Decentralized Sliding Mode Control for Output Tracking of Large-Scale Interconnected Systems*

Yueheng Ding[1], Xing-Gang Yan[1], Zehui Mao[2], Bin Jiang[2], Sarah K. Spurgeon[3]

Abstract— **In this paper, a class of nonlinear interconnected systems with matched and unmatched uncertainties is considered. The isolated subsystem dynamics are described by linear systems and a nonlinear component. The matched uncertainties and unmatched unknown interconnection terms are assumed to be bounded by known functions. Based on sliding mode techniques, a state feedback decentralized control scheme is proposed such that the outputs of the controlled interconnected system track given desired signals uniformly ultimately. The desired reference signals are allowed to be time-varying. Using multiple transformations, the considered system is transferred to a new interconnected system with an appropriate structure to facilitate the sliding surface design and the design of a decentralized controller. A set of conditions is proposed to guarantee that the designed controller drives the tracking errors onto the sliding surface. The sliding motion exhibited by the error dynamics is uniformly ultimately bounded. The developed results are applied to a river quality control problem. Simulation results show that the proposed decentralized control strategy is effective and feasible.**

I. INTRODUCTION

Large-scale systems are often mathematically modelled by interconnections between a set of lower-dimensional subsystems. One of the characteristics of such systems is that each subsystem is usually affected by the others due to the presence of the interconnections. It should be noted that large-scale systems are usually widely distributed in space. Thus the designed systems should have a high tolerance of data-loss during data transfer due to broken/unknown interconnections as well as poor communications to guarantee that the controlled large-scale systems exhibit the required degree of robustness. The control problem for large-scale interconnected systems is challenging. Compared with centralised control, decentralized control needs local information only, and thus information or data transfer between subsystems is not required. Specifically, when the network linking different subsystems is broken, or the data transfer between subsystems is poor or unstable, a centralised control scheme cannot be implemented. In such cases decentralized control provides advantages over centralized control and is a popular choice in the control of large-scale interconnected systems [17].

Recently, the study of large-scale systems with interconnected terms has made great progress, and many interesting results have been obtained. In [6], a large-scale fuzzy system with unknown interconnections was considered, where matched uncertainties or disturbances are not included. There are also some results for interconnected systems (see, e.g. [7], [5], [11]) which require that the interconnections are matched while unmatched interconnections and/or uncertainties are not involved. Moreover, some large-scale systems are considered as a simple or ideal dynamic model (see, e.g. [14], [13], [4]). The structure of these systems lacks generality because the input only exists within a first-order dynamic equation. Decentralized sliding mode control has been developed in [16] where the considered system is fully nonlinear with a more general structure, but only a stabilization problem is considered; tracking requirements are not addressed.

Trajectory tracking and output tracking are an important topic in both control theory and control applications. Some tracking control results have been obtained in (see. [1], [7]). However most consider systems of a special structure (see [13], [4]). Decentralized tracking control for large-scale systems is considered in [9] where model reference control is investigated. Tracking control for interconnected systems is considered based on adaptive fuzzy techniques in [10]. It should be noted that in both [9] and [10], it is required that the isolated subsystems are linear.

Sliding mode control is very popular in dealing with complex systems with uncertainties due to its unique control characteristics ([18], [19]). On the one hand, the sliding mode dynamics are often composed of a reduced-order system when compared with the original system ([17], [2]), which may simplify the corresponding system analysis and design. On the other hand, sliding mode control is totally robust to matched uncertainty and disturbances. This has resulted in the sliding mode control method being widely applied to deal with tracking problems, and many results have been achieved. Trajectory tracking control schemes based on sliding mode techniques are proposed for specific vehicles in (see. [20], [15]). An output tracking sliding mode control is designed in [12] where the considered system is linear. Although tracking control for nonlinear systems with uncertainties is considered in [3] where event-triggered tracking is considered, only matched disturbances are considered. In [21], a tracking problem for a class of large-scale systems with interconnections is considered using

*This work was supported by the National Natural Science Foundation of China under Grants 61922042 and 62020106003, China Scholarship Council for 3 years' study at the University of Kent, and Qing Lan Project.

[1]Yueheng Ding and Xing-Gang Yan are with the School of Engineering and Digital Arts, University of Kent, Canterbury CT2 7NT, UK, yd84@kent.ac.uk; x.yan@kent.ac.uk

[2]Zehui Mao and Bin Jiang are with the College of Automation Engineering, Nanjing University of Aeronautics and Astronautics, Nanjing 210016, P. R. China, zehuimao@nuaa.edu.cn; binjiang@nuaa.edu.cn

[3]Sarah K. Spurgeon is with the Department of Electronic and Electrical Engineering, University College London, London WC1E 7JE, UK, s.spurgeon@ucl.ac.uk

sliding mode control. However, it is required that the reference signals are constant. It should be emphasised that the results concerning output tracking for large-scale nonlinear interconnected systems with unknown interconnections are very few, specifically when the ideal reference signals are time-varying.

In this paper, a class of nonlinear interconnected systems is considered where both unknown matched uncertainty and unmatched nonlinear interconnections are considered. Suitable coordinate transformations are introduced to transfer the nominal subsystems within the interconnected system to systems with special structure. This separates each subsystem of the transformed system into two parts to facilitate the system analysis and control design for output tracking. Then the tracking error dynamic systems are developed, and the sliding surface based on the tracking error system is designed. A set of conditions is proposed to guarantee the uniform ultimate boundedness of the corresponding sliding motion. A decentralized sliding mode control scheme is proposed to drive the nonlinear interconnected systems to the designed sliding surface. Finally, the obtained results are applied to a river quality control problem to show the practicability and feasibility of the proposed approach.

II. System description and basic assumptions

Consider a nonlinear large-scale system formed by N interconnected subsystems as follows:

$$\begin{aligned} \dot{x}_i &= A_i x_i + f_i(x_i) + B_i(u_i + \Delta g_i(x_i)) + h_i(x) \\ y_i &= C_i x_i \qquad i = 1,2,...,N \end{aligned} \tag{1}$$

where $x = col(x_1, x_2, ..., x_N)$, $x_i \in R^{n_i}$, $u_i \in R^{m_i}$ and $y_i \in R^{m_i}$ represent the states, inputs and outputs of the ith subsystem respectively and $m_i < n_i$. The triple (A_i, B_i, C_i) represents constant matrices of appropriate dimensions with B_i and C_i of full rank. The functions $f_i(x_i)$ represent known nonlinear terms in the ith subsystem, and the matched uncertainty of the ith isolated subsystem is denoted by $\Delta g_i(x_i)$. The terms $h_i(x)$ represent the system interconnections including all unmatched uncertainties. All the nonlinear functions are assumed to be continuous in their arguments to guarantee the existence of solutions of the controlled system (1).

The object of this paper is, for a given desired signal $y_{id}(t)$, to design a decentralized sliding mode control

$$u_i = u_i(t, x_i(t), y_{id}(t))$$

such that the system output $y_i(t)$ of controlled system (1) can track the desired signal $y_{id}(t)$, i.e. the tracking errors $y_i(t) - y_{id}(t)$ are uniformly ultimately bounded for $i = 1,2,...,N$, while all the state variables of system (1) are bounded.

Remark 1. It should be noted that in this paper, it is required that system (1) is square, that is, the dimension of each subsystem output is equal to the dimension of the corresponding subsystem input. However, the developed results can be easily extended to the case when the dimension of the subsystem output is greater than the dimension of the subsystem input by slight modification.

In order to deal with the tracking problem stated above, some assumptions are imposed on the considered interconnected system (1).

Assumption 1. The pair (A_i, B_i) is controllable and $\text{rank}(C_i B_i) = m_i$ for $i = 1,2,...,N$.

This follows from the work in [17], [2]. Under Assumption 1, there exists a coordinate transformation $z_i = T_i x_i$ such that the triple (A_i, B_i, C_i) with respect to the new coordinates z_i has the following structure

$$\begin{bmatrix} A_{i11} & A_{i12} \\ A_{i21} & A_{i22} \end{bmatrix}, \quad \begin{bmatrix} 0 \\ B_{i2} \end{bmatrix}, \quad \begin{bmatrix} 0 & C_{i2} \end{bmatrix}$$

where $A_{i11} \in R^{(n_i - m_i)\times(n_i - m_i)}$, the square matrices $B_{i2} \in R^{m_i \times m_i}$ and $C_{i2} \in R^{m_i \times m_i}$ are nonsingular for $i = 1,2,...,N$.

Assumption 2. Suppose that $f_i(x_i)$ has the decomposition $f_i(x_i) = \Gamma_i(x_i)x_i$, where $\Gamma_i \in R^{n_i \times n_i}$ is a continuous function matrix for $i = 1,2,...,N$.

Remark 2. If $f_i(0) = 0$ and f_i is sufficiently smooth, then the decomposition $f_i(x_i) = \Gamma_i(x_i)x_i$ is guaranteed. Therefore, the limitation to $f_i(x_i)$ in Assumption 2 is not strict.

Assumption 3. There exist known continuous functions $\rho_i(\cdot)$ such that

$\| \Delta g_i(x_i) \| \leq \rho_i(x_i)$ for $i = 1,2,...,N$.

Assumption 4. The desired output signal $y_{id}(t)$ is differentiable and satisfies

(i). $\| y_{id}(t) \| \leq L_{i1}$

(ii). $\| \dot{y}_{id}(t) \| \leq L_{i2}$

for $t \in [0, \infty)$, where L_{i1} and L_{i2} are known constants for $i = 1,2,...,N$.

Remark 3. Assumption 4 is a limitation on the desired output signals $y_{id}(t)$. It is required that the desired output signal $y_{id}(t)$ and its derivative $\dot{y}_{id}(t)$ are bounded. This assumption is quite standard and can be satisfied in most practical cases.

III. System structure analysis

Consider the nonlinear interconnected system in (1). Under Assumption 1, there exists a nonsingular coordinate transformation $z_i = T_i x_i$ such that in the new coordinates $z = col(z_1, z_2, ..., z_N)$, system (1) has the following form

$$\begin{aligned} \dot{z}_i &= \begin{bmatrix} A_{i11} & A_{i12} \\ A_{i21} & A_{i22} \end{bmatrix} z_i + \begin{bmatrix} F_{i1}(z_i) \\ F_{i2}(z_i) \end{bmatrix} + \begin{bmatrix} 0 \\ B_{i2} \end{bmatrix} (u_i + \Delta G_i(z_i)) \\ &\quad + \begin{bmatrix} H_{i1}(z) \\ H_{i2}(z) \end{bmatrix} \\ y_i &= \begin{bmatrix} 0 & I_{i2} \end{bmatrix} z_i, \qquad i = 1,2,...,N \end{aligned} \tag{2}$$

where A_{i11} is stable, the square sub-matrices $B_{i2} \in R^{m_i \times m_i}$ are nonsingular. $I_{i2} \in R^{m_i \times m_i}$ is an identity matrix, $col(F_{i1}, F_{i2}) = T_i f_i(x_i)|_{x_i = T_i^{-1} z_i}$ and $F_{i1}(z_i) \in R^{n_i - m_i}$, $F_{i2}(z_i) \in R^{m_i}$. $\Delta G_i(z_i) = T_i \Delta g_i(x_i)|_{x_i = T_i^{-1} z_i}$, $col(H_{i1}(z), H_{i2}(z)) = T_i h_i(x)|_{x = T^{-1} z}$ and $H_{i1}(z) \in R^{n_i - m_i}$, $H_{i2}(z) \in R^{m_i}$. The coordinate transformation $T := col(T_1, T_2, ..., T_N)$.

Since A_{i11} is stable for $i = 1,2,...,N$, for any $Q_i > 0$, the following *Lyapunov* equation has a unique solution $P_i > 0$ such that

$$A_{i11}^{\top} P_i + P_i A_{i11} = -Q_i, \qquad i = 1,2,...,N. \tag{3}$$

Now, in order to fully exploit the structural characteristics, partition $z_i = col(z_{i1}, z_{i2})$ with $z_{i1} \in R^{n_i - m_i}$ and $z_{i2} \in R^{m_i}$. It follows that in the new coordinate z, system (2) has the following form

$$\begin{aligned} \dot{z}_{i1} &= A_{i11}z_{i1} + A_{i12}y_i + F_{i1}(z_{i1}, y_i) \\ &\quad + H_{i1}(z_{11}, y_1, ..., z_{N1}, y_N) \quad (4) \\ \dot{y}_i &= A_{i21}z_{i1} + A_{i22}y_i + F_{i2}(z_{i1}, y_i) + B_{i2}(u_i + \Delta G_i(z_i)) \\ &\quad + H_{i2}(z_{11}, y_1, ..., z_{N1}, y_N) \qquad i = 1, 2, ..., N \quad (5) \end{aligned}$$

From system (2) and Assumption 2,

$$\begin{aligned} col(F_{i1}, F_{i2}) &= T_i f_i(x_i)|_{x_i = T_i^{-1} z_i} \\ &= T_i \Gamma_i(x_i)|_{x_i = T_i^{-1} z_i} T_i^{-1} col(z_{1i}, y_i) \quad (6) \end{aligned}$$

It follows from (6) that the functions $F_{i1}(z_{i1}, y_i)$ in system (4)-(5) can be described by

$$\Gamma_{i11}(z_{i1}, y_i)z_{i1} + \Gamma_{i12}(z_{i1}, y_i)y_i = F_{i1}(z_{i1}, y_i) \qquad (7)$$

where $\Gamma_{i11}(\cdot)$ and $\Gamma_{i12}(\cdot)$ are defined by

$$\begin{bmatrix} \Gamma_{i11}(\cdot) & \Gamma_{i12}(\cdot) \\ \star & \star \end{bmatrix} = T_i \Gamma_i(x_i)|_{x_i = T_i^{-1} z_i} T_i^{-1}$$

and the $\star$s are function matrices which it is not necessary to specify. Therefore, (4) can be described by

$$\begin{aligned} \dot{z}_{i1} =& A_{i11}z_{i1} + A_{i12}y_i + \Gamma_{i11}(z_{i1}, y_i)z_{i1} + \Gamma_{i12}(z_{i1}, y_i)y_i \\ &+ H_{i1}(z_{11}, y_1, ..., z_{N1}, y_N) \end{aligned} \qquad (8)$$

where $\Gamma_{i11}(\cdot)$ and $\Gamma_{i12}(\cdot)$ satisfy (7).

IV. Sliding mode tracking control design

A. Sliding Surface Design

Consider the desired output signal $y_{id}(t)$ satisfying Assumption 4. Then for system (1), the output tracking errors e_i are defined by:

$$e_i(t) = y_i(t) - y_{id}(t) \qquad i = 1, 2, ..., N \qquad (9)$$

and their first-time derivative is:

$$\dot{e}_i(t) = \dot{y}_i(t) - \dot{y}_{id}(t) \qquad i = 1, 2, ..., N \qquad (10)$$

Combining (4)-(5), a new system comprising z_{i1} and e_i can be developed:

$$\begin{aligned} \dot{z}_{i1} &= A_{i11}z_{i1} + A_{i12}y_i + \Gamma_{i11}(z_{i1}, y_i)z_{i1} + \Gamma_{i12}(z_{i1}, y_i)y_i \\ &\quad + H_{i1}(z_{11}, y_1, ..., z_{N1}, y_N) \quad (11) \\ \dot{e}_i &= A_{i21}z_{i1} + A_{i22}(e_i + y_{id}) + F_{i2}(z_{i1}, y_i) + B_{i2}(u_i + \\ &\quad \Delta G_i(z_{i1}, y_i)) + H_{i2}(z_{11}, y_1, ..., z_{N1}, y_N) - \dot{y}_{id}(t) \quad (12) \end{aligned}$$

for $i = 1, 2, ..., N$.

Assumption 5. It is straightforward to find a function $\gamma_i(\cdot)$ such that the following inequalities

$$\begin{aligned} &\|H_{i1}(z_{11}, y_1, ..., z_{N1}, y_N)\| \\ &\le \gamma_{i1}(T^{-1}(z_{11}, y_1, ..., z_{N1}, y_N))(\sum_{j=1}^{N} \| z_{j1} \| + \sum_{j=1}^{N} \| y_j \|) \end{aligned} \qquad (13)$$

$$\begin{aligned} &\|H_{i2}(z_{11}, y_1, ..., z_{N1}, y_N)\| \\ &\le \gamma_{i2}(T^{-1}(z_{11}, y_1, ..., z_{N1}, y_N))(\sum_{j=1}^{N} \| z_{j1} \| + \sum_{j=1}^{N} \| y_j \|) \end{aligned} \qquad (14)$$

hold for $i = 1, 2, ..., N$.

For the system (11)-(12), define the following sliding surface

$$col(e_1, e_2, ..., e_N) = 0 \qquad (15)$$

Then, the sliding mode dynamics have the following form according to the structure of (11)-(12):

$$\begin{aligned} \dot{z}_{i1} &= A_{i11}z_{i1} + A_{i12}y_{id} + \Gamma_{i11}(z_{i1}, y_{id})z_{i1} \\ &\quad + \Gamma_{i12}(z_{i1}, y_{id})y_{id} + H_{i1}(z_{11}, y_{1d}, ..., z_{N1}, y_{Nd}) \end{aligned} \qquad (16)$$

for $i = 1, 2, ..., N$.

Remark 4. From (13) in Assumption 5, when the states reach the sliding surface, it follows that

$$\begin{aligned} &\|H_{i1}(z_{11}, y_{1d}, ..., z_{N1}, y_{Nd})\| \\ &\le \gamma_{i1}(T^{-1}(z_{11}, y_{1d}, ..., z_{N1}, y_{Nd}))(\sum_{j=1}^{N} \| z_{j1} \| + \sum_{j=1}^{N} \| y_{jd} \|) \end{aligned} \qquad (17)$$

hold for $i = 1, 2, ..., N$.

Obviously, the sliding surface (16) is a reduced-order interconnected system composed of N subsystems whose dimension is $n_i - m_i$.

Theorem 1: Consider the sliding mode dynamics given in (16). Under Assumptions 1-5, the sliding mode is uniformly ultimately bounded if there exists a domain

$$\Omega = \{(z_{11}, z_{21}, ..., z_{N1}) | \, \| z_{i1} \| \le c_i, \; i = 1, 2, ..., N\}$$

for some constants $c_i > 0$ such that $M^{\mathsf{T}} + M > 0$ in $\Omega \backslash \{0\}$ where $M := (m_{ij})_{N \times N}$ and

$$m_{ij} = \begin{cases} \lambda_{min}(Q_i) - \| R_i(\cdot) \| - 2 \| P_i \| \gamma_{i1}(\cdot), & i = j \\ -2 \| P_i \| \gamma_{i1}(\cdot), & i \ne j \end{cases} \qquad (18)$$

with P_i and Q_i satisfying (3), and

$$R_i(\cdot) := \Gamma_{i11}(z_{i1}, y_{id})^{\mathsf{T}} P_i + P_i \Gamma_{i11}(z_{i1}, y_{id})$$

where $\Gamma_{i11}(z_{i1}, y_i)$ is given by (2) and $\gamma_{i1}(\cdot)$ is determined by (17).

Proof: It is necessary to prove the system (16) is uniformly ultimately bounded. For system (16), consider the following *Lyapunov* function candidate

$$V(z_{11}, z_{21}, ..., z_{N1}) = \sum_{i=1}^{N} (z_{i1})^{\mathsf{T}} P_i z_{i1} \qquad (19)$$

where P_i satisfies (3).

Then, the time derivative of $V(z_{11}, z_{21}, ..., z_{N1})$ along the trajectories of system (16) is given by

$$
\begin{aligned}
&\dot{V}(z_{11}, z_{21}, ..., z_{N1}) \\
&= \sum_{i=1}^{N} [(\dot{z}_{i1})^{\top} P_i z_{i1} + z_{i1}^{\top} P_i \dot{z}_{i1}] \\
&= \sum_{i=1}^{N} [z_{i1}^{\top} A_{i11}^{\top} P_i z_{i1} + y_{id}^{\top} A_{i12}^{\top} P_i z_{i1} + z_{i1}^{\top} \Gamma_{i11}(z_{i1}, y_{id})^{\top} P_i z_{i1} \\
&\quad + y_{id}^{\top} \Gamma_{i12}(z_{i1}, y_{id})^{\top} P_i z_{i1} + H_{i1}(z_{11}, y_{1d}, ..., z_{N1}, y_{Nd})^{\top} P_i z_{i1} \\
&\quad + z_{i1}^{\top} P_i A_{i11} z_{i1} + z_{i1}^{\top} P_i A_{i12} y_{id} + z_{i1}^{\top} P_i \Gamma_{i11}(z_{i1}, y_{id}) z_{i1} \\
&\quad + z_{i1}^{\top} P_i \Gamma_{i12}(z_{i1}, y_{id}) y_{id} + z_{i1}^{\top} P_i H_{i1}(z_{11}, y_{1d}, ..., z_{N1}, y_{Nd})] \\
&= \sum_{i=1}^{N} \{-z_{i1}^{\top} Q_i z_{i1} + z_{i1}^{\top} [\Gamma_{i11}(z_{i1}, y_{id})^{\top} P_i + P_i \Gamma_{i11}(z_{i1}, y_{id})] z_{i1} \\
&\quad + 2 z_{i1}^{\top} P_i A_{i12} y_{id} + 2 z_{i1}^{\top} P_i \Gamma_{i12}(z_{i1}, y_{id}) y_{id} \\
&\quad + 2 z_{i1}^{\top} P_i H_{i1}(z_{11}, y_{1d}, ..., z_{N1}, y_{Nd})\}
\end{aligned} \tag{20}
$$

where (3) is used to establish the above. By (17) and (i) in Assumption 4, it follows that

$$
\begin{aligned}
&\dot{V}(z_{11}, z_{21}, ..., z_{N1}) \\
&\leq \sum_{i=1}^{N} \{-\lambda_{min}(Q_i) \| z_{i1} \|^2 + \| \Gamma_{i11}(z_{i1}, y_{id})^{\top} P_i \\
&\quad + P_i \Gamma_{i11}(z_{i1}, y_{id}) \| \| z_{i1} \|^2 + 2 \| z_{i1} \| \| P_i \| \| A_{i12} y_{id} \| \\
&\quad + 2 \| z_{i1} \| \| P_i \| \| \Gamma_{i12}(z_{i1}, y_{id}) y_{id} \| \\
&\quad + 2 \| z_{i1} \| \| P_i \| \| H_{i1}(z_{11}, y_{1d}, ..., z_{N1}, y_{Nd}) \| \} \\
&= -\sum_{i=1}^{N} \left\{ \lambda_{min}(Q_i) - \| R_i(\cdot) \| - 2 \| P_i \| \gamma_{i1}(\cdot) \right\} \| z_{i1} \|^2 \\
&\quad + 2 \sum_{i=1}^{N} \sum_{\substack{j=1 \\ j \neq i}}^{N} \| P_i \| \| z_{i1} \| \gamma_{i1}(\cdot) (\| z_{j1} \| + L_{i1}) \\
&\quad + 2 \sum_{i=1}^{N} (\| A_{i12} y_{id} \| + \| \Gamma_{i12}(z_{i1}, y_{id}) y_{id} \|) \cdot \| P_i \| \| z_{i1} \| \\
&\leq -\frac{1}{2} \lambda_{min}(M^{\top} + M) \sum_{i=1}^{N} \| z_{i1} \|^2 + 2 \sum_{i=1}^{N} (\| A_{i12} y_{id} \| \\
&\quad + \| \Gamma_{i12}(z_{i1}, y_{id}) y_{id} \| + \gamma_{i1}(\cdot) L_{i1}) \cdot \| P_i \| \| z_{i1} \| \\
&= -\frac{1}{2} \sum_{i=1}^{N} \{\lambda_{min}(M^{\top} + M) \| z_{i1} \| - 4(\| A_{i12} y_{id} \| \\
&\quad + \| \Gamma_{i12}(z_{i1}, y_{id}) y_{id} \| + \gamma_{i1}(\cdot) L_{i1}) \| P_i \| \} \| z_{i1} \|
\end{aligned} \tag{21}
$$

where the matrix M is defined in (18). Under Assumption 4, $\| y_{id}(t) \| \leq L_{i1}$. It is clear to check $\dot{V} \leq 0$, if

$$
\| z_{i1} \| \geq \frac{4(\| A_{i12} L_{i1} \| + \| \Gamma_{i12}(z_{i1}, y_i) L_{i1} \| + \gamma_{i1}(\cdot) L_{i1}) \| P_i \|}{\lambda_{min}(M^{\top} + M)}
$$

for $i = 1, 2, .., N$. Hence, the conclusion follows. ■

V. Decentralized sliding mode control

For the interconnected system (1), the reachability condition [17], [16] is described by

$$
\sum_{i=1}^{N} \frac{e_i^{\top}(t) \dot{e}_i(t)}{\| e_i(t) \|} < 0 \tag{22}
$$

The following control law is then proposed

$$
\begin{aligned}
u_i = & -B_{i2}^{-1} \frac{e_i}{\| e_i \|} \{ \| A_{i21} z_{i1} \| + \| A_{i22} y_i \| + \| F_{i2}(z_{i1}, y_i) \| \\
& + \| B_{i2} \| \rho_i(z_{i1}, y_i) + k_i(z_{i1}, y_i) + L_{i2} \}
\end{aligned} \tag{23}
$$

for $i = 1, 2, ..., N$, where e_i and L_{i2} are defined by (9) and Assumption 4, respectively. $k_i(z_{i1}, y_i)$ is a control gain to be designed later.

Theorem 2: Consider the nonlinear interconnected system (11)–(12). Under Assumptions 2-5, the controller (23) drives the system (11)–(12) to the composite sliding surface (16) and maintains a sliding motion on it if the controller gains $k_i(z_{i1}, y_i)$ satisfy

$$
\sum_{i=1}^{N} k_i(z_{i1}, y_i) > \sum_{i=1}^{N} \gamma_{i2}(\cdot) \sum_{j=1}^{N} (\| z_{j1} \| + \| y_i \|) \tag{24}
$$

where γ_{i2} are defined by Assumption 5.

Proof: It is necessary to prove that the reachability condition (22) is satisfied. From (12) and Assumption 2,

$$
\begin{aligned}
\dot{e}_i = & A_{i21} z_{i1} + A_{i22} y_i + F_{i2}(z_{i1}, y_i) + B_{i2}(u_i \\
& + \Delta G_i(z_{i1}, y_i)) + H_{i2}(z_{11}, y_1, ..., z_{N1}, y_N) - \dot{y}_{id}
\end{aligned} \tag{25}
$$

for $i = 1, 2, ..., N$. From (23)-(25), it follows

$$
\begin{aligned}
\frac{e_i^{\top} \dot{e}_i}{\| e_i \|} = & \frac{e_i^{\top}}{\| e_i \|} \{A_{i21} z_{i1} + A_{i22} y_i + F_{i2}(z_{i1}, y_i) \\
& + B_{i2} \Delta G_i((z_{i1}, y_i)) + H_{i2}(z_{11}, y_1, ..., z_{N1}, y_N) - \dot{y}_{id}\} \\
& - \| A_{i21} z_{i1} \| - \| A_{i22} y_i \| - \| F_{i2}(z_{i1}, y_i) \| \\
& - \| B_{i2} \| \rho_i(z_{i1}, y_i) - k_i(z_{i1}, y_i) - L_{i2}
\end{aligned} \tag{26}
$$

It is clear to see

$$
\begin{aligned}
& \| A_{i21} z_{i1} + A_{i22} y_i + F_{i2}(z_{i1}, y_i) \| \\
& \quad \leq \| A_{i21} z_{i1} \| + \| A_{i22} y_i \| + \| F_{i2}(z_{i1}, y_i) \|
\end{aligned} \tag{27}
$$

From Assumptions 3-5,

$$
\| B_{i2} \Delta G_i(z_{i1}, y_i) \| \leq \| B_{i2} \| \rho_i(z_{i1}, y_i) \tag{28}
$$

$$
\| H_{i2}(z_{11}, y_1, ..., z_{N1}, y_N) \| \leq \gamma_{i2}(\cdot) \sum_{j=1}^{N} (\| z_{j1} \| + \| y_i \|) \tag{29}
$$

$$
\| \dot{y}_{id} \| \leq L_{i2} \tag{30}
$$

Substituting the above four inequalities (27)-(30) into (26), it follows

$$
\sum_{i=1}^{N} \frac{e_i^{\top}(t) \dot{e}_i(t)}{\| e_i(t) \|} < -\sum_{i=1}^{N} k_i(z_{i1}, y_i) + \sum_{i=1}^{N} \gamma_{i2}(\cdot) \sum_{j=1}^{N} \| z_{j1} \|
$$

If $k_i(z_{i1}, y_i)$ is chosen to satisfy (24), then the reachability condition (22) is satisfied.

Hence, the result follows. ■

Remark 5. Theorem 1 shows that the sliding mode (16) which is an interconnected system, is uniformly ultimately bounded. Theorem 2 shows that the designed control (23) can drive the considered system (11)–(12) to the sliding surface (15). According to the sliding mode theory, Theorems 1 and 2 show that the controlled systems (11)–(12) are uniformly ultimately bounded.

From Remark 5, it follows that the closed-loop systems formed by applying the control (24) to the systems (11)–(12) are uniformly ultimately bounded, which implies that the variables $\|z_{i1}(t)\|$ and $\|e_i(t)\|$ are bounded for $i = 1,2,\ldots,N$. Further, from $e_i(t) = y_i(t) - y_{id}(t)$ and Assumption 4 which guaratees that $y_{id}(t)$ is bounded, it is straightforward to see that the $y_i(t)$ are bounded due to $y_i(t) = e_i(t) + y_{id}(t)$, for $i = 1,2,\ldots,N$. Therefore, all the state variables of the system (4)–(5) are bounded. This shows that the designed decentralized control (24) can not only make the system outputs track the desired reference signals but also keep all the state variables of the system (4)–(5) bounded.

VI. Application to Water Quality Control

In this section, the decentralized control scheme developed in this paper will be applied to a river pollution problem [8]. The water quality of a river is mainly dependent upon the concentrations of oxygen and pollutants. In a simplified manner, this problem can be stated as the task of controlling the pollutants discharged at different places along the river in such a way that the river pollution remains within a given tolerance.

Assume that the river has two regions and each region has a sewage station. Then, the river pollution system can be described by a nonlinear interconnected system as follows (see, [17])

$$\dot{x}_1 = \underbrace{\begin{bmatrix}-1.32 & 0\\ -0.32 & -1.2\end{bmatrix}}_{A_1} x_1 + \underbrace{\begin{bmatrix}1\\0\end{bmatrix}}_{B_1}(u_1 + \Delta g_1(\cdot)) + \underbrace{\begin{bmatrix}sin(x_{21})\\0\end{bmatrix}}_{h_1} \quad y_1 = \underbrace{\begin{bmatrix}1 & 0\end{bmatrix}}_{C_1} x_1 \tag{31}$$

$$\dot{x}_2 = \underbrace{\begin{bmatrix}-1.32 & 0\\ -0.32 & -1.2\end{bmatrix}}_{A_2} x_2 + \underbrace{\begin{bmatrix}1\\0\end{bmatrix}}_{B_2}(u_2 + \Delta g_2(\cdot)) + \underbrace{\begin{bmatrix}-0.9x_{11}\\0\end{bmatrix}}_{h_2} \quad y_2 = \underbrace{\begin{bmatrix}1 & 0\end{bmatrix}}_{C_2} x_2 \tag{32}$$

where $x_1 := col(x_{11}, x_{12})$ and $x_2 := col(x_{21}, x_{22})$. The variables x_{i1} and x_{i2} for $i = 1,2.$ represent the concentration of biochemical oxygen demand (BOD) and the concentration of dissolved oxygen respectively, the controllers u_i are the BOD of the effluent discharge into the river, Δg_i represent any matched uncertainties and h_i represent interconnections respectively for $i = 1,2$. It is assumed that the concentrations of BOD for the two regions are measurable.

In this example, according to (1) the nonlinear term $f_i(x_1) = 0$, so Assumption 2 can be ignored. The matched uncertainties $\Delta g_1(\cdot)$ and $\Delta g_2(\cdot)$ are assumed to satisfy

$$\Delta g_1(\cdot) = -13.2x_{11} \quad \Delta g_2(\cdot) = \cos^2(x_{21}) \tag{33}$$

According to (31)-(32), the interconnections satisfy

$$\| h_1 \| \le 1 \cdot |x_{21}| \quad \| h_2 \| \le |0.9 \cdot x_{11}| \tag{34}$$

Combining (33)-(34), it is clear that Assumption 3 is satisfied.

Moreover, it can be verified that $rank(C_iB_i) = 1 = m_i$ for $i = 1,2$. So the Assumption 1 is satisfied.

Some suitable coordinate transformation matrices T_i are introduced as below: $(z_i = T_i x_i)$

$$T_1 = T_2 = \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}.$$

Then, the system (31)-(32) in z coordinates can be given by

$$\dot{z}_1 = \underbrace{\begin{bmatrix}-1.2 & -0.32\\ 0 & -1.32\end{bmatrix}}_{\hat{A}_1} z_1 + \underbrace{\begin{bmatrix}0\\1\end{bmatrix}}_{\hat{B}_1}(u_1 - 13.2z_{12}) + \underbrace{\begin{bmatrix}0\\ \sin(z_{22})\end{bmatrix}}_{H_1} \quad y_1 = \underbrace{\begin{bmatrix}0 & 1\end{bmatrix}}_{\hat{C}_1} z_1 \tag{35}$$

$$\dot{z}_2 = \underbrace{\begin{bmatrix}-1.2 & -0.32\\ 0 & -1.32\end{bmatrix}}_{\hat{A}_2} z_2 + \underbrace{\begin{bmatrix}0\\1\end{bmatrix}}_{\hat{B}_2}(u_2 + \cos^2(z_{22})) + \underbrace{\begin{bmatrix}0\\ -0.9z_{12}\end{bmatrix}}_{H_2} \quad y_2 = \underbrace{\begin{bmatrix}0 & 1\end{bmatrix}}_{\hat{C}_2} z_2 \tag{36}$$

and the sliding surfaces S_i are: $\dot{z}_{i1} = -1.2z_{i1} - 0.32z_{i2}$, $i = 1,2$.

For simulation purposes, the initial states are chosen as $z_1(0) = col(0,1)$ and $z_2(0) = col(0,0)$, and the desired output signals y_{id} are chosen as: $y_{1d} = 2 \cdot e^{-t}$, $y_{2d} = \sin(0.5t) + 1$.

It is clear that Assumption 4 is satisfied. Let $L_{12} = 2$, $L_{22} = 0.5$.

From (23), the proposed sliding mode controllers are as follows:

$$u_1 = -\frac{y_1 - y_{1d}}{|y_1 - y_{1d}|}(|1.32z_{12}| + |13.2z_{12}| + 3) \tag{37}$$

$$u_2 = -\frac{y_2 - y_{2d}}{|y_2 - y_{2d}|}(|1.32z_{22}| + |\cos^2(z_{22})| + 2.3) \tag{38}$$

According to (3), choose $Q_1 = Q_2 = 1$. Combining (31)-(32), $A_{i11} = -1.2$ for $i = 1,2$. Then $P_1 = P_2 = 0.416$. By direct calculation, it follows from (18) that

$$M^\top + M = \begin{bmatrix}-1.664\gamma_{11} + 2 & -0.832(\gamma_{11} + \gamma_{21})\\ -0.832(\gamma_{11} + \gamma_{21}) & -1.664\gamma_{21} + 2\end{bmatrix}$$

According to (17), (35) and (36), $\gamma_{11} = 6 \cdot \sin(z_{11})$, $\gamma_{21} = 3 \cdot \cos(z_{21}) - 2$. By direct verification, it is straightforward to check that $M^\top + M > 0$, if $|z_{11}| \le d_1 = 5.2$, $|z_{21}| \le d_2 = 3.9$.

According to (21) for this example: $\dot{V}(z_{11}, z_{21}) \le 0$, if $|z_{11}| \ge 0.3$ and $|z_{21}| \ge 0.25$. Therefore, the system (31)-(32) is uniformly ultimately bounded.

The tracking results are shown in Fig. 1. The concentration of biochemical oxygen demand (BOD) of each subsystem y_i can track the ideal reference y_{id} using the controller from (37)-(38) even in the presence of uncertainties. The time responses of the states of the system (31)-(32) are shown in Fig. 2. This shows that the system states are bounded. Simulation results demonstrate that the method developed in this paper is effective.

Fig. 1. Time responses of system outputs and desired outputs.

Fig. 2. Time responses of system state variables.

VII. CONCLUSIONS

This paper has presented a sliding mode control strategy to deal with the output tracking problem of a class of large-scale systems with unmatched unknown nonlinear interconnections. The desired reference signals are allowed to be time-varying. A decentralized sliding mode control scheme has been proposed to satisfy the reachability condition. This drives the interconnected system onto the pre-designed sliding surface. A set of conditions is developed to guarantee that the output tracking errors are uniformly ultimately bounded while all the state variables of the interconnected system are bounded. The application of the developed result to a river pollution control system has demonstrated that the proposed approach is effective and practicable.

REFERENCES

[1] H. Cai and G. Hu. Distributed tracking control of an interconnected leader–follower multiagent system. *IEEE Transactions on Automatic Control*, 62(7):3494–3501, 2017.

[2] C. Edwards and S. K. Spurgeon. *Sliding Mode Control: Theory and Applications*. Taylor and Francis Ltd, London, 1998.

[3] Z. Farzad and S. Mohammad Hossein. On event-triggered tracking for non-linear SISO systems via sliding mode control. *IMA Journal of Mathematical Control and Information*, 37(1):105–119, 2018.

[4] Y. Q. Han and H. S. Yan. Observer-based multi-dimensional taylor network decentralised adaptive tracking control of large-scale stochastic nonlinear systems. *International Journal of Control*, 93:1605–1618, 2020.

[5] H. Huerta, A. G. Loukianov, and J. M. Cañedo. Passivity sliding mode control of large-scale power systems. *IEEE Transactions on Control Systems Technology*, 27(3):1219–1227, 2019.

[6] H. S. Kim, J. B. Park, and Y. H. Joo. Decentralized sampled-data tracking control of large-scale fuzzy systems: An exact discretization approach. *IEEE Access*, 5:12668–12681, 2017.

[7] C. Liu, H. G. Zhang, G. Y. Xiao, and S. X. Sun. Integral reinforcement learning based decentralized optimal tracking control of unknown nonlinear large-scale interconnected systems with constrained-input. *Neurocomputing*, 323:1–11, 2019.

[8] J. Lunze. *Feedback Control of Large-Scale Systems*. Bookmundo Direct, 2020.

[9] P. R. Pagilla, R. V. Dwivedula, and N. B. Siraskar. A decentralized model reference adaptive controller for large-scale systems. *IEEE/ASME Transactions on Mechatronics*, 12(2):154–163, 2007.

[10] X. Ren, G. Yang, and X. Li. Global adaptive fuzzy distributed tracking control for interconnected nonlinear systems with communication constraints. *IEEE Transactions on Fuzzy Systems*, 28(2):333–345, 2020.

[11] G. Rinaldi, P. P. Menon, C. Edwards, and A. Ferrara. Variable gains decentralized super-twisting sliding mode controllers for large-scale modular systems. In *2019 18th European Control Conference (ECC)*, pages 3577–3582, 2019.

[12] J. E. Ruiz-Duarte and A. G. Loukianov. Approximate causal output tracking for linear perturbed systems via sliding mode control. *International Journal of Control 93*, page 993–1004, 2020.

[13] L. N. Tan. Distributed h optimal tracking control for strict-feedback nonlinear large-scale systems with disturbances and saturating actuators. *IEEE Transactions on Systems, Man, and Cybernetics: Systems*, 50(11):4719–4731, 2020.

[14] M. Wan and Y. Yin. Adaptive decentralized output feedback tracking control for large-scale interconnected systems with time-varying output constraints. *Mathematical Problems in Engineering*, pages 1–14, 2020.

[15] Y. Wei, Z. Zheng, Q. Li, Z. Jiang, and P. Yang. Robust tracking control of an underwater vehicle and manipulator system based on double closed-loop integral sliding mode. *International Journal of Advanced Robotic Systems 17*, 2020.

[16] X. G. Yan, C. Edwards, and S. K. Spurgeon. Decentralised robust sliding mode control for a class of nonlinear interconnected systems by static output feedback. *Automatica*, 40(4):613–620, 2004.

[17] X. G. Yan and C. Edwards S. K. Spurgeon. *Variable structure control of complex systems : analysis and design*. Cham: Springer, 2017.

[18] X. G. Yan, S. K. Spurgeon, and C. Edwards. Memoryless static output feedback sliding mode control for nonlinear systems with delayed disturbances. *IEEE Transactions on Automatic Control*, 59(7):1906–1912, 2014.

[19] D. Yao, H. Li, R. Lu, and Y. Shi. Distributed sliding-mode tracking control of second-order nonlinear multiagent systems: An event-triggered approach. *IEEE Transactions on Cybernetics*, 50(9):3892–3902, 2020.

[20] Y. Zhao, X. Sun, G. Wang, and Y. Fan. Adaptive backstepping sliding mode tracking control for underactuated unmanned surface vehicle with disturbances and input saturation. *IEEE Access*, 9:1304–1312, 2021.

[21] Guoqiang Zhu, Linlin Nie, Zhe Lv, Lingfang Sun, Xiuyu Zhang, and Chenliang Wang. Adaptive fuzzy dynamic surface sliding mode control of large-scale power systems with prescribe output tracking performance. *ISA Transactions*, 99:305–321, 2020.