Online Appendices of "Peak-hour Pricing under Negative Externality: Impact of Customer Flexibility and Competitive Asymmetry"

Appendix A: Proofs

Proof of Lemma 1. Recall from (1) that the customer's utilities from shopping in peak and normal hours are, respectively,

$$u_{A,peak} = \alpha V_A - \gamma[\lambda r_{A,peak}] - \delta_A p_A, \quad u_{A,norm} = V_A - \gamma[\lambda r_{A,norm}] - p_A.$$

Here, $r_{A,peak} \in (0,1)$ represents the "initial belief" about the average percentage of customers who shops during peak hour. Let q_j be customer j's percentage of his shopping trips that takes place during peak hours. Recall that we had normalized the market size of infinitesimal customers to 1. For clarity, instead of 1, we let N be the population size so that:

$$r_{A,peak} \equiv \lim_{N\to\infty} \frac{1}{N}\sum_{j=1}^{N} q_j.$$

Now, consider an infinitesimal customer i who wants to maximize his expected utility by randomizing between peak and normal shopping hours via q_i, where his expected utility is:

$$\begin{aligned} u_i(q_i) &= q_i u_{A,peak} + (1-q_i)u_{A,norm} \\ &= q_i\Big(\alpha V_A - \gamma\lambda \lim_{N\to\infty}\frac{1}{N}\Big(\sum_{j\neq i} q_j + q_i\Big) - \delta_A p_A\Big) + (1-q_i)\Big(V_A - \gamma\lambda \lim_{N\to\infty}\frac{1}{N}\Big(\sum_{j\neq i}(1-q_j) + (1-q_i)\Big) - p_A\Big) \\ &= q_i\Big(\alpha V_A - \gamma[\lambda r_{A,peak}] - \delta_A p_A\Big) + (1-q_i)\Big(V_A - \gamma[\lambda r_{A,norm}] - p_A\Big). \end{aligned}$$

Because customer i is infinitesimal, customer i's shopping choice q_i will not affect the belief or actual realization of the *average* $r_{A,peak}$.

In this case, if $\alpha V_A - \gamma[\lambda r_{A,peak}] - \delta_A p_A < V_A - \gamma[\lambda(1-r_{A,peak})] - p_A$, then $q_i = 0$ would maximize his expected utility. In fact, everybody would choose $q_i = 0$, and so cannot be a Nash equilibrium, because players beliefs about others' strategies are not correct in equilibrium.

On the other hand, if $\alpha V_A - \gamma[\lambda r_{A,peak}] - \delta_A p_A > V_A - \gamma[\lambda(1-r_{A,peak})] - p_A$, then $q_i = 1$ would maximize his expected utility. Again, for all customer i, $q_i = 1$, and this cannot be a Nash equilibrium.

Hence, the only possible Nash equilibrium occurs in a mixed strategy where consumers are indifferent between shopping in peak hour and normal hour, i.e., $\alpha V_A - \gamma[\lambda r_{A,peak}] - \delta_A p_A = V_A - \gamma[\lambda(1-r_{A,peak})] - p_A$. This implies that $r_{A,peak}$ solves:

$$\begin{aligned} u_{A,peak} = u_{A,norm} &\Leftrightarrow \alpha V_A - \gamma[\lambda r_{A,peak}] - \delta_A p_A = V_A - \gamma[\lambda(1-r_{A,peak})] - p_A \\ &\Leftrightarrow (\alpha-1)V_A - (\delta_A - 1)p_A = \gamma\lambda(r_{A,peak} - (1-r_{A,peak})). \end{aligned}$$

Hence, we get $r^*_{A,peak} = \frac{1}{2} + \frac{(\alpha-1)V_A - (\delta_A-1)p_A}{2\gamma\lambda}$. By Assumption 2,

$$(\alpha-1)V_A < \lambda\gamma \Rightarrow (\alpha-1)V_A - (\delta_A-1)p_A < \lambda\gamma \Leftrightarrow \frac{(\alpha-1)V_A - (\delta_A-1)p_A}{2\gamma\lambda} < \frac{1}{2} \Leftrightarrow r^*_{A,peak} < 1.$$

Therefore, $r^*_{A,peak}$ can be interpreted as a symmetric mixed strategy under which every customer j will follow so that $q^*_j = r^*_{A,peak}$.

To show that this is a Nash mixed equilibrium strategy, consider customer j deviates from q^*_j by choosing $q_j = q^*_j + \epsilon$. Because customer j is infinitesimal, his expected utility is $q_j \cdot u^*_{A,peak} + (1-q_j)\cdot u^*_{A,norm} = u^*$.

Hence, any deviation from the this strategy does not lead to an improvement in his expected utility, so he has no incentive to deviate. Therefore, we can conclude that q_j^* is an equilibrium. □

Proof of Proposition 1. The profit function $\pi(\delta)$ is quadratic. Taking the first order conditions and rearranging the terms,

$$\frac{\partial \pi_A(\delta_A)}{\partial \delta_A} = 0 \Leftrightarrow \frac{p_A\lambda}{2} + \frac{p_A(\alpha-1)V_A}{2\gamma} + \frac{p_A^2}{2\gamma} - \frac{\partial}{\partial \delta_A}\left[\frac{\delta_A p_A(\delta_A - 1)p_A}{2\gamma}\right] = 0 \Leftrightarrow \delta_A^* = 1 + \frac{\lambda\gamma + (\alpha-1)V_A}{2p_A}.$$

Clearly δ_A^* is increasing in γ. The same analysis can be applied for firm B. □

Proof of Lemma 2. Recall that the customer's utilities from shopping in peak and normal hours in stores A and B are, respectively,

$$u_{A,peak} = \alpha V_A - \gamma[\beta r_{A,peak}] - \delta_A p_A, \qquad u_{B,peak} = \alpha V_B - \gamma[\beta r_{B,peak}] - \delta_B p_B,$$
$$u_{A,norm} = V_A - \gamma(1-\beta) r_{A,norm} - p_A, \qquad u_{B,norm} = V_B - \gamma(1-\beta) r_{B,norm} - p_B.$$

Here, $r_{A,peak}$ and $r_{A,norm} \in (0,1)$ represent the "initial belief" about the average percentage of customers who shop during peak and normal hour respectively. As customers only have store flexibility, we also have $r_{B,peak} = 1 - r_{A,peak}$ and $r_{B,norm} = 1 - r_{A,norm}$. We will derive the equilibrium for peak-hour customers, $r_{A,peak}^*$. The case with normal-hour customers, $r_{A,norm}^*$ is omitted to avoid repetition.

Recall that we had normalized the population size to 1. Let q_j be customer j's percentage of his shopping trips that take place in store A. For clarity, instead of 1, we let N be the population size so that:

$$r_{A,peak} \equiv \lim_{N\to\infty} \frac{1}{N}\sum_{j=1}^{N} q_j, \qquad r_{B,peak} \equiv \lim_{N\to\infty} \frac{1}{N}\sum_{j=1}^{N}(1-q_j) = 1 - r_{A,peak}$$

An infinitesimal customer i wants to maximize his expected utility by randomizing between store A and B in peak hours via q_i, where his expected utility is:

$$\begin{aligned}
u_i(q_i) &= q_i u_{A,peak} + (1-q_i)u_{B,peak} \\
&= q_i\Big(\alpha V_A - \gamma\beta \lim_{N\to\infty}\frac{1}{N}\Big(\sum_{j\neq i} q_j + q_i\Big) - \delta_A p_A\Big) + (1-q_i)\Big(\alpha V_B - \gamma \lim_{N\to\infty}\beta\frac{1}{N}\Big(\sum_{j\neq i}(1-q_j) + (1-q_i)\Big) - \delta_B p_B\Big) \\
&= q_i\Big(\alpha V_A - \gamma[\beta r_{A,peak}] - \delta_A p_A\Big) + (1-q_i)\Big(\alpha V_B - \gamma[\beta r_{B,peak}] - \delta_B p_B\Big) \\
&= q_i\Big(\alpha V_A - \gamma[\beta r_{A,peak}] - \delta_A p_A\Big) + (1-q_i)\Big(\alpha V_B - \gamma[\beta(1 - r_{A,peak})] - \delta_B p_B\Big),
\end{aligned}$$

where the third equality is because customer i is infinitesimal, and his shopping choice q_i will not impact the belief or actual realization of the *average* $r_{A,peak}$.

If $\alpha V_A - \gamma[\beta r_{A,peak}] - \delta_A p_A < \alpha V_B - \gamma[\beta(1 - r_{A,peak})] - \delta_B p_B$, then $q_i = 0$ would maximize his expected utility. In fact, everybody would choose $q_i = 0$, and so cannot be a Nash equilibrium, because players beliefs about others' strategies are not correct in equilibrium.

On the other hand, if $\alpha V_A - \gamma[\beta r_{A,peak}] - \delta_A p_A > \alpha V_B - \gamma[\beta(1 - r_{A,peak})] - \delta_B p_B$, then $q_i = 1$ would maximize his expected utility. Again, for all customer i, $q_i = 1$, and this cannot be a Nash equilibrium.

Hence, the only possible Nash equilibrium occurs in a mixed strategy where customers are indifferent between shopping in peak hour and normal hour, i.e., $\alpha V_A - \gamma[\beta r_{A,peak}] - \delta_A p_A = \alpha V_B - \gamma[\beta(1 - r_{A,peak})] - \delta_B p_B$. This implies that $r_{A,peak}$ solves:

$$\begin{aligned}
u_{A,peak} = u_{B,peak} &\Leftrightarrow \alpha V_A - \gamma[\beta r_{A,peak}] - \delta_A p_A = \alpha V_B - \gamma[\beta(1 - r_{A,peak})] - \delta_B p_B \\
&\Leftrightarrow r_{A,peak}^*(\delta_A, \delta_B) = \frac{1}{2} + \frac{\alpha(V_A - V_B) - (\delta_A p_A - \delta_B p_B)}{2\gamma\beta}.
\end{aligned}$$

By Assumption 3, we have

$$\alpha(V_A - V_B) \leq \gamma\beta \Rightarrow \alpha(V_A - V_B) - (\delta_A p_A - \delta_B p_B) < \gamma\beta \Leftrightarrow \frac{\alpha(V_A - V_B) - (\delta_A p_A - \delta_B p_B)}{2\gamma\beta} < \frac{1}{2} \Leftrightarrow r^*_{A,peak} < 1.$$

Hence, $r^*_{A,peak}$ can be interpreted as a symmetric mixed strategy under which each customer j chooses q^*_j so that $q^*_j = r^*_{A,peak}$.

To show that this is a Nash mixed equilibrium strategy, consider customer j deviates from q^*_j by choosing $q_j = q^*_j + \epsilon$. Because customer j is infinitesimal, his expected utility is $q_j \cdot u^*_{A,peak} + (1 - q_j) \cdot u^*_{B,peak} = u^*$. Hence, any deviation from the this strategy does not lead to an improvement in his expected utility, so he has no incentive to deviate. Therefore, we can conclude that q^*_j is an equilibrium. ☐

Each firm's best-response (in terms of its peak-hour multiplier) given the competing firm's peak-hour multiplier is stated in the following lemma:

Lemma A.1 **(Best-Response Peak-Hour Multipliers – Store Flexibility Only)**. *Suppose that Assumptions 1 and 3 hold. Then firm A's and firm B's best-response peak-hour multipliers satisfy the following:*

$$\delta^{br}_A(\delta_B) = \max\left\{1, \frac{\beta\gamma + \alpha(V_A - V_B)}{2p_A} + \frac{p_B}{2p_A}\delta_B\right\},$$
$$\delta^{br}_B(\delta_A) = \max\left\{1, \frac{\beta\gamma - \alpha(V_A - V_B)}{2p_B} + \frac{p_A}{2p_B}\delta_A\right\}.$$

Proof of Lemma A.1. Under Condition 3, the profit function $\pi_i(\delta_i|\delta_{-i})$ for $i \in \{A, B\}$ is quadratic. We consider the best response of firm A given firm B's peak-period rate. We have

$$\pi_A(\delta_A|\delta_B) = \delta_A p_A \beta r^*_{A,peak} = \delta_A p_A \left(\frac{\beta}{2} + \frac{\alpha(V_A - V_B) - (\delta_A p_A - \delta_B p_B)}{2\gamma}\right).$$

Taking the first order condition and rearranging the terms,

$$\frac{\partial \pi_A(\delta_A|\delta_B)}{\partial \delta_A} = 0 \Leftrightarrow \frac{p_A\beta}{2} + \frac{\alpha(V_A - V_B)p_A}{2\gamma} - \frac{(\delta_A p_A - \delta_B p_B)p_A + \delta_A p_A^2}{2\gamma} = 0$$
$$\Leftrightarrow \delta^*_A(\delta_B) = \frac{\gamma\beta + \alpha(V_A - V_B) + (\delta_B p_B)}{2p_A}.$$

Since the peak period rate cannot be less than 1, we have the maximum operator. The same analysis can be repeated to find the expressions for $\delta^{br}_B(\delta_A)$. ☐

Proof of Proposition 2. Both best response functions of Lemma A.1 have the form,

$$\delta^{br}_A(\delta_B) = \max\{1, a + b\delta_B\}, \quad \delta^{br}_B(\delta_A) = \max\{1, u + v\delta_A\},$$

where $a \equiv \frac{\gamma\beta + \alpha(V_A - V_B)}{2p_A}$, $b \equiv \frac{p_B}{2p_A}$, $u \equiv \frac{\gamma\beta - \alpha(V_A - V_B)}{2p_B}$, and $v \equiv \frac{p_A}{2p_B}$.

The equilibrium fixed point occurs at the intersection between $\delta^{br}_A(\delta_B)$ and inverse of $\delta^{br}_B(\delta_A)$. Because $\delta^{br}_A(\delta_B)$ has smaller slope than the inverted $\delta^{br}_B(\delta_A)$, as illustrated in Figure A-1, there are four possible fixed points:

(i) If $a + b < 1$ and $u + v < 1$ (upper left panel of Figure A-1), then the fixed point $(\delta^*_A, \delta^*_B) = (1, 1)$.

(ii) If $a + b < 1$ and $1 < u + v < y$, where y is the inflection point of $\delta^{br}_A(\delta_B)$ (lower left panel of Figure A-1), then the fixed point $(\delta^*_A, \delta^*_B) = (1, u + v)$.

(iii) If $1 < a + b < x$ and $u + v < 1$, where x is the inflection point of $\delta^{br}_B(\delta_A)$ (upper right panel of Figure A-1), then the fixed point $(\delta^*_A, \delta^*_B) = (a + b, 1)$.

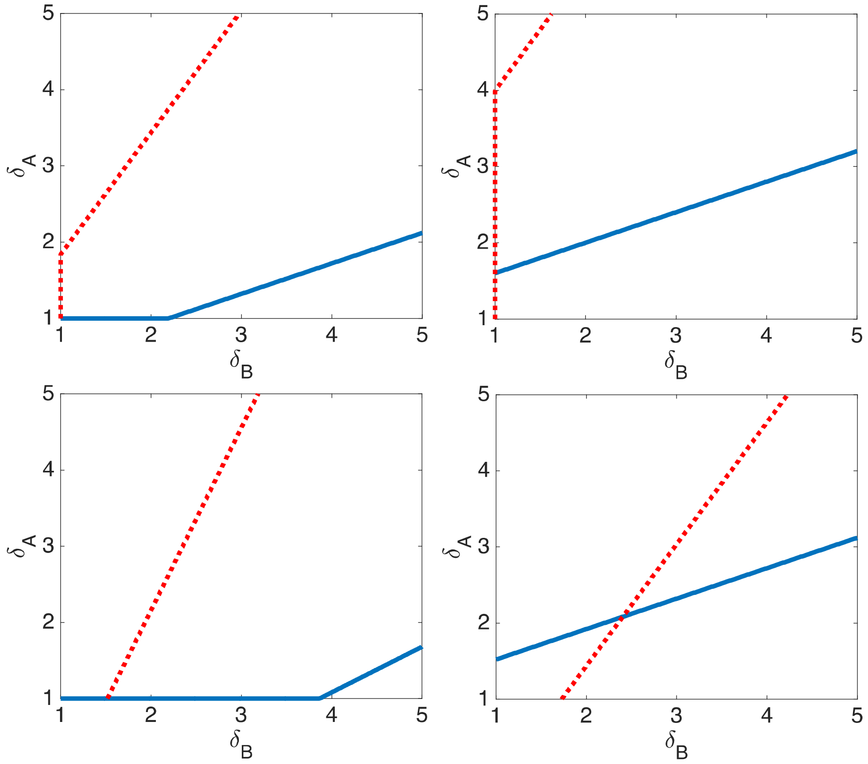


Figure A-1 **Four possible fixed points. The solid curve represents $\delta_A^{br}(\delta_B)$ and the dotted curve represents the inverse of $\delta_B^{br}(\delta_A)$.**

(iv) Otherwise, (lower right panel of Figure A-1), the fixed point (δ_A^*, δ_B^*) occurs at a non-corner solution, i.e., at the intersection of the solid curve with slope $\frac{p_B}{2p_A}$ and the dotted curve with slope $\frac{2p_B}{p_A}$. Solving for δ_A^*,

$$\delta_A^* = \delta_A^{br}(\delta_B^*) = \frac{\beta\gamma + \alpha(V_A - V_B)}{2p_A} + \frac{p_B}{2p_A}\frac{\beta\gamma - \alpha(V_A - V_B) + \delta_A p_A}{2p_B}$$

$$\Leftrightarrow \ \frac{3}{4}\delta_A = \frac{2\beta\gamma + 2\alpha(V_A - V_B) + \beta\gamma - \alpha(V_A - V_B)}{4p_A} \ \Leftrightarrow \ \delta_A = \frac{\beta\gamma + \frac{\alpha(V_A - V_B)}{3}}{p_A}.$$

Plugging in the value for δ_A^* into the best response function for firm B,

$$\delta_B^* = \delta_B^{br}(\delta_A^*) = \frac{\beta\gamma - \alpha(V_A - V_B)}{2p_B} + \frac{p_A}{2p_B}\left(\frac{\beta\gamma + \frac{\alpha(V_A - V_B)}{3}}{p_A}\right) = \frac{2\beta\gamma - \frac{2}{3}\alpha(V_A - V_B)}{2p_B} = \frac{3\beta\gamma - \alpha(V_A - V_B)}{3p_B}.$$

The equilibrium peak period multipliers are determined by four curves $a + b = 1$, $a + b = x$, $u + v = 1$, and $u + v = y$, which in terms of $\nu \equiv \frac{V_A}{V_B}$ and $\rho \equiv \frac{p_A}{p_B}$, are

$$\begin{aligned}
a + b = 1 &\Leftrightarrow \beta\gamma + \alpha(\nu - 1)V_B + p_B = 2\rho p_B &&\Leftrightarrow \nu = \frac{-\beta\gamma + \alpha V_B - p_B}{\alpha V_B} + \frac{2p_B}{\alpha V_B}\rho, \\
a + b = x &\Leftrightarrow \beta\gamma = p_B + \frac{\alpha(\nu - 1)V_B}{3} &&\Leftrightarrow \nu = \frac{3\beta\gamma + \alpha V_B - 3p_B}{\alpha V_B}, \\
u + v = 1 &\Leftrightarrow \beta\gamma - \alpha(\nu - 1)V_B + \rho p_B = 2p_B &&\Leftrightarrow \nu = \frac{\beta\gamma + \alpha V_B - 2p_B}{\alpha V_B} + \frac{p_B}{\alpha V_B}\rho, \\
u + v = y &\Leftrightarrow \beta\gamma = \rho p_B - \frac{\alpha(\nu - 1)V_B}{3} &&\Leftrightarrow \nu = \frac{-3\beta\gamma + \alpha V_B}{\alpha V_B} + \frac{3p_B}{\alpha V_B}\rho.
\end{aligned}$$

These four lines have four different slopes, and one can verify that they all intersect at

$$(\tau,\eta) = \left(\frac{2\beta\gamma}{p_B} - 1,\ 1 + \frac{3}{\alpha V_B}(\beta\gamma - p_B)\right).$$

Moreover, we observe that in case (ii), the condition $a+b<1$ is implied by $1<u+v<y$; and in case (iii) $1<a+b<x$ implies $u+v<1$. Thus, we have our result after removing the redundant conditions. □

Proof of Corollary 1. (i) First, from Proof of Propositions 2, the four lines with four different slopes uniquely intersect at (τ,η). If $\beta\gamma \le p_B$, then both $\tau \le 1$ and $\eta \le 1$ and it is not possible for both $\delta_A^* > 1$ and $\delta_B^* > 1$. Next, we observe from the condition for $(\delta_A^* > 1, \delta_B^* > 1)$ — i.e., $\nu \le \frac{3\beta\gamma + \alpha V_B - 3p_B}{\alpha V_B}$ and $\nu \ge \frac{-3\beta\gamma + \alpha V_B}{\alpha V_B} + \frac{3p_B}{\alpha V_B}\rho$, that an increase in γ enlarges this region. Similarly we observe from the condition for $(\delta_A^* = 1, \delta_B^* = 1)$ — i.e., $\nu \le \frac{-\beta\gamma + \alpha V_B - p_B}{\alpha V_B} + \frac{2p_B}{\alpha V_B}\rho$ and $\nu \ge \frac{\beta\gamma + \alpha V_B - 2p_B}{\alpha V_B} + \frac{p_B}{\alpha V_B}\rho$ that an increase in γ shrinks this region.

(ii) It is clear from the expressions for $\delta_A^* > 1$ and $\delta_B^* > 1$ from Proposition 2. □

Proof of Lemma 3. Recall that the customer's utilities from shopping in peak and normal hours in stores A and B are, respectively,

$$u_{A,peak} = \alpha V_A - \gamma q_{A,peak} - \delta_A p_A, \qquad u_{B,peak} = \alpha V_B - \gamma q_{B,peak} - \delta_B p_B,$$
$$u_{A,norm} = V_A - \gamma q_{A,norm} - p_A, \qquad u_{B,norm} = V_B - \gamma q_{B,norm} - p_B.$$

Here, $q_{A,peak}$, $q_{A,norm}$, $q_{B,peak}$ and $q_{B,norm} \in (0,1)$ represent the initial belief about the average percentage of customers who shop during peak and normal hour in stores A and B,respectively.

Let q_j, q_k and q_l be customer i's percentage of his shopping trips that take place in peak hours in store A, normal hours in store A and peak hours in store B, respectively. Recall that we had normalized the population size to 1. For clarity, instead of 1, we let N be the population size so that:

$$q_{A,peak} \equiv \lim_{N\to\infty} \frac{1}{N}\sum_{j=1}^{N} q_j, \qquad q_{A,norm} \equiv \lim_{N\to\infty} \frac{1}{N}\sum_{k=1}^{N} q_k, \qquad q_{B,peak} \equiv \lim_{N\to\infty} \frac{1}{N}\sum_{l=1}^{N} q_l$$
$$q_{B,norm} \equiv \lim_{N\to\infty} \frac{1}{N}\sum_{j,k,l=1}^{N} (1 - q_j - q_k - q_l) = 1 - q_{A,peak} - q_{A,norm} - q_{B,peak},$$

where the final expression is because $q_{A,peak} + q_{A,norm} + q_{B,peak} + q_{B,norm} = 1$ by Assumption 1.

An infinitesimal customer i wants to maximize his expected utility by randomizing between peak and normal hours, and store A and B, where his expected utility is:

$$\begin{aligned}
u_i(q_i) &= q^i_{A,peak} u_{A,peak} + q^i_{A,norm} u_{A,norm} + q^i_{B,peak} u_{B,peak} + (1 - q^i_{A,peak} - q^i_{A,norm} - q^i_{B,peak}) u_{B,norm} \\
&= q^i_{A,peak}\left(\alpha V_A - \gamma \lim_{N\to\infty}\frac{1}{N}\left(\sum_{j\neq i} q_j + q^i_{A,peak}\right) - \delta_A p_A\right) + q^i_{A,norm}\left(V_A - \gamma \lim_{N\to\infty}\frac{1}{N}\left(\sum_{k\neq i} q_k + q^i_{A,norm}\right) - p_A\right) \\
&\quad + q^i_{B,peak}\left(\alpha V_B - \gamma \lim_{N\to\infty}\frac{1}{N}\left(\sum_{l\neq i} q_l + q^i_{B,peak}\right) - \delta_B p_B\right) + \left(1 - q^i_{A,peak} - q^i_{A,norm} - q^i_{B,peak}\right) \times \\
&\qquad \left(V_B - \gamma \lim_{N\to\infty}\frac{1}{N}\left(\sum_{j,k,l\neq i} (1 - q_j - q_k - q_l) + (1 - q^i_{A,peak} - q^i_{A,norm} - q^i_{B,peak})\right) - p_B\right) \\
&= q^i_{A,peak}\left(\alpha V_A - \gamma[q_{A,peak}] - \delta_A p_A\right) + q^i_{A,norm}\left(V_A - \gamma[q_{A,norm}] - p_A\right) \\
&\quad + q^i_{B,peak}\left(\alpha V_B - \gamma[q_{B,peak}] - \delta_B p_B\right) + q^i_{B,norm}\left(V_B - \gamma[(1 - q_{A,peak} - q_{A,norm} - q_{B,peak}] - p_B\right).
\end{aligned}$$

Because customer i is infinitesimal, customer i's shopping choice q_i will not impact the belief or actual realization of the *average* $q_{A,peak}$, $q_{A,norm}$, $q_{B,peak}$ and $q_{B,norm}$.

The expression for $u_i(q_i)$ is a linear function of q_i. Among the four utilities, $\alpha V_A - \gamma[q_{A,peak}] - \delta_A p_A$, $V_A - \gamma[q_{A,norm}] - p_A$, $\alpha V_B - \gamma[q_{B,peak}] - \delta_B p_B$, and $V_B - \gamma[(1 - q_{A,peak} - q_{A,norm} - q_{B,peak}] - p_B$, if $\alpha V_A - \gamma[q_{A,peak}] - \delta_A p_A$ is the largest, then $q^i_{A,peak} = 1$ would maximize his expected utility. In fact, everybody would choose $q^i_{A,peak} = 1$, and so cannot be a Nash equilibrium, because players beliefs about others' strategies are not correct in equilibrium. Following the same logic, if any of the other three utilities is the largest, it is easy to observe that, all customers would choose the corresponding $q_i = 1$, and this cannot be a Nash equilibrium.

Hence, the only possible Nash equilibrium occurs in a mixed strategy where customers are indifferent between shopping in peak hours and normal hours or in store A and B, i.e., the four utilities are equal. Thus, we have system of three equations:

$$\alpha V_A - \gamma q_{A,peak} - \delta_A p_A = \alpha V_B - \gamma q_{B,peak} - \delta_B p_B, \tag{A-1}$$

$$\alpha V_A - \gamma q_{A,peak} - \delta_A p_A = V_A - \gamma q_{A,norm} - p_A, \tag{A-2}$$

$$V_A - \gamma q_{A,norm} - p_A = V_B - \gamma(1 - q_{A,peak} - q_{B,peak} - q_{A,norm}) - p_B. \tag{A-3}$$

Solving the system of equations (A-1)–(A-3), we have:

$$\begin{aligned}
q^*_{A,peak}(\delta_A, \delta_B) &= \frac{\gamma + (3\alpha - 1)V_A - (1+\alpha)V_B + p_A + (1+\delta_B)p_B}{4\gamma} - \frac{3p_A}{4\gamma}\delta_A, \\
q^*_{A,norm}(\delta_A, \delta_B) &= \frac{\gamma + (3 - \alpha)V_A - (1+\alpha)V_B - 3p_A + (1+\delta_B)p_B}{4\gamma} + \frac{p_A}{4\gamma}\delta_A, \\
q^*_{B,peak}(\delta_A, \delta_B) &= \frac{\gamma - (1+\alpha)V_A - (1 - 3\alpha)V_B + (1+\delta_A)p_A + p_B}{4\gamma} - \frac{3p_B}{4\gamma}\delta_B, \\
q^*_{B,norm}(\delta_A, \delta_B) &= \frac{\gamma - (1+\alpha)V_A - (\alpha - 3)V_B + (1+\delta_A)p_A - 3p_B}{4\gamma} + \frac{p_B}{4\gamma}\delta_B.
\end{aligned}$$

By Assumption 4,

$$\begin{aligned}
&2(\alpha - 1)V_A + (1+\alpha)(V_A - V_B) < 3\gamma \ \Leftrightarrow \ 3\alpha V_A - V_A - V_B - \alpha V_B < 3\gamma \\
&\Rightarrow \ (3\alpha - 1)V_A - (1+\alpha)V_B - 2(p_A - p_B) < 3\gamma \\
&\Leftrightarrow \ \gamma + (3\alpha - 1)V_A - (1+\alpha)V_B - 2p_A + 2p_B < 4\gamma \\
&\Leftrightarrow \ \left.\frac{\gamma + (3\alpha - 1)V_A - (1+\alpha)V_B + p_A + (1+\delta_B)p_B - 3\delta_A p_A}{4\gamma}\right|_{(\delta_A, \delta_B) = (1,1)} < 1.
\end{aligned}$$

where the implication is since $p_A \geq p_B$. This is the condition for $q^*_{A,peak} < 1$ assuming no peak-period pricing occurs ($\delta_A = \delta_B = 1$). If $q^*_{A,peak} < 1$, then $q^*_{A,norm} < 1$ because peak attracts more demand. Also, $q^*_{B,peak} < 1$ because $V_A \geq V_B$. Thus if this condition holds under this restricted setting, it holds for all optimal setting.

Hence, $q^*_{A,peak}$, $q^*_{B,peak}$, $q^*_{A,norm}$ and $q^*_{B,norm}$ can be interpreted as a symmetric mixed strategy under which each customer j chooses q^*_j so that $q^*_j = q^*_{A,peak}$, $q^*_k = q^*_{A,norm}$, $q^*_l = q^*_{B,peak}$, $1 - q^*_j - q^*_k - q^*_l = q^*_{B,norm}$.

To show that this is a Nash mixed equilibrium strategy, consider customer j deviates from q^*_j by choosing $q_j = q^*_j + \epsilon$. Because customer j is infinitesimal, his expected utility is $q_j \cdot u^*_{A,peak} + q_k \cdot u^*_{A,norm} + q_l \cdot u^*_{B,peak} + (1 - q_j - q_k - q_l) \cdot u^*_{B,norm} = u^*$. Hence, any deviation from the this strategy does not lead to an improvement in his expected utility, so he has no incentive to deviate. Therefore, we can conclude that $q^*_{A,peak}, q^*_{A,norm}, q^*_{B,peak}$, and $q^*_{B,norm}$ are equilibrium. $\square$

Each firm's best-response peak-hour multipliers as given in the following lemma, which resembles Lemma A.1.

Lemma A.2 **(Best-Response Peak-Hour Multipliers – Full Flexibility)**. *Suppose that Assumptions 1 and 4 hold. Then firm A's and B's best-response peak-hour multipliers satisfy the following:*

$$\delta_A^{br}(\delta_B) = \max\left\{1,\ \frac{\gamma+(3\alpha-1)V_A-(1+\alpha)V_B+2p_A+p_B}{6p_A}+\frac{p_B}{6p_A}\delta_B\right\},$$
$$\delta_B^{br}(\delta_A) = \max\left\{1,\ \frac{\gamma-(1+\alpha)V_A-(1-3\alpha)V_B+p_A+2p_B}{6p_B}+\frac{p_A}{6p_B}\delta_A\right\}.$$

By simultaneously considering these two best-response peak-hour multipliers, we can obtain the equilibrium peak-hour multipliers as given in the following proposition.

Proof of Lemma A.2. For simplicity, let us denote

$$q_{A,peak} \triangleq (1')-\frac{3p_A}{4\gamma}\delta_A,\ q_{B,peak} \triangleq (2')+\frac{p_A}{4\gamma}\delta_A,\ q_{A,norm} \triangleq (3')-\frac{3p_B}{4\gamma}\delta_B,\ q_{B,norm} \triangleq (4')+\frac{p_B}{4\gamma}\delta_B.$$

The profit function of firm A is

$$\pi_A(\delta_A,\delta_B) = \delta_A p_A\left((1')-\frac{3p_A}{4\gamma}\delta_A\right)+p_A\left((2')+\frac{p_A}{4\gamma}\delta_A\right) = -\frac{3p_A^2}{4\gamma}\delta_A^2+\left[(1')p_A+\frac{p_A^2}{4\gamma}\right]\delta_A+p_A(2'),$$

which is quadratic function of δ_A. Taking the first order conditions and rearranging the terms, firm A's best response peak period multiplier is

$$\frac{\partial\pi_A(\delta_A,\delta_B)}{\partial\delta_A}=0 \Leftrightarrow -\frac{3p_A^2}{2\gamma}\delta_A+\left[(1')p_A+\frac{p_A^2}{4\gamma}\right]=0 \Leftrightarrow \delta_A=\frac{2\gamma}{3p_A}(1')+\frac{2\gamma}{3p_A^2}\left[\frac{p_A^2}{4\gamma}\right]=\frac{2\gamma}{3p_A}(1')+\frac{p_A}{6p_A}$$
$$\Leftrightarrow \delta_A=\frac{\gamma+(3\alpha-1)V_A-(1+\alpha)V_B+2p_A+p_B}{6p_A}+\frac{p_B}{6p_A}\delta_B.$$

Similarly, the profit function of firm B is:

$$\pi_B(\delta_A,\delta_B) = \delta_B p_B\left((3')-\frac{3p_B}{4\gamma}\delta_B\right)+p_B\left((4')+\frac{p_B}{4\gamma}\delta_B\right) = -\frac{3p_B^2}{4\gamma}\delta_B^2+\left[(3')p_B+\frac{p_B^2}{4\gamma}\right]\delta_B+p_B(4'),$$

which is quadratic in δ_B. Taking the first order conditions and rearranging the terms, we have

$$\frac{\partial\pi_B(\delta_A,\delta_B)}{\partial\delta_B}=0 \Leftrightarrow -\frac{3p_B^2}{2\gamma}\delta_B+\left[(3')p_B+\frac{p_B^2}{4\gamma}\right]=0 \Leftrightarrow \delta_B=\frac{2\gamma}{3p_B}(3')+\frac{2\gamma}{3p_B^2}\left[\frac{p_B^2}{4\gamma}\right]=\frac{2\gamma}{3p_B}(3')+\frac{p_B}{6p_B}$$
$$\Leftrightarrow \delta_B=\frac{\gamma-(1+\alpha)V_A-(1-3\alpha)V_B+p_A+2p_B}{6p_B}+\frac{p_A}{6p_B}\delta_A.$$

Since the peak rate cannot be less than 1, we have the maximum operators for both expressions. □

Proof of Proposition 3. We deal with best responses of the form,

$$\delta_A^{br}(\delta_B)=\max\{1,a+b\delta_B\},\quad \delta_B^{br}(\delta_A)=\max\{1,u+v\delta_A\},$$

where $a\equiv\frac{\gamma+(3\alpha-1)V_A-(1+\alpha)V_B+2p_A+p_B}{6p_A}$, $b\equiv\frac{p_B}{6p_A}$, $u\equiv\frac{\gamma-(1+\alpha)V_A-(1-3\alpha)V_B+2p_B+p_A}{6p_B}$, and $v\equiv\frac{p_A}{6p_B}$. The equilibrium (δ_A^*,δ_B^*) occurs when $\delta_A^*=\delta_A^{br}(\delta_B^*)$ and $\delta_B^*=\delta_B^{br}(\delta_A^*)$, i.e., at the intersection between $\delta_A^{br}(\delta_B)$ and inverse of $\delta_B^{br}(\delta_A)$. Because $\delta_A^{br}(\delta_B)$ has smaller slope than the inverted $\delta_B^{br}(\delta_A)$, as illustrated in Figure A-1, there are four possible fixed points.

(i) If $a+b<1$ and $u+v<1$ (upper left panel of Figure A-1), then the fixed point $(\delta_A^*,\delta_B^*)=(1,1)$.

(ii) If $a+b<1$ and $1<u+v<y$, where y is the inflection point of $\delta_A^{br}(\delta_B)$ (lower left panel of Figure A-1), then the fixed point $(\delta_A^*,\delta_B^*)=(1,u+v)$.

(iii) If $1<a+b<x$ and $u+v<1$, where x is the inflection point of $\delta_B^{br}(\delta_A)$ (upper right panel of Figure A-1), then the fixed point $(\delta_A^*,\delta_B^*)=(a+b,1)$.

(iv) Otherwise, in all other cases (lower right panel of Figure A-1), the fixed point (δ_A^*, δ_B^*) occurs at a non corner solution, which can be found by solving the system of equations,

$$\delta_A^*(\delta_B) \triangleq K_A + \frac{p_B}{6p_A}\delta_B, \qquad \delta_B^*(\delta_A) \triangleq K_B + \frac{p_A}{6p_B}\delta_A,$$

where $K_A \equiv \frac{\gamma+(3\alpha-1)V_A-(1+\alpha)V_B+2p_A+p_B}{6p_A}$ and $K_B \equiv \frac{\gamma-(1+\alpha)V_A-(1-3\alpha)V_B+p_A+2p_B}{6p_B}$.

$$\delta_A = K_A + \frac{p_B}{6p_A}\left(K_B + \frac{p_A}{6p_B}\delta_A\right) \Leftrightarrow \delta_A^* = \frac{36}{35}\left(K_A + \frac{p_B}{6p_A}K_B\right) = \frac{36}{35}K_A + \frac{6p_B}{35p_A}K_B;$$

$$\delta_B^* = K_B + \frac{p_A}{6p_B}\left(\frac{36}{35}K_A + \frac{6p_B}{35p_A}K_B\right) = \frac{36}{35}K_B + \frac{6p_A}{35p_B}K_A.$$

We have the expressions after substituting the expressions for K_A and K_B.

The equilibrium peak period multipliers are determined by four curves $a+b=1$, $a+b=x$, $u+v=1$, and $u+v=y$, which in terms of $\nu \equiv \frac{V_A}{V_B}$ and $\rho \equiv \frac{p_A}{p_B}$ are

$$a+b=1 \Leftrightarrow \gamma+(3\alpha-1)V_A-(1+\alpha)V_B+2p_A+2p_B = 6p_A \Leftrightarrow \nu = \frac{-\gamma+(1+\alpha)V_B-2p_B}{(3\alpha-1)V_B} + \frac{4p_B}{(3\alpha-1)V_B}\rho,$$

$$a+b=x \Leftrightarrow \frac{\gamma+(3\alpha-1)V_A-(1+\alpha)V_B+2p_A+2p_B}{6p_A} = \frac{6p_B-\gamma+(1+\alpha)V_A+(1-3\alpha)V_B-p_A-2p_B}{p_A}$$

$$\Leftrightarrow \nu = \frac{7\gamma+(17\alpha-7)V_B-22p_B}{(3\alpha+7)V_B} + \frac{8p_B}{(3\alpha+7)V_B}\rho,$$

$$u+v=1 \Leftrightarrow \gamma-(1+\alpha)V_A-(1-3\alpha)V_B+2p_A+2p_B = 6p_B \Leftrightarrow \nu = \frac{\gamma-(1-3\alpha)V_B-4p_B}{(1+\alpha)V_B} + \frac{2p_B}{(1+\alpha)V_B}\rho,$$

$$u+v=y \Leftrightarrow \frac{\gamma-(1+\alpha)V_A-(1-3\alpha)V_B+2p_A+2p_B}{6p_B} = \frac{6p_A-\gamma-(3\alpha-1)V_A+(1+\alpha)V_B-2p_A-p_B}{p_B}$$

$$\Leftrightarrow \nu = \frac{-7\gamma+(3\alpha+7)V_B-8p_B}{(17\alpha-7)V_B} + \frac{22p_B}{(17\alpha-7)V_B}\rho.$$

These four lines have four different slopes and one can verify that they all intersect at:

$$(\hat{\tau},\hat{\eta}) = \left(\frac{2\alpha\gamma+4\alpha(\alpha-1)V_B+(3-5\alpha)p_B}{(3-\alpha)p_B}, \frac{3\gamma+(5\alpha-3)V_B-6p_B}{(3-\alpha)V_B}\right).$$

One can also verify that that for case (ii), the condition $a+b<1$ is implied by $1<u+v<y$; and for case (iii) $1<a+b<x$ implies $u+v<1$. The result follows upon removing the redundant conditions. □

Proof of Corollary 2. (i) From Proof of Proposition 3, all four lines have four different slopes that uniquely intersect at $(\hat{\tau},\hat{\eta})$. If $\frac{\gamma}{2}+(\alpha-1)V_B \le p_B$, then

$$\frac{\gamma}{2}+(\alpha-1)V_B < p_B \Leftrightarrow 2\alpha\gamma+4\alpha(\alpha-1)V_B+(3-5\alpha)p_B < (3-\alpha)p_B \Leftrightarrow \hat{\tau}<1,$$

$$\frac{\gamma}{2}+(\alpha-1)V_B < p_B \Leftrightarrow 3\gamma+(5\alpha-3)V_B-6p_B < (3-\alpha)V_B \Leftrightarrow \hat{\eta}<1,$$

and the region where both $\delta_A^*>1$ and $\delta_B^*>1$ does not exist.

We observe from the condition for $(\delta_A^*>1, \delta_B^*>1)$ — i.e., $\nu \ge \frac{-\gamma+(1+\alpha)V_B-2p_B}{(3\alpha-1)V_B} + \frac{4p_B}{(3\alpha-1)V_B}\rho$ and $\nu \ge \frac{7\gamma+(17\alpha-7)V_B-22p_B}{(3\alpha+7)V_B} + \frac{8p_B}{(3\alpha+7)V_B}\rho$ — that an increase in γ enlarges this region. Similarly we observe from the condition for $(\delta_A^*=1, \delta_B^*=1)$ — i.e., $\nu \le \frac{-\gamma+(1+\alpha)V_B-2p_B}{(3\alpha-1)V_B} + \frac{4p_B}{(3\alpha-1)V_B}\rho$ and $\nu \ge \frac{\gamma-(1-3\alpha)V_B-4p_B}{(1+\alpha)V_B} + \frac{2p_B}{(1+\alpha)V_B}\rho$ — that an increase in γ shrinks this region.

(ii) It is clear from the equilibrium expressions for $\delta_A^*>1$ and $\delta_B^*>1$. □

Proof of Corollary 3. Suppose that $\tau=\hat{\tau}$ and $\eta=\hat{\eta}$. Then the $(\delta_A^*>1, \delta_B^*>1)$ region is narrower with time flexibility because the slopes of the upper and lower boundaries are respectively $\frac{8p_B}{(3+7\alpha)V_B}>0$ and $\frac{22p_B}{(17\alpha-7)V_B} < \frac{3p_B}{\alpha V_B}$. Also, the $(\delta_A^*=1, \delta_B^*=1)$ region is narrower with time flexibility because the slopes of the lower and upper boundaries are respectively $\frac{4p_B}{(3\alpha-1)V_B} < \frac{2p_B}{\alpha V_B}$ and $\frac{2p_B}{(1+\alpha)V_B} > \frac{p_B}{\alpha V_B}$. □

Proof of Proposition 4. Proposition 4 is a special case (when $\delta_B = 1$) of Proposition 5 (when $\delta_B > 1$), thus the proof is omitted. □

Proof of Proposition 5. Give p_B and δ_B, firm A optimizes over p_A and δ_A to maximize its total profit from normal hours and peak hours. The optimal p_A and δ_A can be obtained by applying first-order conditions on firm A's profit function. We next show the key steps in the three cases, time flexibility, store flexibility and time and store flexibility, respectively.

Time Flexibility Only Similar to §3.1, by applying Lemma 1, we can obtain the equilibrium proportion, $r_{A,peak}(p_A,\delta_A) = \frac{1}{2} + \frac{(\alpha-1)V_A-(\delta_A-1)p_A}{2\gamma\lambda}$. Firm A then maximizes below profit by deciding p_A and δ_A.

$$\begin{aligned}\pi_A(p_A,\delta_A) &= \delta_A p_A \lambda r_{A,peak}(p_A,\delta_A) + p_A\lambda(1-r_{A,peak}(p_A,\delta_A)),\\ &= \delta_A p_A \lambda\Big(\frac{1}{2}+\frac{(\alpha-1)V_A-(\delta_A-1)p_A}{2\gamma\lambda}\Big) + p_A\lambda\Big(\frac{1}{2}-\frac{(\alpha-1)V_A-(\delta_A-1)p_A}{2\gamma\lambda}\Big).\end{aligned}$$

It is easy to verify that the profit function is concave, thus by applying the first order conditions, we obtain

$$\frac{\partial \pi_A}{\partial \delta_A} = 0 \Leftrightarrow \delta_A = 1 + \frac{\lambda\gamma + (\alpha-1)V_A}{2p_A}, \quad \frac{\partial \pi_A}{\partial p_A} = 0 \Leftrightarrow p_A = \frac{\lambda\gamma(1+\delta_A)}{2(\delta_A-1)^2} + \frac{(\alpha-1)V_A}{2(\delta_A-1)}.$$

In the normal period, considering all customers will buy the product as indicated by Assumption 1, firm A will charge the highest possible p_A at the upper bounds defined by Assumption 1 to extract the maximum profit, which is $p_A = \max\{V_A-\gamma, \frac{\alpha V_A-\gamma}{\delta_A}\}$. Knowing that $\delta_A = 1 + \frac{\lambda\gamma+(\alpha-1)V_A}{2p_A}$, one can verify that $V_A - \gamma \le \frac{\alpha V_A-\gamma}{\delta_A}$ by applying Assumption 2. Thus, store A's optimal pricing strategy is:

$$\delta_A^* = 1 + \frac{(\alpha-1)V_A + \gamma\alpha}{(\alpha+1)V_A - \gamma(2+\lambda)}, \quad p_A^* = \frac{(\alpha+1)V_A-\gamma(2+\lambda)}{2}.$$

As $p_A^* > 0$, it is clear that $\delta_A^* > 1$.

Store Flexibility Only Similar to §3.2, a known proportion β of the customers must shop during peak hours and $(1-\beta)$ during normal hours. Let $q_{A,peak} = \beta r_{A,peak}$ and $q_{A,norm} = (1-\beta)r_{A,norm}$ denote the proportions of peak- and normal-hour customers, respectively, who shop in store A. Hence, $q_{B,peak} = \beta(1-r_{A,peak})$ and $q_{B,norm} = (1-\beta)(1-r_{A,norm})$. The customer's utilities for shopping at store A and store B are as follows:

$$\begin{aligned} u_{A,peak} &= \alpha V_A - \gamma[\beta r_{A,peak}] - \delta_A p_A, & u_{B,peak} &= \alpha V_B - \gamma[\beta(1-r_{A,peak})] - \delta_B p_B,\\ u_{A,norm} &= V_A - \gamma(1-\beta)r_{A,norm} - p_A, & u_{B,norm} &= V_B - \gamma(1-\beta)(1-r_{A,norm}) - p_B.\end{aligned}$$

Each customer makes a decision regarding whether to shop at store A or at store B during the peak hours. By applying Lemma 2, we find find the mixed strategy equilibrium,

$$\begin{aligned} u_{A,peak} = u_{B,peak} &\Leftrightarrow r^*_{A,peak}(\delta_A,p_A) = \frac{1}{2} + \frac{\alpha(V_A-V_B)-(\delta_A p_A - \delta_B p_B)}{2\gamma\beta},\\ u_{A,norm} = u_{B,norm} &\Leftrightarrow r^*_{A,norm}(p_A) = \frac{1}{2} + \frac{V_A - V_B - p_A + p_B}{2\gamma(1-\beta)}.\end{aligned}$$

Firm A solve the following profit maximizing problem, which is also a concave function:

$$\pi_A(p_A,\delta_A) = \delta_A p_A \beta r^*_{A,peak}(p_A,\delta_A) + p_A(1-\beta)r^*_{A,norm}(p_A,\delta_A)).$$

Applying first-order conditions yields the following non-corner optimal solutions:

$$\delta_A^* = \frac{\alpha(V_A-V_B)+\gamma\beta+\delta_B p_B}{V_A - V_B + \gamma(1-\beta) + p_B}, \quad p_A^* = \frac{V_A - V_B + \gamma(1-\beta)+p_B}{2}.$$

Note that, p_A^* is increasing in p_B. And δ_A^* should be bounded by $\delta_A^* \ge 1$, thus $\delta_A^* > 1$ only when $(\alpha-1)(V_A - V_B) > \gamma(1-2\beta)$. Also, for $\delta_A^* > 1$, it is straightforward that δ_A^* decreases with p_B.

Store and Time Flexibility Similar to §3.3, from Table 1, each customer's decision regarding when and where to shop hinges on the following net utilities:

$$u_{A,peak} = \alpha V_A - \gamma q_{A,peak} - \delta_A p_A, \quad u_{B,peak} = \alpha V_B - \gamma q_{B,peak} - \delta_B p_B,$$
$$u_{A,norm} = V_A - \gamma q_{A,norm} - p_A, \qquad u_{B,norm} = V_B - \gamma q_{B,norm} - p_B.$$

By applying Lemma 3, thus, we have system of four equations and four unknowns, which determines the equilibrium proportions:

$$\begin{aligned}
&q_{A,peak} + q_{B,peak} + q_{A,norm} + q_{B,norm} = 1\\
&u_{A,peak} = u_{B,peak} \Leftrightarrow \alpha V_A - \gamma q_{A,peak} - \delta_A p_A = \alpha V_B - \gamma q_{B,peak} - \delta_B p_B,\\
&u_{A,peak} = u_{A,norm} \Leftrightarrow \alpha V_A - \gamma q_{A,peak} - \delta_A p_A = V_A - \gamma q_{A,norm} - p_A,\\
&u_{A,norm} = u_{B,norm} \Leftrightarrow V_A - \gamma q_{A,norm} - p_A = V_B - \gamma q_{B,norm} - p_B.
\end{aligned}$$

This yields the following equilibrium proportions $q_{i,j}$:

$$\begin{aligned}
q^*_{A,peak}(\delta_A, p_A) &= \frac{\gamma + (3\alpha - 1)V_A - (1+\alpha)V_B + p_A + (1+\delta_B)p_B}{4\gamma} - \frac{3p_A}{4\gamma}\delta_A,\\
q^*_{A,norm}(\delta_A, p_A) &= \frac{\gamma + (3-\alpha)V_A - (1+\alpha)V_B - 3p_A + (1+\delta_B)p_B}{4\gamma} + \frac{p_A}{4\gamma}\delta_A,\\
q^*_{B,peak}(\delta_A, p_A) &= \frac{\gamma - (1+\alpha)V_A - (1-3\alpha)V_B + (1+\delta_A)p_A + p_B}{4\gamma} - \frac{3p_B}{4\gamma}\delta_B,\\
q^*_{B,norm}(\delta_A, p_A) &= \frac{\gamma - (1+\alpha)V_A - (\alpha-3)V_B + (1+\delta_A)p_A - 3p_B}{4\gamma} + \frac{p_B}{4\gamma}\delta_B.
\end{aligned}$$

By using the equilibrium proportions of customers who shop at the different stores during the different hours as given above, we can express firm A's profit function as

$$\pi_A(\delta_A, p_A) = \delta_A p_A \cdot q^*_{A,peak}(\delta_A, p_A) + p_A \cdot q^*_{A,norm}(\delta_A, p_A).$$

By applying the first-order conditions, we obtain firm A's optimal pricing strategy:

$$\delta^*_A = 1 + \frac{2(\alpha-1)V_A}{2V_A - (\alpha+1)V_B + (1+\delta_B)p_B + \gamma}, \quad p^*_A = \frac{2V_A - (\alpha+1)V_B + (1+\delta_B)p_B + \gamma}{4}.$$

It is clear that $\delta^*_A > 1$, as $p^*_A > 0$. □

Proof of Proposition 6. We next prove the three cases, time flexibility, store flexibility and time and store flexibility, corresponding to Proposition 6(i), (ii) and (iii), respectively.

Time Flexibility Only A customer with $V_A \in [\gamma - \epsilon, \gamma + \epsilon] \triangleq [V_{A\ell}, V_{Ah}]$ has the following utilities,

$$u_{A,peak}(V_A) = \alpha V_A - \gamma\lambda r_{A,peak} - \delta_A p_A, \text{ and } u_{A,norm}(V_A) = V_A - \gamma\lambda(1 - r_{A,peak}) - p_A,$$

where $r_{A,peak}$ represents the equilibrium proportion of loyal customers who purchase during the peak period (to be determined). This customer will purchase in the peak-hour if and only if,

$$u_{A,peak}(V_A) > u_{A,norm}(V_A) \Leftrightarrow (\alpha-1)V_A - (\delta_A - 1)p_A + \gamma\lambda > 2\gamma\lambda r_{A,peak}.$$

We observe that a customer with higher V_A is more likely to buy during the peak times. In equilibrium, there exists a customer with V^*_A who is indifferent between purchasing in peak or normal period. All customers with $V_A > V^*_A$ will purchase during the peak period, and the rest during the normal period, i.e.,

$$r_{A,peak} = \frac{1}{2} + \frac{(\alpha-1)V^*_A - (\delta_A - 1)p_A}{2\gamma\lambda}, \qquad r_{A,peak} = \frac{V_{Ah} - V^*_A}{V_{Ah} - V_{A\ell}}.$$

Substituting the expression for V_A^* from the second expression into the first equation, we have

$$r_{A,peak} = \frac{1}{2} + \frac{(\alpha-1)[V_{Ah} - (V_{Ah} - V_{A\ell})r_{A,peak}] - (\delta_A - 1)p_A}{2\gamma\lambda} \;\Leftrightarrow\; r^*_{A,peak} = \frac{\gamma\lambda + (\alpha-1)V_{Ah} - (\delta_A - 1)p_A}{2\gamma\lambda + (\alpha-1)(V_{Ah} - V_{A\ell})}.$$

We have the profit function,

$$\begin{aligned}\pi_A(\delta_A) &= \delta_A p_A \lambda r^*_{A,peak}(\delta_A) + p_A\lambda(1 - r^*_{A,peak}(\delta_A)) \\ &= \delta_A p_A \lambda \left[\frac{\gamma\lambda + (\alpha-1)V_{Ah} - (\delta_A - 1)p_A}{2\gamma\lambda + (\alpha-1)(V_{Ah} - V_{A\ell})}\right] + p_A\lambda\left[1 - \frac{\gamma\lambda + (\alpha-1)V_{Ah} - (\delta_A - 1)p_A}{2\gamma\lambda + (\alpha-1)(V_{Ah} - V_{A\ell})}\right].\end{aligned}$$

Taking the first order condition,

$$\frac{\partial \pi_A(\delta_A)}{\partial \delta_A} = p_A\lambda \cdot \frac{\gamma\lambda + (\alpha-1)V_{Ah} - (\delta_A - 1)p_A}{2\gamma\lambda + (\alpha-1)(V_{Ah} - V_{A\ell})} + \frac{\delta_A p_A \lambda \cdot (-p_A)}{2\gamma\lambda + (\alpha-1)(V_{Ah} - V_{A\ell})} + \frac{p_A\lambda \cdot p_A}{2\gamma\lambda + (\alpha-1)(V_{Ah} - V_{A\ell}} = 0$$

$$\Leftrightarrow\; \delta_A^* = \frac{p_A\lambda(\gamma\lambda + (\alpha-1)V_{Ah} + 2p_A)}{2p_A\lambda p_A} = 1 + \frac{\gamma\lambda + (\alpha-1)V_{Ah}}{2p_A} = 1 + \frac{\gamma\lambda + (\alpha-1)V_B(\nu+\epsilon)}{2p_A}.$$

Therefore, Proposition 6(i) holds.

Store Flexibility Only A customer with relative valuation $V_A \in [V_{A\ell},\ V_{Ah}]$ compares and chooses where to shop based on the following two utilities

$$u_{A,peak}(V_A) = \alpha V_A - \gamma\beta r_{A,peak} - \delta_A p_A, \text{ and } u_{B,peak}(V_A) = \alpha V_B - \gamma\beta(1 - r_{A,peak}) - \delta_B p_B,$$

where $r_{A,peak}$ denotes the equilibrium proportion of peak time shoppers shopping in store A.

In equilibrium, there exists a customer with $V_A = V_A^*$ whose $u_{A,peak}(V_A^*) = u_{B,peak}(V_A)$. For all customers with $V_A > V_A^*$, they will shop at A because $u_{A,peak}(V_A) > u_{B,peak}(V_A)$; the rest will shop at B, i.e.,

$$\alpha V_A^* - \gamma\beta r_{A,peak} - \delta_A p_A = \alpha V_B - \gamma\beta(1 - r_{A,peak}) - \delta_B p_B, \qquad r_{A,peak} = \frac{V_{Ah} - V_A^*}{V_{Ah} - V_{A\ell}}.$$

Substituting the expression for V_A^* from the second expression to the first, we have

$$\begin{aligned}&\alpha\{V_{Ah} - r_{A,peak}(V_{Ah} - V_{A\ell})\} - \gamma\beta r_{A,peak} - \delta_A p_A = \alpha V - \gamma + \gamma\beta r_{A,peak} - \delta_B p_B \\ &\Leftrightarrow\; r^*_{A,peak}(\delta_A, \delta_B) = \frac{1}{2} + \frac{\alpha\left(\frac{V_{Ah}+V_{A\ell}}{2} - V_B\right) - (\delta_A p_A - \delta_B p_B)}{2\beta\gamma + \alpha(V_{Ah} - V_{A\ell})} = \frac{1}{2} + \frac{(\nu-1)\alpha V_B - (\delta_A p_A - \delta_B p_B)}{2\beta\gamma + \alpha 2\epsilon V_B}.\end{aligned}$$

We have firm A's profit,

$$\pi_A = \delta_A p_A \beta r^*_{A,peak}(\delta_A, \delta_B) = \delta_A p_A \left(\frac{\beta}{2} + \frac{(\nu-1)\alpha V_B - (\delta_A p_A - \delta_B p_B)}{2\gamma + \alpha 2\epsilon V_B/\beta}\right).$$

Each firm seeks to maximize their profit $\pi_A = \delta_A p_A \beta r^*_{A,peak}(\delta_A, \delta_B)$ (or $\pi_B = \delta_B p_B \beta(1 - r^*_{A,peak}(\delta_A, \delta_B))$). Taking the first order conditions,

$$\frac{\partial \pi_A}{\partial \delta_A} = \frac{p_A\beta}{2} + \frac{p_A(\nu-1)\alpha V_B + \delta_B p_B}{2\gamma + (2\epsilon)\alpha V_B/\beta} - \frac{2p_A^2}{2\gamma + (2\epsilon)\alpha V_B \beta}\delta_A = 0 \;\Leftrightarrow\; \delta_A^* = \frac{\beta\gamma + (\nu + \epsilon - 1)\alpha V_B}{2p_A} + \frac{p_B}{2p_A}\delta_B.$$

Similarly, we have firm B's profit

$$\pi_B = \delta_B p_B \beta(1 - r^*_{A,peak}(\delta_A, \delta_B)) = \delta_B p_B \left(\frac{\beta}{2} - \frac{(\nu-1)\alpha V_B - (\delta_A p_A - \delta_B p_B)}{2\gamma + (2\epsilon)\alpha V_B/\beta}\right).$$

Taking the first order conditions,

$$\frac{\partial \pi_B}{\partial \delta_B} = \frac{p_B\beta}{2} - \frac{p_B(\nu-1)\alpha V_B - \delta_A p_A}{2\gamma + (2\epsilon)\alpha V_B/\beta} - \frac{2p_B^2\delta_B}{2\gamma + (2\epsilon)\alpha V_B/\beta} = 0 \;\Leftrightarrow\; \delta_B^* = \frac{\beta\gamma - \alpha V_B(\nu - \epsilon - 1)}{2p_B} + \frac{p_A}{2p_B}\delta_A.$$

Thus, if V_A is distributed uniformly between $[V_{A\ell},\ V_{Ah}]$, taking the maximization operator because $\delta_A, \delta_B \geq 1$, the best responses are

$$\delta_A^{br}(\delta_B) = \max\left\{1,\ \frac{\beta\gamma + \alpha(V_{Ah} - V_B)}{2p_A} + \frac{p_B}{2p_A}\delta_B\right\}, \quad \delta_B^{br}(\delta_A) = \max\left\{1,\ \frac{\beta\gamma - \alpha(V_{A\ell} - V_B)}{2p_B} + \frac{p_A}{2p_B}\delta_A\right\}.$$

Note that the best response peak period multiplier for firm A is a function of the upper bound on V_A, V_{Ah}, where as that for firm B is a function of the lower bound on V_A, $V_{A\ell}$.

The expression for these best responses take the same form as those in Lemma A.1, i.e., $\delta_A^{br}(\delta_B) = \max\{1,\ \bar{a} + b\delta_B\}$ and $\delta_B^{br}(\delta_A) = \max\{1,\ \bar{u} + v\delta_A\}$. Thus, following the same procedure of proving Proposition 2 (but accounting for $\bar{a} \neq a$ and $\bar{u} \neq u$), we can show that the four regions in Figure 5 are defined by the four curves:

$$\begin{aligned}
\bar{a} + b = 1 &\Leftrightarrow \nu = \frac{-\beta\gamma + \alpha V_B - p_B}{\alpha V_B} - \epsilon + \frac{2p_B}{\alpha V_B}\rho, \\
\bar{a} + b = x &\Leftrightarrow \nu = \frac{3\beta\gamma + \alpha V_B - 3p_B}{\alpha V_B} + 3\epsilon, \\
\bar{u} + v = 1 &\Leftrightarrow \nu = \frac{\beta\gamma + \alpha V_B - 2p_B}{\alpha V_B} + \epsilon + \frac{p_B}{\alpha V_B}\rho, \\
\bar{u} + v = y &\Leftrightarrow \nu = \frac{-3\beta\gamma + \alpha V_B}{\alpha V_B} - 3\epsilon + \frac{3p_B}{\alpha V_B}\rho.
\end{aligned}$$

Moreover, the four lines have different slopes and intersect uniquely at

$$(\tilde{\tau}, \tilde{\eta}) = \left(\frac{2(\beta\gamma + \epsilon\alpha V_B)}{p_B} - 1,\ 1 + \frac{3(\beta\gamma - p_B + \epsilon\alpha V_B)}{\alpha V_B}\right).$$

Thus, we have our result after simplifying the conditions accordingly.

For the region $(\delta_A^* > 1, \delta_B^* > 1)$, we observe that the line defining the upper boundary of the region, $\nu \leq \frac{3\beta\gamma + \alpha V_B - 3p_B}{\alpha V_B} + 3\epsilon$, increases in ϵ, while the line defining the lower boundary of the region, $\nu \geq \frac{-3\beta\gamma + \alpha V_B}{\alpha V_B} - 3\epsilon + \frac{3p_B}{\alpha V_B}\rho$, decreases in ϵ. Similarly, for the region $(\delta_A^* = 1, \delta_B^* = 1)$, we observe that the line defining the upper boundary of the region, $\nu \leq \frac{-\beta\gamma + \alpha V_B - p_B}{\alpha V_B} - \epsilon + \frac{2p_B}{\alpha V_B}\rho$, decreases in ϵ and the line defining the lower boundary of the region, $\nu \geq \frac{\beta\gamma + \alpha V_B - 2p_B}{\alpha V_B} + \epsilon + \frac{p_B}{\alpha V_B}\rho$, increases in ϵ. The impact of ϵ on the values of δ_A^* and δ_B^* are clear from the expression. Therefore, Proposition 6(ii) follows.

Both Store and Time Flexibility In this setting, a customer with $V_A \in [V_{A\ell}, V_{Ah}]$ compares between four utilities and chooses when and where to shop:

$$\begin{aligned}
u_{A,peak}(V_A) &= \alpha V_A - \gamma q_{A,peak} - \delta_A p_A; \qquad & u_{B,peak}(V_A) &= \alpha V_B - \gamma q_{B,peak} - \delta_B p_B; \\
u_{A,norm}(V_A) &= V_A - \gamma q_{A,norm} - p_A; & u_{B,norm}(V_A) &= V_B - \gamma q_{B,norm} - p_B.
\end{aligned}$$

In equilibrium, there exists a customer $V_A = V_A^*$ where all four utilities are equal. For customer whose $V_A > V_A^*$, she will purchase from A (either peak or normal period) because doing so has higher utility than purchasing from B (either in peak or normal period respectively). Similarly, for a customer whose $V_A < V_A^*$, he will purchase from B (either in peak or normal period) because purchasing from A (either in peak or normal period respectively) has lower utility that purchasing from B. Because the proportion must sum to 1, we have the following system of four equations and four unknowns $(V_A^*, q_{A,peak}, q_{B,peak}, q_{A,norm})$,

$$\begin{aligned}
\alpha V_A^* - \gamma q_{A,peak} - \delta_A p_A &= \alpha V_B - \gamma q_{B,peak} - \delta_B p_B, \\
\alpha V_A^* - \gamma q_{A,peak} - \delta_A p_A &= V_A^* - \gamma q_{A,norm} - p_A, \\
V_A^* - \gamma q_{A,norm} - p_A &= V_B - \gamma(1 - q_{A,peak} - q_{A,norm} - q_{B,peak}) - p_B, \\
q_{A,peak} + q_{A,norm} &= \frac{V_{Ah} - V_A^*}{V_{Ah} - V_{A\ell}}.
\end{aligned}$$

This linear system of equations can be expressed in the form

$$\begin{bmatrix} \alpha & -\gamma & \gamma & 0 \\ \alpha-1 & -\gamma & 0 & \gamma \\ 1 & -\gamma & -\gamma & -2\gamma \\ 1 & 2\epsilon & 0 & 2\epsilon \end{bmatrix} \cdot \begin{bmatrix} V_A^* \\ q_{A,peak} \\ q_{B,peak} \\ q_{A,norm} \end{bmatrix} = \begin{bmatrix} \alpha V_B + \delta_A p_A - \delta_B p_B \\ (\delta_A - 1)p_A \\ V_B - \gamma + p_A - p_B \\ \nu + \epsilon \end{bmatrix},$$

and unique $(V_A^*, q_{A,peak}, q_{B,peak}, q_{A,norm})$ can be solved by inverting the matrix

$$\begin{bmatrix} V_A^* \\ q_{A,peak} \\ q_{B,peak} \\ q_{A,norm} \end{bmatrix} = \begin{bmatrix} \alpha & -\gamma & \gamma & 0 \\ \alpha-1 & -\gamma & 0 & \gamma \\ 1 & -\gamma & -\gamma & -2\gamma \\ 1 & 2\epsilon & 0 & 2\epsilon \end{bmatrix}^{-1} \cdot \begin{bmatrix} \alpha V_B + \delta_A p_A - \delta_B p_B \\ (\delta_A - 1)p_A \\ V_B - \gamma + p_A - p_B \\ \nu + \epsilon \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\epsilon}{\gamma+(1+\alpha)\epsilon} & 0 & \frac{\epsilon}{\gamma+(1+\alpha)\epsilon} & \frac{\gamma}{\gamma+(1+\alpha)\epsilon} \\ -\frac{\gamma+2(1-\alpha)\epsilon}{4(\gamma^2+\gamma(1+\alpha)\epsilon)} & -\frac{1}{2\gamma} & -\frac{\gamma+2(1-\alpha)\epsilon}{4(\gamma^2+\gamma(1+\alpha)\epsilon)} & -\frac{(1-3\alpha)}{4(\gamma+(1+\alpha)\epsilon)} \\ \frac{3\gamma+2(1+\alpha)\epsilon}{4(\gamma^2+\gamma(1+\alpha)\epsilon)} & -\frac{1}{2\gamma} & -\frac{\gamma+2(1+\alpha)\epsilon}{4(\gamma^2+\gamma(1+\alpha)\epsilon)} & -\frac{(1+\alpha)}{4(\gamma+(1+\alpha)\epsilon)} \\ -\frac{\gamma-2(1-\alpha)\epsilon}{4(\gamma^2+\gamma(1+\alpha)\epsilon)} & \frac{1}{2\gamma} & -\frac{\gamma-2(1-\alpha)\epsilon}{4(\gamma^2+\gamma(1+\alpha)\epsilon)} & \frac{(3-\alpha)}{4(\gamma+(1+\alpha)\epsilon)} \end{bmatrix} \begin{bmatrix} \alpha V_B + \delta_A p_A - \delta_B p_B \\ (\delta_A - 1)p_A \\ V_B - \gamma + p_A - p_B \\ \nu + \epsilon \end{bmatrix}.$$

Thus we have closed form expressions for the equilibrium demand and profit functions for each firm. Thus, following the same procedure as Proposition 3, we can (i) find the best response peak period multipliers for each firm, (ii) characterize the four curves that define the four regions in which one firm, neither firm, or both firm employs peak period pricing, and (iii) find the comparative statics with respect to the level heterogeneity ϵ. The expressions however are cumbersome and their algebraic derivations are omitted for brevity. □

Proof of Proposition 7. In this proof, we only provide the key steps and more details can be found in Sec. B of Appendix. Using the same approach as presented in §5.1, a customer with $\gamma \sim U[\gamma-\epsilon, \gamma+\epsilon] \triangleq U[\gamma_l, \gamma_h]$ compares the utilities from buy in normal or peak hours, then the equilibrium ratio for peak hours can be obtained:

$$r^*_{A,peak}(\delta_A) = \frac{(4\epsilon-2\gamma)\lambda + \sqrt{(4\epsilon-2\gamma)^2\lambda^2 + 16\lambda\epsilon[\lambda(\gamma-\epsilon)+(\alpha-1)V_A-(\delta_A-1)p_A]}}{8\lambda\epsilon}.$$

Therefore, firm A's the profit function is,

$$\pi_A(\delta_A) = \delta_A p_A \lambda r^*_{A,peak}(\delta_A) + p_A\lambda(1 - r^*_{A,peak}(\delta_A)).$$

It is easy to verify that the profit function is concave in δ_A. Taking the first-order conditions yields,

$$\frac{\partial \pi_A(\delta_A)}{\partial \delta_A} = 0 \;\Leftrightarrow\; r^*_{A,peak}(\delta_A) = (1-\delta_A)\frac{\partial r^*_{A,peak}(\delta_A)}{\partial \delta_A}.$$

The first order condition uniquely determines the optimal δ_A^*. Under Assumptions 1 and 2, $r^*_{A,peak} \geq 0$ and it is easy to verify that $\frac{\partial r^*_{A,peak}(\delta_A)}{\partial \delta_A} < 0$, therefore, $\delta_A^* > 1$. □

Appendix B: Detailed Analysis for Heterogeneity in Congestion Aversion Level γ

In this section, we provide detailed results and analysis for the case when customers have heterogeneous congestion aversion coefficient γ. Specifically, we let $\gamma \sim U[\gamma-\epsilon, \gamma+\epsilon] \triangleq U[\gamma_l, \gamma_h]$. Noting that $\epsilon > 0$ represents the level of heterogeneity, we shall examine the impact of heterogeneity in congestion aversion coefficient ϵ on the firms' decision to adopt time-base pricing in this section. We next analyze three scenarios, time flexibility only, store flexibility only, and both time and store flexibility, separately.

Time Flexibility Only Using the same approach as presented in §3.1, we consider the setting where a proportion λ of loyal customers shop at store A (and $(1-\lambda)$ at store B). However, customers are heterogeneous in their congestion aversion coefficient γ. For simplicity, we focus on customers who shop at firm A and firm A's peak-period pricing decision.

A customer with $\gamma \sim U[\gamma-\epsilon, \gamma+\epsilon] \triangleq U[\gamma_l, \gamma_h]$ has the following utilities,

$$u_{A,peak}(\gamma) = \alpha V_A - \gamma\lambda r_{A,peak} - \delta_A p_A, \text{ and } u_{A,norm}(\gamma) = V_A - \gamma\lambda(1-r_{A,peak}) - p_A,$$

where $r_{A,peak}$ represents the equilibrium ratio of customers who purchase during the peak hours (to be determined). This customer will purchase in the peak hours if and only if,

$$u_{A,peak}(\gamma) > u_{A,norm}(\gamma) \Leftrightarrow (\alpha-1)V_A - (\delta_A-1)p_A + \gamma\lambda > 2\gamma\lambda r_{A,peak}.$$

We observe that a customer with lower γ is more likely to buy during the peak hours. In equilibrium, there exists a customer with γ^* who is indifferent between purchasing in peak or normal period. All customers with $\gamma < \gamma^*$ will purchase during the peak period, and the rest during the normal hours, i.e.,

$$r_{A,peak} = \frac{1}{2} + \frac{(\alpha-1)V_A - (\delta_A-1)p_A}{2\gamma^*\lambda}, \qquad r_{A,peak} = \frac{\gamma^* - \gamma_l}{\gamma_h - \gamma_l}.$$

Substituting the expression for γ^* from the second expression into the first equation, we have

$$r_{A,peak} = \frac{1}{2} + \frac{(\alpha-1)V_A - (\delta_A-1)p_A}{2\lambda[r_{A,peak}(\gamma_h-\gamma_l)+\gamma_l]}$$

Solving for $r_{A,peak}$ yields,

$$\begin{aligned} r^*_{A,peak}(\delta_A) &= \frac{(\gamma_h - 3\gamma_l)\lambda + \sqrt{(\gamma_h-3\gamma_l)^2\lambda^2 + 8\lambda(\gamma_h-\gamma_l)[\lambda\gamma_l + (\alpha-1)V_A - (\delta_A-1)p_A]}}{4\lambda(\gamma_h-\gamma_l)} \\ &= \frac{(4\epsilon-2\gamma)\lambda + \sqrt{(4\epsilon-2\gamma)^2\lambda^2 + 16\lambda\epsilon[\lambda(\gamma-\epsilon) + (\alpha-1)V_A - (\delta_A-1)p_A]}}{8\lambda\epsilon}. \end{aligned}$$

Note that, there are two roots from solving the quadratic equation. As $r^*_{A,peak}(\delta_A) \in [0,1]$, the negative root is omitted.

Therefore, firm A's the profit function can be written as,

$$\pi_A(\delta_A) = \delta_A p_A \lambda r^*_{A,peak}(\delta_A) + p_A\lambda(1 - r^*_{A,peak}(\delta_A)).$$

It is easy to verify that the profit function is concave in δ_A. Taking the first-order condition yields,

$$\frac{\partial \pi_A(\delta_A)}{\partial \delta_A} = 0 \quad \Leftrightarrow \quad r^*_{A,peak}(\delta_A) = (1-\delta_A)\frac{\partial r^*_{A,peak}(\delta_A)}{\partial \delta_A}.$$

The first-order condition uniquely determines the optimal δ^*_A, which we solved using the software MATLAB.

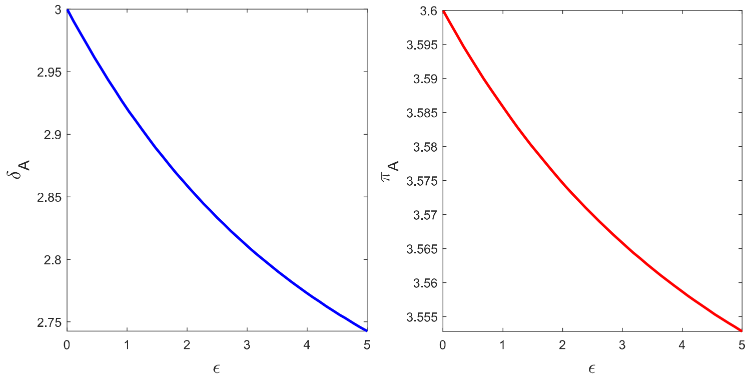
Under Assumptions 1 and 2,the optimal peak hour multiplier, $\delta^*_A > 1$. And,

$$\delta^*_A = 1 + \frac{(4\epsilon-2\gamma)\cdot\# + 24(\alpha-1)\epsilon V_A + \lambda((\gamma+\epsilon)^2 + 6(\gamma+\epsilon)(\gamma-\epsilon) - 3(\gamma-\epsilon)^2)}{36 p_A \epsilon}.$$

where $\# = \sqrt{\lambda[12(\alpha-1)\epsilon V_A + \lambda((\gamma+\epsilon)^2 + 3(\gamma-\epsilon)^2)]}$

Also, it is obvious that $r^*_{A,peak} \geq 0$ and $\frac{\partial r^*_{A,peak}(\delta_A)}{\partial \delta_A} < 0$, therefore, $\delta^*_A > 1$.This means a firm should employ peak-hour pricing. Hence, the presence of customer valuation heterogeneity $\epsilon > 0$ does not influence whether or not peak-hour pricing will occur. However, the impact of ϵ on the optimal δ^*_A and π^*_A is complicated. The relationships are no longer linear, which are visualized in below figure.

Figure A-2 **Equilibrium peak-period multiplier and profit (δ_A^*, π_A^*) as a function of ϵ under heterogeneous γ and time flexibility only.**



Note. Parameters: $V_A = 10$, $p_A = 2$, $\alpha = 1.3$, $\gamma = 5$.

Store Flexibility Only Following the same approach as presented in §3.2, we assume that a proportion β of the customers are peak-hour shoppers. However, they can choose between firm A and B. We examine how the presence of heterogeneity in γ, $\epsilon > 0$, influences the region boundaries in Figure 3, as presented in §3.2. We can express the equilibrium peak-hour multipliers as follows. A customer with $\gamma \sim U[\gamma-\epsilon, \gamma+\epsilon] \triangleq U[\gamma_l, \gamma_h]$ compares and chooses where to shop based on the following two utilities

$$u_{A,peak}(\gamma) = \alpha V_A - \gamma\beta r_{A,peak} - \delta_A p_A, \text{ and } u_{B,peak}(\gamma) = \alpha V_B - \gamma\beta(1 - r_{A,peak}) - \delta_B p_B,$$

where $r_{A,peak}$ denotes the REE proportion of peak time shoppers shopping in store A.

In equilibrium, there exists a customer with $\gamma = \gamma^*$ whose $u_{A,peak}(\gamma^*) = u_{B,peak}(\gamma^*)$. For all customers with $\gamma < \gamma^*$, they will shop at A because $u_{A,peak} > u_{B,peak}$; the rest will shop at B, i.e.,

$$\alpha V_A - \gamma^*\beta r_{A,peak} - \delta_A p_A = \alpha V_B - \gamma^*\beta(1 - r_{A,peak}) - \delta_B p_B, \qquad r_{A,peak} = \frac{\gamma^* - \gamma_l}{\gamma_h - \gamma_l}.$$

Substituting the expression for γ^* from the second expression to the first, we have

$$\alpha(V_A - V_B) - (\delta_A p_A - \delta_B p_B) = \beta(2r_{A,peak} - 1)[r_{A,peak}(\gamma_h - \gamma_l) + \gamma_l]$$

Solving for $r^*_{A,peak}$ yields,

$$\begin{aligned} r^*_{A,peak}(\delta_A) &= \frac{(\gamma_h - 3\gamma_l)\beta + \sqrt{(\gamma_h - 3\gamma_l)^2\beta^2 + 8\beta(\gamma_h - \gamma_l)[\beta\gamma_l + \alpha(V_A - V_B) - (\delta_A p_A - \delta_B p_B)]}}{4\beta(\gamma_h - \gamma_l)} \\ &= \frac{(4\epsilon - 2\gamma)\beta + \sqrt{(4\epsilon - 2\gamma)^2\beta^2 + 16\beta\epsilon[\beta(\gamma - \epsilon) + \alpha(V_A - V_B) - (\delta_A p_A - \delta_B p_B)]}}{8\beta\epsilon}. \end{aligned}$$

Note that, there are two roots from solving the quadratic equation. As $r^*_{A,peak}(\delta_A) \in [0,1]$, the negative root is omitted.

Each firm seeks to maximize their profits,

$$\pi_A(\delta_A) = \delta_A p_A \beta r^*_{A,peak}(\delta_A, \delta_B), \text{ and } \pi_B(\delta_B) = \delta_B p_B \beta(1 - r^*_{A,peak}(\delta_A, \delta_B)).$$

Taking the first-order conditions yields,

$$\begin{aligned} \frac{\partial \pi_A}{\partial \delta_A} &= \frac{2p_A\beta(2\epsilon - 1\gamma) + \#1}{8\epsilon} - \frac{\beta\delta_A p_A^2}{\#1} = 0 \\ \frac{\partial \pi_B}{\partial \delta_B} &= p_B\beta\big[1 - \frac{\beta(4\epsilon - 2\gamma) + \#1}{8\beta\epsilon}\big] - \frac{\beta\delta_B p_B^2}{\#1} = 0. \end{aligned}$$

where $\#1 = \sqrt{\beta[4\beta\gamma^2 + 16\epsilon(\alpha(V_A - V_B) - (\delta_A p_A - \delta_B p_B))]}$. Note that the two implicit best-response functions from the first-order conditions should be bounded by $\delta_A, \delta_B \geq 1$.

The non-corner optimal solutions of δ_A^*, δ_B^* are obtained by MATLAB and given below,

$$\delta_A^* = \frac{\epsilon - 2\gamma}{2p_A\epsilon}\left(\frac{4}{5}\beta\gamma + \#2\right) + \#3, \quad \delta_B^* = \frac{2\gamma + 20\epsilon}{40p_B\epsilon}\left(\frac{4}{5}\beta\gamma + \#2\right) - \#4.$$

where $\#2 = \frac{\sqrt{\frac{4\beta[36\beta\gamma^2 + 80\alpha\epsilon(V_A - V_B)]}{25}}}{2}$, $\#3 = \frac{4\beta\gamma^2 + 16\alpha\epsilon(V_A - V_B)}{40p_A\epsilon}$, and $\#4 = \frac{4\beta\gamma^2 + 16\alpha\epsilon(V_A - V_B)}{40p_B\epsilon}$.

Due to the complicated expression of the best response functions, we numerically analyze the optimal solutions and the corresponding conditions, by following the same steps as in the Proof of Proposition 2. The results are visualized on left panel of Figure 6.

Both Store and Time Flexibility We now consider how the heterogeneity in γ influences the peak-hour multipliers in the setting where customers are flexible in both store choice and shopping time. In this setting, a customer with $\gamma \sim U[\gamma - \epsilon, \gamma + \epsilon] \triangleq U[\gamma_l, \gamma_h]$ compares between four utilities and chooses when and where to shop:

$$u_{A,peak}(\gamma) = \alpha V_A - \gamma q_{A,peak} - \delta_A p_A; \qquad u_{B,peak}(\gamma) = \alpha V_B - \gamma q_{B,peak} - \delta_B p_B;$$
$$u_{A,norm}(\gamma) = V_A - \gamma q_{A,norm} - p_A; \qquad u_{B,norm}(\gamma) = V_B - \gamma q_{B,norm} - p_B.$$

In equilibrium, there exists a customer $\gamma = \gamma^*$ where all four utilities are equal. For customer whose $\gamma < \gamma^*$, she will purchase from firm A (either peak or normal period) because doing so has higher utility than purchasing from firm B (either in peak or normal period respectively). Similarly, for a customer whose $\gamma > \gamma^*$, he will purchase from B (either in peak or normal period) because purchasing from A (either in peak or normal period respectively) has lower utility that purchasing from B. Because the proportions must sum to 1, we have the following system of four equations and four unknowns $(\gamma^*, q_{A,peak}, q_{B,peak}, q_{A,norm})$,

$$\begin{aligned}
\alpha V_A - \gamma^* q_{A,peak} - \delta_A p_A &= \alpha V_B - \gamma^* q_{B,peak} - \delta_B p_B, \\
\alpha V_A - \gamma^* q_{A,peak} - \delta_A p_A &= V_A - \gamma^* q_{A,norm} - p_A, \\
V_A - \gamma^* q_{A,norm} - p_A &= V_B - \gamma^*(1 - q_{A,peak} - q_{A,norm} - q_{B,peak}) - p_B, \\
q_{A,peak} + q_{A,norm} &= \frac{\gamma^* - \gamma_l}{\gamma_h - \gamma_l}.
\end{aligned}$$

This system of equations determine the unique optimal solutions of $(\gamma^*, q_{A,peak}^*, q_{B,peak}^*, q_{A,norm}^*, q_{B,norm}^*)$, which can be obtained in closed-form using MATLAB. Thus we have closed form expressions for the demand and profit functions for each firm in equilibrium.

$$\begin{aligned}
\pi_A(\delta_A, \delta_B) &= \delta_A p_A \cdot q_{A,peak}^*(\delta_A, \delta_B) + p_A \cdot q_{A,norm}^*(\delta_A, \delta_B), \\
\pi_B(\delta_A, \delta_B) &= \delta_B p_B \cdot q_{B,peak}^*(\delta_A, \delta_B) + p_B \cdot q_{B,norm}^*(\delta_A, \delta_B).
\end{aligned}$$

Due to the complex forms of the demand proportions and profit functions, it is unable to solve the best-response functions of firm A and B even in MATLAB. Therefore, instead of obtaining the best-response functions of firm A and B then numerically analyzing the optimal δ_A^* and δ_B^*, we follow below steps to observe the firms' optimal peak-hour pricing strategy. We first replace some parameters with real values, i.e., we let $V_B = 10$, $p_B = 2$, $\epsilon = 2$, $\alpha = 1.2$, $\gamma = 5$. Second, we obtain the best-response functions, $\delta_A(\delta_B)$ and $\delta_B(\delta_A)$, which now only involve the parameters V_A and p_A. Note that, these two best-response functions should also be bounded by $\delta_A, \delta_B \geq 1$. Third, we search over the space of $V_A \in [10, 35]$ and $p_A \in [2, 20]$ (equivalent to

$\nu \in [1, 3.5]$ and $\rho \in [1, 10]$, which are the ranges of y-axis and x-axis in Figure 6), to find the optimal pairs of δ_A^* and δ_B^* that maximize the firms' profit functions. Finally, based on whether δ_A^* and δ_B^* are equal to or larger than 1, we plot the four curves that determine the regions for four different types of optimal solution pairs. This leads to the right panel of Figure 6.