Online Supplement to: "Efficient Compromising"

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In this online appendix we prove the analytical results mentioned in Section 6 of the main paper. The structure of this appendix is as follows: In Appendix A we characterize the optimal cropped triangle rule. In Appendix B we show that the second-best in the public goods problem can be implemented using an incentive compatible cropped triangle rule.

Appendix A: Welfare Maximization Among Incentive Compatible Cropped Triangle Rules

The claim that we prove in this appendix concerns the maximization of expected welfare with weight $\lambda = 0.5$ among all incentive compatible cropped triangle rules. To maximize expected welfare among all these rules we shall restrict attention to cropped triangle rules that satisfy the following symmetry condition: $f_A(t, t') = f_C(t', t)$ and $f_B(t, t') = f_B(t', t)$ for all $(t, t') \in [0, 1]^2$. To see that this is without loss of generality consider any incentive compatible decision rule (f_A, f_B, f_C), and define a new, symmetric decision rule by swapping the roles of players 1 and 2 and alternatives A and C with probability 0.5. Thus, with probability 0.5 the original rule is applied, and with probability 0.5 player 1 finds himself in the role of player 2 in the original rule. But since the original rule was incentive compatible for players 1 and 2, so is the new rule. Moreover, with equal welfare weights, expected welfare remains unchanged. Note that the argument is not restricted to cropped triangle rules but is general. For cropped triangle rules, moreover, the function f_B is symmetric by construction. The above argument only establishes that for cropped triangle rules it is without loss of generality to assume that also the functions f_A and f_C satisfy the symmetry condition.

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The structure of our argument will be as follows. First, we establish a necessary condition that all functions f_B that are part of a symmetric, incentive compatible cropped triangle rule have to satisfy. Then we determine the welfare maximizing choice of f_B where f_B is as in Figure 3 of the main paper and satisfies the necessary conditions. Thus, we solve a relaxed optimization problem. Finally, we show that the solution satisfies the constraints of the original optimization problem, that is, that we can construct functions f_A and f_C that make the rule f_B incentive compatible.

If a function f_B is part of a symmetric, incentive compatible cropped triangle rule, then there must be interim probability functions p_i mapping types into interim probabilities of preferred outcomes that make the rule f_B incentive compatible. For given f_B the functions p_i are determined by Lemma 3, part (ii) (see main paper), once we have fixed the boundary values $p_i(1)$. By the symmetry assumption, $p_i(1)$ will be the same for both i, and we denote it for simplicity by p. Our necessary condition will be that it must be possible to find some $p \in [0, 1]$ such that, if we substitute this p for $p_i(1)$ in Lemma 3, part (ii), we obtain functions $p_i(t_i)$ that satisfy together with f_B the "ex ante adding up" constraint, i.e. the ex ante expected probability of the compromise and the ex ante expected values of $p_1(t_1)$ and $p_2(t_2)$ add up to one.

To work out the necessary condition in detail, we first note that for cropped triangle rules, the interim probability of the compromise is:

$$q_i(t_i) = \begin{cases} 0 & \text{if } 0 \le t_i \le c, \\ t_i - a & \text{if } c \le t_i \le 1 + a - c, \\ 1 - c & \text{if } 1 + a - c \le t_i \le 1. \end{cases}$$

The ex ante probability of the compromise, that is the ex ante expected value of q_i (for i either 1 or 2) can most easily be calculated as the size of the shaded area in Figure 3 of the main paper, which is:

$$\frac{1}{2}(1-a)^2 - (c-a)^2$$

where the size of the shaded area in Figure 3 of the main paper was determined as the size of a rectangular triangle with two sides of length $1-a$ minus the size of the two smaller triangles that are "cropped" in Figure 3 of the main paper, and that are rectangular with two sides of length $c-a$.

Next, we infer, using Lemma 3 of the main paper, the interim probabilities of the most preferred alternatives p_i. Obviously, if $1+a-c \leq t_i \leq 1$, incentive compatibility requires:

$$\begin{aligned} p_i(t_i) &= p_i(1) \\ &= p. \end{aligned}$$

If $c \leq t_i \leq 1+a-c$ we have

$$\begin{aligned} p_i(t_i) &= p_i(1) + q_i(1) - q_i(t_i)t_i - \int_{t_i}^{1} q_i(s_i)ds_i \\ &= p + (1-c) - (t_i - a)t_i \\ &\quad - \int_{t_i}^{1+a-c} (s_i - a)ds_i - (1-(1+a-c))(1-c) \end{aligned}$$

where the integral was calculated in two parts, and the second part equals the size of a rectangle with sides of length $1-(1+a-c)$ and $1-c$. We continue the calculation as follows:

$$\begin{aligned} &= p + (1-c) - (t_i - a)t_i \\ &\quad - \left[\frac{1}{2}(s_i)^2 - as_i\right]_{t_i}^{1+a-c} - (c-a)(1-c) \\ &= p + (1-c) - (t_i - a)t_i \\ &\quad - \frac{1}{2}(1+a-c)^2 + a(1+a-c) \\ &\quad + \frac{1}{2}(t_i)^2 - at_i - (c-a)(1-c) \\ &= p + \frac{1}{2}(1+a-c)^2 - \frac{1}{2}(t_i)^2 \end{aligned}$$

For $0 \leq t_i \leq c$ we have:

$$
\begin{aligned}
p_i(t_i) &= p_i(1) + q_i(1) - q_i(t_i)t_i - \int_{t_i}^{1} q_i(s_i)ds_i \\
&= p + (1-c) - \frac{1}{2}(1-a)^2 + (c-a)^2
\end{aligned}
$$

where the integral was calculated as the size of a rectangular triangle with two sides of length $1-a$ minus the size of the two smaller triangles that are "cropped" in Figure 3 of the main paper.

Now we are in a position to determine the ex ante expected value of p_i:

$$
\begin{aligned}
& p + c\left[(1-c) - \frac{1}{2}(1-a)^2 + (c-a)^2\right] \\
& + \int_{c}^{1+a-c} \frac{1}{2}(1+a-c)^2 - \frac{1}{2}(t_i)^2 \, dt_i \\
= \; & p + c\left[(1-c) - \frac{1}{2}(1-a)^2 + (c-a)^2\right] \\
& + (1+a-2c)\frac{1}{2}(1+a-c)^2 - \frac{1}{6}\left[(t_i)^3\right]_c^{1+a-c} \\
= \; & p + c\left[(1-c) - \frac{1}{2}(1-a)^2 + (c-a)^2\right] \\
& + (1+a-2c)\frac{1}{2}(1+a-c)^2 - \frac{1}{6}(1+a-c)^3 + \frac{1}{6}c^3 \\
= \; & \frac{1}{3} + p - c + c^2 + \frac{1}{3}c^3 + a + a^2 + \frac{1}{3}a^3 - 2ac - a^2c
\end{aligned}
$$

where the last step was verified by MATHEMATICA.

The necessary condition with which we shall work is now that twice this value, plus the ex ante expected value of q_i must equal 1:

$$
\begin{aligned}
\frac{2}{3} + 2p - 2c + 2c^2 + \frac{2}{3}c^3 + 2a + 2a^2 + \frac{2}{3}a^3 - 4ac - 2a^2c & \\
+\frac{1}{2}(1-a)^2 - (c-a)^2 &= 1 \Leftrightarrow \\
-\frac{1}{12} - \frac{1}{2}a - \frac{3}{4}a^2 - \frac{1}{3}a^3 + c + ac + a^2c - \frac{1}{2}c^2 - \frac{1}{3}c^3 &= p
\end{aligned}
$$

where the last step was again verified by MATHEMATICA. The constraint that we shall work with when maximizing expected welfare is now that there must be some $p \in [0,1]$ such that the above equation holds. This is the same as the requirement that the left hand side of the above equation is contained in $[0, 1]$. In the following, we denote this expression by $E(a, c)$.

We seek to determine the welfare-maximizing choice of a and c subject to the condition $E(a, c) \in [0, 1]$. We proceed in two steps. We first ask which choices, if any, of $c \in \left[a, \frac{1+a}{2}\right]$ are optimal for given $a \in [0, 1]$. Then we ask which choice of a is best.

Note that for given a welfare is maximized by choosing c as small as possible. The smallest admissible value of c is $c = a$. If for this choice of c we have $E(a, c) \in [0, 1]$, then it is the optimal choice.

$$\begin{aligned} E(a,a) &\in [0,1] \Leftrightarrow \\ -\frac{1}{12} - \frac{1}{2}a - \frac{3}{4}a^2 - \frac{1}{3}a^3 + a + a^2 + a^3 - \frac{1}{2}a^2 - \frac{1}{3}a^3 &\in [0,1] \Leftrightarrow \\ -\frac{1}{12} + \frac{1}{2}a - \frac{1}{4}a^2 + \frac{1}{3}a^3 &\in [0,1]. \end{aligned}$$

Numerically, we can determine using MATHEMATICA that this is the case if and only if

$$a \geq 0.178846.$$

For smaller values of a MATHEMATICA shows that we have: $E(a, a) < 0$. On the other hand, $E(a, \frac{1+a}{2}) > 0$ where $c = \frac{1+a}{2}$ is the largest admissible value of c. The proof is as follows:

$$\begin{aligned} E(a, \frac{1+a}{2}) &= -\frac{1}{12} - \frac{1}{2}a - \frac{3}{4}a^2 - \frac{1}{3}a^3 + \frac{1+a}{2} + a\frac{1+a}{2} \\ &\quad + a^2\frac{1+a}{2} - \frac{1}{2}\left(\frac{1+a}{2}\right)^2 - \frac{1}{3}\left(\frac{1+a}{2}\right)^3 \\ &= \frac{1}{8}\left(2 + a + 12a^2 + 13a^3\right) > 0 \end{aligned}$$

where the simplification in the last step was obtained using MATHEMATICA. By the continuity of $E(a, c)$ in c we can now conclude that there is a smallest $c \in [a, \frac{1+a}{2}]$ such that $E(a, c) = 0$. This c is the optimal choice, given a.

To determine the optimal choice of a, we first prove that the optimal choice of c increases in a. This is obvious for $a \geq 0.178846$. For smaller values of a it follows from the fact that the value of $E(a, c)$ decreases in a. To show this we calculate:

$$\begin{aligned}
\frac{\partial E}{\partial a} &= -\frac{1}{2} - \frac{3}{2}a - a^2 + c + 2ac \\
&\leq -\frac{1}{2} - \frac{3}{2}a - a^2 + \frac{1+a}{2} + 2a\frac{1+a}{2} \\
&= 0.
\end{aligned}$$

As the optimal c is increasing in a, it follows immediately that expected welfare is decreasing in a, assuming that for each a the optimal c is chosen. Therefore, the optimal choice of a is $a = 0$. The corresponding choice of c is the smallest c for which $E(0, c) = 0$. This equation is equivalent to:

$$-\frac{1}{12} + c - \frac{1}{2}c^2 - \frac{1}{3}c^3 = 0.$$

MATHEMATICA shows that there is a unique solution c^* in $[0, 1]$ of this equation, and that it is: $c^* \approx 0.087373$.

We have now solved the relaxed maximization problem, and we complete the argument by constructing functions f_A and f_C that make the optimal f_B incentive compatible. We define f_A as follows:

$$f_A(t_1, t_2) = \begin{cases}
\frac{1}{2} & \text{if } t_1 \leq c^* \text{ and } t_2 \leq c^*, \\
1 - \frac{(1-c^*-t_2)(t_2-c^*)}{2c^*} & \text{if } t_1 \leq c^* \text{ and } c^* \leq t_2 \leq 1 - c^* \\
1 & \text{if } t_1 \leq c^* \text{ and } t_2 > 1 - c^* \\
\frac{(1-c^*-t_1)(t_1-c^*)}{2c^*} & \text{if } c^* < t_1 \leq 1 - c^* \text{ and } t_2 \leq c^*, \\
\frac{1}{2} & \text{if } c^* < t_1 \leq 1 - c^* \text{ and } c^* < t_2 \leq 1 - t_1, \\
0 & \text{otherwise.}
\end{cases}$$

The function f_C is defined symmetrically, and we omit the formal definition. The construction of f_A is shown in Figure 1. In Figure 1 we refer to a function h. We define for every $t \in [0, 1]$:

$$h(t) \equiv \frac{(1 - c^* - t)(t - c^*)}{2c^*}.$$

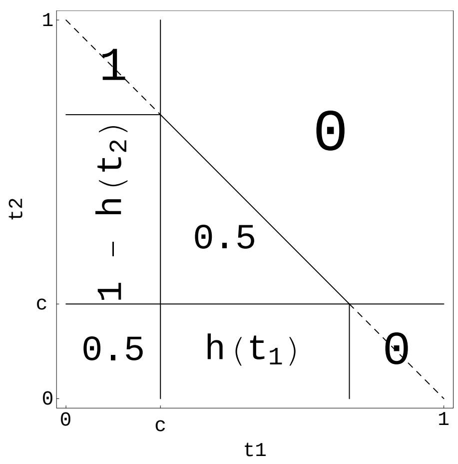

Figure 1: The probability of alternative A under the optimal cropped triangle rule

To check that what we have defined are actually probabilities we need to verify that

$$h(t) \in [0,1] \text{ for all } t \in [c^*, 1-c^*].$$

It is obvious that $h(t)$ is non-negative for all relevant t. Plotting it in MATHEMATICA one can verify that it is never more than 1. We also need to check that the probabilities that we have defined add up to 1 for every type vector. This is obvious.

It remains to verify that these probabilities give rise to the interim probabilities $p_i(t_i)$ that make the decision rule incentive compatible. Clearly, the implied interim probabilities of the compromise $q_i(t_i)$ are monotonically increasing in type t_i. It remains to verify that the interim probabilities of the preferred alternatives are those required by part (ii) of Lemma 3 in the main paper. We have:

$$\begin{aligned}
p_i(t_i) &= p = 0 \text{ when } 1 - c^* \leq t_i \leq 1. \\
p_i(t_i) &= c^* h(t_i) + \frac{1}{2}(1 - c^* - t_i) \\
&= \frac{(1-c^*-t_i)(t_i - c^*)}{2} + \frac{1}{2}(1 - c^* - t_i) \\
&= \frac{(1-c^*-t_i)(1-c^*+t_i)}{2} \\
&= \frac{1}{2}(1-c^*)^2 - \frac{1}{2}(t_i)^2 \text{ when } c^* \leq t_i \leq 1 - c^*.
\end{aligned}$$

In these first two cases we thus obtain the expressions that are required by incentive compatibility and that were derived above. For the remaining, third, case: $0 \leq t_i \leq c^*$, no calculation is needed. The conclusion can be derived from two observations. First, the total probability of the preferred alternative that is assigned by our rule in this case is ex ante the same as required by incentive compatibility. This is because our mechanism clearly has the property that ex ante probabilities add up to 1. Indeed, it also has this property ex post. Thus, the probability assigned to the preferred alternative ex ante if $0 \leq t_i \leq c^*$ is 1 minus the probability assigned to the preferred alternative if $t > c^*$. Moreover, the probabilities assigned to the preferred alternative and that are required by incentive compatibility add up to 1. This is indeed the constraint under which we determined the optimal mechanism. Because for $t_i > c^*$ our mechanism assigns the correct probabilities to the preferred alternative, the same must be true ex ante if $0 \leq t_i \leq c^*$. The second observation is that incentive compatibility requires the probability assigned to the preferred alternative to be constant for $0 \leq t_i \leq c^*$. Our mechanism has this property. Therefore, it must assign exactly the probabilities required by incentive compatibility to the preferred alternative for $0 \leq t_i \leq c^*$.

Appendix B: The Second Best Public Goods Rule as a Cropped Triangle Rule

In this appendix we prove the claim in Section 6 of the main paper that the function f_B that corresponds to the second-best in the public goods problem with equal welfare weights and uniform type distribution can be implemented as an incentive compatible cropped triangle rule. Recall that the second-best public goods decision rule corresponds to a function f_B of the type described in Figure 3 of the main paper with parameters $a = c = 0.25$. Clearly, this rule implies that q_i is increasing in t_i for $i = 1, 2$. By Lemma 3 we therefore have an incentive compatible decision rule if and only if the interim probabilities of the preferred alternatives satisfy:

$$p_i(t_i) = \begin{cases} p_i(1) + \frac{3}{4} - t_i\left(t_i - \frac{1}{2}\right) - \int\limits_{t_i}^{1} \left(s_i - \frac{1}{4}\right) ds_i & \\ = p_i(1) + \frac{1}{2} - \frac{1}{2}\left(t_i\right)^2 & \text{if } t_i \geq 0.25 \\ p_i(1) + \frac{1}{2} - \frac{1}{2} \cdot \left(\frac{1}{4}\right)^2 & \\ = p_i(1) + \frac{15}{32} & \text{if } t_i < 0.25 \end{cases}$$

We can achieve incentive compatibility by defining f_A by:

$$f_A(t_1,t_2) = \begin{cases} 0.5 & \text{if } t_1, t_2 < 0.25 \\ \frac{1}{8} - 2\left(t_1\right)^2 + 2t_1 & \text{if } t_1 \geq 0.25, t_2 < 0.25 \\ \frac{7}{8} + 2\left(t_2\right)^2 - 2t_2 & \text{if } t_1 < 0.25, t_2 \geq 0.25 \\ 0.5 & \text{if } t_1, t_2 \geq 0.25, t_1 + t_2 < 1.25 \\ 0 & \text{if } t_1 + t_2 \geq 1.25 \end{cases}$$

and defining f_C analogously. It is trivial to verify that $f_A(t_1,t_2) \in [0,1]$, $f_B(t_1,t_2) \in [0,1]$ and $f_A(t_1,t_2) + f_B(t_1,t_2) + f_C(t_1,t_2) = 1$ for all $(t_1,t_2) \in [0,1]^2$. It remains to check the incentive compatibility constraint. Note first that

$$p_i(1) = \frac{1}{32}$$

for $i = 1,2$. Therefore, for $t_i \geq 0.25$, we need to check that:

$$p_i(t_i) = \frac{17}{32} - 0.5\left(t_i\right)^2.$$

We calculate:

$$\begin{aligned} p_i(t_i) &= \frac{1}{4}\left(\frac{1}{8} - 2\left(t_i\right)^2 + 2t_i\right) + (1.25 - t_i - 0.25)\frac{1}{2} \\ &= \frac{17}{32} - \frac{1}{2}\left(t_i\right)^2. \end{aligned}$$

For $t_i < 0.25$ we need to check:

$$p_i(t_i) = \frac{1}{2}.$$

We calculate:

$$\begin{aligned} p_i(t_i) &= \frac{1}{4}\cdot\frac{1}{2} + \int_{\frac{1}{4}}^{1} \frac{7}{8} + 2\left(t_j\right)2 - 2t_j dt_j \\ &= \frac{1}{2}. \end{aligned}$$