# Distributed Static Output Feedback Robust Model Predictive Control for Process Networks

Shuzhan Zhang[1], Dongya Zhao[1*], Sarah K. Spurgeon[2]
1. College of New Energy, China University of Petroleum (East China), Qingdao, China, 266580;
2. Department of Electronic & Electrical Engineering, University College London, Torrington Place, London WC1E 7JE, U. K.
∗ Corresponding author's Email: dyzhao@upc.edu.cn

**Abstract**

This paper formulates a distributed static output feedback robust model predictive control for process networks to solve problems relating to unmeasured states and interconnected couplings. The initial conditions on the couplings are predicted by previous information and the boundedness of the predicted error is proved. In light of the static output feedback design conditions, the distributed static output feedback robust model predictive control is designed by transforming an infinite time optimization problem into a tractably solved one. The solvability of the optimization problem and the stability are proved to underpin the proposed approach. Simulations and an experimental case study are provided to validate the effectiveness of the proposed approach.

**Index Terms**

**Key Words:** robust model predictive control, distributed static output feedback, process networks

## I. Introduction

Modern industrial chemical plants consist of many process units arranged in a complex structure which produces a process network [1], such as a multistage extraction process [2], a gas boiler heating system [3] and an ethanol production process [4]. Such a process network contains many subsystems interacting with each other through mass and energy interconnections [5] and these interconnections result in strong coupling among subsystems. Plant operations are usually limited by the actuators and process equipment and such hard constraints should be considered within any control design [6], [7]. In addition, modeling error and external disturbances are inevitable in physical processes, which makes it essential to consider the system uncertainty within the design approach[8]. Achieving optimality is a major challenge in the control of process networks [9].

In practice, there are broadly three control frameworks for process networks. Centralized control is widely used to regulate process variables at desired values traditionally [10], but process networks are usually large-scale systems where the implementation of centralized methods might be hindered by the necessity for large computational load [11]. Decentralized control is often used to reduce the computational load in process networks. However, this may result in limited capacity for optimizing the control performance [12], [13]. More recently, distributed control schemes have been adopted to approximate the centralized objective by designing a local controller associated with each subsystem and exchanging information among all subsystems [14], [15], [16]. For process networks, distributed robust model predictive control (RMPC) is a good candidate due to its inherent ability to handle constraints and uncertainties [17].

A significant purpose of distributed control for process networks is to reduce the computational load in the presence of couplings when compared with a centralized control approach. Distributed control seeks to achieve global stabilisation and/or performance. Distributed RMPC is a very suitable candidate control strategy for this problem because it can accommodate the actions of other actuators in computing the control action of local actuators in real time [18]. This paradigm has been extensively used for process networks [19], [18], [20], [21], [22]. For example, distributed RMPC has been used in [23] to control process networks with a parallel structure.

However, most of the existing literature on distributed RMPC is based on the assumption that all the states are available for controller implementation. Note that the system states cannot be fully measured in many practical cases [24], [25]. In this case, an approach to distributed output feedback RMPC (OFRMPC) is desirable. The majority of such distributed OFRMPC approaches adopt dynamic output feedback and frequently an observer approach is used [26], [27], including a tube-based minimax observer [28], a Luenberger observer [29] and a moving horizon observer [30]. Due to the implementation of the state observer the system order is typically increased [31], [32], [33], which in turn may increase the controller complexity, produce more difficult stability analysis and even reduce control accuracy [34], [35], [33]. When compared with such a distributed dynamic OFRMPC, distributed static OFRMPC does not increase the order of the system and is more straightforward both in terms of design and implementation.

Although a large amount of research has been carried out on the development of static output feedback controllers [36], [37], there is relatively little literature considering distributed static output feedback control. This existing work is mainly focused on multi-agent systems [38], [39], [40], [41], [42]. Note that the mass and energy interconnections have not been considered in these frameworks for distributed control of multi-agent systems and these methods cannot be applied to process networks directly. It is thus appropriate to consider the design of distributed static OFRMPC for process networks, which can deal with the interconnection terms, the presence of uncertainty and also achieve optimal performance in the presence of constraints. To the best of the authors' knowledge, there are no distributed static OFRMPC approaches yet proposed in the literature.

It is challenging to design a distributed static OFRMPC for process networks. The coupling between subsystems has to be considered to guarantee global performance and an assumption of static output feedback control increases the challenge still further. The initial conditions are required at each prediction step to guarantee the performance of a distributed static OFRMPC. When the states are not available for measurement, the initial values must be predicted. The error between the predicted value and the real value of the initial conditions may render a recursive solution infeasible. Careful consideration of these prediction errors is required in the analysis. In addition, the constraints, the uncertainties and the nonconvexity of the static output feedback problem should be considered to derive an explicit distributed static OFRMPC.

Motivated by the above description, a distributed static OFRMPC algorithm is initially proposed for a discrete time linear system subject to uncertainties. The main contributions of this paper are as follows: (1) The state and input couplings between subsystems are explicitly considered to ensure global performance. An iterative algorithm is designed to take care of the coupling terms where the state requirement is transformed into a requirement on the initial conditions. (2) The initial conditions on the couplings as well as the OFRMPC are both predicted from previous information and the boundedness of the predicted error is proved. (3) In light of the static output feedback design conditions in [43] and [44], a non-convex distributed static OFRMPC is designed by transforming an infinite time optimization problem into a tractably solvable one. Effectively the solution to the original problem is transformed to the solution of a set of LMIs. In addition, the overall control objective is to further guarantee the global performance by designing the distributed static output model predictive control. After solving these issues, a step by step control algorithm is presented to facilitate the practical design and the closed-loop system stability is analyzed. Compared with distributed state feedback RMPC, the problem of unavailable state information is solved in this paper by deploying a static output feedback scheme. Compared with centralized static OFRPMC, the computational time can be reduced by the proposed approach with almost no reduction in the global performance. Compared with distributed dynamic OFRMPC, the order of the system is not increased by using the proposed approach so that the calculation load can be reduced whilst maintaining high levels of performance. Moreover, the proposed method can cope with the couplings and constraints inherent in process networks more effectively than existing distributed static output feedback control methods that were designed for multi-agent systems.

The paper is organized as follows. In Section II, the problem is formulated and the essential assumptions and definitions are given. A distributed static OFRMPC algorithm is proposed and its stability is addressed in Section III. The results of simulations and an experimental trial are demonstrated in Section IV to validate the proposed approach. Finally, some conclusions are drawn in Section V.

## II. Problem formulation

Consider a linear discrete time system composed of $N$ subsystems coupled via states and inputs. Subsystem $i$ can receive information from all the other $(N-1)$ subsystems and the dynamic model with uncertainties for the $i$th subsystem is given by the following equation:

$$\begin{aligned} x_i(k+1) &= A_{ii}x_i(k) + B_{ii}u_i(k) + D_iw_i(k) + \sum_{j=1, j\neq i}^{N} [A_{ij}x_j(k) + B_{ij}u_j(k)] \\ y_i(k) &= C_ix_i(k) + v_i(k) \end{aligned} \tag{1}$$

where $x_i(k) \in \mathbb{X}_i \subseteq \mathbb{R}^{n_i}$ is the state vector, $u_i(k) \in \mathbb{U}_i \subseteq \mathbb{R}^{m_i}$ is the control input and $y_i(k) \in \mathbb{Y}_i \subseteq \mathbb{R}^{p_i}$ is the output vector. $w_i(k) \in \mathbb{W}_i \subseteq \mathbb{R}^{w_i}$ is the unknown disturbance and $v_i(k) \in \mathbb{V}_i \subseteq \mathbb{R}^{p_i}$ is the unknown noise. $\mathbb{X}_i$, $\mathbb{U}_i$ and $\mathbb{Y}_i$ are polyhedral and polytopic constraint sets, respectively, $\mathbb{W}_i$ and $\mathbb{V}_i$ are $\mathbb{C}$-sets. $A_{ii} \in \mathbb{R}^{n_i\times n_i}$, $B_{ii} \in \mathbb{R}^{n_i\times m_i}$, $A_{ij} \in \mathbb{R}^{n_i\times n_j}$, $B_{ij} \in \mathbb{R}^{n_i\times m_j}$, $C_i \in \mathbb{R}^{p_i\times n_i}$, $D_i \in \mathbb{R}^{n_i\times w_i}$, $i = 1, \cdots, N$, $j = 1, \cdots, N$.

For the $i$th subsystem (1), the interconnection term $\sum_{j=1, j\neq i}^{N} [A_{ij}x_j(k) + B_{ij}u_j(k)]$ caused by the coupling between mass and energy appears. Note that $x_j(k)$ and $u_j(k)$ are in the coupling term, and will not be available for calculation of the current control for system (1) because the state $x_j(k)$ cannot be measured and the control $u_i(k)$ also requires $u_j(k)$. Design complexity thus arises due to the couplings as well as the unknown states. The unknown disturbance and the unknown noise are bounded as:

$$w_i(k)^T w_i(k) \leq x_i(k)^T x_i(k) \tag{2}$$

$$v_i(k)^T v_i(k) \leq x_i(k)^T C_i^T C_i x_i(k) \tag{3}$$

Note that $w_i(k)$ and $v_i(k)$ are inevitable in process networks and can be handled by distributed RMPC effectively. However the upper bounds of $w_i(k)$ and $v_i(k)$ are state-dependent and it is more complex to deal with them within a static output feedback scheme. For system (1), the following assumptions are made:

*Assumption 1:* The pair $(A_{ii}, B_{ii})$ is assumed to be controllable.

*Assumption 2:* $C_i$ has full row rank and $B_{ii}$ has full column rank. There exists an invertible matrix $G_i \in \mathbb{R}^{n_i \times p_i}$ and a unit matrix $I$ such that:

$$C_i G_i = \left[ \begin{array}{cc} I & 0 \end{array} \right] \tag{4}$$

*Assumption 3:* $rank(C_i B_{ii}) = m_i$ and there are no invariant zeros of the triple $(A_{ii}; B_{ii}; C_i)$ outside the unit disk.

*Remark 1:* Assumption 1 is the basic assumption for controllability of system (1) and Assumption 2 incorporates the solvability conditions for solution of a static output feedback control problem. Assumption 3 puts forward necessary and sufficient conditions on system (1) which are independent of the state-space representation. For the first condition in 3, $rank(C_i B_{ii}) = m_i$ means the nominal subsystem is relative degree one. For the second condition in 3, there are no invariant zeros of the triple $(A_{ii}; B_{ii}; C_i)$ outside the unit disk, which ensures the nominal subsystem is minimum phase. Under these assumptions, the static output feedback gain $K_i$ exists and a distributed static OFRMPC can be designed. The conditions for existence of a static output feedback gain for centralized continuous systems have been presented in [43]; $rank(CB) = m$ and no invariant zeros of the triple $(A; B; C)$ lie in $\mathbb{C}_+$. These conditions have been extended for a discrete-time linear system in [44].

A distributed static OFRMPC law can be designed:

$$u_i(k) = K_i y_i(k) \tag{5}$$

where $K_i \in \mathbb{R}^{m_i \times p_i}$ is the static output feedback gain matrix.

The following performance index is selected for distributed static OFRMPC design:

$$\min_{u_i(k)} \max_{w_i(k), v_i(k)} J_i(k) \tag{6}$$

s.t.

$$\begin{array}{c} u_i(k) \in \mathbb{U}_i \\ y_i(k) \in \mathbb{Y}_i \end{array} \tag{7}$$

where

$$\begin{aligned} J_i(k) = & \sum_{l=0}^{\infty} [x_i(k+l\,|k)^T H_i^1 x_i(k+l\,|k) + u_i(k+l\,|k)^T H_i^2 u_i(k+l\,|k)] \\ & + \sum_{j=1, j\neq i}^{N} \sum_{l=0}^{\infty} [x_j(k+l\,|k)^T H_j^1 x_j(k+l\,|k) + u_j(k+l\,|k)^T H_j^2 u_j(k+l\,|k)] \end{aligned} \tag{8}$$

and $H_i^1 \in \mathbb{R}^{n_i \times n_i}$, $H_i^2 \in \mathbb{R}^{m_i \times m_i}$, $H_j^1 \in \mathbb{R}^{n_j \times n_j}$ and $H_j^2 \in \mathbb{R}^{m_j \times m_j}$ are weighted symmetric positive definite matrices. Here $x_i(k+l\,|k)$ represents the prediction of the state vector of the system at time $k+l$ given the information available at time $k$ based on the prediction model: $x_i(k+l+1\,|k) = A_{ii}x_i(k+l\,|k) + B_{ii}u_i(k+l\,|k) + \sum_{j=1, j\neq i}^{N} [A_{ij}x_j(k+l\,|k) + B_{ij}u_j(k+l\,|k)]$.

In (7), the input constraints and the output constraints are considered together; these need to be handled carefully in the control design if the performance is not to be adversely affected. Equation (6) is a "min-max" optimization which is not trivial to solve because the performance index (8) is an infinite horizon optimization and also because $x_i(k)$ and $x_j(k)$ are not measurable. At the time period $k$, $x_i(k+0\,|k)$ and $x_j(k+0\,|k)$ are needed to predict the future $J_i(k)$ in the control calculation. These initial conditions have to be predicted which leads to design difficulty and analysis challenges.

At the time period $k$, the closed loop equation is given by:

$$x_i(k+1) = A_{ii}x_i(k) + B_{ii}K_iC_ix_i(k) + B_{ii}K_iv_i(k) + D_iw_i(k) + \sum_{j=1, j\neq i}^{N} [A_{ij}x_j(k) + B_{ij}K_jC_jx_j(k) + B_{ij}K_jv_j(k)] \tag{9}$$

The control objective is to design a distributed static OFRMPC (5) for system (1) by solving the optimization problem (6) under disturbance (2), noise (3) and constraints (7).

*Remark 2:* In system (1), both the state and input couplings are considered. This formulation is general and can be written in the following centralized form:

$$\begin{bmatrix} x_1(k+1) \\ \vdots \\ x_i(k+1) \\ \vdots \\ x_N(k+1) \end{bmatrix} = \begin{bmatrix} A_{11} & \cdots & A_{1i} & \cdots & A_{1N} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ A_{i1} & \cdots & A_{ii} & \cdots & A_{iN} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ A_{N1} & \cdots & A_{Ni} & \cdots & A_{NN} \end{bmatrix} \begin{bmatrix} x_1(k) \\ \vdots \\ x_i(k) \\ \vdots \\ x_N(k) \end{bmatrix} + \begin{bmatrix} B_{11} & \cdots & B_{1i} & \cdots & B_{1N} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ B_{i1} & \cdots & B_{ii} & \cdots & B_{iN} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ B_{N1} & \cdots & B_{Ni} & \cdots & B_{NN} \end{bmatrix} \begin{bmatrix} u_1(k) \\ \vdots \\ u_i(k) \\ \vdots \\ u_N(k) \end{bmatrix}$$
$$+ \begin{bmatrix} D_1 & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & D_i & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & D_N \end{bmatrix} \begin{bmatrix} w_1(k) \\ \vdots \\ w_i(k) \\ \vdots \\ w_N(k) \end{bmatrix}$$

$$\begin{bmatrix} y_1(k) \\ \vdots \\ y_i(k) \\ \vdots \\ y_N(k) \end{bmatrix} = \begin{bmatrix} C_1 & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & C_i & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & C_N \end{bmatrix} \begin{bmatrix} x_1(k) \\ \vdots \\ x_i(k) \\ \vdots \\ x_N(k) \end{bmatrix} + \begin{bmatrix} v_1(k) \\ \vdots \\ v_i(k) \\ \vdots \\ v_N(k) \end{bmatrix}$$

Hence system (1) can approximate a centralized objective by designing a distributed static OFRMPC. When compared with the centralized static OFRMPC, the proposed method only requires the calculation of a local control $u_i$, which will reduce the computational time.

*Remark 3:* In the existing distributed static output feedback control methods, the state and input couplings $\sum_{j=1, j\neq i}^{N} [A_{ij}x_j(k) + B_{ij}u_j(k)]$ are usually not considered. In this paper, these couplings will be handled directly by the proposed method. Although the design complexity is increased, the global performance is guaranteed by introducing these couplings. Moreover, the performance index of the distributed static OFRMPC which is given in (8) considers the overall control objective for the entire plant since it takes into account the goals of the other controllers. This will ensure global performance is achieved.

## III. Static output feedback robust distributed model predictive control

### A. *Static output feedback robust distributed model predictive control algorithm*

The performance index in (8) has an infinite time horizon. In order to design the distributed static OFRMPC law (5), the optimization problem (6)-(7) should be transformed into a solvable problem. Rewrite (8) as:

$$J_i(k) = \sum_{l=0}^{\infty} [x(k+l\,|k)^T H^1 x(k+l\,|k) + u(k+l\,|k)^T H^2 u(k+l\,|k)] \tag{10}$$

where

$$x(k+l\,|k) = \begin{bmatrix} x_1(k+l\,|k)^T & \cdots & x_i(k+l\,|k)^T & \cdots & x_N(k+l\,|k)^T \end{bmatrix}^T$$
$$u(k+l\,|k) = \begin{bmatrix} u_1(k+l\,|k)^T & \cdots & u_i(k+l\,|k)^T & \cdots & u_N(k+l\,|k)^T \end{bmatrix}^T$$
$$H^1 = diag\{H_1^1, \cdots, H_i^1, \cdots, H_N^1\}$$
$$H_2 = diag\{H_1^2, \cdots, H_i^2, \cdots, H_N^2\}$$

and $x(k+l\,|k) \in \mathbb{R}^{(n_1+n_2+\cdots+n_N)}$, $u(k+l\,|k) \in \mathbb{R}^{(m_1+m_2+\cdots+m_N)}$, $H^1 \in \mathbb{R}^{(n_1+n_2+\cdots+n_N)\times(n_1+n_2+\cdots+n_N)}$ and $H^2 \in \mathbb{R}^{(m_1+m_2+\cdots+m_N)\times(m_1+m_2+\cdots+m_N)}$.

It is assumed that there exists a quadratic function $V_i(x_i(k\,|k))$:

$$V_i(x_i(k\,|k)) = x_i(k\,|k)^T P_i x_i(k\,|k) \tag{11}$$

where $P_i \in \mathbb{R}^{(n_i \times n_i)}$ is a symmetric positive definite matrix. When $k \to \infty$, if $x_i(\infty\,|k) \to 0$, then $V_i(x_i(\infty\,|k)) \to 0$. Assume that at time step $k$ and $l \geq 0$, the quadratic function $V_i(k)$ satisfies the following robust inequality constraints:

$$V_i(x_i(k+l+1\,|k)) - V_i(x_i(k+l\,|k)) \leq -[x(k+l\,|k)^T H^1 x(k+l\,|k) + u(k+l\,|k)^T H^2 u(k+l\,|k)] \tag{12}$$

Add the two sides of (12) from $l = 0$ to $l = \infty$:

$$- V_i(x_i(k|k)) \leq -J_i(k) \tag{13}$$

Then, the following inequality can be given:

$$\max_{w_i(k),v_i(k)} J_i(k) \leq V_i(x_i(k|k)) \tag{14}$$

It can be seen from (14) that $V_i(x_i(k|k))$ is essentially an upper bound of $J_i(k)$ and the infinite time domain optimization problem (6) is transformed into a solvable finite time problem. It is necessary to find the optimal solution to the upper bound function $V_i(x_i(k|k))$ under the constraints (12):

$$\min_{u_i(k)} V_i(x_i(k|k)), s.t.(7), (12) \tag{15}$$

Then the optimization problem (6) is solved by appealing to the following Theorem.

*Theorem 1:* For system (1) under Assumptions 1, 2 and 3, if there exist scalars $\gamma_i$, $\alpha_{u_i}$, $\alpha_{y_i}$ and positive definite matrices $Y_i > 0$, $M_i > 0$, $S_i > 0$, $Y_{i1}$, $M_{i11}$, then, the static output feedback robust distributed model predictive control $u_i(k) = K_i y_i(k)$ can be designed in terms of the following LMIs:

$$\min_{\gamma_i, M_i, S_i, Y_i, Y_{i1}, M_{i11}} \gamma_i \tag{16}$$

$$s.t. \begin{bmatrix} 1 & x_i(k|k)^T \\ x_i(k|k) & S_i \end{bmatrix} \geq 0 \tag{17}$$

$$\begin{bmatrix} {E_i}^T S_i E & (A_i S + B_i Y)^T & (B_i Y)^T & S(H^{1^{1/2}})^T & \sqrt{2}(H^{2^{1/2}} Y)^T & S(D_i E_i)^T \\ A_i S + B_i Y & S_i & 0 & 0 & 0 & 0 \\ B_i Y & 0 & S_i & 0 & 0 & 0 \\ H^{1^{1/2}} S & 0 & 0 & {\gamma_i}^{-1} I & 0 & 0 \\ \sqrt{2} H^{2^{1/2}} Y & 0 & 0 & 0 & {\gamma_i}^{-1} I & 0 \\ D_i E_i S & 0 & 0 & 0 & 0 & S_i \end{bmatrix} \geq 0 \tag{18}$$

$$\begin{bmatrix} \alpha_{u_i} I & \sqrt{2} Y_i \\ \sqrt{2} Y_i^T & S_i \end{bmatrix} \geq 0 \tag{19}$$

$$\begin{bmatrix} \alpha_{y_i} I & \sqrt{2}(A_i S + B_i Y) \\ \sqrt{2}(S A_i^T + Y^T B_i^T) & S \end{bmatrix} \geq 0 \tag{20}$$

where $K_i = Y_{i1} {M_{i11}}^{-1}$, $M_i = \begin{bmatrix} M_{i11} & 0 \\ M_{i21} & M_{i22} \end{bmatrix}$, $Y_i = \begin{bmatrix} Y_{i1} & 0 \end{bmatrix}$, $S = diag\{S_1, \cdots, S_i, \cdots, S_N\}$, $Y = diag\{Y_1, \cdots, Y_i, \cdots, Y_N\}$, $E_i = diag\left\{ \begin{matrix} {Z_i}^- & I_i & Z_i^+ \end{matrix} \right\} \in R^{n_i \times (n_1 + \cdots + n_N)}$, $I_i \in R^{n_i \times n_i}$, ${Z_i}^- \in R^{n_i \times (n_1 + \cdots + n_{i-1})}$ and $Z_i^+ \in R^{n_i \times (n_{i+1} + \cdots + n_N)}$ are zero matrices, $M_1 = diag\{M_{11}, \cdots, M_{i1}, \cdots, M_{N1}\}$, $K = diag\{K_1, \cdots, K_i, \cdots, K_N\}$,

$$A_i = \begin{bmatrix} A_{i1} & \cdots & A_{ii} & \cdots & A_{iN} \end{bmatrix}$$

$$B_i = \begin{bmatrix} B_{i1} & \cdots & B_{ii} & \cdots & B_{iN} \end{bmatrix}$$

**Proof**:

The optimization problem (15) is equivalent to the following equations:

$$x_i(k|k)^T P_i x_i(k|k) \leq \gamma_i \tag{21}$$

Define $S_i = \gamma_i P_i^{-1}$, then:

$$x_i(k|k)^T {S_i}^{-1} x_i(k|k) \leq 1 \tag{22}$$

According to the Schur Complement Lemma, (22) can be converted into:

$$\begin{gathered} \min_{\gamma_i, S_i, Y_i, M_i} \gamma_i \\ s.t. \begin{bmatrix} 1 & x_i(k|k)^T \\ x_i(k|k) & S_i \end{bmatrix} \geq 0 \end{gathered} \tag{23}$$

Rearrange the closed loop system (9):

$$
\begin{aligned}
x_i(k+1) &= (A_{ii}+B_{ii}K_iC_i)x_i(k)+B_{ii}K_iv_i(k)+D_iw_i(k)+\sum_{j=1,j\neq i}^{N}[A_{ij}x_j(k)+B_{ij}K_jC_jx_j(k)+B_{ij}K_jv_j(k)] \\
&= (A_i+B_iKC)x(k)+B_iKv(k)+D_iw_i(k)
\end{aligned} \tag{24}
$$

where $v(k)=\begin{bmatrix} v_1(k)^T & \cdots & v_i(k)^T & \cdots & v_N(k)^T \end{bmatrix}^T$, $C=diag\{C_1,\cdots,C_i,\cdots,C_N\}$.
Let $L_i(k)={x_i}^T(k+1)P_ix_i(k+1)-{x_i}^T(k)P_ix_i(k)+x^T(k)H^1x(k)+u^T(k)H^2u(k)$ and substitute (24) into (12):

$$
\begin{aligned}
L_i(k) &= [(A_i+B_iKC)x(k)+B_iKv(k)+D_iw_i(k)]^TP_i[(A_i+B_iKC)x(k)+B_iKv(k)+D_iw_i(k)] \\
&\quad -{x_i}^T(k)P_ix_i(k)+x^T(k)H^1x(k)+u^T(k)H^2u(k) \\
&\leq [(A_i+B_iKC)x(k)]^TP_i[(A_i+B_iKC)x(k)]+[B_iKv(k)]^TP_i[B_iKv(k)]+[D_iw_i(k)]^TP_i[D_iw_i(k)] \\
&\quad -{x_i}^T(k)P_ix_i(k)+x^T(k)H^1x(k)+u^T(k)H^2u(k)
\end{aligned} \tag{25}
$$

Consider (2), (3):

$$
\begin{aligned}
L_i(k) &\leq [(A_i+B_iKC)x(k)]^TP_i[(A_i+B_iKC)x(k)]+[B_iKCx(k)]^TP_i[B_iKCx(k)]+[D_ix_i(k)]^TP_i[D_ix_i(k)] \\
&\quad -{x_i}^T(k)P_ix_i(k)+x^T(k)H^1x(k)+u^T(k)H^2u(k)
\end{aligned} \tag{26}
$$

Substitute (5) into (26):

$$
\begin{aligned}
L_i(k) &\leq [(A_i+B_iKC)x(k)]^TP_i[(A_i+B_iKC)x(k)]+[B_iKCx(k)]^TP_i[B_iKCx(k)]+[D_ix_i(k)]^TP_i[D_ix_i(k)] \\
&\quad +x^T(k)H^1x(k)+2[KCx(k)]^TH^2[KCx(k)]-{x_i}^T(k)P_ix_i(k) \\
&\leq x^T(k)[(A_i+B_iKC)^TP_i(A_i+B_iKC)+(B_iKC)^TP_i(B_iKC) \\
&\quad +H^1+2(KC)^TH^2(KC)+{E_i}^T({D_i}^TP_iD_i-P_i)E_i]x(k)\leq 0
\end{aligned} \tag{27}
$$

Noting $P_i=\gamma_i{S_i}^{-1}$, then according to the Schur Complement Lemma, (27) can be converted into:

$$
\begin{bmatrix}
{E_i}^T{S_i}^{-1}E_i & (A_i+B_iKC)^T & (B_iKC)^T & ({H^1}^{1/2})^T & \sqrt{2}({H^2}^{1/2}KC)^T & (D_iE_i)^T \\
A_i+B_iKC & S_i & 0 & 0 & 0 & 0 \\
B_iKC & 0 & S_i & 0 & 0 & 0 \\
{H^1}^{1/2} & 0 & 0 & {\gamma_i}^{-1}I & 0 & 0 \\
\sqrt{2}{H^2}^{1/2}KC & 0 & 0 & 0 & {\gamma_i}^{-1}I & 0 \\
D_iE_i & 0 & 0 & 0 & 0 & S_i
\end{bmatrix} \geq 0 \tag{28}
$$

Multiplying both sides of (28) by diagonal matrix $diag\{S,I,I,I,I,I\}$ yields:

$$
\begin{bmatrix}
{E_i}^TS_iE_i & S(A_i+B_iKC)^T & S(B_iKC)^T & S({H^1}^{1/2})^T & \sqrt{2}S({H^2}^{1/2}KC)^T & S(D_iE_i)^T \\
A_iS+B_iKCS & S_i & 0 & 0 & 0 & 0 \\
B_iKCS & 0 & S_i & 0 & 0 & 0 \\
{H^1}^{1/2}S & 0 & 0 & {\gamma_i}^{-1}I & 0 & 0 \\
\sqrt{2}{H^2}^{1/2}KCS & 0 & 0 & 0 & {\gamma_i}^{-1}I & 0 \\
D_iE_iS & 0 & 0 & 0 & 0 & S_i
\end{bmatrix} \geq 0 \tag{29}
$$

Let $S_i=G_iM_i$, $KCS=diag\{K_1C_1S_1,\cdots,K_iC_iS_i,\cdots,K_1C_1S_N\}$, according to Assumption 2:

$$
\begin{aligned}
K_iC_iS_i &= K_iC_iG_iM_i = K_i\begin{bmatrix} I & 0 \end{bmatrix}M_i \\
&= K_i\begin{bmatrix} M_{i11} & 0 \end{bmatrix} = \begin{bmatrix} Y_{i1} & 0 \end{bmatrix}
\end{aligned} \tag{30}
$$

Hence, $KCS=Y$, (29) is equivalent to (18), $K_i=Y_{i1}{M_{i11}}^{-1}$ and $u_i(k)=Y_{i1}{M_{i11}}^{-1}y_i(k)$.

The constraints (7) need to be handled. For $u_i(k) \in \mathbb{U}_i$, assume that there exists a scalar $\alpha_{u_i}$ making:

$$\|u_i(k+l\,|k)\| = \alpha_{u_i} \tag{31}$$

Then,

$$\begin{aligned} &\max_{l\geq 0}\|u_i(k+l\,|k)\| = \max_{l\geq 0}\|K_i y_i(k+l\,|k)\| = \max_{l\geq 0}\|K_i[C_i x_i(k+l\,|k) + v_i(k+l\,|k)]\| \\ &\leq \max_{l\geq 0}[\|K_i C_i x_i(k+l\,|k)\| + \|K_i v_i(k+l\,|k)\|] \leq 2\max_{v_i\in\mathbb{V}_i}\|K_i C_i x_i(k+l\,|k)\| \end{aligned} \tag{32}$$

Consider $K_i C_i = Y_i S_i^{-1}$

$$\max_{v_i\in\mathbb{V}_i}\left\|Y_i {S_i}^{-1} x_i(k+l\,|k)\right\| \leq \frac{1}{2}\alpha_{u_i} \tag{33}$$

Let $z_i = {S_i}^{-\frac{1}{2}} x_i$ which is a common definition [45]. It then follows that:

$$\max_{x_i\in_i}\left\|Y_i {S_i}^{-1} x_i(k+l\,|k)\right\| = \max_{{z_i}^T z_i\in_i}\left\|{S_i}^{-\frac{1}{2}} {Y_i}^T Y_i {S_i}^{-\frac{1}{2}} x_i(k+l\,|k)\right\| = \lambda_{\max}({S_i}^{-\frac{1}{2}} {Y_i}^T Y_i {S_i}^{-\frac{1}{2}}) \leq \frac{1}{2}\alpha_{u_i} \tag{34}$$

According to the Schur Complement Lemma, (34) can be converted into:

$$\begin{bmatrix} \alpha_{u_i} I & Y_i\sqrt{2}{S_i}^{-\frac{1}{2}} \\ {S_i}^{-\frac{1}{2}}\sqrt{2}Y_i^T & I \end{bmatrix} \geq 0 \tag{35}$$

Multiplying both sides of (35) by diagonal matrix $diag(I, {S_i}^{\frac{1}{2}})$ yields:

$$\begin{bmatrix} \alpha_{u_i} I & \sqrt{2}Y_i \\ \sqrt{2}Y_i^T & S_i \end{bmatrix} \geq 0 \tag{36}$$

which is equivalent to (19). For $Y_i(k) \in \mathbb{Y}_i$, it can also be transformed into the following form:

$$\|y_i(k+l\,|k)\| = \alpha_{y_i} \tag{37}$$

Then, using a similar procedure as used for the input constraint, (38) can be obtained:

$$\begin{bmatrix} \alpha_{y_i} I & \sqrt{2}(A_i S + B_i Y) \\ \sqrt{2}(S A_i^T + Y^T B_i^T) & S \end{bmatrix} \geq 0 \tag{38}$$

Hence, (7) is guaranteed. Q.E.D.

By considering Theorem 1, problem (6)-(8) can be solved by the LMI optimization problem:

$$\min_{\gamma_i, M_i, S_i, Y_i, G_{i1}, \alpha_{u_i}, \alpha_{y_i}} \gamma_i \qquad s.t. (17), (18), (19), (20) \tag{39}$$

*Remark 4:* From the proof of Theorem 1, it can be seen that all states are written in a quadratic form using the robust inequality constraints (12). According to (2) and (3), the disturbance and noise are handled in (27). The couplings and uncertainties are solved by LMI (18). Under Assumption 2, the non-convex static output feedback design is converted to a convex optimization problem and sufficient conditions are presented. The explicit distributed static OFRMPC law is given by Theorem 1, but also the constraints are handled by the LMIs (19) and (20).

*Remark 5:* In Theorem 1, inequality (17) must use the state information $x_i(k\,|k)$ as the initial value for iterative calculation. Note that the iterative calculation does not require high accuracy of the initial value, so the value of the current state $x_i(k\,|k)$ can be replaced by a predicted state $\bar{x}_i(k\,|k-1)$, which means $x_i(k\,|k) \approx \bar{x}_i(k\,|k-1)$. $\bar{x}_i(k\,|k-1)$ is predicted by previous information. At time $k$, $u_i(k-l), l = 1, \cdots, k$, and $x_i(0)$ for all $i = 1, \cdots, N$ are available. It follows that $\bar{x}_i(k\,|k-1)$ can be obtained from this information based on the nominal prediction model: $\bar{x}_i(k\,|k-1) = A_i[A^{k-1}x(0) + \sum_{l=0}^{k-2} A^l B u(k-l-2)] + B_i u(k-1)$, where $A$ and $B$ have been defined in Remark 2 and $A_i$ and $B_i$ have been defined in Theorem 1. The predicted error is defined as: $\bar{e}_i(k) = x_i(k\,|k) - \bar{x}_i(k\,|k-1)$. The recursive feasibility cannot be guaranteed if the predicted error is not bounded, since an unbounded error will result in the non-coincidence of the information between two adjacent steps. This problem will be dealt with in the next subsection

*Remark 6:* If for the $i$th subsystem, inequality (18) is solved in a distributed fashion, it can be seen from (28) that the local controller can take advantage of the actions of other actuators in computing its own control action in real time. Take an element in inequality (28) as an example, e.g., $A_i + B_i K C$ where $A_i$, $B_i$ and $C$ are all known matrices. $K = diag\{K_1, \cdots, K_i, \cdots, K_N\}$, $K_j (j = 1, 2, ..., N, j \neq i)$ refers to the solution obtained from the previous iteration which is computed by the $j$th local controller. The $i$th local controller only needs to calculate $K_i$ during the current iteration which can reduce the computational time and will be more effective than applying a centralized method.

The step by step algorithm is given as follows:

*Algorithm 1:*

Step 1 (Initialization): At control interval $k = 0$, set $K_i = 0$.

Step 2 (Updating): At the beginning of the control interval ($k$), all the controllers exchange their output measurements and initial estimates $K_i$, set the iteration $t = 0$ and $K_i$=${K_i}^{(t=0)}$, where $t$ is the iteration number.

Step 3 (Iterations): At time step $k$, solve $N$ LMI problems (35) in parallel to obtain the matrices $\gamma_i^{(t)}$, $M_i^{(t)}$, $S_i^{(t)}$, $Y_i^{(t)}$, $G_{i1}^{(t)}$, ${\alpha_{u_i}}^{(t)}$, ${\alpha_{y_i}}^{(t)}$ and feedback gain $K_i^{(t)}$. Check the convergence for a specified error tolerance $e_i$ which is defined by the user. If $K_i^{(t)}$ satisfies:

$$\left\| {K_i}^{(t)} - {K_i}^{(t-1)} \right\| \le e_i$$

then, go to step 4. Otherwise continue to iterate, exchange the solution $K_i$ and set $t = t + 1$.

Step 4 (Implementation): Apply the control $u_i(k) = K_i(k)y_i(k)$ to the corresponding subsystems. Go to the control interval $k = k + 1$, return to step 2 and repeat the procedure.

*Remark 7:* Algorithm 1 can be thought of as a robust model predictive algorithm. The feasibility of this method can be ensured by adopting an infinite prediction horizon. It is necessary that the system is asymptotically stable or the pair $(A_{ii}, B_{ii})$ is controllable. In addition, for a given matrix $C_i$, the choice of $G_i$ is not unique. In this paper, define $G_i = \left[ \begin{array}{cc} C_i^T(C_iC_i^T)^{-1} & C_i^\dagger \end{array} \right]$ where $C_i^\dagger$ is the orthogonal basis of the zero space of the matrix $C_i$. $G_i$ satisfies $C_iG_i = \left[ \begin{array}{cc} I & 0 \end{array} \right]$ and guarantees the feasibility of Step 3 in Algorithm 1.

### *B. Closed loop system stability analysis*

The predicted error has been defined as:

$$\bar{e}_i(k) = x_i(k\,|k) - \bar{x}_i(k\,|k-1) \tag{40}$$

The boundedness of $\bar{e}_i(k)$ is proved by the following proposition:

*Proposition 1:* For each subsystem (1) with $z_i = S_i^{-\frac{1}{2}}x_i$ and $K_iC_i = Y_iS_i^{-1}$, if there exist positive definite matrices $Y_i > 0$ and $S_i > 0$ such that $\|Y_i\|$ and $\|S_i\|$ are bounded, then the predicted error is bounded.

**Proof**: Substitute the closed-loop system (9) into (40):

$$\begin{aligned} \bar{e}_i(k) &= x_i(k\,|k) - x_i(k\,|k-1) \\ &= x_i(k\,|k) - (A_i + B_iKC)x(k-1\,|k-1) + B_iKv(k-1\,|k-1) + D_iw_i(k-1\,|k-1) \end{aligned} \tag{41}$$

Using the bounds on the disturbance and the noise from equations (2) and (3):

$$\begin{aligned} \|\bar{e}_i(k)\| &= \|x_i(k\,|k) - (A_i + B_iKC)x(k-1\,|k-1) + B_iKv(k-1\,|k-1) + D_iw_i(k-1\,|k-1)\| \\ &\le \|x_i(k\,|k)\| + \|(A_i + B_iKC)x(k-1\,|k-1)\| + \|B_iKv(k-1\,|k-1)\| + \|D_iw_i(k-1\,|k-1)\| \\ &\le \|x_i(k\,|k)\| + \|(A_i + B_iKC)x(k-1\,|k-1)\| + \|B_iKCx(k-1\,|k-1)\| + \|D_ix_i(k-1\,|k-1)\| \end{aligned} \tag{42}$$

Using $z = S^{-\frac{1}{2}}x$, $KC = YS^{-1}$, it can be obtained that:

$$\begin{aligned} \|\bar{e}_i(k)\| &\le \|x_i(k\,|k)\| + \|(A_i + B_iKC)x(k-1\,|k-1)\| + \|B_iKCx(k-1\,|k-1)\| + \|D_ix_i(k-1\,|k-1)\| \\ &= \left\|{S_i}^{\frac{1}{2}}z_i(k)\right\| + \left\|(A_i + B_iKC)S^{\frac{1}{2}}z(k-1)\right\| + \left\|B_iKCS^{\frac{1}{2}}z(k-1)\right\| + \left\|D_i{S_i}^{\frac{1}{2}}z_i(k-1)\right\| \\ &\le \left\|{S_i}^{\frac{1}{2}}\right\|\|z_i(k)\| + \left\|A_iS^{\frac{1}{2}} + B_iKCS^{\frac{1}{2}}\right\|\|z(k-1)\| + \left\|B_iKCS^{\frac{1}{2}}\right\|\|z(k-1)\| + \left\|D_i{S_i}^{\frac{1}{2}}\right\|\|z_i(k-1)\| \\ &= \left\|{S_i}^{\frac{1}{2}}\right\|\|z_i(k)\| + \left\|A_iS^{\frac{1}{2}} + B_iYS^{-\frac{1}{2}}\right\|\|z(k-1)\| + \left\|B_iYS^{-\frac{1}{2}}\right\|\|z(k-1)\| + \left\|D_i{S_i}^{\frac{1}{2}}\right\|\|z_i(k-1)\| \end{aligned} \tag{43}$$

From (43), it can be seen that $\|z_i(k)\| \le 1$ and $\|z_i(k-1)\| \le 1$. If $\|Y_i\|$ and $\|S_i\|$ are bounded, $\|Y\|$ and $\|S\|$ are bounded, then $\left\|S_i^{\frac{1}{2}}\right\|$, $\left\|A_iS^{\frac{1}{2}} + B_iYS^{-\frac{1}{2}}\right\|$, $\left\|B_iYS^{-\frac{1}{2}}\right\|$ and $\left\|D_iS_i^{\frac{1}{2}}\right\|$ are bounded. Hence the predicted error $\bar{e}(k)$ is bounded. Q.E.D.

The boundedness of the predicted error is proved by Proposition 1 and the recursive feasibility can be guaranteed. To analyze the stability of the closed-loop system, the following lemma is given:

*Lemma 1:* Every feasible solution of optimization problem (39) in time $k$ is still feasible in time $N(N > k)$ [46].

*Theorem 2:* For system (1), under the distributed static OFRMPC gain $K_i$ which is given by Theorem 1, if the optimization problem (39) has a feasible solution, then the closed loop system in (9) will be asymptotically stable.

**Proof**:

According to Lemma 1, the optimization problem (39) is always feasible. The quadratic function has been defined in (11):

$$V_i(x_i(k\,|k)) = x_i(k\,|k)^TP_ix_i(k\,|k)$$

It follows that:

$$\Delta V_i(x_i(k\,|k)) = V_i(x_i(k+1\,|k)) - V_i(x_i(k\,|k)) < -[x(k\,|k)^T H^1 x(k\,|k) + u(k\,|k)^T H^2 u(k\,|k)] < 0 \tag{44}$$

where $H^1$ and $H^2$ have been defined after (10) and are symmetric positive definite matrices. Hence after $K_i$ has been implemented at time $k$, the closed-loop system is asymptotically stable. Q.E.D.

## IV. Case Study

In this section, the proposed method is verified both by simulation and experiment. The first simulation seeks to verify the effectiveness of the proposed approach in stabilising a given system and the results are compared with those achieved by using the distributed dynamic OFRMPC method and the centralized static OFRMPC method. Then, a simulation model consisting of twenty subsystems is used to illustrate the ability of the proposed method in dealing with the couplings in complex process networks. In this case the proposed method is compared with centralized static OFRMPC. Finally, a continuous stirred tank reactor (CSTR) experiment is used to further validate the proposed approach.

### *A. Stabilization Problem*

Consider system (1). Assume that there are two discrete time linear subsystems where the system matrices are given by:

$$\begin{array}{lll} A_{11} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} & A_{12} = \begin{bmatrix} 0.26 & 0.28 \\ 0 & 0.25 \end{bmatrix} & \\ A_{22} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} & A_{21} = \begin{bmatrix} 0.26 & 0.28 \\ 0 & 0.25 \end{bmatrix} & \\ B_{11} = \begin{bmatrix} 1 \\ 0.5 \end{bmatrix} & B_{12} = \begin{bmatrix} 0.03 \\ 0.15 \end{bmatrix} & D_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ B_{22} = \begin{bmatrix} 1 \\ 0.5 \end{bmatrix} & B_{21} = \begin{bmatrix} 0.03 \\ 0.15 \end{bmatrix} & D_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ C_1 = \begin{bmatrix} 1 & 0 \end{bmatrix} & C_2 = \begin{bmatrix} 1 & 0 \end{bmatrix} & H_1^1 = 2I \quad H_2^1 = I \quad H_1^2 = 2I \quad H_2^2 = I \end{array}$$

Rewrite the above distributed system in a centralized form:

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} \begin{bmatrix} u_1(k) \\ u_2(k) \end{bmatrix} + \begin{bmatrix} D_1 & 0 \\ 0 & D_2 \end{bmatrix} \begin{bmatrix} w_1(k) \\ w_2(k) \end{bmatrix}$$
$$\begin{bmatrix} y_1(k) \\ y_2(k) \end{bmatrix} = \begin{bmatrix} C_1 & 0 \\ 0 & C_2 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} v_1(k) \\ v_2(k) \end{bmatrix}$$

For the proposed method, the initial value of the control matrices is given by: $M_1 = M_2 = \begin{bmatrix} 0.5426 & 0 \\ -0.01054 & 0.4942 \end{bmatrix}, Y_1 = Y_2 = \begin{bmatrix} 0.9412 & 0 \end{bmatrix}$. For the distributed dynamic OFRMPC, the observer is of Luenberger type and the observer poles are chosen as $\lambda_1 = \begin{bmatrix} 0.7921 & 0.5952 \end{bmatrix}$ and $\lambda_2 = \begin{bmatrix} 0.6853 & 0.5024 \end{bmatrix}$. The corresponding observer gains are given by: $L_{dynamic1} = \begin{bmatrix} 2.6 & -5.7 \end{bmatrix}^T$, $L_{dynamic2} = \begin{bmatrix} 3.5 & -6.7 \end{bmatrix}^T$. For the centralized static OFRMPC, $M = diag\{M_1, M_2\}$ and $Y = diag\{Y_1, Y_2\}$. To ensure that the initial state is in the initial feasible region, as in [47], the initial states of the two subsystems are selected as $x_1(0) = \begin{bmatrix} 3 & -1 \end{bmatrix}^T$ and $x_2(0) = \begin{bmatrix} 2 & -1 \end{bmatrix}^T$. The control horizon is 1, the prediction horizon is 10 and the simulation length is 50. Consider the presence of uncertainties as described in (2) and (3) where $w_1(k) = w_2(k) = sqrt(0.1) \times randn(2,1)$ are random disturbances satisfying the Gaussian distribution and $v_1(k) = v_2(k) = sqrt(0.1) \times randn(2,1)$ are random noises satisfying the Gaussian distribution.

In the following simulations, the solid line presents the results when the system is controlled by the distributed dynamic OFRMPC, the dashed line presents the results when the system is controlled by the proposed method and the dash-dotted line gives the results when the system is controlled by a centralized static OFRMPC. The state variables are shown in Fig. 1 and Fig. 2. It is seen that both states of the two subsystems are stabilized by all three methods. The static OFRMPC methods stabilizes the states more rapidly. Since the static method uses only the output information directly and does not need to observe the states, it improves the control accuracy. The output signals of the two subsystems are shown in Fig. 3 and Fig. 4 The performance is shown in Fig. 5 and Fig. 6. It can be seen that the proposed method has very similar performance when compared with the centralized approach. However the computational time of the proposed method is only 0.54s while the centralized static OFRMPC uses 1.67s. It is shown that the couplings are handled effectively by the proposed method, so that the global performance can approximate that of the centralized one with a reduction in computational time. From Fig. 7 and Fig. 8, it can be seen that the control inputs corresponding to the dynamic control are larger than for the static control. This validates the effectiveness of the proposed approach. The constraints and uncertainties are handled and the distributed RMPC can work well using the proposed static output feedback scheme. The proposed approach has better performance than the dynamic approach when the states are not available and reduces the computational time when compared to the centralized method. The control performance is very similar in both cases.

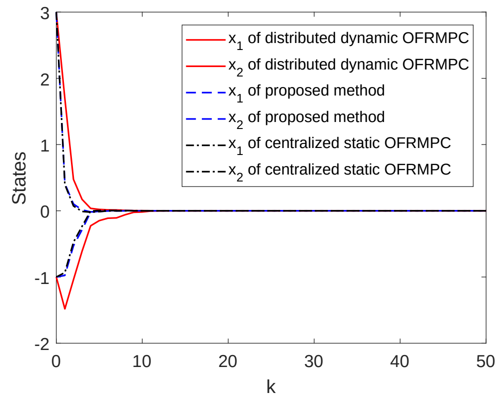


Fig. 1: States of the three methods for subsystem 1.

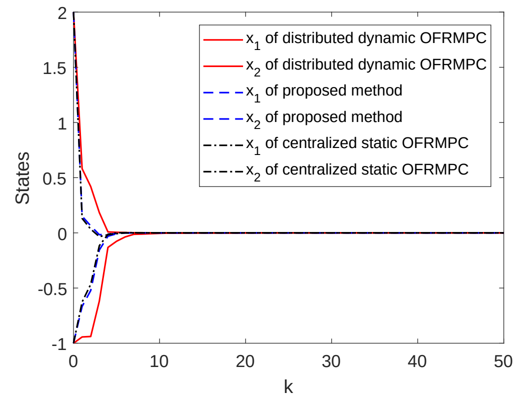


Fig. 2: States of the three methods for subsystem 2.

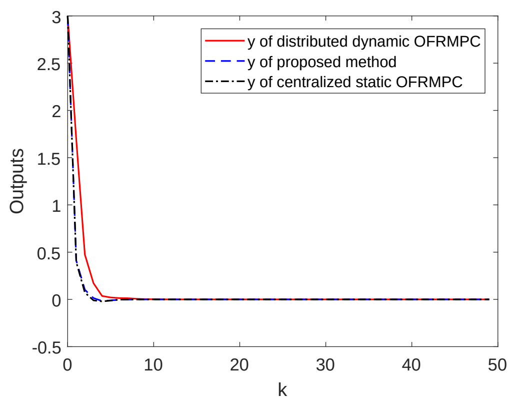


Fig. 3: Outputs of the three methods for subsystem 1.

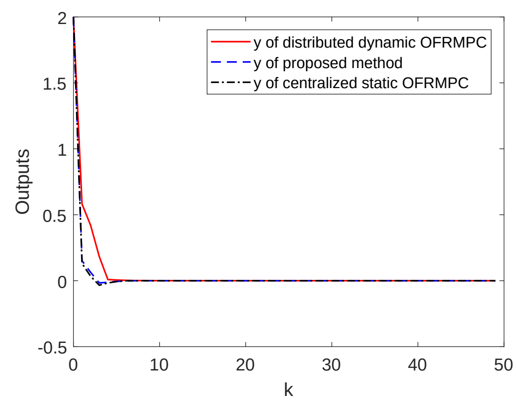


Fig. 4: Outputs of the three methods for subsystem 2.

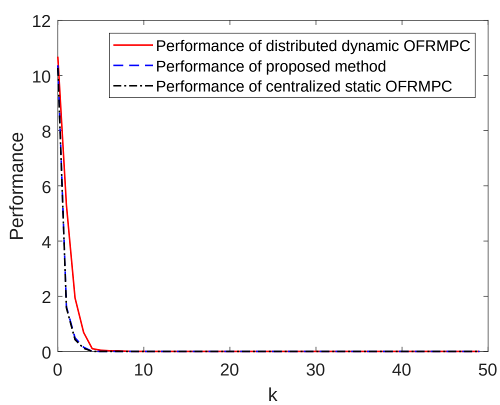


Fig. 5: Performance of the three methods for subsystem 1.

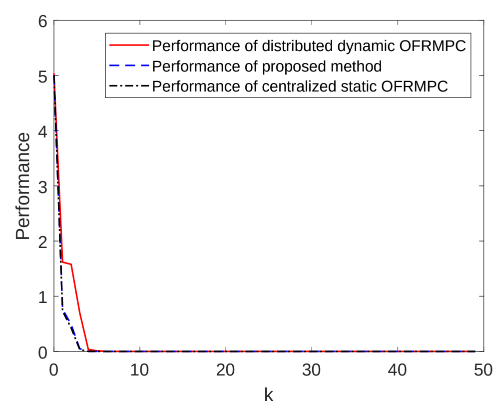


Fig. 6: Performance of the three methods for subsystem 2.

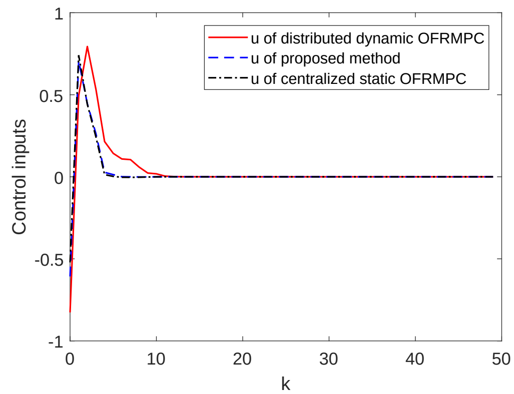


Fig. 7: Inputs of the three methods for subsystem 1.

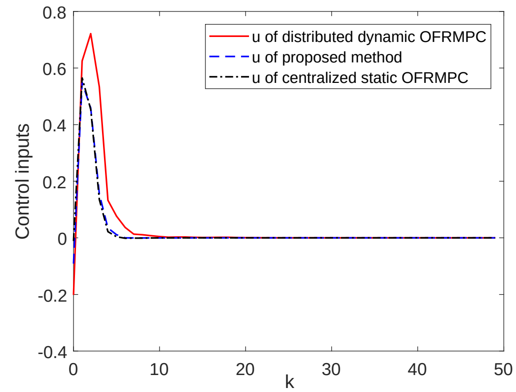


Fig. 8: Inputs of the three methods for subsystem 2.

When compared with the static OFRMPC case, the response of the dynamic control is slower because the observer has to converge. Note that if the observer gains were made larger, the transient observer effects could be expected to reduce, but the larger observer gains may cause degradation of the control performance since it will increase the magnitude of the states.

### B. Stabilization Problem for twenty subsystems

Consider system (1) and assume that there are twenty discrete time linear subsystems. The system matrices are given by:

$$A_{ii} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \quad A_{ij} = \begin{bmatrix} 0.26 & 0.28 \\ 0 & 0.25 \end{bmatrix}$$
$$B_{ii} = \begin{bmatrix} 1 \\ 0.5 \end{bmatrix} \quad B_{ij} = \begin{bmatrix} 0.03 \\ 0.15 \end{bmatrix} \quad D_i = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$
$$C_i = \begin{bmatrix} 1 & 0 \end{bmatrix} \quad H_i^1 = 2I \quad H_i^2 = I$$

where $i = 1, \cdots, 20$, $j = 1, \cdots, 20$ and $i \neq j$. For the proposed method, the initial value of the control matrices can be given by: $M_i = \begin{bmatrix} 0.5426 & 0 \\ -0.01054 & 0.4942 \end{bmatrix}$, $i = 1, \cdots, 20$, $Y_i = \begin{bmatrix} 0.9412 & 0 \end{bmatrix}$, $i = 1, \cdots, 20$. For the centralized static OFRMPC, $M = diag\{M_1, \cdots, M_{20}\}$ and $Y = diag\{Y_1, \cdots, Y_{20}\}$. The initial states of subsystems are selected as $x_i(0) = \begin{bmatrix} 4 & -1 \end{bmatrix}^T$, $i = 1, \cdots, 20$. The control horizon is 1, the prediction horizon is 10 and the simulation length is 50. Consider the presence of uncertainties as described in (2) and (3) where $w_i(k) = sqrt(0.1) \times randn(2, 1)$, $i = 1, \cdots, 20$, is a random disturbance satisfying the Gaussian distribution and $v_i(k) = sqrt(0.1) \times randn(2, 1)$, $i = 1, \cdots, 20$, is a random noise satisfying the Gaussian distribution.

In the following simulations, the dashed line presents the results when the system is controlled by the proposed method and the dash-dotted line gives the results when the system is controlled by the centralized static OFRMPC. The results of the 1st, 6th, 12th and 18th subsystems are selected randomly from the 20 subsystems and displayed from Fig.9 to Fig.12. To show the ability of the approach to deal with couplings, the sum of the performance for all the subsystems (i.e., $J = \sum_{i=1}^{20} \sum_{k=0}^{50} [x_i(k)^T {H_i}^1 x_i(k) + u_i(k)^T {H_i}^2 u_i(k)]$) is compared. The performance of the proposed method is 547.3624 while that of the centralized static OFRMPC is 543.9621. From Fig.9 to Fig.12, it can be seen that the proposed method can achieve a very similar performance to that achieved by centralized static OFRMPC. However the computational time of the proposed method is only 12.34s while that of the centralized static OFRMPC is 28.61s. This illustrates the proposed approach has the ability to deal effectively with the couplings with a shorter computational time.

### C. CSTR System Experiment

The effectiveness of the proposed approach is further validated by an experimental trial. The Process Modelling and Control Group at the China University of Petroleum (East China) has developed an experimental rig which is shown in Figure 13. The operational interface of the rig is shown in Figure 14. The four reactors, labelled R101, R102, R103, R104, can be connected in numerous ways for control validation and testing (series, parallel, series and parallel). The chemical reaction is carried out after feeding. The equipment can be configured to implement continuous operation as well as enable measurement and control of the flow, liquid level and temperature. V111 is the header tank which contains acetic ether and V112 is the header tank containing sodium hydroxide. These raw materials are processed in the CSTR simultaneously [23]. The reaction function is $A + B = C + D$ and the output of this experiment is the temperature of the reactor. Only two reactors (R101 and R102) are

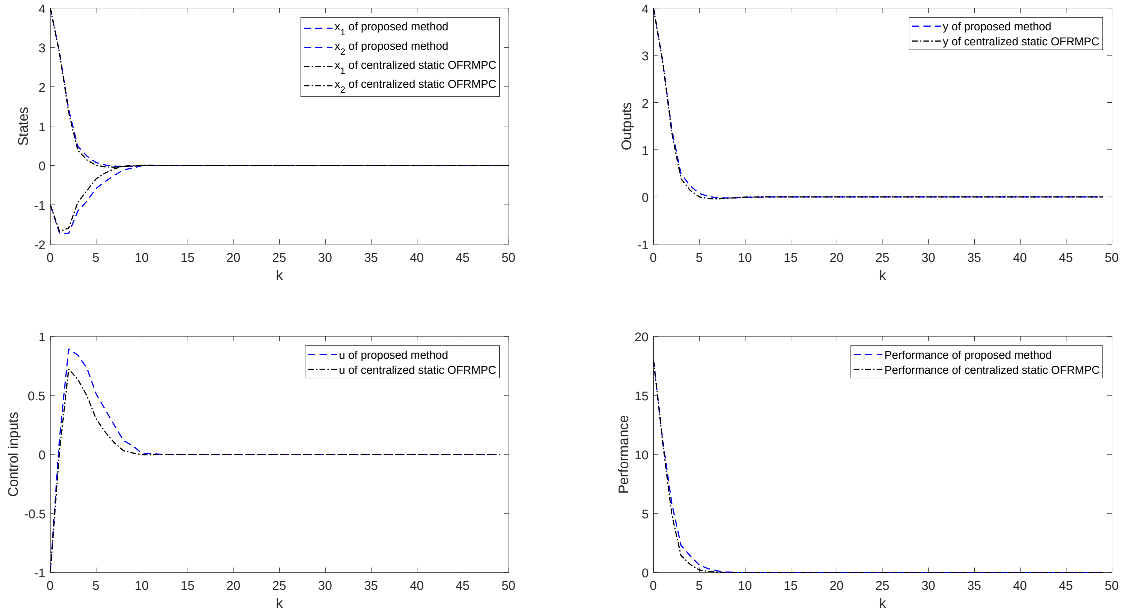


Fig. 9: The results of the 1st subsystem

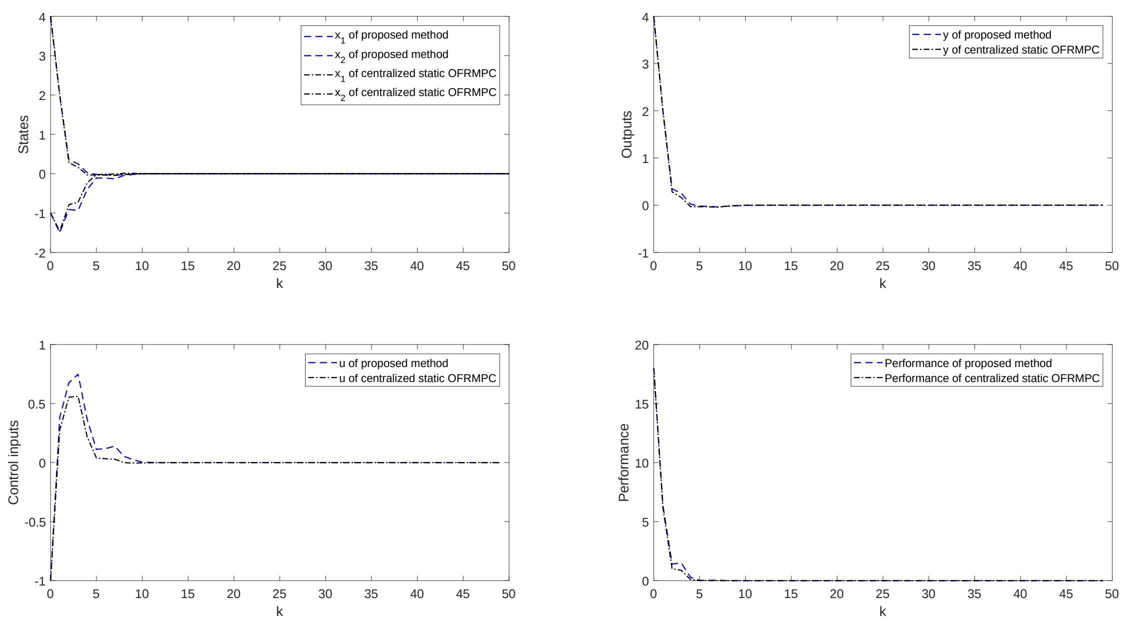


Fig. 10: The results of the 6th subsystem

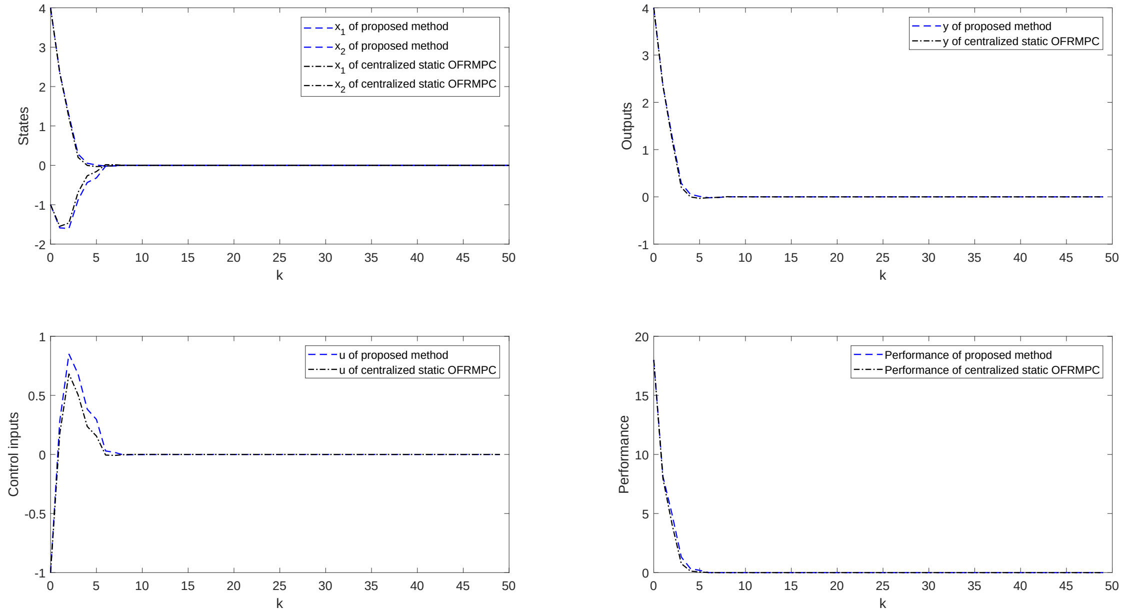


Fig. 11: The results of the 12th subsystem

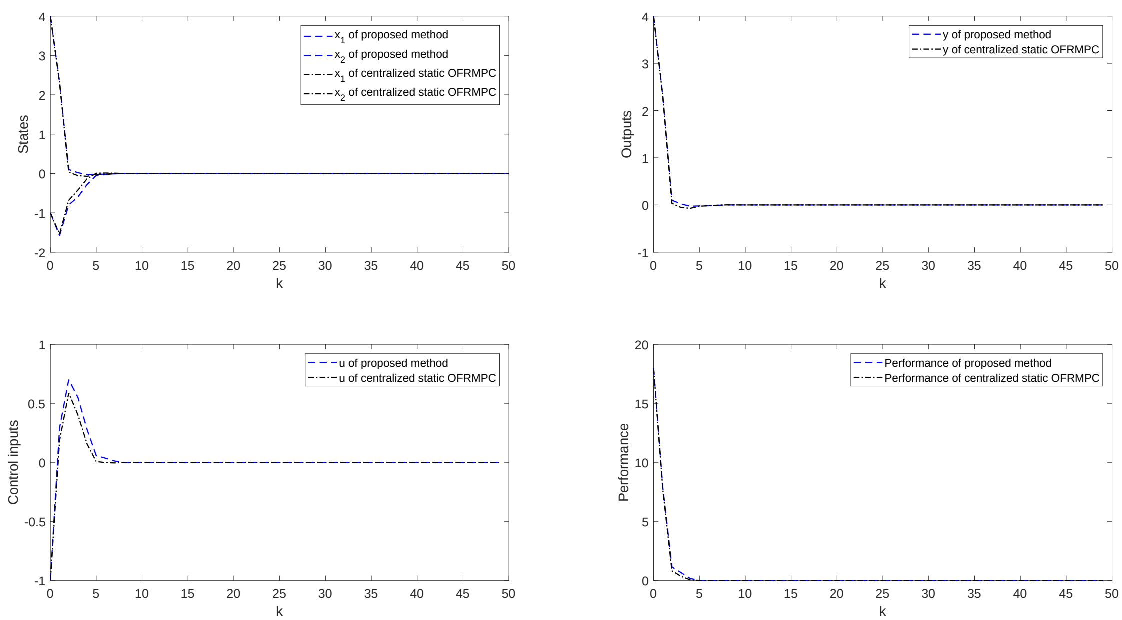


Fig. 12: The results of the 18th subsystem

Fig. 13: Experimental equipment

used in these experiments. The nonlinear dynamic equations of the plant model have been given in [23]. Choosing a sampling interval of $T_s = 0.0025h$, the nominal discrete time linear state space model of the plant has the form:

$$\begin{aligned} x_1(k+1) = A_{11}x_1(k) + B_{11}u_1(k) + A_{12}x_2(k) + B_{12}u_2(k), y_1(k) = C_1x_1(k) \\ x_2(k+1) = A_{22}x_2(k) + B_{22}u_2(k) + A_{21}x_1(k) + B_{21}u_1(k), y_2(k) = C_2x_2(k) \end{aligned} \tag{45}$$

where $x_1$ and $u_1$ are the state and input vectors for R101, respectively, while $x_2$ and $u_2$ are the state and manipulated input vectors for R102. The system matrices are given by

$$\begin{aligned}
&A_{11} = \begin{bmatrix} 0.9600 & 0.0039 \\ -0.2488 & 0.8902 \end{bmatrix}, A_{12} = \begin{bmatrix} 0.0722 & 0.0002 \\ -0.0134 & 0.0773 \end{bmatrix}, \\
&A_{22} = \begin{bmatrix} 0.8312 & 0.0024 \\ -0.0235 & 0.5627 \end{bmatrix}, A_{21} = \begin{bmatrix} 0.0657 & 0.0002 \\ -0.0201 & 0.0645 \end{bmatrix}, \\
&B_{11} = \begin{bmatrix} 0.0072 \\ -0.0009 \end{bmatrix}, B_{12} = \begin{bmatrix} 0.0722 \\ -0.0134 \end{bmatrix}, D_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \\
&B_{22} = \begin{bmatrix} 0.0097 \\ -0.0005 \end{bmatrix}, B_{21} = \begin{bmatrix} 0.0247 \\ -0.0200 \end{bmatrix}, D_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \\
&C_1 = \begin{bmatrix} 1 & 0 \end{bmatrix}, C_2 = \begin{bmatrix} 1 & 0 \end{bmatrix}, H_1^1 = I, H_2^1 = I, H_1^2 = I, H_2^2 = I.
\end{aligned}$$

The initial value of the control matrices can be obtained as: $M_1 = \begin{bmatrix} 0.0345 & 0 \\ -0.0018 & 0.1035 \end{bmatrix}, M_2 = \begin{bmatrix} 0.0292 & 0 \\ -0.0017 & 0.1297 \end{bmatrix}$, $Y_1 = \begin{bmatrix} 0.2469 & 0 \end{bmatrix}, Y_2 = \begin{bmatrix} 0.2716 & 0 \end{bmatrix}$. For the dynamic OFRMPC, the observer gains are given as follows : $L_{dynamic1} = [4.7 \quad 1.6]^T$, $L_{dynamic2} = [4.6 \quad 1.7]^T$. The control horizon is 1, the predicted horizon is 20 and the sampling time is 0.4s. The $w_1(k) \subseteq \mathbb{R}^2$ and $w_2(k) \subseteq \mathbb{R}^2$ are random disturbances satisfying the Gaussian distribution, $v_1(k) \subseteq \mathbb{R}^2$ and $v_2(k) \subseteq \mathbb{R}^2$ are random noises satisfying a Gaussian distribution. For the two reactors R101 and R102, the desired set-points across the three experiments are all 30 degrees Celsius. The initial temperatures of R101 and R102 are 22.6 degrees Celsius and 24.6 degrees Celsius, respectively, for the distributed dynamic OFRMPC experiment. The initial temperatures of R101 and R102 are 24.5 degrees Celsius and 23.6 degrees Celsius, respectively, for the proposed method. The initial temperatures of R101 and R102 are 25.5 degrees Celsius and 25.1 degrees Celsius, respectively, for the centralized static OFRMPC experiment. The initial temperatures are different in the three experiments, because they were performed on different days and the temperature is affected by many environmental factors, such as the weather and the water temperature. The control objective is to increase the temperature of the tank from the initial temperature to 30 degrees Celsius by manipulating the flow of water in the jacket. The temperature tracking results are shown in Fig. 15, Fig. 16 and Fig. 17. Comparing Fig. 15 with Fig. 16, it can be seen that the proposed method has better control performance than that exhibited by the dynamic control. From Fig. 16 and Fig. 17, it can be seen that the proposed method has very similar control performance to that obtained by the centralized static OFRMPC.

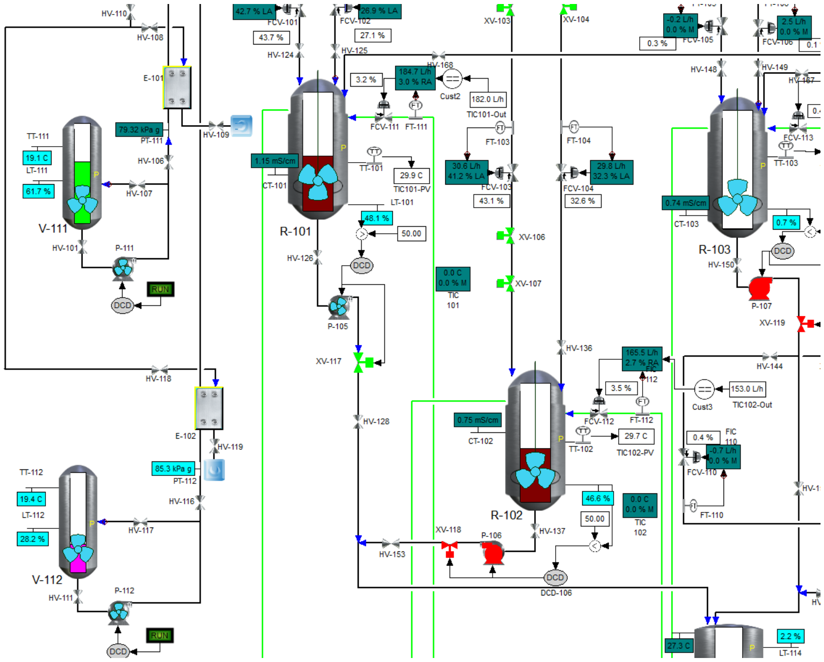


Fig. 14: Parallel R101 and R102.

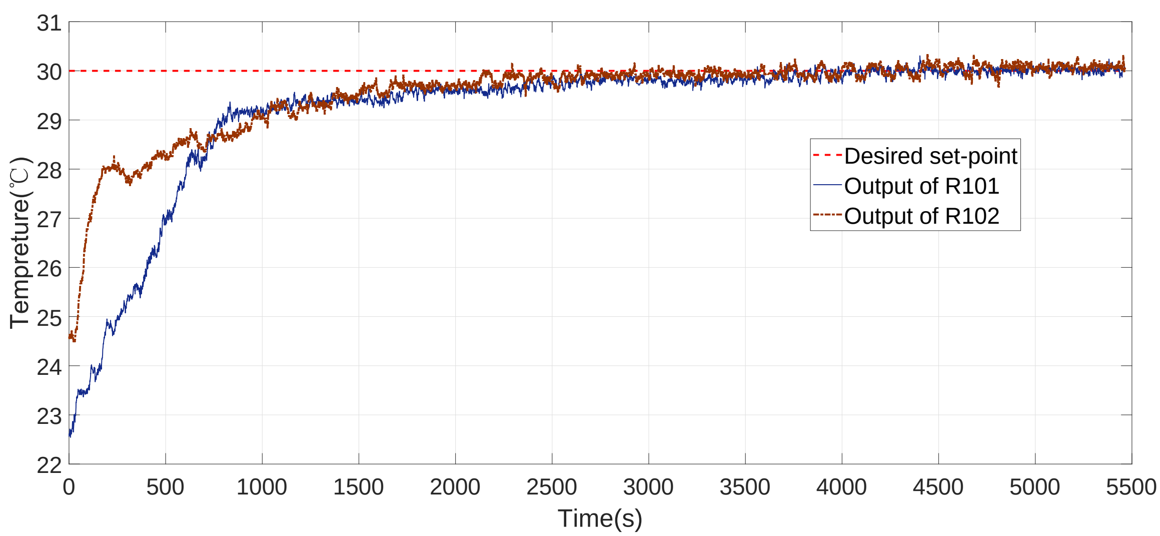


Fig. 15: Tracking performance when CSTRs are controlled by distributed dynamic OFRMPC.

## V. Conclusion

This paper has proposed an algorithm to implement a distributed static OFRMPC strategy for the case when the system states are not measurable. Necessary and sufficient conditions are imposed to ensure the existence of a static output feedback control law. The initial conditions of the current states for all subsystems are predicted by the previous information to ensure the distributed RMPC can operate within a static output feedback framework. The state and input couplings are considered and an overall performance index is designed to guarantee global performance on the premise of reducing the computational load. The control law is designed with an explicit expression under the constraints and uncertainties. An iterative LMI approach is used to solve the distributed static OFRMPC optimization problem. The simulation results show that the proposed approach can effectively deal with a discrete time linear system with state and input couplings in process networks. It can achieve very similar performance to a centralized static OFRMPC while using less computational time and exhibiting better performance than a distributed dynamic OFRMPC. The results of an experimental trial further illustrate that the proposed approach is suitable for process networks when the states are unmeasured.

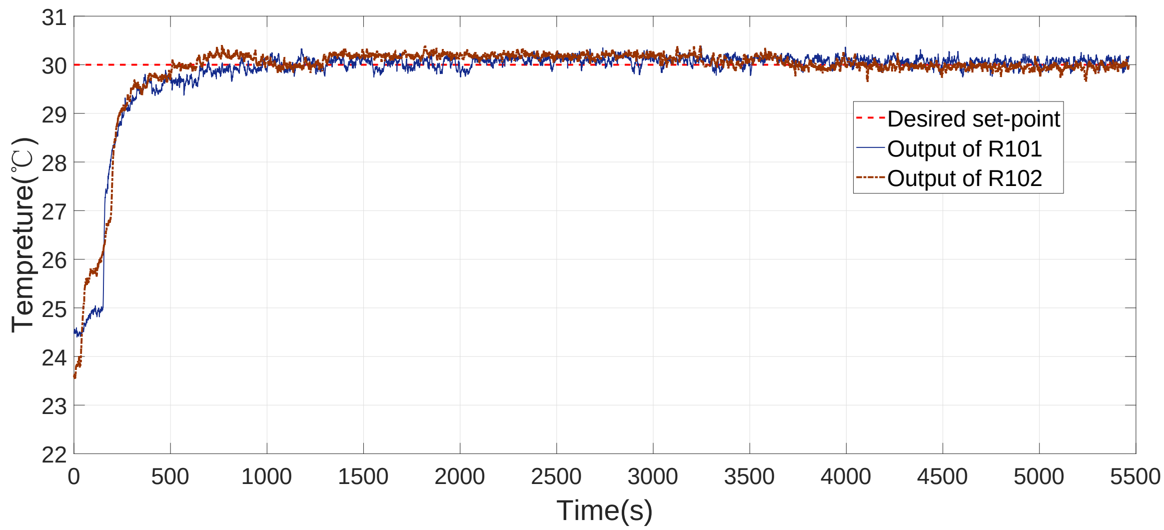


Fig. 16: Tracking performance when CSTRs are controlled by proposed method.

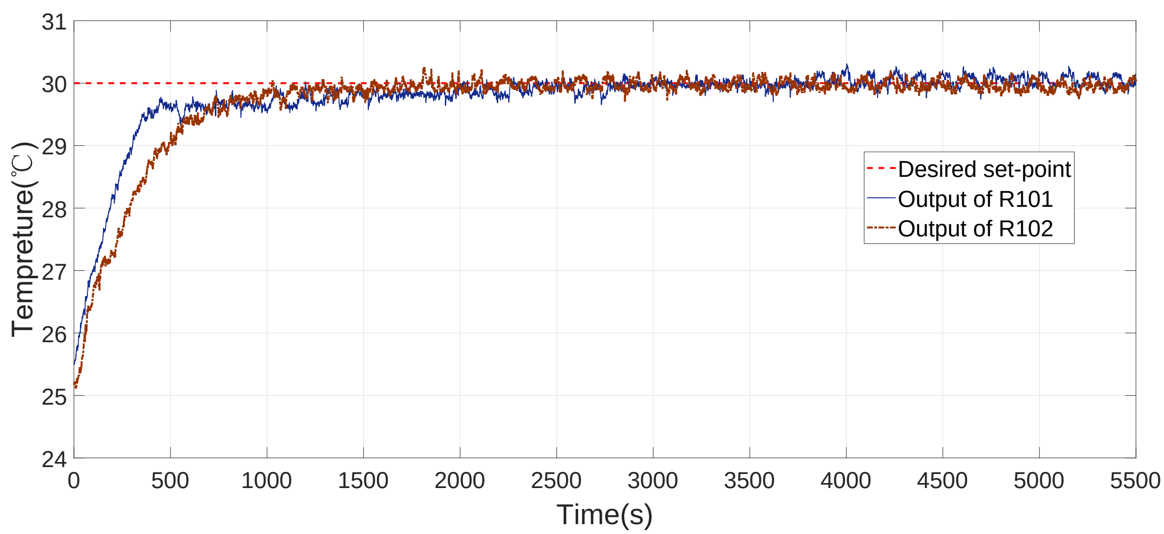


Fig. 17: Tracking performance when CSTRs are controlled by centralized static OFRMPC.

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