Convergence Rate Analysis of Galerkin Approximation of Inverse Potential Problem[*]

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Abstract

In this work we analyze the inverse problem of recovering a space-dependent potential coefficient in an elliptic / parabolic problem from distributed observation. We establish novel (weighted) conditional stability estimates under very mild conditions on the problem data. Then we provide an error analysis of a standard reconstruction scheme based on the standard output least-squares formulation with Tikhonov regularization (by an H^1-seminorm penalty), which is then discretized by the Galerkin finite element method with continuous piecewise linear finite elements in space (and also backward Euler method in time for parabolic problems). We present a detailed error analysis of the discrete scheme, and provide convergence rates in a weighted $L^2(\Omega)$ for discrete approximations with respect to the exact potential. The error bounds explicitly depend on the noise level, regularization parameter and discretization parameter(s). Under suitable conditions, we also derive error estimates in the standard $L^2(\Omega)$ and interior L^2 norms. The analysis employs sharp a priori error estimates and nonstandard test functions. Several numerical experiments are given to complement the theoretical analysis.

Keywords: inverse problems, parameter identification, Tikhonov regularization, error estimate

1   Introduction

In this work, we study the inverse problem of recovering a space-dependent potential coefficient in elliptic and parabolic equations. Let $\Omega \subset \mathbb{R}^d$ $(d = 1, 2, 3)$ be a simply connected convex polyhedral domain with a boundary $\partial\Omega$. Then the governing equation in the elliptic and parabolic cases are given respectively by

$$\begin{cases} -\Delta u + qu = f, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

and

$$\begin{cases} \partial_t u - \Delta u + qu = f, & \text{in } \Omega \times (0, T), \\ u = 0, & \text{on } \partial\Omega \times (0, T), \\ u(\cdot, 0) = u_0, & \text{in } \Omega, \end{cases} \tag{1.2}$$

where $T > 0$ is the final time. The functions f and u_0 in (1.1) and (1.2) are the source and initial data, respectively. The space-dependent potential q belongs to the admissible set K such that

$$K = \{q \in L^\infty(\Omega) : c_0 \le q(x) \le c_1 \text{ a.e. in } \Omega\},$$

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with $0 \le c_0 < c_1 < \infty$. To explicitly indicate the dependence of the solution u to problems (1.1) and (1.2) on the potential q, we write $u(q)$. Further, we are given the observational data z^δ on Ω or $\Omega \times (T_0, T)$:

$$\begin{cases} z^\delta(x) = u(q^\dagger)(x) + \xi(x), & x \in \Omega, & \text{elliptic} \\ z^\delta(x,t) = u(q^\dagger)(x,t) + \xi(x,t), & (x,t) \in \Omega \times (T_0, T), & \text{parabolic} \end{cases}$$

where $u(q^\dagger)$ denotes the exact data (corresponding to the exact potential $q^\dagger$), $0 \le T_0 < T$, and ξ denotes the measurement noise. The accuracy of the data z^δ is measured by the noise level $\delta = \|u(q^\dagger) - z^\delta\|_{L^2(\Omega)}$ or $\delta = \|u(q^\dagger) - z^\delta\|_{L^2(T_0,T;L^2(\Omega))}$, in the elliptic and parabolic cases, respectively. The inverse potential problem is to recover the potential q from the noisy observation z^δ. It arises in several practical applications, where q represents the radiativity coefficient in heat conduction [41] and perfusion coefficient in Pennes' bio-heat equation in human physiology [33, 37] (see [35, 43] for experimental studies) and the elliptic case also in quantitative dynamic elastography [13].

The inverse potential problem is ill-posed, which poses challenges to construct accurate and stable numerical approximations. A number of reconstruction methods have been designed to overcome the ill-posed nature, with the most prominent one being Tikhonov regularization [18, 26]. In practical computation, one still needs to discretize the continuous regularized formulation. This is often achieved by the Galerkin finite element method (FEM) when the domain Ω is irregular and the problem data (u_0 and f) have only limited regularity. This strategy has been widely used [41, 16, 42]. Yamamoto and Zou [41] proved the convergence of the discrete approximations in the parabolic case. However, the convergence rates of discrete approximations are generally very challenging to obtain, due to the *inherent nonconvexity* of the regularized functional, which itself stems from the high degree of *nonlinearity* of the parameter-to-state map, despite the PDEs (1.1) and (1.2) being linear. Indeed, this has been a long standing issue for the numerical analysis of many nonlinear inverse problems, e.g., parameter identifications for PDEs. So far there have been only very few error bounds on discrete approximations, despite the fact that such an analysis would provide useful guidelines for choosing suitable discretization parameters. For the related inverse conductivity problem, the works [39, 27] derived error bounds in a weighted $L^2(\Omega)$ norm by employing a special test function for elliptic and parabolic cases, and the latter work [27] also gives the standard $L^2(\Omega)$ error estimates with the help of a positivity condition on the weighted function.

In this work we study the concerned elliptic and parabolic inverse potential problems, and contribute in the following two aspects. First, we establish novel conditional stability estimates for the concerned inverse problem, including both weighted $L^2(\Omega)$ and standard $L^2(\Omega)$ stability. The latter is obtained under a certain positivity condition, which can be verified for a class of problem data. The derivation is purely variational, using only a nonstandard test function, and extends directly to the error analysis. Our analysis strategy is similar to that in the interesting works [6, 3], which are concerned with recovering the diffusion coefficient from the internal measurements. Second, we derive novel weighted $L^2(\Omega)$ error bounds for the discrete approximations under very mild regularity conditions on the problem data and unknown coefficient q as well as the standard $L^2(\Omega)$ error bounds under some positivity condition. Note that the analysis does not employ standard source type conditions. Instead, it is achieved by a novel choice of the test function in the weak formulation, as in the conditional stability analysis, adapting the stability argument to the discrete setting, which allows us to bypass the standard source condition. To the best of our knowledge, these results represent the first error bounds for the discrete approximations for the inverse potential problem. Further, we provide several numerical experiments to complement the theoretical analysis.

Now we review existing works on the analysis and numerics of the inverse potential problem. Several uniqueness and stability results have been obtained [34, 14, 28, 4, 13]. Choulli and Yamamoto [14] proved the uniqueness of recovering the potential $q^\dagger$, initial condition and boundary coefficient from terminal measurement, and also gave a stability result under a smallness condition. In the parabolic case, Beretta and Cavaterra [4] proved the unique recovery of the potential $q(x)$ from the time-averaged observation. More recently, Choulli [13] derived a new stability estimate in the elliptic case. The well-posedness of the continuous regularized formulation has been analyzed for both elliptic / parabolic cases [19, 41, 16, 42], and convergence rates with respect to the noise level δ were obtained under various conditions. In the 1D elliptic case, Engl et al [19, Example 3.1] derived a convergence rate of the regularized approximation by Tikhonov regularization under the standard source condition with a small sourcewise representer. Hao and Quyen [24] presented a different approach without explicitly using the source condition. More recently, Chen et

al [11] proved conditional stability of the inverse problem in negative Sobolev spaces, which allows deriving variational inequality type source conditions and also showing convergence rates of the regularized solutions, and the authors studied both elliptic and parabolic cases. Klibanov, Li, and Zhang [29] presented an interesting convexification method for the inverse problem which allows proving globle convergence despite the nonlinearity of the inverse problem (and actually the method allows recovering a time-independent source simultaneously). This work extends the current literature with new stability analysis and error analysis of discrete approximations Broadly speaking, the present work is along the line of research which connects stability analysis with error analysis of discrete schemes (see, e.g., [9, 10]) and convergence rate with conditional stability (see, e.g., [12, 40]).

The rest of the paper is organized as follows. In Section 2 we present novel conditional stability estimates for the concerned inverse problems. In Sections 3 and 4, we describe the regularized formulations and their finite element discretizations, and derive novel error bounds on the discrete approximations, for the elliptic and parabolic cases, respectively. In Section 5, we present one- and two-dimensional numerical experiments to complement the theoretical analysis. We conclude with some useful notation. For any $m \geq 0$ and $p \geq 1$, we denote by $W^{m,p}(\Omega)$ the standard Sobolev spaces of order m, equipped with the norm $\|\cdot\|_{W^{m,p}(\Omega)}$ and also write $H^m(\Omega)$ and $H_0^m(\Omega)$ with the norm $\|\cdot\|_{H^m(\Omega)}$ when $p = 2$ [1]. We denote the $L^2(\Omega)$ inner product by $(\cdot, \cdot)$. We also use Bochner spaces: For a Banach space B, we define by

$$H^m(0, T; B) = \{v : v(\cdot, t) \in B \text{ for a.e. } t \in (0, T) \text{ and } \|v\|_{H^m(0,T;B)} < \infty\}.$$

The space $L^\infty(0, T; B)$ is defined similarly. Throughout, we denote by C a generic positive constant not necessarily the same at each occurrence but always independent of the discretization parameters h and τ, the noise level δ and the regularization parameter α.

2 Conditional stability estimates

In this section, we present novel conditional stability estimates for the concerned inverse problem. The analysis will also inspire the error analysis of the discrete approximations in Sections 3 and 4.

2.1 Elliptic inverse problem

We have the following conditional stability results in weighted and standard $L^2(\Omega)$ norms.

Theorem 2.1. *Suppose that $q_1, q_2 \in K \cap H^1(\Omega)$ and $f \in L^2(\Omega)$, with $\|\nabla q_1\|_{L^2(\Omega)}, \|\nabla q_2\|_{L^2(\Omega)} \leq c_q$. Let $u(q_1)$ and $u(q_2)$ be the corresponding weak solutions of problem (1.1). Then there exists a constant C depends on c_q such that*

$$\|(q_1 - q_2)u(q_1)\|_{L^2(\Omega)} \leq C\|u(q_1) - u(q_2)\|_{H^1(\Omega)}^{\frac{1}{2}}.$$

Moreover, if there exists a $\beta \geq 0$ such that

$$u(q_1)(x) \geq C\mathrm{dist}(x, \partial\Omega)^\beta \quad a.e. \text{ in } \Omega, \tag{2.1}$$

then with a constant C depending on c_q, the following estimate holds

$$\|q_1 - q_2\|_{L^2(\Omega)} \leq C\|u(q_1) - u(q_2)\|_{H^1(\Omega)}^{\frac{1}{2(1+2\beta)}}.$$

Proof. By the weak formulations of $u(q_1)$ and $u(q_2)$, for any $\varphi \in H_0^1(\Omega)$

$$((q_1 - q_2)u(q_1), \varphi) = -(\nabla(u(q_1) - u(q_2)), \nabla\varphi) - (q_2(u(q_1) - u(q_2)), \varphi) =: \mathrm{I}.$$

Let $\varphi = (q_1 - q_2)u(q_1)$. Note that $\nabla\varphi = (\nabla q_1 - \nabla q_2)u(q_1) + (q_1 - q_2)\nabla u(q_1)$. Since $q_1 \in K \cap H^1(\Omega)$, elliptic regularity theory implies $u(q_1) \in H^2(\Omega)$, and by Sobolev embedding theorem, we have $u(q_1) \in L^\infty(\Omega)$ for $d = 1, 2, 3$. Then we have $\|\varphi\|_{L^2(\Omega)} \leq C$ and $\|\nabla\varphi\|_{L^2(\Omega)} \leq C$, i.e., $\varphi \in H_0^1(\Omega)$. Now by the Cauchy-Schwarz inequality, we obtain the first assertion by

$$|\mathrm{I}| \leq C(\|\nabla(u(q_1) - u(q_2))\|_{L^2(\Omega)} + \|u(q_1) - u(q_2)\|_{L^2(\Omega)}) \leq C\|u(q_1) - u(q_2)\|_{H^1(\Omega)}. \tag{2.2}$$

3

Next we decompose the domain Ω into two disjoint sets $\Omega = \Omega_\rho \cup \Omega_\rho^c$, with $\Omega_\rho = \{x \in \Omega : \text{dist}(x, \partial\Omega) \geq \rho\}$ and $\Omega_\rho^c = \Omega \setminus \Omega_\rho$, with the constant $\rho > 0$ to be chosen. On the subdomain Ω_ρ, we have

$$\int_{\Omega_\rho} (q_1 - q_2)^2 \mathrm{d}x = \rho^{-2\beta} \int_{\Omega_\rho} (q_1 - q_2)^2 \rho^{2\beta} \mathrm{d}x \leq \rho^{-2\beta} \int_{\Omega_\rho} (q_1 - q_2)^2 \text{dist}(x, \partial\Omega)^{2\beta} \mathrm{d}x$$

$$\leq C\rho^{-2\beta} \int_{\Omega_\rho} (q_1 - q_2)^2 u(q_1)^2 \mathrm{d}x \leq C\rho^{-2\beta} \|u(q_1) - u(q_2)\|_{H^1(\Omega)}.$$

Meanwhile, by the box constraint of K, we have $\int_{\Omega_\rho^c} (q_1 - q_2)^2 \mathrm{d}x \leq C|\Omega_\rho^c| \leq C\rho$. Then the desired result follows by balancing the last two estimates with ρ. $\qquad\square$

The positivity condition (2.1) quantifies the decay rate of the solution $u(q_1)$ to zero as $\text{dist}(x, \partial\Omega) \to 0^+$ (due to the presence of a zero Dirichlet boundary condition). It can be verified under suitable conditions on the source f. This requires the following property of Green's function for the elliptic problem. The notation $B(x, r)$ denotes the ball centered at $x \in \mathbb{R}^d$ with a radius r.

Theorem 2.2. *Let the diffusion coefficient $a \in L^\infty(\Omega)$ with a strictly positive lower bound over Ω. For any $y \in \Omega$ and $r > 0$, let $G_q(x) := G_q(x, y) \in H^1(\Omega \setminus B(y, r)) \cap W_0^{1,1}(\Omega)$ be Green's function for the elliptic operator $-\text{div}(a\nabla\cdot) + qI$ (with a zero Dirichlet boundary condition). Then for $d \geq 2$, the following estimate holds*

$$G_q(x, y) \geq C|x - y|^{2-d}, \quad \text{for } |x - y| \leq \tfrac{1}{2}\text{dist}(x, \partial\Omega).$$

Proof. When $q \equiv 0$, the result is well known for $d \geq 3$ [31, 23]. We prove the slightly more general case for completeness. Let $\rho(x) = \text{dist}(x, \partial\Omega)$. Since the operator $-\text{div}(a\nabla\cdot) + qI$ is self-adjoint, there holds $G_q(x, y) = G_q(y, x)$. It suffices to prove

$$G_q(x, y) \geq C|x - y|^{2-d}, \quad \text{for } |x - y| \leq \tfrac{1}{2}\rho(y). \tag{2.3}$$

By definition, we have

$$\int_\Omega a(z)\nabla G_q(z, y) \cdot \nabla\varphi(z) + q(z)G_q(z, y)\varphi(z) \, \mathrm{d}z = \varphi(y), \quad \forall\varphi \in C_0^\infty(\Omega). \tag{2.4}$$

Let $r := |x - y|$. Consider a cut-off function $\varphi_1 \in C_0^\infty(\Omega)$ with the following properties: $\varphi_1 \equiv 1$ on $B(y, r) \cap (\Omega \setminus B(y, \frac{r}{2}))$ and $\varphi_1 \equiv 0$ on $(\Omega \setminus B(y, \frac{3r}{2})) \cup B(y, \frac{r}{4})$, meanwhile $0 \leq \varphi_1 \leq 1$ and $|\nabla\varphi_1| \leq Cr^{-1}$. By inserting a test function $(\varphi_1(z))^2 G_q(z, y)$ into (2.4) and applying the boundedness and uniform ellipticity of the operator and the Cauchy-Schwarz inequality, since $G_q(z) := G_q(z, y) \in H^1(\Omega \setminus B(y, \frac{r}{4}))$, we derive

$$\int_{\frac{r}{4} \leq |z-y| \leq \frac{3r}{2}} \varphi_1(z)^2 |\nabla G_q(z, y)|^2 \, \mathrm{d}z \leq C \int_{\frac{r}{4}|z-y| \leq \frac{3r}{2}} \varphi_1(z) G_q(z, y)|\nabla\varphi_1(z)||\nabla G_q(z, y)| \, \mathrm{d}z$$

$$\leq C\Big(\int_{\frac{r}{4} \leq |z-y| \leq \frac{3r}{2}} \varphi_1(z)^2 |\nabla G_q(z, y)|^2 \, \mathrm{d}z\Big)^{\frac{1}{2}} \Big(\int_{\frac{r}{4} \leq |z-y| \leq \frac{3r}{2}} G_q(z, y)^2 |\nabla\varphi_1(z)|^2 \, \mathrm{d}z\Big)^{\frac{1}{2}}.$$

This and the construction of φ_1 lead to

$$\int_{\frac{r}{2} \leq |z-y| \leq r} |\nabla G_q(z)|^2 \mathrm{d}z \leq Cr^{d-2} \sup_{\frac{r}{4} \leq |z-y| \leq \frac{3r}{2}} |G_q(z, y)|^2. \tag{2.5}$$

Since $G_q(x, y) \in W_0^{1,1}(\Omega)$ and $q \leq c_1$, we can choose a sufficiently small radius $r_0 := r_0(c_1) < \frac{r}{3}$ such that

$$\int_{z \in B(y, r_0)} G_q(z, y) \, \mathrm{d}z \leq \tfrac{1}{2c_1}.$$

Replacing the radius r by r_0 and repeating the argument of (2.5) yield

$$\int_{\frac{r_0}{2} \leq |z-y| \leq r_0} |\nabla G_q(z)|^2 \mathrm{d}z \leq Cr_0^{d-2} \sup_{\frac{r_0}{4} \leq |z-y| \leq \frac{3r_0}{2}} |G_q(z, y)|^2.$$

Let $\varphi_2 \in C_0^\infty(\Omega)$ be a test function of (2.4) satisfying $\varphi_2 \equiv 1$ on $B(y, \frac{r_0}{2})$ and $\varphi_2 \equiv 0$ on $((\Omega \setminus B(y, r_0)) \cap$
$B(y, \frac{r}{2})) \cup (\Omega \setminus B(y, r))$, with $0 \leq \varphi_2 \leq 1$ on Ω, $|\nabla \varphi_2| \leq C r_0^{-1}$ on $B(y, r_0)$ and $|\nabla \varphi_2| \leq C r^{-1}$ on $(\Omega \setminus$
$B(y, r/2)) \cap B(y, r)$. It follows from the boundedness of the operator, and the last three estimates that

$$
\begin{aligned}
1 &= \int_\Omega a(z) \nabla G_q(z,y) \cdot \nabla \varphi_2(z) + q(z) G_q(z,y) \varphi_2(z) \, \mathrm{d}z \\
&\leq \int_{\frac{r_0}{2} \leq |z-y| \leq r_0} a(z) \nabla G_q(z,y) \cdot \nabla \varphi_2(z) \, \mathrm{d}z + \int_{\frac{r}{2} \leq |z-y| \leq r} a(z) \nabla G_q(z,y) \cdot \nabla \varphi_2(z) \, \mathrm{d}z \\
&\quad + \frac{1}{2} + \int_{\frac{r}{2} \leq |z-y| \leq r} q(z) G_q(z,y) \varphi_2(z) \, \mathrm{d}z \\
&\leq C r_0^{d-2} \sup_{\frac{r_0}{4} \leq |z-y| \leq \frac{3r_0}{2}} G_q(z,y) + C r^{d-2} \sup_{\frac{r}{4} \leq |z-y| \leq \frac{3r}{2}} G_q(z,y) + \frac{1}{2} + C r^d \sup_{\frac{r}{2} \leq |z-y| \leq r} G_q(z,y) \\
&\leq C r^{d-2} \sup_{\frac{r_0}{4} \leq |z-y| \leq \frac{3r}{2}} G_q(z,y) + \frac{1}{2} \leq C r^{d-2} \inf_{\frac{r_0}{4} \leq |z-y| \leq \frac{3r}{2}} G_q(z,y) + \frac{1}{2} \leq C|x-y|^{d-2} G_q(x,y) + \frac{1}{2},
\end{aligned}
$$

where we have used Harnack's inequality [22, p. 189] for Green's function $G_q(z)$ on the compact subset
$\{z \in \Omega : \frac{r_0}{4} \leq |z-y| \leq \frac{3r}{2}\} \subset\subset \Omega$, with the constant C depending on the d, c_1 and Ω. This completes the
proof of the theorem. $\qquad\square$

Remark 2.1. *When $d = 1$, i.e., $\Omega = (a,b)$ with $-\infty < a < b < \infty$, Green's function $G_{c_1}(x,y)$ of the
operator $-\Delta + c_1 I$ is explicitly given by*

$$
G_{c_1}(x,y) = \begin{cases}
-\dfrac{e^{2\sqrt{c_1}a}(e^{2\sqrt{c_1}y} - e^{2\sqrt{c_1}b})}{2\sqrt{c_1}(e^{\sqrt{c_1}(2a+y)} - e^{\sqrt{c_1}(2b+y)})} \cdot (e^{-\sqrt{c_1}x} - e^{\sqrt{c_1}(x-2a)}), & a \leq x \leq y, \\[3mm]
-\dfrac{e^{2\sqrt{c_1}b}(e^{2\sqrt{c_1}y} - e^{2\sqrt{c_1}a})}{2\sqrt{c_1}(e^{\sqrt{c_1}(2a+y)} - e^{\sqrt{c_1}(2b+y)})} \cdot (e^{-\sqrt{c_1}x} - e^{\sqrt{c_1}(x-2b)}), & y \leq x \leq b.
\end{cases}
\tag{2.6}
$$

*Now consider the asymptotics of the function $G_{c_1}(x,y)$ near the boundary. Let y be close to the point a and
$|x-y| \leq \frac{1}{2}(y-a)$. Since $e^x - 1 \geq x$ on $\mathbb{R}$, we have*

$$
G_{c_1}(x,y) \begin{cases}
\simeq e^{\sqrt{c_1}(x-2a)} - e^{-\sqrt{c_1}x} \geq C(x-a) \geq C|x-y|, & a \leq x \leq y, \\[2mm]
\simeq e^{2\sqrt{c_1}y} - e^{2\sqrt{c_1}a} \geq C(y-a) \geq C|x-y|, & a \leq y \leq x,
\end{cases}
$$

*where the symbol " $\simeq$ " denotes by " $=$ " up to a positive constant depending on Ω and c_1. A similar result
holds when y is close to the point b. Since the operator $-\Delta + c_1 I$ is self-adjoint, we have*

$$
G_{c_1}(x,y) \geq C|x-y| \quad \text{for} \quad |x-y| \leq \tfrac{1}{2}\rho(x) := \mathrm{dist}(x, \partial\Omega).
$$

*That is, the assertion in Theorem 2.2 holds also for $d = 1$. For a general potential $q \in K \cap H^1(\Omega)$, by the
weak maximum principle, for any fixed y, we have $G_q(x,y) \geq G_{c_1}(x,y)$ a.e. $x \in \Omega$, and thus the desired
assertion follows.*

Now we can state a sufficient condition for the positivity condition (2.1).

Proposition 2.1. *Let $q \in H^1(\Omega) \cap K$ and $f \geq c_f > 0$ a.e. in Ω. Then condition (2.1) holds with $\beta = 2$.*

Proof. Recall that for every $y \in \Omega$, there exists a unique Green's function $G_q(\cdot, y) \in H^1(\Omega \setminus B(y,r)) \cap W_0^{1,1}(\Omega)$
for the elliptic operator $-\Delta + qI$, such that

$$
\int_\Omega \nabla G_q(x,y) \cdot \nabla \varphi(x) + q G_q(x,y) \varphi(x) \, \mathrm{d}x = \varphi(y), \quad \forall \varphi \in C_0^\infty(\Omega).
$$

By Theorem 2.2, we have

$$
G_q(x,y) \geq C|x-y|^{-(d-2)} \quad \text{for } |x-y| \leq \tfrac{1}{2}\rho(x) := \mathrm{dist}(x, \partial\Omega), \quad d \geq 2.
$$

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Now for any $x \in \Omega$, let $B(x, \frac{\rho(x)}{2}) \subset \Omega$ be the ball centered at x with a radius $\frac{\rho(x)}{2}$. Since $G_q(x, y) \geq 0$, $x, y \in \Omega$, we have

$$u(q)(x) = \int_\Omega G_q(x, y) f(y) \mathrm{d}y \geq \int_{B(x, \frac{\rho(x)}{2})} f(y) G_q(x, y) \mathrm{d}y$$

$$\geq c_f C \int_{B(x, \frac{\rho(x)}{2})} |x - y|^{-(d-2)} \mathrm{d}y \geq C\rho^2(x) = C\mathrm{dist}(x, \partial\Omega)^2,$$

and thus the desired result follows directly. $\qquad\square$

Remark 2.2. *The argument for deriving the standard $L^2(\Omega)$ estimate relies heavily on the weighted $L^2(\Omega)$ estimate and the positivity condition (2.1). Note that such an analysis strategy could also be applied to other elliptic inverse problems [25, 30] (when equipped with alternative conditions for removing the weighted function).*

2.2 Parabolic inverse problem

The next result gives a conditional stability estimate for the parabolic inverse problem.

Theorem 2.3. *Suppose that $q_1, q_2 \in K \cap H^1(\Omega)$ with $\|\nabla q_1\|_{L^2(\Omega)}, \|\nabla q_2\|_{L^2(\Omega)} \leq c_q$, $u_0 \in H^2(\Omega) \cap H_0^1(\Omega)$ and $f \in H^1(0, T; L^2(\Omega))$. Let $u(q_1)$ and $u(q_2)$ be the corresponding weak solutions of problem (1.2). Then with $T_0 \leq s \leq t \leq T$, there exists a constant C depending on c_q such that*

$$\int_{T_0}^T \int_{T_0}^t \int_s^t \|(q_1 - q_2)u(q_1)(\xi)\|_{L^2(\Omega)} \, \mathrm{d}\xi \mathrm{d}s \mathrm{d}t \leq C\|u(q_1) - u(q_2)\|_{L^2(T_0, T; H^1(\Omega))}^{\frac{1}{2}}.$$

Moreover, if there exists a $\beta \geq 0$ such that

$$u(q_1)(x, t) \geq C\mathrm{dist}(x, \partial\Omega)^\beta \quad a.e. \text{ in } \Omega \tag{2.7}$$

for any $t \in (T_0, T)$, then there exists a constant C depending on c_q such that

$$\|q_1 - q_2\|_{L^2(\Omega)} \leq C\|u(q_1) - u(q_2)\|_{L^2(T_0, T; H^1(\Omega))}^{\frac{1}{2(1+2\beta)}}.$$

Proof. By the weak formulations of $u(q_1)$ and $u(q_2)$, for any $\varphi \in H_0^1(\Omega)$

$$((q_1 - q_2)u(q_1), \varphi) = -(\partial_\xi u(q_1) - \partial_\xi u(q_2), \varphi) - (\nabla(u(q_1) - u(q_2)), \nabla\varphi) - (q_2(u(q_1) - u(q_2)), \varphi) =: \mathrm{I}_1 + \mathrm{I}_2 + \mathrm{I}_3.$$

Let $\varphi = (q_1 - q_2)u(q_1)$. Then $\nabla\varphi = (\nabla q_1 - \nabla q_2)u(q_1) + (q_1 - q_2)\nabla u(q_1)$. By the standard parabolic regularity theory [20], problem (4.2) has a unique solution $u(q_1) \in H^1(0, T; H_0^1(\Omega)) \cap L^\infty(0, T; H^2(\Omega))$, and then by Sobolev embedding theorem [1], $u(q_1) \in L^\infty(0, T; L^\infty(\Omega))$. Then there holds $\|\varphi\|_{L^2(\Omega)} \leq C$ and $\|\nabla\varphi\|_{L^2(\Omega)} \leq C$. Thus we have $\varphi \in H_0^1(\Omega)$. Meanwhile, the Cauchy-Schwarz inequality and the box constraint yield

$$\int_{T_0}^T |\mathrm{I}_2| \, \mathrm{d}t \leq C\|\nabla(u(q_1) - u(q_2))\|_{L^2(T_0, T; L^2(\Omega))} \quad \text{and} \quad \int_{T_0}^T |\mathrm{I}_3| \mathrm{d}t \leq C\|u(q_1) - u(q_2)\|_{L^2(T_0, T; L^2(\Omega))}.$$

It remains to bound the term I_1. By integration by parts, we have

$$\int_s^t (\partial_\xi u(q_1)(\xi) - \partial_\xi u(q_2)(\xi), \varphi(\xi)) \, \mathrm{d}\xi = (u(q_1)(t) - u(q_2)(t), \varphi(t)) - (u(q_1)(s) - u(q_2)(s), \varphi(s))$$

$$- \int_s^t (u(q_1)(\xi) - u(q_2)(\xi), \partial_\xi\varphi(\xi)) \, \mathrm{d}\xi.$$

For the first two terms, by the Cauchy-Schwarz inequality, since $\|\varphi(t)\|_{L^2(\Omega)} \leq C$, we have

$$\left| \int_{T_0}^T \int_{T_0}^t (u(q_1) - u(q_2), \varphi)(t) - (u(q_1) - u(q_2), \varphi)(s) \, \mathrm{d}s \mathrm{d}t \right| \leq C\|u(q_1) - u(q_2)\|_{L^2(T_0, T; L^2(\Omega))}.$$

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Next by the regularity $u \in H^1(0, T; L^2(\Omega))$, and the box constraint in K, we have $\partial_\xi \varphi = (q_1 - q_2)\partial_\xi u(q_1)(\xi) \in L^2(0, T; L^2(\Omega))$. Using Cauchy-Schwarz inequality leads to

$$\Big|\int_{T_0}^{T}\int_{T_0}^{t}\int_{s}^{t}(u(q_1)(\xi) - u(q_2)(\xi), \partial_\xi\varphi(\xi))\,\mathrm{d}\xi\mathrm{d}s\mathrm{d}t\Big| \leq C\Big(\int_{T_0}^{T}\|u(q_1) - u(q_2)\|_{L^2(\Omega)}^2\,\mathrm{d}t\Big)^{\frac{1}{2}}\Big(\int_{T_0}^{T}\|\partial_t\varphi\|_{L^2(\Omega)}^2\,\mathrm{d}t\Big)^{\frac{1}{2}}$$
$$\leq C\|u(q_1) - u(q_2)\|_{L^2(T_0, T; L^2(\Omega))}\|\partial_t u\|_{L^2(T_0, T; L^2(\Omega))}.$$

These estimates directly imply the first desired assertion. Under the positivity condition (2.7), the $L^2(\Omega)$ estimate follows from the argument of Theorem 2.1. □

The next result gives a sufficient condition on the positivity condition (2.7).

Proposition 2.2. *Let $q \in H^1(\Omega) \cap K$, the source $f \geq c_f > 0$ and $\partial_t f \leq 0$ a.e. in $\Omega \times (0, T)$, $u_0 \geq 0$ and $f(0) + \Delta u_0 - qu_0 \leq 0$ a.e. in Ω. Then there exists a positive constant C, depending only on c_0, c_1, c_f and Ω, such that the positivity condition (2.7) holds with $\beta = 2$.*

Proof. Since $f \geq 0$ and $u_0 \geq 0$, the standard parabolic maximum principle (see, e.g., [32, 21]) implies $u \geq 0$, a.e. in $\Omega \times [0, T]$. Let $w := \partial_t u$, which satisfies

$$\begin{cases} \partial_t w - \Delta w + qw = \partial_t f, & \text{in } \Omega \times (0, T), \\ w = 0, & \text{on } \partial\Omega \times (0, T), \\ w(0) = f(0) + \Delta u_0 - qu_0, & \text{in } \Omega. \end{cases}$$

Since $\partial_t f \leq 0$ a.e. in $\Omega \times (0, T)$ and $f(0) + \Delta u_0 - qu_0 \leq 0$ a.e. in Ω, the parabolic maximum principle yields $w \leq 0$ a.e. in $\Omega \times [0, T]$. It suffices to prove that (2.7) with $\beta = 2$ holds for any $t \in (0, T]$. By fixing $t \in [T_0, T]$, we have $f(t) - w(t) \in L^2(\Omega)$. Then consider the following elliptic problem

$$\begin{cases} -\Delta u(t) + qu(t) = f(t) - w(t), & \text{in } \Omega, \\ u(t) = 0, & \text{on } \partial\Omega. \end{cases}$$

By the property of Green's function $G_q(x, y)$ in Theorem 2.2, there holds

$$u(q)(x, t) = \int_\Omega G_q(x, y)(f(y, t) - w(y, t))\mathrm{d}y \geq \int_{B(x, \frac{\rho(x)}{2})} f(y, t)G_q(x, y)\mathrm{d}y$$
$$\geq c_f C \int_{B(x, \frac{\rho(x)}{2})} |x - y|^{-(d-2)}\mathrm{d}y \geq C\rho^2(x) = C\mathrm{dist}(x, \partial\Omega)^2,$$

for any $x \in \Omega$ and $t \in [T_0, T]$, i.e., the positivity condition (2.7) with $\beta = 2$ holds. □

3 Error analysis for the elliptic inverse problem

Now we formulate the regularized output least-squares formulation for the elliptic inverse problem, discretize the continuous formulation by the Galerkin FEM with continuous piecewise linear elements, and provide a complete error analysis.

3.1 Regularization problem and its FEM approximation

To reconstruct the coefficient q, we employ the standard Tikhonov regularization with an $H^1(\Omega)$ seminorm penalty [18, 26], minimizes the following regularized functional:

$$\min_{q \in K} J_\alpha(q) = \frac{1}{2}\|u(q) - z^\delta\|_{L^2(\Omega)}^2 + \frac{\alpha}{2}\|\nabla q\|_{L^2(\Omega)}^2, \tag{3.1}$$

where $u(q) \in H_0^1(\Omega)$ satisfies

$$(\nabla u(q), \nabla\varphi) + (qu(q), \varphi) = (f, \varphi), \quad \forall\varphi \in H_0^1(\Omega). \tag{3.2}$$

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Recall that the given data $z^\delta \in L^2(\Omega)$ is noisy with a noise level δ relative to the exact data $u(q^\dagger)$ (corresponding to the exact radiativity $q^\dagger$), i.e., $\|u(q^\dagger) - z^\delta\|_{L^2(\Omega)} = \delta$. The continuous problem (3.1)–(3.2) is well-posed in the sense that it has at least one global minimizer, and the minimizer is continuous with respect to the perturbations in the data, and further as the noise level tends to zero, the sequence of minimizers converges to the exact solution in $H^1(\Omega)$ (if α is chosen properly) [19, 18, 26].

To discretize problem (3.1)–(3.2), we employ the standard Galerkin FEM [7]. Let $\mathcal{T}_h$ be a shape regular quasi-uniform simplicial triangulation of the domain Ω, with a grid size h. On the triangulation $\mathcal{T}_h$, we define the conforming piecewise linear finite element spaces V_h and V_{h0} by

$$V_h := \{v_h \in H^1(\Omega) : v_h|_T \text{ is a linear polynomial, } \forall T \in \mathcal{T}_h\}$$

and $V_{h0} := V_h \cap H_0^1(\Omega)$. We use the spaces V_{h0} and V_h to approximate the state u and the parameter q, respectively. The following inverse inequality holds in the finite element space V_{h0} [7]

$$\|v_h\|_{H^1(\Omega)} \le Ch^{-1}\|v_h\|_{L^2(\Omega)}, \quad \forall v_h \in V_{h0}. \tag{3.3}$$

We denote by P_h the standard $L^2(\Omega)$-projection operator associated with the finite element space V_{h0}. Then it is known that for $s = 1, 2$ [15, 36]:

$$\|v - P_h v\|_{L^2(\Omega)} + h\|\nabla(v - P_h v)\|_{L^2(\Omega)} \le Ch^s\|v\|_{H^s(\Omega)}, \quad \forall v \in H^s(\Omega) \cap H_0^1(\Omega). \tag{3.4}$$

Let Π_h be the Lagrange interpolation operator associated with the finite element space V_h. It satisfies the following error estimate for $s = 1, 2$ and $1 \le p \le \infty$ (with $sp > d$):

$$\|v - \Pi_h v\|_{L^p(\Omega)} + h\|v - \Pi_h v\|_{W^{1,p}(\Omega)} \le ch^s\|v\|_{W^{s,p}(\Omega)}, \quad \forall v \in W^{s,p}(\Omega). \tag{3.5}$$

Now we can state the finite element approximation of problem (3.1)-(3.2):

$$\min_{q_h \in K_h} J_{\alpha,h}(q_h) = \frac{1}{2}\|u_h(q_h) - z^\delta\|_{L^2(\Omega)}^2 + \frac{\alpha}{2}\|\nabla q_h\|_{L^2(\Omega)}^2, \tag{3.6}$$

with $K_h = K \cap V_h$, and $u_h(q_h) \in V_{h0}$ satisfying

$$(\nabla u_h(q_h), \nabla \varphi_h) + (q_h u_h(q_h), \varphi_h) = (f, \varphi_h), \quad \forall \varphi_h \in V_{h0}. \tag{3.7}$$

The discrete problem is well-posed: there exists at least one global minimizer $q_h^* \in K_h$ to problem (3.6)–(3.7), and it depends continuously on the data perturbation. The main objective of this work is to bound the error $q_h^* - q^\dagger$ of the approximation q_h^*.

3.2 Error estimates

Now we derive a weighted $L^2(\Omega)$ estimate of the error $q^\dagger - q_h^*$, under the following assumption. By the standard elliptic regularity theory [20] and Sobolev embedding theorem [1], Assumption 3.1 implies $u(q^\dagger) \in H^2(\Omega) \hookrightarrow L^\infty(\Omega)$, for $d = 1, 2, 3$.

Assumption 3.1. $q^\dagger \in H^2(\Omega) \cap K$ and $f \in L^2(\Omega)$.

Next we give the main result of this section, i.e., a novel weighted $L^2(\Omega)$ error estimate.

Theorem 3.2. *Let Assumption 3.1 be fulfilled. Let $q^\dagger \in K$ be the exact potential, $u(q^\dagger)$ be the solution of problem (3.2), and $q_h^* \in K_h$ be a minimizer of problem (3.6)-(3.7). Then with $\eta = h^2 + \delta + \alpha^{\frac{1}{2}}$, there holds*

$$\|(q^\dagger - q_h^*)u(q^\dagger)\|_{L^2(\Omega)} \le C(h^{\frac{1}{2}} + \alpha^{\frac{1}{4}} + \min(h^{\frac{1}{2}} + h^{-\frac{1}{2}}\eta^{\frac{1}{2}}, 1))\alpha^{-\frac{1}{4}}\eta^{\frac{1}{2}}, \tag{3.8}$$

where the constant C depends only on $q^\dagger$.

The proof employs two preliminary estimates.

Lemma 3.1. *Let Assumption 3.1 be fulfilled. Then there holds*

$$\|u(q^\dagger) - u_h(\Pi_h q^\dagger)\|_{L^2(\Omega)} + h\|\nabla(u(q^\dagger) - u_h(\Pi_h q^\dagger))\|_{L^2(\Omega)} \leq Ch^2. \tag{3.9}$$

Proof. Céa's Lemma and the standard duality argument imply

$$\|u(q^\dagger) - u_h(q^\dagger)\|_{L^2(\Omega)} + h\|\nabla(u(q^\dagger) - u_h(q^\dagger))\|_{L^2(\Omega)} \leq Ch^2\|u(q^\dagger)\|_{H^2(\Omega)} \leq Ch^2. \tag{3.10}$$

Then it suffices to bound $w_h = u_h(\Pi_h q^\dagger) - u_h(q^\dagger)$. Clearly, w_h satisfies for any $\varphi_h \in V_{h0}$

$$(\nabla w_h, \nabla \varphi_h) + ((\Pi_h q^\dagger) w_h, \varphi_h) = ((q^\dagger - \Pi_h q^\dagger) u_h(q^\dagger), \varphi_h)$$
$$= ((q^\dagger - \Pi_h q^\dagger) u(q^\dagger), \varphi_h) + ((q^\dagger - \Pi_h q^\dagger)(u(q^\dagger) - u_h(q^\dagger)), \varphi_h) =: \mathrm{I}.$$

Letting $\varphi_h = w_h$, the estimate (3.10), and the approximation property of Π_h in (3.5) give

$$|\mathrm{I}| \leq Ch^2\big(\|q\|_{H^2(\Omega)}\|u(q)\|_{L^\infty(\Omega)}\|w_h\|_{L^2(\Omega)} + \|q\|_{L^\infty(\Omega)}\|u(q)\|_{H^2(\Omega)}\|w_h\|_{L^2(\Omega)}\big).$$

Consequently, we have

$$\|\nabla w_h\|_{L^2(\Omega)}^2 + c_0\|w_h\|_{L^2(\Omega)}^2 \leq Ch^2\|w_h\|_{L^2(\Omega)}.$$

Since $c_0 \geq 0$, this and Poincare's inequality lead to

$$\|\nabla w_h\|_{L^2(\Omega)} \leq Ch^2.$$

This, Poincaré's inequality, (3.10) and the triangle inequality imply the assertion. $\qquad\square$

The next result gives a crucial a priori bound on $\|\nabla q_h^*\|_{L^2(\Omega)}$ and state the approximation error $\|u(q^\dagger) - u_h(q_h^*)\|_{L^2(\Omega)}$. Note that this estimate allows bounding $\|\nabla q_h^*\|_{L^2(\Omega)}$ a priori by $C\alpha^{-\frac{1}{2}}(h^2 + \delta + \alpha^{\frac{1}{2}})$, where the constant C depends only on the a priori regularity of the exact potential $q^\dagger$. This estimate shows explicitly the delicate interplay of the parameters h, α and δ and it will play a central role in the error analysis below.

Lemma 3.2. *Let the assumption in Theorem 3.2 be fulfilled. Then the following estimate holds*

$$\|u(q^\dagger) - u_h(q_h^*)\|_{L^2(\Omega)} + \alpha^{\frac{1}{2}}\|\nabla q_h^*\|_{L^2(\Omega)} \leq C(h^2 + \delta + \alpha^{\frac{1}{2}}). \tag{3.11}$$

Proof. Since q_h^* is the minimizer of the system (3.6)-(3.7) and $\Pi_h q^\dagger \in K_h$, we have

$$J_{\alpha,h}(q_h^*) \leq J_{\alpha,h}(\Pi_h q^\dagger).$$

Then the definition of δ and Lemma 3.1 imply

$$\|u_h(q_h^*) - z^\delta\|_{L^2(\Omega)}^2 + \alpha\|\nabla q_h^*\|_{L^2(\Omega)}^2 \leq \|u_h(\Pi_h q^\dagger) - z^\delta\|_{L^2(\Omega)}^2 + \alpha\|\nabla \Pi_h q^\dagger\|_{L^2(\Omega)}^2$$
$$\leq C(\|u_h(\Pi_h q^\dagger) - u(q^\dagger)\|_{L^2(\Omega)}^2 + \|u(q^\dagger) - z^\delta\|_{L^2(\Omega)}^2 + \alpha\|\nabla \Pi_h q^\dagger\|_{L^2(\Omega)}^2)$$
$$\leq C(h^4 + \delta^2 + \alpha),$$

where the last line follows from the inequality (3.5) and the inverse inequality (3.3), i.e., $\|\nabla \Pi_h q^\dagger\|_{L^2(\Omega)} \leq C\|q^\dagger\|_{H^1(\Omega)}$. Then by the triangle inequality, we deduce

$$\|u(q^\dagger) - u_h(q_h^*)\|_{L^2(\Omega)}^2 + \alpha\|\nabla q_h^*\|_{L^2(\Omega)}^2 \leq C(\|u(q^\dagger) - z^\delta\|_{L^2(\Omega)}^2 + \|u_h(q_h^*) - z^\delta\|_{L^2(\Omega)}^2 + \alpha\|\nabla q_h^*\|_{L^2(\Omega)}^2)$$
$$\leq C(h^4 + \delta^2 + \alpha).$$

This completes the proof of the lemma. $\qquad\square$

Now we can prove Theorem 3.2, which relies heavily on the novel test function $\varphi = (q^\dagger - q_h^*)u(q^\dagger)$, and the overall strategy is inspired by the conditional stability analysis in Section 2.

Proof. For any $\varphi \in H_0^1(\Omega)$, it follows from the weak formulations of $u(q^\dagger)$ and $u_h(q_h^*)$ that

$$\begin{aligned}
((q^\dagger - q_h^*)u(q^\dagger), \varphi) &= ((q^\dagger - q_h^*)u(q^\dagger), \varphi - P_h\varphi) + ((q^\dagger - q_h^*)u(q^\dagger), P_h\varphi) \\
&= ((q^\dagger - q_h^*)u(q^\dagger), \varphi - P_h\varphi) + (\nabla(u_h(q_h^*) - u(q^\dagger)), \nabla P_h\varphi) + (q_h^*(u_h(q_h^*) - u(q^\dagger)), P_h\varphi) \\
&=: \mathrm{I}_1 + \mathrm{I}_2 + \mathrm{I}_3.
\end{aligned}$$

Let $\varphi = (q^\dagger - q_h^*)u(q^\dagger)$. Then direct computation gives $\nabla\varphi = u(q^\dagger)(\nabla q^\dagger - \nabla q_h^*) + (q^\dagger - q_h^*)\nabla u(q^\dagger)$. By Assumption 3.1 and the a priori regularity $u(q^\dagger) \in H^2(\Omega) \hookrightarrow L^\infty(\Omega)$ for $d = 1, 2, 3$, we have

$$\|\varphi\|_{L^2(\Omega)} \leq C \quad \text{and} \quad \|\nabla\varphi\|_{L^2(\Omega)} \leq C(1 + \|\nabla q_h^*\|_{L^2(\Omega)}).$$

Thus, $\varphi \in H_0^1(\Omega)$. We bound the three terms. By Lemma 3.2 and (3.4), we bound the term I_1 by

$$|\mathrm{I}_1| \leq Ch\|\nabla\varphi\|_{L^2(\Omega)} \leq Ch(1 + \|\nabla q_h^*\|_{L^2(\Omega)}) \leq Ch(1 + \alpha^{-\frac{1}{2}}\eta) \leq Ch\alpha^{-\frac{1}{2}}\eta,$$

where the constant C depends on $q^\dagger$. By applying Lemma 3.2 and (3.4) again and also the inverse inequality (3.3), the term I_2 can be bounded by

$$\begin{aligned}
|\mathrm{I}_2| &\leq \|\nabla(u(q^\dagger) - u_h(q_h^*))\|_{L^2(\Omega)}\|\nabla\varphi\|_{L^2(\Omega)} \\
&\leq (\|\nabla(u(q^\dagger) - P_hu(q^\dagger))\|_{L^2(\Omega)} + Ch^{-1}\|P_hu(q^\dagger) - u_h(q_h^*)\|_{L^2(\Omega)})\|\nabla\varphi\|_{L^2(\Omega)} \\
&\leq C(h + h^{-1}\|u(q^\dagger) - u_h(q_h^*)\|_{L^2(\Omega)})\|\nabla\varphi\|_{L^2(\Omega)} \\
&\leq C(h + h^{-1}\eta)(1 + \|\nabla q_h^*\|_{L^2(\Omega)}) \leq C(h + h^{-1}\eta)\alpha^{-\frac{1}{2}}\eta.
\end{aligned}$$

Meanwhile, by the a priori estimate, we have

$$\|\nabla(u(q^\dagger) - u_h(q_h^*))\|_{L^2(\Omega)} \leq C.$$

Thus, we obtain

$$|\mathrm{I}_2| \leq C\min(h + h^{-1}\eta, 1)\alpha^{-\frac{1}{2}}\eta.$$

Finally, the bound on the term I_3 follows from Lemma 3.2 and the $L^2(\Omega)$ stability of P_h by

$$|\mathrm{I}_3| \leq C\|P_h\varphi\|_{L^2(\Omega)}\|u(q^\dagger) - u_h(q_h^*)\|_{L^2(\Omega)} \leq C\eta.$$

Then the desired estimate follows from the bounds on I_i. $\qquad\square$

From Theorem 3.2, we can derive two $L^2(\Omega)$ estimates on the error $q^\dagger - q_h^*$. First, we give an interior L^2-error estimate, by means of maximum principle.

Corollary 3.1. *Let the assumptions in Theorem 3.2 be fulfilled and the source $f \not\equiv 0$ be nonnegative a.e. in Ω. Then for any compact subset $\Omega' \subset\subset \Omega$ with $\mathrm{dist}(\overline{\Omega'}, \partial\Omega) > 0$, there exists a positive constant C, depending on $\mathrm{dist}(\overline{\Omega'}, \partial\Omega)$ and $q^\dagger$, such that*

$$\|q^\dagger - q_h^*\|_{L^2(\Omega')} \leq C(h^{\frac{1}{2}} + \alpha^{\frac{1}{4}} + \min(h^{\frac{1}{2}} + h^{-\frac{1}{2}}\eta^{\frac{1}{2}}, 1))\alpha^{-\frac{1}{4}}\eta^{\frac{1}{2}}.$$

Proof. Let $w \in H^2(\Omega) \cap H_0^1(\Omega)$ be the solution of problem (3.2) with q replaced by c_1. Since $f \geq 0$ a.e. in Ω, by the maximum principle [8], $u(q^\dagger) \geq 0$. Note that $u(q^\dagger) - w$ satisfies

$$(\nabla(u(q^\dagger) - w), \nabla\varphi) + (q^\dagger(u(q^\dagger) - w), \varphi) = ((c_1 - q^\dagger)u(q^\dagger), \varphi), \quad \forall\varphi \in H_0^1(\Omega).$$

By the maximum principle [8], $u(q^\dagger) - w \geq 0$ a.e. in Ω. Meanwhile, by Sobolev embedding theorem [1], $H^2(\Omega) \hookrightarrow C(\overline{\Omega})$, we have $u(q^\dagger) \geq w \geq 0$ in Ω. Then by Theorem 3.2

$$\|(q^\dagger - q_h^*)w\|_{L^2(\Omega)} \leq C(h^{\frac{1}{2}} + \alpha^{\frac{1}{4}} + \min(h^{\frac{1}{2}} + h^{-\frac{1}{2}}\eta^{\frac{1}{2}}, 1))\alpha^{-\frac{1}{4}}\eta^{\frac{1}{2}}.$$

Note that for any $\Omega' \subset\subset \Omega$, by [38, Theorem 1] and $w \in C(\overline{\Omega})$, there exists a positive constant C depending on $\mathrm{dist}(\overline{\Omega'}, \partial\Omega)$ such that $w \geq C > 0$ in Ω'. The desired estimate follows directly. $\qquad\square$

The next result gives an $L^2(\Omega)$ error estimate under the positivity condition (2.1).

Corollary 3.2. *Let the conditions in Theorem 3.2 be fulfilled, and condition (2.1) holds. Then there holds*

$$\|q^\dagger - q_h^*\|_{L^2(\Omega)} \leq C((h^{\frac{1}{2}} + \alpha^{\frac{1}{4}} + \min(h^{\frac{1}{2}} + h^{-\frac{1}{2}}\eta^{\frac{1}{2}}, 1))\alpha^{-\frac{1}{4}}\eta^{\frac{1}{2}})^{\frac{1}{1+2\beta}}.$$

Proof. The proof is identical with that in Theorem 2.1, and hence omitted. $\qquad\square$

Remark 3.1. *The error estimates provide useful guidelines for choosing the algorithmic parameters, e.g., α and h, in order to achieve the best possible convergence rates of the discrete approximations q_h^* in terms of the noise level δ. Indeed, by properly balancing the terms in the upper bound, we should choose $\alpha \sim \delta^2$ and $h \sim \delta^{\frac{1}{2}}$ so as to obtain the following interior estimates: $\|q^\dagger - q_h^*\|_{L^2(\Omega')} \leq C\delta^{\frac{1}{4}}$. Similarly, there holds $\|q_h - q^\dagger\|_{L^2(\Omega)} \leq C\delta^{\frac{1}{4(1+2\beta)}}$, under the positivity condition (2.1). Note that the choice $\alpha \sim \delta^2$ contrasts sharply with that in standard regularization theory [18, 26] which typically assumes a slower decay than δ^2, but it is actually the most common choice when using conditional stability estimates [12, 17]. It is instructive to compare the rate with the conditional stability estimate in Theorem 2.1. Specifically, by the a prior regularity estimate $u(q) \in H^2(\Omega)$ for $q \in K \cap H^1(\Omega)$ and the Gagliardo-Nirenberg interpolation inequality*

$$\|v\|_{H^1(\Omega)} \leq C\|v\|_{L^2(\Omega)}^{\frac{1}{2}}\|v\|_{H^2(\Omega)}^{\frac{1}{2}},$$

we have

$$\|q_1 - q_2\|_{L^2(\Omega)} \leq C\|u(q_1) - u(q_2)\|_{L^2(\Omega)}^{\frac{1}{4(1+2\beta)}}, \quad \forall q_1, q_2 \in K \cap H^1(\Omega).$$

Thus the error estimate in Corollary 3.2 is consistent with the conditional stability estimate in Theorem 2.1.

4 Error analysis for the parabolic inverse problem

Now we turn to the convergence analysis for parabolic systems.

4.1 Regularized formulation and its FEM approximation

Like the elliptic case in Section 3, we consider the inverse problem of recovering a space-dependent potential q from the following noisy distributed observation z^δ

$$z^\delta(x, t) = u(q^\dagger)(x, t) + \xi(x, t), \quad \text{in } \Omega \times (T_0, T),$$

where $0 \leq T_0 < T$, and the accuracy of z^δ is measured by the noise level δ, defined by $\delta = \|u(q) - z^\delta\|_{L^2(T_0, T; L^2(\Omega))}$. To reconstruct the potential q from the data z^δ, we employ the standard Tikhonov regularization, which minimizes the following regularized functional:

$$\min_{q \in K} J_\alpha(q) = \frac{1}{2}\|u(q)(t) - z^\delta(t)\|_{L^2(T_0, T; L^2(\Omega))}^2 + \frac{\alpha}{2}\|\nabla q\|_{L^2(\Omega)}^2, \tag{4.1}$$

where $u(t) \equiv u(q)(t) \in H_0^1(\Omega)$ with $u(0) = u_0$ satisfies

$$(\partial_t u(t), \varphi) + (\nabla u(t), \nabla\varphi) + (qu(t), \varphi) = (f, \varphi), \quad \forall \varphi \in H_0^1(\Omega), \text{ a.e. } t \in (0, T). \tag{4.2}$$

Similar to the elliptic case, the well-posedness of the continuous formulation (4.1)–(4.2) can be shown easily using the direct method in calculus of variation [41].

For the spatial discretization of problem (4.1)–(4.2), we apply the Galerkin FEM described in Section 3, i.e., to approximate the state variable $u \in H_0^1(\Omega)$ by V_{h0} and the unknown potential q by the space V_h. For the time discretization, we employ the backward Euler method on a uniform time grid. We divide the time interval $(0, T)$ into N equal subintervals with a time step size τ and the time grids $t^n = n\tau$, $n = 0, \dots, N$. Further, we denote by $v^n = v(t^n)$ and define the backward difference quotient $\partial_\tau v^n$ and and the piecewise constant L^2-projection in the cell (t^{n-1}, t^n) by

$$\partial_\tau v^n := \tau^{-1}(v^n - v^{n-1}) \quad \text{and} \quad \bar{v}^n := \tau^{-1}\int_{t^{n-1}}^{t^n} v(t)\,\mathrm{d}t,$$

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respectively. Throughout, we assume that T_0 is at a grid point, with $N_0 = T_0/\tau$ for some $N_0 \in \mathbb{N}$ (which depends on the step size τ).

Then the fully discrete problem for problem (4.1)-(4.2) reads:

$$\min_{q_h \in K_h} J_{\alpha,h,\tau}(q_h) = \frac{\tau}{2} \sum_{n=N_0}^{N} \|u_h^n(q_h) - z_n^\delta\|_{L^2(\Omega)}^2 + \frac{\alpha}{2} \|\nabla q_h\|_{L^2(\Omega)}^2, \tag{4.3}$$

with $z_n^\delta = \tau^{-1} \int_{t_{n-1}}^{t_n} z^\delta(t)\mathrm{d}t$, $K_h = K \cap V_h$, where $u_h^n \equiv u_h^n(q_h) \in V_{h0}$ satisfies $u_h^0(q_h) = P_h u_0$ and

$$(\partial_\tau u_h^n, \varphi_h) + (\nabla u_h^n, \nabla \varphi_h) + (q_h u_h^n, \varphi_h) = (f^n, \varphi_h), \quad \forall \varphi_h \in V_{h0}, \ n = 1, \dots, N. \tag{4.4}$$

Problem (4.3)—(4.4) is a finite-dimensional constrained optimization problem, and the existence of a minimizer q_h^* follows easily in view of the norm equivalence in finite-dimensional spaces. Below we give an error analysis of the approximation q_h^*.

In the analysis, we use extensively the Ritz projection operator $R_h : H_0^1(\Omega) \to V_{h0}$ defined by

$$(\nabla u(q), \nabla \varphi_h) + (qu(q), \varphi_h) = (\nabla R_h u(q), \nabla \varphi_h) + (q R_h u(q), \varphi_h), \quad \forall \varphi_h \in V_{h0}.$$

The following approximation result holds [36]:

$$\|v - R_h v\|_{L^2(\Omega)} + h\|v - R_h v\|_{H^1(\Omega)} \le Ch^2 \|v\|_{H^2(\Omega)}, \quad \forall v \in H^2(\Omega) \cap H_0^1(\Omega). \tag{4.5}$$

4.2 Error estimates

First, we derive a weighted $L^2(\Omega)$-error estimates of the approximation q_h^*. Throughout we make following assumption on the problem data. Under Assumption 4.1, by the standard parabolic regularity theory [20], the weak solution $u(q^\dagger) \in L^2(0,T; H_0^1(\Omega)) \cap H^1(0,T; H^{-1}(\Omega))$ to problem (4.2) belongs to $H^1(0,T; H_0^1(\Omega)) \cap L^\infty(0,T; H^2(\Omega))$, and then by Sobolev embedding theorem [1], $u(q^\dagger) \in L^\infty(0,T; L^\infty(\Omega))$.

Assumption 4.1. $q^\dagger \in H^2(\Omega) \cap K$, $u_0 \in H^2(\Omega) \cap H_0^1(\Omega)$ and $f \in H^1(0,T; L^2(\Omega))$.

Now we can state the main result of this section, i.e., a weighted $L^2(\Omega)$ error estimate.

Theorem 4.2. Let Assumption 4.1 be fulfilled. Let $q^\dagger \in K$ be the exact coefficient, $u(q^\dagger)$ be the solution of problem (4.2), and $q_h^* \in K_h$ be a minimizer of problem (4.3)-(4.4). Then with $\eta = \tau + h^2 + \delta + \alpha^{\frac{1}{2}}$, there exists a constant C depending on $q^\dagger$ such that

$$\tau^3 \sum_{j=N_0+1}^{N} \sum_{i=N_0+1}^{j} \sum_{n=i}^{j} \|(q^\dagger - q_h^*)u^n(q^\dagger)\|_{L^2(\Omega)} \le C(\tau^{\frac{1}{2}} + h^{\frac{1}{2}} + \alpha^{\frac{1}{4}} + \min(h^{\frac{1}{2}} + h^{-\frac{1}{2}}\eta^{\frac{1}{2}}, 1))\alpha^{-\frac{1}{4}}\eta^{\frac{1}{2}}. \tag{4.6}$$

The overall proof strategy is similar to the elliptic case in Section 3, but it is more involved due to the presence of time derivative. It relies heavily on the following preliminary results. The lengthy proofs are similar to the elliptic case, and hence are deferred to the appendix.

Lemma 4.1. Let Assumption 4.1 be fulfilled. Then for sufficiently small τ, there holds

$$\tau \sum_{n=1}^{N} \|u^n(q^\dagger) - u_h^n(q^\dagger)\|_{L^2(\Omega)}^2 \le C(\tau^2 + h^4). \tag{4.7}$$

Lemma 4.2. Let Assumption 4.1 be fulfilled. Then for small τ, there holds

$$\tau \sum_{n=1}^{N} \|u^n(q^\dagger) - u_h^n(\Pi_h q^\dagger)\|_{L^2(\Omega)}^2 \le C(\tau^2 + h^4). \tag{4.8}$$

Lemma 4.3. Let the assumption in Theorem 4.2 be fulfilled. Then there holds

$$\tau \sum_{n=N_0}^{N} \|u^n(q^\dagger) - u_h^n(q_h^*)\|_{L^2(\Omega)}^2 + \alpha\|\nabla q_h^*\|_{L^2(\Omega)}^2 \le C(\tau^2 + h^4 + \delta^2 + \alpha).$$

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Now we can state the proof of Theorem 4.2.

Proof. Let $u \equiv u(q^\dagger)$. For any $\varphi \in H_0^1(\Omega)$, we have the following splitting

$$((q^\dagger - q_h^*)u^n, \varphi) = ((q^\dagger - q_h^*)u^n, \varphi - P_h\varphi) + ((q^\dagger - q_h^*)u^n, P_h\varphi)$$
$$= ((q^\dagger - q_h^*)u^n, \varphi - P_h\varphi) + (\partial_\tau u_h^n(q_h^*) - \partial_t u^n, P_h\varphi)$$
$$+ (\nabla(u_h^n(q_h^*) - u^n), \nabla P_h\varphi) + (q_h^*(u_h^n(q_h^*) - u^n), P_h\varphi) =: \sum_{i=1}^4 \mathrm{I}_i^n.$$

Let $\varphi = \varphi^n = (q^\dagger - q_h^*)u^n$. Then $\nabla\varphi^n = (\nabla q^\dagger - \nabla q_h^*)u^n + (q^\dagger - q_h^*)\nabla u^n$. By Assumption 4.1 and the a priori regularity $u \in L^\infty(0, T; L^\infty(\Omega))$, there holds

$$\|\varphi^n\| \leq C \quad \text{and} \quad \|\nabla\varphi^n\|_{L^2(\Omega)} \leq C(1 + \|\nabla q_h^*\|_{L^2(\Omega)}), \quad n = 1, \ldots, N, \tag{4.9}$$

which implies $\varphi^n \in H_0^1(\Omega)$. Using Assumption 4.1, Lemma 4.3, and repeating the argument of Theorem 3.2, we deduce

$$\sum_{n=N_0}^N |\mathrm{I}_1^n| \leq Ch\alpha^{-\frac{1}{2}}\eta, \quad \tau \sum_{n=N_0}^N |\mathrm{I}_3^n| \leq C\min(h + h^{-1}\eta, 1))\alpha^{-\frac{1}{2}}\eta, \text{ and } \tau \sum_{n=N_0}^N |\mathrm{I}_4^n| \leq C\eta,$$

where the constant C depends on $q^\dagger$. To estimate I_2^n, we split it into

$$\mathrm{I}_2^n = (\partial_\tau(u_h^n(q_h^*) - u^n), P_h\varphi^n) + (\partial_\tau u^n - \partial_t u^n, P_h\varphi^n) =: \mathrm{I}_{2,1}^n + \mathrm{I}_{2,2}^n.$$

Let $\varphi = P_h\varphi^n$ in (A.5) and (4.2). By Assumption 4.1, $f \in H^1(0, T; L^2(\Omega))$, we have the a priori regularity $u \in H^1(0, T; H^1(\Omega))$. Consequently, the proof of Lemma 4.1 yields

$$\tau \sum_{n=N_0}^N \|\bar{f}^n - f^n\|_{L^2(\Omega)}^2 \leq C\tau^2, \quad \tau \sum_{n=N_0}^N \|\bar{u}^n - u^n\|_{H^1(\Omega)}^2 \leq C\tau^2, \quad \tau \sum_{n=N_0}^N \|\bar{u}^n - u^n\|_{L^2(\Omega)}^2 \leq C\tau^2.$$

This, the $L^2(\Omega)$-stability of P_h in (3.4), the Cauchy-Schwarz inequality and (4.9) imply

$$\left| \tau \sum_{n=N_0}^N \mathrm{I}_{2,2}^n \right| = \left| \tau \sum_{n=N_0}^N (\bar{f}^n - f^n, P_h\varphi^n) - \tau \sum_{n=N_0}^N (\nabla(\bar{u}^n - u^n), \nabla P_h\varphi^n) - \tau \sum_{n=N_0}^N (q^\dagger(\bar{u}^n - u^n), P_h\varphi^n) \right|$$

$$\leq C\left(\tau \sum_{n=N_0}^N \|\bar{f}^n - f^n\|_{L^2(\Omega)}^2\right)^{\frac{1}{2}} + \left(\tau \sum_{n=N_0}^N \|\bar{u}^n - u^n\|_{H^1(\Omega)}^2\right)^{\frac{1}{2}} \max_{n=N_0,\ldots,N} \|\nabla\varphi^n\|_{L^2(\Omega)}$$

$$+ \left(\tau \sum_{n=N_0}^N \|\bar{u}^n - u^n\|_{L^2(\Omega)}^2\right)^{\frac{1}{2}} \leq C\tau\alpha^{-\frac{1}{2}}\eta.$$

Next we bound the term $\mathrm{I}_{2,1}^n$. For $N_0 + 1 \leq i \leq j \leq N$, the summation by parts formula leads to

$$\tau \sum_{n=i}^j \mathrm{I}_{2,1}^n = (u_h^j(q_h^*) - u^j, P_h\varphi^j) - (u_h^{i-1}(q_h^*) - u^{i-1}, P_h\varphi^{i-1}) - \tau \sum_{n=i}^j (u_h^{n-1}(q_h^*) - u^{n-1}, \partial_\tau P_h\varphi^n).$$

Using Assumption 4.1, Lemma 4.3, the $L^2(\Omega)$ stability of P_h and the Cauchy-Schwarz inequality, we have

$$\left| \tau^2 \sum_{j=N_0+1}^N \sum_{i=N_0+1}^j (u_h^j(q_h^*) - u^j, P_h\varphi^j) - (u_h^{i-1}(q_h^*) - u^{i-1}, P_h\varphi^{i-1}) \right|$$

$$\leq C\left(\tau \sum_{n=N_0}^N \|u_h^n(q_h^*) - u^n\|_{L^2(\Omega)}^2\right)^{\frac{1}{2}} \leq C\eta.$$

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Next, by the $L^2(\Omega)$ stability of P_h in (3.4), the box constraint and the Cauchy-Schwarz inequality, we have

$$\|\partial_\tau P_h \varphi^n\|_{L^2(\Omega)} = \left\|P_h\left(\tau^{-1}\int_{t^{n-1}}^{t^n}(q^\dagger - q_h^*)\partial_t u \mathrm{d}t\right)\right\|_{L^2(\Omega)} \le C\left\|\tau^{-\frac{1}{2}}\left(\int_{t^{n-1}}^{t^n}|\partial_t u|^2\mathrm{d}t\right)^{\frac{1}{2}}\right\|_{L^2(\Omega)}.$$

Since $u \in H^1(0,T;H_0^1(\Omega))$, we deduce

$$\tau\sum_{n=N_0+1}^{N}\|\partial_\tau P_h\varphi^n\|_{L^2(\Omega)}^2 \le \sum_{n=N_0+1}^{N}\|\partial_t u\|_{L^2(t^{n-1},t^n;L^2(\Omega))}^2 = \|\partial_t u\|_{L^2(T_0,T;L^2(\Omega))}^2 \le C.$$

This and the Cauchy-Schwarz inequality imply

$$\left|\tau\sum_{n=i}^{j}(u_h^{n-1}(q_h^*)-u^{n-1},\partial_\tau P_h\varphi^n)\right| \le \tau\sum_{n=i}^{j}\|u_h^{n-1}(q_h^*)-u^{n-1}\|_{L^2(\Omega)}\|\partial_\tau P_h\varphi^n\|_{L^2(\Omega)}$$

$$\le \left(\tau\sum_{n=N_0+1}^{N}\|u_h^{n-1}(q_h^*)-u^{n-1}\|_{L^2(\Omega)}^2\right)^{\frac{1}{2}}\left(\tau\sum_{n=N_0+1}^{N}\|\partial_\tau P_h\varphi^n\|_{L^2(\Omega)}^2\right)^{\frac{1}{2}} \le C\eta,$$

and hence there holds

$$\left|\tau^3\sum_{j=N_0+1}^{N}\sum_{i=N_0+1}^{j}\sum_{n=i}^{j}(u_h^{n-1}(q_h^*)-u^{n-1},\partial_\tau P_h\varphi^n)\right| \le C\eta.$$

Combining the preceding estimates gives

$$\tau^3\sum_{j=N_0+1}^{N}\sum_{i=N_0+1}^{j}\sum_{n=i}^{j}\|(q^\dagger-q_h^*)u^n\|_{L^2(\Omega)}^2 \le C(\tau+h+\alpha^{\frac{1}{2}}+\min(h+h^{-1}\eta,1))\alpha^{-\frac{1}{2}}\eta.$$

This completes the proof of the theorem. $\qquad\square$

Now we establish interior and global L^2 error estimations.

Corollary 4.1. *Let Assumption 4.1 be fulfilled, the source $f \not\equiv 0$ be nonnegative a.e. in $\Omega \times (0,T)$ and $u_0 \ge 0$ a.e. in Ω. Then for any compact subset $\Omega' \subset\subset \Omega$ with $\mathrm{dist}(\overline{\Omega'},\partial\Omega) > 0$, there exists a positive constant C, depending on $\mathrm{dist}(\overline{\Omega'},\partial\Omega)$ and $q^\dagger$, such that*

$$\|q^\dagger - q_h^*\|_{L^2(\Omega')} \le C(\tau^{\frac{1}{2}}+h^{\frac{1}{2}}+\alpha^{\frac{1}{4}}+\min(h^{\frac{1}{2}}+h^{-\frac{1}{2}}\eta^{\frac{1}{2}},1))\alpha^{-\frac{1}{4}}\eta^{\frac{1}{2}}.$$

Proof. By the standard parabolic maximum principle (see, e.g., [32, 21]), $f \ge 0$ a.e. in $\Omega \times (0,T)$ and $u_0 \ge 0$ in Ω imply $u(q^\dagger) \ge 0$ a.e. in $\Omega \times (0,T)$. Let $w \in L^2(0,T;H_0^1(\Omega)) \cap H^1(0,T;H^{-1}(\Omega))$ with $w(0) = \frac{u_0}{2}$ solve

$$(\partial_t w,\varphi)+(\nabla w,\nabla\varphi)+(c_1 w,\varphi) = (f,\varphi), \quad \forall\varphi \in H_0^1(\Omega),\ \text{a.e. } t \in (0,T).$$

Then $v = u(q^\dagger) - w$ satisfies $v(0) = \frac{u_0}{2}$ and

$$(\partial_t v,\varphi)+(\nabla v,\nabla\varphi)+(q^\dagger v,\varphi) = ((c_1-q^\dagger)u(q^\dagger),\varphi), \quad \forall\varphi \in H_0^1(\Omega),\ \text{a.e. } t \in (0,T).$$

Then the weak maximum principle for parabolic problem implies $v \ge 0$, i.e., $u(q^\dagger) \ge w$ a.e. in $\Omega \times (0,T)$. By Theorem 4.2, there holds

$$\tau^3\sum_{j=N_0+1}^{N}\sum_{i=N_0+1}^{j}\sum_{n=i}^{j}\|(q^\dagger-q_h^*)w^n\|_{L^2(\Omega)} \le C(\tau^{\frac{1}{2}}+h^{\frac{1}{2}}+\alpha^{\frac{1}{4}}+\min(h^{\frac{1}{2}}+h^{-\frac{1}{2}}\eta^{\frac{1}{2}},1))\alpha^{-\frac{1}{4}}\eta^{\frac{1}{2}}.$$

Then by [5, Theorem 2.4], we have $w \in C((0,T];C(\overline{\Omega}))$ and $w > 0$ in $\Omega \times (0,T)$. The assertion follows from the continuity of w and the compactness of Ω'. $\qquad\square$

We also have the following global $L^2(\Omega)$ estimate by the positive condition (2.7).

Corollary 4.2. *Let conditions in Theorem 4.2 be fulfilled and condition (2.7) hold. Then there holds*

$$\|q^\dagger - q_h^*\|_{L^2(\Omega)} \le C((\tau^{\frac{1}{2}}+h^{\frac{1}{2}}+\alpha^{\frac{1}{4}}+\min(h^{\frac{1}{2}}+h^{-\frac{1}{2}}\eta^{\frac{1}{2}},1))\alpha^{-\frac{1}{4}}\eta^{\frac{1}{2}})^{\frac{1}{(1+2\beta)}}.$$

Remark 4.1. *For the choice $\alpha \sim \delta^2$, $\tau \sim \delta$ and $h \sim \delta^{\frac{1}{2}}$, we obtain the following interior estimate: $\|q^\dagger - q_h^*\|_{L^2(\Omega')} \le C\delta^{\frac{1}{4}}$. Likewise, under the positivity condition, there holds $\|q^\dagger - q_h^*\|_{L^2(\Omega)} \le C\delta^{\frac{1}{4(1+2\beta)}}$.*

5 Numerical experiments and discussions

Now we present numerical experiments for elliptic and parabolic inverse problems to illustrate the analysis. We solve the discrete optimization system by the conjugate gradient method [2], with the gradient computed by the standard adjoint technique. Despite the regularized functional being nonconvex, the method converges relatively robustly with the given initial guess, and the convergence is achieved within tens of iterations. The lower and upper bounds of the box constraint K are taken to be 0.4 and 2, respectively. In the elliptic case, the noise data z^δ is generated by

$$z^\delta(x) = u(q^\dagger)(x) + \epsilon \|u(q^\dagger)\|_{L^\infty(\Omega)}\xi(x), \quad x \in \Omega,$$

where ξ follows the standard normal Gaussian distribution and $\epsilon > 0$ is the relative noise level. The noise data $z^\delta(x,t)$ in the parabolic case is generated similarly. In the computation, we set the parameters $\alpha \sim \delta^2$, $h \sim \sqrt{\delta}$ and $\tau \sim \delta$ according to Remarks 3.1 and 4.1.

The first test is about the elliptic case.

Example 5.1. *We consider the following two cases.*

(a) $\Omega = (0,1)$, $q = 1 + x(1-x)\sin(2\pi x)$ *and* $f \equiv 1$.

(b) $\Omega = (0,1)^2$, $q = 1 + y(1-y)\sin(\pi x)$ *and* $f \equiv 1$.

To study the convergence behavior of the discrete approximation q_h^*, we employ two different metrics, i.e., $e_q = \|q^\dagger - q_h^*\|_{L^2(\Omega)}$ and $e_u = \|u(q^\dagger) - u_h(q_h^*)\|_{L^2(\Omega)}$. Note that the error analysis provides a convergence $O(\delta^{\frac{1}{4}})$ at best (in the interior) for e_q, and the state approximation e_u is predicted to be $O(\delta)$. The numerical results for Example 5.1 are presented in Table 1. The convergence of e_q and e_u can be observed clearly as the noise level $\delta \to 0$, more precisely, in the one-dimensional case, with the behavior $e_q \sim \delta^{0.54}$ and $e_u \sim \delta^{1.01}$. These observations remain valid for the two-dimensional test in (b). The empirical rate of e_q is much faster than the theoretical one, indicating potential suboptimality of the theoretical results in Corollaries 3.1 and 3.2. Figs. 1 and 2 show the numerical reconstruction for the examples with different noise levels. These plots clearly show the convergence of the reconstruction q_h^* as the noise level δ tends to zero. Note the accuracy near the boundary is a bit worse in all cases.

Table 1: Numerical results for Example 5.1.

(a) initialized with $\alpha = 2.00$e-7 and $h = 2.00$e-2

ϵ	2.00e-2	5.00e-3	1.25e-3	3.13e-4	7.81e-5	1.95e-5	rate
e_q	1.30e-1	1.20e-1	2.85e-2	2.57e-2	1.02e-2	5.98e-3	0.54
e_u	2.71e-4	6.81e-5	1.80e-5	3.43e-6	9.78e-7	1.72e-7	1.01

(b) initialized with $\alpha = 1.00$e-6 and $h = 1.00$e-1

ϵ	1.00e-2	2.50e-3	6.25e-4	4.00e-4	1.00e-4	rate
e_q	1.16e-1	9.14e-2	7.07e-2	5.81e-2	3.55e-2	0.33
e_u	8.36e-5	3.41e-5	1.30e-5	7.88e-6	1.60e-6	0.99

The second test is about the parabolic case.

Example 5.2. *We consider the following two cases.*

(a) $\Omega = (0,1)$, $T_0 = 0$, $T = 0.01$, $q = 1 + \sin(2\pi x)/2$, $u_0 = \sin(\pi x)$ *and* $f \equiv 1$.

(b) $\Omega = (0,1)^2$, $T_0 = 0$, $T = 0.01$, $q = 1 + \sin(\pi x)\sin(\pi y)/2$, $u_0 = \sin(\pi x)\sin(\pi y)$ *and* $f \equiv 1$.
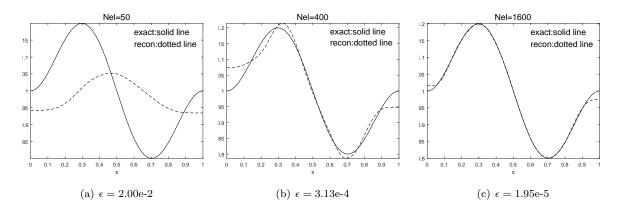
(a) $\epsilon = 2.00\text{e-}2$ (b) $\epsilon = 3.13\text{e-}4$ (c) $\epsilon = 1.95\text{e-}5$

Figure 1: Numerical reconstructions for Example 5.1(a).

(a) exact (b) $\epsilon = 1.00\text{e-}4$ (c) $\epsilon = 3.91\text{e-}5$
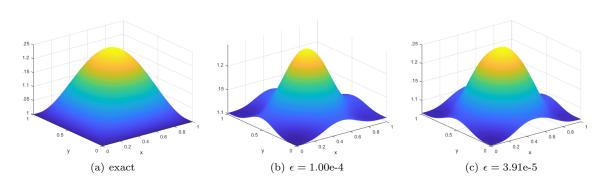
Figure 2: Numerical reconstructions for Example 5.1(b).

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Like in the elliptic case, we monitor the following two metrics, i.e., $e_q = \|q^\dagger - q_h^*\|_{L^2(\Omega)}$ and $e_u = (\tau \sum_{n=N_0}^{N} \|u^n(q^\dagger) - u_h^n(q_h^*)\|_{L^2(\Omega)}^2)^{\frac{1}{2}}$. The numerical results for Example 5.2 are presented in Table 2 and Figs. 3 and 4. Overall the convergence behaivour is very similar to the elliptic case: A steady convergence of both e_q and e_u is observed; The convergence rate of e_u is slightly faster than first order; but the convergence rate of e_q is again much higher than theoretical one.

Table 2: Numerical results for Example 5.2.

(a) initialized with $\alpha = 1.00\text{e-}8$ and $h = 2.00\text{e-}2$

ϵ	1.00e-2	2.50e-3	6.25e-4	1.56e-4	3.91e-5	rate
e_q	6.78e-1	1.73e-1	1.06e-1	6.69e-2	4.19e-2	0.50
e_u	1.61e-4	1.62e-5	3.68e-6	1.24e-6	2.31e-7	1.18

(b) initialized with $\alpha = 1.00\text{e-}6$ and $h = 1.00\text{e-}1$

ϵ	1.00e-2	2.50e-3	4.00e-4	1.00e-4	rate
e_q	9.35e-1	8.73e-1	4.11e-1	2.22e-1	0.30
e_u	4.31e-4	4.63e-5	2.15e-5	4.36e-6	1.06

(a) $\epsilon = 2.50\text{e-}3$ (b) $\epsilon = 1.56\text{e-}4$ (c) $\epsilon = 3.91\text{e-}5$

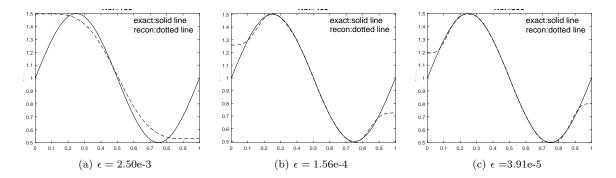Figure 3: Numerical reconstructions for Example 5.2(a).

A Proofs of technical lemmas

In this appendix, we collect the proof of Lemmas 4.1, 4.2 and 4.3. First we prove Lemma 4.1.

Proof. Let $u \equiv u(q^\dagger)$ and $u_h^n \equiv u_h^n(q^\dagger)$. Under the data regularity in Assumption 4.1, we have $u \in H^1(0, T; H_0^1(\Omega))$. Consequently,

$$|u^n - \bar{u}^n| = \left| \tau^{-1} \int_{t^{n-1}}^{t^n} \int_t^{t^n} \partial_s u(s) \mathrm{d}s \mathrm{d}t \right| \leq \tau^{\frac{1}{2}} \left(\int_{t^{n-1}}^{t^n} |\partial_t u|^2 \mathrm{d}t \right)^{\frac{1}{2}},$$

and thus the following *a priori* estimate holds

$$\tau \sum_{n=1}^{N} \|u^n - \bar{u}^n\|_{L^2(\Omega)}^2 \leq \tau^2 \sum_{n=1}^{N} \|\partial_t u\|_{L^2(t^{n-1}, t^n; L^2(\Omega))}^2 = \tau^2 \|\partial_t u\|_{L^2(0, T; L^2(\Omega))}^2 \leq C\tau^2.$$

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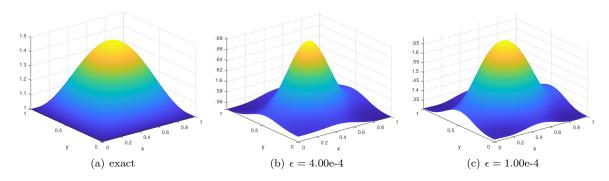(a) exact (b) $\epsilon = 4.00\text{e-}4$ (c) $\epsilon = 1.00\text{e-}4$

Figure 4: Numerical reconstructions for Example 5.2(b).

Thus it suffices to estimate $\tau \sum_{n=1}^{N} \|\bar{u}^n - u_h^n\|_{L^2(\Omega)}^2$. We introduce a discrete dual problem: find $\{w_h^{n-1}\}_{n=N}^{1} \subset V_{h0}$ with $w_h^N = 0$ such that

$$(-\partial_\tau w_h^n, \varphi_h) + (\nabla w_h^{n-1}, \nabla \varphi_h) + (q^\dagger w_h^{n-1}, \varphi_h) = (\bar{u}^n - u_h^n, \varphi_h), \quad \forall \varphi_h \in V_{h0}. \tag{A.1}$$

By letting $\varphi_h = -\tau \partial_\tau w_h^n$ in (A.1), and the Cauchy-Schwarz inequality, we have

$$\tau \|\partial_\tau w_h^n\|_{L^2(\Omega)}^2 + \tfrac{1}{2}(\|\nabla w_h^{n-1}\|_{L^2(\Omega)}^2 - \|\nabla w_h^n\|_{L^2(\Omega)}^2 + \|q^{\dagger \frac{1}{2}} w_h^{n-1}\|_{L^2(\Omega)}^2 - \|q^{\dagger \frac{1}{2}} w_h^n\|_{L^2(\Omega)}^2)$$
$$\leq C\tau \|\bar{u}^n - u_h^n\|_{L^2(\Omega)}^2 + \tfrac{\tau}{2}\|\partial_\tau w_h^n\|_{L^2(\Omega)}^2.$$

Since $w_h^N = 0$ and $\nabla w_h^N = 0$, summing the inequality over n from 1 to N gives

$$\tau \sum_{n=1}^{N} \|\partial_\tau w_h^n\|_{L^2(\Omega)}^2 \leq C\tau \sum_{n=1}^{N} \|\bar{u}^n - u_h^n\|_{L^2(\Omega)}^2. \tag{A.2}$$

Similarly, by letting $\varphi_h = 2w_h^{n-1}$ in (A.1), and applying the Cauchy-Schwarz inequality, we deduce

$$\tau^{-1}(\|w_h^{n-1}\|_{L^2(\Omega)}^2 - \|w_h^n\|_{L^2(\Omega)}^2) \leq \|\bar{u}^n - u_h^n\|_{L^2(\Omega)}^2 + \|w_h^{n-1}\|_{L^2(\Omega)}^2.$$

Then summing this inequality over n from N to any $k \in \{N, N-1 \dots, 1\}$ yields

$$\|w_h^{k-1}\|_{L^2(\Omega)}^2 \leq \tau \sum_{n=N}^{k} \|w_h^{n-1}\|_{L^2(\Omega)}^2 + \tau \sum_{n=N}^{k} \|\bar{u}^n - u_h^n\|_{L^2(\Omega)}^2.$$

Then for small τ, the discrete Gronwall's inequality leads to

$$\|w_h^{k-1}\|_{L^2(\Omega)}^2 \leq C\tau \sum_{n=1}^{N} \|\bar{u}^n - u_h^n\|_{L^2(\Omega)}^2, \quad \forall k \in \{N, N-1, \dots, 1\}. \tag{A.3}$$

The estimates (A.2) and (A.3) together imply

$$\max_{1 \leq n \leq N} \|w_h^{n-1}\|_{L^2(\Omega)}^2 + \tau \sum_{n=1}^{N} \|\partial_\tau w_h^n\|_{L^2(\Omega)}^2 \leq C\tau \sum_{n=1}^{N} \|\bar{u}^n - u_h^n\|_{L^2(\Omega)}^2. \tag{A.4}$$

Next, using the Ritz projection R_h, we have the following splitting

$$\tau \sum_{n=1}^{N} \|\bar{u}^n - u_h^n\|_{L^2(\Omega)}^2 = \tau \sum_{n=1}^{N} (\bar{u}^n - u_h^n, \bar{u}^n - R_h \bar{u}^n) + \tau \sum_{n=1}^{N} (\bar{u}^n - u_h^n, R_h \bar{u}^n - u_h^n).$$

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Substituting $\varphi_h = \tau(R_h \bar{u}^n - u_h^n)$ into (A.1), using the definition of Ritz projection R_h and summing over n from 1 to N lead to

$$\tau \sum_{n=1}^{N} \|\bar{u}^n - u_h^n\|_{L^2(\Omega)}^2 = \tau \sum_{n=1}^{N} (\bar{u}^n - u_h^n, \bar{u}^n - R_h \bar{u}^n) + \tau \sum_{n=1}^{N} (-\partial_\tau w_h^n, R_h \bar{u}^n - u^n)$$

$$+ \tau \sum_{n=1}^{N} (-\partial_\tau w_h^n, u^n - u_h^n) + \tau \sum_{n=1}^{N} (\nabla w_h^{n-1}, \nabla(\bar{u}^n - u_h^n))$$

$$+ \tau \sum_{n=1}^{N} (q^\dagger w_h^{n-1}, \bar{u}^n - u_h^n) =: \sum_{i=1}^{5} \mathrm{I}_i.$$

Thus, it suffices to bound the summands I_i, $i = 1, \ldots, 5$. Meanwhile, integrating the identity (4.2) on the interval (t^{n-1}, t^n) for $n = 1, \ldots, N$ gives

$$(\partial_\tau u^n, \varphi) + (\nabla \bar{u}^n, \nabla \varphi) + (q^\dagger \bar{u}^n, \varphi) = (\bar{f}^n, \varphi), \quad \forall \varphi \in H_0^1(\Omega). \tag{A.5}$$

Letting $\varphi = w_h^{n-1}$ in (A.5) and $\varphi_h = w_h^{n-1}$ in (4.4) (associated with $q^\dagger$ instead of q_h, i.e., the finite element problem for u_h^n), we deduce

$$\tau \sum_{n=1}^{N} (\partial_\tau(u^n - u_h^n), w_h^{n-1}) + \tau \sum_{n=1}^{N} (\nabla(\bar{u}^n - u_h^n), \nabla w_h^{n-1}) \tag{A.6}$$

$$+ \tau \sum_{n=1}^{N} (q^\dagger(\bar{u}^n - u_h^n), w_h^{n-1}) = \tau \sum_{n=1}^{N} (\bar{f}^n - f^n, w_h^{n-1}).$$

Meanwhile, since $w_h^N = 0$ and $u_h^0 = P_h u_0$, by the summation by parts formula, we deduce

$$\mathrm{I}_3 = (u_0 - P_h u_0, w_h^0) + \tau \sum_{n=1}^{N} (\partial_\tau(u^n - u_h^n), w_h^{n-1}). \tag{A.7}$$

It follows from (A.6) and (A.7) that

$$\sum_{i=3}^{5} \mathrm{I}_i = (u_0 - P_h u_0, w_h^0) + \tau \sum_{n=1}^{N} (\bar{f}^n - f^n, w_h^{n-1}).$$

It remains to bound the terms separately. First, by (A.4),

$$|(u_0 - P_h u_0, w_h^0)| \le \|u_0 - P_h u_0\|_{L^2(\Omega)} \|w_h^0\|_{L^2(\Omega)} \le C h^2 \|u_0\|_{H^2(\Omega)} \left(\tau \sum_{n=1}^{N} \|\bar{u}^n - u_h^n\|_{L^2(\Omega)}^2 \right)^{\frac{1}{2}}.$$

Meanwhile, since $\tau \sum_{n=1}^{N} \|\bar{f}^n - f^n\|_{L^2(\Omega)}^2 \le \tau^2 \|\partial_t f\|_{L^2(0,T;L^2(\Omega))}$, we obtain from (A.4) that

$$\left| \tau \sum_{n=1}^{N} (\bar{f}^n - f^n, w_h^{n-1}) \right| \le \left(\tau \sum_{n=1}^{N} \|\bar{f}^n - f^n\|_{L^2(\Omega)}^2 \right)^{\frac{1}{2}} \left(\tau \sum_{n=1}^{N} \|\bar{u}^n - u_h^n\|_{L^2(\Omega)}^2 \right)^{\frac{1}{2}}$$

$$\le C\tau \left(\tau \sum_{n=1}^{N} \|\bar{u}^n - u_h^n\|_{L^2(\Omega)}^2 \right)^{\frac{1}{2}}.$$

Further, it follows directly from (4.5), the a priori regularity $u \in L^2(0,T;H^2(\Omega)) \cap H^1(0,T;L^2(\Omega))$ and the estimate (A.4) that

$$|\mathrm{I}_1| \le C h^2 \|u\|_{L^2(0,T;H^2(\Omega))} \left(\tau \sum_{n=1}^{N} \|\bar{u}^n - u_h^n\|_{L^2(\Omega)}^2 \right)^{\frac{1}{2}},$$

$$|I_2| = \left| \tau \sum_{n=1}^{N} (-\partial_\tau w_h^n, R_h \bar{u}^n - \bar{u}^n) + \tau \sum_{n=1}^{N} (-\partial_\tau w_h^n, \bar{u}^n - u^n) \right|$$

$$\leq Ch^2 \tau \sum_{n=1}^{N} \|\partial_\tau w_h^n\|_{L^2(\Omega)} \|\bar{u}^n\|_{H^2(\Omega)} + \tau \sum_{n=1}^{N} \|\partial_\tau w_h^n\|_{L^2(\Omega)} \|\bar{u}^n - u^n\|_{L^2(\Omega)}$$

$$\leq C(\tau + h^2)(\tau \sum_{n=1}^{N} \|\bar{u}^n - u_h^n\|_{L^2(\Omega)}^2)^{\frac{1}{2}}.$$

These estimates together imply

$$\tau \sum_{n=1}^{N} \|\bar{u}^n - u_h^n\|_{L^2(\Omega)}^2 \leq C(\tau^2 + h^4).$$

The desired estimate (4.7) follows directly by the triangle inequality. $\qquad\square$

Next we give the proof of Lemma 4.2.

Proof. By Lemma 4.1, it suffices to prove

$$\tau \sum_{n=1}^{N} \|u_h^n(q^\dagger) - u_h^n(\Pi_h q^\dagger)\|_{L^2(\Omega)}^2 \leq C(\tau^2 + h^4).$$

Let $\rho_h^n := u_h^n(q^\dagger) - u_h^n(\Pi_h q^\dagger)$ for $n = 1, 2, \ldots, N$. Then ρ_h^n satisfies for any $\varphi_h \in V_{h0}$

$$(\partial_\tau \rho_h^n, \varphi_h) + (\nabla \rho_h^n, \nabla \varphi_h) + (\Pi_h q^\dagger \rho_h^n, \varphi_h) = ((\Pi_h q^\dagger - q^\dagger) u_h^n(q^\dagger), \varphi_h) \qquad (A.8)$$
$$= ((\Pi_h q^\dagger - q^\dagger)(u_h^n(q^\dagger) - u^n(q^\dagger)), \varphi_h) + ((\Pi_h q^\dagger - q^\dagger) u^n(q^\dagger), \varphi_h).$$

Letting $\varphi_h = 2\rho_h^n$ in (A.8), by (3.5), Assumption 4.1, the regularity $\|u(q^\dagger)\|_{L^\infty(0,T;L^\infty(\Omega))} \leq C$ and Young's inequality, there holds

$$\tau^{-1}(\|\rho_h^n\|_{L^2(\Omega)}^2 - \|\rho_h^{n-1}\|_{L^2(\Omega)}^2) \leq C\|q^\dagger\|_{L^\infty(\Omega)} \|u^n(q^\dagger) - u_h^n(q^\dagger)\|_{L^2(\Omega)} \|\rho_h^n\|_{L^2(\Omega)}$$
$$+ Ch^2 \|q^\dagger\|_{H^2(\Omega)} \|u(q^\dagger)\|_{L^\infty(0,T;L^\infty(\Omega))}) \|\rho_h^n\|_{L^2(\Omega)}$$
$$\leq C\|u^n(q^\dagger) - u_h^n(q^\dagger)\|_{L^2(\Omega)}^2 + Ch^4 + \|\rho_h^n\|_{L^2(\Omega)}^2.$$

Summing the inequality over n from 1 to any $k \in \{1, 2 \ldots, N\}$ and noting $\rho_h^0 = 0$ and Lemma 4.1, we have

$$\|\rho_h^k\|_{L^2(\Omega)}^2 \leq C\tau \sum_{n=1}^{k} \|\rho_h^n\|_{L^2(\Omega)}^2 + C(\tau^2 + h^4),$$

and thus for small τ, the discrete Gronwall's inequality leads directly to

$$\|\rho_h^k\|_{L^2(\Omega)}^2 \leq C(\tau^2 + h^4), \quad k = 1, 2, \ldots, N.$$

This completes the proof of the lemma. $\qquad\square$

Last we give the proof of Lemma 4.3.

Proof. First we prove an elementary estimate:

$$\tau \sum_{n=N_0}^{N} \|u^n(q^\dagger) - z_n^\delta\|_{L^2(\Omega)}^2 \leq C(\tau^2 + \delta^2). \qquad (A.9)$$

Indeed, by the proof of Lemma 4.1, we have $\tau \sum_{n=N_0}^{N} \|u^n(q^\dagger) - \bar{u}^n(q^\dagger)\|_{L^2(\Omega)}^2 \leq C\tau^2$. Meanwhile, direct computation leads to

$$|\bar{u}^n(q^\dagger) - z_n^\delta| = \left| \tau^{-1} \int_{t^{n-1}}^{t^n} u(q^\dagger)(t) - z^\delta(t) dt \right| \leq \tau^{-\frac{1}{2}} \left(\int_{t^{n-1}}^{t^n} |u(q^\dagger)(t) - z^\delta(t)|^2 dt \right)^{\frac{1}{2}},$$

and hence
$$\tau \sum_{n=N_0}^{N} \|\bar{u}^n(q^\dagger) - z_n^\delta\|_{L^2(\Omega)}^2 \leq \|u(q^\dagger) - z^\delta\|_{L^2(T_0,T;L^2(\Omega))}^2 = \delta^2.$$

The claim (A.9) follows from triangle inequality. Since q_h^* is the minimizer of the system (4.3)-(4.4) and $\Pi_h q^\dagger \in K_h$, we have
$$J_{\alpha,h,\tau}(q_h^*) \leq J_{\alpha,h,\tau}(\Pi_h q^\dagger).$$

Then by the inequality (3.5), $\|\nabla \Pi_h q^\dagger\|_{L^2(\Omega)} \leq C\|q^\dagger\|_{H^1(\Omega)}$, and thus there holds

$$\tau \sum_{n=N_0}^{N} \|u_h^n(q_h^*) - z_n^\delta\|_{L^2(\Omega)}^2 + \alpha\|\nabla q_h^*\|_{L^2(\Omega)}^2 \leq \tau \sum_{n=N_0}^{N} \|u_h^n(\Pi_h q^\dagger) - z_n^\delta\|_{L^2(\Omega)}^2 + \alpha\|\nabla \Pi_h q^\dagger\|_{L^2(\Omega)}^2$$

$$\leq C\tau \sum_{n=N_0}^{N} \|u_h^n(\Pi_h q^\dagger) - u^n(q^\dagger)\|_{L^2(\Omega)}^2 + C\tau \sum_{n=N_0}^{N} \|u^n(q^\dagger) - z_n^\delta\|_{L^2(\Omega)}^2 + C\alpha \leq C(\tau^2 + h^4 + \delta^2 + \alpha).$$

where the last step is due to Lemma 4.2 and (A.9). Then by the triangle inequality and (A.9), we obtain

$$\tau \sum_{n=N_0}^{N} \|u^n(q^\dagger) - u_h^n(q_h^*)\|_{L^2(\Omega)}^2 + \alpha\|\nabla q_h^*\|_{L^2(\Omega)}^2$$
$$\leq C\Big(\tau \sum_{n=N_0}^{N} \|u^n(q^\dagger) - z_n^\delta\|_{L^2(\Omega)}^2 + \tau \sum_{n=N_0}^{N} \|z_n^\delta - u_h^n(q_h^*)\|_{L^2(\Omega)}^2 + \alpha\|\nabla q_h^*\|_{L^2(\Omega)}^2\Big)$$
$$\leq C(\tau^2 + h^4 + \delta^2 + \alpha).$$

This completes the proof of the lemma. $\qquad\square$

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