Arbitrary Pole Placement with Sliding Mode Control

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***Abstract*— This paper considers the problem of placing all the poles arbitrarily for a linear time-invariant plant with (the linear part of) sliding mode control. We solve this problem in two ways. In the first approach, we design a sliding mode control by specifying the desired pole locations. The closed-loop system under this control law has all eigenvalues at the desired places. In the second approach, the sliding mode control is designed from a given state feedback gain so that all the poles of the closed-loop system are placed at the same location as that of the state feedback controller. Here, we provide a necessary and sufficient condition for the existence of a linear gain using the sliding mode control to achieve the desired pole assignment. This condition is always fulfilled for the single input case whereas it is only applicable for certain multi-input scenarios that meet the conditions stated in the paper. In both the approaches, one can place the closed-loop poles with the proposed sliding mode control at any arbitrary location in the left half of the complex plane, unlike with traditional design, where m poles are at the origin with m being the number of control inputs. A numerical example illustrates the proposed design methodology for sliding mode control.**

I. INTRODUCTION

A typical sliding mode based controller consists of two essential steps: design of a sliding manifold followed by the design of a control law to ensure the system trajectories move to lie on the chosen manifold. The motion of the system trajectories restricted to this manifold is called the *sliding mode*, and the control law which maintains this motion is termed as the *sliding mode control*. By selecting a stable sliding surface, the motion on the sliding manifold can be made stable. The key outcome of this design is that the plant response becomes insensitive to a class of disturbance signals that are implicit in the input channels [1]. This feature has been explored in many classic works on sliding mode control; it can be found in [2], [3] and references cited therein.

Stability of the sliding mode dynamics is guaranteed by placing all the poles at appropriate locations. As the order of this dynamics is $n-m$, only $n-m$ poles can be placed at any desired places, where n and m are the number of state and input variables, respectively. To achieve this, traditionally the full-order plant is transformed into a canonical structure, the so-called *regular form*, and the design is completed by extracting a reduced-order dynamics. This provides a transparent structure for selecting the sliding surface parameters at the cost of placing only $n-m$ poles. On the contrary, our goal in this paper is to propose a method for designing the sliding mode control to place all the closed-loop poles at any arbitrary location.

Although the problem considered in this paper is quite different, it has some connection with a few works appearing in the literature. In [1] and later in [4], the sliding hyperplane is chosen as the left eigenvector of the closed-loop system by assuming all the poles are in the left half of the complex plane. This method again guarantees only placement of reduced-order system poles. But, the advantage is that it avoids transformation of the system into regular form. The same motivation was adopted in the late nineties by proposing a full-order based design of a sliding hyperplane (see [5], [6], [7]). Here, Lyapunov based construction of a sliding hyperplane is presented to analyze the stability directly with the full-order plant. The readers can refer to [8], [9], [10], [11] and references cited therein. However, in all these techniques, the eigenvalues of the reduced-order system are placed at any desired locations. An exception is found in [12], where the controller stabilizes the plant robustly by placing all the poles at any desired place. However, the controller does not have an equivalent control structure so it may induce sliding motion only for sufficiently large switching gain. In [13], the controller can place all the poles in the left-half of complex plane by adding a linear proportional term. But, the design of this additional term for a given state-feedback gain is not presented.

In this paper, we propose a new method of designing the sliding hyperplane such that the linear part of sliding mode control places the closed-loop poles at the same location as that of a given state-feedback controller. We solve this problem in two ways. In the first case, the pole locations are specified whereas in the second case a given state-feedback gain is used as the basis for design. In both cases, the controller consists of a linear term that is designed to place the remaining poles. A necessary and sufficient condition is given for the existence of this linear term in the control law. It is observed that the solution to the problem always exists for a single input system. On the other hand, this may not be true in the case of multi-input systems because of the nonuniqueness of state-feedback gain. Our result gives a verifiable condition to determine whether arbitrary pole placement is possible with sliding mode control. Finally, we construct a common Lyapunov function for the full order plant to analyze the stability during both reaching and sliding phases. This proposed approach can offer some benefits over

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traditional sliding mode control design. When the sliding motion takes place, the switching function rejects the disturbance completely with its equivalent value being equal to zero. As a result, the closed-loop system behaves like the nominal system, i.e., without disturbance, and therefore, the performance can be compared with that of a linear state-feedback controller.

The paper is organized as follows: Section II gives the problem statement. We state the main results of this paper in Section III. The stability of the closed-loop system is discussed in Section IV. We elaborate the design methodology with a numerical example in Section V, which is followed by some concluding remarks in Section VI.

II. PROBLEM STATEMENT

Consider a linear time-invariant system

$$\dot{x} = Ax + B(u + d), \tag{1}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$ and $d \in \mathbb{R}^m$ are the state, control input, and disturbance input vectors, respectively. The class of disturbances satisfying the above criteria are called *matched uncertainty* An appropriately designed sliding mode control will reject matched disturbances when the system is in the sliding mode. The following assumptions are made.

Assumption 1: The system (1) is controllable.

Assumption 2: The disturbance input is bounded by a known constant $d_0 > 0$, i.e., $\|d(t)\| \leq d_0$ for all $t \geq 0$.

Sliding mode control is a very popular method for controlling the plant (1) as the closed-loop system renders the equilibrium point (the origin) of the system asymptotic stable. The design steps involve first synthesis of a stable manifold, known as the sliding surface, followed by the choice of a discontinuous control law that brings the trajectory to this manifold in finite-time. Denote $C \in \mathbb{R}^{m \times n}$ as the sliding surface parameter and then $s : \mathbb{R}^n \to \mathbb{R}^m$ with

$$s \equiv s(x) = Cx$$

as the sliding variable. One can construct the sliding manifold as the set of all points meeting the condition $s = Cx = 0$, which is given by

$$\mathscr{S} = \{x \in \mathbb{R}^n : s = Cx = 0\}. \tag{2}$$

The sliding manifold is said to be stable if the system trajectory restricted to the set $\mathscr{S}$ converges to zero asymptotically. Thus, the proper design of C is paid significant attention in the sliding mode literature. The control law to enforce the sliding motion on the system is given by Utkin ([14]) as

$$u = -(CB)^{-1}(CAx + K\text{sign}(s)) \tag{3}$$

where $K > \|CB\|d_0$ is called the switching gain. Here, the invertibility of CB is always guaranteed by the inherent structure of C. Indeed, this is also an essential criterion for the existence of sliding mode control.

Note that the equality $Cx = 0$ establishes an algebraic relation between the state components revealing the state-feedback structure of a reduced-order (or sliding mode) dynamics. In the reduced-order design with the control law (3), only $n - m$ poles are placed at any arbitrary location, whereas m poles are at zero, see [2] (the order of the reduced-order system is $n - m$).

In this paper, we address the problem of how to place all the poles at any arbitrary location (hence, in the left half of the complex plane) by proper choice of sliding surface parameter and the control law. We state this more formally below.

Problem 1: Design a sliding mode control such that all the poles of the closed-loop system are arbitrarily placed and a sliding mode is enforced on the closed-loop system.

III. DESIGN OF SLIDING MODE CONTROL

This section develops the sliding mode control by specifying the pole locations. Unlike the traditional method, we specify here all n eigenvalues for the closed-loop system. The control law will be designed to place the eigenvalues at these specified places.

A. Design of Sliding Hyperplane

Without loss of generality, the system (1) is assumed to be in the regular form as

$$\dot{x}_1 = A_{11}x_1 + A_{12}x_2 \tag{4a}$$

$$\dot{x}_2 = A_{21}x_1 + A_{22}x_2 + u + d \tag{4b}$$

where $x_1 \in \mathbb{R}^{n-m}$ and $x_2 \in \mathbb{R}^m$ are the first $n - m$ and the last m components of the transformed state vector, respectively. The system block matrices are of appropriate dimensions. With the state components, the sliding hyperplane can be expressed as

$$x_2 = -C_1 x_1,$$

which can be viewed as a virtual feedback law for the subsystem (4a). If the matrix C_1 is chosen to ensure that $A_{11} - A_{12}C_1$ is Hurwitz, the stability of the sliding mode dynamics can be guaranteed. Thus, we select C_1 using any standard methods to place $n - m$ poles of the closed-loop system at the desired location. Denote $C = \begin{bmatrix} C_1 & I_m \end{bmatrix}$. Then, the corresponding sliding manifold can be given by (2).

B. Pole Location Based Design

The sliding mode control law consists of a linear term and a switching term. Traditionally, the linear part of the controller places the poles of the closed-loop system whereas the switching part is responsible for rejecting the matched disturbance. However, it is important to note that the equivalent control (linear part) can place $n - m$ poles arbitrarily anywhere and m poles are located at the origin. We will modify the linear term by including an additional element so that all the poles can be placed arbitrarily. Note that this design methodology has been used in [9] to place the remaining m poles of the closed-loop system. Here, we use the same trick to design the sliding mode control law.

Let $M \in \mathbb{R}^{m\times m}$ be the matrix whose eigenvalues are located at the desired m eigenvalues of the closed-loop system. Then, we propose the following control law

$$u = -CAx + MCx - K\text{sign}(P_2 s), \tag{5}$$

where $K > 0$ is the switching gain to be designed later, and $P_2 = P_2^\top > 0$ as a solution to the following equation

$$M^\top P_2 + P_2 M = -Q_2 \tag{6}$$

for a given $Q_2 = Q_2^\top > 0$. One of the important observations is that the proposed control law reduces to

$$u = -CAx - K\text{sign}(P_2 s)$$

on $\mathscr{S}$. This shows that equivalent control is only applied to the plant on the sliding manifold.

Remark 1: The control law (5) has appeared in the literature in the context of the proportional reaching law [13]. When the control law is designed based on this reaching law, an additional linear term appears in the controller that enhances the convergence rate to the sliding manifold. However, the only difference is that the eigenvalues of M in our case are restricted to desired locations whereas the term is chosen arbitrarily in the proportional reaching law. Although this does not give any significant difference in the structure, the analysis shows that this term contributes m eigenvalues of the closed-loop system.

The closed-loop system can be written with the proposed control law (5) as

$$\dot{x} = A_c x - BK\text{sign}(P_2 s) + Bd, \tag{7}$$

where $A_c = A - B(CA - MC)$. We can transform the system (7) into new coordinates that show the closed-loop system has all poles at the desired locations. A change of coordinates $x \mapsto Tx$ is given where

$$T = \begin{bmatrix} I_{n-m} & 0 \\ C_1 & I_m \end{bmatrix},$$

and the system (7) can be written as

$$\dot{x}_1 = (A_{11} - A_{12}C_1)x_1 + A_{12}s \tag{8a}$$

$$\dot{s} = Ms - K\text{sign}(P_2 s) + d. \tag{8b}$$

It is evident from the above that $\sigma(A_c) = \sigma(A_{11} - A_{12}C_1) \cup \sigma(M)$, where $\sigma(\cdot)$ denotes the spectrum of the matrix. Moreover, these are placed at the desired locations. From now onwards, we will consider the system (8) to analyze the stability.

C. State-Feedback Gain Based Design

The additional linear term in the sliding mode control can also be designed from a given state-feedback control gain. For the single input case, the solution to this problem always exists because the feedback gain is unique. However, for the multi-input case, the state-feedback gain is non-unique and thus the problem of existence of sliding mode control gain may not be solvable always. We provide the necessary and sufficient conditions for solving the Problem 1.

Proposition 1: Consider the system (4) and the control (5). Let $F \in \mathbb{R}^{m\times n}$ be a given stabilizing state-feedback gain for the pair (A, B). Then, the gain $M \in \mathbb{R}^{m\times m}$ exists if and only if there is an $L \in \mathbb{R}^{m\times m}$ such that

$$CA - F = LC. \tag{9}$$

Proof: The sliding mode control gain exists if and only if the matrices M and F satisfy $CA - MC = F$. We consider two cases separately. In the first case, none of the closed-loop eigenvalues are at zero and the second case considers at least one eigenvalue is at zero.

In the first case, the matrix M is invertible. So, the existence of a linear control gain is equivalent to

$$C = M^{-1}(CA - F) \iff CA - F = MC$$

which gives $L = M$. To consider the second case, we assume without loss of generality that M is diagonal. So, we have

$$M = \begin{bmatrix} M_{11} & 0 \\ 0 & M_{22} \end{bmatrix}$$

where $M_{11} \in \mathbb{R}^{r\times r}$ is an invertible matrix for some $r \geq 0$. Note that $M_{22} = 0$ because all nonzero eigenvalues are contained in M_{11}. Then, the equality $CA - MC = F$ can be written as

$$\begin{bmatrix} M_{11} & 0 \\ 0 & M_{22} \end{bmatrix} \begin{bmatrix} \widetilde{C}_1 \\ \widetilde{C}_2 \end{bmatrix} = \begin{bmatrix} \widetilde{C}_1 A - F_1 \\ \widetilde{C}_2 A - F_2 \end{bmatrix}$$

where $\widetilde{C}_1$ and $\widetilde{C}_2$ are the first r and last $m-r$ rows of C, respectively, and similarly F_1 and F_2 are the r and $m-r$ rows of F. Clearly, $\widetilde{C}_1 = M_{11}^{-1}(\widetilde{C}_1 A - F_1) \iff \widetilde{C}_1 A - F_1 = M_{11}\widetilde{C}_1$. By taking

$$L = \begin{bmatrix} M_{11} & 0 \\ 0 & 0 \end{bmatrix}$$

we prove our result. ■

Remark 2: The restrictive condition in Proposition 1 stems from the fact that the state-feedback gain is non-unique for the multi-input case. The existence of M can be guaranteed if and only if F satisfies (9). Otherwise, M cannot be designed for the given linear state-feedback gain. However, the equality (9) always holds for any F in single input case, i.e., $m = 1$. This is because the state-feedback gain is unique for the single-input system.

Remark 3: The above design methodology based on the state-feedback gain can have additional advantages over the existing techniques. As the gains of the linear part of the sliding mode controller (5) and the state-feedback controller ($u = -Fx$) are equal, the controller (5) can achieve a comparable performance with that of a state-feedback controller when the sliding mode takes place. For instance, if F is obtained by the LQR method and a sliding mode is enforced, the closed-loop system with the sliding mode control becomes

$$\dot{x} = A_c x = (A - BF)x$$

because $K\text{sign}(P_2 s)|_{\text{eq}} = d$, where $K\text{sign}(P_2 s)|_{\text{eq}}$ denotes the equivalent value of the switching term. Then, the system will have the same performance as that of the LQR controller during the sliding phase. Note that the similar observation is also found in the integral sliding mode at additional cost of adding the nominal dynamics [15].

IV. STABILITY ANALYSIS

This section presents the stability analysis of the closed-loop system with the proposed design methodology. We construct here a Lyapunov function for the full-order closed-loop system and analyze its behavior from the initial point. Our analysis shows that the energy function decreases along the solutions of (7).

A. System Stability

The main result is as follows.

Theorem 1: Consider the system (1) and the control law (5). Then, the system trajectories converge to zero asymptotically if the switching gain

$$K > d_0. \tag{10}$$

Proof: Since $A_{11} - A_{12}C_1$ is a Hurwitz matrix, there always exists a symmetric and positive definite matrix $P_1 \in \mathbb{R}^{(n-m)\times(n-m)}$ such that

$$(A_{11} - A_{12}C_1)^\top P_1 + P_1(A_{11} - A_{12}C_1) = -Q_1$$

for any $Q_1 = Q_1^\top > 0$. Let $\overline{Q}_2 \in \mathbb{R}^{m\times m}$ be any symmetric and positive definite matrix. Then, the matrix

$$Q_2 = \overline{Q}_2 + A_{12}^\top P_1 Q_1^{-1} P_1 A_{12}$$

is also symmetric and positive definite. Then, using the value of Q_2, we solve (6) for the matrix $P_2 > 0$.

Now, we study the behavior of closed-loop system (8) under the control law (5). To analyze its stability, we construct the Lyapunov function $V(x_1, s) = x_1^\top P_1 x_1 + s^\top P_2 s$ and study its convergence. The time derivative of V is

$$\begin{aligned} \dot{V}(x_1,s) &= x_1^\top((A_{11} - A_{12}C_1)^\top P_1 + P_1(A_{11} - A_{12}C_1))x_1 \\ &\quad + x_1^\top P_1 A_{12} s + s^\top A_{12}^\top P_1 x_1 + s^\top(M^\top P_2 + P_2 M)s \\ &\quad - 2s^\top P_2 K \operatorname{sign}(P_2 s) - 2s^\top P_2 d \\ &= -x_1^\top Q_1 x_1 + x_1^\top P_1 A_{12} s + s^\top A_{12}^\top P_1 x_1 - s^\top Q_2 s \\ &\quad - 2K\|P_2 s\|_1 - 2s^\top P_2 d \\ &\le \begin{bmatrix} x_1^\top & s^\top \end{bmatrix} \begin{bmatrix} -Q_1 & P_1 A_{12} \\ A_{12}^\top P_1 & -Q_2 \end{bmatrix} \begin{bmatrix} x_1 \\ s \end{bmatrix} \\ &\quad - 2K\|P_2 s\| + 2d_0\|P_2 s\|. \end{aligned}$$

Our goal now is to show the symmetric matrix in the first term is negative definite. Simple matrix manipulation reveals that the matrix in the above inequality can be factorized as follows

$$\begin{bmatrix} -Q_1 & P_1 A_{12} \\ A_{12}^\top P_1 & -Q_2 \end{bmatrix} = \begin{bmatrix} I_{n-m} & 0 \\ -A_{12}^\top P_1 Q_1^{-1} & I_m \end{bmatrix} \times \begin{bmatrix} -Q_1 & 0 \\ 0 & -\overline{Q}_2 \end{bmatrix} \begin{bmatrix} I_{n-m} & -Q_1^{-1} P_1 A_{12} \\ 0 & I_m \end{bmatrix}.$$

indicating it is a negative definite matrix. Applying this identity together with the condition (11), the above inequality can be reduced to

$$\begin{aligned} \dot{V}(x_1,s) &\le -\begin{bmatrix} \bar{x}_1^\top & s^\top \end{bmatrix} \begin{bmatrix} Q_1 & 0 \\ 0 & \overline{Q}_2 \end{bmatrix} \begin{bmatrix} \bar{x}_1 \\ s \end{bmatrix} - 2\eta\|P_2 s\| \\ &\le -\min\{\lambda_{\min}(Q_1), \lambda_{\min}(\overline{Q}_2)\} \left\| \begin{bmatrix} \bar{x}_1 \\ s \end{bmatrix} \right\|^2 - 2\eta\|P_2 s\| \\ &< 0 \end{aligned}$$

where $\bar{x}_1 := x_1 - Q_1^{-1} P_1 A_{12} s$, $K \ge d_0 + \eta$ for some $\eta > 0$ and $\lambda_{\min}(\cdot)$ is the minimum eigenvalue of the given matrix. This shows that the Lyapunov function $V(Tx)$ converges monotonically to zero provided $Tx \neq 0$. In other words, the equilibrium point $x = 0$ is asymptotically stable. ■

Remark 4: Robust stabilization against matched uncertainties is considered in [12]. The control law consists of a Lyapunov function based discontinuous term to reject the disturbance while the linear component stabilizes the nominal part of the dynamics. Like in our case, the linear control also places all poles in the left half of the complex plane. But, the main difference from our present work is that the control law does not reduce to Utkin's equivalent control when the gain does not satisfy (9). Despite that, the sliding motion can be enforced if the switching gain of the controller is sufficiently large. Many contributions in sliding mode literature, e.g., [6] have explored this design methodology for the design of a sliding mode based control law.

B. Existence of Sliding Mode

The sliding motion can be shown to exist with our proposed control law following standard arguments.

Proposition 2: Consider the system (1) and the control law (5). The controller enforces the sliding motion on the manifold $\mathscr{S}$ if the switching gain satisfies

$$K > d_0. \tag{11}$$

Proof: Recall that the closed-loop sliding variable dynamics is given by (8b). Consider the Lyapunov function $V_s(s) = s^\top P_2 s$. Then, the directional directive of $V_s(s)$ along the solutions of (8b) is given by

$$\begin{aligned} \dot{V}_s(s) &= \dot{s}^\top P_2 s + s^\top P_2 \dot{s} \\ &= s^\top(M^\top P_2 + P_2 M)s - 2K\|P_2 s\|_1 + 2s^\top P_2 d \\ &\le -s^\top Q_2 s - 2K\|P_2 s\| + 2d_0\|P_2 s\|, \end{aligned}$$

where we used $\|s\|_1 \ge \|s\|$. Using (11), we see from the above inequality that

$$\dot{V}_s(s) \le -2\eta\|P_2 s\| \le -2\eta\sqrt{\lambda_{\min}(P_2)}\sqrt{V_s(s)}$$

for some $\eta > 0$. Solving the above differential inequality using the Comparison Lemma ([12]), one can show that $V_s(s(t)) = 0$ for all $t \ge \tau := [\sqrt{V_s(s(0))}/(\eta\sqrt{\lambda_{\min}(P_2)})]$. In other words, $s(t) = 0$ for all $t \ge \tau$. Therefore, the sliding mode is established in the system, and the proof is completed. ■

The proposed control law has an inherent property of exhibiting a sliding motion. The system trajectories during this mode become *insensitive* to the matched perturbations. This property is used to show the asymptotic convergence of the plant state despite the presence of disturbance signals.

V. A NUMERICAL EXAMPLE

We illustrate the proposed design methodology with the following system parameters

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

The bounded disturbance input is taken as $d = \sin(10t)$. Since all eigenvalues are at the origin, the open-loop system is unstable. But, observing that the system is controllable, there always exists a state-feedback control gain F such that $A - BF$ is Hurwitz. By following the standard pole placement method, we construct an F to place the eigenvalues of $A - BF$ at $-1, -2, -3$, i.e., $\sigma(A - BF) = \{-1, -2, -3\}$, and it is given by

$$F = \begin{bmatrix} 6 & 11 & 6 \end{bmatrix}.$$

Our goal here is to design a sliding mode control law that also places the poles at the same location as that of the above state-feedback gain. We follow the steps suggested in Subsection III-B for the synthesis of the control gain. The sliding hyperplane can be designed by considering the pole placement problem for the reduced order system

$$A_{11} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad A_{12} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

As (A_{11}, A_{12}) is controllable, a vector $C_1^\top \in \mathbb{R}^2$ can be obtained for which the matrix $A_{11} - A_{12}C_1$ is Hurwitz. We choose C_1 to place the poles at -1 and -2 and it is given by $C_1 = \begin{bmatrix} 2 & 3 \end{bmatrix}$. Thus, we have

$$C = \begin{bmatrix} C_1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 3 & 1 \end{bmatrix}.$$

Let us choose the design parameters as follows. For $Q_1 = I_2$, the Lyapunov equation $(A_{11} - A_{12}C_1)^\top P_1 + P_1(A_{11} - A_{12}C_1) = -I_2$ gives

$$P_1 = \begin{bmatrix} 1.25 & 0.25 \\ 0.25 & 0.25 \end{bmatrix}.$$

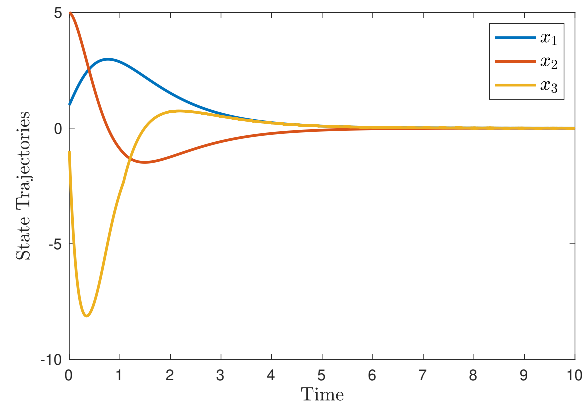
Then, we choose $\overline{Q}_2 = 2$, and subsequently, we obtain $Q_2 = 2.125$.

A. Design via Pole Placement

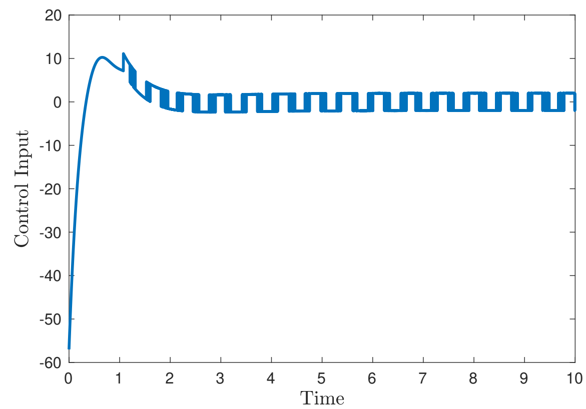
We choose $M = -3$. Then, we solve equation (6) with the above Q_2 that gives $P_2 = 0.3542$. It can be verified that $CA - MC = F$. Then, the sliding mode control law becomes

$$u = -\begin{bmatrix} 6 & 11 & 6 \end{bmatrix} x - K \operatorname{sign}(P_2 s)$$

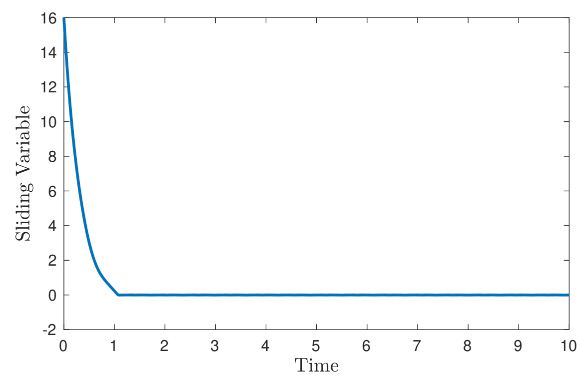
where $K = 2$.



(a) State trajectory versus time.



(b) Control input versus time.



(c) Sliding trajectory versus time.

Fig. 1. The response of the closed-loop system.

B. Design via State-Feedback Gain

Since the state-feedback gain is unique, the control parameter M exists if and only if $CA - F = MC$ by Proposition 1. In this case, it can be observed that

$$CA - F = \begin{bmatrix} -6 & -9 & -3 \end{bmatrix} = -3 \begin{bmatrix} 2 & 3 & 1 \end{bmatrix}.$$

Thus, the control law can be designed using the value of M in a similar manner.

With the above design parameters, we now simulate the closed-loop system. Take $x_0 = \begin{bmatrix} 1 & 5 & -1 \end{bmatrix}$. Fig. 1 plots the

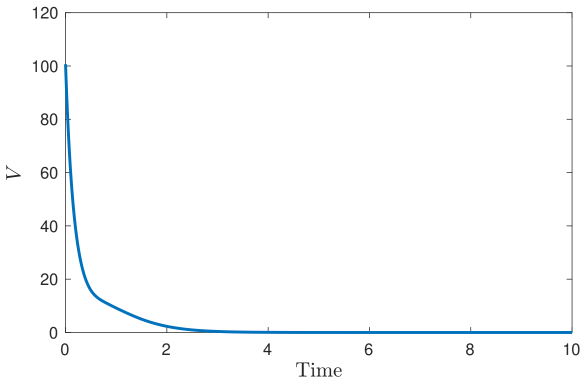


Fig. 2. The plot of Lyapunov function.

response of the system. The state trajectories converge to the equilibrium point asymptotically as it is shown in Theorem 1. This is achieved by making the sliding variable zero in a finite-time, which is depicted in Fig. 1. Moreover, the control input also becomes discontinuous during the sliding phase. Unlike the classical approach, we construct here a common Lyapunov function to analyze the convergence of the state trajectories. The Lyapunov function given in Theorem 1 converges monotonically towards zero. Fig. 2 shows the plot of the Lyapunov function with time, verifying our claim. This analysis may be useful in analyzing the performance of sliding mode control particularly in the constrained scenario. For example, it is often required to obtain the region of attraction under bounded control. Here, the proposed Lyapunov function can be useful in estimating these domains.

VI. CONCLUSION

We presented a design methodology for sliding mode control, which can place the poles of the closed-loop system arbitrarily. Two approaches were adopted to solve this problem. First, the poles are directly placed by specifying the pole locations. In the latter, we design a sliding mode control from a given a state-feedback gain. Moreover, the stability of the closed-loop system is analyzed by constructing a common Lyapunov function for the reaching and sliding phase dynamics. Finally, a numerical problem is considered to illustrate the design technique.

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