IMA Journal of Numerical Analysis (2016) Page 1 of 16
doi:10.1093/imanum/drn000

$L^2(H^1_\gamma)$ Finite Element Convergence for Degenerate Isotropic Hamilton–Jacobi–Bellman Equations

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[Received on 16 September 2016]

In this paper we study the convergence of monotone $P1$ finite element methods for fully nonlinear Hamilton–Jacobi–Bellman equations with degenerate, isotropic diffusions. The main result is strong convergence of the numerical solutions in a weighted Sobolev space $L^2(H^1_\gamma(\Omega))$ to the viscosity solution without assuming uniform parabolicity of the HJB operator.

Keywords: finite element methods, degenerate partial differential equations, Hamilton–Jacobi–Bellman equations, viscosity solutions.

1. Introduction

Hamilton-Jacobi-Bellman (HJB) equations characterise the value functions of optimal control problems. For a wide range of control problems one can compute optimal control policies from the *partial derivatives* of the value function.

An important tool in the analysis of HJB equations and their numerical approximations is the concept of viscosity solutions. Its definition is based on sign information on function values of candidate solutions, leading typically to proofs of L^∞ convergence of numerical methods, cf. Barles & Souganidis (1991). It is more difficult to prove convergence in other norms if solely the concept of viscosity solutions is used.

The approach with weak solution, familiar from semilinear differential equations, in the context of Hamilton-Jacobi-Bellman equations is delicate because often uniqueness cannot be ensured. However, we believe that combining the notions of viscosity and weak solution is attractive for numerical analysis: the former to deal with uniqueness and the latter to study convergence of partial derivatives.

In Jensen & Smears (2013) the uniform convergence of $P1$ finite element approximations to the viscosity solutions of isotropic, degenerate parabolic HJB equations was shown. In addition $L^2(H^1)$ convergence was demonstrated, under the assumption that the HJB equation is uniformly parabolic. In this paper we remove the assumption of uniform parabolicity and verify that strong convergence in weighted $L^2(H^1_\gamma)$ spaces can be maintained, see Theorem 7.1 below. Also a condition in (Jensen & Smears , 2013, Assumption 7.1) that the d-dimensional Lebesgue measure of the boundary of the zero level set of the value function has to vanish is not needed anymore.

Our approach uses coercivity properties of the HJB operator. An alternative technique to control derivative terms is proposed in Smears & Süli (2014), where HJB equations satisfying Cordes condi-

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tions are discretised. A general review of the recent advances in the discretisation of fully nonlinear equations is given in Feng, Glowinsky & Neilan (2013).

In Section 2 we introduce the Bellman equation, while in Section 3 the numerical method is defined. Section 4 is concerned with uniform convergence to the value function. In Section 5 a nonlinear projection operator is constructed and analysed, which preserves positivity and boundary conditions of the viscosity solution. Section 6 is concerned with the coercivity of the continuous and discrete Bellman operators. The main result of strong convergence in a weighted Sobolev space is proved in Section 7. Finally, in Section 8 assumptions of the prior analysis are translated into concrete parameter values for the method of artificial viscosity.

2. Problem statement

Let Ω be a bounded polygonal domain in $\mathbb{R}^d$, $d \geqslant 2$. Let A be a compact metric space and let

$$A \to C(\overline{\Omega}) \times C(\overline{\Omega}, \mathbb{R}^d) \times C(\overline{\Omega}) \times C(\overline{\Omega}),\ \alpha \mapsto (a^\alpha, b^\alpha, c^\alpha, f^\alpha)$$

be continuous, such that the families of functions $\{a^\alpha\}_{\alpha\in A}$, $\{b^\alpha\}_{\alpha\in A}$, $\{c^\alpha\}_{\alpha\in A}$ and $\{f^\alpha\}_{\alpha\in A}$ are equicontinuous. Consider the bounded linear operators

$$L^\alpha :\ H^2(\Omega) \to L^2(\Omega),\ w \mapsto -a^\alpha \Delta w + b^\alpha \cdot \nabla w + c^\alpha w, \qquad \alpha \in A.$$

We assume that $a^\alpha \geqslant 0$, i.e. that all L^α are of degenerate elliptic. Similarly, we require that $c^\alpha \geqslant 0$. Furthermore, suppose that pointwise $f^\alpha \geqslant 0$. Observe that

$$\sup_{\alpha\in A} \| (a^\alpha, b^\alpha, c^\alpha, f^\alpha) \|_{C(\overline{\Omega})\times C(\overline{\Omega},\mathbb{R}^d)\times C(\overline{\Omega})\times C(\overline{\Omega})} < \infty, \tag{2.1}$$

and also $\sup_{\alpha\in A} \|L^\alpha\|_{H^2(\Omega)\to L^2(\Omega)} < \infty$. Let the final-time data $v_T \in C(\overline{\Omega})$ be non-negative, that is $v_T \geqslant 0$ on $\overline{\Omega}$, and let v_T satisfy homogeneous boundary conditions on $\partial\Omega$. For smooth w, let

$$Hw := \sup_\alpha (L^\alpha w - f^\alpha),$$

where the supremum is applied pointwise. The HJB equation considered is

$$-\partial_t v + Hv = 0 \qquad \text{in } \Omega_T := (0,T)\times\Omega, \tag{2.2a}$$

$$v = 0 \qquad \text{on } (0,T)\times\partial\Omega, \tag{2.2b}$$

$$v = v_T \qquad \text{on } \{T\}\times\overline{\Omega}. \tag{2.2c}$$

DEFINITION 2.1 (Barles & Perthame (1988); Fleming & Soner (2006)) An upper semi-continuous (lower semi-continuous) function $v : [0,T]\times\overline{\Omega} \to \mathbb{R}$ is a viscosity subsolution (supersolution) of

$$-\partial_t v + Hv = 0 \quad \text{on } \Omega_T, \tag{2.3}$$

if for any $w \in C^\infty(\mathbb{R}\times\mathbb{R}^d)$ such that $v - w$ has a strict local maximum (minimum) at $(t,x) \in (0,T)\times\Omega$ with $v(t,x) = w(t,x)$, gives $-\partial_t w(t,x) + Hw(t,x) \leqslant 0$, (greater than or equal to 0). If $v \in C([0,T]\times\overline{\Omega})$ is both a viscosity subsolution and a supersolution of equation (2.3), then v is called a viscosity solution.

The viscosity solution of (2.2) is understood to be a viscosity solution of the PDE (2.2a), in the sense of Definition 2.1, that satisfies pointwise the boundary conditions (2.2b) and (2.2c). Owing to the definition of viscosity solutions, v is a continuous function.

3. The numerical scheme

Let V_i, $i \in \mathbb{N}$, be a sequence of piecewise linear, conforming, shape-regular finite element spaces with nodes y_i^ℓ. Here ℓ is the index ranging over the nodes of the finite element mesh $\mathscr{T}_i$. Let $V_i^0 \subset V_i$ be the subspace of functions which satisfy homogeneous Dirichlet conditions on $\partial\Omega$. It is convenient to assume that $y_i^\ell \in \Omega$ for $\ell \leqslant N_i := \dim V_i^0$; i.e. the index ℓ first ranges over internal nodes and then over boundary nodes. The associated hat functions are denoted ϕ_i^ℓ; that is $\phi_i^\ell \in V_i$ and $\phi_i^\ell(y_i^l) = 1$ if $l = \ell$, otherwise $\phi_i^\ell(y_i^l) = 0$. Set $\hat{\phi}_i^\ell := \phi_i^\ell / \|\phi_i^\ell\|_{L^1(\Omega)}$. Thus, the ϕ_i^ℓ are normalised in the L^∞ norm whilst the $\hat{\phi}_i^\ell$ are normalised in the L^1 norm. The mesh size, i.e. the largest diameter of an element, is denoted Δx_i. It is assumed that $\Delta x_i \to 0$ as $i \to \infty$.

Let h_i be the (uniform) time step size used in conjunction with V_i, with $T/h_i \in \mathbb{N}$, and let s_i^k be the kth time step at the refinement level i. It is assumed that $h_i \to 0$ as $i \to \infty$. The set of time steps is $S_i := \{s_i^k : k = 0, \dots, T/h_i\}$. Let the ℓth entry of $d_i w(s_i^k, \cdot)$ be

$$(d_i w(s_i^k, \cdot))_\ell = \frac{w(s_i^{k+1}, y_i^\ell) - w(s_i^k, y_i^\ell)}{h_i}.$$

For each α and i, we introduce operators E_i^α and I_i^α to break L^α into an explicit and implicit part:

$$E_i^\alpha : H^2(\Omega) \to L^2(\Omega),\ w \mapsto -\bar{a}_i^\alpha \Delta w + \bar{b}_i^\alpha \cdot \nabla w + \bar{c}_i^\alpha w,$$
$$I_i^\alpha : H^2(\Omega) \to L^2(\Omega),\ w \mapsto -\bar{\bar{a}}_i^\alpha \Delta w + \bar{\bar{b}}_i^\alpha \cdot \nabla w + \bar{\bar{c}}_i^\alpha w,$$

with continuous

$$\begin{aligned} A &\to C(\overline{\Omega}) \times C(\overline{\Omega}, \mathbb{R}^d) \times C(\overline{\Omega}), \quad \alpha \mapsto (\bar{a}_i^\alpha, \bar{b}_i^\alpha, \bar{c}_i^\alpha), \\ A &\to C(\overline{\Omega}) \times C(\overline{\Omega}, \mathbb{R}^d) \times C(\overline{\Omega}), \quad \alpha \mapsto (\bar{\bar{a}}_i^\alpha, \bar{\bar{b}}_i^\alpha, \bar{\bar{c}}_i^\alpha). \end{aligned} \tag{3.1}$$

It is required that $\bar{c}_i^\alpha$ and $\bar{\bar{c}}_i^\alpha$ are non-negative and that there is $C \in \mathbb{R}$ such that

$$\|\bar{c}_i^\alpha\|_{L^\infty} + \|\bar{\bar{c}}_i^\alpha\|_{L^\infty} \leqslant C, \qquad \forall i \in \mathbb{N},\ \forall \alpha \in A. \tag{3.2}$$

Also, find for each i a non-negative f_i^α which approximates f^α: $f_i^\alpha \approx f^\alpha$. The conceptual statements $L^\alpha \approx E_i^\alpha + I_i^\alpha$ and $f^\alpha \approx f_i^\alpha$ are made precise as follows:

ASSUMPTION 3.1 For all sequences of nodes $(y_i^\ell)_{i \in \mathbb{N}}$, where in general $\ell = \ell(i)$ depends on i:

$$\begin{aligned} \lim_{i\to\infty} \sup_{\alpha \in A} \Big(\big\| a^\alpha - \big(\bar{a}_i^\alpha(y_i^\ell) + \bar{\bar{a}}_i^\alpha(y_i^\ell)\big) \big\|_{L^\infty(\operatorname{supp} \hat{\phi}_i^\ell)} + \big\| b^\alpha - \big(\bar{b}_i^\alpha + \bar{\bar{b}}_i^\alpha\big) \big\|_{L^\infty(\Omega, \mathbb{R}^d)} \\ + \big\| c^\alpha - \big(\bar{c}_i^\alpha + \bar{\bar{c}}_i^\alpha\big) \big\|_{L^\infty(\Omega)} + \big\| f^\alpha - f_i^\alpha \big\|_{L^\infty(\Omega)} \Big) = 0. \end{aligned}$$

Let $\langle \cdot, \cdot \rangle$ denote the standard inner product for both of the spaces $L^2(\Omega)$ and $L^2(\Omega, \mathbb{R}^d)$, the two cases being distinguished by the arguments of the inner product. Consider the following discretisation of E_i^α and I_i^α by operators E_i^α and I_i^α that map $H^1(\Omega)$ to $\mathbb{R}^{N_i}$: for $w \in H^1(\Omega)$, $\ell \in \{1, \dots, N_i = \dim V_i^0\}$,

$$(\mathsf{E}_i^\alpha w)_\ell := \bar{a}_i^\alpha(y_i^\ell) \langle \nabla w, \nabla \hat{\phi}_i^\ell \rangle + \langle \bar{b}_i^\alpha \cdot \nabla w + \bar{c}_i^\alpha w, \hat{\phi}_i^\ell \rangle, \tag{3.3a}$$

$$(\mathsf{I}_i^\alpha w)_\ell := \bar{\bar{a}}_i^\alpha(y_i^\ell) \langle \nabla w, \nabla \hat{\phi}_i^\ell \rangle + \langle \bar{\bar{b}}_i^\alpha \cdot \nabla w + \bar{\bar{c}}_i^\alpha w, \hat{\phi}_i^\ell \rangle, \tag{3.3b}$$

$$(\mathsf{F}_i^\alpha)_\ell := \langle f_i^\alpha, \hat{\phi}_i^\ell \rangle. \tag{3.3c}$$

Throughout this work, we identify E_i^α and I_i^α, when restricted to V_i, with their matrix representations with respect to the nodal basis $\{\phi_i^\ell\}_\ell$. Under this basis, the nodal evaluation operator $w \mapsto w(y_i^\ell)$ corresponds to the identity matrix Id.

We now define the numerical scheme for (2.2). Obtain the numerical solution $v_i(T,\cdot) \in V_i^0$ by nodal interpolation of v_T. Then, for each $k \in \{0,\ldots,T/h_i - 1\}$, the numerical solution $v_i(s_i^k,\cdot) \in V_i^0$ is defined inductively by

$$-d_i v_i(s_i^k,\cdot) + \sup_{\alpha \in A} \big(\mathsf{E}_i^\alpha v_i(s_i^{k+1},\cdot) + \mathsf{I}_i^\alpha v_i(s_i^k,\cdot) - \mathsf{F}_i^\alpha \big) = 0. \tag{3.4}$$

4. Review of monotonicity and uniform convergence

The proof of gradient convergence in weighted spaces, given in Section 7, is based on the non-negativity of numerical solutions and uniform convergence to the viscosity solution.

ASSUMPTION 4.1 For each $\alpha \in A$, assume that E_i^α, restricted to V_i, has non-positive off-diagonal entries. Let h_i be small enough so that $h_i\mathsf{E}_i^\alpha - \mathsf{Id}$ is monotone for every α, i.e. so that all entries of all $h_i\mathsf{E}_i^\alpha - \mathsf{Id}$ are non-positive. For each α, suppose that for all $v \in V_i$ such that v has a non-positive local minimum at the internal node y_i^ℓ, we have $(\mathsf{I}_i^\alpha v)_\ell \leqslant 0$.

It was shown in (Jensen & Smears , 2013, Theorem 3.1) that Assumption 4.1 implies the existence of a unique numerical solution v_i of (3.4) and that v_i is non-negative.

Let $t = \vartheta s_i^k + (1-\vartheta)s_i^{k+1} \in [s_i^k, s_i^{k+1}]$ lie between two time steps, $\vartheta \in [0,1]$. Then we interpret $v_i(t,\cdot)$ as the linear interpolant between $v_i(s_i^k,\cdot)$ and $v_i(s_i^{k+1},\cdot)$:

$$v_i(t,\cdot) = \vartheta v_i(s_i^k,\cdot) + (1-\vartheta) v_i(s_i^{k+1},\cdot). \tag{4.1}$$

ASSUMPTION 4.2 The Hamilton–Jacobi–Bellman problem (2.2) has a unique viscosity solution v and

$$\lim_{i\to\infty} \|v_i - v\|_{L^\infty(\Omega_T)} = 0. \tag{4.2}$$

In Jensen & Smears (2013) it was demonstrated that Assumption 4.2, that is uniform convergence, holds if the following conditions are satisfied:

1. *Orthogonal projection:* Suppose there exist linear mappings $P_i : C([0,T], H^1(\Omega)) \to [0,T] \times V_i$ which satisfy for all $\hat{\phi}_i^\ell \in V_i^0$

$$\langle \nabla P_i w(t,\cdot), \nabla \hat{\phi}_i^\ell \rangle = \langle \nabla w(t,\cdot), \nabla \hat{\phi}_i^\ell \rangle, \quad \forall t \in [0,T], \tag{4.3}$$

and there is a constant $C \geqslant 0$ such that for every $w \in C^\infty(\mathbb{R}^d)$ and $i \in \mathbb{N}$,

$$\|P_i w\|_{W^{1,\infty}(\Omega)} \leqslant C\,\|w\|_{W^{1,\infty}(\Omega)} \quad \text{and} \quad \lim_{i\to\infty} \|P_i w - w\|_{W^{1,\infty}(\Omega)} = 0. \tag{4.4}$$

2. *Boundary control:* For each $\alpha \in A$, we define $v_i^\alpha : S_i \to V_i^0$ to be the numerical solution of the linear evolution problem associated to the control α with homogeneous Dirichlet conditions: $v_i^\alpha(T,\cdot) = v_i(T,\cdot)$ is the interpolant of v_T, and for each $k \in \{0,\ldots,T/h_i - 1\}$,

$$(h_i \mathsf{I}_i^\alpha + \mathsf{Id})\, v_i^\alpha(s_i^k,\cdot) = -(h_i \mathsf{E}_i^\alpha - \mathsf{Id})\, v_i^\alpha(s_i^{k+1},\cdot) + h_i \mathsf{F}_i^\alpha. \tag{4.5}$$

Suppose that for each $(t,x) \in [0,T] \times \partial\Omega$

$$\inf_{\alpha \in A} \sup_{(s_i^k, y_i^\ell) \to (t,x)} \limsup_{i \to \infty} v_i^\alpha(s_i^k, y_i^\ell) = 0,$$

where the supremum is taken over the set of all sequences of nodes which converge to (t,x).

3. *Comparison:* Let $\overline{v}$ be a lower semi-continuous supersolution with $\overline{v}(T,\cdot) = v_T$ and $\overline{v}|_{[0,T]\times\partial\Omega} = 0$. Similarly, let $\underline{v}$ be an upper semi-continuous subsolution with $\underline{v}|_{[0,T]\times\partial\Omega} = 0$ and $\underline{v}(T,\cdot) = v_T$. Then $\underline{v} \leqslant \overline{v}$.

5. Projection into the approximation space

For shorthand, let $W = W^{1,d+1+\varepsilon}(\Omega_T) \cap L^2((0,T), H_0^1(\Omega)) \subset C(\overline{\Omega_T})$ with $\varepsilon > 0$. We also use the discrete spaces

$$W_i := \{v \in C([0,T], V_i^0) : v|_{[s_i^k, s_i^{k+1}]\times\Omega} \text{ is affine in time}\},$$

which means that functions in W_i have the form of (4.1) between two time-steps. Observe that $W_i \subset W$ for all $i \in \mathbb{N}$.

We introduce the cut-off operation

$$C_i : W \to W,\ w \mapsto \max\{w - \|v - v_i\|_{L^\infty(\Omega_T)}, 0\}.$$

Furthermore, we denote the nodal interpolant from W onto W_i by $\mathscr{I}_i$, meaning that $\mathscr{I}_i w(s_i^k, y_i^\ell) = w(s_i^k, y_i^\ell)$ at all time steps s_i^k and spatial nodes y_i^ℓ. Finally we define $Q_i = \mathscr{I}_i \circ C_i$. Thus Q_i is a mapping of the type $W \to W_i$. Observe that $Q_i v \in W_i$ satisfies homogeneous boundary conditions. Furthermore, from $C_i v \leqslant v_i$ and the monotonicity of the nodal interpolation operator it follows that $Q_i v \leqslant v_i$. The stability of the max operation and $\mathscr{I}_i$ gives, cf. (Ern & Guermond , 2004, Corollary 1.110) with $\ell = 0$ and $p = d+1+\varepsilon$,

$$\begin{aligned} \|Q_i v\|_W &\lesssim \|v\|_W, \\ \|Q_i v(T,\cdot)\|_{L^\infty(\Omega)} &\lesssim \|v(T,\cdot)\|_{L^\infty(\Omega)}. \end{aligned} \tag{5.1}$$

LEMMA 5.1 Suppose that $v \in W$. The sequence $Q_i v$ consists of non-negative functions, satisfying homogeneous Dirichlet boundary conditions and $Q_i v \leqslant v_i$. Moreover, the sequence converges strongly in W and $L^\infty(\Omega_T)$ to v.

Proof. The convergence of $Q_i v$ to v in W remains. We break the proof into two steps by means of the triangle inequality:

$$\|Q_i v - v\|_W \leqslant \|Q_i v - \mathscr{I}_i v\|_W + \|\mathscr{I}_i v - v\|_W.$$

Step 1. Observe that

$$v - C_i v = \min\{\|v - v_i\|_{L^\infty(\Omega_T)}, v\}.$$

Thus $\|v - C_i v\|_{L^\infty(\Omega_T)} \to 0$ as $i \to \infty$. Moreover, denoting the gradient in time and space by $\nabla_{(t,x)}$,

$$\|\nabla_{(t,x)}(v - C_i v)\|_{L^{d+1+\varepsilon}(\Omega_T)} = \|\nabla_{(t,x)} v\|_{L^{d+1+\varepsilon}(\Gamma_i)},$$

where $\Gamma_i = \{x \in \Omega_T : v(x) \in (0, \|v - v_i\|_{L^\infty(\Omega_T)})\}$ as we have $\nabla_{(t,x)}(v - C_i v) = 0$ in $\Omega_T \setminus \Gamma_i$. Because $\bigcap_i \Gamma_i = \emptyset$ it follows that $\|v - C_i v\|_W \to 0$ as $i \to \infty$. Hence, owing to (5.1), $\|Q_i v - \mathscr{I}_i v\|_W = \|\mathscr{I}_i(C_i v - v)\|_W$ vanishes as $i \to \infty$.

Step 2. Let $\delta > 0$. Recalling the density of smooth functions in W there is a $v_\delta \in C^\infty(\overline{\Omega_T})$ such that $\|v_\delta - v\|_W \leqslant \delta$. For $t \in [0,T]$ we know (Ern & Guermond , 2004, Corollary 1.110) that

$$\|\mathscr{I}_i v_\delta(t,\cdot) - v_\delta(t,\cdot)\|_{W^{1,d+1+\varepsilon}(\Omega)} \lesssim \Delta x_i \|v_\delta(t,\cdot)\|_{W^{2,d+1+\varepsilon}(\Omega)}.$$

and similarly for $x \in \mathscr{T}_i$

$$\|\mathscr{I}_i v_\delta(\cdot,x) - v_\delta(\cdot,x)\|_{W^{1,d+1+\varepsilon}([0,T])} \lesssim h_i \|v_\delta(\cdot,x)\|_{W^{2,d+1+\varepsilon}([0,T])}.$$

We conclude that for i sufficiently large

$$\|\mathscr{I}_i v - v\|_W \leqslant \|\mathscr{I}_i(v - v_\delta)\|_W + \|\mathscr{I}_i v_\delta - v_\delta\|_W + \|v_\delta - v\|_W \lesssim \delta,$$

using once again (5.1). □

We shall also require a super-approximation result on nodal interpolation in weighted Sobolev spaces. We cite Theorem 2.1 in Demlow, Guzman & Schatz (2011), with the notation adapted to this paper.

LEMMA 5.2 Let T be a mesh element with diameter $\Delta x \leqslant 1$ and $\mathscr{I}$ be the nodal interpolation operator of T. Given $g \in W^{2,\infty}(T)$ there exists a value K, depending on $\|g\|_{W^{2,\infty}(T)}$ and the shape regularity of T and the dimension d, such that for *affine* functions w

$$\|g^2 w - \mathscr{I}(g^2 w)\|_{H^1(T)} \leqslant K \Delta x \big(\|\nabla(gw)\|_{L^2(T)} + \|w\|_{L^2(T)}\big).$$

6. Coercivity properties of the Hamilton–Jacobi–Bellman operator and its discretisation

Owing to the non-negativity of v, for each $\hat{\alpha} \in A$, we formally have for the exact solution

$$\partial_t v + \sup_\alpha (L^\alpha v - f^\alpha) = 0 \implies \partial_t v + L^{\hat{\alpha}} v \leqslant f^{\hat{\alpha}} \implies \langle \partial_t v, v\rangle + \langle L^{\hat{\alpha}} v, v\rangle \leqslant \langle f^{\hat{\alpha}}, v\rangle. \qquad (6.1)$$

Furthermore, if there exists an $\hat{\alpha} \in A$ such that $a^{\hat{\alpha}} \in W^{2,\infty}(\Omega)$ and $c^{\hat{\alpha}} - \frac{1}{2}(\nabla \cdot b^{\hat{\alpha}} + \Delta a^{\hat{\alpha}}) \geqslant 0$ we have for $w \in H^1_0(\Omega)$

$$\langle L^{\hat{\alpha}} w, w\rangle = \langle a^{\hat{\alpha}} \nabla w, \nabla w\rangle + \langle (c^{\hat{\alpha}} - \tfrac{1}{2}(\nabla \cdot b^{\hat{\alpha}} + \Delta a^{\hat{\alpha}}))w, w\rangle, \qquad (6.2)$$

thus giving in combination with (6.1) control on v in a Sobolev space weighted by $a^{\hat{\alpha}}$. We intend to build the gradient convergence proof upon a bound similar to (6.1), with the differential operator replaced by its discretisation and v by $v_i - Q_i v$.

Fix an arbitrary $\alpha \in A$. It is useful to view E_i^α and I_i^α as bilinear forms on $H^1(\Omega) \times V_i$. Functions $u \in V_i$ have the nodal representation

$$u(y) = \sum_\ell u(y_i^\ell)\, \phi_i^\ell(y).$$

To test with functions other than $\hat{\phi}_i^\ell$ we introduce the following bilinear form as a partially discrete operation: for $w \in H^1(\Omega)$ and $u \in V_i$

$$\langle\langle \mathsf{E}_i^\alpha w, u\rangle\rangle := \sum_\ell u(y_i^\ell)\big(\bar{a}_i^\alpha(y_i^\ell)\langle \nabla w, \nabla \phi_i^\ell\rangle + \langle \bar{b}_i^\alpha \cdot \nabla w + \bar{c}_i^\alpha\, w, \phi_i^\ell\rangle\big).$$

We use the corresponding interpretation for $\langle\langle \mathsf{I}_i^\alpha w, u\rangle\rangle$ and also

$$\langle\langle w,u\rangle\rangle = \langle\langle \mathsf{Id}\, w,u\rangle\rangle = \sum_\ell w(y_i^\ell)\, u(y_i^\ell) \|\phi_i^\ell\|_{L^1(\Omega)},$$
$$\langle\langle \mathsf{F}_i^\alpha, u\rangle\rangle = \sum_\ell u(y_i^\ell)\, \langle f_i^\alpha, \phi_i^\ell\rangle = \langle f_i^\alpha, u\rangle.$$

Let $H^1_\gamma(\Omega)$ be the closure of $C_0^\infty(\Omega)$ in the norm

$$\|v\|_\gamma^2 := \int_\Omega v^2 \gamma \,\mathrm{d}x + \int_\Omega |\nabla v|^2 \gamma \,\mathrm{d}x,$$

where $\gamma : \Omega \to \mathbb{R}$ is a non-negative $L^\infty(\Omega)$ function. We write $H^1_\alpha(\Omega)$ as abbreviation of $H^1_{a^\alpha}(\Omega)$ and $H^1_i(\Omega)$ for $H^1_{\gamma_i}(\Omega)$ where γ_i is a weight depending on $i \in \mathbb{N}$.

We now formulate a discrete analogue of (6.1) with (6.2): Consider that there exists an $\alpha \in A$ and weights γ_i and a $C' > 0$ such that for all $i \in \mathbb{N}$

$$\begin{aligned}
&|w|^2_{L^2((0,T),H^1_i(\Omega))} \\
&\lesssim \sum_{k=0}^{\frac{T}{h_i}-1} \Big(\Big\langle\!\Big\langle \big(h_i \mathsf{E}_i^\alpha - \mathsf{Id}\big) w(s_i^{k+1},\cdot) + \big(h_i \mathsf{I}_i^\alpha + \mathsf{Id}\big) w(s_i^k,\cdot), w(s_i^k,\cdot)\Big\rangle\!\Big\rangle \Big) \\
&\qquad + \tfrac{1}{2} \langle\langle w(T,\cdot), w(T,\cdot)\rangle\rangle + C' \|w(T,\cdot)\|^2_{H^1(\Omega)} \qquad (6.3)\\
&\overset{(*)}{=} \sum_{k=0}^{\frac{T}{h_i}-1} \Big(h_i \Big\langle\!\Big\langle \mathsf{E}_i^\alpha w(s_i^{k+1},\cdot) + \mathsf{I}_i^\alpha w(s_i^k,\cdot), w(s_i^k,\cdot)\Big\rangle\!\Big\rangle + \tfrac{1}{2} \langle\langle w(s_i^{k+1},\cdot) - w(s_i^k,\cdot), w(s_i^{k+1},\cdot) - w(s_i^k,\cdot)\rangle\rangle \Big) \\
&\qquad + \tfrac{1}{2} \langle\langle w(0,\cdot), w(0,\cdot)\rangle\rangle + C' \|w(T,\cdot)\|^2_{H^1(\Omega)}
\end{aligned}$$

for all $w \in W_i$ with $w \geqslant 0$ and $i \in \mathbb{N}$, where $(*)$ is a simple reformulation in terms of a telescope sum and

$$|w|^2_{L^2((0,T),H^1_i(\Omega))} = \int_0^T \int_\Omega |\nabla w|^2 \gamma_i \,\mathrm{d}x\,\mathrm{d}t.$$

The bound (6.3) is a property of the numerical scheme, which should in spirit be derived analogously to (6.1) and (6.2), once the parameters of the scheme are selected suitably. In the following Example 6.1 this derivation is shown for the fully implicit setting. In Section 8.2 the bound is established for a wider range of explicit-implicit methods.

EXAMPLE 6.1 Suppose that there is an $\alpha \in A$ such that $\sqrt{a^\alpha} \in W^{2,\infty}(\Omega)$ and $c^\alpha - \frac{1}{2}(\nabla \cdot b^\alpha + \Delta a^\alpha) \geqslant 0$. Choosing a fully implicit scheme with $I_i^\alpha = L^\alpha + 2K\,\Delta x_i\,(\|\sqrt{a^\alpha}\|_{W^{1,\infty}(\Omega)} + 1)\,\Delta$ and $E^\alpha = 0$, the highest order term in $\langle\langle \mathsf{I}_i^\alpha w, w\rangle\rangle$ is at time s_i^k:

$$\sum_\ell w(s_i^k, y_i^\ell)\bar{\bar{a}}_i^\alpha(y_i^\ell) \langle \nabla w(s_i^k,\cdot), \nabla \phi_i^\ell\rangle = \langle \nabla w(s_i^k,\cdot), \nabla \mathscr{I}_i(\bar{\bar{a}}_i^\alpha(s_i^k,\cdot) w(s_i^k,\cdot))\rangle \qquad (6.4)$$

with $w \in W_i$ as weight and numerical diffusion coefficient

$$\gamma_i := \bar{\bar{a}}_i^\alpha = a^\alpha + 2K\,\Delta x_i\,(\|\sqrt{a^\alpha}\|_{W^{1,\infty}(\Omega)} + 1). \qquad (6.5)$$

According to Lemma 5.2, for i sufficiently large,

$$\begin{aligned}|\langle\nabla w,\nabla\mathscr{I}_i(a^\alpha w)\rangle-\langle\nabla w,\nabla(a^\alpha w)\rangle| &\leqslant \|\nabla w\|_{L^2(\Omega)}\cdot\|\mathscr{I}_i(a^\alpha w)-a^\alpha w\|_{H^1(\Omega)}\\ &\leqslant K\,\Delta x_i\,\|\nabla w\|_{L^2(\Omega)}\big(\|\nabla(\sqrt{a^\alpha}w)\|_{L^2(\Omega)}+\|w\|_{L^2(\Omega)}\big)\\ &\leqslant K\,\Delta x_i\,(\|\sqrt{a^\alpha}\|_{W^{1,\infty}(\Omega)}+1)\,\|w\|^2_{H^1(\Omega)}.\end{aligned}\tag{6.6}$$

Therefore, adding and subtracting $\langle\nabla w,\nabla(a^\alpha w)\rangle$ from (6.4), the triangle inequality and (6.6) give

$$\begin{aligned}\langle\!\langle \mathsf{I}_i^\alpha w,w\rangle\!\rangle &= \big(\langle\nabla w,\nabla\mathscr{I}_i(a^\alpha w)-\langle\nabla w,\nabla(a^\alpha w)\rangle\big)+\langle\nabla w,\nabla(a^\alpha w)\rangle\\ &\quad+2K\,\Delta x_i\,(\|\sqrt{a^\alpha}\|_{W^{1,\infty}(\Omega)}+1)\,\langle\nabla w,\nabla w\rangle+\langle b^\alpha\cdot\nabla w+c^\alpha\,w,w\rangle\\ &\geqslant \langle L^\alpha w,w\rangle+K\,\Delta x_i\,(\|\sqrt{a^\alpha}\|_{W^{1,\infty}(\Omega)}+1)\,\langle\nabla w,\nabla w\rangle\geqslant\frac{1}{2}\,|w|^2_{H_i^1(\Omega)},\end{aligned}$$

implying (6.3) as the reformulation $(*)$ shows. We used here that $\bar{\bar{a}}_i^\alpha-a^\alpha$ is constant and therefore unaffected by nodal interpolation. □

Due to the definition of the numerical method and the non-negativity of the v_i, if (6.3) holds then

$$\begin{aligned}|v_i|^2_{L^2(H_i^1)} &\lesssim \sum_{k=0}^{\frac{T}{h_i}-1}\Big(\big\langle\!\big\langle\big(h_i\mathsf{E}_i^\alpha-\mathsf{Id}\big)v_i(s_i^{k+1},\cdot)+\big(h_i\mathsf{I}_i^\alpha+\mathsf{Id}\big)v_i(s_i^k,\cdot),v_i(s_i^k,\cdot)\big\rangle\!\big\rangle\Big)\\ &\qquad+\tfrac{1}{2}\langle\!\langle v_i(T,\cdot),v_i(T,\cdot)\rangle\!\rangle+C'\|v_i(T,\cdot)\|^2_{H^1(\Omega)}\\ &\leqslant \sum_{k=0}^{\frac{T}{h_i}-1}\big\langle\!\big\langle h_i\mathsf{F}_i^\alpha,v_i(s_i^k,\cdot)\big\rangle\!\big\rangle+\tfrac{1}{2}\langle\!\langle v_i(T,\cdot),v_i(T,\cdot)\rangle\!\rangle+C'\|v_i(T,\cdot)\|^2_{H^1(\Omega)}\\ &\lesssim (T\,\|f_i^\alpha\|_{L^1(\Omega)}+1)\,\|v_i\|_{L^\infty([0,T]\times\Omega)}+C'\|v_i(T,\cdot)\|^2_{H^1(\Omega)},.\end{aligned}\tag{6.7}$$

Using stability of nodal interpolation to bound $\|v_i(T,\cdot)\|_{H^1(\Omega)}$ in terms of the final time data, we see that (6.3) guarantees stability in the $L^2(H_i^1)$-norm.

7. Weighted gradient convergence

The following theorem is the main result of the paper. It lists three assumptions: the first, on the second derivatives of $\sqrt{\bar{a}_i^\alpha}$ and $\sqrt{\bar{\bar{a}}_i^\alpha}$, is needed for the application of Lemma 5.2, the second for (7.5) and the third in (7.2).

THEOREM 7.1 Suppose the value function v belongs to the space W and there is an $\alpha\in A$ and weights $\gamma,\gamma_i\in L^\infty(\Omega)$, $i\in\mathbb{N}$, such that

1. $\|\sqrt{\bar{a}_i^\alpha}\|_{W^{2,\infty}(\Omega)}$ and $\|\sqrt{\bar{\bar{a}}_i^\alpha}\|_{W^{2,\infty}(\Omega)}$ are uniformly bounded in i,
2. there is a $C>0$ such that $\bar{a}_i^\alpha\leqslant C\gamma_i$ and $\bar{\bar{a}}_i^\alpha\leqslant C\gamma_i$ for all $i\in\mathbb{N}$,
3. the coercivity condition (6.3) and $\gamma\lesssim\gamma_i$ are satisfied for large i.

Then the numerical solutions converge to the viscosity solution v strongly in $L^2([0,T],H^1_\gamma(\Omega))$.

Proof. *Step 1:* Let us assume for a moment that the approximations $Q_i v \in W_i$ satisfy

$$\sum_{k=0}^{\frac{T}{h_i}-1} \left\langle\!\!\left\langle \big(h_i \mathsf{E}_i^\alpha - \mathsf{Id}\big) Q_i v(s_i^{k+1},\cdot) + \big(h_i \mathsf{I}_i^\alpha + \mathsf{Id}\big) Q_i v(s_i^k,\cdot), (v_i - Q_i v)(s_i^k,\cdot) \right\rangle\!\!\right\rangle \to 0. \tag{7.1}$$

We discuss the validity of (7.1) in Step 2 below.

With $\xi^k = v_i(s_i^k,\cdot) - Q_i v(s_i^k,\cdot)$,

$$\begin{aligned}
&|v_i - Q_i v|^2_{L^2(H_i^1)} \\
&\overset{(6.3)}{\lesssim} \sum_{k=0}^{\frac{T}{h_i}-1} \left\langle\!\!\left\langle \big(h_i \mathsf{E}_i^\alpha - \mathsf{Id}\big)\xi^{k+1} + \big(h_i \mathsf{I}_i^\alpha + \mathsf{Id}\big)\xi^k, \xi^k \right\rangle\!\!\right\rangle + \tfrac{1}{2}\langle\langle \xi^{T/h_i}, \xi^{T/h_i}\rangle\rangle + C' \|\xi^{T/h_i}\|^2_{H^1(\Omega)} \\
&= \sum_{k=0}^{\frac{T}{h_i}-1} \left\langle\!\!\left\langle \big(h_i \mathsf{E}_i^\alpha - \mathsf{Id}\big) v_i(s_i^{k+1},\cdot) + \big(h_i \mathsf{I}_i^\alpha + \mathsf{Id}\big) v_i(s_i^k,\cdot), \xi^k \right\rangle\!\!\right\rangle + \tfrac{1}{2}\langle\langle \xi^{T/h_i}, \xi^{T/h_i}\rangle\rangle \\
&\quad - \sum_{k=0}^{\frac{T}{h_i}-1} \left\langle\!\!\left\langle \big(h_i \mathsf{E}_i^\alpha - \mathsf{Id}\big) Q_i v(s_i^{k+1},\cdot) + \big(h_i \mathsf{I}_i^\alpha + \mathsf{Id}\big) Q_i v(s_i^k,\cdot), \xi^k \right\rangle\!\!\right\rangle + C' \|\xi^{T/h_i}\|^2_{H^1(\Omega)} \\
&\overset{(*)}{\leqslant} \sum_{k=0}^{\frac{T}{h_i}-1} \left\langle\!\!\left\langle h_i \mathsf{F}_i^\alpha, \xi^k \right\rangle\!\!\right\rangle - \sum_{k=0}^{\frac{T}{h_i}-1} \left\langle\!\!\left\langle \big(h_i \mathsf{E}_i^\alpha - \mathsf{Id}\big) Q_i v(s_i^{k+1},\cdot) + \big(h_i \mathsf{I}_i^\alpha + \mathsf{Id}\big) Q_i v(s_i^k,\cdot), \xi^k \right\rangle\!\!\right\rangle \\
&\quad + \tfrac{1}{2}\langle\langle \xi^{T/h_i}, \xi^{T/h_i}\rangle\rangle + C' \|\xi^{T/h_i}\|^2_{H^1(\Omega)},
\end{aligned} \tag{7.2}$$

using in $(*)$ the numerical scheme and that, due to the assumptions on the Q_i, the sign of $v_i - Q_i v$ is known. Since

$$\begin{aligned}
\sum_{k=0}^{\frac{T}{h_i}-1} \left\langle\!\!\left\langle h_i \mathsf{F}_i^\alpha, \xi^k \right\rangle\!\!\right\rangle &\leqslant \|f_i^\alpha\|_{L^2(\Omega)} \sum_{k=0}^{\frac{T}{h_i}-1} h_i \big(\|v_i(s_i^k,\cdot) - v(s_i^k,\cdot)\|_{L^2(\Omega)} + \|v(s_i^k,\cdot) - Q_i v(s_i^k,\cdot)\|_{L^2(\Omega)} \big) \\
&\lesssim \|f_i^\alpha\|_{L^2(\Omega)} \big(\|v_i - v\|_{L^2(\Omega_T)} + \|v - Q_i v\|_{L^2(\Omega_T)} \big),
\end{aligned}$$

the first term in (7.2) vanishes as $i \to \infty$. The second term vanishes due to (7.1). For the two last terms recall Step 2 in the proof of Lemma 5.1 and that v_i is the interpolant of v at time T. Hence

$$|v_i - v|_{L^2([0,T],H_i^1(\Omega))} \to 0$$

as $i \to \infty$.

Step 2: It remains to show (7.1). The terms connected to the time derivative in (7.1) vanish in the limit as

$$\begin{aligned}
\sum_{k=0}^{\frac{T}{h_i}-1} \left\langle\!\!\left\langle Q_i v(s_i^{k+1},\cdot) - Q_i v(s_i^k,\cdot), \xi^k \right\rangle\!\!\right\rangle &= \sum_{k=0}^{\frac{T}{h_i}-1} h_i \left\langle\!\!\left\langle (\partial_t Q_i v)|_{(s_i^k, s_i^{k+1})}, \xi^k \right\rangle\!\!\right\rangle \\
&\lesssim \|\partial_t v\|_{L^1(\Omega_T)} \|\xi^k\|_{L^\infty(\Omega_T)},
\end{aligned} \tag{7.3}$$

using the uniform convergence in ξ^k. Recall that $\langle\langle \mathsf{I}_i^\alpha Q_i v(s_i^k,\cdot),\xi^k\rangle\rangle$ is equal to

$$\sum_\ell (v_i - Q_i v)(s_i^k, y_i^\ell)\Big(\bar{\bar{a}}_i^\alpha(y_i^\ell)\langle \nabla Q_i v(s_i^k,\cdot),\nabla\phi_i^\ell\rangle + \langle \bar{\bar{b}}_i^\alpha\cdot\nabla Q_i v(s_i^k,\cdot) + \bar{\bar{c}}_i^\alpha\, Q_i v(s_i^k,\cdot),\phi_i^\ell\rangle\Big).$$

The lower-order terms vanish due to the uniform convergence of $v_i - Q_i v$ to 0 and the bound

$$\sup_i \|\bar{\bar{b}}_i^\alpha\cdot\nabla Q_i v(s_i^k,\cdot) + \bar{\bar{c}}_i^\alpha\, Q_i v(s_i^k,\cdot)\|_{L^1(\Omega)} < \infty.$$

We note that

$$\sum_\ell (v_i - Q_i v)(s_i^k, y_i^\ell)\,\bar{\bar{a}}_i^\alpha(y_i^\ell)\,\langle \nabla Q_i v(s_i^k,\cdot),\nabla\phi_i^\ell\rangle = \langle \nabla Q_i v(s_i^k,\cdot),\nabla\mathscr{I}_i(\bar{\bar{a}}_i^\alpha(v_i - Q_i v))(s_i^k,\cdot)\rangle,$$

so that in (7.1) the implicit part of the second-order term becomes

$$\sum_{k=0}^{\frac{T}{h_i}-1} h_i\,\langle \nabla Q_i v(s_i^k,\cdot),\nabla\mathscr{I}_i(\bar{\bar{a}}_i^\alpha(v_i - Q_i v))(s_i^k,\cdot)\rangle = \int_0^T \langle\, \mathscr{J}_i\nabla Q_i v,\, \mathscr{J}_i\nabla\mathscr{I}_i(\bar{\bar{a}}_i^\alpha(v_i - Q_i v))\rangle\,\mathrm{d}t, \tag{7.4}$$

where $\mathscr{J}_i$ maps any $w : [0,T]\to L^2(\Omega;\mathbb{R}^d)$ onto the step function $(\mathscr{J}_i w)|_{[s_i^k,s_i^{k+1})} \equiv w(s_i^k,\cdot)$. Observe that $\mathscr{J}_i\nabla Q_i v$ converges strongly in $L^2(\Omega_T;\mathbb{R}^d)$. Owing to Lemma 5.2, we have the chain of inequalities

$$\begin{aligned}\|\nabla\mathscr{I}_i(\bar{\bar{a}}_i^\alpha\xi)\|_{L^2(\Omega;\mathbb{R}^d)} &\leqslant \|\nabla(\bar{\bar{a}}_i^\alpha\xi)\|_{L^2(\Omega;\mathbb{R}^d)} + \|\nabla(\bar{\bar{a}}_i^\alpha\xi - \mathscr{I}_i(\bar{\bar{a}}_i^\alpha\xi))\|_{L^2(\Omega;\mathbb{R}^d)}\\ &\leqslant \|\nabla(\bar{\bar{a}}_i^\alpha\xi)\|_{L^2(\Omega;\mathbb{R}^d)} + K\,\Delta x_i\big(\|\nabla(\sqrt{\bar{\bar{a}}_i^\alpha}\xi)\|_{L^2(\Omega)} + \|\xi\|_{L^2(\Omega)}\big)\\ &\leqslant (\|\sqrt{\bar{\bar{a}}_i^\alpha}\|_{L^2(\Omega)} + K\,\Delta x_i)\|\sqrt{\bar{\bar{a}}_i^\alpha}\nabla\xi\|_{L^2(\Omega;\mathbb{R}^d)} + (\|\sqrt{\bar{\bar{a}}_i^\alpha}\|_{H^1(\Omega)}^2 + K\,\Delta x_i)\|\xi\|_{L^2(\Omega)}\end{aligned}$$

at a time $s_i^k\in[0,T)$. Observe that

$$\|\sqrt{\bar{\bar{a}}_i^\alpha}\nabla\xi\|_{L^2(\Omega;\mathbb{R}^d)} \lesssim |\xi|_{H_i^1(\Omega)} \tag{7.5}$$

by the hypotheses of the theorem. Therefore, in combination with Assumption 4.2 as well as (5.1) and (6.7), we obtain an $L^\infty(L^2)$ bound over $\mathscr{J}_i\nabla\mathscr{I}_i(\bar{\bar{a}}_i^\alpha(v_i - Q_i v))$ in (7.4). The convergence

$$\lim_{i\to\infty}\int_0^T \langle w, \mathscr{J}_i\nabla\mathscr{I}_i(\bar{\bar{a}}_i^\alpha(v_i - Q_i v))\rangle\mathrm{d}t = -\lim_{i\to\infty}\int_0^T \langle \nabla\cdot w, J_I\mathscr{I}_i(\bar{\bar{a}}_i^\alpha(v_i - Q_i v))\rangle\mathrm{d}t = 0$$

with test functions w in the dense subset $C_0^1(\Omega_T;\mathbb{R}^d)$ gives weak convergence of $\nabla\mathscr{I}_i(\bar{\bar{a}}_i^\alpha(v_i - Q_i v))$ in $L^2(\Omega_T;\mathbb{R}^d)$, see (Yoshida , 1980, p. 121). Combining weak and strong convergence, it is ensured that (7.4) converges to 0 as $i\to\infty$, cf. (Zeidler , 1990, Prop. 21.23). A similar argument shows that $\sum_k h_i\,\langle\langle \mathsf{E}_i^\alpha Q_i v(s_i^{k+1},\cdot),\xi^k\rangle\rangle$ vanishes in the limit. Therefore we proved (7.1). □

The Sobolev regularity of the value functions is for example discussed in Fleming & Soner (2006) and Wei, Wu & Zhao (2014) and Zhou (2015).

We conclude the section with a brief sketch how the convergence of the optimal controls of the discrete problem to the optimal controls of the continuous problem might be established on the basis of Theorem 7.1.

EXAMPLE 7.2 Let $\alpha_0 : \Omega \to [0,1]$ be a smooth, non-negative function and set $\alpha_1(x) = 2\,\alpha_0(x)$ for all $x \in \Omega$. Let B be the unit ball. We consider the equation

$$-v_t + (\sup_{\alpha\in[\alpha_0(x),\alpha_1(x)]} \alpha\,\Delta v) + |\nabla v| = -v_t + (\sup_{\alpha\in[\alpha_0(x),\alpha_1(x)]} \alpha\,\Delta v) + (\sup_{\beta\in\partial B} \beta\cdot\nabla v)$$
$$= -v_t + \sup_{(\alpha,\beta)\in[\alpha_0(x),\alpha_1(x)]\times\partial B} (\alpha\,\Delta v + \beta\cdot\nabla v) = f, \qquad (7.6)$$

which may degenerate where $\alpha_0(x) = 0$. Now let $(v_i)_i$ be a sequence of finite element solutions converging to the value function v as in Theorem 7.1, supposing a fully implicit time discretisation as in Example 6.1. We may omit for this example the further specification of the problem such as the boundary conditions; however, we expect that the hypotheses of Theorem 7.1 are met.

We let $\gamma = \alpha_0$ and suppose that on some non-empty open set N we have the bound $\alpha_0(x) \geqslant \delta$ from below with $\delta > 0$. As ∇v_h converges in $L^{d+1+\varepsilon}$ to ∇v on $[0,T]\times N$ it follows that the piecewise linear functions with the values $\langle \nabla v_h, \hat{\phi}_i^\ell\rangle$ at the nodes y_i^ℓ converge to ∇v on $[0,T]\times N$ as well.

The optimal control of the first-order term at y_i^ℓ is $\beta_i^\ell = \langle \nabla v_h, \hat{\phi}_i^\ell\rangle / |\langle \nabla v_h, \hat{\phi}_i^\ell\rangle|$. If we interpret β_i as the piecewise linear function which interpolates the β_i^ℓ at the nodes, then β_i converges on $[0,T]\times N$ in $L^{d+1+\varepsilon}$ to the optimal control $\nabla v/|\nabla v|$ of the continuous problem, assuming no division by zero occurs on N (which would correspond to the case that $\nabla v = 0$ and any $\beta \in \partial B$ is an optimal control).

Described as the Bang-bang principle, one typically sees regions where the optimal control α of the continuous problem is either equal to α_0 or α_1, otherwise $\Delta v = 0$ and any $\alpha \in [\alpha_0, \alpha_1]$ is optimal; see (Jensen & Smears , 2012, Section 6) for a related example. In this spirit let us suppose that there is a non-empty open set $(s,t)\times M \subset [0,T]\times N$ on which α_0 is the unique optimal control of the continuous problem.

We claim that then there cannot be a non-empty open subset $(s_h,t_h)\times M_h$ of $(s,t)\times M$ on which α_1 is the optimal control of the discrete problem after sufficient refinement. More precisely we claim that there is no $(s_h,t_h)\times M_h$ such that for all nodes $y_i^\ell \in M_h$ and times $s_i^k \in (s_h,t_h)$ for all large i we have

$$-d_i v_i(s_i^k,\cdot) + \alpha_1\,\langle \nabla v_i, \nabla\hat{\phi}_i^\ell\rangle = \langle \hat{f}_i^\ell, \hat{\phi}_i^\ell\rangle, \qquad (7.7)$$

where $\hat{f}_i^\ell = f_i^\ell - |\langle \nabla v_h, \hat{\phi}_i^\ell\rangle|$. Similarly, v solves on M, with $\hat{f}(x) = f(x) - |\nabla v(x)|$, the linear equation

$$-v_t - \alpha_0\,\Delta v = \hat{f}. \qquad (7.8)$$

So suppose there was an M_h on which (7.7) holds. For simplicity let us assume that M_h is the union of elements, at any level i of refinement and time s_i^k; thus the boundary of M_h consists of edges of elements. Similarly we suppose that s_h and t_h are discrete time steps. Then we can use v_i on the boundary of $(s_h,t_h)\times M_h$ as Dirichlet data, converging in L^∞ to v. Together with the Dirichlet data, (7.7) may be viewed as a localised initial boundary value problem. Linear finite element analysis and '$\hat{f}_i^\ell \to \hat{f}$' imply the convergence of the v_i to the solution of

$$-w_t - \alpha_1\,\Delta w = \hat{f},$$

which is different from v, giving a contradiction.

8. The method of artificial diffusion

We illustrate now a way of choosing the coefficients of E_i^α and I_i^α in order to satisfy the assumptions of the above analysis.

For all $\alpha \in A$ we need to impose the conditions to ensure uniform convergence. For one $\hat{\alpha} \in A$ we wish to enforce terms to guarantee convergence in a Sobolev space with a weight associated to $L^{\hat{\alpha}}$.

REMARK 8.1 One could define a set $B \subset A$ of multiple indices $\hat{\alpha}$ for which one wishes to derive Sobolev norm bounds, which are associated to different weights $\gamma^{\hat{\alpha}}$ and $\gamma_i^{\hat{\alpha}}$. The analysis generalises directly.

Given a function $g : \Omega \to \mathbb{R}^d$ and an element K of the mesh $\mathscr{T}_i$, we denote

$$|g|_K := \Big(\sum_{j=1}^{d} \|g_j\|_{L^\infty(K)}^2 \Big)^{\frac{1}{2}}.$$

If g is elementwise constant then $|g|_K$ is simply the Euclidean norm of g on K. Let Δx_K denote the diameter of K. We assume that the meshes $\mathscr{T}_i$ are strictly acute, cf. Burman & Ern (2002), in the sense that there exists $\vartheta \in (0, \pi/2)$ such that for all $i \in \mathbb{N}$:

$$\nabla \phi_i^\ell \cdot \nabla \phi_i^l \big|_K \leqslant -\sin(\vartheta)\, |\nabla \phi_i^\ell|_K \, |\nabla \phi_i^l|_K \qquad \forall \ell, l \leqslant \dim V_i,\ \ell \neq l,\ \forall K \in \mathscr{T}_i. \tag{8.1}$$

We choose a splitting of the form

$$a^\alpha = \tilde{a}^\alpha + \tilde{\tilde{a}}^\alpha, \quad b^\alpha = \tilde{b}^\alpha + \tilde{\tilde{b}}^\alpha, \quad c^\alpha = \tilde{c}^\alpha + \tilde{\tilde{c}}^\alpha,$$

which does not depend on $i \in \mathbb{N}$. It is generally necessary to add artificial diffusion to the second-order coefficients, that means that we generally need to determine i-dependent coefficients $\bar{a}_i^\alpha \geqslant \tilde{a}^\alpha$ and $\bar{\bar{a}}_i^\alpha \geqslant \tilde{\tilde{a}}^\alpha$. It can also be necessary to construct $\bar{\bar{c}}_i^\alpha \geqslant \tilde{\tilde{c}}^\alpha$. For the other lower-order terms we can use the above splitting directly: For all $i \in \mathbb{N}$ set

$$\bar{b}_i^\alpha = \tilde{b}^\alpha, \bar{\bar{b}}_i^\alpha = \tilde{\tilde{b}}^\alpha, \bar{c}_i^\alpha = \tilde{c}^\alpha \text{ and also } f_i^\alpha = f^\alpha,$$

where all terms are in $C(\overline{\Omega})$, $\tilde{a}^\alpha$ and $\tilde{\tilde{a}}^\alpha$ are non-negative and all $\tilde{c}^\alpha$ and $\tilde{\tilde{c}}^\alpha$ are non-negative and satisfy inequality (3.2).

For instance, one could discretise symmetric terms implicitly, i.e. $a^\alpha = \tilde{\tilde{a}}^\alpha$ and $c^\alpha = \tilde{\tilde{c}}^\alpha$, and screw-symmetric terms explicitly, i.e. $b^\alpha = \tilde{b}^\alpha$. With this approach the coercivity properties of L^α are well incorporated. Alternatively, if there is coercivity which is uniform in α with respect to a useful weight γ, then it is interesting to select $\mathsf{I}_i^\alpha = \mathsf{I}_i$ independently of α because in this case only a linear system needs to be solved at each time step while an $\mathsf{O}(\Delta x_i)$ time-step may be preserved. We also refer to Jensen & Smears (2012) for further illustrations of operator splittings.

To obtain uniform convergence we select the non-negative parameters $\bar{\nu}_i^\alpha$ and $\bar{\bar{\nu}}_i^\alpha$ such that for all K that have y_i^ℓ as vertex:

$$\big(|\bar{b}_i^\alpha|_K + \Delta x_K \|\bar{c}_i^\alpha\|_{L^\infty(K)} \big) \leqslant \bar{\nu}_i^\alpha \, \sin(\vartheta)\, |\nabla \hat{\phi}_i^\ell|_K \, \mathrm{vol}(K), \tag{8.2a}$$

$$\big(|\bar{\bar{b}}_i^\alpha|_K + \Delta x_K \|\bar{\bar{c}}_i^\alpha\|_{L^\infty(K)} \big) \leqslant \bar{\bar{\nu}}_i^\alpha \, \sin(\vartheta)\, |\nabla \hat{\phi}_i^\ell|_K \, \mathrm{vol}(K). \tag{8.2b}$$

We now turn to convergence in the weighted Sobolev norm associated with the control $\hat{\alpha}$. We require that $\sqrt{\tilde{a}^{\hat{\alpha}}}, \sqrt{\tilde{\tilde{a}}^{\hat{\alpha}}} \in W^{2,\infty}(\Omega)$ as well as

$$\begin{aligned} \tilde{\tilde{c}}^{\hat{\alpha}} - \tfrac{1}{2}(\nabla \cdot \tilde{\tilde{b}}^{\hat{\alpha}} + \Delta \tilde{\tilde{a}}^{\hat{\alpha}}) &\geqslant 0, \\ \tilde{c}^{\hat{\alpha}} - \tfrac{1}{2}(\nabla \cdot \tilde{b}^{\hat{\alpha}} + \Delta \tilde{a}^{\hat{\alpha}}) &\geqslant 0, \\ \gamma := \tilde{\tilde{a}}^{\hat{\alpha}} - \tilde{a}^{\hat{\alpha}} &\geqslant 0 \end{aligned} \tag{8.3}$$

so that

$$\tilde{a}^{\hat{\alpha}} \lesssim \gamma \qquad \text{and} \qquad \tilde{\tilde{a}}^{\hat{\alpha}} \lesssim \gamma. \tag{8.4}$$

We choose $\bar{a}_i^{\hat{\alpha}}$ and $\bar{\bar{a}}_i^{\hat{\alpha}}$ both in $C(\overline{\Omega})$ such that at the nodes

$$\left.\begin{aligned} \bar{a}_i^{\hat{\alpha}}(y_i^\ell) &\geqslant \tilde{a}^{\hat{\alpha}}(y_i^\ell) + \bar{\nu}_i^{\hat{\alpha}} \\ \bar{\bar{a}}_i^{\hat{\alpha}}(y_i^\ell) &\geqslant \tilde{\tilde{a}}^{\hat{\alpha}}(y_i^\ell) + \mu_i \end{aligned}\right\} \tag{8.5}$$

such that for large i

$$\mu_i \geqslant \max\Big\{ \bar{\bar{\nu}}_i^{\hat{\alpha}}, 2\,K\,\Delta x_i \Big(3\|\sqrt{\bar{a}_i^{\hat{\alpha}}}\|_{W^{1,\infty}(\Omega)} + \|\sqrt{\bar{\bar{a}}_i^{\hat{\alpha}}}\|_{W^{1,\infty}(\Omega)} + 3\Big) + 2\,h_i\|\nabla\tilde{a}^\alpha + \tilde{b}^{\hat{\alpha}}\|^2_{L^\infty(\Omega)}\Big\}. \tag{8.6}$$

Notice the recursive nature of the definition of $\bar{\bar{a}}_i^{\hat{\alpha}}$ which appears also on the right-hand side of (8.6). For Δx_i small enough it is clear that μ_i can be chosen so that is satisfies (8.6). Finally, we set

$$\bar{\bar{c}}_i^{\hat{\alpha}} = \tilde{\tilde{c}}^{\hat{\alpha}} + K\,\Delta x_i \Big(\|\sqrt{\bar{a}_i^{\hat{\alpha}}}\|_{W^{1,\infty}(\Omega)} + 1\Big) + \tfrac{h_i}{2}\|\tilde{c}^{\hat{\alpha}}\|^2_{L^\infty(\Omega)}. \tag{8.7}$$

Note that if the diffusion and reaction terms are approximated implicitly then $\bar{\bar{c}}_i^{\hat{\alpha}} = \tilde{\tilde{c}}^{\hat{\alpha}}$. Sections 8.1 and 8.2 below show that, under suitable time-step restrictions, (8.3), (8.4) and (8.5) give convergence.

8.1 *Verification of uniform convergence*

Suppose that $w \in V_i$ has a non-positive local minimum at an interior node y_i^ℓ. It was shown in Jensen & Smears (2013) that then

$$(\mathsf{E}_i^\alpha w)_\ell \leqslant 0, \qquad (\mathsf{I}_i^\alpha w)_\ell \leqslant 0, \tag{8.8}$$

and, if $\bar{\nu}_i^\alpha$ and $\bar{\bar{\nu}}_i^\alpha$ are chosen optimally, then for $K \in \mathscr{T}_i$

$$\left.\begin{aligned} \bar{\nu}_i^\alpha &= \mathsf{O}\big(\sup_K\big\{|\bar{b}_i^\alpha|_K \Delta x_K + \|\bar{c}_i^\alpha\|_{L^\infty(K)}\Delta x_K^2\big\}\big), \\ \bar{\bar{\nu}}_i^\alpha &= \mathsf{O}\big(\sup_K\big\{|\bar{\bar{b}}_i^\alpha|_K \Delta x_K + \|\bar{\bar{c}}_i^\alpha\|_{L^\infty(K)}\Delta x_K^2\big\}\big). \end{aligned}\right\} \tag{8.9}$$

Note that (8.9) is consistent with Assumption 3.1. We turn to time step restrictions for semi-implicit and explicit methods which give Assumption 4.1. The non-positivity of the diagonal terms of $h_i\mathsf{E}_i^\alpha - \mathsf{Id}$ expands to

$$\begin{aligned} 1 &\geqslant h_i\Big(\bar{a}_i^\alpha(y_i^\ell)\langle\nabla\phi_i^\ell, \nabla\hat{\phi}_i^\ell\rangle + \langle\bar{b}_i^\alpha\cdot\nabla\phi_i^\ell + \bar{c}_i^\alpha\,\phi_i^\ell, \hat{\phi}_i^\ell\rangle\Big) \\ &= h_i\Big(\mathsf{O}\big(\bar{a}_i^\alpha\,\Delta x_K^{-2}\big) + \mathsf{O}\big(|\bar{b}_i^\alpha|_K\,\Delta x_K^{-1}\big) + \mathsf{O}\big(\bar{c}_i^\alpha\big)\Big). \end{aligned}$$

Therefore the time step restriction imposed by L^α is $h_i \lesssim \inf_K\big(\Delta x_K^2/\bar{a}_i^\alpha(y_i^\ell)\big)$, $y_i^\ell \in \overline{K}$, if there is a non-zero $\tilde{a}^\alpha$ and i is large. It is $h_i \lesssim \inf_K\big(\Delta x_K/|\bar{b}_i^\alpha(y_i^\ell)|_K\big)$ if all $\bar{a}_i^\alpha = 0$, $i \in \mathbb{N}$, and there are non-zero $\bar{b}_i^\alpha$, and is $\mathsf{O}(1)$ if all $\bar{a}_i^\alpha$ and $\bar{b}_i^\alpha$ vanish. There is no restriction if also all $\bar{c}_i^\alpha$ are zero. If the scheme is fully implicit, there are no time-step restrictions.

Assumption 4.2 holds if there is an orthogonal projection, boundary control and comparison, see page 4. The former is essentially a quasi-uniformity assumption on the mesh, cf. Demlow, Leykekhman, Schatz and Wahlbin (2012), the latter two on the boundary value problem. In fact, the comparison principle is one of the building blocks in the theory of viscosity solutions.

8.2 *Verification of Sobolev convergence*

The main step of this section is the proof that (6.3) is satisfied. We begin with the examination of the explicit terms. With $w^{k+1} = w(s_i^{k+1}, \cdot)$ and $w^k = w(s_i^k, \cdot)$,

$$\langle\!\langle \mathsf{E}_i^{\hat\alpha} w^{k+1}, w^k \rangle\!\rangle = \langle \nabla w^{k+1}, \nabla \mathscr{I}_i(\bar a_i^{\hat\alpha} w^k)\rangle + \langle \tilde b^{\hat\alpha} \cdot \nabla w^{k+1} + \tilde c^{\hat\alpha}\, w^{k+1}, w^{k+1} + (w^k - w^{k+1})\rangle, \tag{8.10}$$

using the interpolation operator as in (6.4). We also find, as in (6.6), for large i

$$|\langle \nabla w^{k+1}, \nabla \mathscr{I}_i(\bar a_i^{\hat\alpha} w^k)\rangle - \langle \nabla w^{k+1}, \nabla \bar a_i^{\hat\alpha} w^k\rangle| \leqslant K\, \Delta x_i \big(\big\| \sqrt{\bar a_i^{\hat\alpha}} \big\|_{W^{1,\infty}(\Omega)} + 1 \big)\, |w^{k+1}|_{H^1(\Omega)}\, \|w^k\|_{H^1(\Omega)}.$$

Therefore,

$$\begin{aligned} \langle \nabla w^{k+1}, \nabla \mathscr{I}_i(\bar a_i^{\hat\alpha} w^k)\rangle &\geqslant \big(\langle \nabla w^{k+1}, \nabla \mathscr{I}_i(\bar a_i^{\hat\alpha} w^k)\rangle - \langle \nabla w^{k+1}, \nabla \bar a_i^{\hat\alpha} w^k\rangle \big) + \langle \nabla w^{k+1}, \nabla \bar a_i^{\hat\alpha} w^k\rangle \\ &\geqslant \langle \nabla w^{k+1}, \bar a_i^{\hat\alpha} \nabla w^k\rangle + \langle \nabla w^{k+1}, w^{k+1} \nabla \bar a_i^{\hat\alpha}\rangle + \langle \nabla w^{k+1}, (w^k - w^{k+1}) \nabla \bar a_i^{\hat\alpha}\rangle \\ &\quad - K\, \Delta x_i \big(\big\| \sqrt{\bar a_i^{\hat\alpha}} \big\|_{W^{1,\infty}(\Omega)} + 1 \big)\, |w^{k+1}|_{H^1(\Omega)}\, \|w^k\|_{H^1(\Omega)}. \end{aligned}$$

Recalling (8.3) and using that $\tilde a^{\hat\alpha} - \bar a_i^{\hat\alpha}$ is constant, we see that

$$\langle \nabla w^{k+1}, w^{k+1} \nabla \bar a_i^{\hat\alpha}\rangle + \langle \tilde b^{\hat\alpha} \cdot \nabla w^{k+1} + \tilde c^{\hat\alpha}\, w^{k+1}, w^{k+1}\rangle = \langle (\tilde c^{\hat\alpha} - \tfrac{1}{2}(\nabla \cdot \tilde b^{\hat\alpha} + \Delta \tilde a^{\hat\alpha})) w^{k+1}, w^{k+1}\rangle \geqslant 0.$$

Now

$$\begin{aligned} & |\langle (\nabla \bar a_i^{\hat\alpha} + \tilde b^{\hat\alpha}) \cdot \nabla w^{k+1} + \tilde c^{\hat\alpha}\, w^{k+1}, w^k - w^{k+1}\rangle| \\ \leqslant & \tfrac{h_i}{2} \big(\|\nabla \tilde a^{\alpha} + \tilde b^{\hat\alpha}\|_{L^\infty}^2\, \|\nabla w^{k+1}\|_{L^2}^2 + \|\tilde c^{\hat\alpha}\|_{L^\infty}^2\, \|w^{k+1}\|_{L^2}^2 \big) + \tfrac{1}{2h_i} \|w^k - w^{k+1}\|_{L^2}^2 \\ \leqslant & \tfrac{h_i}{2} \big(\|\nabla \tilde a^{\alpha} + \tilde b^{\hat\alpha}\|_{L^\infty}^2\, \|\nabla w^{k+1}\|_{L^2}^2 + \|\tilde c^{\hat\alpha}\|_{L^\infty}^2\, \|w^{k+1}\|_{L^2}^2 \big) + \tfrac{1}{2h_i} \langle\!\langle w^k - w^{k+1}, w^k - w^{k+1} \rangle\!\rangle \end{aligned}$$

where we can use Jensen's inequality as $\sum_\ell \phi_i^\ell(x) = 1$ is a convex combination:

$$\begin{aligned} \|w^k - w^{k+1}\|_{L^2}^2 &= \int_\Omega \Big(\sum_\ell \big(w^k(y_i^\ell) - w^{k+1}(y_i^\ell) \big) \phi_i^\ell(x) \Big)^2 \mathrm{d}x \\ &\leqslant \int_\Omega \sum_\ell \big(w^k(y_i^\ell) - w^{k+1}(y_i^\ell) \big)^2 \phi_i^\ell(x) \mathrm{d}x = \langle\!\langle w^k - w^{k+1}, w^k - w^{k+1} \rangle\!\rangle. \end{aligned}$$

We summarise

$$\begin{aligned} \langle\!\langle \mathsf{E}_i^{\hat\alpha} w^{k+1}, w^k \rangle\!\rangle \geqslant & -\tfrac{1}{2} \big\| \sqrt{\bar a_i^{\hat\alpha}} \nabla w^{k+1} \big\|_{L^2}^2 - \tfrac{1}{2} \big\| \sqrt{\bar a_i^{\hat\alpha}} \nabla w^k \big\|_{L^2}^2 - \tfrac{1}{2h_i} \langle\!\langle w^k - w^{k+1}, w^k - w^{k+1} \rangle\!\rangle \\ & - K\, \Delta x_i \big(\big\| \sqrt{\bar a_i^{\hat\alpha}} \big\|_{W^{1,\infty}(\Omega)} + 1 \big) \big(\tfrac{1}{2} |w^{k+1}|_{H^1(\Omega)}^2 + \tfrac{1}{2} \|w^k\|_{H^1(\Omega)}^2 \big) \\ & - \tfrac{h_i}{2} \big(\|\nabla \tilde a^{\alpha} + \tilde b^{\hat\alpha}\|_{L^\infty(\Omega)}^2\, \|\nabla w^{k+1}\|_{L^2(\Omega)}^2 + \|\tilde c^{\hat\alpha}\|_{L^\infty(\Omega)}^2\, \|w^{k+1}\|_{L^2(\Omega)}^2 \big). \end{aligned} \tag{8.11}$$

Now to the implicit terms. Recall (8.7), where $\bar{\bar c}_i^{\hat\alpha}$ is increased compared to $\tilde c^{\hat\alpha}$ to provide additional control with the L^2 scalar product. Also compare (8.6) with the artificial diffusion introduced in (6.5).

Then, as in Example 6.1, for large i,

$$\begin{aligned}\langle\!\langle \mathsf{I}_i^{\hat{\alpha}} w^k, w^k \rangle\!\rangle \geqslant{}& \langle \tilde{\tilde{a}}^{\hat{\alpha}} \Delta w^k, \Delta w^k \rangle + \langle (\tilde{\tilde{c}}^{\hat{\alpha}} - \tfrac{1}{2}(\nabla \cdot \tilde{\tilde{b}}^{\hat{\alpha}} + \Delta \tilde{\tilde{a}}^{\hat{\alpha}})) w^k, w^k \rangle \\ &+ \Big(K \Delta x_i \Big(3 \| \sqrt{\bar{a}_i^{\hat{\alpha}}} \|_{W^{1,\infty}(\Omega)} + \| \sqrt{\bar{\bar{a}}_i^{\hat{\alpha}}} \|_{W^{1,\infty}(\Omega)} + 3 \Big) + h_i \| \nabla \tilde{a}^{\alpha} + \tilde{b}^{\hat{\alpha}} \|^2_{L^\infty(\Omega)} \Big) \langle \nabla w^k, \nabla w^k \rangle \\ &+ \Big(K \Delta x_i \Big(\| \sqrt{\bar{a}_i^{\hat{\alpha}}} \|_{W^{1,\infty}(\Omega)} + 1 \Big) + h_i \| \tilde{c}^{\hat{\alpha}} \|^2_{L^\infty(\Omega)} \Big) \langle w^k, w^k \rangle.\end{aligned} \tag{8.12}$$

We observe that nearly all terms on the right-hand side of (8.11) can be bounded by corresponding terms in (8.12) of time k or $k+1$, also owing to (8.3). The exception are $\langle\!\langle w^k - w^{k+1}, w^k - w^{k+1} \rangle\!\rangle$ and when w is evaluated at the final time. For the former we point to the occurrence of this term in (6.3). For the latter we choose C' such that

$$\begin{aligned}C' \| w^{T/h_i} \|^2_{H^1(\Omega)} \geqslant{}& \tfrac{1}{2} \big\| \sqrt{\bar{a}_i^{\hat{\alpha}}} \nabla w^{T/h_i} \big\|^2_{L^2(\Omega)} + \tfrac{K\,\Delta x_i}{2} \big(\big\| \sqrt{\bar{a}_i^{\hat{\alpha}}} \big\|_{W^{1,\infty}(\Omega)} + 1 \big) |w^{T/h_i}|^2_{H^1(\Omega)} + h_i \| \sqrt{\gamma_i} \nabla w^{T/h_i} \|^2_{L^2(\Omega)} \\ &+ \tfrac{h_i}{2} \big(\| \nabla \tilde{a}^{\alpha} + \tilde{b}^{\hat{\alpha}} \|^2_{L^\infty(\Omega)} \| \nabla w^{T/h_i} \|^2_{L^2(\Omega)} + \| \tilde{c}^{\hat{\alpha}} \|^2_{L^\infty(\Omega)} \| w^{T/h_i} \|^2_{L^2(\Omega)} \big).\end{aligned}$$

Then we obtain (6.3) with $\gamma_i = \gamma + \frac{\mu_i}{3}$:

$$\begin{aligned}&\sum_{k=0}^{\frac{T}{h_i}-1} \Big(h_i \langle\!\langle \mathsf{E}_i^{\alpha} w^{k+1} + \mathsf{I}_i^{\alpha} w^k, w^k \rangle\!\rangle + \tfrac{1}{2} \langle\!\langle w^{k+1} - w^k, w^{k+1} - w^k \rangle\!\rangle \Big) + \tfrac{1}{2} \langle\!\langle w^0, w^0 \rangle\!\rangle + C' \| w^{T/h_i} \|^2_{L^2(\Omega)} \\ &\geqslant \sum_{k=0}^{\frac{T}{h_i}} \Big(h_i \langle \gamma_i \nabla w^k, \nabla w^k \rangle \Big) + \tfrac{1}{2} \langle\!\langle w^0, w^0 \rangle\!\rangle \geqslant |w|^2_{L^2((0,T),H_i^1(\Omega))},\end{aligned}$$

where the last inequality follows because the composite trapezium rule is exact for functions in W_i. Finally, observe that $\gamma \lesssim \gamma_i$ and that by (8.4) there is a $C > 0$ such that $\bar{a}_i^{\alpha} \leqslant C\gamma_i$ and $\bar{\bar{a}}_i^{\alpha} \leqslant C\gamma_i$ for all $i \in \mathbb{N}$, as required in the statement of Theorem 7.1.

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