Sliding Mode Based Decentralized Tracking Control of Underactuated Four-Body Systems

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***Abstract*—In this paper, output tracking control of four-body systems with flexible links is considered, where only the first and last bodies are with actuators. The systems are mathematically modelled based on the Newton's second law, which is described as an interconnected system with two subsystems. Due to the mechanical wear and changes of the external environment, it is necessary to consider uncertainties in the systems which are assumed to be bounded by known functions. Based on sliding mode techniques, a decentralized control scheme is proposed such that the outputs of the controlled systems can track the desired signals uniformly ultimately. Using specific coordinate transformations, the considered systems are transformed to a new system with a special structure to facilitate the design of sliding surfaces and sliding mode controllers. A set of conditions is developed under which the designed decentralized controller drives the tracking errors onto the sliding surface and stay on the surface thereafter while the system states are bounded. For simulation purposes, the desired signals are selected as part of a hyperbolic function which is time-varying and in consistence with the real situations. Simulation demonstrates that the proposed control strategy is effective and feasible.**

***Index Terms*—Decentralized control, Large-scale system, Sliding mode control, Tracking control.**

I. Introduction

Control of large-scale interconnected systems is a popular research topic in the area of system control. Such a class of systems widely exists in our real life, for example, a coupled inverted pendulum, river quality control, high-speed transaction and flight control etc. (see, [21], [13]). These systems have received great attention and many results have been obtained (see, e.g. [17], [22], [3]).

A large-scale system can be modelled by interconnections of a set of lower-dimensional subsystems, and each subsystem is usually affected by the others due to the interactions between these subsystems. Underactuated mechanical systems actually belong to a class of large-scale systems with some special structure. The controller design for such a mechanical system is more challenging when compared with that of normal systems due to less actuators. However, this type of systems has its own advantages such as light weight, low cost and small energy consumption [9]. Therefore, an underactuated system is purposely introduced in some of practical engineering systems, such as aircraft, underwater vehicles and humanoid robots, for which the control problem has attracted much attention (see, e.g. [5], [6], [10]).

In this paper, the tracking control of a class of underactuated four-body systems with unmatched uncertainties is considered. The four bodies are connected by springs and dampers shown in Fig. 1 where only the first body and fourth body are controlled by actuators installed on them [14]. It is well-known that multi-body systems are usually composed of several bodies by interconnections that have similar structure as many other physical systems (see, e.g. [2], [7], [16]). In recent decades, there have been many researchers working on the control of multi-body systems. A neural network-based adaptive control of multi-link robots in the joint space is studied in [19]. In [11], an absolute coordinate based method is considered to deal with the dynamics and control of a rigid-exible multi-body system with cylindrical clearance joints. Also, a robust PD control scheme [8] is proposed for exible joint robots based on a disturbance observer, and He and Sun proposed a boundary feedback control [4] for the exible robotic manipulator to achieve the desired angular position tracking. Among all these existing works, the considered systems are usually required to be full-actuated. This has motivated the study of underactuated multi-body systems.

Trajectory tracking and output tracking are very popular and important topics in both control theory and control engineering, and many tracking control results have been obtained (see, e.g. [1], [12]). In [20], an adaptive fuzzy technique based tracking control of large-scale systems is considered and a kind

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of decentralized tracking control for large-scale systems is studied in [18] where model reference control is investigated. It should be noticed that the work obtained in [20] and [18] requires the isolated subsystems have a typical structure. So far, the study on output tracking control of underactuated multi-body systems using sliding mode control is very limited. Sliding mode control is a popular method in dealing with complex systems with uncertainties because this type of control is totally robust to matched terms. Therefore, the sliding mode control based method has been widely applied in system tracking control. In [23], a tracking problem for a class of large-scale systems with interconnections is considered using sliding mode techniques where desired signals are required to be constant. It should be emphasised that the research on decentralized tracking control for large-scale underactuated multi-body systems using sliding mode control is very few, specifically when the desired signals are time-varying.

In this paper, the four-body underactuator mechanical system shown in Fig.1 is considered. The mathmatical model is from [15] where the centralised control is focused although the tracking problem is considered. The four-body system is described by an interconnected system where unmatched uncertainties are considered. Also, a different suitable coordinate transformation is introduced to transfer the original system to an interconnected system with two low-dimensional subsystems connected by springs and dampings when compared with the coordinates in [15]. Then, the sliding surface is designed in terms of the tracking errors between system outputs and corresponding desired signals. A set of conditions is proposed to guarantee the uniformly ultimately boundedness of the corresponding sliding motion. A decentralized sliding mode control scheme is proposed to drive the states of the transformed interconnected system to the designed sliding surface. Finally, simulation study on a four-bodies dynamic system is presented to demonstrate the proposed approaches.

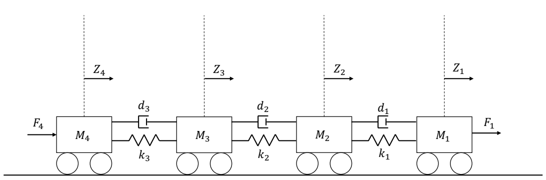

Fig. 1. Under-actuated four-body system structure

II. FOUR-BODY SYSTEM DESCRIPTION

In this paper, for a square matrix A, $\lambda_{min}(A)$ means the minimum eigenvalue of matrix A. $\|\cdot\|$ demotes the Euclidean norm or its induced norm and $col(\cdot)$ is a column matrix. The considered benchmark system has been shown in Fig.1 which is composed of four mass bodies connected by springs and dampers. From the *Newton-second* law and the work in [15] , the dynamical equations of the four-body system in Fig.1 are given by

$$\begin{aligned} M_1\ddot{z}_1(t) =& F_1(t) - k_1(z_1(t)-z_2(t)) \\ & - d_1(\dot{z}_1(t)-\dot{z}_2(t)) - b_{r_1}\dot{z}_1(t) \end{aligned} \tag{1}$$

$$\begin{aligned} M_2\ddot{z}_2(t) =& -k_2(z_2(t)-z_3(t)) - k_1(z_2(t)-z_1(t)) \\ & - d_2(\dot{z}_2(t)-\dot{z}_3(t)) - d_1(\dot{z}_2(t)-\dot{z}_1(t)) - b_{r_2}\dot{z}_2(t) \end{aligned} \tag{2}$$

$$\begin{aligned} M_3\ddot{z}_3(t) =& -k_3(z_3(t)-z_4(t)) - k_2(z_3(t)-z_2(t)) \\ & - d_3(\dot{z}_3(t)-\dot{z}_4(t)) - d_2(\dot{z}_3(t)-\dot{z}_2(t)) - b_{r_3}\dot{z}_3(t) \end{aligned} \tag{3}$$

$$\begin{aligned} M_4\ddot{z}_4(t) =& F_4(t) - k_3(z_4(t)-z_3(t)) \\ & - d_3(\dot{z}_4(t)-\dot{z}_3(t)) - b_{r_4}\dot{z}_4(t) \end{aligned} \tag{4}$$

where M_i is the mass of the i-th body, k_i and d_i are the spring and damping constants respectively for $i = 1,2,3,4$. b_{r_i} denotes the corresponding friction parameter. z_i, z_{i+1} and $\dot{z}_i$, $\dot{z}_{i+1}$ are the displacements and the corresponding speeds of the i-th and $(i+1)$-th bodies. F_1 and F_4 are the tractive forces acting on the 1st and 4-th bodies respectively.

III. SYSTEM STRUCTURE ANALYSIS

In this paper, a more practical situation is considered. On one hand, it is assumed that there exist some uncertainties in the isolated subsystems which might be caused by the variations of spring and damping parameters. On the other hand, it is assumed that there exist some uncertainties in the interconnections which might be caused by external factors like temperature change, external disturbances and so on.

For the system (1)-(4), choose the following coordinate transformation:

$$\begin{aligned} & \left[x_{11}, x_{12}, x_{13}, x_{14}, x_{21}, x_{22}, x_{23}, x_{24}\right]^T \\ & = \left[z_1(t), \dot{z}_1(t), z_2(t), \dot{z}_2(t), z_3(t), \dot{z}_3(t), z_4(t), \dot{z}_4(t)\right]^T \end{aligned} \tag{5}$$

and let the inputs $col(u_1,u_2) := col(F_1(t),F_4(t))$. It is assumed that the velocities of four-bodies $\dot{z}_1, \dot{z}_2, \dot{z}_3$ and $\dot{z}_4$ are measurable which are taken as system outputs.

Furthermore, an additional feedback transformation is introduced as:

$$\begin{aligned} u_1 &= k_1x_{11} + (d_1+b_{r_1})x_{12} - k_1x_{13} - d_1x_{14} + M_1v_1 \\ u_2 &= -k_3x_{21} - d_3x_{22} + k_3x_{23} + (d_3+b_{r_4})x_{24} + M_4v_2 \end{aligned} \tag{6}$$

where v_1 and v_2 are the new control inputs which will be designed later. Then, the system (1)-(4) can be described by

$$\begin{cases} \dot{x}_{11} = x_{12} \\ \dot{x}_{12} = v_1 + \Delta\Phi_{11}(x_1) \\ \dot{x}_{13} = x_{14} \\ \dot{x}_{14} = \dfrac{k_1}{M_2}x_{11} + \dfrac{d_1}{M_2}x_{12} - \dfrac{(k_1+k_2)}{M_2}x_{13} \\ \qquad - \dfrac{(d_1+d_2+b_{r_2})}{M_2}x_{14} + (\dfrac{k_2}{M_2}x_{21} + \dfrac{d_2}{M_2}x_{22}) + \Delta\Psi_{12}(x) \\ y_1 = x_{12} \end{cases} \tag{7}$$

$$\begin{cases} \dot{x}_{21} = x_{22} \\ \dot{x}_{22} = -\dfrac{(k_2+k_3)}{M_3}x_{21} - \dfrac{(d_2+d_3+b_{r_3})}{M_3}x_{22} + \dfrac{k_3}{M_3}x_{23} \\ \quad + \dfrac{d_3}{M_3}x_{24} + (\dfrac{k_2}{M_3}x_{13} + \dfrac{d_2}{M_3}x_{14}) + \Delta\Psi_{21}(x) \\ \dot{x}_{23} = x_{24} \\ \dot{x}_{24} = v_2 + \Delta\Phi_{22}(x_2) \\ y_2 = x_{24} \end{cases} \tag{8}$$

where x_{11}, x_{13}, x_{21} and x_{23} represent the displacements of the four bodies. x_{12}, x_{14}, x_{22} and x_{24} are the corresponding speeds. u_1 and $u_2 \in R$ are the inputs. $x_1 = col(x_{11}, x_{12}, x_{13}, x_{14})$, $x_2 = col(x_{21}, x_{22}, x_{23}, x_{24})$ and $x = col(x_{11}, x_{12}, x_{13}, x_{14}, x_{21}, x_{22}, x_{23}, x_{24})$. $\Delta\Phi_{11}$ and $\Delta\Phi_{22}$ represent the uncertainties in the corresponding isolated subsystem. And $\Delta\Psi_{12}$, $\Delta\Psi_{21}$ denote uncertainties in the interconnections.

The object of this paper is, for a given desired output signal $y_d(t)$, to design a decentralized sliding mode control such that the speeds of the first and the fourth bodies can track the same desired signal $y_d(t)$, i.e.

$$\lim_{t\to\infty} |y_i(t) - y_d(t)| = 0, \qquad i = 1,2$$

In the meanwhile, the variables $z_1(t) - z_2(t)$, $z_1(t) - z_4(t)$, $z_3(t) - z_4(t)$, $z_2(t) - z_3(t)$, $\dot{z}_2(t)$ and $\dot{z}_3(t)$ are bounded.

Assumption 3.1: The desired output signal $y_d(t)$ is smooth and

$$|\dot{y}_d(t)| \leq L$$

where L is a known constant for all $t \in [0, \infty)$.

Assumption 3.2: There exist some known continuous functions $\rho_i(\cdot)$ and $\mu_i(\cdot)$ such that
(i). $|\Delta\Phi_{11}| \leq \rho_1(x_1)\|x_1\|$, $\quad |\Delta\Phi_{22}| \leq \rho_2(x_2)\|x_2\|$.
(ii). $|\Delta\Psi_{12}| \leq \mu_1(x)\|x\|$, $\quad |\Delta\Psi_{21}| \leq \mu_2(x)\|x\|$.

Remark 1: The Assumption 3.2 implies that the uncertainties considered in this work are bounded. The boundedness of $\Delta\Phi$ is only related to the local information because it is assumed to be caused by the changes of spring and/or damping parameters which only affect the corresponding subsystem dynamics. The interconnected uncertainties $\Delta\Psi$ are assumed to be bounded by a function of the whole system states because the uncertainties are considered to be caused by some external factors which might affect all the dynamical performance of the four bodies.

If the speeds do not go to zero, it means that some of the states in (7)-(8) may not be bounded. For this situation, an appropriate coordinate transformation T is introduced as follows

$$\begin{cases} \xi_{11} = x_{11} \\ \xi_{12} = x_{11} - x_{13} \\ \xi_{13} = x_{14} \\ e_1 = x_{12} - y_d \end{cases} \qquad \begin{cases} \xi_{21} = x_{11} - x_{23} \\ \xi_{22} = x_{21} - x_{23} \\ \xi_{23} = x_{22} \\ e_2 = x_{24} - y_d. \end{cases} \tag{9}$$

The system dynamic (7)-(8) under the coordinate $col(\xi, e)$ can be shown in the following sliding mode standard form:

$$\dot{\xi}_i = A_{i11}\xi_i + A_{i12}e_i + H_{i1}(\xi, e, \dot{y}_d, y_d) \tag{10}$$

$$\dot{e}_i = A_{i21}\xi_i + A_{i22}e_i + B_i v_i + H_{i2}(\xi, e, \dot{y}_d, y_d) \tag{11}$$

for $i = 1,2$ where

$$A_{111} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & \frac{(k_1+k_2)}{M_2} & -\frac{(d_1+d_2+b_{r_2})}{M_2} \end{bmatrix} \quad A_{112} = \begin{bmatrix} 1 \\ 1 \\ \frac{d_1}{M_2} \end{bmatrix}$$
$$H_{11} = \begin{bmatrix} y_d \\ y_d \\ \frac{d_1}{M_2}y_d + \frac{k_2}{M_2}(\xi_{22} - \xi_{21}) + \frac{d_2}{M_2}\xi_{23} + \Pi_{12}(\xi, e) \end{bmatrix} \tag{12}$$
$$B_1 = 1 \quad H_{12} = \Gamma_{11}(\xi_1, e_1) - \dot{y}_d$$

$$A_{211} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ \frac{k_2}{M_3} & -\frac{(k_2+k_3)}{M_3} & -\frac{(d_2+d_3+b_{r_3})}{M_3} \end{bmatrix} \quad A_{212} = \begin{bmatrix} -1 \\ -1 \\ \frac{d_3}{M_3} \end{bmatrix}$$
$$H_{21} = \begin{bmatrix} e_1 \\ y_d \\ \frac{d_3}{M_3}y_d - \frac{k_2}{M_3}\xi_{12} + \frac{d_2}{M_3}\xi_{13} + \Pi_{21}(\xi, e) \end{bmatrix}$$
$$B_2 = 1 \quad H_{22} = \Gamma_{22}(\xi_2, e_2) - \dot{y}_d \tag{13}$$

and

$$\begin{aligned} \Gamma_{11}(\xi_1, e_1) &= T\Delta\Phi_{11}|_{col(\xi_1, e_1) = Tx_1} \\ \Pi_{12}(\xi, e) &= T\Delta\Psi_{12}|_{col(\xi, e) = Tx} \\ \Pi_{21}(\xi, e) &= T\Delta\Psi_{21}|_{col(\xi, e) = Tx} \\ \Gamma_{22}(\xi_2, e_2) &= T\Delta\Phi_{22}|_{col(\xi_2, e_2) = Tx_2} \end{aligned} \tag{14}$$

where $\xi = col(\xi_1, \xi_2)$, $\xi_1 = col(\xi_{11}, \xi_{12}, \xi_{13})$, $\xi_2 = col(\xi_{21}, \xi_{22}, \xi_{23})$, and $e = col(e_1, e_2)$. The transformation T has been defined in (9).

IV. SLIDING MODE STABILITY ANALYSIS

A. Sliding Surface Design

It is assumed that the desired signal $y_d(t)$ satisfying the Assumption 3.1. Then the tracking errors e_i are defined as $e_i = y_i(t) - y_d(t)$ for $i = 1,2$.

For the system (10)-(11), the sliding mode surface is defined as

$$\left[e_1, e_2\right]^T = 0. \tag{15}$$

Then, the sliding mode dynamics have the following form according to the structure of (10)-(11):

$$\dot{\xi}_s = A_{11}\xi_s + H_1(\xi, 0, y_d) \tag{16}$$

where $\xi_s = col(\xi_{11s}, \xi_{12s}, \xi_{13s}, \xi_{21s}, \xi_{22s}, \xi_{23s})$ is the considered system states and $H_1 = col(H_{11}, H_{12})$. ξ_{1is} and ξ_{2is} for $i = 1,2,3$ are the sliding mode variables when the system (10)-(11) moving on the sliding surface (15), and

$$A_{11} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & \frac{(k_1+k_2)}{M_2} & d_{122} & -\frac{k_2}{M_2} & \frac{k_2}{M_2} & \frac{d_2}{M_2} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & -\frac{k_2}{M_3} & \frac{d_2}{M_3} & \frac{k_2}{M_3} & -\frac{(k_2+k_3)}{M_3} & d_{233} \end{bmatrix} \tag{17}$$

$$H_1 = col\Big(y_d, y_d, \frac{d_1}{M_2}y_d + \Pi_{12}(\xi,e), 0, y_d, \frac{d_3}{M_3}y_d + \Pi_{21}(\xi,e)\Big) \tag{18}$$

where $d_{122} = -\frac{(d_1+d_2+b_{r_2})}{M_2}$ and $d_{233} = -\frac{(d_2+d_3+b_{r_3})}{M_3}$.

Remark 2: When the sliding motion occurs, $e = 0$. Combining with the inequaliy $\|x\| \le \sum_{i=1}^{4}|x_{1i}| + \sum_{i=1}^{4}|x_{2i}|$ and Assumption 3.2, it follows that the uncertainties $\Pi_{12}(\xi,e)$ and $\Pi_{21}(\xi,e)$ in (18) satisfy

$$\begin{aligned} \Pi_{12}(\xi,e) &\le \mu_1(T^{-1}(\xi,0))(\sum_{i=1}^{3}|\xi_{1is}| + \sum_{i=1}^{3}|\xi_{2is}|) \\ \Pi_{21}(\xi,e) &\le \mu_2(T^{-1}(\xi,0))(\sum_{i=1}^{3}|\xi_{1is}| + \sum_{i=1}^{3}|\xi_{2is}|). \end{aligned} \tag{19}$$

B. Sliding motion stability

The coordinate transformation (9) is nonsingular. Thus, the stability of (7)-(8) is equivalent to that of (10)-(11).

After calculating, the matrix A_{11} in (16) has two zero eigenvalues, corresponding to the dynamic equations

$$\dot{\xi}_{11s} = y_d, \quad i.e., \quad \dot{z}_1(t) = y_d \tag{20}$$

$$\dot{\xi}_{21s} = 0, \quad i.e., \quad \dot{z}_1(t) - \dot{z}_4(t) = 0. \tag{21}$$

Remark 3: For (20), it could get $\dot{\xi}_{11s} = \dot{x}_{11} = x_{12} = y_d$. As x_{12} and x_{11} represent the speed and displacement of the first body, respectively. The displacement trajectory x_{11} tracks a desired speed trajectory. Then, the state $x_{12} = y_d$ satisfies the system performance requirement. Hence, in the analysis of the system stability of (16), the state variable ξ_{11s} is separated from the rest of the state variables in (16) and only the boundedness of the partial state vector $[\xi_{12s}, \xi_{13s}, \xi_{21s}, \xi_{22s}, \xi_{23s}]$ is considered.

Remark 4: From (21), it could get $\xi_{21s} = z_1(t) - z_4(t) = z_1(0) - z_4(0)$, which means the displacement between the first and fourth bodies is constant and equal to the displacement at the initial time. Therefore, in the analysis of the system stability of (16), the state variable ξ_{21s} is eliminated as well. Finally, only the further partial state vector $\bar{\xi}_s = col(\xi_{12s}, \xi_{13s}, \xi_{22s}, \xi_{23s})$ is considered.

After the above analysis, remove the first and fourth row and column. The new distribution matrix $\bar{A}_{11}$ corresponding to the new state $\bar{\xi}_s$ can be shown as

$$\dot{\bar{\xi}}_s = \bar{A}_{11}\bar{\xi}_s + \bar{H}_1(\xi, 0, y_d) \tag{22}$$

where

$$\bar{A}_{11} = \begin{bmatrix} 0 & -1 & 0 & 0 \\ \frac{(k_1+k_2)}{M_2} & -\frac{(d_1+d_2+b_{r_2})}{M_2} & \frac{k_2}{M_2} & \frac{d_2}{M_2} \\ 0 & 0 & 0 & 1 \\ -\frac{k_2}{M_3} & \frac{d_2}{M_3} & -\frac{(k_2+k_3)}{M_3} & -\frac{(d_2+d_3+b_{r_3})}{M_3} \end{bmatrix} \tag{23}$$

$$\bar{H}_1 = col\Big(y_d, \frac{d_1}{M_2}y_d + \Pi_{12}(\xi,e), y_d, \frac{d_3}{M_3}y_d + \Pi_{21}(\xi,e)\Big). \tag{24}$$

Remark 5: According to the $\Pi_{12}(\xi,e)$ and $\Pi_{21}(\xi,e)$ in (19), the $\bar{H}_1$ in (24) can be shown:

$$\bar{H}_1 \le \bar{H}_{11} + \bar{H}_{12} \tag{25}$$

where

$$\bar{H}_{11} = \begin{bmatrix} 0 \\ \mu_1(T^{-1}(\bar{\xi}_s,0))(|\xi_{12s}| + |\xi_{13s}| + |\xi_{22s}| + |\xi_{23s}|) \\ 0 \\ \mu_2(T^{-1}(\bar{\xi}_s,0))(|\xi_{12s}| + |\xi_{13s}| + |\xi_{22s}| + |\xi_{23s}|) \end{bmatrix} \tag{26}$$

$$\bar{H}_{12} = col\Big(y_d, \frac{d_1}{M_2}y_d, y_d, \frac{d_3}{M_3}y_d\Big). \tag{27}$$

Next, a stability result will be presented for the interconnected system (16).

Assumption 4.1: The matrix $\bar{A}_{11}$ in (22) is Hurwitz stable.

From Assumption 4.1, it follows that for any positive definite matrix Q, the following *Lyapunov* equation has a unique solution $P > 0$

$$\bar{A}_{11}^{\top}P + P\bar{A}_{11} = -Q. \tag{28}$$

Theorem 1: Consider the sliding mode dynamics given in (16), under Assumptions 3.1-3.2, the sliding mode is uniformly ultimately bounded if there exists a domain

$$\Omega = \{(\xi_{12s}, \xi_{13s}, \xi_{22s}, \xi_{23s}) | \|\bar{\xi}_s\| \le c\} \tag{29}$$

for the constant $c > 0$ such that $M^{\top} + M > 0$ in $\Omega \backslash \{0\}$, where

$$M = \begin{bmatrix} q & 0 & 0 & 0 \\ p_2 & q+p_2 & p_2 & p_2 \\ 0 & 0 & q & 0 \\ p_3 & p_3 & p_3 & q+p_3 \end{bmatrix} \tag{30}$$

where $q = \lambda_{min}(Q)$, $p_2 = -2\|P\|\mu_1(\cdot)$ and $p_3 = -2\|P\|\mu_2(\cdot)$.

Proof 1: From the above analysis, all that needs to be proved is that (22) is uniformly ultimately bounded. For (22), consider the following *Lyapunov* function candidate

$$V(\bar{\xi}_s) = (\bar{\xi}_s)^{\top}P\bar{\xi}_s \tag{31}$$

where P satisfies (28). Then, the time derivate of $V(\bar{\xi}_s)$ along the trajectories of system (22) is given by

$$\begin{aligned} \dot{V}(\bar{\xi}_s) =& \dot{\bar{\xi}}_s^{\top}P\bar{\xi}_s + \bar{\xi}_s^{\top}P\dot{\bar{\xi}}_s \\ =&[\bar{A}_{11}\bar{\xi}_s + \bar{H}_1(\xi,0,y_d)]^{\top}P\bar{\xi}_s + \bar{\xi}_s^{\top}P[\bar{A}_{11}\bar{\xi}_s + \bar{H}_1(\xi,0,y_d)] \\ =&\bar{\xi}_s^{\top}\bar{A}_{11}^{\top}P\bar{\xi}_s + \bar{\xi}_s^{\top}P\bar{A}_{11}\bar{\xi}_s + \bar{H}_1(\xi,0,y_d)^{\top}P\bar{\xi}_s \\ &+ \bar{\xi}_s^{\top}P\bar{H}_1(\xi,0,y_d) \end{aligned} \tag{32}$$

where (28) is used to establish the above. Combining with the $\bar{H}_1$ in (25), it follows that

$$\begin{aligned} \dot{V}(\bar{\xi}_s) &\le -\bar{\xi}_s^{\top}Q\bar{\xi}_s + 2\bar{\xi}_s^{\top}P\bar{H}_1(\xi,0,y_d) \\ &\le -\lambda_{min}(Q)\|\bar{\xi}_s\|^2 + 2\|\bar{\xi}_s\|\|P\|\bar{H}_1(\xi,0,y_d) \\ &\le -\frac{1}{2}\lambda_{min}(M^{\top}+M)\|\bar{\xi}_s\|^2 + 2\|\bar{\xi}_s\|\|P\|\|\bar{H}_{12}\| \\ &\le -\frac{1}{2}[\lambda_{min}(M^{\top}+M)\|\bar{\xi}_s\| - 4\|P\|\|\bar{H}_{12}\|]\|\bar{\xi}_s\|. \end{aligned} \tag{33}$$

It is clear to see that $\dot{V} \le 0$, if $\|\bar{\xi}_s\| \ge \frac{4\|P\|\|\bar{H}_{12}\|}{\lambda_{min}(M^{\top}+M)}$. Hence, the conclusion follows.

V. DECENTRALIZED SLIDING MODE CONTROL DESIGN

The objective is now to design a decentralized sliding mode control such that the system states are driven to the sliding surface. For the interconnected system (11), the reachability condition is described by

$$\sum_{i=1}^{2} \frac{e_i(t)\dot{e}_i(t)}{|e_i(t)|} < 0. \tag{34}$$

Then, the following control law is proposed

$$v_i = -\frac{e_i}{|e_i|}[K_i(\xi_i, e_i) + L]. \tag{35}$$

Theorem 2: For the interconnected system (10)-(11), under Assumptions 3.1-3.2, the controller (35) drives the system (10)-(11) to sliding surface (15) and maintains a sliding motion on it thereafter if the controller gain $K_i(\xi_i, e_i)$ in (35) satisfies

$$\begin{aligned}\sum_{i=1}^{2} K_i(\xi_i, e_i) >& \rho_1(T^{-1}(\xi_1, e_1))(\sum_{i=1}^{3}|\xi_{1i}| + |e_1|) + \\ & \rho_2(T^{-1}(\xi_2, e_2))(\sum_{i=1}^{3}|\xi_{2i}| + |e_2|)\end{aligned} \tag{36}$$

where ρ_1 and ρ_2 are defined by Assumption 3.2, and ξ_i and e_i for $i = 1, 2$ are mentioned in (9).

Proof 2: From the analysis above, all need to be proved is that the composite reachability condition (34) is satisfied. From the error dynamics in (11), it follows

$$\dot{e}_i = v_i + \Gamma_{ii}(\xi_i, e_i) - \dot{y}_d \tag{37}$$

for $i = 1, 2$. From (35)-(37),

$$\frac{e_i\dot{e}_i}{|e_i|} = \frac{e_i}{|e_i|}[\Gamma_{ii}(\xi_i, e_i) - \dot{y}_d] - K_i(\xi_i, e_i) - L. \tag{38}$$

Combining with the Assumption 3.1, $|\dot{y}_d| \leq L$. According to (14), Assumption 3.2 and transformation (9)

$$\begin{aligned}|\Gamma_{11}| &\leq \rho_1(T^{-1}(\xi_1, e_1))(\sum_{i=1}^{3}|\xi_{1i}| + |e_1|) \\ |\Gamma_{22}| &\leq \rho_2(T^{-1}(\xi_2, e_2))(\sum_{i=1}^{3}|\xi_{2i}| + |e_2|).\end{aligned} \tag{39}$$

Substituting the above inequalities (39) into (38), it follows

$$\begin{aligned}\sum_{i=1}^{2} \frac{e_i(t)\dot{e}_i(t)}{|e_i(t)|} < &-\sum_{i=1}^{2} K_i(\xi_i, e_i) + \rho_1(T^{-1}(\xi_1, e_1))(\sum_{i=1}^{3}|\xi_{1i}| \\ &+ |e_1|) + \rho_2(T^{-1}(\xi_2, e_2))(\sum_{i=1}^{3}|\xi_{2i}| + |e_2|).\end{aligned} \tag{40}$$

Then, if $K_i(\xi_i, e_i)$ is chosen to satisfy (36), it follows that the reachability condition (34) is satisfied. Hence, the result follows.

Remark 6: Theorem 1 shows that the sliding dynamic (16) is uniformly ultimately bounded. Theorem 2 shows that the designed control (35) can drive the considered system (10)-(11) to the sliding surface (15). From the sliding mode theory, Theorems 1 and 2 together show that the considered system (10)-(11) is uniformly ultimately bounded.

From Remark 6, it follows that the closed-loop systems formed by applying the control (35) to system (10)-(11) is uniformly ultimately bounded, which implies that the variables $\|\xi\|$ and $\|e\|$ are bounded. Further, from $e_i = y_i - y_d$ and the Assumption 3.1, it is straightforward to see that y_i are bounded as well due to $y_i = e_i + y_d$.

Therefore, all the state variables of the system (10)-(11) are bounded. This shows that the proposed decentralized control (35) can not only make the system outputs track the desired signal but also keep all the state variables of the system $col(\xi, e)$ bounded.

VI. SIMULATION STUDY

In this section, simulation study is to be carried out to demonstrate the obtained results above. For simulation purpose, the desired speed signal is introduced as follows:

$$y_d(t) = \tanh(t). \ t \geq 0 \tag{41}$$

which is consistant with the practical case.

Consider the four bodies connected by the spring and damper as shown in Fig.1 where the first and the fourth bodies are controlled by actuators. It is assumed that $M_1 = M_4 = 126000$ kg, $M_2 = M_3 = 101090$ kg, $b_{r_1} = b_{r_2} = b_{r_3} = b_{r_4} = 1.08 \times 10^{-4}$ Ns/(mkg), $k_1 = k_3 = 100 \times 10^6$ N/m, $k_2 = 30 \times 10^6$ N/m, $d_1 = d_3 = 80 \times 10^4$ Ns/m, $d_2 = 40 \times 10^4$ Ns/m. The initial conditions are chosen as $x(0) = [0 \ 1.3 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0.4]^\top$.

By direct calculation, the eigenvalues of the matrix $\bar{A}_{11}$ are $-7.9137 \pm 38.9887i$ and $-3.9569 \pm 31.2019i$. All of the eigenvalues have their negative real part which means the matrix $\bar{A}_{11}$ satisfies the Assumption 4.1. In this case, once the proposed controller (35) drives the system onto the sliding surface, the tracking errors e_i will be zero. So the sliding dynamics in (22) will be uniformly ultimately bounded, which means the distance between the adjacent bodies will be bounded and the speed of four bodies will track the desired signal (41).

For simulation purpose, the uncertainties are assumed as

$$|\Delta\Phi_{11}| \leq \alpha_{11}|x_{12}| + \alpha_{12}|x_{14}| \tag{42}$$

$$|\Delta\Phi_{22}| \leq \alpha_{21}|x_{22}| + \alpha_{22}|x_{24}| \tag{43}$$

$$|\Delta\Psi_{12}| \leq sin^2(x_{11} + x_{21})\|x\| \tag{44}$$

$$|\Delta\Psi_{21}| \leq \frac{3}{2}cos^2(x_{13} + x_{22})\|x\| \tag{45}$$

where $\alpha_{11}, \alpha_{12}, \alpha_{21}$ and α_{22} are some known constants.

The proposed controller v_i according to the above analysis (35) is given by $v_i = -\frac{e_i}{|e_i|}[K_i(\xi_i, e_i) + L]$, for $i = 1, 2$ where $e_i = y_i(t) - \tanh(t)$. L is chosen as 1 and according to Assumption 3.2 and (36), the controller gain $K_1(\cdot)$ and $K_2(\cdot)$ are set as

$$\begin{aligned}K_1(\xi_1, e_1, t) &> \alpha_{11}|e_1 + \tanh(t)| + \alpha_{12}|\xi_5 + \tanh(t)| \\ K_2(\xi_2, e_2, t) &> \alpha_{21}|\xi_6 + \tanh(t)| + \alpha_{22}|e_2 + \tanh(t)|\end{aligned} \tag{46}$$

where $\alpha_{11} = 1, \alpha_{12} = 2, \alpha_{21} = 1$ and $\alpha_{22} = 2$.

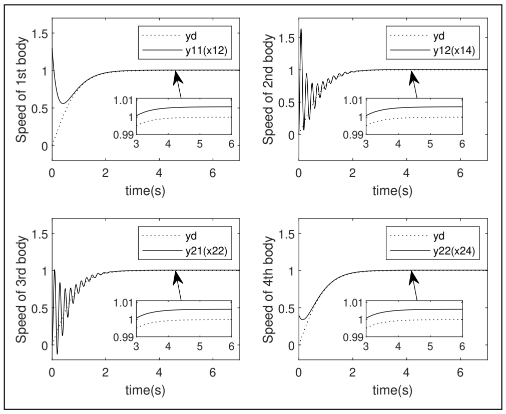

Fig. 2. Time responses of bodies' speed.

According to (28), choose Q as an identity matrix. And according to (19), the $\mu(\cdot)$ in (44)-(45) are as follows

$$\begin{aligned} \mu_1(\cdot) &= sin^2(2\xi_{11} - \xi_{21} + \xi_{22}) \\ \mu_2(\cdot) &= \frac{3}{2}cos^2(\xi_{11} - \xi_{12} + \xi_{23}). \end{aligned} \tag{47}$$

By direct calculation, it follows from (30) that

$$\begin{aligned} &M^\top + M \\ &= \begin{bmatrix} 2 & -a\mu_1 & 0 & -a\mu_2 \\ -a\mu_1 & 2-2a\mu_1 & -a\mu_1 & -a\mu_1 - a\mu_2 \\ 0 & -a\mu_1 & 2 & -a\mu_2 \\ -a\mu_2 & -a\mu_1 - a\mu_2 & -a\mu_2 & 2-2a\mu_2 \end{bmatrix} \end{aligned} \tag{48}$$

where $a = 125.13$. After direct verification, it is easy to show $M^\top + M > 0$, when $\|\bar{\xi}_s\| \le 0.02176$. And for this example, $\dot{V}(\bar{\xi}_s) \le 0$, if $\|\bar{\xi}_s\| \ge 0.0018$. Therefore, the considered system is uniformly ultimately bounded.

The tracking results are shown in Fig.2. The speeds of the first and fourth bodies can track the desired signal $y_d(t)$, despite the existence of the interconnections and uncertainties caused by the spring and damping. And the middle two bodies also have similar tracking performance when the system was stable. The simulation demonstrates that the result developed in this paper is effective.

VII. CONCLUSIONS

A sliding mode based decentralized tracking control of a class of underactuated multi-body systems with unknown uncertainties has been proposed in this paper. The desired signal is allowed to be time-varying. A decentralized sliding mode control scheme has been proposed to satisfy the reachability condition to drive the interconnected close-loop system onto the pre-designed sliding surface. Some assumptions are introduced to make sure that the system states are uniformly ultimately bounded. The simulations of the developed results to the four-body system have demonstrated that the proposed approach is effective and practicable.

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