

## Proof-theoretic Semantics in Sheaves (Extended Abstract)

David Pym                      Eike Ritter                      Edmund Robinson  
University College London    University of Birmingham    Queen Mary University of London

In proof-theoretic semantics [6], model-theoretic validity is replaced by proof-theoretic validity. Validity of formulae is defined inductively from a base giving the validity of atoms using inductive clauses derived from proof-theoretic rules. A key aim is to show completeness of the proof rules without any requirement for formal models. Establishing this for propositional intuitionistic logic (IPL) raises some technical and conceptual issues [2, 3, 5].

We relate the (complete) base-extension semantics of [5] to categorical proof theory and sheaf-theoretic semantics (e.g., [1]). For the latter, propositions are interpreted as functors from a category of bases to the lattice  $\{\{\top\}, \emptyset\}$ . This set of functors forms the truth values of a topos of functors from bases to **Set**. There are two critical aspects: the stability of interpretation under extension of bases lands us in the world of Kripke models, and the non-standard interpretation of disjunction is revealed to come from a Grothendieck topology.

**Base-extension Semantics in Presheaves.** Sandqvist [5] gives a base-extension proof-theoretic semantics for IPL for which natural deduction is sound and complete. A base  $\mathcal{B}$  is set of atomic rules (for  $\vdash_{\mathcal{B}}$ ) as in Definition 1, which also defines the application of base rules, and satisfaction in a base ( $\Vdash_{\mathcal{B}}$ ). Roman  $p, P$ , etc. denote atoms and sets of atoms; Greek  $\phi, \Gamma$ , etc. denote formulae and sets of formulae.

**Definition 1 (Sandqvist’s Semantics)** *Base rules  $\mathcal{R}$ , application of base rules, and satisfaction of formulae in a (possibly finite) countable base  $\mathcal{B}$  of rules  $\mathcal{R}$  are defined as follows:*

$$\frac{[P_1] \quad \dots \quad [P_n]}{r} \mathcal{R} \quad \begin{array}{l} \text{(Ref)} \quad P, p \vdash_{\mathcal{B}} p \\ \text{(App}_{\mathcal{R}}) \quad \text{if } ((P_1 \Rightarrow q_1), \dots, (P_n \Rightarrow q_n)) \Rightarrow r \text{ and, for all } i \in [1, n], \\ P, P_i \vdash_{\mathcal{B}} q_i, \text{ then } P \vdash_{\mathcal{B}} r \end{array}$$
  

$$\begin{array}{ll} \text{(At)} \quad \text{for atomic } p, \Vdash_{\mathcal{B}} p \text{ iff } \vdash_{\mathcal{B}} p & \text{(}\vee\text{)} \quad \Vdash_{\mathcal{B}} \phi \vee \psi \text{ iff, for every atomic } p \text{ and every } \mathcal{C} \supseteq \mathcal{B}, \\ & \text{if } \phi \Vdash_{\mathcal{C}} p \text{ and } \psi \Vdash_{\mathcal{C}} p, \text{ then } \Vdash_{\mathcal{C}} p \\ \text{(}\supset\text{)} \quad \Vdash_{\mathcal{B}} \phi \supset \psi \text{ iff } \phi \Vdash_{\mathcal{B}} \psi & \text{(}\perp\text{)} \quad \Vdash_{\mathcal{B}} \perp \text{ iff, for all atomic } p, \Vdash_{\mathcal{B}} p \\ \text{(}\wedge\text{)} \quad \Vdash_{\mathcal{B}} \phi \wedge \psi \text{ iff } \Vdash_{\mathcal{B}} \phi \text{ and } \Vdash_{\mathcal{B}} \psi & \text{(Inf)} \quad \text{for } \Theta \neq \emptyset, \Theta \Vdash_{\mathcal{B}} \phi \text{ iff, for every } \mathcal{C} \supseteq \mathcal{B}, \text{ if } \Vdash_{\mathcal{C}} \theta \\ & \text{for every } \theta \in \Theta, \text{ then } \Vdash_{\mathcal{C}} \phi \end{array}$$

There is a substitution (cut) operation on bases that maps derivations  $P \vdash_{\mathcal{B}} p$  and  $p, Q \vdash_{\mathcal{B}} q$  to a derivation  $P, Q \vdash_{\mathcal{B}} q$ .

Key to understanding our categorical formulation is the Yoneda lemma (see [1]): let  $\mathcal{C}$  be a locally small category, let **Set** be the category of sets, and  $F \in [\mathcal{C}^{op}, \mathbf{Set}]$  (the category of presheaves over  $\mathcal{C}$ ); then, for each object  $C$  of  $\mathcal{C}$ , with  $h^C = \text{hom}(-, C)$ , the natural transformations  $\text{Nat}(h^C, F) \equiv \text{hom}(\text{hom}(-, C), F) \cong F(C)$ .

We give a category-theoretic formulation of proof-theoretic validity using presheaves (i.e., functors  $F \in [\mathcal{W}^{op}, \mathbf{Set}]$ ), where  $\mathcal{W}$  has objects pairs  $(\mathcal{B}, P)$  and morphisms are given by conclusions of the base and derivations in the larger base. Composition is given by substitution.

Define a functor  $\llbracket \phi \rrbracket: \mathcal{W}^{op} \rightarrow \mathbf{Set}$  by induction over the structure of  $\phi$  as follows: the base case  $\llbracket p \rrbracket(\mathcal{B}, P)$  is the set of derivations  $P \vdash_{\mathcal{B}} p$ .  $\llbracket p \rrbracket$  applied to morphisms is given by substitution. The definition is extended to the connectives homomorphically. A key step is the use of the Yoneda lemma to define the (hom-set) interpretation of  $\supset$ , which is used to define the interpretation of Sandqvist's (elimination-style) semantics for  $\vee$  (see also below). Thus we establish the formal functoriality and naturality of Sandqvist's semantics.

**Theorem 2 (Soundness & Completeness)** *Define (cf. [5])  $\Gamma \Vdash \phi$  as: for all  $\mathcal{B}$ , if  $\Vdash_{\mathcal{B}} \psi$  for all  $\psi \in \Gamma$ , then  $\Vdash_{\mathcal{B}} \phi$ . Then  $\Gamma \vdash \phi$  (in natural deduction for IPL, cf. [5]) iff  $\Gamma \Vdash \phi$ .*

The proof of soundness uses the existence of a natural transformation corresponding to  $\Vdash$ :  $\Gamma \vdash \phi$  iff there exists a natural transformation from  $\llbracket \Gamma \rrbracket$  to  $\llbracket \phi \rrbracket$ . The proof of completeness uses a special base, as in [5], which is extended via  $\llbracket - \rrbracket$  to the full consequence relation.

**Sheaves and Disjunction.** Standard Kripke semantics interprets both conjunction and disjunction pointwise (i.e., on each base, in proof-theoretic semantics [3]), while it relies on the extension ordering for implication (cf. the discussion of Goldfarb's semantics in [3]). This is a result of the requirement that the set of bases validating any proposition should be closed under extension: propositions do not become untrue if we are given additional atomic information. But there is an issue over the interpretation of disjunction. A standard constructive view is that the proof of a disjunction should resolve to a proof of one of the disjuncts. This is not obviously stable under extension of information and obtaining a pointwise disjunction reflecting this viewpoint is the hardest part of the proof of completeness of standard Kripke models for IPL. We show that Sandqvist's approach avoids this difficulty by using a Grothendieck topology.

In this section, we ignore differences between derivations, and interpret propositions as truth values in the topos  $\mathcal{S} = [\mathcal{W}^{op}, \mathbf{Set}]$ . These can be identified with subfunctors of the constant singleton functor  $\{\top\}$  (cf. [1]). Atomic propositions are interpreted in  $\mathcal{S}$  via  $\llbracket p \rrbracket(\mathcal{B}, P) = \{\top \mid P \vdash_{\mathcal{B}} p\} = \{\top \mid P \Vdash_{\mathcal{B}} p\}$ . Sandqvist's satisfaction conditions for conjunction and implication correspond to the internal interpretation of the logic in the topos  $\mathcal{S}$ , but his conditions for disjunction and false do not.

For each atomic proposition  $p$ , we form an internal operator on truth values:  $j_p(\omega) = (\omega \supset \llbracket p \rrbracket) \supset \llbracket p \rrbracket$ . The set of atomic propositions internalizes as the constant functor:  $At(\mathcal{B}) = \{p \mid p \text{ is atomic}\}$ . Consider the function on truth values that is the internal interpretation of  $j(\omega) = \forall p \in At. j_p(\omega) = \forall p \in At. (\omega \supset \llbracket p \rrbracket) \supset \llbracket p \rrbracket$ . This is a Lawvere-Tierney topology — that is, the internalization of a Grothendieck topology — and each  $\llbracket p \rrbracket$  is  $j$ -closed.

Sandqvist's satisfaction conditions correspond exactly to the standard interpretation of connectives in the topos of sheaves for this topology.

**Proposition 3** *For any proposition  $\phi$ , and any world  $W = (\mathcal{B}, P)$ ,  $P \Vdash_{\mathcal{B}} \phi$  iff  $\llbracket \phi \rrbracket(\mathcal{B}, P) = \{\top\}$ , where  $\llbracket \phi \rrbracket$  is the standard interpretation of  $\phi$  in  $Sh_j(\mathcal{S})$ .*

This follows from the closure of sheaves under conjunction and implication, intuitionistic equivalence of  $((\phi \vee \psi) \supset p) \supset p$  and  $((\phi \supset p) \wedge (\psi \supset p)) \supset p$ , and expansion of definitions.

This sheaf model can be seen as a continuation semantics in which a complete proof-search [4] is the proof of an atomic proposition. Using a topology for this results in a disjunction being valid iff a point is covered by refinements on each of which one of the disjuncts holds — cf. Beth’s semantics (see, e.g., [1]).

## References

- [1] J. Lambek and P. Scott. *Introduction to higher-order categorical logic*. Cambridge studies in advanced mathematics 7, Cambridge University Press, 1986.
- [2] T. Piecha. Completeness in Proof-theoretic Semantics. In: *Advances in Proof-theoretic Semantics*, T. Piecha P. Schroeder-Heister (eds), Trends in Logic 43, 231-252, Springer 2016.
- [3] T. Piecha and P. Schroeder-Heister. Incompleteness of Intuitionistic Propositional Logic with Respect to Proof-Theoretic Semantics. *Studia Logica* 107, 233-246, 2019.
- [4] D. Pym and E. Ritter. *Reductive Logic and Proof-search: Proof Theory, Semantics, and Control*. OUP, 2004
- [5] T. Sandqvist. Base-extension semantics for intuitionistic sentential logic. *Logic Journal of the IGPL*, 23(5), 719-731, 2015.
- [6] P. Schroeder-Heister. Proof-theoretic versus Model-theoretic Consequence. In: *The Logica Yearbook 2007*. Prague: Filosofia 2008, 187-200.