \dot{\tilde{\zeta}}_{n1}(t) + \frac{1}{\bar{\gamma}_{n2}} \tilde{\zeta}_{n2}(t) \dot{\tilde{\zeta}}_{n2}(t) \\ &\leq \sigma^T(x) \frac{\partial \sigma}{\partial x} h_n(x, t) u(t) + \sigma^T(x) \frac{\partial \sigma}{\partial x} \psi_n(x, t) \\ &- l(x, t) \|\sigma(x)\| - \|\sigma(x)\| \Phi(x, t) \Theta(x, t) + \frac{1}{\bar{\gamma}_{n1}} \tilde{\vartheta}_{n1}(t) \dot{\tilde{\vartheta}}_{n1}(t) \\ &+ \frac{1}{\bar{\gamma}_{n2}} \tilde{\vartheta}_{n2}(t) \dot{\tilde{\vartheta}}_{n2}(t) + \frac{1}{\bar{\gamma}_{n1}} \tilde{\zeta}_{n1}(t) \dot{\tilde{\zeta}}_{n1}(t) + \frac{1}{\bar{\gamma}_{n2}} \tilde{\zeta}_{n2}(t) \dot{\tilde{\zeta}}_{n2}(t) \\ &= -\sigma^T(x) \frac{\partial \sigma}{\partial x} h_n(x, t) g_n^{-1}(x, t) \left(\frac{\partial \sigma}{\partial x} f_n(x, t) + \phi_n(x, t) \right) \\ &- \sigma^T(x) \text{sgn}(\sigma(x, t)) \frac{\partial \sigma}{\partial x} h_n(x, t) g_n^{-1}(x, t) \bar{l}(x, t) \\ &- \sigma^T(x) \text{sgn}(\sigma(x, t)) \frac{\partial \sigma}{\partial x} h_n(x, t) g_n^{-1}(x, t) \Phi(x, t) \Theta(x, t) \\ &- \bar{l}(x, t) \|\sigma(x)\| + \sigma^T(x) \frac{\partial \sigma}{\partial x} \psi_n(x, t) - \|\sigma(x)\| \Phi(x, t) \Theta(x, t) \\ &- \|\sigma(x)\| \left\| \frac{\partial \sigma}{\partial x_1} \right\| (\vartheta_1 \varpi_1(x_1, x_{21}) + \zeta_1) \\ &- \|\sigma(x)\| \left(\left\| \frac{\partial \sigma}{\partial x_{21}} \right\| + \left\| \frac{\partial \sigma}{\partial x_{22}} \right\| \right) (\vartheta_2 \varpi_2(x) + \zeta_2) \\ &+ \|\sigma(x)\| \left\| \frac{\partial \sigma}{\partial x_1} \right\| (\vartheta_1 \varpi_1(x_1, x_{21}) + \zeta_1) \\ &+ \|\sigma(x)\| \left(\left\| \frac{\partial \sigma}{\partial x_{21}} \right\| + \left\| \frac{\partial \sigma}{\partial x_{22}} \right\| \right) (\vartheta_2 \varpi_2(x) + \zeta_2) \\ &+ \frac{1}{\bar{\gamma}_{n1}} \tilde{\vartheta}_{n1}(t) \dot{\tilde{\vartheta}}_{n1}(t) + \frac{1}{\bar{\gamma}_{n2}} \tilde{\vartheta}_{n2}(t) \dot{\tilde{\vartheta}}_{n2}(t) \\ &+ \frac{1}{\bar{\gamma}_{n1}} \tilde{\zeta}_{n1}(t) \dot{\tilde{\zeta}}_{n1}(t) + \frac{1}{\bar{\gamma}_{n2}} \tilde{\zeta}_{n2}(t) \dot{\tilde{\zeta}}_{n2}(t). \end{aligned} \quad (61)$$

As $\frac{\partial \sigma}{\partial x} \psi_n(x, t) = \frac{\partial \sigma}{\partial x_1} \psi_1(x_1, x_{21}, t) + \frac{\partial \sigma}{\partial x_{21}} \psi_{21}(x, t) +$

$\frac{\partial \sigma}{\partial x_{22}} \psi_{22}(x, t)$, it follows that

$$\begin{aligned} &\left\| \frac{\partial \sigma}{\partial x} \right\| \|\psi_n(x, t)\| \\ &\leq \left\| \frac{\partial \sigma}{\partial x_1} \right\| \|\psi_1(x_1, x_{21}, t)\| + \left\| \frac{\partial \sigma}{\partial x_{21}} \right\| \|\psi_{21}(x, t)\| \\ &+ \left\| \frac{\partial \sigma}{\partial x_{22}} \right\| \|\psi_{22}(x, t)\|. \end{aligned} \quad (62)$$

Further, using (31)-(32), (62) implies

$$\begin{aligned} \sigma^T(x) \frac{\partial \sigma}{\partial x} \psi_n(x, t) - \|\sigma(x)\| \left\| \frac{\partial \sigma}{\partial x_1} \right\| (\vartheta_1 \varpi_1(x_1, x_{21}) + \zeta_1) \\ - \|\sigma(x)\| \left(\left\| \frac{\partial \sigma}{\partial x_{21}} \right\| + \left\| \frac{\partial \sigma}{\partial x_{22}} \right\| \right) (\vartheta_2 \varpi_2(x) + \zeta_2) \leq 0. \end{aligned} \quad (63)$$

Similarly, the following result holds for $\left\| \frac{\partial \sigma}{\partial x} h_n(x, t) \right\|$

$$\begin{aligned} \left\| \frac{\partial \sigma}{\partial x} h_n(x, t) \right\| &\leq \left\| \frac{\partial \sigma}{\partial x_1} h_1(x_1, x_{21}, t) \right\| \|\Delta_1(x_1, x_{21}, t)\| \\ &+ \left\| \frac{\partial \sigma}{\partial x_{21}} h_{21}(x, t) \right\| \|\Delta_{21}(x, t)\| \\ &+ \left\| \frac{\partial \sigma}{\partial x_{22}} h_{22}(x, t) \right\| \|\Delta_{22}(x, t)\| \leq \Pi, \end{aligned} \quad (64)$$

where Π is defined in (54).

Then, it follows that

$$\begin{aligned} \dot{V}_r &\leq \left\| \sigma^T(x) \frac{\partial \sigma}{\partial x} h_n(x, t) g_n^{-1}(x, t) \left(\frac{\partial \sigma}{\partial x} f_n(x, t) + \phi_n(x, t) \right) \right\| \\ &- \|\sigma(x)\| \left\| \frac{\partial \sigma}{\partial x} h_n(x, t) g_n^{-1}(x, t) \right\| \bar{l}(x, t) - \bar{l}(x, t) \|\sigma(x)\| \\ &+ \|\sigma(x)\| \left\| \frac{\partial \sigma}{\partial x} h_n(x, t) g_n^{-1}(x, t) \right\| \Phi(x, t) \Theta(x, t) \\ &+ \|\sigma(x)\| \left\| \frac{\partial \sigma}{\partial x_1} \right\| (\tilde{\vartheta}_{n1}(t) \varpi_1(x_1, x_{21}) + \tilde{\zeta}_{n1}(t)) \\ &+ \|\sigma(x)\| \left(\left\| \frac{\partial \sigma}{\partial x_{21}} \right\| + \left\| \frac{\partial \sigma}{\partial x_{22}} \right\| \right) (\tilde{\vartheta}_{n2}(t) \varpi_2(x) + \tilde{\zeta}_{n2}(t)) \\ &+ \frac{1}{\bar{\gamma}_{n1}} \tilde{\vartheta}_{n1}(t) (-\dot{\tilde{\vartheta}}_{n1}(t)) + \frac{1}{\bar{\gamma}_{n2}} \tilde{\vartheta}_{n2}(t) (-\dot{\tilde{\vartheta}}_{n2}(t)) \\ &+ \frac{1}{\bar{\gamma}_{n1}} \tilde{\zeta}_{n1}(t) (-\dot{\tilde{\zeta}}_{n1}(t)) + \frac{1}{\bar{\gamma}_{n2}} \tilde{\zeta}_{n2}(t) (-\dot{\tilde{\zeta}}_{n2}(t)) \\ &\leq -\|\sigma(x)\| \left(\bar{l}(x, t) - \left\| \frac{\partial \sigma}{\partial x} h_n(x, t) \right\| \|g_n^{-1}(x, t)\| \right. \\ &\quad \left. \times \left(\left\| \frac{\partial \sigma}{\partial x} f_n(x, t) + \phi_n(x, t) \right\| + \Phi(x, t) \Theta(x, t) \right. \right. \\ &\quad \left. \left. + \bar{l}(x, t) \right) \right). \end{aligned} \quad (65)$$

Applying (64) and (53) into (65),

$$\dot{V}_r \leq -\epsilon \|\sigma(x)\|, \quad \epsilon > 0, \quad (66)$$

which means that the dynamic system consisting of (59) and (55)-(58) is uniformly ultimately stable and its solutions are uniformly ultimately bounded. Therefore, all the variables $\tilde{\vartheta}_{n1}(t)$, $\tilde{\vartheta}_{n2}(t)$, $\tilde{\zeta}_{n1}(t)$, and $\tilde{\zeta}_{n2}(t)$ are bounded. Further, (66) implies $\sigma(x) \in L_2$. So, from the Barbálat Lemma, for any initial condition $\|\sigma(x)\| > 0$, the dynamical system (24)-(26) can be driven to the sliding surface $\sigma(x) = 0$. \square

V. SIMULATION EXAMPLE

To verify the proposed method, a simulation study is conducted on a HIRM aircraft taken from [19]. After coordinate transformation, the aircraft system has the form of (24)-(26), with $h_1(x, t) = \begin{bmatrix} 0 \\ x_{21} \end{bmatrix}$, $h_{21}(x, t) = [0.2961x_{12}, 0.5891x_{21}]$, $h_{22}(x, t) = [0.0381x_{12}, 0.5795x_{22}]$, $\|\Delta_1(x, t)\| = \|\Delta_2(x, t)\| \leq 0.1|\sin x_{11}|$, $\|\psi_1(x, t)\| = \|[0, \psi_{12}(x, t)]^T\| \leq 0.68|\sin(-9.8x_{12})|$, $\|\psi_2(x, t)\| = \|\psi_{21}(x, t), \psi_{22}(x, t)\|^T \leq 0.53|\sin(-8.6x_{22})|$.

Using Assumption 2 and considering that the states should converge to zero, the sliding function is designed as $\sigma(x) = [3x_{21} \ 2x_{22} + x_{11} + \hat{\vartheta}_1(t)|\sin(-9.8x_{12})|]^T$. Then, on the sliding surface, it follows that $x_{21} = 0$ and $h_1(t, x) = 0$. When the system is limited to the sliding surface, the dynamics are given by:

$$\begin{aligned} \dot{x}_{11}(t) &= -0.5004x_{11}(t) + 25.1574x_{12}(t) \\ &\quad - 1.008\hat{\vartheta}_1(t)|\sin(-9.8x_{12})| + \psi_{11}(x_{12}, t), \end{aligned} \quad (67)$$

$$\begin{aligned} \dot{x}_{12}(t) &= -0.356x_{11} - 1.6948x_{12} + a(x_{11}, x_{12}) \\ &\quad + 0.022\hat{\vartheta}_1(t)|\sin(-9.8x_{12})| + \psi_{12}(x_{12}, t), \end{aligned} \quad (68)$$

where $a(x_{11}, x_{12}) = \sin(-9.6849x_{12} + 0.0066(x_{11} + \hat{\vartheta}_1(t)|\sin(-9.8x_{12})|))/(1 + 0.7650x_{12} + 0.00025(x_{11} + \hat{\vartheta}_1(t)|\sin(-9.8x_{12})|))$.

For the sliding mode dynamics (67)-(68), define the candidate Lyapunov function as $V(t, x_{11}, x_{12}) = [x_{11} \ x_{12}]P[x_{11} \ x_{12}]^T$, where $P = \begin{bmatrix} 1.2145 & 13.6428 \\ 13.6428 & 202.9155 \end{bmatrix}$. According to (50), the sliding mode controller can be obtained with parameters $\hat{\vartheta}_1(t)$, $\hat{\vartheta}_{n1}(t)$ and $\hat{\vartheta}_{n2}(t)$ being updated by (39), (55) and (57). For simulation purposes, the initial condition is chosen as $[0.2, \ 0.12, \ 5, \ 0.16]^T$ and $v_0 = 268$. To eliminate the chattering caused by the discontinuous controller due to the sign function, the boundary layer method proposed in [2], [20], [21] is used, in which, the discontinuous sign function is approximated by the continuous saturation function $\frac{\sigma(x, t)}{\|\sigma(x, t)\| + 0.008}$.

The time response of the states, i.e., current airspeed, angle of attack, pitch rate and pitch angle, are shown in Fig. 1. It can be seen that the airspeed converges to the desired set-point, and the other states are close to zero in the steady state. Moreover, all the adaptive parameters are also bounded, which are shown in Fig. 2. From the results, it is evident that the proposed adaptive sliding mode control can ensure that the system exhibits the expected performance even in the presence of input distribution and system uncertainties.

VI. CONCLUSIONS

In this paper, a sliding mode controller is designed for a class of uncertain nonlinear systems in a generalized form. A new sliding surface design method is proposed, and the stability of the resulting sliding mode dynamics is analyzed. For the considered nonlinear system in the presence of matched and mismatched uncertainties including uncertainty in the input distribution, a sliding mode controller is constructed to ensure that the dynamical system reaches the sliding mode in finite time. An adaptive sliding mode controller is further developed for the case when the bounds on the system uncertainties are unknown. Detailed discussion of the closed-loop system performance is presented and simulation on a HIRM aircraft system verifies the effectiveness of the proposed sliding mode control approach.

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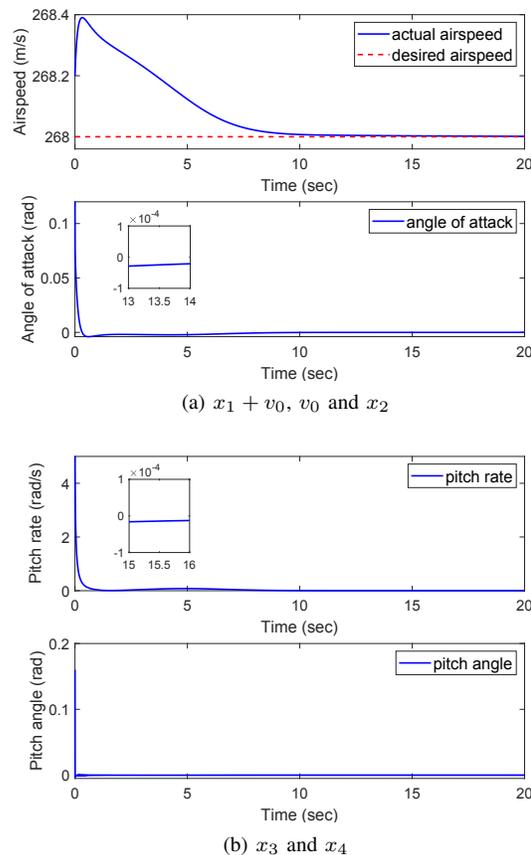


Fig. 1. The time responses of the HIRM aircraft system

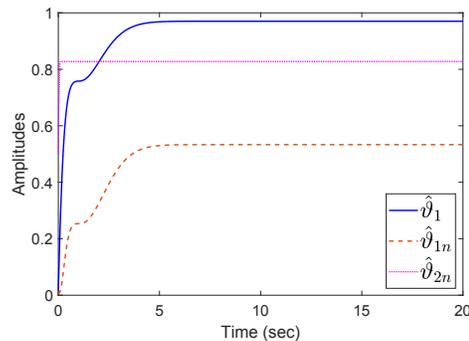


Fig. 2. The time responses of adaptive parameters

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