Closed-form expressions for spatial correlation parameters for performance analysis of fluid antenna systems

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The emerging fluid antenna technology enables a high-density position-switchable antenna in a small space to obtain enormous performance gains for wireless communications. To understand the theoretical performance of fluid antenna systems, it is important to account for the strong spatial correlation over the different positions (referred to as 'ports'). Previous works used a classical, generalised correlation model to characterise the channel correlation among the ports but were limited by the lack of degree of freedom of the model to imitate the correlation structures in an actual antenna. In this letter, it is proposed to use a common correlation parameter and to choose it by setting the correlation coefficient of any two ports to be the same as the average correlation coefficient of an actual fluid antenna taking up a linear space. A closed-form expression for the spatial correlation parameter is first derived assuming that the number of ports is large, and it is illustrated that the correlation parameter depends only on the size of the fluid antenna but not the port density. Simpler expressions are then obtained for small and large sizes of fluid antenna. The resulting model is finally used to study the performance of fluid antenna systems. Simulation results based on the proposed model are provided to confirm the promising performance of fluid antenna in single and multiuser environments.

Introduction: With the astronomical number of internet-of-things (IoT) devices looming, massive connectivity has become a necessary feature in 5G and beyond mobile communications [1, 2]. Recently, reconfigurable fluid antenna technologies have emerged to provide a novel way to adapt the antenna for ultimate agility [3]. One appealing application is to realise a position-switchable antenna which allows the radiating element to change its location (referred to as 'port') with very high resolution [4]. In refs. [5, 6], Wong et al. investigated the use of such antenna and reported that massive diversity and multiplexing gains were possible if the number of ports, N, was large. There was also the suggestion that hundreds of users could share the same radio channel for massive connectivity without multiuser signal optimisation and coordination, all dealt with by their fluid antennas.

The analysis in refs. [5, 6] is based on the generalised correlation model in ref. [7] which parameterises the fading channel seen at the kth port as

$$g_k=\sigma\left(\sqrt{1-\mu_k^2}x_k+\mu_k x_0\right)+j\sigma\left(\sqrt{1-\mu_k^2}y_k+\mu_k y_0\right),\quad \text{for } k=1,2,\dots,N, \tag{1}$$

where $x_0,x_1,\dots,x_N,y_0,y_1,\dots,y_N$ are all independent and identically distributed (i.i.d.) Gaussian random variables with zero mean and variance of 0.5, and the parameters $\{\mu_k\}$ serve to specify the correlation structures of the channels over the ports. Note that in refs. [5, 6], the first port was treated as an auxiliary reference port to ease the mathematical analysis. In other words, the $(N+1)$-port model in refs. [5, 6] is basically the same as the model (1) with N ports. In the studies, μ_k was chosen to be

$$\mu_k=J_0\left(\frac{2\pi kW}{N-1}\right) \tag{2}$$

in an attempt to reflect the correlation structure between the kth port and the reference port, as if the distance between them is $\frac{kW\lambda}{N-1}$ where λ is the wavelength and W is the size of fluid antenna normalised by λ. In ref. (2), $J_0(\cdot)$ denotes the zero-order Bessel function of the first kind.

While Equation (2) is a sensible way to try to incorporate spatial correlation into the ports, reminiscent of a linear fluid antenna design, a closer look at Equation (2) reveals that it does not represent the expected spatial correlation structure and the main weakness of Equation (2) is that the correlation between any two ports cannot be observed without the reference port, which means that if two ports are independent to the reference port, then they will be uncorrelated to each other too by the model. This could overestimate the performance of fluid antenna systems. In the next section, we study this problem and propose a simple solution that allows us to connect any two ports via the correlation parameter μ_k without the need of a designated reference port while preserving the mathematical tractability of the model (1).

Spatial correlation parameter optimisation: To highlight the issue of Equation (2), we know that based on the model (1), the correlation coefficient between any two ports, k and ℓ ($k\neq\ell$), should be set as

$$\rho_{k,\ell}=\mu_k\mu_\ell=J_0\left(\frac{2\pi(k-\ell)W}{N-1}\right), \tag{3}$$

in which the right-hand-side accounts for the spatial correlation expected for two ports with a distance of $\frac{(k-\ell)W\lambda}{N-1}$ apart. Unfortunately, no solution for $\{\mu_k\}$ is possible because there are $N(N-1)/2$ equations but we have only N parameters. To fully overcome this problem, one would have to model each channel g_k by a linear combination of N i.i.d. complex Gaussian random variables (instead of two in Equation (1)). In so doing, the increase in the number of parameters could permit setting their values to match (3).[1] Nonetheless, this would make the problem much more difficult than it already is, and the performance analysis is not known to be possible. Hence, our priority is to keep the tractability of Equation (1) while achieving close to (3) so that the model emulates the correlation structures in practice.

To this end, we propose to use a common correlation parameter μ (i.e. $\mu_k=\mu\ \forall k$) in Equation (1), which will link all the ports together without a reference port or any port is a reference to any other port. Furthermore, we choose μ^2 as the absolute of the average of the correlation coefficients by (3), i.e.

$$\mu^2=\left|\frac{2}{N(N-1)}\sum_{k=1}^{N-1}(N-k)J_0\left(\frac{2\pi kW}{N-1}\right)\right|, \tag{4}$$

which is basically Equation (3) after setting $\mu_k=\mu_\ell=\mu$ and the right-hand-side is replaced by the mean of the correlation coefficients required.

The following theorem presents a closed-form expression for μ as N goes to infinity, which succinctly connects the size of fluid antenna to the correlation parameter that can be useful for performance analysis.

Theorem 1. *As $N\to\infty$, the correlation parameter μ is given by*

$$\mu=\sqrt{2}\sqrt{{}_1F_2\left(\frac{1}{2};1,\frac{3}{2};-\pi^2W^2\right)-\frac{J_1(2\pi W)}{2\pi W}}, \tag{5}$$

where ${}_aF_b(\cdot;\cdot;\cdot)$ represents the generalised hypergeometric function and $J_1(\cdot)$ is the first-order Bessel function of the first kind.

Proof. Considering the right-hand-side of Equation (4) when N is very large, and ignoring the $|\cdot|$ sign, we have

$$\begin{aligned}&\text{RHS of Equation (4)}\\ &\overset{(a)}{=}2\sum_{k=1}^{N-1}J_0\left(\frac{2\pi kW}{N-1}\right)\frac{1}{N-1}-2\sum_{k=1}^{N-1}J_0\left(\frac{2\pi kW}{N-1}\right)\frac{k}{N-1}\frac{1}{N}\\ &\overset{(b)}{\approx}2\int_0^1 J_0(2\pi Wx)\mathrm{d}x-2\int_0^1 xJ_0(2\pi Wx)\mathrm{d}x,\end{aligned} \tag{6}$$

[1]One may attempt to use copula theory to handle the channel correlation over the ports but the involvement of the inverse of the cumulative density function of the marginal seems to prohibit the performance analysis from deducing any insightful results. Note that performance analysis of fluid antenna systems using copula theory is not understood.

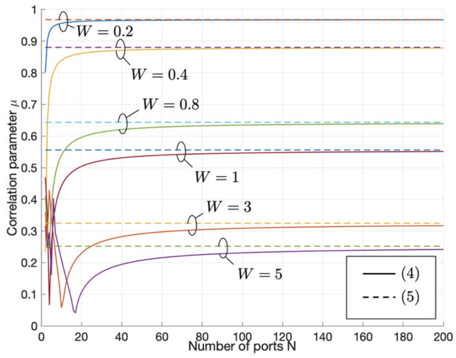

Fig. 1 *Convergence of Equations (4) to (5) as a function of N for various W*

where (a) separates the sum into two sums and (b) treats $\frac{k}{N-1} \to x$ and $\frac{1}{N} = \frac{1}{N-1} \to dx$ and summation becomes integration as $N \to \infty$. Using the property of Bessel function and taking the square root gives Equation (5). □

Figure 1 provides the numerical results to demonstrate the convergence performance of Equation (5) as N increases by comparing the values of μ obtained by Equations (4) and (5). The results show that Equation (4) converges quickly to Equation (5) when W is small. Even when W is relatively large, the expression (5) still provides a fair estimate of the required correlation parameter. Note that if W is large but N is very small, the correlation parameter using Equation (4) looks chaotic due to the fluctuating nature of Bessel function. It is worth pointing out that the current state-of-the-art position-switchable fluid antenna in ref. [4] permits a very large N which is only limited by the resolution of the digital pump. As a result, Equation (5) should provide a valid parameter for the channel to characterise the correlation structure as a function of W. On the other hand, the results in the figure indicate that the proposed approach is intuitive. As expected, the larger the value of W, the smaller the value of μ.

For different ranges of W, the expression of μ can be simplified. Such results are presented in the following theorem.

Theorem 2. *The correlation parameter μ can be approximated by*

$$\mu \approx \mu_{\text{approx}} = \begin{cases} 1 - \dfrac{\pi^2 W^2}{12}, & \textit{for small } W, \\[2ex] \dfrac{1}{\sqrt{\pi W}}, & \textit{for large } W. \end{cases} \tag{7}$$

Proof. For small W, we can express the hypergeometric function and the Bessel function in Equation (5) using Taylor series about 0, respectively, as

$${}_1F_2\left(\frac{1}{2}; 1, \frac{3}{2}; -\pi^2 W^2\right) = 1 - \frac{\pi^2 W^2}{3} + \frac{\pi^4 W^4}{20} + O(W^6) \tag{8}$$

and

$$\frac{J_1(2\pi W)}{2\pi W} = \frac{1}{2} - \frac{\pi^2 W^2}{4} + \frac{\pi^4 W^4}{24} + O(W^6), \tag{9}$$

where $O(\cdot)$ denotes the big-O notation. Using the above results gives

$$\mu^2 = 1 - \frac{\pi^2 W^2}{6} + \frac{\pi^4 W^4}{60} + O(W^6). \tag{10}$$

Finally, taking the square root and using the approximation $\sqrt{1-x} \approx 1 - \frac{x}{2}$ for small x gives the desired result. In the case for large W, we obtain the Taylor series in terms of W^{-1} and drop the higher-order terms. □

We show the values of μ and μ_{approx} using different expressions (some up to W^4 and some to W^2 terms for small W) in Table 1. The results show that for large W, the approximation is always good. On the other hand, for small W, evidently, the accuracy of μ_{approx} is better if the W^4 terms are included. However, if $W \le 0.6$, the approximation using up to W^2 terms is accurate although more terms need to be used for $0.7 \le W < 1$.

Table 1. The values of the spatial correlation parameter, μ

W	μ using (5)	μ_{approx} (up to W^4)	μ_{approx} (up to W^2)
0.1	0.991822593869	0.991823072839	0.991775329666
0.2	0.967853346139	0.967884400684	0.967101318663
0.3	0.929749599165	0.930110832790	0.925977966992
0.4	0.880167631939	0.882253796523	0.868405274652
0.5	0.822599623583	0.830803398193	0.794383241644
0.6	0.761132019759	0.786274362146	0.703911867967
0.7	0.700102898574	0.764055639247	0.596991153622*
0.8	0.643630243996	0.782445903567*	0.473621098609*
0.9	0.595012605684	0.856020920486*	0.333801702926*

W	μ using (5)	μ_{approx} (up to $(W^{-1})^{0.5}$)
1.0	0.556107207025	0.564189583548
1.5	0.464519898111	0.460658865962
2.0	0.396664784074	0.398942280401
2.5	0.358309669745	0.356824823231
3.0	0.324684221336	0.325735007935
3.5	0.302349968533	0.301572017546
4.0	0.281493112792	0.282094791774
4.5	0.266438813578	0.265961520268
5.0	0.251924182354	0.252313252202

*In these cases, the approximation appears to be inaccurate.

Performance analysis for fluid antenna systems: Here, we apply the results in Theorem 2 to a few theoretical results in [6] so that we can link several key performance metrics of a fluid antenna system to the size of the fluid antenna and easily see its impact. The fluid antenna system concerned is a device-to-device (D2D) multiuser system with each receiver equipped with a fluid antenna, i.e. a U-to-U interference channel with U being the number of users sharing the same radio channel. Each receiver suffers from $U-1$ interferers and switches its port for maximising the symbol signal-to-interference ratio (SIR). In ref. [8], this approach was referred to as fast fluid antenna multiple access (FAMA). We present our new results below when all users and channels are i.i.d. and have a common SIR target γ.

Theorem 3. *To achieve a multiplexing gain m ($\le U$) with an SIR target γ, each user needs to have the number of ports, N, satisfying*

$$N \gtrsim \begin{cases} \dfrac{6m\gamma}{\pi^2 W^2}, & \textit{for small } W, \\[2ex] \dfrac{m\gamma}{1 - \frac{1}{\pi W}}, & \textit{for large } W. \end{cases} \tag{11}$$

Proof. Replacing N by $N+1$ and substituting the results in Equation (2) to [6, Equation (27)] give the desired result. Note that U has been assumed to be large. □

Theorem 4. *The multiplexing gain of FAMA can be approximated by*

$$m \approx \begin{cases} \min\left\{\dfrac{\pi^2 N W^2}{6\gamma}, U\right\}, & \textit{for small } W, \\[2ex] \min\left\{\dfrac{N}{\gamma}\left(1 - \dfrac{1}{\pi W}\right), U\right\}, & \textit{for large } W. \end{cases} \tag{12}$$

Proof. Using Equation (11) as an approximation and rearranging the terms obtain the required result, which completes the proof. □

The two theorems above provide an intuitive understanding of the fluid antenna system. In particular, if we inspect Equation (12) and consider the situation where m has not reached its limit U, then we see that while increasing W helps enhancing m, it has a diminishing return because $m \to \frac{N}{\gamma}$ quickly and further increase in W will have little effect. On the other hand, if W is small, W has an effect of countering the number of ports N in delivering m. For example, if $W = 0.1$, we will need 100 times larger N to obtain the same m. For the special case of $W = 0.5$, we have

$$m \approx \frac{\pi^2}{24}\left(\frac{N}{\gamma}\right) \doteq 0.41\left(\frac{N}{\gamma}\right), \tag{13}$$

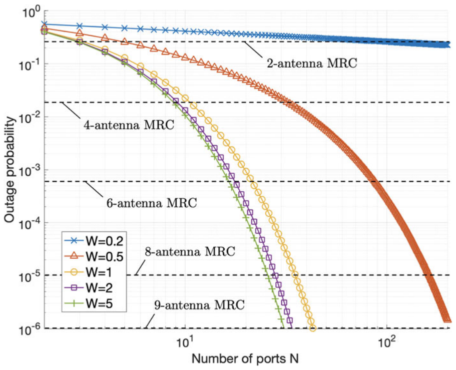

Fig. 2 *Outage probability performance for single-user fluid antenna systems using Equation (16) against the number of ports N for various size W with* $\frac{\gamma}{\Gamma}=1$

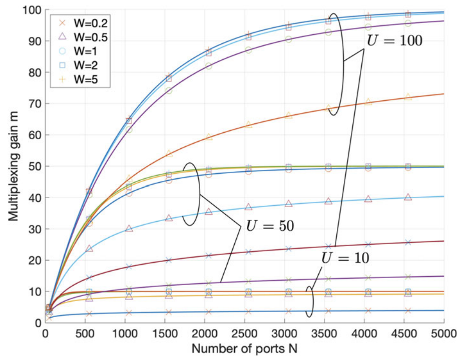

Fig. 3 *Multiplexing gain for FAMA systems using Equation (18) against the number of ports N for various size W and number of users U with* $\gamma=10$

while for $W=1$, we get

$$m \approx \left(1-\frac{1}{\pi}\right)\left(\frac{N}{\gamma}\right) \doteq 0.68\left(\frac{N}{\gamma}\right). \tag{14}$$

Based on this discussion, one can define the multiplexing efficiency as

$$\eta \triangleq \frac{m}{\left(\frac{N}{\gamma}\right)} \approx \begin{cases} \dfrac{\pi^2 W^2}{6}, & \text{for small } W, \\ 1-\dfrac{1}{\pi W}, & \text{for } W \geq 1, \end{cases} \quad \text{when } N \text{ is very large.} \tag{15}$$

As such, according to the above calculations (with approximations), a fluid antenna with $W=0.5$ and $W=1$ will have an efficiency of 41% and 68%, respectively. Note also that as $W\to\infty$, $\eta\to 1$ and it has a 100% efficiency. Moreover, this efficiency estimate is only useful in the linear region of m, i.e. when m behaves as a linear function of N. When N is extremely large, m will plateau and the efficiency will decrease as N continues to increase. Lastly, the achievable m is always discounted by the target SIR γ.

Outage probability and multiplexing gain results: So far, we have already established a spatial correlation model via μ whose value can be obtained based on the size of the fluid antenna, W, using Equation (5). The model provides a better linkage to the correlation structures of the channels over the ports for a given W. In this section, we applied this model to re-evaluate the performance of single-user fluid antenna system and FAMA. In particular, for the single-user case, the outage probability becomes [5, Theorem 3]

$$p_{\rm su}(\gamma)=\int_0^\infty e^{-t}\left[1-Q_1\left(\sqrt{\frac{2\mu^2}{1-\mu^2}}\sqrt{t},\sqrt{\frac{2}{1-\mu^2}}\sqrt{\frac{\gamma}{\Gamma}}\right)\right]^N {\rm d}t, \tag{16}$$

in which Γ is the average signal-to-noise ratio (SNR) for each port and $Q_1(\cdot,\cdot)$ is the first-order Marcum Q-function. Also, γ becomes the target for SNR. On the other hand, for FAMA, the outage probability for a typical user is given by Equation (17) (see top of this page), where $a=\sqrt{(U-1)\gamma}$ and the multiplexing gain can be exactly evaluated as [6, Theorem 1]

$$p_{\rm fama}(\gamma)=\int_0^\infty e^{-z}\int_0^\infty e^{-t}\left[1+\left(\frac{a^2}{a^2+1}\right)e^{-\left(\frac{1}{a^2+1}\right)\frac{\mu^2}{1-\mu^2}(a^2z+t)}I_0\left(\frac{a}{a^2+1}\left(\frac{2\mu^2}{1-\mu^2}\right)\sqrt{zt}\right)\right.$$
$$\left.-Q_1\left(\frac{1}{\sqrt{a^2+1}}\sqrt{\frac{2\mu^2}{1-\mu^2}}\sqrt{t},\frac{a}{\sqrt{a^2+1}}\sqrt{\frac{2\mu^2}{1-\mu^2}}\sqrt{z}\right)\right]^N {\rm d}t{\rm d}z, \tag{17}$$

$$m=(1-p_{\rm fama}(\gamma))U. \tag{18}$$

Both Equations (16) and (17) can be evaluated numerically using `Integral` and `Integral2` in `MATLAB`, respectively.

Figure 2 provides the outage probability results of a single-user fluid antenna system of different size W and N when $\frac{\gamma}{\Gamma}=1$. Comparing the results with that in Figures 3 and 4 of ref. [5], it appears that ref. [5] overestimates the performance especially when W is small although the differences for large W are mild. The new results offer a much more reasonable evaluation of the performance by more accurately modelling the spatial correlation among the ports. In particular, we can now notice that with $W=0.2$, the correlation appears to be really strong that greatly limits the performance even if N is large, which is not observable in ref. [5]. That said, the conclusion about the massive diversity of fluid antenna is still valid as the fluid antenna system can meet or even exceed the maximum ratio combining (MRC) system with many uncorrelated antennas, as seen in Figure 2.

We now turn our attention to the multiuser FAMA system by observing the results in Figure 3 where the multiplexing gains for different W and N are illustrated when $\gamma=10$. Our interest is to compare the results with that in Figure 7 of ref. [6]. For $W\geq 1$, the results in ref. [6] in fact did not overestimate the performance, mainly because they were based on a lower bound. However, when W is small, e.g. 0.2 and 0.5, the size limitation begins to show up in the new results, which cannot be detected in [6]. In addition, we see that $W=0.5$ can bring the biggest jump in the multiplexing gain when increasing the size but $W=1$ appears to already achieve over 95% of the maximum possible multiplexing gain if N is large enough. Also, Equation (15) can accurately estimate the multiplexing efficiency of FAMA in the linear region. Finally, though the size W matters in the performance of FAMA, the fact that FAMA can accommodate hundreds of users on the same radio channel for massive connectivity is confirmed if N is sufficiently large.

Conclusion: This letter identified the limitation of the conventional spatial correlation model used in the performance evaluation of the emerging fluid antenna systems and proposed a new way to set the correlation parameter by using a closed-form expression which can characterise the correlation over all the ports effectively as a function of the size of the fluid antenna while maintaining the tractability of the traditional model. The new model also permitted analysis to illustrate clearly how the various performance of fluid antenna links with the size, which has not been possible before. Using simulation results, we were able to confirm that fluid antenna systems can obtain enormous diversity and multiplexing gains although $W<0.5$ would result in a severe penalty in compromising the number of ports available.

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